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# *"On the Integrability of Demand Functions: from Antonelli to Hurwicz-Uzawa"*

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**Abstract**

In this thesis we walk a historical path along the main stages of the "integrability of demand" theory, starting with the work of G. B. Antonelli and arriving to "Preference, Utility and Demand" by Hurwicz and Uzawa. In doing this we examine the main results of this branch of Microeconomics both from a theoretical and a mathematical point of view. Furthermore we determine the relation between the utility maximization framework and the revealed preference approach on the one side and the links between this latter and "individual decision making" theory on the other side.

The theory of the "Integrability of demand" deals with the problem of recovering a utility function from a consumer demand by assuming specific properties on this latter. The idea is that, by using a process of mathematical integration, it is possible to prove the existence of a function from which the demand could be derived. This problem was first mathematically handled by G. B. Antonelli in 1886 in his "Sulla Teoria Matematica della Economia Politica" (cfr. [1]).

From that moment onwards, this theory became the object of the study of many economists as it was revealed as a fundamental part of microeconomic theory in a sense that we will specify hereinafter.

The classical approach to the "individual decision making" theory in Microeconomics uses the utility maximization framework. We consider an economy with $n$ *goods* whose prices are given by a positive real numbers vector. In this economy an agent is endowed with a certain income and he will choose his demand taking into consideration all these parameters. It is possible for the consumer to express his preference through a so-called utility function, in a way which we will explain later on. The preferred consumptions are now the solution of the maximization of the utility function subject to the budget constraint.

In the lights of this result the theory of "integrability" gains importance for two main reasons.

First, is it possible to obtain a utility function from the demand? That is, does it make sense to move in a opposite sense with respect to the process described above?

Second, it is useful to answer to the critics of the utility maximization approach. In fact ≪the utility maximization approach in the study of consumer behavior is sometimes criticized as a lot of scholars think the notion of utility is a psychological measurement and cannot be observed, and thus, they think the demand function from utility maximization is meaningless≫ (cfr. [8]).

In this thesis we will answer to both questions. In fact we will present Theorem 7.2 to analyze the first one. What about the second? The integrability result tells us that a utility function can be derived from observable data on demand although the utility function is not directly observable.

I will follow two lines in developing this sketch on the "Integrability of demand" theory. On the one hand I will go through an historical approach, while on the other I will develop some mathematical-technical aspects in such a way that the two paths complete each other.

The first section is a sketch of the "individual decision making" theory mentioned above. Some familiar results are recalled, such as the definition of utility function (cfr. Def. 1.1) and its properties or the "Utility Maximization Problem" (cfr. 1.2). In Proposition 1.8 two fundamental properties of the so called "Slutsky Matrix" are introduced (symmetry and negative semidefiniteness). Throughout this work we will refer often to these two concepts. We assume the notions presented in Section 1 to be familiar to the reader; therefore we will not dwell in this sense on this subject. Anyone interested in exploring this topic can refer to "Microeconomic Theory" by A. Mas-Colell et al. (cfr. [14]). Using the definitions and the results proposed in chapter 1, it will be better clarified how the problem of integrability appears. In fact the reader can notice that the equations in (1.4) allow us to get the utility function we are looking for as the integral of the inverse demand function, where the marginal utility of wealth represents the corresponding integrating factor. By analogy we can get the indirect utility function from the equations in (1.5) as the integral of the demand, or the expenditure function as the exact integral of the hicksian demand as showed in (1.9). These three integration processes are possible only when some specific hypoteses are satisfied. We will consider all the details during the discussion of this thesis.

Let me also highlight that the notation adopted in this first chapter will be maintained as far as possible throughout all the work.

After this mandatory introductory chapter we will present “Sulla Teoria Matematica dell’Economia Politica” by Giovan Battista Antonelli, which is nowadays considered the first fundamental contribution to the development of the “Integrability of demand” theory. G. B. Antonelli (San Miniato,1858-Cassano Spinola,1944) was student of mathematics at the Scuola Normale Superiore di Pisa and then he studied on to graduate as engineer at the Politecnico di Milano. He left us as legacy two theoretical works: the “Nota sulle relazioni indipendenti tra le coordinate di una forma fondamentale in uno spazio di quante si vogliono dimensioni e sulla forma normale di una funzione omogenea di essa” and “Sulla Teoria Matematica della Economia Politica” (1886). It is likely to believe that his work remained unknown to most of the economists for a long time, at least until some distinguished scholars such as Allen, Hicks, Georgescu-Roegen, Samuelson, Houthakker direct their studies to the “Integrability” problem. Vilfredo Pareto faced, without much success, the topic in its “Manuel d’èconomie politique” (cfr. [15]) in a way that is probably completely independent from the work of Antonelli. V. Volterra took part to the discussion on integrability of demand bringing some criticisms to the work of Pareto in his [23]. After that we have to wait 30/40 years before the integrability conditions (cfr. 2.23) presented by the Tuscan mathematician are recalled. In the second chapter of this thesis we will present in detail the theoretical-mathematical description used in the “Sulla Teoria Matematica della Economia Politica”. In this introduction we just want to stress how the author was ingenious in posing, more or less consciously, the problem of the derivation of utility from visible data.

We will fill the time gap between between Antonelli and the other "integrability problem authors" by presenting two chapters in which we describe the main results of the "Revealed Preference" theory. The idea is to build a great circle, or even better, a large triangle through which we will be able to link several fundamental results. The reader is welcome to analyze chapter 3 and 4 in this sense. The importance of the "Revealed Preference" theory will emerge along the path and will be re-emphasized in the last chapter, where the design I was talking about, will be completely clarified. No wonder if the pioneer of this branch of microeconomics was the same Samuelson mentioned above. The American economist in his "A Note on the Pure Theory of Consumer's Behavior" (cfr. [17]) introduced the "Weak Axiom of Revealed Preference" (WARP) by saying that ≪if an individual selects batch one over batch two, he does not at the same time select two over one≫ . The idea behind this definition is simple. If we suppose that an agent chooses a certain consumption $x$ over $y$ when both bundles of goods are affordable, he has somehow revealed his preference for $x$ over $y$. Once again this approach could be seen as one possible answer to the "utility maximization" framework critics in a sense which we are going to specify further on. Let us just mention that many negative comments has been made moving in the direction according to which ≪Instead of replacing "metaphysical" terms such as "desire" and "purpose," (embodied in the concept of utility) he used it to legitimate them by giving them operational definitions≫(cfr. [8]). In Proposition 3.2 we show the equivalence between the WARP and the "compensated law of demand" under usual hypotheses. This property cannot be extended to the differential case. In this sense in Proposition 3.3 we prove that WARP implies the negative semidefiniteness of the Slutsky Matrix, while through Example 3.1 the reader can immediately infer that the opposite implication is not always valid. We notice that some further assumptions would be required. In this respect we cited the work of Mas-Colell et al. (cfr. [14]) where the equivalence between the negative semidifiniteness of the Slutsky Matrix and the Weak Axiom is proved under the hypothesis of the symmetry of the same matrix. This assumption, with the exception of the case of a 2-goods economy where it is superfluous (cfr. Proposition 3.5), is revealed to be crucial in the general framework. It is important for the reader to keep this result in mind as it will be used when we will summarize all our analysis in the last chapter of the thesis. Meanwhile we want to highlight the economical meaning of the hypotheses made so far. The symmetry of the Slutsky Matrix may be seen as the equality in the change of the compensated demand of a certain good with respect to a change in price of another good on the one hand and the the change of the compensated demand of the latter with respect to a change in price of the former on the other hand. Furthermore the intuition behind the negative semidefiniteness of the Matrix is the negative effect on the compensated demand of one good with respect to its own price together with the largest weight of this effect over the effect on the demand of the same good with respect to the prices of the other goods.

At this point we will make a further jump in order to close this large parenthesis on "revealed preference". Our choice is to present the work by H. Uzawa (cfr. Ch.1 of [3]) as we retain it to be complete and exhaustive in a way we will specify hereinafter. Meanwhile we want to mention some historical-theoretical aspects that we consider fundamental to have a clear framework. When Samuelson introduced the Weak Axiom his idea was to find a way to go through the path drawn by the utility maximization theory along a parallel track. What immediately emerged is that assuming the validity of the weak axiom in consumer behavior is a condition too weak to imply utility maximization. In this respect H. S. Houthakker introduced in [10] a generalization of WARP. The Strong Axiom (SARP), in some sense, extends the rationality incorporated in the WARP to a broader meaning. In fact if on the one hand through the Weak Axiom the consumer "reveals" his choice between two available consumption bundle $x, y$ ($xRy$) on the other hand through the SARP the agent expresses the same preference by using a finite sequence of relations. Indeed we can assume the existence of some intermediate goods $x^0, x^1...x^s$ for which holds $xRx^0$, $x^0Rx^1$, ..., $x^sRy$. In this case we will say that $x$ is "indirectly" revealed preferred to $y$ ($xR^*y$). In chapter 4 we essentially create a "bridge between two worlds": on the one bank the consumer's demand, on the other one his preference. In fact Theorem 4.1 shows how the indirect revealed preference relation $R^*$ may be deduced (cfr. Definition 4.4) from a given demand function. The other way round, given a preference relation, Theorem 4.6 guarantees the existence of a derived demand function. The Strong Axiom plays a fundamental role, indeed it is a requirement in Theorem 4.1 and it is implied in Theorem 4.6. In this respect it seems inappropriate to omit a discussion on [10] by Houthakker. Our decision is related to the fact that Lemma 4.1 is completely based on the procedure used by the Dutch economist and we retain it superfluous to repeat this kind of reasoning. The other hypotheses required in Theorem 4.1 are usual assumptions on the demand. Note that we provide a proof of the theorem with weaker conditions (cfr. Note 4.2) with respect to those used by Uzawa. In the same chapter Theorem 4.4 gives an other important result in "Revealed Preference" theory. In fact it explains the extent to which WARP and SARP are equivalent. Let us recall that in [4] D. Gale provides an example where he proved that the two concepts are different. The equivalence holds only in the case of a 2-*goods-economy* as it is shown by Rose in [16].

We then return to the analysis of "the integrability of demand" by presenting the work of N. Georgerscu-Roegen (cfr. [5]). Let us just recall that the results we are going to present would not be possible without some fundamental contributions, such as those of Slutsky in [21] and Hicks & Allen in [9]. When [5] was published in 1936 it did not gain much success. In fact we can retain Georgescu's work a little bit "pioneristic". Samuelson, in some sense, revalued [5] when he mentions that he was inspired by this work in completing his [18]. The approach of the Roumanian statistician is original and deeply different from that of his predecessors. Chapter five of this thesis is an attempt to read [5] in a familiar way maintaining as much as possible the originality of the author. Georgescu makes some hypothesis on the demand of the consumer and starting from this point he tries to construct some "indifference surfaces", representing the agent's preferences. The idea behind all his work is to consider the behaviour of the consumer around a point, through what we can call a "local analysis". We present both a graphical framework (2-*goods-economy*) and a more mathematical-theoretical one (for the case of 3 or more goods). Given a point $M$ (in the plane, space, etc.), which represents the demand of the consumer, we look for the "preference/nonpreference/indifference" directions. The idea is that the consumer, when it is possible, would move on the preference directions. These movements are in some sense prevent by the constriction due to the budget plane passing through the point we are considering. By assuming some continuity/convexity hypotheses (cfr. Note 5.1, Note 5.2) it is possible to establish a connection between this budget plane (or better the infinitesimal-near point to $M$) and the preference directions (cfr. 5.16). We will construct the "indifference surfaces" by "shaping" them on the indifference directions. We can ask how this work is linked to the [1] by Antonelli. The mathematical Appendix of [5] provides a more traditional analysis of the sufficient conditions for the integrability problem. The integral of the differential form in (5.2) is exactly the solution of the system of differential equations (cfr. 2.7) the Tuscan mathematician was looking for. Despite this premise we must highlight that the two authors arrive to two different conclusions. Antonelli gives his "integrability conditions"(cfr. 2.23) in term of the symmetry of the Slutsky Matrix while Georgescu takes into consideration some assumptions on the negative semidefiniteness of the same matrix (cfr. Note 5.3). Please note that the work, where the Slutsky equation is presented (cfr. [21]), is posterior with respect to Antonelli's (cfr. [1]). We ask the reader to make a leap in time and check the connection between the equations in 2.23 and the symmetry of the Slutsky Matrix we considered above.

We want to mention that in 1946 J. Ville published "The Existence-Conditions of a Total Utility Function", which is an attempt to solve the problem of recovering a utility function from a consumer's demand. In this work the author starts by assuming some hypoteses such as homogeneity, budget exhaustion and continuity, and he proves that the integrability conditions of an expression like (6.8) is equivalent to the absence of closed contours along which this expression is constant (cfr. pp.127-128, [22]). The interested reader can deepen this analysis by analyzing the English version of [22] reviewed by K. J. Arrow in 1947.

Chapter 6 presents an analysis of [18] by Samuelson. Since we decide to consider a pure-mathematical approach we want to mention the main economic aspects behind the work in this introduction. First, let us note that we consider [18] as the most complete and conscious introduction to the "integrability problem", at least among those presented up to that moment. The author begin his work by considering the problem of integrability in the case of a 2-*goods-economy* (prices are fixed at $p_1^A, p_2^A$). In this framework we have a consumer, endowed with a positive income $M^A$, choosing a certain consumption bundle, defined as $A$. Under these hypotheses it is possible to construct the budget line, passing through $A$, as a straight line with slope equal to the price ratio $-\frac{p_1^A}{p_2^A}$ . At this point, we can repeat this construction for every point $B, C, ...$ chosen by the consumer when the income is given by $M^B, M^C, ...$ and prices are $(p_1^B, p_2^B), (p_1^C, p_2^C), ...$ Now, as Samuelson states, ≪If the observed price ratio $\frac{p_1}{p_2}$ is given as the following continuous and differentiable function of the two goods, $B(x_1, x_2)$, then mathematical analysis assures us that the differential equation $\frac{dx_2}{dx_1} = -B(x_1, x_2)$ gives rise to a unique family of curves. In two dimension there is no integrability problem... the order of consumption, in the sense of the path along which the consumer actually moves behind the scenes of the market-place, has nothing to do with the problem of integrability versus non-integrability$\gg$. This last specification wants to eliminate any possible misunderstandings regarding the two different concepts of $order\ of\ consumptions$ and *path-dependency* in the integrals (cfr. [18]). We want to stress that the basic idea behind this reasoning is that we can think of the "indifference curves" we are looking for as the "envelope" of a family of budget lines. Next step will be the case of a $3 - goods - economy$. The situation in this framework could become a little bit more tricky. We proceed in a way that is analogous to the 2-*goods-economy* case. In fact we will still be able to observe the preferred consumptions once prices and income are given. Hence it will be possible to construct the "budget planes" as the $3 - dimensions$ case of the "budget lines". Before trying to answer the question "can the indifference surfaces be thought as the envelope of a family of budget planes?" we want to clarify some specification on the nature of the demand we are considering. In fact in what we have exposed so far it is assumed that the consumer is able to decide which is his preferred consumption. In particular he should choose, for any situation, if a certain option $A$ is better/indifferent/worse than an other possible $B$ (complete preference). Furthermore we request that a transitivity condition is valid. As we already said the 3-*dimension-case* is not so obvious. In chapter 6 we present Samuelson's integrability conditions (cfr. 6.16). It will be possible to show that an envelope of a family of budget planes gives origin to these surfaces. At this point we just want to propose the intuition behind the process. Given a demand triple $A$ and the corresponding budget plane, then by using Samuelson's words $\ll$We need only indicate at $A$ a little button, or better still a little thumbtack, whose back or head lies in the budget plane and whose point tell us which is the $preferred$ direction". At this point our integrability problem would be to find if it is somehow possible to "join together" these thumbtack to construct a what we called "indifference surface"$\gg$. We invite the reader to reflect on the equivalence between the "integrability conditions" proposed by Antonelli (2.23) and those presented by Samuelson (cfr. 6.16). Despite the fact that the second ones are the result of a more complete and thoughtful reasoning the results obtained are the same in the two cases. Samuelson "integrability conditions" essentially require the symmetry of the Slutsky Matrix. Chapter 7 shows how this result is incomplete as it "bypasses" the crucial assumption of the negative semidefiniteness of the same matrix.

Chapter 6 of [3] represents the end of the dispute on the "integrability of demand" problem, as it presented an exhaustive answer to our initial questions. The framework of the analysis is the most general one, that is, we consider an *n-goods-economy* where prices are given in the positive octant and income is a positive real number. The authors assume some usual hypotheses on the consumer's demand function, such as single-valuedness, budget exhaustion and differentiability. In Theorem 7.1 the symmetry and the negative semidefiniteness of the Slutsky Matrix is deduced starting from constrained preference maximization, while in Theorem 7.2 the existence of a utility function is obtained starting from the assumptions of the the symmetry and the negative semidefiniteness of the same matrix. In particular this last theorem is the key through which we can conclude all the analysis made so far. As the mathematical framework is completely developed in chapter 7 we want to stress the main economical-theoretical implications behind this fundamental result. To this end let us consider one by one the hypotheses required. As we mentioned above we consider the usual assumptions on single-valuedness, budget exhaustion and differentiability of the demand. These hypotheses are "usual" in the sense that, if removed, we would not be able to present the problem in (7.16). Before analyzing the other assumptions we invite the reader to reflect on what kind of problem we are facing and on all the implications deriving from this fact. Indeed the integrability problem is essentially an existence problem, a fact that obliges us to move along a "constructing proof". The assumption on differentiability should be analyzed under this light. Any student familiar with calculus will recognize this hypothesis as crucial in showing the existence of a solution for the differential system in 7.16. At this point the hypotheses on the symmetry of the Slutsky Matrix and the lipschitzianity one will gain the important role of guaranteeing the unicity of the solution (cfr. Lemma 7.1, Lemma 7.2). The idea is that our "unique solution" will be used to define the utility function we are looking for. Next step will be to prove that what we have obtained may really be considered as a "utility function". Let us just note that we believe that Samuelson in [18] left this point open and this is why his work cannot be considered exhaustive. The hypothesis on the negative semidefiniteness of the Slutsky takes a fundamental role in this sense. In Lemma 7.4 before and in Lemma 7.7, Lemma 7.8 after it is shown how under this assumptions it is possible to consider the function we take as our "utility function" as well-defined. In fact we show that the value of the function introduced in (7.43) is independent from the system of prices/income considered (cfr. Lemma 7.7). What we obtained is exactly what we were looking for: a utility function from which the demand is derived. Our decision is to completely focus on this result and to provide the proof of Theorem 7.2 in this respect. We limit ourselves to provide the statement of Theorem 7.3, Theorem 7.4 and Theorem 7.5. These results regard essentially the properties that are possible to deduce for the utility function defined in Theorem 7.2.

As our problem can be considered as solved in the last chapter of this thesis we will summarize all the results obtained so far and put them all together in order to provide the reader a complete framework of analysis. We thought this chapter as "something in its own right", in such a way that it could be read as separate from all the rest. We omit all the mathematical procedures to make it "elegant", but full of intuitions.

CONTENTS:

# 1 Preliminary Results

We will briefly recall some fundamental results in microeconomic theory. We will give just a sketch as these themes should be familiar to any economics student.

First, we want to specify some points in the notation we will use.

Let us note that a preference relation is, in general, a binary relation on a set of alternatives $X$. We will denote by $xRy$ the relation $x$ is "*weakly*" preferred with respect to $y$. The expression $xPy$ will instead be used to express a "*strong*" preference for $x$ over $y$.

What about the connection between these two concepts?

Starting from $R$ we can derive two other relations:

i) The "strong" preference relation mentioned above, by defining

$$xPy \, iff \, xRy \, but \, not \, yRx;$$

ii) the "indifference" relation, $\sim$, by defining

$$x \sim y \, iff \, xRy \, and \, yRx.$$

By analogy it is possible to define the "weak" preference relation using the "strong" one. In fact it is possible to define the former as:

j) $xRy \, iff \, y\overline{P}x \, (= x\overline{P}^{-1}y),$

where $y\overline{P}x$ stands for "$y$ is not strongly preferred to $x$".

We want to specify that in all the thesis the "weak preference relation" is considered as primitive.

We will consider a 1-*agent* economy with $n$ consumption goods, whose prices are given by a vector $p \in \mathbb{R}^n_{++}$, and the consumer's income is a positive real number $M$. Some modifications will be introduced here and there when necessary.

**Definition 1.1** A function $u : X \subseteq \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ is a "utility function" representing the preference relation $R$ iff

$$xRy\, iff\, u(x) \geq u(y),\ for\, every\, x, y \in X.$$

**Proposition 1.1** Let $u(\,)$ be a "utility function" representing the preference relation $R$, then

$$R\, is\, rational, \tag{1.1}$$

where *rational* means that it satisfies:

1. Completeness
2. Transitivity

**Proof** The previous results are immediate consequences of Definition 1.1 and the properties of real numbers.

What can we say about the other way round? The so-called *Lexicographic Preference* (cfr. chapter 3, [15]) is an example of complete preference relation for which it is not possible to have a corresponding utility function.

**Definition 1.2** The preference relation $R$ is said to be continuous iff for every couple of sequences $(x^n)_{n\in\mathbb{N}}$, $(y^n)_{n\in\mathbb{N}}$ in $X$, with
$\begin{cases} x^n R y^n & for\, every\, n = 0, 1, 2, ... \\ \lim_{n\to\infty} x^n = x,\ \lim_{n\to\infty} y^n = y, & \end{cases},$
it results $xRy$.

**Proposition 1.2** Let $R$ be a rational, continuous preference relation on $X$, then:

there exists a continuous utility function $u\,(\,)$ representing the preference relation. (cfr. [13])

We will not provide the proof of Proposition 1.2.

Note that there are infinitely many utility functions for $R$ and not all continuous. In fact it is sufficient to apply a strictly increasing (not necessarly continuous) transformation to the function $u\,()$ obtained above to get a new utility function representing $R$.

Viceversa it is possible to prove the continuity of the preference relation $R$ starting from the continuity of the utility function $u\,()$.

**Definition 1.3** The preference relation $R$ on $X$ is said to be *locally nonsatiated* if for every $x \in X$ and for every positive real number $\varepsilon$ there exist $y \in X$ such that

$$\parallel y - x \parallel \leq \varepsilon \, and \, y P x.$$

The usual *Utility Maximization Problem* is stated in these terms: we consider an economy where prices are set at $p = (p_1, p_2, ..., p_n) \in \mathbb{R}^n_{++}$ and we assume the consumer has a rational, continuous and locally nonsatiated preference relation with a available income $M > 0$. Let $u\,()$ represents a utility function for preference $R$. Then the agent would face the problem

$$(UM)_{p,M} \begin{cases} \max_{x \in \mathbb{R}^n_+} u(x) \\ s.t. \, px \leq M \end{cases} . \tag{1.2}$$

We will term *demand correspondence* the correspondence that assigns to each couple $(p, M) \in \mathbb{R}^n_{++} \times \mathbb{R}^+$ the set of the solutions of (1.2) and we will denote it as $x\,(p, M)$.

We can use Kuhn-Tucker conditions to solve the systems in (1.2). In particular when we have an interior optimum ($x^* \in \mathbb{R}^n_{++}$) it must be

$$\nabla u(x^*) = \lambda p, \tag{1.3}$$

where $\lambda$ is a so-called Lagrange multiplier.

We can restate (1.3) as:

$$\frac{\frac{\delta u(x(p,w))}{\delta p_1}}{p_1} = ... = \frac{\frac{\delta u(x(p,w))}{\delta p_n}}{p_n} = \lambda = \mu(p,w), \tag{1.4}$$

where we named $\mu(p,w)$ the "marginal utility of wealth".

**Proposition 1.3** Let $x(p,M)$ be the demand correspondence for system (1.2), wher $u(\,)$ is assumed to represent a locally nonsatiated preference relation. Then the following properties hold for every $(p,M) \in \mathbb{R}^n_{++} \times \mathbb{R}^+$:

h) $x(\,)$ is homogeneous of degree zero with respect to $(p,M)$;

hh) $px = M$, for every $x \in x(p,M)$ .

**Definition 1.4** Under the hypotheses made so far for the definition of the demand correspondence we can define the *indirect utility function* $v : \mathbb{R}^n_{++} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ as $v(p,M) = u(x),\ with\, x \in x(p,M)$.

When we differentiate the indirect utility function with respect to one price $p_i$ $(i \in \{1,2,...,n\})$, after some algebraic manipulation, we get:

$$\frac{\delta v(p,w)}{\delta p_i} = -\mu(p,w)x_i(p,w),\ for\, all\, i \in \{1,2,...,n\}\,. \tag{1.5}$$

By analogy when we differentiate the indirect utility function with respect to wealth we have:

$$\frac{\delta v(p,w)}{\delta w} = \mu(p,w) \tag{1.6}$$

In a way completely similar to what we did for (1.2) we can define the *expenditure minimization problem* as

$$(EM)_{p,\bar{u}} \begin{cases} \min_{x \in \mathbb{R}^n_+} px \\ s.t.\, u(x) \geq \bar{u} \end{cases}, \tag{1.7}$$

where $\overline{u}$ is exogenously given.

We will term *hicksian correspondence* he correspondence that assigns to each couple $(p, \overline{u}) \in \mathbb{R}^n_{++} \times \mathbb{R}$ the set of the solutions of (1.3) and we will denote it as $h(p, \overline{u})$.

**Definition 1.5** We will define the *expenditure function* $e : \mathbb{R}^n_{++} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ as $e(p, \overline{u}) = px'$, with $x' \in h(p, \overline{u})$.

**Proposition 1.4** Let $h()$ be the function defined above. Then the following properties hold:

g) $h()$ is homogeneous of degree zero in $p$;

gg) $u(x) = \overline{u}$, for every $x \in h(p, \overline{u})$ .

**Proposition 1.5** Let $e()$ be the function defined above. Then the following properties hold:

k) $e()$ is homogeneous of degree one in $(p, \overline{u})$;

kk) $e()$ is increasing in $\overline{u}$ and nondecreasing in $p_i$, $i \in \{1, ..., n\}$;

kkk) $e()$ is concave in $p$;

kkkk) $e()$ is continuous in $p, u$.

In constrained optimization, it is often possible to convert the primal problem (i.e. the original form of the optimization problem, as the one presented in UM) to a dual form, which is termed the dual problem (EM). The following theorem is a "duality theorem" in this sense.

**Theorem 1.1** Suppose that $u()$ is a continuous utility function representing a locally nonsatiated preference relation $R$ defined on the consumption set $X = \mathbb{R}^n_+$ and the price vector is $p \in \mathbb{R}^n_{++}$. We have:

j) if $x^*$ is optimal in the UM when wealth is $w > 0$, then $x^*$ is optimal in the EM when the required utility level is $u(x^*)$. Moreover, the minimized expenditure level in this EM is exactly $w$;

jj) if $x^*$ is optimal in the EM when the required utility level is $\overline{u} > u(0)$, then $x^*$ is optimal in the UM when wealth is $px^*$. Moreover, the maximized utility level in this UM is exactly $\overline{u}$.

Using Theorem 1.1, we can relate the hicksian and the walrasian demand correspondences as follows:

$$h(p, \overline{u}) = x(p, e(p, \overline{u}))\ and\ x(p, w) = h(p, v(p, w)) \tag{1.8}$$

Using the first equality in 1.8 and the properties of the expenditure function it is possible to get:

$$\frac{\delta e(p, \overline{u})}{\delta p_i} = h(p, \overline{u}),\ for\ every\ i \in \{1, 2, ..., n\}\,. \tag{1.9}$$

**Definition 1.6.I** A preference relation $R$ on $X$ is said to be *convex* if for every $x, y, z \in X$, with $yPx$, $zPx$, it is

$$[\alpha y + (1-\alpha)z]Px,\ for\ any\ 0 \leq \alpha \leq 1.$$

**Definition 1.6.II** A preference relation $R$ on $X$ is said to be *strictly convex* if for every $x, y, z \in X$, with $yRx$, $zRx$, $y \neq z$ it is

$$[\alpha y + (1-\alpha)z]Px,\ for\ any\ 0 < \alpha < 1.$$

**Note 1.1** The hypothesis of strict convexity is fundamental in the sense that it allows us to handle single-valued functions instead of multi-valued functions. In fact when strict convexity holds we have that the problems

(1.2) and (1.3) have only one solution and we can consider $x(p, M)$ and $h(p, \overline{u})$ as functions in the usual sense.

**Proposition 1.6** Let's suppose that $u\,()$ is a continuous utility function representing a rational, locally nonsatiated, strictly convex preference relation $R$. Let $h\,(\,,\overline{u}) = (h_1\,(\,,\overline{u})\,, h_2\,(\,,\overline{u})\,, ..., h_n\,(\,,\overline{u}))$ be continuously differentiable in $(p, \overline{u})$. Then

i) $\qquad D_p h(p, \overline{u}) = D^2_{pp} e(p, \overline{u});$

ii) $\qquad D_p h(p, \overline{u})$ is negative semidefinite;

iii) $\qquad D_p h(p, \overline{u})$ is symmetric;

iv) $\qquad D_p h(p, \overline{u}) p = 0,$

where $D_p h(p, \overline{u}) = \left\| \begin{array}{cccc} \frac{dh_1}{dp_1} & \frac{dh_1}{dp_2} & .... & \frac{dh_1}{dp_n} \\ \frac{dh_2}{dp_1} & \frac{dh_2}{dp_2} & ... & \frac{dh_2}{dp_n} \\ ... & & ... & \\ \frac{dh_n}{dp_1} & & & \frac{dh_n}{dp_n} \end{array} \right\|.$

**Proposition 1.7** Let us suppose that $u\,()$ is a continuous utility function representing a regular, locally nonsatiated, strictly convex preference relation $R$. Then for all $(p, M)$ and $\overline{u} = v(p, M)$ it is

$$S_{ij}\,(p, M) = \frac{dh_i(p, \overline{u})}{dp_j} = \frac{\delta x_i(p, M)}{\delta p_j} + \frac{\delta x_i(p, M)}{\delta M} x_j(p, M),$$

$$for\, every\, i, j \in \{1, 2, ..., n\}\,.\ (Slutsky\, Equation) \tag{1.10}$$

**Proposition 1.8** Let's suppose that $u\,()$ is a continuous utility function representing a regular, locally nonsatiated, strictly convex preference relation $R$. Then the Slutsky Matrix $S(p, M) = [S_{ij}\,(p, M)]_{i,j=1,...,n}$ satisfies

j) Symmetry;

jj) Negative Semidefiniteness;

jjj) $S(p, M)\, p = 0$.

It is intuitive to deduce Proposition 1.8 as a direct consequence of Proposition 1.6 and 1.7.

We will provide a Proof for a proposition equivalent to Proposition 1.8 as presented by Uzawa and Hurwicz in [3].

## 2 "Sulla Teoria Matematica della Economia Politica", G.B. Antonelli

The problem of the "Integrability of demand arguments" starts with consumer demand functions having some properties that would be implied by constrained utility maximization were they generated from that source. Using a process of mathematical integration, the arguments then proceed to demonstrate the existence of utility functions from which those demand functions could be derived. This problem concerns whether one can "recover" a preference ordering that generates the given demand function. In 1886, G. B. Antonelli deals with the problem from a mathematical point of view.

Antonelli opened his *Sulla Teoria Matematica dell'Economia politica* (cfr. [1]) saying "Molti economisti hanno sostenuto e sostengono essere impossibile di trattare i problemi della Economia politica per mezzo dell'Analisi matematica. Malgrado la loro asserzione *a priori* molti e svariatissimi tentativi di questo genere furono pubblicati fino da molti anni;...Queste riflessioni generali mostrano che non è vano il tentare una Teoria matematica dell'Economia Politica, anche se in un primo studio si debbono supporre condizioni ed ipotesi in parte più semplici, o non conformi completamente alla realtà. Nel caso particolare è poi bene riflettere che a questi studi si riconnettono dei problemi per la cui risoluzione nessun metodo scientifico si possiede, e relativamente ai quali dei risultati anche approssimativi sarebbero di grande vantaggio." As sustained by J. A. Schumpeter in [19] Antonelli's work represents "a remarkable performance that seems to anticipate later work in some important point".

We want to present the main contents of the *Sulla Teoria Matematica dell'Economia politica* based on the commentary of G. Demaria and G. Ricci (cfr. [1]), authors of the notes of the reprinted edition of Antonelli's work.

We will focus our attention on the particular case of a 3 goods economy trying to generalize the main conclusions to the more complete framework of an *n-goods-economy*. When we consider 3 goods we face a market with a triple of prices (each one referred to a single good) and a wealth level. The two parameters we use will be the prices ratios. For each couple of these values, given the initial agent endowment, we have a preferred triple of consumed goods. The idea is to pass from these choices of quantities to a map of indifference surface. In order to make this step the so called *integrability conditions* must be satisfied. As in the commentary of G. Ricci we split Antonelli Postulate into two different assumptions (cfr. P.1 and P.2). The idea is that they are valid in different situations and they bear different implications. When P.1 is verified we can write the *integrability conditions* but only when both P.1 and P.2 are valid it is possible to construct the map of $U$.

**THE MODEL**

Let us consider a market with $n = 3$ commodities. The general case will be an extension of this particular situation.

We denote with $a_1$, $a_2$, $a_3$ the quantity of the 3 goods. Prices are assumed to be positive and equal to $p_1$, $p_2$, $p_3$. The consumer has an initial endowment $\overline{a}_1, \overline{a}_2, \overline{a}_3$ implying that he is to face a budget constraint $\lambda = p_1\overline{a}_1+p_2\overline{a}_2+p_3\overline{a}_3$.

Let us compute the ratio between prices $q_1 = \frac{p_1}{p_1} = 1$, $q_2 = \frac{p_2}{p_1}$, $q_3 = \frac{p_3}{p_1}$. The agent will have a net demand/supply (depending on the sign) function $C_1$, $C_2$, $C_3$, each corresponding to one of the three commodities. The condition expressed by the budget constraint will be reflected in the demand/supply function as

$$p_1C_1 + p_2C_2 + p_3C_3 = 0; \tag{2.1}$$

the preferred consumption vector will be

$$A_1 = \overline{a}_1 + C_1,\ A_2 = \overline{a}_2 + C_2,\ A_3 = \overline{a}_3 + C_3. \tag{2.2}$$

Hence, given $\lambda$ and vector $p$ the individual can choose between all the triple $(a_1, a_2, a_3)$ satisfying

$$p_1 a_1 + p_2 a_2 + p_3 a_3 = \lambda. \tag{2.3}$$

Let's introduce Antonelli's first Postulate

P.1 for each $\lambda$, $(p_1, p_2, p_3)$ the consumer chooses the three quantities $(a_1, a_2, a_3)$ in a unique way in order to get the preferred triple.

We want to present a geometric interpretation of the assumptions made so far.

We can think of the mathematical formula (2.3) as the equation of a plane (the generalization to hyperplanes in the case of more than 3 commodities will be straightforward) whose direction cosines are represented by the price vector $p$ and $\lambda$ gives us a measure of the distance from the origin.

If the prices are fixed we get the optimal choice of the consumer by varying the level of $\lambda$ and using P.1 we can describe a curve $\Gamma$ with equation

$$A_1 = A_1(\lambda, \overline{p}_1, \overline{p}_2, \overline{p}_3),\ A_2 = A_2(\lambda, \overline{p}_1, \overline{p}_2, \overline{p}_3),\ A_3 = A_3(\lambda, \overline{p}_1, \overline{p}_2, \overline{p}_3) \tag{2.4}$$

What the author assumed is essentially that:

- consumer choices are homogeneous of degree zero with respect to (prices, income);
- whatever initial situation $(\overline{a}_1, \overline{a}_2, \overline{a}_3)$, such that $\sum_{i=1}^{n} p_i \overline{a}_i = \lambda$, given prices, the consumer chooses the same preferred triple $(A_1, A_2, A_3)$.

Hence, using (2.2) and (2.3) we have

$$\begin{cases} C_i = -a_i + A_i(\lambda, p_1, p_2, p_3) \quad i = 1, 2, 3; \\ \sum_{i=1}^{3} p_i A_i = \sum_{i=1}^{3} p_i a_i = \lambda \end{cases} \tag{2.5}$$

where $(a_1, a_2, a_3)$, as specified in the second equality are chosen as "initial points".

By differentiating (2.3) and (2.4) with respect to $a_i$ $(i = 1, 2, 3)$ we get

$$\frac{\delta\lambda}{\delta a_i} = p_i, \ \frac{\delta A_k}{\delta a_i} = \frac{\delta A_k}{\delta\lambda}\frac{\delta\lambda}{\delta a_i} = p_i\frac{\delta A_k}{\delta\lambda}, \ k = 1, 2, 3$$

Hence from (2.5)

$$\frac{\delta C_i}{\delta a_i} = -1 + \frac{\delta A_i}{\delta\lambda}p_i, \ \frac{\delta C_i}{\delta a_j} = \frac{\delta A_i}{\delta\lambda}p_j, \ i, j = 1, 2, 3 \, i \neq j,$$

and solving for $\frac{\delta A_i}{\delta\lambda}$ we get $\frac{(1+\frac{\delta C_i}{\delta a_i})}{p_i} = \frac{\frac{\delta C_i}{\delta a_j}}{p_j}$, $i, j = 1, 2, 3 \, i \neq j$, which is equivalent to the following system of differential equations

$$\begin{cases} \frac{(1+\frac{\delta C_1}{\delta a_1})}{p_1} = \frac{\frac{\delta C_1}{\delta a_2}}{p_2} = \frac{\frac{\delta C_1}{\delta a_3}}{p_3} \\ \frac{\frac{\delta C_2}{\delta a_1}}{p_1} = \frac{(1+\frac{\delta C_2}{\delta a_2})}{p_2} = \frac{\frac{\delta C_2}{\delta a_3}}{p_3} \\ \frac{\frac{\delta C_3}{\delta a_1}}{p_1} = \frac{\frac{\delta C_3}{\delta a_2}}{p_2} = \frac{(1+\frac{\delta C_3}{\delta a_3})}{p_3} \end{cases} \tag{2.6}$$

which in case of $n$ commodity bundles is

$$\begin{cases} \frac{(1+\frac{\delta C_1}{\delta a_1})}{p_1} = \frac{\frac{\delta C_1}{\delta a_2}}{p_2} = \dots\dots\dots = \frac{\frac{\delta C_1}{\delta a_n}}{p_n} \\ \frac{\frac{\delta C_2}{\delta a_1}}{p_1} = \frac{(1+\frac{\delta C_2}{\delta a_2})}{p_2} = \dots\dots\dots = \frac{\frac{\delta C_2}{\delta a_n}}{p_n} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{\frac{\delta C_n}{\delta a_1}}{p_1} = \frac{\frac{\delta C_n}{\delta a_2}}{p_2} = \dots\dots\dots = \frac{(1+\frac{\delta C_3}{\delta a_3})}{p_3} \end{cases} \tag{2.7}$$

The system (2.6) with the equation in (2.1) guarantees sufficient condition which $C_1$, , $C_2$, $C_3$ must satisfy.

When we make prices vary, considering P.1, we get a curve $\Gamma(p_1, p_2, p_3)$ given by the equations in (2.4).

Without any loss of generality, we can pass from considering $(p_1, p_2, p_3)$ as independent variable to $(q_1, q_2, q_3) = (1, q_2, q_3)$. Hence, from (2.4) we get a new system of equations

$$A_1 = A_1(\lambda, q_2, q_3),\ A_2 = A_2(\lambda, q_2, q_3),\ A_3 = A_3(\lambda, q_2, q_3). \tag{2.8}$$

Let us now consider the Jacobian matrix

$$J_{\lambda,q_2,q_3}(A_1, A_2, A_3) = \begin{Vmatrix} \frac{\delta A_1}{\delta \lambda} & \frac{\delta A_2}{\delta \lambda} & \frac{\delta A_3}{\delta \lambda} \\ \frac{\delta A_1}{\delta q_2} & \frac{\delta A_2}{\delta q_2} & \frac{\delta A_3}{\delta q_2} \\ \frac{\delta A_1}{\delta q_3} & \frac{\delta A_2}{\delta q_3} & \frac{\delta A_2}{\delta q_3} \end{Vmatrix}, \tag{2.9}$$

and the partial Jacobian matrix

$$J_{q_2,q_3}(A_1, A_2, A_3) = \begin{Vmatrix} \frac{\delta A_1}{\delta q_2} & \frac{\delta A_2}{\delta q_2} & \frac{\delta A_3}{\delta q_2} \\ \frac{\delta A_1}{\delta q_3} & \frac{\delta A_2}{\delta q_3} & \frac{\delta A_2}{\delta q_3} \end{Vmatrix}. \tag{2.10}$$

When we consider the case of $n$ consumption goods we have

$$J_{\lambda,q_2,q_3,..,q_n}(A_1, A_2, ..., A_n) = \begin{Vmatrix} \frac{\delta A_1}{\delta \lambda} & \frac{\delta A_2}{\delta \lambda} & \cdots & \frac{\delta A_n}{\delta \lambda} \\ \frac{\delta A_1}{\delta q_2} & \frac{\delta A_2}{\delta q_2} & \cdots & \frac{\delta A_n}{\delta q_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\delta A_1}{\delta q_n} & \frac{\delta A_2}{\delta q_n} & \cdots & \frac{\delta A_n}{\delta q_n} \end{Vmatrix}. \tag{2.11}$$

When the matrix (2.10) has rank=2 or, by analogy, (2.11) is full rank it is possible to apply the implicit function theorem to the system of equation (2.8), so that there exist two functions $Q_2\,()$ and $Q_3()$ such that

$$q_2 = Q_2(A_1, A_2, A_3),\ q_3 = Q_3(A_1, A_2, A_3), \tag{2.12}$$

and in the general case

$$\begin{cases} q_2 = Q_2(A_1, A_2, ..., A_n) \\ q_3 = Q_3(A_1, A_2, ..., A_n) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ q_n = Q_n(A_1, A_2, ..., A_n) \end{cases} . \tag{2.13}$$

We now consider the second Antonelli Postulate:

P.2 There exists a function $U(\,)$ of the quantity $(a_1, a_2, a_3)$ that, given the price vector $p$ and the level $\lambda$, assumes the maximum value on the point $(A_1, A_2, A_3)$, that is the preferred triple.

We will proceed as following:

we maximize the utility function on the plane characterized by $(p_1, p_2, p_3)$ and $\lambda$ and then we impose that the value that maximizes the utility function is exactly the triple $(A_1, A_2, A_3)$.

Let use the usual Lagrangian multipliers method.

$$\mathcal{L}(\mu, A_1, A_2, A_3) = U(A_1, A_2, A_3) - \mu(p_1 A_1 + p_2 A_2 + p_3 A_3 - \lambda)$$

F.O.C.s read

$$\frac{\delta \mathcal{L}}{\delta A_1} = \frac{\delta \mathcal{L}}{\delta A_2} = \frac{\delta \mathcal{L}}{\delta A_3} = \frac{\delta \mathcal{L}}{\delta \mu} = 0,$$

that is

$$\frac{\delta U}{\delta A_1} - \mu p_1 = 0, \; \frac{\delta U}{\delta A_2} - \mu p_2 = 0, \; \frac{\delta U}{\delta A_3} - \mu p_3 = 0,$$

equivalent to the following conditions:

$$\frac{\delta U}{\delta A_1} = \mu p_1, \; \frac{\delta U}{\delta A_2} = \mu p_2, \; \frac{\delta U}{\delta A_3} = \mu p_3. \tag{2.14}$$

Rearranging (2.14) we have:

$$\frac{\frac{\delta U}{\delta A_1}}{\frac{\delta U}{\delta A_2}} = \frac{p_1}{p_2} = \frac{1}{q_2}, \ \frac{\frac{\delta U}{\delta A_1}}{\frac{\delta U}{\delta A_3}} = \frac{p_1}{p_3} = \frac{1}{q_3}. \tag{2.15}$$

Since we know that the maximum is realized when equations (2.12) are satisfied, we get

$$\frac{\delta U}{\delta A_1} = \frac{1}{Q_2}\frac{\delta U}{\delta A_2}, \ \frac{\delta U}{\delta A_1} = \frac{1}{Q_3}\frac{\delta U}{\delta A_3}. \tag{2.16}$$

**Note 2.1** Equations (2.16) can be stated in the general case ($n$ commodities) as

$$\frac{\delta U}{\delta A_1} = \frac{1}{Q_2}\frac{\delta U}{\delta A_2}, \ \frac{\delta U}{\delta A_1} = \frac{1}{Q_3}\frac{\delta U}{\delta A_3}, \ \dots, \ \frac{\delta U}{\delta A_1} = \frac{1}{Q_n}\frac{\delta U}{\delta A_n}. \tag{2.17}$$

Hence when the function $U$ exists conditions (2.16) must be satisfied.

We want to find some kind of *integrability conditions* for the equations in (2.16). Let's assume that both $Q_2(\,)$ and $Q_3(\,)$ admit first order continuous partial derivatives; then differentiating both sides of the two equations with respect to $A_1$we get the system:

$$\begin{cases} \frac{\delta^2 U}{\delta A_1 \delta A_2} = Q_2 \frac{\delta^2 U}{\delta A_1^2} + \frac{\delta Q_2}{\delta A_1}\frac{\delta U}{\delta A_1} \\ \frac{\delta^2 U}{\delta A_3 \delta A_1} = Q_3 \frac{\delta^2 U}{\delta A_1^2} + \frac{\delta Q_3}{\delta A_1}\frac{\delta U}{\delta A_1} \end{cases}. \tag{2.18}$$

Now, taking the derivative of the first equation in (2.16) with respect to $A_3$ and of the second one with respect to $A_2$ we have:

$$\begin{cases} \frac{\delta^2 U}{\delta A_2 \delta A_3} = Q_2 \frac{\delta^2 U}{\delta A_1 \delta A_3} + \frac{\delta Q_2}{\delta A_3}\frac{\delta U}{\delta A_1} \\ \frac{\delta^2 U}{\delta A_3 \delta A_2} = Q_3 \frac{\delta^2 U}{\delta A_1 \delta A_2} + \frac{\delta Q_3}{\delta A_2}\frac{\delta U}{\delta A_1} \end{cases}. \tag{2.19}$$

When we assume the utility function $U(\,)$ have continuous mixed derivatives on its domain (cfr. Schwarz Theorem, [20]) we have from (2.19):

$$Q_2\frac{\delta^2 U}{\delta A_1\delta A_3}+\frac{\delta Q_2}{\delta A_3}\frac{\delta U}{\delta A_1}=\frac{\delta^2 U}{\delta A_2\delta A_3}=$$
$$=\frac{\delta^2 U}{\delta A_3\delta A_2}=Q_3\frac{\delta^2 U}{\delta A_1\delta A_2}+\frac{\delta Q_3}{\delta A_2}\frac{\delta U}{\delta A_1},$$

implying

$$Q_2\frac{\delta^2 U}{\delta A_1\delta A_3}-Q_3\frac{\delta^2 U}{\delta A_1\delta A_2}+\frac{\delta U}{\delta A_1}(\frac{\delta Q_2}{\delta A_3}-\frac{\delta Q_3}{\delta A_2})=0. \tag{2.20}$$

Substituting the equations in (2.18) into (2.20) we have (under Schwartz Th. hypoteses):

$$Q_2(Q_3\frac{\delta^2 U}{\delta A_1^2}+\frac{\delta Q_3}{\delta A_1}\frac{\delta U}{\delta A_1})-Q_3(Q_2\frac{\delta^2 U}{\delta A_1^2}+\frac{\delta Q_2}{\delta A_1}\frac{\delta U}{\delta A_1})+\frac{\delta U}{\delta A_1}(\frac{\delta Q_2}{\delta A_3}-\frac{\delta Q_3}{\delta A_2})=0,$$

from which we get

$$\frac{\delta U}{\delta A_1}(\frac{\delta Q_3}{\delta A_1}Q_2-\frac{\delta Q_2}{\delta A_1}Q_3+\frac{\delta Q_2}{\delta A_3}-\frac{\delta Q_3}{\delta A_2})=0. \tag{2.21}$$

If we exclude the trivial case $\frac{\delta U}{\delta A_1}=0$ equation (2.21) will give the *integrability condition*

$$\frac{\delta Q_3}{\delta A_1}Q_2-\frac{\delta Q_2}{\delta A_1}Q_3+\frac{\delta Q_2}{\delta A_3}-\frac{\delta Q_3}{\delta A_2}=0. \tag{2.22}$$

**Note 2.2** It is possible to generalize equation (2.22) for the case of $n$ commodities bundles. We have to extend the equation obtained for the indices 2 and 3 to all pairs $(k,l)$ ranging in $(2,3,...,n)$ getting the following system of equations:

$$\frac{\delta Q_l}{\delta A_1}Q_k-\frac{\delta Q_k}{\delta A_1}Q_l+\frac{\delta Q_k}{\delta A_l}-\frac{\delta Q_l}{\delta A_k}=0,\ k,l=2,3,...,n \tag{2.23}$$

Equations (2.23) represent the $\begin{pmatrix} n-1 \\ 2 \end{pmatrix}$ *integrability conditions* we were looking for.

Note that these conditions are only necessary. The reader will have a more complete framework when Theorem 7.2 will be presented.

# 3 "Revealed Preference" theory

In studying consumer behavior we usually start from a model where we have a *preference-based-approach*. The agent is assumed to have a certain mental structure (preferences) that allows him to make choices. Preferences, in this case, are not observable. Starting from making some assumptions on this unknown preference structure we try to predict the consumer demand, that is of course observable. In this context we want to present the opposite process, that is, we will focus on the available data we have.

The main idea is that the consumer, in some sense, grants his explicit preference to a choice inside the set of all alternatives. Through this choice he provides critical information about his tastes. In this respect we speak of *Revealed Preferences*.

Samuelson in [17] introduced the the Weak Axiom of Revealed Preference (WARP) as follows:

**Definition 3.1** WARP holds $iff$ for every price vector $p^0, p^1$, income $M^0, M^1$ and single-valued demanded consumption bundles $x^0 \neq x^1$ satisfying $p^0 x^1 \leq p^0 x^0$ it is

$$p^1 x^0 > p^1 x^1 \tag{3.1}$$

This kind of assumption guarantees that the consumer is in some sense coherent. In fact when WARP is valid the agent grants his preference to a certain consumption bundle over an other one whenever both are available. This property eliminates the possibility that the consumer chooses a certain bundle $x^0$ over $x^1$ in one situation and then $x^1$ over $x^0$ in a second period.

Let us now suppose that WARP holds. We want to investigate the consequences we can obtain from this assumption.

We will denote by the vector $x(p, M) = (x_1(p, M), x_2(p, M), ..., x_n(p, M))$ consumer's choice of good $1, 2, ..., n$, when prices are given by $p = (p_1, p_2, ..., p_n)$

and the available income is $M$. It is obvious to see that if we assume that the agent use all his income and WARP is valid the function $x(p, M)$ is homogeneous of degree 0 in prices. In fact let us consider the following:

**Proposition 3.1** Let us be given a price vector $p = (p_1, p_2, ..., p_n)$ and a positive income$M$. If

i) all income is spent;

ii) (3.1) is verified;

then for each $i = 1, 2, ..., n$ $x_i(p, M) = x_i(tp, tM)$, *for every positive* $t$.

**Proof** Let's suppose that $x^0$ is chosen when the system of prices is $p^0$ and income is given by $M^0$,

$$x^0 = x^0(p^0, M^0), \tag{3.2}$$

while $x^1$ is chosen at $p^1 = tp^0$, $M^1 = tp^0$, where $t$ is a positive real number,

$$x^1 = x^1(p^1, M^1) = x^1(tp^0, tM^0). \tag{3.3}$$

Using i)

$$tp^0x^1 = p^1x^1 = M^1 = tM^0 = tp^0x^0 = p^1x^0. \tag{3.4}$$

Hence, from (4):

$$\begin{cases} p^0x^1 = p^0x^0 \\ p^1x^1 = p^1x^0 \end{cases} . \tag{3.5}$$

Ab absurdo, let's assume that $x^0$ and $x^1$ are different consumption bundles. Using ii) in the first equation of (3.5) we have

$$p^1x^0 > p^1x^1,$$

that is in contradiction with the second equation in (3.5). Hence, it is $x^0 = x^1$, that is, the function is homogenous of degree 0.

$\square$

We want now to show a fundamental property deriving from WARP: the substitution effect of own price changes cannot be positive. In this sense we will show that the substitution matrix is negative semidefinite. We proceed through two main steps. At first we work with a finite variation of prices and demand, then we will deal with the differential case. Let us recall the following definitions.

**Definition 3.2** Let $A \in \mathbb{C}^{nxn}$ be a (symmetric) quadratic matrix, $A$ is said to be $negative\ (semi)definite\ iff$

$$x^* A x <_{(\leq)} 0,\ for\, every\, vector\, x \in \mathbb{C}^n,\ x \neq 0, \tag{3.6}$$

where $x^*$ is the conjugate transpose of $x$.

**Note 3.1** the definition of a $negative\ (semi)definite$ matrix is usually given for symmetric matrix. In our case we would not make this kind of assumption when it is not required.

When the consumer faces a variation of prices he will generally change his choices due to budget constraint problems.

We will define a *compensated price change* as following

**Definition 3.3** A couple $(p^1, M^1)$ is said to be a *compensated price change* from $(p, M)$ if

$$p^1 x(p, M) = M^1. \tag{3.7}$$

We will say $(p^1, M^1) \in \Phi(p, M)\ iff\ (p^1, M^1)$ is a *compensated price change* from $(p, M)$.

The idea is that at the new system of prices and income the consumer is allowed to pursue (by spending all his income) the same bundle he chose at the initial situation.

Let us show the negative semidefinitness of the substitution matrix through the following proposition

**Proposition 3.2** Given a price vector $p = (p_1, p_2, ..., p_n)$ and a positive income $M$. Let

j) $x(p, M)$ be homogeneous of degree 0;

jj) all income be spent;

then it is

$$WARP\, is\, valid\, iff$$

$$for\, any\, (p^1, M^1) \in \Phi(p, M)\, it\, is \tag{3.8}$$

$$(p^1 - p)(x^1 - x) \leq 0,\, with\, <\, for\, x^1 \neq x,$$

where $x^1 = x(p^1, M^1)$, $x = x(p, M)$

**Proof** Let us prove the two implications.

- “The only if part”

When $x = x^1$ we have $(p^1 - p)(x^1 - x) = 0$ and the result is obvious. Hence, let us suppose $x \neq x^1$. We can rewrite the expression in (3.8) as $p^1(x^1 - x) - p(x^1 - x)$. Using jj) and the definition of *compensated price change* it is

$$p^1(x^1 - x) - p(x^1 - x) = M^1 - M^1 - p(x^1 - x) = -p(x^1 - x). \tag{3.9}$$

By definition (3.7) we have that the consumer can choose $x$ at the price, income couple $(p^1, M^1)$, so that it is possible to use WARP to obtain

$$px^1 > M. \tag{3.10}$$

From (3.9), using again jj), we have:

$$-p(x^1 - x) = px - px^1 = M - px^1 < 0. \tag{3.11}$$

(3.11) gives the proof of the "only if part" of the proposition.

- "The if part"

Let us first prove that WARP holds when it is considered the "compensated price change".

In order to show that WARP is valid let assume

$$px^1 \leq px, \tag{3.12}$$

that is $px^1 - px \leq 0$, equivalent to $p(x^1 - x) \leq 0$. Using the hypothesis in (3.8) we have $(p^1 - p)(x^1 - x) < 0$, implying

$$p^1(x^1 - x) < p(x^1 - x) \leq 0. \tag{3.13}$$

Hence

$$p^1x^1 < p^1x, \; i.e. \, WARP \, holds. \tag{3.14}$$

We will proceed using an ab absurdo proof. By supposing that WARP is violated we will construct a specific compensated price change which makes (3.8) fail.

Let's consider two pairs $(p^1.M^1)$, $(p^2, M^2)$ which do not satisfy WARP. Then let us consider $x^1 = x\,(p^1.M^1)$, $x^2 = x\,(p^2, M^2)$. As WARP is not valid it is $p^1x^2 \leq p^1x^1$, $p^2x^1 \leq p^2x^2$, or by using jj)

$$p^1x^2 \leq M^1,\, p^2x^1 \leq M^2. \tag{3.15}$$

When one of the inequalities in (3.15) is satisfied as equality we fall once again in the previous case ( the compensated price change one).

Hence it must be

$$p^1x^2 < M^1,\, p^2x^1 < M^2. \tag{3.16}$$

Let's choose a convex combination of the prices $p^1, p^2$ $p = \alpha p^1 + (1-\alpha)p^2$ in such a way that it results

$$px^1 = px^2, \tag{3.17}$$

and let $M = px^1 = px^2$.

Using (3.16), jj) and the definition of $p$ we have

$$\alpha M^1 + (1-\alpha)M^2 > \alpha p^1x^1 + (1-\alpha)p^2x^1 = M =$$

$$= px(p,M) = \alpha p^1x(p,M) + (1-\alpha)p^2x(p,M). \tag{3.18}$$

Two possible cases arise:

1. $p^1x(p,M) < M^1$ or

2. $p^2x(p,M) < M^2$.

Let's consider 1. We have that, from (3.17), $px^1 = M$ and $p^1x(p,M) \leq M^1$. We immediately see that we fall once again in the case of the violation of WARP for the compensated price.

When we analyze case 2. we get the same result. Hence we get the "if part".

□

We can restate the expression $(p^1 - p)(x^1 - x) \leq 0$, (for every compensated price changes) by defining the *compensated law of demand* :

$$\triangle p \triangle x \leq 0. \tag{3.19}$$

What we have seen so far is the equivalence between WARP and the compensated law of demand.

The next step will be consider the generalization of this result to the "differential case".

We want to prove

**Proposition 3.3** Let assume $x(p, M)$ to be differentiable. If

j) all income be spent;

jj) WARP is valid;

then the Slutsky Matrix $S(p, M)$ is negative semidefinite, where $S(p, M) = D_p x(p, M) + D_M x(p, M) x(p, M)$.

**Proof** We can restate (3.19) as

$$dpdx \leq 0, \tag{3.20}$$

where $dp$ represents a differential change in prices and what we called a compensated change can now be expressed as

$$dM = x(p, M)dp. \tag{3.21}$$

Let us consider the total differentiation of $x(p, M)$, that is

$$dx = D_p x(p, M)dp + D_M x(p, M)dM. \tag{3.22}$$

Substituting (3.21) in (3.22) we get

$$dx = [D_p x(p, M) + D_M x(p, M)x(p, M)]dp, \tag{3.23}$$

and multiplying both sides of (3.23) by $dp$, in consideration of (3.20), it is

$$dpdx = dp[D_p x(p, M) + D_M x(p, M)x(p, M)]dp \leq 0. \tag{3.24}$$

□

We have substantially proved by (3.24) that WARP implies the negative semidefinitness of the *Slutsky Matrix*.

The implication of Proposition 3.2 is unique, in the sense that some further assumptions must be considered in order to deduce WARP from (3.24). We will present this framework later (cfr. Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4, Example 3.1, Example 3.2).

Let's also note that in general we do not have the symmetry of the Slutsky Matrix by using the assumptions made so far.

What can we say in this respect?

We would like to present first this preliminary result:

**Proposition 3.4** Let us assume $x(p, M)$ to be differentiable. If

j) all income be spent;

jj) the Slutsky Matrix $S(p, M)$ is symmetric

then $x(p, M)$is homogeneous of degree 0.

**Proof** j) is equivalent to ask $px(p, M) = M$, which gives by differentiating with respect to $p_i$, $i = 1, 2, ..., n$

$$x_i(p, M) + \sum_{j=1}^{n} p_j \frac{\delta x_j(p, M)}{\delta p_i} = 0, \tag{3.25}$$

and with respect to M

$$\sum_{j=1}^{n} p_j \frac{\delta x_j(p,M)}{\delta M} = 1. \tag{3.26}$$

Let's introduce, for a fixed couple $(p, M)$ the following function

$$f_i(t) = x_i(tp, tM),\ for\, t \in \mathbb{R}^+, \tag{3.27}$$

where $i \in \{1, 2..., n\}$.

If we prove that the function defined in (3.27) is constant in $t$ we show that $x_i(p, M)$ is homogeneous of degree 0 for every $i \in \{1, 2..., n\}$ ,which is equivalent to say that $x(p, M)$ is homogeneous of degree 0.

Let us compute

$$\frac{df_i(t)}{dt} = \sum_{j=1}^{n} p_j \frac{\delta x_i(tp,tM)}{\delta p_j} + M \frac{\delta x_i(tp,tM)}{\delta M}. \tag{3.28}$$

Using j) we have $tpx(tp, tM) = tM$, that can be restated, dividing both sides by $t$, as

$$\sum_{j=1}^{n} p_j x_j(tp,tM) = M. \tag{3.29}$$

Using (3.29) into (3.28) we have that

$$\frac{df_i(t)}{dt} = \sum_{j=1}^{n} p_j \frac{\delta x_i(tp,tM)}{\delta p_j} + \sum_{j=1}^{n} p_j x_j(tp,tM) \frac{\delta x_i(tp,tM)}{\delta M} =$$

$$\sum_{j=1}^{n} p_j [\frac{\delta x_i(tp,tM)}{\delta p_j} + x_j(tp,tM) \frac{\delta x_i(tp,tM)}{\delta M}], \tag{3.30}$$

which gives exactly the $(i, j)th$ term of the Slutsky Matrix.

From jj)

$$\frac{df_i(t)}{dt} = \sum_{j=1}^{n} p_j[\frac{\delta x_i(tp,tM)}{\delta p_j} + x_j(tp,tM)\frac{\delta x_i(tp,tM)}{\delta M}] =$$

$$= \sum_{j=1}^{n} p_j[\frac{\delta x_j(tp,tM)}{\delta p_i} + x_i(tp,tM)\frac{\delta x_j(tp,tM)}{\delta M}] =$$

$$= \sum_{j=1}^{n} p_j\frac{\delta x_j(tp,tM)}{\delta p_i} + \sum_{j=1}^{n} p_j x_i(tp,tM)\frac{\delta x_j(tp,tM)}{\delta M}. \tag{3.31}$$

Multiplying and dividing (3.31) by $t$, we get

$$\frac{df_i(t)}{dt} = \frac{1}{t}[\sum_{j=1}^{n} tp_j\frac{\delta x_j(tp,tM)}{\delta p_i}] + \frac{1}{t}x_i(tp,tM)[\sum_{j=1}^{n} tp_j\frac{\delta x_j(tp,tM)}{\delta M}].$$

Just replacing the two expressions in square brackets with (3.25) and (3.26) we have

$$\frac{df_i(t)}{dt} = \frac{1}{t}[-x_i(tp,tM)] + \frac{1}{t}x_i(tp,tM)[1] = 0. \tag{3.32}$$

(3.32) says that $f_i$ is constant. Hence we have the result.

□

Let us consider what can we say about the opposite implication. That is, what do we have to request to get the symmetry of the *Slutsky Matrix*?

We start from the following proposition:

**Proposition 3.5** Let's consider a consumer in an economy with only two goods $(n = 2)$.

Let $x(p, M)$ be differentiable. If

i) $x(p,M)$ is homogeneous of degree 0;

ii) all the budget is spent

then the matrix $S(p, M)$ is symmetric.

In order to prove this Proposition we will use the following

**Proposition 3.6** Let $x(p, M)$ be differentiable. If

i) $x(p,M)$ is homogeneous of degree 0;

ii) all the budget is spent;

then

$$\begin{cases} pS(p,M) = 0 \\ S(p,M)p = 0 \end{cases} \tag{3.33}$$

**Proof** Remember that from ii) we have that (3.25) and (3.26) are verified. Furthermore i) is equivalent to impose that the expression in (3.28) is equal to 0, i.e.

$$\sum_{j=1}^{n} p_j \frac{\delta x_i}{\delta p_j} + M \frac{\delta x_i}{\delta M} = 0. \tag{3.34}$$

By definition it is

$$S_{ij}(p,M) = \frac{\delta x_i}{\delta p_j} + x_j \frac{\delta x_i}{\delta M}, \; S_{ji}(p,M) = \frac{\delta x_j}{\delta p_i} + x_i \frac{\delta x_j}{\delta M}. \tag{3.35}$$

The first expression in (3.33) can be restated as

$$\sum_{j=1}^{n} p_j S_{ij}(p,M) = \sum_{j=1}^{n} p_j [\frac{\delta x_i}{\delta p_j} + x_j \frac{\delta x_i}{\delta M}] = \sum_{j=1}^{n} p_j \frac{\delta x_i}{\delta p_j} + \sum_{j=1}^{n} p_j x_j \frac{\delta x_i}{\delta M},$$

which using (3.34) gives

$$pS(p,M) = -M \frac{\delta x_i}{\delta M} + \sum_{j=1}^{n} p_j x_j \frac{\delta x_i}{\delta M} = (-M + \sum_{j=1}^{n} p_j x_j) \frac{\delta x_i}{\delta M} = 0.$$

The last equality is a direct consequence of ii).

In order to verify that the second formula is valid, let us compute $S(p, M)p$. We have

$$S(p, M)p = \sum_{j=1}^{n} S_{ji}(p, M)p_j = \sum_{j=1}^{n} [\frac{\delta x_j}{\delta p_i} + x_i \frac{\delta x_j}{\delta M}]p_j. \tag{3.36}$$

Using (3.26) in (3.36) we have $\sum_{j=1}^{n} [\frac{\delta x_j}{\delta p_i} + x_i \frac{\delta x_j}{\delta M}]p_j = \sum_{j=1}^{n} \frac{\delta x_j}{\delta p_i} p_j + x_i \sum_{j=1}^{n} \frac{\delta x_j}{\delta M} p_j = \sum_{j=1}^{n} \frac{\delta x_j}{\delta p_i} p_j + x_i$, and from (3.25) we immediately get $S(p, M)p = 0$.

Hence we have that both the expression in (3.33) are verified.

□

We will now give the proof of Proposition 3.4 as a direct consequence of Proposition 3.5:

**Proof** When it is $n = 2$ the Slutsky Matrix is given by

$$S(p, M) = \left\| \begin{array}{cc} S_{11}(p, M) & S_{12}(p, M) \\ S_{21}(p, M) & S_{22}(p, M) \end{array} \right\|.$$

Let us observe that it is possible to apply Proposition 3.5. Hence we have:

$$pS(p, M) = 0, \ S(p, M)p = 0. \tag{3.37}$$

Let's restate (37) as these two systems of equations:

$$\begin{cases} S_{11}(p, M)p_1 + S_{12}(p, M)p_2 = 0 \\ S_{21}(p, M)p_1 + S_{22}(p, M)p_2 = 0 \end{cases}, \tag{3.38}$$

$$\begin{cases} p_1 S_{11}(p, M) + p_2 S_{21}(p, M) = 0 \\ p_1 S_{12}(p, M) + p_2 S_{22}(p, M) = 0 \end{cases}. \tag{3.39}$$

Using (38) and (39) we get the following equalities

$$\frac{p_2}{p_1} = -\frac{S_{11}}{S_{12}} = -\frac{S_{21}}{S_{22}} = -\frac{S_{11}}{S_{21}} = -\frac{S_{12}}{S_{22}}.$$

Hence, rearranging, we have $S_{12}(p, M) = S_{21}(p, M)$, which is exactly the symmetry of the Slutsky Matrix.

$\square$

We want to present an exercise to show that the result obtained in Proposition 3.4 is not generalizable to the case of a generic $n \geq 3$.

In particular we will deal with a situation of a 3-goods economy. After the proof of the homogeneity of the demand and of the condition of budget exhaustion we will show that the Slutsky Matrix associated is not symmetric. Furthermore we will prove the negative semidefiniteness of the matrix in order to link this concept with that of WARP. In particular, we want to show that even the validity of the Weak Axiom is not sufficient to guarantee the symmetry of the Slutsky Matrix.

**Example 3.1** Let consider a prices vector $(p_1, p_2, p_3) \in \mathbb{R}^3_{++}$and a level of income $M \geq 0$.

We have

$$\begin{cases} x_1(p, M) = \frac{(p_2 - p_1)}{p_3}; \\ x_2(p, M) = -\frac{p_2}{p_3}; \\ x_3(p, M) = \frac{(p_2 - p_1)^2}{p_3^2} + \frac{M}{p_3}; \end{cases}$$

It is straightforward to see that $x(\,)$ is homogenous of degree zero.

In fact

$$x_1(tp, tM) = \frac{t(p_2 - p_1)}{tp_3} = x_1(p, M);$$

$$x_2(tp, tM) = -\frac{tp_2}{tp_3} = x_2(p, M);$$

$$x_3(tp, tM) = \frac{t^2(p_2 - p_1)^2}{t^2 p_3^2} + \frac{tM}{tp_3}; = x_3(p, M);$$

In order to verify the budget exhaustion condition let's compute $px(p, M) = \frac{p_1p_2 - p_1^2 - p_2^2 + (p_2-p_1)^2 + p_3M}{p_3}$, which gives exactly

$$px(p, M) = M.$$

Since $x(\,)$ is differentiable we can obtain the *Slutsky Matrix* for our example, that is, just remembering (3.35):

$$S(p, M) = \left\| \begin{matrix} -\frac{1}{p_3} & \frac{1}{p_3} & \frac{p_1-p_2}{p_3^2} \\ 0 & -\frac{1}{p_3} & \frac{p_2}{p_3^2} \\ \frac{p_1-p_2}{p_3^2} & \frac{p_2-2p_1}{p_3^2} & \frac{(p_1-p_2)^2}{p_3^3} \end{matrix} \right\|. \tag{3.40}$$

It is evident that it results $S_{12}(p, M) \neq S_{21}(p, M)$, which is sufficient to say that the matrix defined in (3.40) is not symmetric. We will now show the negative semidefiniteness of (3.40) by using the famous result on the minor of $S(p, M)$ (cfr. [20]).

We have:

1. $(-1)\ \left| -\frac{1}{p_3} \right| \ = \frac{1}{p_3} > 0;$

2. $(-1)^2 \begin{vmatrix} -\frac{1}{p_3} & \frac{1}{p_3} \\ 0 & -\frac{1}{p_3} \end{vmatrix} = \frac{1}{p_3^2} > 0;$

3. $(-1)^3 \begin{vmatrix} -\frac{1}{p_3} & \frac{1}{p_3} & \frac{p_1-p_2}{p_3^2} \\ 0 & -\frac{1}{p_3} & \frac{p_2}{p_3^2} \\ \frac{p_1-p_2}{p_3^2} & \frac{p_2-2p_1}{p_3^2} & \frac{(p_1-p_2)^2}{p_3^3} \end{vmatrix} = \frac{p_2p_1}{p_3^5} > 0.$

$\square$

In order to have a complete view on this framework we will make reference to the paper by A. Mas-Colell, H.F. Sonnenschein and R. Kihlstrom

"The demand theory of the WARP" (cfr. [14]). We will always assume the hypotheses of

i) homogeneity of degree 0 of the demand;

ii) budget exhaustion.

We already proved in Proposition 3.2 how we can deduce the negative semidifiniteness of the Slutsky matrix from WARP. The authors in this paper make a further step by introducing a new definition. We say that the demand satisfies the *weak-weak axiom* (WWA) condition if

$$for\, every\, p^0, p^1,\, when\, x \neq x',\, if\, p^0x^1 < p^0x^0\, then\, it\, is\, p^1x^0 > p^1x^1.$$

We immediately get that **WWA** is implied by the **WA** (weak axiom).

Let us mention the 3 main results of the paper:

**Theorem 3.1** If $x(\,)$ satisfies **WWA** then the Slutsky Matrix associated is $negative\ semidefinite$ (**NSD**).

**Theorem 3.2** If the Slutsky Matrix associated to the demand function $x(\,)$ is $negative\ definite$ (**ND**), then $x(\,)$ satisfies **WA**.

**Theorem 3.3** If the Slutsky Matrix associated to the demand function $x(\,)$ is $negative\ definite$ (**ND**), then $x(\,)$ satisfies **WA**.

We can summarize all these results through the following conceptual map:

$$\begin{array}{ccc} \mathbf{WA} & \overset{\nLeftarrow}{\Rightarrow}(\beta) & \mathbf{WWA} \\ \Uparrow(\alpha) & & \Updownarrow(\gamma) \\ \mathbf{ND} & \overset{\nLeftarrow}{\Rightarrow}(\theta) & \mathbf{NSD} \end{array} . \tag{3.41}$$

The implication in ($\theta$) is an immediate consequence of the definition of negative definite/semidefinite matrix. The authors also present the following example to show that the implication in ($\beta$) is valid only in one direction.

**Example 3.2** Let us consider a vector prices $(p_1, p_2, p_3) \in \mathbb{R}^3_{++}$and a level of income $M \geq 0$.

We have

$$\begin{cases} x_1(p, M) = \frac{p_2}{p_3}; \\ x_2(p, M) = -\frac{p_1}{p_3}; \\ x_3(p, M) = \frac{M}{p_3}; \end{cases}$$

It is straightforward to see that $x(\,)$ is homogenous of degree zero and the budget exhaustion is valid.

In order to prove that **WWA** holds we can show that the *Slutsky Matrix* is negative semidefinite.

Let us consider

$$S(p, M) = \left\| \begin{matrix} 0 & \frac{1}{p_3} & -\frac{p_2}{p_3^2} \\ -\frac{1}{p_3} & 0 & -\frac{p_1}{p_3^2} \\ \frac{p_2}{p_3^2} & -\frac{p_1}{p_3^2} & 0 \end{matrix} \right\|. \tag{3.42}$$

We can prove the negative semidefiniteness of the matrix considering the same criterion used in Example 3.1.

We want to show now that the **WA** is not valid. Let's consider the two following prices vectors $p^0 = (1, 1, 1)$ $p^1 = (2, 1, 1)$, and income $M^0 = 1$, $M^1 = 2$ We have $x^0\,(p, M) = (1, -1, 1)$ and $x^1(p, M) = (1, -2, 2)$.

It is

$$p^0 x^1 = 1 = p^0 x^0,$$

and

$$p^1 x^0 = 2 = p^1 x^1.$$

Hence, WARP is violated.

□

Please note that for (3.42) it results $S_{12}(p, M) \neq S_{21}(p, M)$, which is sufficient to have the non-symmetry of the *Slutsky Matrix*. It is shown by Hurwicz and Uzawa that if it is taken by assumption that the *Slutsky Matrix* is symmetric the **WA** is completely equivalent to the **WWA**.

Under the hypothesis of Symmetry we have:

$$\begin{array}{ccc} \mathbf{WA} & \Leftrightarrow (\beta) & \mathbf{WWA} \\ \Uparrow (\alpha) & & \Updownarrow (\gamma) \\ \mathbf{ND} & \Rightarrow (\theta) & \mathbf{NSD} \end{array} . \tag{3.43}$$

# 4 "Utility, Demand and Preference", Ch.1, H. Uzawa

In this chapter we still focus on the "Revealed Preference" theory by analyzing chapter 1 of [3] by H. Uzawa. A generalization of Theorem 1 (cfr. Chapter 1 of [3]) will be proposed (cfr. Theorem 4.1). In this preamble we will bore the reader with an other reference to some of the concepts analyzed in the previous chapter. We deem it necessary to better converge the reader on the topic.

Paul A. Samuelson in [17] introduced the Weak Axiom of Revealed Preference (WARP).

Let us suppose we were in a situation 0 (price and income) where commodity bundle $x^1$ could be chosen but commodity $x^0$ actually has been chosen ($x^o$ is revealed preferred to $x^1$) then at the price and income situation 1 at which commodity bundle $x^1$ is chosen it is impossible to choose commodity bundle $x^0$($x^1$ is not revealed preferred to $x^0$).

$x^0$ revealed preferred to $x^1$ means that $x^0$ is chosen when both $x^0$ and $x^1$ are affordable. For $x^1$ not to be revealed to $x^0$ means that when $x^1$ is chosen then $x^0$ must not to be affordable; that is, the cost of $x^0$ must exceed the cost of $x^1$ at all prices $x^1$ is chosen. Suppose that $x^0$ is revealed preferred to $x^1$ at price system $p^0$ and that $x^1$ is chosen at some other price $p^1$. Then $WARP$ can formally be expressed as:

$$p^0x^1 \leq p^0x^0 \Rightarrow p^1x^0 > p^1x^1 \tag{4.1}$$

What the weak axiom indicates is that if $x^1$ is chosen at some price system $p^1$, then $x^0$ will be more expensive than $x^1$ at prices $p^1$.

A generalization of WARP is introduced by Houthakker in "Revealed Preference and the Utility Function" (cfr. [10]).

Houthakker's contribution was to recognize that one needs to extend the "direct" revealed preference relation to what he called the "indirect" revealed preference relation. We say that $x^0$ is "indirectly" revealed preferred to $x^s$ if there exists a finite sequence of commodity bundles $x^0, x^1...x^s$ such that $x^t$ is "directly" revealed preferred to commodity bundle $x^{t+1}$ for every $t = 0, 1, ..., s-1$.

It is easy to prove that SARP implies WARP.

We will consider in this section some regularity condition for demand function under which the converse implication is verified.

Rose in [16] offered a formal argument that the Strong Axiom and the Weak Axiom were equivalent in two dimensions, providing a rigorous, algebraic foundation for Samuelson's earlier graphic exposure.

**Definition 4.1** Given the vector price and the income the *budgetset* $X(p, M)$ is the set of all commodity bundles whose market values evaluated at $p$ do not exceed income $M$: $X(p, M) = \{x = (x_1, x_2, ..., x_n)\, such\, that\, x \epsilon \Omega\, and\, px \leq M\}$.

In this chapter we will denote the demand function as $h(p, M)$ in place of $x(p, M)$. This symbolism is the one used by Uzawa in his work. Note that no differences exist between the function we introduced in chapter 1 and the one we will introduce in the following definition. The reader is only invited to take into consideration the properties assumed for the demand here and there.

**Demand function** A function $x = h(p, M) = h_1(p, M), h_2(p, M)...h_n(p, M)$ is a *demand function* if the following conditions are satisfied:

D.I $x = (h(p, M))$ is a commodity bundle in $\Omega$ for any given price vector $p = (p_1, p_2, ...p_n)$ and income $M$.

D.II Any commodity bundle $x$ is chosen for a suitable price vector $p = (p_1, p_2, ...p_n)$ and income $M$, i.e. $x = h(p, M)$.

D.III $x = (h(p, M))$ satisfies the budget equation $ph(p, M) = M$, for all positive price vector and income.

We will sometimes consider the following stringent condition in place of D.II

D.II' For any commodity bundle $x = (h(p, M))$ the price vector $p = (p_1, p_2, ...p_n)$ at which $x$ is chosen, exists and is uniquely determined except for the multiplication by a scalar.

We will furthermore assume some regularity condition for the demand function. In particular we will use

D.IV $h(p, M)$ is a lipschitzian function with respect to $M$.

**Note 4.1** The condition presented in D.IV is formally expressed as following:

there exist two positive real numbers $\epsilon$ and $L$ such that for all $p$ satisfying $\parallel p - p^0 \parallel < \epsilon$ and all $M^I, M^{II}$ with $\mid M^I - M^0 \mid < \epsilon$ and $\mid M^{II} - M^0 \mid < \epsilon$ it holds

$$\parallel h(p, M^{II}) - h(p, M^I) \parallel \leq L \mid M^{II} - M^I \mid; \tag{4.2}$$

$h(p, M)$ is said to be a lipschitzian function with respect to $M$ in $(p^0, M^0)$ .

**Definition 4.2** Let $f(x)$ be defined on an interval $I$ and suppose we can find two positive constants $L$ and $\alpha$ such that

$$\mid f(x^1) - f(x^2) \mid \leq L \mid x^1 - x^2 \mid^\alpha, \; for\, every\, x^1, x^2 \epsilon I \tag{4.3}$$

Then $f$ is said to satisfy Holder condition of order $\alpha$ and we say that $f \epsilon Lip(\alpha)$.

If $f \epsilon Lip(1)$ it is said to be Lipschitz continuous

**Proposition 4.1** Let $f : \Theta \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with:

1. $f \epsilon C^1(\Theta, \mathbb{R}^m)$;
2. $\exists M > 0 \, such\, that \, \max_{y \epsilon \Theta} \parallel J_f(y) \parallel \leq M$

then $f \epsilon Lip(1)$.

It is straightforward to prove the continuity of a Lipschitz continuous function (cfr. [10]).

Using the definition we can also prove that a function satisfying Holder condition is continuous.

Hence, when it is possible we will consider the following condition in place of D.IV:

D.IV* $h(p, M)$ satisfy Holder condition of order $\alpha$ with respect to $M$.

The following proposition gives a class of functions satisfying Lipschitz conditions.

**Proposition 4.2** Let us suppose $h(p, M)$ represents a demand function (satisfying D.I, D.II, D.III) and

1. there are not inferior goods at $(p, M)$;

then $f$ is a Lipschitz continuous function.

**Proof** Let's consider a variation in income $\triangle M$. Using D.III we have

$$ph(p, M) = M \Leftrightarrow \sum_{i=1}^{n} p_i h_i(p, M) = M \tag{4.4}$$

$$ph(p, M + \triangle M) = M + \triangle M \Leftrightarrow \sum_{i=1}^{n} p_i h_i(p, M + \triangle M) = M + \triangle M \tag{4.5}$$

subtracting (3.4) from (3.5) and dividing both sides by the nonzero quantity $\triangle M$ we get

$$\sum_{i=1}^{n} p_i \frac{[h_i(p, M + \triangle M) - h_i(p, M)]}{\triangle M} = 1 \tag{4.6}$$

We can observe that we have all positive terms (from the non inferior goods condition and the positivity of the prices) summing 1.

Hence each term must be not larger than 1, i.e.

$$p_i \frac{[h_i(p, M + \triangle M) - h_i(p, M)]}{\triangle M} \leq 1 \tag{4.7}$$

Rearranging

$$[h_i(p, M + \triangle M) - h_i(p, M)] \leq \frac{1}{p_i} \triangle M \tag{4.8}$$

that is equivalent to the Lipschitz condition for the demand function.

$\square$

As we already said in the introduction of this thesis we will denote by

$$x^0 R x^1$$

the relation $x^0$ is revealed preferred to $x^1$.

Considering assumption D.II on the demand function we get

$$for\, any\, positive\, bundles\, x, y \epsilon \Omega\, such\, that\, x \geq y\, it\, is\, xRy \tag{4.9}$$

We will indicate by

$$x^0 R^* x^1$$

the relation $x^0$ is indirectly revealed preferred to $x^1$.

It is obvious that

$$xR^*y, yR^*z\, implies\, xR^*z\, (transitivity) \tag{4.10}$$

Let us consider 2 systems of prices $p^a$ and $p^b$, for any given positive income $M^a$ we define the function $\rho_{b,a}(M^a)$ as

$$\rho_{b,a}(M^a) = sup\left\{M\, such\, that\, h(p^a, M^a)R^* h(p^b, M)\right\} \tag{4.11}$$

The function $\rho_{b,a}(\,)$ associates to any income $M^a$ at price $p^a$ the income $\rho_{b,a}(M^a)$ as the supremum of those income at price $p^b$ such that the commodity bundle $h(p^a, M^a)$ is indirectly revealed preferred to the corresponding commodity bundle $h(p^b, M)$.

By analogy we introduce

$$\rho^{'}_{b,a}(M^a) = inf\left\{M\, such\, that\, h(p^b, M)R^* h(p^a, M^a)\right\}. \tag{4.12}$$

As consequence of (4.10) we can deduce that $\rho_{b,a}(\,)$ is a non decreasing function of $M^a$. Furthermore we will say that $\rho_{b,a}(\,)$ satisfies the Regularity condition (R) if:

$$for\, any\, given\, price\, systems\, p^a\, and\, p^b$$

$$the\, function\, \rho_{b,a}(\,)\, is\, strictly\, increasing \tag{4.13}$$

It is possible to show that if the Weak Axiom is satisfied and D.I,...,D.IV hold then we can restate (4.13) as:

(R') for any price vectors $p^a$, $p^b$ the function $\rho_{b,a}(M^a)$ is finite and the the function $\rho^{'}_{b,a}(M^a)$ is continuous.

Let us also note that the Weak and Strong Axiom defined by Samuelson and Houthakker respectively may be restated in terms of the preference relations $R$ and $R^*$ introduced above.

In particular WARP is equivalent to

(W) $x^0 R x^1$ implies $x^1 \overline{R} x^0$,

while SARP is given by

(S) $xR^*y$ implies $y\overline{R^*}x$.

These two results are immediate consequences of the definitions.

As we already make with the demand, we decide to follow's Uzawa approach and symbolism. In this respect some definitions or properties of the preference relations may be stated in a different style with respect the one we used so far.

**Preference relations** A binary relation $P$ defined on the set $\Omega$ is called a *preference relation* when the following axioms are satisfied:

P.I Irreflexibility: for any $x \in \Omega$, we have $x\overline{P}x$;

P.II Transitivity: for any $x, y, z \in \Omega$, with $xPy$, $yPz$ it is $xPz$;

P.III Monotonicity: for any $x, y \in \Omega$, such that $x \geq y$ we have $xPy$;

P.IV (Houthakker) Convexity: for any $x, y \in \Omega$, such that $x\overline{P}y$, we have $[(1-\lambda)x + \lambda y]Px$ for all $0 < \lambda < 1$;

P.V L-Continuity: for any $x^0 \in \Omega$ the set $\{x\, such\, that\, x\epsilon\Omega\, and\, x^0Px\}$ is an open set in $\Omega$.

We can consider the following in place of P.I

P.I' Asymmetry: for any $x, y \in \Omega$, if $xPy$ we have $y\overline{P}x$;

The equivalence in considering P.I' instead of P.I is straightforward. In fact, ab absurdo, suppose that there exists $y$ such that it does not hold $y\overline{P}y$, that is $yPy$. Then for asymmetry we would have a contradiction. The other way round let us suppose ab absurdo that asymmetry does not hold. Then there exist $x, y \in \Omega$ such that $xPy$ and $yPx$. Using transitivity we get $xPx$, which contradicts the hypotesis of irreflexivity.

Notice that in order to have the usual definition of Continuity we have to add to axiom P.V the assumption of U-continuity (where U and L stands for Upper and Lower, respectively):

P.VI U-continuity: for any $x^0 \epsilon \Omega$ the set $\{x\, such\, that\, x \epsilon \Omega\, and\, xPx^0\}$ is an open set in $\Omega$.

P.V and P.VI may be substituted by requesting the complementary sets defined above to be closed in $\Omega$.

We want to open a small parenthesis on Preference Relation theory in order to make it easier for the reader to interpret some results otherwise unfamiliar. In particular we will make some comparison between the hypotheses in P.I, P.II and P.III and the assumptions on the preference relation $R$ introduced in chapter 1 of this thesis.

**Definition 4.2.I (cfr. [2])** A (weak) preference relation $R$ on $\Omega$ is said to be *regular* if it is complete, transitive and reflexive.

A (strong) preference relation is said to be "negative transitive" $iff$

$$for\, every\, x, y, z \in \Omega,\; with\, x\overline{P}y\, and\, y\overline{P}z\, it\, is\, x\overline{P}z$$

**Definition 4.2.II** A (strong) preference relation $P$ on $\Omega$ is said to be $u - regular$ if it is asymmetric and negative transitive.

The two following propositions (cfr. [11]) explain the connection between the two previous definition

**Proposition 4.3** Given a $u - regular$ strong preference relation $P$, the strong relation $R$ derived from the weak one (cfr. the Introduction of Chapter 1)

$$xRy\, iff\, (y\overline{P}x)$$

is a *regular* preference

**Proposition 4.4** Given a regular preference $R$, define $P$ and $\sim$ to be the asymmetric and symmetric parts of $R$ as in chapter 1.

Then $P$ is a $u - regular$ preference and $\sim$ is an equivalence relation (reflexive, symmetric, and transitive).

**Proposition 4.5** Given a $u - regular$ preference $P$, let $\rho(P)$ denote the regular preference $\overline{P}^{-1}$ induced by $P$; and given a *regular* preference $R$, let $\sigma(R)$ denote the $u - regular$ strong preference $P$ induced by $R$. Then

$$P = \sigma(\rho(P)),\ while$$

$$R = \rho(\sigma(R)).$$

Note that the *regular* and $u - regular$ properties are fundamental to get the results of Proposition 4.5.

Let us come back to our initial problem

**Definition 4.3** Let us consider a price vector $p$ and an income $M$, and let $P$ be a preference relation as defined above. A commodity bundle $x^0$ is said to be *optimum* with respect to the preference relation $P$ in the budget set $X(p, M)$ if $x^0 \in X(p, M)$ and for any $x \in \Omega$, with $xPx^0$ it is $x \notin X(p, M)$. The idea is that if I have an *optimum* $x^0$ there exists no commodity bundles in the budget set which are preferred to $x^0$.

**Definition 4.4** A demand function $h(p, M)$ is defined as derived from a preference relations $P$ if for every price vector $p$ and positive income $M$ the commodity bundle $h(p, M)$ is optimum with respect to $P$ in the budget set $X(p, M)$.

**Theorem 4.1** Let $h(p, M)$ be a demand function satisfying D.I,...,D.IV and the SARP. Then the indirect revealed preference relation $R^*$, generated by $h(p, M)$, is a preference relation on the set $\Omega$ of all positive commodity bundles (i.e. $R^*$ satisfies P.I,...,P.V ) and the demand function $h(p, M)$ is derived from $R^*$.

**Proof** In order to prove the theorem we have to show that the preference $R^*$ satisfies the 5 conditions that characterize a revealed preference relation. The final step is to prove that the demand function $h(p, M)$ is derived from the indirect revealed preference relation ordering $R^*$.

We can immediately derive P.II, P.III and P.I' through the properties of the demand function. We have to prove that also P.IV and P.V are satisfied.

In order to prove P.IV we need the following:

**Lemma 4.1** Let the demand function $h(p, M)$ satisfy D.I, D.II, D.III, D.IV* and SARP. Then, for any price vector $p^a$and $p^b$, we have:

$$h(p^a, M^a)R^*h(p^b, M)\, for\, all\, M < \rho_{b,a}(M^a), \tag{4.14}$$

and

$$h(p^b, M)R^*h(p^a, M^a)\, for\, all\, M > \rho_{b,a}(M^a), \tag{4.15}$$

where $\rho_{b,a}(\,)$ is defined in (4.11)

**Proof** We can deduce (4.14) by simply using the definition of $\rho_{b,a}(\,)$ introduced in (4.11).

In order to show that (4.15) is valid we want

$$\rho_{b,a}(M^a) = \rho'_{b,a}(M^a)\, for\, all\, M^a. \tag{4.16}$$

In fact in this case we could restate (4.15) as

$$h(p^b, M)R^*h(p^a, M^a)\, for\, all\, M > \rho'_{b,a}(M^a), \tag{4.17}$$

which can be seen as a direct consequence of the definition of $\rho'_{b,a}(\,)$ in (4.12).

Hence our proof will be focused on showing the validity of (4.16). A process analogous to what Houthakker used in [10] will be considered.

For any $S \in \mathbb{N}$ let us consider the following two sequences defined recursively as:

$$\overline{M^{0S}} = M^a,\ \overline{x^{0S}} = x^a = h(p^a, M^a)$$

$$\overline{M^{K+1,S}} = p^{\frac{(K+1)}{S}} \overline{x^{K,S}},\ \overline{x^{KS}} = h(p^{\frac{K}{S}}, \overline{M^{KS}}),\ (k = 0, 1, 2, ..., S-1), \quad (4.18)$$

where

$$p^t = p^a + t(p^b - p^a),\ 0 \leq t \leq 1. \quad (4.19)$$

While it is

$$M^{0,S} = \overline{M^{0S}},$$

$$M^{KS} = p^{\frac{K}{S}} x^{K+1,S},\ x^{K+1,S} = h(p^{\frac{(K+1)}{S}}, M^{K+1,S}),\ (k = 0, 1, 2, ..., S-1). \quad (4.20)$$

Using D.III in (4.20) we have that

$$\lim_{M^{K+1,S} \to 0} x^{K+1,S} = 0,\ \lim_{M^{K+1,S} \to \infty} x^{K+1,S} = \infty, \quad (4.21)$$

in fact the price vector always ranges between $p^a$ and $p^b$. (4.21) implies

$$\lim_{M^{K+1,S} \to 0} M^{K,S} = 0,\ \lim_{M^{K+1,S} \to \infty} M^{K,S} = \infty, \quad (4.22)$$

(4.22) together with the continuity hypothesis of $h(\,)$ guarantees the existence of a solution for (4.20).

Let us suppose, ab absurdo, that

$$there\, exists\, \widetilde{M^a}\, such\, that\, \rho'_{b,a}(\widetilde{M^a}) > \rho_{b,a}(\widetilde{M^a}); \tag{4.23}$$

then we would arrive to a chain of relation such as

$$h(p^b, M)R^* h(p^a, \widetilde{M^a})R^* h(p^b, M),$$

which obviously contradicts the SARP (S).

We can state

$$\rho_{b,a}(M^a) \leq {\rho'}_{b,a}(M^a)\, for\, all\, M^a. \tag{4.24}$$

Proceeding through an ab absurdo reasoning it is furthermore possible to prove that

$$M^{SS} \leq \rho_{b,a}(M^a)\, and\, {\rho'}_{b,a}(M^a) \leq \overline{M^{SS}}. \tag{4.25}$$

From (4.24) and (4.25) we can write the following sequence of disequalities:

$$M^{SS} \leq \rho_{b,a}(M^a) \leq {\rho'}_{b,a}(M^a) \leq \overline{M^{SS}}. \tag{4.26}$$

Since the two sequences in (4.18) and (4.20) are defined for every $S \in \mathbb{N}$ (4.16) is verified if the disequalities in (4.26) hold as equality when $S$ diverges, that is if

$$\lim_{S \to \infty} (\overline{M^{SS}} - M^{SS}) = 0. \tag{4.27}$$

Taking into account D.III and the definition in (4.18), (4.20) we can compute:

$$\overline{M^{K+1,S}} - \overline{M^{K,S}} = p^{\frac{(K+1)}{S}} \overline{x^{K,S}} - p^{\frac{K}{S}} \overline{x^{K,S}} = \frac{1}{S}(p^b - p^a)\overline{x^{K,S}}, \qquad (4.28)$$

$$M^{K+1,S} - M^{K,S} = p^{\frac{(K+1)}{S}} x^{K+1,S} - p^{\frac{K}{S}} x^{K+1,S} = \frac{1}{S}(p^b - p^a)x^{K+1,S}. \quad (4.29)$$

By defining $\nu^{K,S} = \overline{M^{K,S}} - M^{K,S}$ we get, subtracting (4.29) from (4.28)

$$\nu^{K+1,S} - \nu^{K,S} = \frac{1}{S}(p^b - p^a)(\overline{x^{K,S}} - x^{K+1,S}),\ for\, any\, K = 0, 1, ..., S-1. \quad (4.30)$$

For any $j \in \{0, 1, ..., S-1\}$ we can compute

$$\sum_{K=0}^{j-1} \nu^{K+1,S} - \nu^{K,S} = \nu^{1,S} - \nu^{0,S} + \nu^{2,S} - \nu^{1,S} + ... + \nu^{j,S} - \nu^{j-1,S} =$$

$$= \nu^{j,S} - \nu^{0,S} = \nu^{j,S}, \qquad (4.31)$$

where the last equality is consequence of $\nu^{0,S} = 0$, and the first one can be restated as

$$\nu^{j,S} = \frac{1}{S}(p^b - p^a)\left((x^a - x^{jS}) + \sum_{K=1}^{j-1} \overline{x^{K,S}} - x^{K,S}\right). \qquad (4.32)$$

Since $x^a R^* x^{j,S}$, we have from (S)

$$p^{\frac{j}{S}} x^a \geq p^{\frac{j}{S}} x^{j,S}. \qquad (4.33)$$

We define $\overline{p_j} = max\left\{p_j^a, p_j^b\right\},\ j = 1, 2, ..., n$ and $\underline{p_j} = min\left\{p_j^a, p_j^b\right\},\ j = 1, 2, ..., n$.

From (4.33) we can deduce: $\overline{p}x^a \geq \underline{p}x^{j,S}$.

For any $j$, independently from $S$ we can say

$$x^{j,S} \in \Gamma = \left\{ x \, such \, that \, x \geq 0, \, \overline{p}x^a \geq \underline{p}x^{j,S} \right\}.$$

Let

$$A = \max_{x \epsilon \Gamma} \mid (p^b - p^a)(x^a - x) \mid; \tag{4.34}$$

by considering that $h(p, M) \epsilon Lip(\alpha)$ with respect to $M$,

$$\mid \overline{x^{KS}} - x^{KS} \mid = \mid h(p^{K/S}, \overline{M^{KS}}) - h(p^{K/S}, M^{KS}) \mid \leq L \mid \overline{M^{KS}} - M^{KS} \mid^{\alpha}, \tag{4.35}$$

where $L$ and $\alpha$ are two positive real numbers.

We can use (4.31),(4.34) and (4.35) to get

$$v^{jS} \leq \frac{1}{S} \left\{ A + B(v^{1S} + ... + v^{j-1,S}) \right\} \; j = 1, 2, ..., S; \tag{4.36}$$

where $B^{\alpha} = L \mid p^b - p^a \mid^{\alpha}$ .

Hence, we can get the following recursive formula

$$v^{jS} \leq \frac{A}{S}(1 + \frac{B^{\alpha}}{S})^{j-1} \; j = 1, 2, ..., S, \tag{4.37}$$

and in particular,

$$v^{SS} \leq \frac{A}{S}(1 + \frac{B^{\alpha}}{S})^{S-1}. \tag{4.38}$$

Since $\lim_{S \to \infty}(1 + \frac{B^{\alpha}}{S})^{S-1} = e^{B^{\alpha}}$, we have $\lim_{S \to \infty} v^{SS} = 0$, that is exactly (4.27).

□

**Note 4.2** The proof of Lemma 4.1 we have provided is based on the one reported by Uzawa in [3]. The main difference is that we extend the theorem to functions satisfying Holder condition and not only Lipschitz-continuous functions. A large class of functions falls into the first category and not in the second.

**Proof (P.IV)** We consider two goods $x^a = h(p^a, M^a)$, $x^b = h(p^b, M^b)$ satisfying the hypotheses of P.IV, that is:

$$\overline{x^a R^* x^b},\ x^a \neq x^b, \tag{4.39}$$

Let $x^c = h(p^c, M^c)$ be a convex combination of $x^a$ and $x^b$ :

$$x^c = (1-c)x^a + cx^b,\ c \in (0,1).$$

Let us consider two possible case:

i) $\qquad p^c x^b \geq p^c x^a;$

ii) $\qquad p^c x^b < p^c x^a;$

In case i) we have $p^c x^c = (1-c)p^c x^a + cp^c x^b \geq p^c x^a$, while in case ii) it is $p^c x^c = (1-c)p^c x^a + cp^c x^b > p^c x^b$.

In case i) it is $x^c R x^a$, in fact $p^a x^c = (1-c)p^a x^a + cp^a x^b > p^a x^a$, where the last inequality is consequence of (4.39). In case ii), by analogy, using the continuity in $M$ of $h(p, M)$ we have the existence of a positive real number $\epsilon$ small enough such that:

$$x^c R h(p^b, M^b + \epsilon). \tag{4.40}$$

Considering (4.39) and Lemma 4.1 we have:

$$h(p^b, M^b + \epsilon) R^* x^a,\ for\, all\, positive\, \epsilon. \tag{4.41}$$

Using (4.40), (4.41) and the transitivity of $R$ we have

$$x^c R^* x^a.$$

□

**Proof (P.V)** Let us consider any positive commodity bundle $x^b$ satisfying $x^0 R^* x^b$. Two possible cases arise:

j) $x^0 R x^b$;

jj) there exists $x^1$ such that $x^0 R^* x^1$, $x^1 R^* x^b$;

We can consider case j) as part of jj) when we assume $x^0 = x^1$.

Hence we have

$$p^1 x^1 \geq p^1 x^b,\ x^1 \neq x^b, \tag{4.42}$$

being $p^1$ the price vector at which $x^1$ is chosen.

Let

$$x^2 = \frac{x^1}{2} + \frac{x^b}{2},$$

and compute $p^1 x^2 = p^1 \frac{x^1}{2} + p^1 \frac{x^b}{2}$. It is, by using (4.42),

$$p^1 x^1 \geq p^1 x^2,\ x^1 \neq x^2. \tag{4.43}$$

When we consider WARP for $x^1, x^2$, from (4.43) we immediately get

$$p^2 x^1 > p^2 x^2, \tag{4.44}$$

where $p^2$ represents the price vector at which $x^2$ is chosen.

Using the definition of $x^2$ and the disequality in (4.44) we get

$$p^2 x^2 = p^2 \frac{x^1}{2} + p^2 \frac{x^b}{2} > p^2 \frac{x^2}{2} + p^2 \frac{x^b}{2},$$

implying:

$$p^2x^2 > p^2x^b. \tag{4.45}$$

At this point we can say that there exists a neighborhood $I^b$ of commodity bundles $x^b$ such that:

$$p^2x^2 > p^2x,\ for\, all\, x \in I^b;$$

hence

$$x^0R^*x,\ for\, all\, x \in I^b, \tag{4.46}$$

which is exactly the continuity hypothesis in P.V.

□

What we get so far is that that if the demand function satisfies axioms D.I, D.II, D.III, D.IV* then for the preference $R^*$ P.I, P.II, P.III, P.IV, P.V are all valid. In order to complete the Proof of Theorem 4.1. we left to prove that the demand function $h(p, M)$ is derived from $R^*$. The condition according to which the demand function can be said as derived from a certain preference relation is stated in Definition 4.4 in this chapter.

Let $x^0$ be the choice of the consumer when prices and income are given by the couple $(p^0, M^0)$, that is $x^0 = h(p^0, M^0)$. For any other commodity bundles $x \neq x^o$ on the budget set it must result $x^0Rx$, which obviously implies $x^0R^*x$. Viceversa let's consider a certain $x^0$ in the budget set defined by $p^0$ and $M^0$ such that $x^0R^*x$, for all $x \neq x^o$. Then it must be $x^0 = h(p^0, M^0)$. In fact, if ab absurdo we suppose the existence of $\hat{x} \neq x^0$ in the budget set such that $\hat{x} = h(p^0, M^0)$ we would violate the SARP.

Our last step consists on showing the unicity of the preference relation considered.

Let us consider $R'$ as any preference relation defined on the support of $R^*$, from which $h(p, M)$ is derived. When $xR^*y$ we must have, by definition,

$xR'y$. Using the transitivity hypothesis, it must be

$$xR^*y \, implies \, xR'y. \tag{4.47}$$

Viceversa let us consider two consumption bundles $x, y$ satisfying $x\overline{R^*}y$. By using the construction of Lemma 4.1, we can consider a sequence $(y^n)_{n\in\mathbb{N}}$ with

$$\lim_{n\to\infty} y^n = y, \, and \, y^n R^* x, \, for \, any \, n \in \mathbb{N}.$$

The hypothesis of continuity in P.V (sequence continuity) and (4.47) guarantees $x\overline{R'}y$. Hence we have

$$x\overline{R^*}y \, implies \, x\overline{R'}y. \tag{4.48}$$

The unicity of the preference relation is straightforward from (4.47) and (4.48).

□

The two following theorems focus on the properties we can deduce for the demand function. We will not present the proofs.

**Theorem 4.2** Let a demand function $h(p, M)$ satisfy D.I,..., D.IV. Then the SARP implies the continuity of the demand function $h(p, M)$ with respect to price vector $p$ and $M$.

**Theorem 4.3** Let $h(p, M)$ be a demand function satisfying D.I, D.II', D.III, D.IV and the SARP. Then the indirect revealed preference relation $R^*$ satisfies Axioms P.I,...,P.VI on $\Omega$.

Theorem 4.4 expresses a fundamental property in "Revealed Preference" theory:

**Theorem 4.4** Let a demand function $h(p, M)$ satisfy D.I,...,D.IV. Then the SARP holds if and only if the WARP and the Regularity Condition (R) are both satisfied.

We would not report the proof as it could be easily deduced by the reader from Lemma 4.1.

The following two theorems show how, given any preference relation on the set $\Omega$, it is possible to derive the corresponding demand function.

**Theorem 4.5** Let $P$ be a preference relation on the set $\Omega$ of all nonnegative commodity bundles satisfying Axioms P.I,...,P.V. Then there exists a demand function $h(p, M)$ which satisfies D.I, D.III and the SARP.

**Theorem 4.6** Let $P$ be a preference relation on the set $\Omega$ of all nonnegative commodities satisfying Axioms P.I,..., P.VI. Then there exists a demand function $h(p, M)$ that is derived from the preference relation $P$. The demand function $h(p, M)$ satisfies D.I, D.II, D.III and the SARP, and it is continuous with respect to price vector $p$.

# 5 "The Pure Theory of Consumer's Behaviour", N. Georgescu-Roegen

At this point we will try to analyze "The Pure Theory of Consumer's Behaviour"(cfr. [5]) by N. Georgescu-Roegen. Samuelson refers to this article by saying "Professor N. Georgescu-Roegen wrote one of the most important clarifications of the problem of integrability and also of the even more subtle problems of transitivity. Until re-reading his article recently I did not realize how it must have stimulated my own work on the subject."

We will focus our attention on the first two sections of the paper as they are in some sense more directly linked with the problem of integrability we are interested in.

Georgescu develops a theory for the construction of "indifference surfaces" starting from four sufficient hypotheses. As a statistician he builds his work with several probabilistic/statistic references in a way that can be considered interesting and for sure original.

We will try to follow his presentation.

**THE MODEL**

In developing his model Georgescu says "Let $S$ be an ordinal and continuous set of combinations, i.e., a set such that any combination $C_r$ belonging to the set may be completely characterized by its rank $r$...."

Let us just recall that in mathematics a "combination" on a given set $X$ can be thought as a particular way in selecting the elements of $X$, where the order of choice is not important.

Hence suppose that $S$ is a set of combinations on a given set $X$, and the elements of $S$ are denoted as $C_r$. Then we should be able to order the combinations through the indices $r$. We will consider the combination $(C_a)$ as "always preferred" to combination $(C_b)$ if $a > b$. $(C_a)$ will be said "preferential". Let us consider the case where the individual faces a third combination $(T)$ . Let's assume $(T)$ is preferred to $(C_b)$ if $b < r$ while $(C_a)$ is preferred to

$(T)$ if $R < a$. Hence we are assuming the possibility of defining two classes of combinations: those preferred to $(T)$ and those to which $(T)$ is preferred. Georgescu, in this respect, considers the following Postulate:

A. There is a unique combination $(C_t)$ that will separate the non-preferred combinations from the preferred ones.

$C_t$ will coincide with $C_r$ and $C_R$.

What we say so far may look a little bit " like an end in itself" and not relevant for our "Integrability Problem". When we deal with the following construction the assumption in A. will be clarified. Meanwhile the reader can thought the Hypothesis in A. as a continuity assumption.

**Note 5.1** We want to provide a possible explanation for the ambiguity the reader will face in using this first Postulate. The author is requesting that our set of combinations is a total order, in such a way that we are able to construct a "ranking" for the considered elements. Since the hypothesis in A. will be used on a continuous set (not countable) our idea is that a more "traditional continuity hypothesis" would fit better in this context. Furthermore it is not completely clear the idea behind the choice of considering the two indices $r$ and $R$. Our opinion is the author looked for the existence of a non-singular set dividing the preferred and non-preferred combinations.

We will now face with the more familiar geometrical approach. Let us consider for simplicity a two dimension consumption space, and define $M(x_1, x_2)$ as the initial position of our agent, where $(x_1, x_2)$ represent the coordinates of $M$.

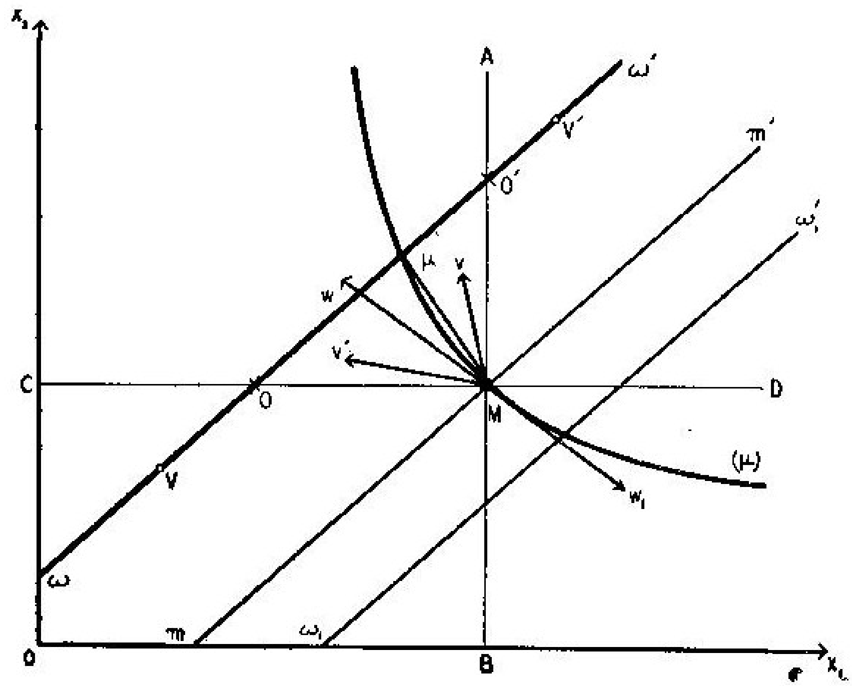


FIGURE 5.1

At this point the author introduces his second Postulate:

B. There is no saturation point.

Once again this hypothesis is stated in a "natural science" way more than as an economical fact. Postulate B. means for that "the individual will prefer to $M$ any other position within the right angle $A\hat{M}D$, and that he will take the trouble to move from $M(x_1, x_2)$ to $M(x_1 + \triangle x_1, x_2 + \triangle x_2)$ if he can do so without any further conditions. On the contrary, $M$ will be preferred to any combination within the angle $C\hat{M}B$"

**Note 5.2** The hypothesis made in B., as the one presented in A., looks in some sense not appropriate. Without any convexity assumption on the demand many results would not be achieved in what follows. The reader should assume convexity when necessary in the rest of this chapter.

Let us start by considering a positive straight line $\overline{\omega\omega'}$ and all the points lying on this set. These combinations form a preferential set. All the points on the north-west of $O'$ such as $V'$ are preferred to $M$, while those on the south-east of $O$ such as $V$ are non-preferred to $M$. Postulate A. guarantees the existence of a unique point $\mu$ representing an indifference combination to $M$. Now let us consider the family of lines parallel to $\overline{\omega\omega'}$ intersecting the $\overline{CD}$ from $O$ to $M$. For each line we can define a point $\mu$ indifferent to $M$. Hence it is possible to define the locus of $\mu$. We can construct the two tangent lines to the locus of $\mu$ in $M$, which are $\overrightarrow{Mw}$ and $\overrightarrow{Mw_1}$, respectively.

Through this construction we find the *prefence* and *nonpreference* directions. The first will be all the directions from $w_1$ up to $w$ obtained through an anticlockwise rotation (e.g. $v$). All the others will be nonproference directions (e.g. $v'$). What we mean by *preference* direction is that the consumer will move in that sense when possible.

At this point the third postulate is presented:

C. The limiting directions $\overrightarrow{Mw}$ and $\overrightarrow{Mw_1}$ are vertically opposite.

The hypotesis in C. is essentially a smoothness assumption. What the author wants to state by postulating C. is that the direction identified by $\overline{ww_1}$ is somehow the "indifference direction". What does this mean? The idea is that the locus of $\mu$ will be the indifference curve we were looking for and the straight line $\overline{ww_1}$ is the tangent to the curve. Notice that we are not yet "authorized" to speak about indifference curves as we have not defined them. What we can say is that "the assumption expresses the fact that the individual will exchange either $x_1$ for $x_2$ or $x_2$ for $x_1$ at any given rate of exchange, with the exception of that rate which equals the slope of the corresponding indifference elements". Roughly speaking, C. guarantees the absence of points of non-differentiability and this assumption guarantees the possibility of proceeding in our construction.

The last assumption refers to the behaviour of the indifference curve when the direction of $\overline{\omega\omega'}$ changes:

D. The indifference direction at any point is uniquely determined.

Let us now suppose that Postulate A, B, C and D are simultaneously valid, then a construction such as that in FIGURE 5.1 can be considered. Consumer's preference will be described through a differential equation as follows:

$$\varphi_1(x_1, x_2)dx_1 + \varphi_2(x_1, x_2)dx_2 = 0,\ with\, \varphi_1(\,), \varphi_2(\,) \geq 0, \tag{5.1}$$

where the inequality is satisfied strictly at least for one of the two functions.

The construction made so far may be traced in the case of three or more goods. When, for example, we have the case of a 3-*goods-economy* we should think of "the case where the choice of the individual is limited to the combinations represented by a plane which passes through a preference direction positive inclined with respect to all coordinate axes, we reach the result that the indifference element is represented by the total differential equation

$$\varphi_1(x_1, x_2, x_3)dx_1 + \varphi_2(x_1, x_2, x_3)dx_2 + \varphi_3(x_1, x_2, x_3)dx_3 = 0, \tag{5.2}$$

which in the $n - goods$ case becomes

$$\sum_{i=1}^{n} \varphi_i(x_1, x_2, ..., x_n)dx_i = 0. \tag{5.3}$$

The main idea is that a direction defined by an increment $(\triangle x_1, \triangle x_2, ..., \triangle x_n)$ is an indifferent direction if $\sum_{i=1}^{n} \varphi_i(x_1, x_2, ..., x_n)\triangle x_i = 0$. At this point the author specifies that some other conditions on the function $\varphi_i(\,)$ should be added in (5.3) to obtain the result we are looking for. If for the case expressed by (5.1) we have not "integrability problem" or at least we solved this problem (cfr. [18]), when we consider a *more-than-*2*-goods-economy* the situation could become a little bit "tricky". In this regard Professor Georgescu introduced a "stability equilibrium concept". Let us be given the point $M$

defined above and consider an indifferent direction defined by $\overrightarrow{MM'}$. That is, $M'$ is chosen is such a way that $\overrightarrow{MM'}$ is an indifference direction for $M$. Now we will face an equilibrium stability situation if the direction defined by $\overrightarrow{MM'}$ is a non-preference direction for $M'$. The condition of stability of an equilibrium in mechanics can be satisfied, roughly speaking, if when we move a body "slightly" from one point of equilibrium to an other it will naturally go back to the first one. Let us analyze in more detail this condition.

As we already did in previous situations (cfr. chapter 1) we will choose the function $\varphi_1(\,)$ as numeraire and we will define $n-1$ new functions given by $B_2 = \frac{\varphi_2(\,)}{\varphi_1(\,)}$, $B_3 = \frac{\varphi_3(\,)}{\varphi_1(\,)}$, ... , $B_n = \frac{\varphi_n(\,)}{\varphi_1(\,)}$.

Dividing (5.3) by $\varphi_1(\,)$ and considering that $\overrightarrow{MM'}$ is an indifference direction, we get

$$\sum_{i=1}^{n} \frac{\varphi_i}{\varphi_1} \triangle x_i = \triangle x_1 + \sum_{i=2}^{n} B_i \triangle x_i = L = 0, \qquad (5.4)$$

with $\triangle x_i$ sufficiently small.

Let us consider the first-order Taylor approximation of $B_1, B_2, ..., B_n$ in $x + \triangle x = (x_1 + \triangle x_1, ..., x + \triangle x_n)$ defined in such a way that it results exactly $M' = M'(x + \triangle x)$. If we do not consider the terms from the second order on, we get

$$B_i(x + \triangle x) \cong B_i(x) + \sum_{j=1}^{n} \frac{\delta B_i(x)}{\delta x_j} \triangle x_j, \; for\, every\, i \in \{2, ..., n\}\,. \qquad (5.5)$$

When we introduced (5.3) we specified how the sign of equality in the equation is equivalent to say that the direction defined by an increment $(\triangle x_1, \triangle x_2, ..., \triangle x_n)$ is an indifferent direction. The condition of the stable equilibrium would be given by analogy requiring that (5.3) holds with strict inequality ($<$) for $M'$ when the direction is given by $\overrightarrow{MM'} = (\triangle x_1, \triangle x_2, ...,$ $\triangle x_n)$. I am sure that no ambiguity arises in using the same symbolism in denoting the general increment $(\triangle x_1, \triangle x_2, ..., \triangle x_n)$ and the specific one for $\overrightarrow{MM'}$.

Hence in our case it must result:

$$\triangle x_1 + \sum_{i=2}^{n} B_i(x + \triangle x)\triangle x_i < 0. \tag{5.6}$$

(5.6) can be restated, in the light of (5.4) and (5.5), as:

$$\sum_{i=2}^{n}\sum_{j=1}^{n} \frac{\delta B_i(x)}{\delta x_j}\triangle x_j \triangle x_i < 0. \tag{5.7}$$

I want to stress that condition (5.7) really looks like a condition of negative definiteness for a certain unspecified matrix $\overline{S}$.

We just confine ourselves to the intuition. Starting from this consideration we analyze how (5.7) can be considered as satisfied when subject to (5.4).

In fact rearranging (5.7) we have:

$$\sum_{i=2}^{n}(\frac{\delta B_i(x)}{\delta x_1}\triangle x_1 + \sum_{j=2}^{n}\frac{\delta B_i(x)}{\delta x_j}\triangle x_j)\triangle x_i =$$

$$\triangle x_1 \sum_{i=2}^{n}\frac{\delta B_i(x)}{\delta x_1}\triangle x_i + \sum_{i=2}^{n}(\sum_{j=2}^{n}\frac{\delta B_i(x)}{\delta x_j}\triangle x_j)\triangle x_i < 0. \tag{5.8}$$

We can obtain, from (5.4), $\triangle x_1 = -\sum_{j=2}^{n} B_j \triangle x_j$, in such a way that (5.8) is equivalent to

$$F = -\sum_{i=2}^{n}(\sum_{j=2}^{n} B_j \frac{\delta B_i(x)}{\delta x_1}\triangle x_j)\triangle x_i + \sum_{i=2}^{n}(\sum_{j=2}^{n}\frac{\delta B_i(x)}{\delta x_j}\triangle x_j)\triangle x_i =$$

$$\sum_{i=2}^{n}\sum_{j=2}^{n}(\frac{\delta B_i(x)}{\delta x_j} - B_j\frac{\delta B_i(x)}{\delta x_1})\triangle x_j \triangle x_i < 0. \tag{5.9}$$

Now the intuition we grasped becomes exploitable. Indeed condition (5.9) is now equivalent to the negative semidefiniteness of the $(n-1) \times (n-1)$ square matrix $S = \| \frac{\delta B_i(x)}{\delta x_j} - B_j \frac{\delta B_i(x)}{\delta x_1} \|_{i,j=2,\ldots,n}$ . It is possible to rewrite the inequality in (5.9) as

$$F = \frac{1}{2} \sum_{i=2}^{n} \sum_{j=2}^{n} f_{ij} \triangle x_j \triangle x_i < 0, \tag{5.10}$$

where it is assumed

$$f_{ij} = f_{ji} = \frac{\delta B_i(x)}{\delta x_j} - B_j \frac{\delta B_i(x)}{\delta x_1} + \frac{\delta B_j(x)}{\delta x_i} - B_i \frac{\delta B_j(x)}{\delta x_1}, \ i, j = 2, 3, ..., n. \tag{5.11}$$

**Note 5.3** The author seems to come, or at least to get very close to the fundamental condition of negative semidefiniteness of the Slutsky Matrix, while he completely "bypasses" any symmetry condition.

Hence Matrix $S$ will be defined as

$$S = \left\| \begin{array}{cccc} f_{22} & f_{23} & \cdots & f_{2n} \\ f_{32} & \cdots & & \\ \cdots & & & \\ f_{n2} & \cdots & & f_{nn} \end{array} \right\|,$$

and using a familiar criterion ( ) we can say that matrix $S$ is negative semidefinite $iff$

$$|f_{22}| < 0; \begin{vmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{vmatrix} > 0; \ ... \tag{5.12}$$

By some tedious algebraic operations that I will not dwell on we arrive to say that "the system in (5.7) subject to (5.4) is satisfied when the principal minors, including always the elements of the first row and column, of the determinant

$$D = \left\| \begin{matrix} 0 & 1 & B_2 & B_3 & ... & B_n \\ 1 & 0 & B_{2,1} & B_{3,1} & ... & B_{n,1} \\ B_2 & B_{2,1} & 2B_{2,2} & B_{2,3}+B_{3,2} & ... & B_{2,n}+B_{n,2} \\ B_3 & B_{3,1} & B_{3,2}+B_{2,3} & 2B_{3,3} & ... & B_{3,n}+B_{n,3} \\ ... & ... & ... & ... & & ... \\ B_n & B_{n,1} & B_{n,2}+B_{2,n} & B_{n,3}+B_{3,n} & ... & 2B_{n,n} \end{matrix} \right\| \tag{5.13}$$

are, starting with the third order, alternatively positive and negative"; i.e,

$$\left| \begin{matrix} 0 & 1 & B_2 \\ 1 & 0 & B_{2,1} \\ B_2 & B_{2,1} & 2B_{2,2} \end{matrix} \right| > 0; \left| \begin{matrix} 0 & 1 & B_2 & B_3 \\ 1 & 0 & B_{2,1} & B_{3,1} \\ B_2 & B_{2,1} & 2B_{2,2} & B_{2,3}+B_{3,2} \\ B_3 & B_{3,1} & B_{3,2}+B_{2,3} & 2B_{3,3} \end{matrix} \right| < 0, .... \tag{5.14}$$

Now let us denote by $\Phi$ the expression in (5.6), and take its second-order Taylor expansion:

$$\Phi = \triangle x_1 + \sum_{i=2}^{n} B_i(x+\triangle x)\triangle x_i \approxeq$$

$$\approxeq L + a^2L^2 - L_1^2 - L_2^2 - ... - L_{n-1}^2, \tag{5.15}$$

where $L_1, L_2, ..., L_{n-1}$ are linear functions of $\triangle x_1, \triangle x_2, ..., \triangle x_n$, $L$ is defined in (5.4) and the second equality in (5.15) is a consequence of (5.14).

We will not dwell with the mathematical details of what follows but it is important to grasp the intuition.

$\Phi = 0$ represents a quadratic and $L = 0$ is its tangent plane. Let us consider a small movement from the origin, then it results:

$$\Phi < 0, \, if \, L \leq 0.$$

Hence we can restate (5.6) as:

$$\triangle x_1 + \sum_{i=2}^{n} B_i(x + \triangle x)\triangle x_i < 0, \, if \, \triangle x_1 + \sum_{i=2}^{n} B_i(x)\triangle x_i \leq 0. \tag{5.16}$$

(5.16) becomes our necessary and sufficient stability condition and can be interpreted as "any direction $(\triangle x_1, \triangle x_2, ..., \triangle x_n)$ which constitutes for $M(x_1, x_2, ..., x_n)$ either an indifference or a non-preference direction, will be a non-preference direction for the infinitesimally-near point $M(x_1+\triangle x_1, x_2+\triangle x_2, ..., x_n + \triangle x_n)$".

# 6 "The Problem of Integrability in Utility Theory", P.A. Samuelson

We want to present the main results reached in "The Problem of Integrability in Utility Theory" by P. A. Samuelson (cfr. [18]). As usual let us consider an economy with $n$ consumption goods. Let's consider the same hypotheses we made so far on the demand function $x(p, M) = (x_1(p, M), x_2(p, M), ..., x_n(p, M))$, i.e.

i) $x(\ )$ is differentiable;

ii) $x(\ )$ is homogeneous of degree zero;

iii) budget exhaustion.

At this point the author introduces a further hypothesis:

iv) the demand function is invertible.

We want to find a mathematical criterion equivalent to assumption iv).

Using i), ii) and iii) from Proposition 3.6 we get

$$S(p,M)p = \left\| \begin{array}{ccc} \frac{dh_1}{dp_1} & \cdots & \frac{dh_1}{dp_n} \\ \cdots & \cdots & \cdots \\ \frac{dh_n}{dp_1} & \cdots & \frac{dh_n}{dp_n} \end{array} \right\| \begin{pmatrix} p_1 \\ \cdots \\ p_n \end{pmatrix} = \tag{6.1}$$

$$= \left\| \begin{array}{ccc} \frac{\delta x_1}{\delta p_1} + \frac{\delta x_1}{\delta M} x_1 & \cdots & \frac{\delta x_1}{\delta p_n} + \frac{\delta x_1}{\delta M} x_1 \\ \cdots & \cdots & \cdots \\ \frac{\delta x_n}{\delta p_1} + \frac{\delta x_1}{\delta M} x_n & \cdots & \frac{\delta x_n}{\delta p_n} + \frac{\delta x_n}{\delta M} x_n \end{array} \right\| \begin{pmatrix} p_1 \\ \cdots \\ p_n \end{pmatrix} = 0$$

Since we assumed positive prices and positive income (6.1) implies $S(p, M)$ to be singular. In fact, ab absurdo, $\det S(p, M) \neq 0$ the nul vector price would be the only solution of the system $Sp = 0$.

Let us consider the implicit function theorem (cfr. [20]) for our framework.

We have $F : \mathbb{R}^{n+n+1} \to \mathbb{R}^n$, where

$$F(x_1, x_2, ..., x_n, p_1, p_2, ...., p_n, M) = \begin{pmatrix} x_1 - x_1(p_1, p_2, ..., p_n, M) \\ ... \\ x_n - x_n(p_1, p_2, ..., p_n, M) \\ M - M \end{pmatrix}$$

and if it results that the matrix

$$\begin{Vmatrix} \frac{\delta x_1}{\delta p_1} & ... & \frac{\delta x_1}{\delta p_n} \\ ... & ... & ... \\ \frac{\delta x_n}{\delta p_1} & ... & \frac{\delta x_n}{\delta p_n} \end{Vmatrix} \, is \, non \, singular, \tag{6.2}$$

(from det$\begin{Vmatrix} \frac{\delta x_1}{\delta p_1} & ... & \frac{\delta x_1}{\delta p_n} & \frac{\delta x_1}{\delta M} \\ ... & ... & ... & ... \\ \frac{\delta x_n}{\delta p_1} & ... & \frac{\delta x_n}{\delta p_n} & \frac{\delta x_n}{\delta M} \\ 0 & ... & 0 & 1 \end{Vmatrix} \neq 0$), then it is possible to get

$$\begin{cases} p_1 = p_1(x_1, x_2, ..., x_n) \\ ... \\ p_n = p_n(x_1, x_2, ..., x_n) \\ M = M(x_1, x_2, ..., x_n) \end{cases}.$$

Hence condition iv) is completely equivalent to (6.2).

Let's consider, for convenience, a new set of variables. By fixing the $n-th$ price as the numeraire we have:

$$(B_1, B_2, ..., B_{n-1}, B_n) = (\frac{p_1}{p_n}, \frac{p_n}{p_n}, ..., \frac{p_{n-1}}{p_n}, \frac{M}{p_n}).$$

In this case when hypothesis iv) holds it is possible to invert the demand function to get

$$\begin{cases} B_1 = B_1(x_1, x_2, ..., x_n) \\ ... \\ B_{n-1} = B_{n-1}(x_1, x_2, ..., x_n) \end{cases} ; \qquad (6.3)$$

and since iii) is valid we get $B_n = \frac{M}{p_n} = B_1x_1 + B_2x_2 + ... + B_{n-1}x_{n-1} + \frac{p_n}{p_n}x_n$.

We can note that the implicit function theorem is valid in this new framework if the determinant of the jacobian of $F^*$ with respect to $B_1$,$B_2$,...,$B_n$ is non-zero, where

$$F^* = \begin{pmatrix} x_1 - x_1(\frac{p_1}{p_n}, ..., \frac{p_{n-1}}{p_n}, \frac{M}{p_n}) \\ ... \\ x_n - x_n(\frac{p_1}{p_n}, ..., \frac{p_{n-1}}{p_n}, \frac{M}{p_n}) \end{pmatrix} = \begin{pmatrix} x_1 - x_1(B_1, ..., B_n) \\ ... \\ x_n - x_n(B_1, ..., B_n) \end{pmatrix}.$$

The implicit function theorem states that

$$B_{x_1,x_2,...,x_n} = -J^{-1}_{B_1,...,B_n}(F)J_{x_1,x_2,...,x_n}(F).$$

Since $J_{x_1,x_2,...,x_n}(F^*)$ is the $n \times n$ identity matrix we have that

$$J^{-1}_{B_1,...,B_n}(F^*) = \begin{pmatrix} \frac{\delta B_1}{\delta x_1} & ... & \frac{\delta B_1}{\delta x_n} \\ ... & ... & ... \\ \frac{\delta B_n}{\delta x_1} & ... & \frac{\delta B_n}{\delta x_n} \end{pmatrix} = \begin{pmatrix} (\frac{\delta B_i}{\delta x_j})_{i,j=1,...,n-1} & (\frac{\delta B_i}{\delta x_n})_{i=1,...n-1} \\ (\frac{\delta B_n}{\delta x_j})_{j=1,...n-1} & \frac{\delta B_n}{\delta x_n} \end{pmatrix},$$

where we use a block-matrix representation in the last equality.

By using some algebraic manipulation we have

$$J^{-1}_{B_1,...,B_n}(F^*) = \left\| \begin{matrix} 1 & 0 & ... & 0 & \\ 0 & 1 & & & \\ ... & & ... & 0 & 0 \\ 0 & ... & 0 & 1 & \\ (x_j)_{j=1,...,n-1} & & & & 1 \end{matrix} \right\| * \left\| \begin{matrix} (\frac{\delta B_i}{\delta x_j} - B_j\frac{\delta B_i}{\delta x_n})_{i,j=1,...,n-1} & (\frac{\delta B_i}{\delta x_n})_{i=1,...n-1} \\ 0 & 1 \end{matrix} \right\| *$$

$$* \left\| \begin{array}{cccc|c} 1 & 0 & ... & 0 & \\ 0 & 1 & & & 0 \\ ... & & ... & 0 & \\ 0 & ... & 0 & 1 & \\ \multicolumn{4}{c}{(B_j)_{j=1,...,n-1}} & 1 \end{array} \right\| . \tag{6.4}$$

It has been shown that if an integrable utility function $U()$ exists then the Slutsky Matrix $S = (\frac{\delta x_i}{\delta p_j} + \frac{\delta x_i}{\delta M} x_j)_{i,j=1,...,n}$ is symmetric, as the equality $\frac{\delta^2 U}{\delta p_i \delta p_j} = \frac{\delta^2 U}{\delta p_j \delta p_i}$ must hold. We can rewrite

$$S = \left\| \begin{array}{cc} (\frac{\delta x_i}{\delta p_j} + \frac{\delta x_i}{\delta M} x_j)_{i,j=1,...,n-1} & (\frac{\delta x_i}{\delta p_n} + \frac{\delta x_i}{\delta M} x_n)_{i=1,...,n-1} \\ (\frac{\delta x_n}{\delta p_j} + \frac{\delta x_n}{\delta M} x_j)_{j=1,...,n-1} & \frac{\delta x_n}{\delta p_n} + \frac{\delta x_n}{\delta M} x_n \end{array} \right\| = \left\| \begin{array}{cc} S_n & s_{in} \\ s_{nj} & s_{nn} \end{array} \right\| . \tag{6.5}$$

From (6.1) we have that S is a singular matrix. Hence it would be sufficient to prove the symmetry of $S_n$. That is we have $\frac{(n-1)(n-2)}{2}$ conditions to verify.

It is possible to show that

$$J_{B_1,...,B_n}(F) = J = \left\| \begin{array}{cc} (\frac{\delta x_i}{\delta p_j})_{i,j=1,...,n-1} & (\frac{\delta x_i}{\delta p_n})_{i=1,...,n-1} \\ (\frac{\delta x_n}{\delta p_j})_{j=1,...,n-1} & \frac{\delta x_n}{\delta p_n} \end{array} \right\| .$$

Hence we can rewrite

$$\left\| \begin{array}{cc} S_n & (\frac{\delta x_i}{\delta p_n})_{i=1,...,n-1} \\ s_{nj} & \frac{\delta x_n}{\delta p_n} \end{array} \right\| = J \left\| \begin{array}{cccc|c} 1 & 0 & ... & 0 & \\ 0 & 1 & & & 0 \\ ... & & ... & 0 & \\ 0 & ... & 0 & 1 & \\ \multicolumn{4}{c}{(x_j)_{j=1,...,n-1}} & 1 \end{array} \right\| ,$$

and by using (6.4) we get

$$\left\| \begin{array}{cc} S_n & (\frac{\delta x_i}{\delta p_n})_{i=1,...,n-1} \\ s_{nj} & \frac{\delta x_n}{\delta p_n} \end{array} \right\| = \left\| \begin{array}{cc} \begin{array}{cccc} 1 & 0 & ... & 0 \\ 0 & 1 & & \\ ... & & ... & 0 \\ 0 & ... & 0 & 1 \end{array} & 0 \\ (B_j)_{j=1,...,n-1} & 1 \end{array} \right\|^{-1} *$$

$$* \left\| \begin{array}{cc} (\frac{\delta B_i}{\delta x_j} - B_j \frac{\delta B_i}{\delta x_n})_{i,j=1,...,n-1} & (\frac{\delta B_i}{\delta x_n})_{i=1,...n-1} \\ 0 & 1 \end{array} \right\|^{-1} =$$

$$= \left\| \begin{array}{cc} [(\frac{\delta B_i}{\delta x_j} - B_j \frac{\delta B_i}{\delta x_n})_{i,j=1,...,n-1}]^{-1} & (\,)_{i=1,...n-1} \\ (\,)_{j=1,...,n-1} & (\,) \end{array} \right\| .$$

In order to prove the symmetry of (6.4) we just have to show that $[(\frac{\delta B_i}{\delta x_j} - B_j \frac{\delta B_i}{\delta x_n})_{i,j=1,...,n-1}]$ is symmetric. What we found is exactly the condition presented by Antonelli (cfr. 2.23).

Using (6.3) we define the following system of differential equations

$$\begin{cases} -\frac{\delta x_n}{\delta x_1} = B_1(x_1, x_2, ..., x_n) \\ ... \\ -\frac{\delta x_n}{\delta x_{n-1}} = B_{n-1}(x_1, x_2, ..., x_n) \end{cases} ; \tag{6.6}$$

and the total differential equation

$$B_1 dx_1 + B_2 dx_2 + ... + B_{n-1} dx_{n-1} + dx_n = 0. \tag{6.7}$$

We term an "indifference path" a path satisfying (6.7). We can consider

$$Q_1 dx_1 + Q_2 dx_2 + ... + Q_{n-1} dx_{n-1} + Q_n dx_n = 0, \tag{6.8}$$

in place of (6.7), where $Q_n(\,)$ is a non-zero function and $Q_i = Q_n B_i$.

Let us recall some fundamental results on the theory of total differential equation.

**Definition 6.1 (cfr. [7])** Let $V$ be a real linear space and let $V^*$ be the space $Hom\,(V, \mathbb{R})$. Then the map $<\cdot,\cdot>: V^* \times V \longrightarrow \mathbb{R}$ associating to each couple $(f, v)$ in $V^* \times V$ the value $f\,(v)$ is said to be the "duality" between $V^*$ and $V$.

**Definition 6.2.I** Let $\Psi$ be an open set of $\mathbb{R}^n$ then $\omega : \Psi \to (\mathbb{R}^n)^*$ is said to be a linear differential form in $\Psi$.

We can restate Definition 6.2.I as

**Definition 6.2.II** Let $\Psi$ be an open set of $\mathbb{R}^n$; then the map $\omega$ defined on $\Psi$ is said to be a differential form if for every $x \in \Psi$ it is associated a linear form $\omega(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$, that is for every vector $x \in \Psi$ we have a vector $g\,(x) = (g_1(x), ...., g_n(x)) \in \mathbb{R}^n$ such that

$$\omega(x)(v) = g(x) \cdot v\, for\, every\, v \in \mathbb{R}^n, \tag{6.9}$$

where $g\,(\,)$ is said to be the coefficients vector of the differential form $\omega$. We will say that $\omega$ is continuous if $g(\,)$ is continuous and by analogy $\omega$ is said to be of class $C^k$ when $g(\,)$ is of class $C^k$.

Let us note that, when we denote by $dx_i$ the $i-th$ linear form associating with every vector $v$ its $i-th$ component, we can rewrite (6.9) in the more usual aspect as:

$$\omega(x_1, ...., x_n) = \sum_{i=1}^{n} g_i(x_1, ..., x_n)dx_i. \tag{6.10}$$

**Definition 6.3** Let $\omega$ be continuous, then we can define the "line integral" on the curve $\gamma \in C^1(x^0, x^1, \Psi)$ as

$$\int_\gamma \omega = \int_a^b g(\gamma(t)) \cdot \gamma'(t)dt, \tag{6.11}$$

where $a, b$ are the extremes of the interval along which the curve $\gamma$ is defined.

**Definition 6.4** Let $\omega$ be a differential form as above and $g : \Psi \longrightarrow \mathbb{R}^n$ the coefficients vector of $\omega$. $\omega$ is said to be an "exact differential" if there exists a function $f : \Psi \longrightarrow \mathbb{R}$ such that $g = \nabla f$ or equivalently $\omega = df$.

**Proposition 6.1 (cfr. [6])** Let $\omega$ be a continuous differential form. Then it is equivalent:

d.i) $\omega$ is an exact differential;

d.ii) $\int_\gamma \omega = 0$, for every $x \in \Psi$ and for every $\gamma \in C^1(x, x, \Psi)$;

d.iii) $\int_\gamma \omega = \int_\mu \omega$, for every $x^0, x^1 \in \Psi$ and for every $\gamma, \mu \in C^1(x^0, x^1, \Psi)$.

d.ii) means that the integral over a closed path is 0 independently from the path, while d.iii) is equivalent to say that any curves defined between the two integration points on which the integral is computed guarantees the same value for the integration.

**Definition 6.5** Under the hypotheses made so far a $C^1$ differential form is said to be "closed" iff:

$$\frac{\delta g_i}{\delta x_j} = \frac{\delta g_j}{\delta x_i}, \; for\, every\, i, j = 1, ..., n.$$

**Lemma 6.1 (Poincaré)** Let $\omega$ be a differential form of class $C^1$. If $\omega$ is closed and defined on an open star domain then $\omega$ is an exact differential.

Let us go back to our "integrability problem".

Referring to expression in (6.8) Samuelson says "Only if $\sum Q^k dx_k = dQ$, is an *exact differential* with $Q^k = \frac{\delta Q}{\delta x_k} = Q_k$ and $Q_{kj} = Q_{jk}$, will such an integral be *always the same* for *different* paths between two specified end-points; and only in this case we can be sure that if we go from point A to point B by one path and return back to A by an other path, the value of the integral will none the less be zero over the round trip, indicating that A is exactly as good as itself and no better."

The link with the Poincaré's Lemma is straightforward.

Instead of asking (6.8) to be an exact differential, we will look for the existence of an integrating factor. That is, we are asking whether there exists a function $I\left(x_1, ..., x_n\right)$ such that

$$I \sum Q^k dx_k = dV \, is \, an \, exact \, integral. \tag{6.12}$$

(6.12) is equivalent to ask for the existence of a set of variable proportional to $B^1, B^2, ..., B^{n-1}$guaranteeing the differential form in (6.7) to be integrable.

In order to have the hypotheses of the Poincaré's Lemma we should ask for

$$\frac{\delta^2 V}{\delta x_j \delta x_i} = \frac{\delta I}{\delta x_i} Q^j + I \frac{\delta Q^j}{\delta x_i} = \frac{\delta I}{\delta x_j} Q^i + I \frac{\delta Q^i}{\delta x_j} = \frac{\delta^2 V}{\delta x_i \delta x_j},$$

$$for \, every \, i, j = 1, ..., n, \; i \neq j. \tag{6.13}$$

or, equivalently,

$$\frac{\delta I}{\delta x_i} Q^j - \frac{\delta I}{\delta x_j} Q^i + I \left( \frac{\delta Q^j}{\delta x_i} - \frac{\delta Q^i}{\delta x_j} \right) = 0. \tag{6.14}$$

After some simple algebraic manipulation (6.14) can be restated as

$$Q^i \left( \frac{\delta Q^j}{\delta x_n} - \frac{\delta Q^n}{\delta x_j} \right) + Q^j \left( \frac{\delta Q^n}{\delta x_i} - \frac{\delta Q^i}{\delta x_n} \right) + Q^n \left( \frac{\delta Q^i}{\delta x_j} - \frac{\delta Q^j}{\delta x_i} \right) = 0,$$

$$for \, every \, i, j = 1, ..., n-1, \; i \neq j. \tag{6.15}$$

Let us now recall that we have previously defined $Q^k = B^k Q^n$, $k \in$

$\{1, 2, ..., n\}$. Hence after some algebraic manipulation (cfr. [24]) we can restate (6.15) as

$$\frac{\delta B^i}{\delta x_j} - B^j \frac{\delta B^i}{\delta x_n} = \frac{\delta B^j}{\delta x_i} - B^i \frac{\delta B^j}{\delta x_n},$$

$$for\, every\, i, j = 1, ..., n-1,\ i \neq j, \tag{6.16}$$

which are exactly Antonelli's integrability conditions (cfr. 2.23).

As we already mentioned in the introduction we want to stress that the conditions in 6.16 are not sufficient to solve our integrability problem. Next chapter will clarify this point.

# 7 "Utility, Demand and Preference", Ch. 6, L. Hurwicz & H. Uzawa

Chapter 6 of [3] by L. Hurwicz and H. Uzawa can be considered as "the final solution" of the "integrability problem". In this section we will focus on this work. Some additional considerations will complete our analysis.

As usual let us consider an economy where an $n - dimension$ vector of consumption goods is represented by $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n_+$ and prices are given by the n-tuple $(p_1, p_2, ..., p_n) \in \mathbb{R}^n_{++} = \Pi$. We will denote by $x = x(p, M)$ the demand function associating to the couple $(p, M)$ (where $M$ measure the income) the consumption bundle $x$. Let's assume price system $p$ and income $M$ range in the set $\Omega = \left\{(p, M) : p \in \mathbb{R}^n_{++},\ M \in \mathbb{R}^n_+\right\}$.

We will consider the following assumptions when necessary:

(A) $x(p, M)$ is a semipositive single-valued function defined for each $(p, M) \in \Omega$;

(B) the budget constraint is satisfied with equality, i.e $px(p, M) = M$, for each $(p, M) \in \Omega$;

(D) $x_i(p, M)$ is differentiable on $\Omega$ for every $i = 1, 2, ..., n$;

(E) For any positive $\alpha', \alpha''$ there exists a positive $K_{\alpha',\alpha''}$ such that, for each $i = 1, 2, ..., n$, it is $\mid \frac{\delta x_i(p,M)}{\delta M} \mid \leq K_{\alpha',\alpha''} \, for \, (p, M) \epsilon \Omega \, and \, p_j \in [\alpha', \alpha'']$, for any $j = 1, 2, ..., n$.

Let us denote by $X$ the image set of the function $x(p, M)$. Using the demand function and assuming (D) it is possible to define the Slutsky matrix as

$$S(p, M) = \ \|S_{i,j}(p, M)\|_{i,j=1,..,n} \ , \tag{7.1}$$

$$where \, S_{i,j}(p, M) = \frac{\delta x_i(p, M)}{\delta p_j} + \frac{\delta x_i(p, M)}{\delta M} x_j(p, M).$$

(7.1) is said to be

(S) *symmetric* if $S_{i,j}(p, M) = S_{j,i}(p, M)$ for every $i, j = 1, 2, ..., n$ and for each $(p, M) \in \Omega$;

(N) *negative semindefinite* if $v^T S(p, M)v \leq 0$ for all $v \epsilon \mathbb{R}^n$ and for all $(p, M) \in \Omega$.

We will now present a theorem guaranteeing property (S) and (N) to be verified starting from assuming some regularity conditions for the utility maximization.

**Theorem 7.1** Let us consider a preference relation $R$ defined on the set of all conceivable commodity bundles with nonnegative vector with the following properties:

P.I* Reflexivity;

P.II* Completeness;

P.III* Transitivity

and let $x(\,)$, satisfying (A), (B) and (D), be the unique maximizer of the preference relation subject to the budget inequality $px \leq M$.

Then the Slutsky matrix $S(p, M)$ satisfies properties (S) and (N).

**Proof** Let $(p^0, M^0)$ be a point in $\Omega$ with $x^0 = x(p^0, M^0)$. We can define the set

$$K = \left\{ x \in \mathbb{R}^n_+ \, such \, that \, x R x^0 \right\}. \tag{7.2}$$

Let us introduce the so called "expenditure function" as

$$\mu(p) = \inf_{x \epsilon K} px \tag{7.3}$$

We first prove the concavity of $\mu(\,)$.

Let's consider two couples $(p, x)$ , $(p^1, x^1)$ satisfying $\mu(p) = px$ and $\mu(p^1) = p^1x^1$ respectively.

Then let's define $p^2 = tp + (1-t)p^1$, $t \in [0, 1]$ as the convex combination of the two prices systems $p, p^1$ and let $p^2x^2 = \mu(p^2)$.

We have $p^2x^2 = tpx^2+(1-t)p^1x^2$, where, in general, it is $\begin{cases} px^2 \geq \mu(p) \\ p^1x^2 \geq \mu(p^1) \end{cases}$.

Hence, using $p^2x^2 = \mu(p^2)$ it is $\mu(p^2) = p^2x^2 \geq t\mu(p) + (1-t)\mu(p^1)$.

From the concavity we can immediately obtain the continuity of $\mu(\,)$.

Now let us define

$$X(p) = x(p, \mu(p))\; for\, all\, p \in \Pi. \tag{7.4}$$

$X(\,)$ is continuous as combination of continuous functions. Now, using (B) we get

$$pX(p) = \mu(p)\; for\, all\, p \in \Pi. \tag{7.5}$$

We want to prove that for any $p' \in \Pi$ it is

$$pX(p) \leq pX(p') \tag{7.6}$$

Let's consider a sequence $(x^n)_{n\epsilon\mathbb{N}}$ such that $x^nPx$, for every $n\epsilon\mathbb{N}$ and $\lim_{n\to\infty} p'x^n = \mu(p') = \inf_{x\epsilon K} p'x$.

By the definition of inf, for every positive real number$\epsilon$, there exists an index $n_\epsilon$ such that $p'x^n < \mu(p') + \epsilon$ for every $n > n^\epsilon$.

Let $x^\epsilon = x(p', \mu(p') + \epsilon)$. Using the definition of demand function, $x^\epsilon$ is the unique maximizer subject to the constraint $p'x \leq \mu(p') + \epsilon$. Therefore, $p'x^n < \mu(p') + \epsilon$ implies $x^\epsilon Px^n$ which means $x^\epsilon \in K$.

Using the continuity of the demand function $x(\,)$ we have $\lim_{\epsilon\to 0} x^\epsilon = X(p')$.

Let us consider the two case:

1) $\qquad X(p) \in K;$

2) $\qquad X(p) \notin K.$

In 1), $X\,(p)$ minimizes over $K$ the function $px$. And since $x^{\epsilon} \in K$, it is $pX\,(p) \leq px^{\epsilon}$ for all $\epsilon$. Taking the limit we get $pX\,(p) \leq pX(p')$.

In 2) $x^0PX\,(p)$. Since $x^nPx^0$, $x^{\epsilon}Px^n$, for the transitivity, $x^{\epsilon}PX(p)$. Ab absurdo suppose $pX\,(p) > pX(p')$. Using the continuity assumption, for small $\epsilon$, it is $pX\,(p) > px^{\epsilon}$. Thus $x^{\epsilon}$ is more desirable than $X\,(p)$ and cheaper at price $p$. We found a contradiction as $X\,(p)$ does not maximize satisfaction subject to the budget constraint. It must be $pX(p) \leq pX(p')$. Hence we have (7.6).

We want to prove that we can define the $\frac{\delta\mu}{\delta p_j}$ for every $j = 1, 2, ..., n$. We have:

$$\frac{\delta\mu}{\delta p_j} = X^j(p)\; for\, each\, j = 1, 2, ..., n. \tag{7.7}$$

Take two system of prices $p$, $(p + \triangle p)$, consider $X(p)$, $X(p + \triangle p)$ respectively.

We have $pX\,(p) \leq pX\,(p + \triangle p)$, that is

$$p\triangle X(p) \geq 0, \tag{7.8}$$

where $\triangle X(p) = X(p + \triangle p) - X(p)$.

Furthermore, considering (7.8), it is

$$\triangle\mu(p) = \mu(p + \triangle p) - \mu(p) = (p + \triangle p)(X(p) + \triangle X(p)) - pX(p) =$$

$$= p\triangle X(p) + \triangle pX(p) + \triangle p\triangle X(p) \geq \triangle pX(p) + \triangle p\triangle X(p).$$

Hence

$$\triangle\mu(p) \geq \triangle p X(p) + \triangle p \triangle X(p). \tag{7.9}$$

Let 's consider a particular vector $\triangle p = (0, ..., 0, \triangle p_j, 0, ..., 0)$, with $\triangle p_j > 0$, using (7.9):

$$\frac{\triangle\mu(p)}{\triangle p_j} \geq X^j(p) + \triangle X^j(p).$$

If we take the limit we have

$$\lim_{\triangle p_j \to 0^+} \frac{\triangle\mu(p)}{\triangle p_j} \geq X^j(p). \tag{7.10}$$

In fact from the continuity of $X()$, $\lim_{\triangle p_j \to 0} \triangle X^j(p) = 0$, where the existence of the limit is guaranteed by the concavity of $\mu$.

When we consider $\triangle p_j < 0$, we have, by analogy,

$$\lim_{\triangle p_j \to 0^-} \frac{\triangle\mu(p)}{\triangle p_j} \leq X^j(p). \tag{7.11}$$

Combining (7.10) and (7.11) we get

$$\lim_{\triangle p_j \to 0^- \leq 0} \frac{\triangle\mu(p)}{\triangle p_j} \leq X^j(p) \leq \lim_{\triangle p_j \to 0^+} \frac{\triangle\mu(p)}{\triangle p_j}. \tag{7.12}$$

Note that since $\mu(p)$ is concave in $p$ it is

$$\lim_{\triangle p_j \to 0^-} \frac{\triangle\mu(p)}{\triangle p_j} \geq \lim_{\triangle p_j \to 0^+} \frac{\triangle\mu(p)}{\triangle p_j}. \tag{7.13}$$

Hence, considering (7.12) we prove the existence of $\frac{\delta\mu}{\delta p_j}$ and the validity of (7.7). The continuity of $\frac{\delta\mu}{\delta p_j}$ is a direct consequence of the continuity of $X^j(p)$.

Now let us take the partial derivative of $X^j()$ with respect to $p_i$:

$$\frac{\delta X^j}{\delta p_i} = \frac{\delta x_j(p, \mu(p))}{\delta p_i} + \frac{\delta x_j(p, \mu(p))}{\delta\mu(p)} \frac{\delta\mu(p)}{\delta p_i} =$$

$$= \frac{\delta x_j(p, \mu(p))}{\delta p_i} + \frac{\delta x_j(p, \mu(p))}{\delta M} x_i(p, \mu(p)); \tag{7.14}$$

this expression is exactly $\frac{\delta^2 \mu}{\delta p^j \delta p^i}$. We want to show that $M^0 = \mu(p^0)$. By definition we have $M^0 \geq \mu(p^0)$ and in order to prove the equality it is sufficient to have $M^0 \not> \mu(p^0)$. Let us suppose that, ab absurdo, $M^0 > \mu(p^0)$, then there exists $x'$ such that $x' P x^0$ and $p^0 x^0 > p^0 x'$, that is a contradiction as we would have $x' \in x(p^0, M^0)$.

Hence for every $(p^0, M^0)$ it is

$$\frac{\delta^2 \mu}{\delta p^j \delta p^i}(p^0) = S_{ji}(p^0, M^0). \tag{7.15}$$

From (7.15), using Young Theorem on the symmetry of second derivatives (cfr. [12]) and the property of concave function we have that the Slutsky matrix just defined is symmetric and negative semidefinite, i.e. property (S) and (N) hold.

□

At this point, we would like to present some theorems guaranteeing the existence of a utility function generating the demand function, starting by assuming some regularity conditions such as the symmetry and the semidefiniteness of the Slutsky matrix.

**Theorem 7.2** Let the demand function $x(p, M)$ satisfy conditions (A), (B), (D) and (E). If the Slutsky matrix satisfies (S) and (N), then there exists a utility function $u(\,)$ defined on the range $X$ of the demand function $x(p, M)$ such that the value of $x(p, M)$ of the demand at $(p, M)$ uniquely maximizes $u(x)$ over the usual budget set.

In order to prove Theorem 7.2 we would present some preliminary results.

**Theorem 7.2.I (cfr. existence Theorem III, Mathematical Appendix, [3] )** For each $i \in \{1, ..., n\}$ let $f_i : \Omega = \Pi \times \Theta \subseteq \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$, with

2.1.i) $\Pi = \{x = (x_1, x_2, ..., x_n)\, such\, that\, x_i > 0,\ i = 1, 2, ..., n\};$

2.1.ii) $\Theta = \{z\, such\, that\, 0 \leq z < \infty\},$

satisfying

(D) for each $i \in \{1, ..., n\} f_i\,(\ ,\ )$ is differentiable in $\Omega$;

(S) $\frac{\delta f_i(x,z)}{\delta x_j} + \frac{\delta f_i(x,z)}{\delta z} f_j(x, z) = \frac{\delta f_j(x,z)}{\delta x_i} + \frac{\delta f_j(x,z)}{\delta z} f_i(x, z)$, for each $i, j \in \{1, ..., n\}$;

(O) for each $i \in \{1, ..., n\} f_i\,(x, 0) = 0$,for any $x \in \Pi$;

(E) for any positive $\alpha', \alpha''$ there exists a positive $K_{\alpha',\alpha''}$ such that, for each $i = 1, 2, ..., n$, it is $\mid \frac{\delta f_i(x,z)}{\delta z} \mid \leq K_{\alpha',\alpha''}\, for\, all\, (x, z) \in \Omega\, and\, x_j \in [\alpha', \alpha''],\ j = 1, 2, ..., n.$

Then for every $(x^0, z^0)$ there exists a unique continuous solution $\omega_{(x^0,z^0)}(x)$ of the following system:

$$(P)_{(x^0,z^0)} \begin{cases} \frac{\delta z}{\delta x_1} = f_1(x_1, x_2, ..., x_n, z) \\ \frac{\delta z}{\delta x_2} = f_2(x_1, x_2, ..., x_n, z) \\ .... \\ \frac{\delta z}{\delta x_n} = f_n(x_1, x_2, ..., x_n, z) \end{cases} . \tag{7.16}$$

**Note 7.1** Let us consider the following system of differential equation:

$$\begin{cases} \frac{\delta M}{\delta p_1} = x_1(p_1, p_2, ..., p_n, M) \\ \frac{\delta M}{\delta p_2} = x_1(p_1, p_2, ..., p_n, M) \\ .... \\ \frac{\delta M}{\delta p_n} = x_n(p_1, p_2, ..., p_n, M) \end{cases} . \tag{7.17}$$

We can observe that (7.17) is equivalent to (7.16) and since all the hypotheses of Theorem 7.2.I are satisfied we can state the following:

**Lemma 7.1** Let the demand function $x(p, M)$ satisfy the conditions (A), (D), (E) and (S). Then the system (7.17) has a unique solution for any "initial condition" $(p^*, M^*) \in \Omega$. That is, there exists a unique continuous function $\mu_{p*M*}(\,)$ such that:

$$\begin{cases} \mu_{p*M*}(p^*) = M^* \\ \frac{\delta \mu_{p*M*}(p)}{\delta p_i} = x_i(p, \mu_{p*M*}(p)) \quad i = 1, 2, ..., n \end{cases}, \; for\, all\, p \in \Pi.$$

Since in Lemma 7.1 the uniqueness of the solution is deduced we have:

**Lemma 7.2** Let $(p', M'), (p'', M'') \in \Omega$ be two "initial conditions" guaranteeing

i) there exists $p^0 \in \Pi$ such that $\mu_{p'M'}(p^0) = \mu_{p''M''}(p^0)$;

then it is

ii) $\mu_{p'M'}(p) = \mu_{p''M''}(p)$, for all $p \in \Pi$.

**Lemma 7.3** Let $(p', M'), (p'', M'') \in \Omega$ be two "initial conditions" guaranteeing

j) there exists $p^0 \in \Pi$ such that $\mu_{p'M'}(p^0) < \mu_{p''M''}(p^0)$;

then it is

jj) $\mu_{p'M'}(p) < \mu_{p''M''}(p)$, for all $p \in \Pi$.

**Proof** Let us suppose, ab absurdo, that there exists $p^0, p^* \in \Pi$ such that $\mu_{p'M'}(p^0) < \mu_{p''M''}(p^0)$ and $\mu_{p'M'}(p^*) \geq \mu_{p''M''}(p^*)$, that is j) is valid and jj) is violated for some value $p^* \in \Pi$.

Let's consider the convex combination of $p^0$ and $p^*$ given by

$$p^t = tp^* + (1 - t)p^0,\; t \in [0, 1],$$

and define the function $\varphi : [0,1] \to \mathbb{R}$ as

$$\varphi(t) = \mu_{p'M'}(p^t) - \mu_{p''M''}(p^t). \tag{7.18}$$

It results that the function in (7.18) is continuous as combination of continuous functions and it is:

$$\varphi(0) = \mu_{p'M'}(p^0) - \mu_{p''M''}(p^0) < 0,$$

$$\varphi(1) = \mu_{p'M'}(p^*) - \mu_{p''M''}(p^*) \geq 0,$$

where the two inequalities hold as immediate consequence of the hypotheses on $p^0, p^* \in \Pi$. As a consequence of the Intermediate Existence Theorem we can state: there exists $\tilde{t} \in (0,1]$ such that $\varphi(\tilde{t}) = 0$, i.e there is a vector $p^{\tilde{t}} = \tilde{t}p^* + (1-\tilde{t})p^0$ with $\mu_{p'M'}(p^{\tilde{t}}) = \mu_{p''M''}(p^{\tilde{t}})$. From Lemma 7.2 we would deduce

$$\mu_{p'M'}(p) = \mu_{p''M''}(p),\ for\,all\,p \in \Pi,$$

which is clearly a contradiction of j).

□

The idea is that the function $\mu_{pM}(p^*)$ will be used to construct the utility function $u(x)$ on the domain given by the image of $x(p,M)$, denoted by $X$.

In fact given an arbitrary fixed $p^*$ we will define for any $x \in X$

$$u(x) = U_{p*}(x) = \mu_{pM}(p^*),$$

where $(p, M)$ represents any couple satisfying $x = x(p, M)$.

We will proceed as follows: first we will prove that the value of $U_{p*}(x)$, for any given $p^*$, is independent from the choice on $(p, M)$ given the validity of the relation $x = x(p, M)$ (cfr. Lemma 7.7). Afterward we will show that $U_{p*}(\,)$ is a utility function (cfr. Lemma 7.8).

Lemma 7.4, Lemma 7.5 and Lemma 7.6 will be used to prove these two results.

**Lemma 7.4** Let us consider two couples $(p^0, M^0)$ , $(p^1, M^1)$ satisfying:

k) $\qquad x^0 = x(p^0, M^0) \neq x^1 = x(p^1, M^1)$;

kk) $\qquad M^1 \geq \mu_{p^0 M^0}(p^1)$.

Then

$$p^0 x^1 > p^0 x^0. \tag{7.19}$$

**Proof** Let us assume:

Case 1) $\qquad$ kk) holds with equality, i.e. $M^1 = \mu_{p^0 M^0}(p^1)$.

Let's consider the convex combination of $p^0, p^1$ as

$$p^t = tp^1 + (1-t)p^0,\ t \in [0,1] \tag{7.20}$$

and define

$$M^t = \mu_{p^0 M^0}(p^t),\ x^t = x(p^t, M^t). \tag{7.21}$$

We will denote by $\psi : [0,1] \longrightarrow \mathbb{R}$ the function

$$\psi(t) = p^0 x^t = p^0 x(tp^1 + (1-t)p^0, \mu_{p^0 M^0}(tp^1 + (1-t)p^0)) \tag{7.22}$$

The expression in (7.22) is differentiable for the hypotheses assumed in Theorem 7.2. Hence we have:

$$\frac{d\psi(t)}{dt} = p^0 \frac{\delta x}{\delta p^t}(p^1 - p^0) + p^0 \frac{\delta x}{\delta \mu}\frac{\delta \mu}{\delta p^t}(p^1 - p^0) =$$

$$= p^0[\frac{\delta x}{\delta p^t} + \frac{\delta x}{\delta \mu}\frac{\delta \mu}{\delta p^t}](p^1 - p^0) = p^0 S(p^t, M^t)(p^1 - p^0), \tag{7.23}$$

where $S(p^t, M^t)$ is the Slutsky Matrix.

By differentiating the budget equation $p^t x^t = M^t$ in $t$ we have:

$$\frac{dp^t}{dt}x^t + p^t\frac{dx^t}{dt} = \frac{dM^t}{dt},$$

that is

$$p^t S(p^t, M^t)(p^1 - p^0) = 0. \tag{7.24}$$

When we subtract (7.24) from (7.23) we have

$$\frac{d\psi(t)}{dt} = p^0 S(p^t, M^t)(p^1 - p^0) - p^t S(p^t, M^t)(p^1 - p^0) =$$

$$= -t(p^1 - p^0)S(p^t, M^t)(p^1 - p^0). \tag{7.25}$$

Using the negative semidefiniteness of the Slutsky Matrix we have that $(p^1 - p^0)S(p^t, M^t)(p^1 - p^0) \leq 0$. Hence

$$\frac{d\psi(t)}{dt} \geq 0, \; for\, every\, t \in [0, 1].$$

It is straightforward to deduce

$$\psi(0) = p^0 x^0 \leq p^0 x^1 = \psi(1), \tag{7.26}$$

and if we prove that $\psi(0) \neq \psi(1)$ we have (7.19).

Let's suppose, ab absurdo, $\psi(0) = \psi(1)$, then it is $\frac{d\psi(t)}{dt} = 0, \; for\, every\, t \in [0, 1]$ which gives

$$(p^1 - p^0)S(p^t, M^t)(p^1 - p^0) = 0 = (p^1 - p^0)\frac{dx^t}{dt}, \tag{7.27}$$

where the second equality is given using the definition of $x^t$.

We get $\frac{dx^t}{dt} = 0$, which implies

$$x^1 = x^0, \tag{7.28}$$

in contradiction with k).

Case 2) kk) holds with strict inequality, i.e. $M^1 > \mu_{p^0M^0}(p^1)$.

We have $M^1 = \mu_{p^1M^1}(p^1)$, hence we can restate kk) as

$$\mu_{p^1M^1}(p^1) > \mu_{p^0M^0}(p^1), \tag{7.29}$$

and applying Lemma 7.3 $\mu_{p^1M^1}(p) > \mu_{p^0M^0}(p),\ for\, all\, p \in \Pi$, and in particular it results

$$\mu_{p^1M^1}(p^0) > \mu_{p^0M^0}(p^0) = M^0. \tag{7.30}$$

If we prove

$$\mu_{p^1M^1}(p^0) \leq p^0x^1, \tag{7.31}$$

from (7.30) we get $p^0x^1 \geq \mu_{p^1M^1}(p^0) > \mu_{p^0M^0}(p^0) = M^0 = p^0x^0$, which is exactly kk).

It is possible to prove (7.31) following a line of argument which is analogous to the one we used in Case 1).

□

Through the next Lemma we prove how WARP holds under the assumptions made so far.

**Lemma 7.5** Let's consider two demand consumptions bundles $x^0 = x(p^0, M^0)$ and $x^1 = x(p^1, M^1)$, satisfying $p^0x^0 \geq p^0x^1$, with $x^0 \neq x^1$.Then it is

$$p^1x^0 > p^1x^1. \tag{7.32}$$

**Proof** The Proof of Lemma 7.5 is an immediate consequence of Lemma 7.3 and Lemma 7.4.

The two following Lemmas guarantee that the utility function we are going to construct is "well defined", in the sense that it will give the same result independently from the "initial conditions" we will ask to be respected.

**Lemma 7.6** Let $x \in X$. Then the set

$$\Xi(x) = \{(p, M) \in \Omega \, such \, that \, x(p, M) = x\}$$

is convex.

**Proof** Let's consider two pairs $(p^0, M^0), (p^1, M^1) \in \Xi(\overline{x})$, or equivalently $x(p^0, M^0) = x(p^1, M^1) = \overline{x}$. From the hypothesis on budget exhaustion we have:

$$p^0\overline{x} = M^0, \; p^1\overline{x} = M^1. \tag{7.33}$$

We can define the convex combination of $p^0, p^1$ and $M^0, M^1$respectively as

$$p(t) = tp^1 + (1 - t)p^0, \tag{7.34}$$

$$M(t) = tM^1 + (1 - t)M^0. \tag{7.35}$$

If we take

$$x(t) = x(p(t), M(t)), \tag{7.36}$$

we can observe, as consequence of (7.33), (7.34), (7.35), that

$$p(t)\overline{x} = tp^1\overline{x} + (1-t)p^0\overline{x} = tM^1 + (1-t)M^0 = M(t) = p(t)x(t). \quad (7.37)$$

Now, if we get that $x(t) = \overline{x},\ for\, every\, t \in [0,1]$ we immediately get the convexity of $\Xi(x)$.

Let us suppose ab absurdo there exists $\tilde{t} \in (0,1)$ such that $x\left(\tilde{t}\right) \neq \overline{x}$, then by the equality in (7.37) and Lemma 7.5 we would have

$$p^0\overline{x} < p^0x(\tilde{t}),\ p^1\overline{x} < p^1x(\tilde{t}). \quad (7.38)$$

Let's first multiply the two equations in (7.38) by $(1-\tilde{t})$ and $\tilde{t}$, respectively and then sum the two new expression:

$$(1-\tilde{t})p^0\overline{x} + \tilde{t}p^1\overline{x} < (1-\tilde{t})p^0x(\tilde{t}) + \tilde{t}p^1x(\tilde{t}). \quad (7.39)$$

(7.39) can be restate as $p(\tilde{t})\overline{x} < p(\tilde{t})x(\tilde{t})$, contradicting (7.37).
□

**Lemma 7.7** Let $x(p^0, M^0) = x(p^1, M^1)$, then

$$\mu_{p^1M^1}(p) = \mu_{p^0M^0}(p),\ for\, all\, p \in \Pi.$$

**Proof** Let $p(t) = tp^1 + (1-t)p^0$, $M(t) = tM^1 + (1-t)M^0$ be defined as in Lemma 7.6. We immediately have

$$x(p(t), M(t)) = x(p^0, M^0),\ t \in [0,1], \quad (7.40)$$

which guarantees $\frac{dx(t)}{dt} = 0$. When we differentiate with respect to $t$ the budget exhaustion condition we get;

$$\frac{dM(t)}{dt} = x(p(t), M(t))\frac{dp(t)}{dt},\ t \in [0,1]. \quad (7.41)$$

(7.41) satisfies the system of equations in (7.17) .

By applying Lemma 7.2 we have:

$$\mu_{p^0 M^0}(p(t)) = M(t),\ t \in [0,1];$$

which gives $\mu_{p^0 M^0}(p^1) = M(1) = M^1$, that can be used to apply Lemma 7.2 once again and get:

$$\mu_{p^0 M^0}(p) = \mu_{p^1 M^1}(p),\ for\, all\, p \in \Pi. \tag{7.42}$$

□

In the following Lemma we will define the utility function we were looking for. Theorem 7.2 is a direct consequence of Lemma 7.8.

**Lemma 7.8** Let the demand function satisfy the hypotheses of Theorem 7.2. Then for any price $p^*$, the function

$$U_{p*}(x) = \mu_{pM}(p^*),\ x \in x(p,M) \tag{7.43}$$

is single-valued on $X$, and for any couple $(p, M)$ it is

$$U_{p*}(x(p,M)) > U_{p*}(x),\ for\, any\, x \in X,\ such\, that$$

$$px \leq M,\ x \neq x(p,M). \tag{7.44}$$

**Proof** Using Lemma 7.7 we can deduce the single-valuedness of the function in (7.43).

Now, let's consider two couples $(p^0, M^0), (p^1, M^1)$, and define $x^0 = x(p^0, M^0),\ x^1 = x(p^1, M^1)$. Let's assume

$$p^0 x^1 \leq M^0,\ x^1 \neq x^0.$$

Using Lemma 7.4 we immediately deduce:

$$\mu_{p^1M^1}(p^1) = M^1 < \mu_{p^0M^0}(p^1),$$

which by Lemma 7.3 implies

$$\mu_{p^1M^1}(p) < \mu_{p^0M^0}(p),\ for\, all\, p \in \Pi. \tag{7.45}$$

When we set $p = p^*$ (7.45) becomes $\mu_{p^1M^1}(p^*) < \mu_{p^0M^0}(p^*)$, which is, by definition, equivalent to

$$U_{p*}(x^0) > U_{p*}(x^1). \tag{7.46}$$

□

Theorem 7.3 shows that the utility indicators presented in Theorem 7.2 define the same ordering for the considered commodities bundles.

**Theorem 7.3** For any two positive price vectors, $p^*$ and $p^{**}$ the function $U_{p*}(x) = \mu(p^*; p, m)$ and $U_{p**}$ (defined by analogy from $U_{p*}(\,)$) induce the same ordering on the range $X$ of the demand function $x(p, M)$.

We say that two real-valued functions, $f\,(a)$ and $g\,(a)$, defined on a set $A$ induce the same ordering on $A$ if: for all $a', a'' \in A$, $f\,(a') > f(a'')\, iff\; g(a') > g(a'')$.

The two following theorems guarantee some regularity conditions for the utility function introduced in Theorem 7.2.

**Theorem 7.4** For any price vector $p^*$, $U_{p*}(x)$ is monotone increasing with respect to the vectorial ordering of $X$, and the indefference sets of the function $U_{p*}$are strictly convex toward to the origin.

**Theorem 7.5** $U_{p*}(x)$ is $upper - semicontinuous$ in $x$, for every choice $p^*$.

We will not provide the Proof of Theorem 7.3, Theorem 7.4 and Theorem 7.5.

# 8 Conclusions

In this final chapter we will summarize the main results we obtained so far in order for the reader to have a complete framework on this work.

Let us first consider the utility maximization problem as presented in 1.2. When the utility function $u(\,)$ is a continuous function representing a regular, locally nonsatiated, strictly convex preference relation the associated (derived from the demand function) Slutsky Matrix is symmetric and negative semidefinite (Proposition 1.8). We obtained an equivalent result in Theorem 7.1. In fact when we consider a preference relation $R$ defined on the set of all conceivable commodity bundles satisfying reflexivity, completeness, transitivity under the hypotheses (A), (B) and (D) (cfr. Chapter 7) on the demand function we get the symmetry (S) and negative semidefinitness (N) of the Slutsky Matrix. The other way round, as we show in Theorem 7.2, if the demand function satisfies conditions (A), (B), (D) and (E) and the Slutsky matrix satisfies (S) and (N) then there exists a utility function $u(\,)$ such that the value of the demand maximizes $u(\,)$ over the usual budget set.

We note that the hypotheses on the demand (Theorem 7.2) in one direction are stronger then those used in the other (Theorem 7.1). Indeed we are obliged to assume (A), (B), (D) and (E) to get the following:

$$\exists\, a\, continuous,\, locally\, nonsatiated,\, convex\, utility\, function$$

$$generating\, the\, demand\, function.$$

$$\Updownarrow \tag{8.1}$$

$$Slutsky\, Matrix\, satisfies\, (S)\, and\, (N).$$

This equivalence is a consequence of the premise above and of Theorem

7.3, Theorem 7.4 and Theorem 7.5.

We mentioned that in [14] Mas-Colell et al. showed that under the hypothesis of the symmetry of the Slutsky Matrix the Weak Axiom and the Strong Axiom are equivalent. Hence we can extend the equivalence in (8.1) as follows:

$$\exists\, a\, continuous,\; locally\, nonsatiated,\; convex\, utiliy\, function$$

$$generating\, the\, demand\, function.$$

$$\Updownarrow \tag{8.2}$$

$$Slutsky\, Matrix\, satisfies\, (S)\, and\, (N).$$

$$\Updownarrow \tag{8.3}$$

$$SARP\, holds$$

$$\Updownarrow \tag{8.4}$$

$$WARP\, and\, (R)\, are\, satisfied$$

where the equivalence in 8.4 is the result of Theorem 4.4 which guarantees that the SARP holds if and only if the WARP and the Regularity Condition (R) are both satised for the demand function subject to condition (A), (B), (D) and (E). At this point the link between the "integrability" theory and the "revealed preference" one should be clear when we take into consideration what we remind hereinafter.

In fact in Chapter 4 we show that, when (A), (B), (D), (E) and SARP are satisfied the indirect revealed preference relation $R^*$ generated by the demand function is a preference relation on the set of all positive commodity bundles (i.e. $R^*$ satises P.I,...,P.V ) and the demand function is derived from $R^*$(cfr. Theorem 4.1). We can move in the opposite direction through Theorem 4.6 where, given a preference relation $P$ on the set of all nonnegative commodities (satisfying Axioms P.I,..., P.VI), it is proved the existence of a demand function that is derived from the preference relation $P$, for which (A), (B), (D) and SARP hold.

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