Dottorato di Ricerca in Matematica. XIX ciclo,
2004-2006.
Università degli studi di Pisa

Tesi di Dottorato di

Luigi Vezzoni

THE GEOMETRY OF SOME SPECIAL $\mathrm{SU}(n)$-STRUCTURES

Direttore della Ricerca: Prof. Paolo de Bartolomeis

Coordinatore del dottorato: Prof. Fabrizio Broglia

Contents

Chapter 1

Introduction

A *Calabi-Yau* manifold is a Kähler manifold endowed with a constant and normalized complex volume form. Equivalently a Calabi-Yau manifold can be defined as a manifold equipped with an integrable $\mathrm{SU}(n)$-structure. The study of this class of manifolds begins with the proof of the celebrated Calabi's conjecture given by Yau in [71, 72]. As a direct consequence of Calabi-Yau theorem we get that any compact simply connected Kähler manifold with vanishing first Chern class admits a Calabi-Yau structure.

Moreover Calabi-Yau manifolds have a central role in string theory. In fact in this physical theory the universe is represented as

$$X \times C$$

where X denotes the Minkowski space and C is a compact Calabi-Yau 3-fold.

Furthermore the Hitchin-Strominger-Yau-Zaslow theory of deformation of Special Lagrangian submanifolds introduced in the study of Mirror Symmetry imposes to consider generalizations of the Calabi-Yau structure (see e.g. [39]).

In [28] de Bartolomeis and Tomassini introduce a natural generalization of Calabi-Yau manifolds to the non-holomorphic case. Namely, a *generalized Calabi-Yau manifold* is a symplectic manifold endowed with a compatible almost complex structure and a normalized complex volume form ε covariant constant with respect to the Chern connection. Special Lagrangian submanifolds of generalized Calabi-Yau manifolds have the important property to have vanishing Maslov class.

In dimension six the definition of generalized Calabi-Yau manifold can be improved by requiring that real part of ε to be closed. Such manifolds are called *special generalized Calabi-Yau manifolds*. In this case the form $\mathfrak{Re}\,\varepsilon$ is a calibration and special Lagrangian submanifolds are calibrated submanifolds (see [37]).

The first problem faced in this thesis is to give a description of the infinitesimal deformations of the almost complex structures on a compact symplectic manifold admitting a generalized Calabi-Yau structure (such structures will be called *admissible*). At the beginning of Chapter 2 we give an example of a non-admissible almost complex structure calibrated by a symplectic form on a nilmanifold (see example 3.9). After that we compute the tangent space to the moduli space of admissible almost complex structures calibrated by the same symplectic form. As a direct application we get that the standard holomorphic structure on the complex torus is not rigid.

In the second part of chapter 3 we take into account 6-dimensional special generalized Calabi-Yau manifolds. In order to determine some Riemannian properties of our manifolds we write down (with the aid of MAPLE) an explicit formula for the scalar curvature and the Ricci tensor of an arbitrary SU(3)-structure (see theorems 3.32, 3.34). As a direct application of our formulae we get that the scalar curvature of a 6-dimensional generalized Calabi-Yau manifold is always non-positive and that the Einstein equation forces a special generalized Calabi-Yau structure to be integrable (also in the non-compact case) (see corollary 3.36). In chapter 4 we take into account special generalized Calabi-Yau structures. In the first part of this chapter we give some examples of special generalized Calabi-Yau structures on compact manifolds and we describe some special Lagrangian submanifolds. After that we prove that the set of the Calabi-Yau structures can be not open in the set of generalized Calabi-Yau structures (see remark 4.10) and we give a compact example of a complex manifold admitting generalized Calabi-Yau structures, but no special generalized Calabi-Yau structures. In section 4.1.3 we study special Lagrangian geometry proving an extension theorem. More precisely, we prove that, given a family of special generalized Calabi-Yau manifolds $(M, \kappa_t, J_t, \varepsilon_t)$ and a compact special Lagrangian submanifold $p_0\colon L \hookrightarrow M$ of $(\kappa_0, J_0, \varepsilon_0)$ under some cohomological conditions there exists a family of special Lagrangian submanifolds $p_t\colon L \hookrightarrow (M, \kappa_t, J_t, \varepsilon_t)$, that extends $p_0\colon L \hookrightarrow M$ (see theorem 4.13). This is an extension of a theorem of Lu Peng (see [45] and also [56]) to the context of special generalized Calabi-Yau manifolds. In the last part of this chapter we generalize the Lu Peng theorem to the 4-dimensional case.

In the last chapter we introduce a generalization of the Calabi-Yau structure to the contest of contact manifolds introducing the definition of *contact Calabi-Yau manifold* . Roughly speaking, a contact Calabi-Yau manifold is a $2n+1$-

dimensional manifold M endowed with a contact form α, a complex structure J on $\xi = \ker\alpha$ calibrated by $\kappa = \frac{1}{2}\,d\alpha$ and a closed complex basic volume form ε. These manifolds are a special class of α-Einstein null-Sasakian manifolds. As a direct consequence of the definition, in a contact Calabi-Yau manifold $(M, \alpha, J, \varepsilon)$ the real part of ε is a calibration. Furthermore, it turns out that a submanifold $p\colon L \hookrightarrow M$ of a contact Calabi-Yau manifold is calibrated by $\mathfrak{Re}\,\varepsilon$ if and only if

$$p^*(\alpha) = 0\,, \quad p^*(\mathfrak{Im}\,\varepsilon) = 0\,.$$

In such a case L is said to be a *special Legendrian submanifold*. We prove that the Moduli space of deformations of special Legendrian submanifolds near a fixed compact one L is a smooth 1-dimensional manifold. Furthermore we study Lu Peng problem in this class of manifolds. Then we classify 5-dimensional nilpotent algebras admitting an invariant contact Calabi-Yau structure and in the last section we generalize our results to the r-contact manifolds.

Acknowledgements: I am grateful to my advisor Paolo de Bartolomeis for suggesting me the problems faced in the present work and for his precious advice. The principal results of this paper have been obtained in collaboration with Adriano Tomassini. I would like to express my gratitude to him. His constant support and encouragement have been very important for me.
The formula describing the Ricci tensor of an SU(3)-manifold in terms of torsion forms has been computed in collaboration to Lucio Bedulli. I would like to thank him. The proof of this formula has been obtained with the aid or MAPLE. I am grateful to Robert Bryant for supplying me with the computer programs he used to perform the symbolic computations in the G_2-case. I am also grateful to Richard Cleyton for suggesting a considerable strengthening of a previous version of Corollary 3.36 and useful discussions.
Finally, I am pleased to thank the referees for valuable remarks and suggestions for a better presentation of the results present in this paper.

Notation: In this thesis we use the following notation:
Given a manifold M, we denote by $\Lambda^r M$ the space of smooth r-forms on M and we set $\Lambda M := \bigoplus_{r=1}^n \Lambda^r M$.
Furthermore when a coframe $\{\alpha_1, \dots, \alpha_n\}$ is given we will denote the r-form $\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_r}$ by $\alpha_{i_1 \dots i_r}$. We use the convention:

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha \,.$$

In the indicial expressions the symbol of sum over repeated indices is sometimes omitted.

Chapter 2

Background

2.1 Symplectic vector spaces

Let V be a $2n$-dimensional real vector space. A *symplectic structure* on V is a skew-symmetric non-degenerate 2-form κ, i.e. a 2-form satisfying $\kappa^n \neq 0$. The pair (V, κ) is called a *symplectic vector space*. A symplectic structure κ gives a duality $\sharp_\kappa \colon V \to V^*$ defined by

$$\sharp_\kappa(v)w = \kappa(v, w)\,,$$

for $v, w \in V$.

Example 2.1. Let us denote by $\{e_1, \dots e_{2n}\}$ the standard basis of $\mathbb{R}^{2n}$ and by $\{e^1, \dots, e^{2n}\}$ the respective dual frame. The 2-form

$$\kappa_0 = \sum_{i=1}^{n} e^{2i-1} \wedge e^{2i}\,,$$

is a symplectic structure on $\mathbb{R}^{2n}$ called the *standard symplectic structure*. The pair $(\mathbb{R}^{2n}, \kappa_0)$ is said to be the *standard symplectic vector space*.

An endomorphism ϕ between two symplectic vector spaces (V_1, κ_1), (V_2, κ_2) is said to be a *symplectomorphism* if

$$\phi^*(\kappa_2) = \kappa_1\,.$$

It is well known that any $2n$-dimensional symplectic vector space (V, κ) is symplectomorphic to $(\mathbb{R}^{2n}, \kappa_0)$.

An endomorphism J of V is said to be a *complex structure* if it satisfies $J^2 = -I$, where I denotes the identity on V. The pair (V, J) is called a *complex vector space*.

Example 2.2. Let $J_0 \in \mathrm{End}(\mathbb{R}^{2n})$ be represented by the matrix

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n denotes the identity of $\mathbb{R}^n$. This endomorphism satisfies $J_0^2 = -I$ and consequently it is a complex structure on $\mathbb{R}^{2n}$. We refer to J_0 as to the *standard complex structure.*

A complex structure J on V gives a natural splitting of $V^{\mathbb{C}} := V \otimes \mathbb{C}$ in $V^{\mathbb{C}} = V_J^{1,0} \oplus V_J^{0,1}$, where $V_J^{1,0}$ is the J-eigenspaces relatively to i and $V_J^{0,1}$ is the J-eigenspace relatively to $-i$. Furthermore if we set

$$\Lambda_J^{1,0} V := (V_J^{1,0})^*\,, \quad \Lambda_J^{0,1} V := (V_J^{0,1})^*$$

and

$$\Lambda_J^{p,q} V := \underbrace{\Lambda_J^{1,0} V \wedge \cdots \wedge \Lambda_J^{1,0} V}_{p-times} \wedge \underbrace{\Lambda_J^{0,1} V \wedge \cdots \wedge \Lambda_J^{0,1} V}_{q-times},$$

we have that the vector space $\Lambda_{\mathbb{C}}^r V^*$ of complex valued r-forms on V splits as

$$\Lambda_{\mathbb{C}}^r V^* = \bigoplus_{p+q=r} \Lambda_J^{p,q} V\,.$$

Moreover the space $\mathrm{End}(V)$ of endomorphisms of V decomposes in

$$\mathrm{End}(V) = \mathrm{End}_J^{1,0}(V) \oplus \mathrm{End}_J^{0,1}(V)\,, \tag{2.1}$$

where

$$\mathrm{End}_J^{1,0}(V) = \{L \in \mathrm{End}(V) \mid J \circ L = L \circ J\}\,, \tag{2.2}$$

$$\mathrm{End}_J^{0,1}(V) = \{F \in \mathrm{End}(V) \mid J \circ F = -F \circ J\}\,. \tag{2.3}$$

A positive-defined scalar product g on a complex vector space (V, J) is said to be *J-Hermitian* if $g(Jv, Jw) = g(v, w)$ for any $v, w \in V$. Note that if h is an arbitrary positive-defined scalar product on V, then the tensor $g := \frac{1}{2}(h + Jh)$ defines a J-Hermitian metric on V. It follows that any complex structure admits a Hermitian metric.

Let (V, κ) be a symplectic vector space. A complex structure J on V is said to be *κ-tamed* if

$$\kappa(v, Jv) > 0$$

for any $v \in V$, $v \neq 0$. Let us denote by $\mathcal{T}_\kappa(V)$ the space of κ-tamed complex structures on V. By the definition we immediately get that this space in an open subset of $\mathrm{End}(V)$. Moreover if $J \in \mathcal{T}_\kappa(V)$ and $Z \in \mathrm{End}(V)$, then $\widetilde{J} = -JZ$ is κ-tamed if and only if

$$Z > 0\,, \quad Z^{-1} = J^{-1}ZJ\,, \tag{2.4}$$

where $Z > 0$ means that

$$g_J(v, Zv) > 0 \text{ for all } v \in V\,, v \neq 0\,.$$

Vice versa if J and $\widetilde{J}$ are κ-tamed the endomorphism $J\widetilde{J}$ satisfies conditions (2.4). It follows that $\mathcal{T}_\kappa(V)$ is parameterized by the set

$$A := \{Z \in \mathrm{End}(V) \text{ satisfying } (2.4)\}\,.$$

Let

$$F\colon A \to \{W \in \mathrm{End}(V) \mid |W| < 1\}\,,$$

be the map defined by

$$F(Z) = (I - Z)(I + Z)^{-1}\,.$$

This map interchanges J and $-J$ and a vector $W \in V$ belongs to the image of F if and only if it satisfies

$$|W| < 1\,, \quad -W = J^{-1}WJ\,.$$

It follows that $\mathrm{Im}(F)$ is a convex space. Hence we have proved that $\mathcal{T}_\kappa(V)$ is contractible.

A κ-tamed complex structure J is said to be κ-*calibrated* (or κ-compatible) if

$$\kappa(Jv, Jw) = \kappa(v, w)\,,$$

for any $v, w \in V$. Let us denote by $\mathcal{C}_\kappa(V) \subset \mathcal{T}_\kappa(V)$ the space of κ-calibrated complex structures on V. Any $J \in \mathcal{T}_\kappa(V)$ induces a positive defined inner product g_J on V by the relation

$$g_J(v, w) := \frac{1}{2}(\kappa(v, Jw) - \kappa(Jv, w))\,. \tag{2.5}$$

Note that if $J \in \mathcal{C}_\kappa(V)$, then g_J is simply obtained by $g_J(v, w) = \kappa(v, Jw)$.

Example 2.3. The standard complex structure J_0 is κ_0-calibrated and it induces the standard inner product on $\mathbb{R}^{2n}$.

The scalar product g_J is obviously J-Hermitian. Consequently g_J induces the *Hodge star operator* $*\colon \Lambda_J^{p,q}V \to \Lambda_J^{n-q,n-p}V$, defined by

$$\alpha \wedge *\overline{\beta} = g_J(\alpha, \overline{\beta})\frac{\kappa^n}{n!}\,.$$

We recall that $*^2 = (-1)^{p+q}I$ and that it is $\mathbb{C}$-linear. In a analogue way κ induces an operator $\bigstar\colon \Lambda^r V^* \to \Lambda^{2n-r}V^*$, called the *symplectic star operator*, by means of the relation

$$\alpha \wedge \bigstar\beta = \kappa(\alpha, \beta)\frac{\kappa^n}{n!}\,.$$

We have the following easy-prove lemma (see e.g. [27])

Lemma 2.4. *These identities hold:*

1. $\bigstar^2 = I$;
2. *if* $J \in \mathcal{C}_\kappa(V)$, *then* $J* = *J = \bigstar$.

The following lemma, proved in [27], will be useful in chapter 3.

Lemma 2.5. *Let* $\zeta \in \Lambda^1 V^*$ *and* $\gamma \in \Lambda^r V^*$; *we have*

$$\bigstar(\zeta \wedge \gamma) = (-1)^r \zeta \wedge \bigstar(\kappa \wedge \gamma) - (-1)^r \bigstar(\kappa \wedge \bigstar(\zeta \wedge \bigstar\gamma))\,. \tag{2.6}$$

2.1.1 Lagrangian subspaces

Let (V, κ) be a symplectic vector space. A subspace $i\colon W \hookrightarrow V$ is said to be *isotropic* if $i^*\kappa = 0$. We have the following lemma (see e.g. [48])

Lemma 2.6. *Let* $i\colon W \hookrightarrow V$ *be an isotropic subspace, then* $dim_{\mathbb{R}} W \leq \frac{1}{2}\, dim_{\mathbb{R}} V$.

We recall the following

Definition 2.7. A subspace $i\colon W \hookrightarrow V$ of a symplectic vector space (V, κ) is said to be *Lagrangian* if

1. it is isotropic ;
2. $dim_{\mathbb{R}} W = \frac{1}{2}\, dim_{\mathbb{R}} V$.

Let $i\colon W \hookrightarrow V$ be a Lagrangian subspace; fix $J \in \mathcal{C}_\kappa(M)$ and consider the metric g_J induced by (κ, J) by (2.5). Since $i^*\kappa = 0$ we have that

$$g_J(w, Jw) = 0 \tag{2.7}$$

for any $w \in W$. Consequently if $\{w_1, \ldots, w_n\}$ is a g_J-orthonormal basis of W, then $\{w_1, \ldots, w_n, Jw_1, \ldots, Jw_n\}$ is an orthonormal basis of the ambient space V. It follows that, if we denote by $\Lambda(V, \kappa)$ the set of the Lagrangian vector subspaces of (V, κ), then

$$\Lambda(V, \kappa) \simeq \mathrm{U}(n)/\mathrm{O}(n)\,.$$

In the sequel we will denote by $\Lambda(n)$ the homogeneous space $\mathrm{U}(n)/\mathrm{O}(n)$.

2.2 Symplectic manifolds and calibrated almost complex structure

Let M be a $2n$-dimensional manifold.

Definition 2.8. An *almost symplectic structure* on M is a non-degenerate 2-form κ. The pair (M, κ) is said to be an almost symplectic manifold.
If further κ is closed, i.e. $d\kappa = 0$, then it is called a *symplectic structure* and (M, κ) a *symplectic manifold*.

As in the linear case an almost symplectic structure κ induces an endomorphism

$$\sharp_\kappa \colon TM \to T^*M\,,$$

given by

$$\sharp_\kappa(X)(Y) = \kappa(X, Y)\,. \tag{2.8}$$

An easy application of Stokes' theorem gives the following

Lemma 2.9. *Let (M, κ) be a compact symplectic manifold and let $b_j(M) := \dim H^j(M, \mathbb{R})$ be the j-th Betti number of M. Then*

$$b_{2i} \neq 0\,,$$

for any $1 \leq i \leq n$.

Let (M_1, κ_1), (M_2, κ_2) be symplectic manifolds; a diffeomorphism $\phi \colon M_1 \to M_2$ satisfying

$$\phi^*(\kappa_1) = \kappa_2$$

is said to be a *symplectomorphism*. Let us denote by $\mathrm{Sp}(M, \kappa)$ the group of the symplectomorphisms of (M, κ). We have the following well-known

Lemma 2.10 (Darboux). *Any symplectic manifold is locally symplectomorphic to* $(\mathbb{R}^{2n}, \kappa_0)$.

Hence there are no local symplectic invariants.

Now we recall the definitions of complex and almost complex structure:

Definition 2.11. An *almost complex structure* on M is an endomorphism J of TM satisfying $J^2 = -I$. The couple (M, J) is called an *almost complex manifold*.

If M admits a holomorphic atlas, then it inherits a canonical almost complex structure J, locally defined by

$$\begin{cases} J(\partial_{x_i}) = \partial_{y_i}\,, & for\ i = 1 \dots n\,, \\ J(\partial_{y_j}) = -\partial_{x_i}\,, & for\ i = 1 \dots n\,, \end{cases}$$

where $\{z_i = x_i + iy_i\}$ is a system of holomorphic coordinates.

Definition 2.12. An almost complex structure is said to be *integrable*, or a *complex structure*, if it is induced by a holomorphic atlas. In this case the couple (M, J) is called a *complex manifold*.

For any almost complex manifold (M, J) it is defined the Nijenhuis tensor of J:

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]\,,$$

for $X, Y \in TM$. Obviously, if J is integrable, the correspondent Nijenhuis tensor vanishes. We have the following important

Theorem 2.13 (Newlander-Nirenberg [53]). *An almost complex structure J is integrable if and only the Nijenhuis tensor N_J vanishes identically.*

Let (M, J) be an almost complex manifold; according with the decomposition of $\Lambda^r_{\mathbb{C}}M$ in $\Lambda^r_{\mathbb{C}}M = \bigoplus_{p+q=r} \Lambda^{p,q}_J M$, the exterior derivative $d\colon \Lambda^r_{\mathbb{C}}M \to \Lambda^{r+1}_{\mathbb{C}}M$ splits as

$$d\colon \Lambda^{p,q}_J M \to \Lambda^{p+2,q-1}_J M \oplus \Lambda^{p+1,q}_J M \oplus \Lambda^{p,q+1}_J M \oplus \Lambda^{p-1,q+2}_J M\,,$$

$$d = A_J + \partial_J + \overline{\partial}_J + \overline{A}_J\,.$$

It is well known that J is integrable if and only if $\overline{\partial}_J^2 = 0$ (or equivalently if and only if $A_J = 0$).

Finally we recall that a complex p-form γ on an almost complex manifold (M, J) is said to be *holomorphic* if it is of type $(p, 0)$ and satisfies

$$\overline{\partial}_J \gamma = 0\,.$$

2.2.1 Hermitian structures

A Riemann metric g on a complex manifold (M, J) is said to be *J-Hermitian* if it is preserved by J. In this case the triple (M, g, J) is said to be an *almost Hermitian manifold*. If further the almost complex structure J is integrable, then (M, g, J) is called a *Hermitian manifold*. If h is an arbitrary metric on M, then the tensor $g := \frac{1}{2}(h + Jh)$ defines a Hermitian metric on (M, J). It follows that any almost complex manifold (M, J) admits Hermitian metrics. An almost Hermitian structure (g, J) induces an almost symplectic structure κ on M given by the relation

$$\kappa(X, Y) := g(JX, Y)\,. \tag{2.9}$$

Let us consider now an almost symplectic manifold (M, κ). An almost complex structure J on M is said to be κ-tamed if for any $x \in M$, J_x is a linear complex structure on T_xM tamed by κ_x. Since the space of the complex structures on a vector space tamed by a linear symplectic structure is contractible, any symplectic manifold (M, κ) admits a κ-tamed almost complex structure. Furthermore the space $\mathcal{T}_\kappa(M)$ of κ-tamed almost complex structures on M is a contractible space, too. A κ-tamed almost complex structure is said to be *κ-calibrated* if

$$\kappa_x(J_xv, J_xw) = \kappa_x(v, w)\,,$$

for any $x \in M$, $v, w \in T_xM$. As in the tamed case we have that any symplectic structure κ admits a κ-calibrated almost complex structure and that the space $\mathcal{C}_\kappa(M)$ of the κ-calibrated almost complex structures is a contractible subspace of $\mathcal{T}_\kappa(M)$.

Any κ-calibrated almost complex structure J induces a Hermitian metric g_J by relation (2.9). Hence one can define an almost Hermitian structure as a couple (κ, J) instead of a couple (g, J).

Notation: From now on, when an almost Hermitian structure (κ, J) is given, we will denote by g_J the induced metric and by ∇ the Levi-Civita connection of g_J.

We have following well-known

Lemma 2.14. *Let (M, κ, J) be an almost Hermitian manifold. The following formula holds*

$$2g_J((\nabla_X J)Y, Z) = d\kappa(X, Y, Z) - d\kappa(X, JY, JZ) + g_J(N_J(Y, Z), JX)\,, \tag{2.10}$$

for any vector fields X, Y, Z *on* M.

In the sequel we will use the following

Lemma 2.15. *Let* (M, J) *be a compact almost complex manifold. Let* $f\colon M \to \mathbb{C}$ *be a holomorphic map, i.e. a map satisfying* $\overline{\partial}_J f = 0$*, then it is constant.*

2.2.2 Symplectic and complex bundles

Definitions of symplectic and complex structure can be easily generalized to fibre bundles. Symplectic bundles will be used in chapter 5.

Let $\pi\colon F \to M$ be a vector bundle on an arbitrary manifold M. A *symplectic structure* on F is by definition a smooth section κ of $F^* \otimes F^*$ such that κ_x is a symplectic structure on the fibre F_x for any $x \in M$. For example an almost symplectic structure on M is a symplectic structure on the tangent bundle TM.

If κ is a symplectic structure on F, then the pair (F, κ) is said to be a *symplectic vector bundle* on M. Let J be an endomorphism of F. If $J^2 = -I_F$, then it is said to be a *complex structure* on F and the pair (F, J) is called a *complex vector bundle*. A complex structure J on F is said to be *calibrated* by a symplectic structure κ if

$$\kappa(J\cdot, J\cdot) = \kappa(\cdot, \cdot)\,, \quad \kappa(\cdot, J\cdot) > 0\,.$$

In this case the tensor $g_J(\cdot,\cdot) := \kappa(\cdot, J\cdot)$ is a J-Hermitian metric on F. Also in this case the space $\mathcal{C}_\kappa(F)$ of κ-calibrated complex structures on F is non-empty and contractible.

2.2.3 The Chern connection

Let (M, κ) be an almost symplectic manifold and let J be a κ-calibrated almost complex structure on M. The pair (κ, J) induces a connection on M, called the *Chern connection*, whose covariant derivative characterized by the following properties

$$\widetilde{\nabla} J = 0\,, \quad \widetilde{\nabla} g_J = 0\,, \quad T^{\widetilde{\nabla}}(JX, Y) = T^{\widetilde{\nabla}}(X, JY)\,,$$

If further κ is a symplectic structure, then the relative Chern connection is simply given by the formula

$$\widetilde{\nabla} := \nabla - \frac{1}{2} J \nabla J\,, \tag{2.11}$$

where ∇ is the Levi Civita connection associated to the metric g_J induced by (κ, J). In this case the torsion of $\widetilde{\nabla}$ reduces to

$$T^{\widetilde{\nabla}} = \frac{1}{4} N_J \,.$$

In the symplectic case we also have that if we denote by $\widetilde{\nabla}^{0,1}$ the (0,1)-part of $\widetilde{\nabla}$, then $\widetilde{\nabla}^{0,1} = \overline{\partial}_J$ (see e.g. [28]).

2.2.4 First Chern class of a symplectic manifold

Let (M, κ) be an almost symplectic manifold. Since $\mathcal{T}_\kappa(M)$ is a contractible space, then the first Chern class of (M, J) does not depend from the choice of $J \in \mathcal{T}_\kappa(M)$. Hence we can define the first Chern class $c_1(M, \kappa) \in H^2(M, \mathbb{C})$ of an almost symplectic manifold (M, κ) as the first Chern class of (M, J), where J is arbitrary element of $\mathcal{T}_\kappa(M)$.
If J belongs to $\mathcal{C}_\kappa(M)$, then the Ricci form $\widetilde{\rho}$ of the respective Chern connection is a closed form. It can be seen that

$$c_1(M, \kappa) = [\frac{1}{2\pi} \widetilde{\rho}] \,.$$

In particular the cohomology class of $\widetilde{\rho}$ does not depend from the choice of $J \in \mathcal{C}_\kappa(M)$.

2.3 Kähler and Calabi-Yau manifolds

In this section we recall the definitions of Kähler and Calabi-Yau manifold and some results which will be useful in the sequel.

2.3.1 Kähler manifolds

Let M be a $2n$-dimensional manifold.

Definition 2.16. A *Kähler structure* on M is a pair (κ, J), where

- κ is a symplectic structure;
- J is a κ-calibrated complex structure.

The triple (M, κ, J) is said to be a *Kähler manifold.*

We have the following

Proposition 2.17. *Let (M, κ, J) be an almost Hermitian manifold. The following facts are equivalent*

1. $\nabla J = 0$;

2. *the Chern connection of (κ, J) coincides with the Levi Civita connection induced by the metric g_J;*

3. *(M, κ, J) is a Kähler manifold.*

Example 2.18. The standard Hermitian space $(\mathbb{R}^{2n}, \kappa_0, J_0)$ is a Kähler manifold.
The complex projective space $\mathbb{CP}^n$ equipped with the standard complex structure and the Fubini-Study metric is a Kähler manifold.

Furthermore we have that any complex submanifold of a Kähler manifold is a Kähler manifold too. It follows that any algebraic manifold is a Kähler manifold.

Now we recall some basic properties of Kähler manifolds:

- The existence of a Kähler structure on a manifold M forces the odd Betti numbers of M to be even.

- If ∇ denotes the Levi-Civita connection of a Kähler metric on a manifold M, then around any $o \in M$ there exists a local $(1,0)$-frame $\{Z_1, \dots, Z_n\}$ satisfying
$$\nabla_i Z_j[o] = \nabla_{\bar{i}} Z_j[o] = \nabla_i Z_{\bar{j}}[o] = 0\,,$$
for any $0 \leq i, j \leq n$.

- A Kähler form κ can be always write locally as $\kappa = i\,\partial_J \overline{\partial}_J \phi$, for some C^∞ map ϕ called the *potential* of κ.

- In a Kähler manifold any exact $(1,1)$-form $\gamma = d\eta$ can be write as $\gamma = i\partial_J \overline{\partial}_J \phi$, for some smooth map ϕ.

- The curvature tensor of a Kähler metric satisfies
$$R(JX, JY) = R(X, Y)\,,$$
for any pair of vector fields (X, Y).

2.3.2 Calabi's conjecture and Calabi-Yau structures

Let (M, J) be a complex manifold admitting Kähler structures and let g be a Kähler metric on (M, J) with Kähler form κ. If ρ denotes the Ricci form of g, then it is a closed and satisfies $c_1(M, J) = \frac{1}{2\pi}[\,\rho\,]$. Hence it is natural to ask which (1,1)-forms representing $c_1(M, J)$ are the Ricci form of some Kähler metric on (M, J). In 1954 Eugenio Calabi proposed the following conjecture

Conjecture: *Let (M, J) be a compact complex manifold admitting Kähler structures and let g be a Kähler metric on (M, J) with Kähler form κ. Let ρ' be a real closed $(1, 1)$-form on M such that $c_1(M, J) = \frac{1}{2\pi}[\,\rho'\,]$. Then there exists a unique (up to homothety) Kähler metric g' on (M, J) with Kähler form κ' such that $[\kappa] = [\kappa'] \in H^2(M, \mathbb{C})$ and ρ' is the Ricci form of g'.*

A complete proof of the Calabi's conjecture was given by Yau in the celebrated papers [71] and [72].

As a direct consequence of the Calabi's conjecture we have the following

Corollary 2.19. *Let (M, J) be a compact complex manifold admitting Kähler structure and with vanishing first Chern class. Then there exists a Ricci-flat Kähler metric on (M, J).*

Furthermore

Lemma 2.20. *Let (M, κ, J) be a $2n$-dimensional Kähler manifold. The following facts are equivalent*

1. *the metric g_J is Ricci-flat;*
2. *the restricted holonomy group of $\mathrm{Hol}^0(\nabla)$ is contained in $\mathrm{SU}(n)$.*

Hence we have the following

Corollary 2.21. *Let (M, κ, J) be a $2n$-dimensional simply connected Kähler manifold. The following facts are equivalent*

1. *the metric g_J is Ricci-flat;*
2. *there exists $\varepsilon \in \Lambda_J^{n,0}M$ satisfying $\nabla\varepsilon = 0$.*

Now we can recall the definition of Calabi-Yau manifold

Definition 2.22. Let M be a $2n$-dimensional manifold. A *Calabi-Yau* structure on M is a triple (κ, J, ε), where

- (κ, J) is a Kähler structure on M;

- $\varepsilon \in \Lambda_J^{n,0} M$ is a nowhere vanishing form satisfying

$$\nabla \varepsilon = 0 \,,$$

  where ∇ is the Levi-Civita connection of the metric g_J associated to (κ, J). The triple $(M, \kappa, J, \varepsilon)$ is called a *Calabi-Yau manifold.*

Summarizing we have

- A Calabi-Yau structure induces a Ricci-flat metric.
- If (M, J) is a compact simply connected complex manifold admitting Kähler metric and with vanishing first Chern, then it is always possible to find a Calabi-Yau structure on M compatible with J.

2.4 G-structures and intrinsic torsion

Let M be a n-dimensional manifold and let $\mathcal{L}(M)$ be the $\mathrm{GL}(n, \mathbb{R})$-principal bundle of the linear frames on M. Let G be a subgroup of $\mathrm{GL}(n, \mathbb{R})$. A *$G$-structure* on M is a reduction $\mathcal{Q}$ of $\mathcal{L}(M)$ with structure group G. Let H be a connection on M; H is said be *compatible* with $\mathcal{Q}$ if $H_{|\mathcal{Q}}$ defines a connection on $\mathcal{Q}$. A linear connection ∇ on TM is said to be *compatible* with a G-structure $\mathcal{Q}$ if it is induced by a $\mathcal{Q}$-admissible connection on $\mathcal{L}(M)$. We have the following

Theorem 2.23. *A connection ∇ on TM is compatible with $\mathcal{Q}$ if and only if it has Holonomy group* $\mathrm{Hol}(\nabla)$ *contained in G.*

Now we can recall the definition of integrable G-structure

Definition 2.24. A G-structure $\mathcal{Q}$ on M is said to be *torsion-free* if there exists a $\mathcal{Q}$-compatible torsion-free connection ∇ on TM.

In order to study the torsion of a G-structure it is useful to introduce the map (with notation of [43])

$$\sigma \colon \mathfrak{g} \otimes (\mathbb{R}^n)^* \to (\mathbb{R}^n)^* \otimes \Lambda^2 (\mathbb{R}^n)^*$$

given by the relation

$$\sigma(v \otimes \alpha \otimes w) := v \otimes \alpha \wedge \beta$$

(where we identify the Lie algebra $\mathfrak{g}$ of G with a subspace of $\mathbb{R}^n \otimes (\mathbb{R}^n)^*$) and the vector spaces

$$W_1 := \mathbb{R}^n \otimes \Lambda^2 (\mathbb{R}^n)^* \,, \quad W_2 := \mathrm{Im}\, \sigma \,, \quad W_3 := W_1 / W_2 \,, \quad W_4 := \ker \sigma \,.$$

Let $\rho_i\colon W_i \to \mathrm{Aut}(W_i)$ be the standard representation and let

$$\rho_i(\mathcal{Q}) := \mathcal{Q} \times W_i/G$$

be the vector bundle associated to W_i. If ∇, ∇' are two connections on TM compatible with $\mathcal{Q}$, then their projections on $\rho_3(\mathcal{Q})$ coincide. So we can define the *intrinsic torsion* $T^i(\mathcal{Q})$ of a G-structure $\mathcal{Q}$ as the projection on $\rho_3(\mathcal{Q})$ of the torsion of an arbitrary connection on TM compatible with $\mathcal{Q}$. It easy to see that if $T^i(\mathcal{Q})$ vanishes, then there exists a torsion-free connection ∇ on TM compatible with $\mathcal{Q}$. Hence the intrinsic torsion of a G-reduction $\mathcal{Q}$ is an obstruction to find torsion-free connections on TM compatible with $\mathcal{Q}$.

Example 2.25. If (M, J) is an almost complex manifold, then the almost complex structure J induce a $\mathrm{GL}(n, \mathbb{C})$-reduction $\mathcal{Q}$ of $\mathcal{L}(M)$. Let g be an arbitrary J-Hermitian metric on M and let $\widetilde{\nabla}$ be the Chern connection of (M, J). Since $\widetilde{\nabla}J = 0$, then $\widetilde{\nabla}$ is a connection compatible with $\mathcal{Q}$. We can write

$$T^{\widetilde{\nabla}} = \frac{1}{4}\,N_J + L_J\,.$$

It is easy to see that $L_J \in \rho_3(\mathcal{Q})$. Consequently the intrinsic torsion of $\mathcal{Q}$ is

$$T^i(\mathcal{Q}) = \frac{1}{4}\,N_J\,.$$

Example 2.26. Let (M, g, J) be an almost Hermitian manifold. The pair (g, J) induces a $\mathrm{U}(n)$-reduction $\mathcal{Q}$ of $\mathcal{L}(M)$. Since

$$\widetilde{\nabla}J = 0\,, \quad \widetilde{\nabla}g = 0\,,$$

then $\widetilde{\nabla}$ is compatible with $\mathcal{Q}$. In this case we have

$$T^i(\mathcal{Q}) = T^{\widetilde{\nabla}}\,,$$

so that the $\mathrm{U}(n)$ reduction is integrable if and only if (g, J) induces a Kähler structure on M.
Note that if $d\kappa = 0$, then the intrinsic torsion of (κ, J, ε) reduces to $\frac{1}{4}\,N_J$.

Chapter 3

SU(n)-structures

In this chapter, which is the core of the present work, we take in consideration SU(n)-structures. In the first section we recall the definition of *generalized Calabi-Yau manifold*, given in [28], which is a natural generalization of the Calabi-Yau structure to the non-holomorphic case. In §3.1.1 we introduce the definition of *κ-admissible almost complex structure*, which simply refers to a κ-calibrated almost complex structure admitting a generalized Calabi-Yau structure. Furthermore, in the spirit of Kodaira-Spencer theory of deformations of complex structures, we study infinitesimal deformations of admissible almost complex structures computing the tangent space to the Moduli Space. In §3.1.4 we perform our computations in the explicit case of the complex torus showing that the standard complex structure is not rigid.
In §3.2 we specialize to the 6-dimensional case. After some algebraic preliminar computations we characterize some special SU(3)-structures in terms of intrinsic torsion and we write down an explicit formula for the scalar curvature and the Ricci tensor of an arbitrary SU(3)-manifold. It turns out that the scalar curvature of the metric induced by a 6-dimensional generalized Calabi-Yau structure is non positive and that the Einstein condition forces a 6-dimensional special generalized Calabi-Yau structure to be integrable.

3.1 Generalized Calabi-Yau manifolds

Let M be a $2n$-dimensional manifold. Since SU(n) is the Lie group of the Endomorphisms of $\mathbb{R}^n$ preserving the standard symplectic structure κ_0, the standard complex structure J_0 and the complex volume form

$$\varepsilon_0 := dz_1 \wedge \cdots \wedge dz_n \,,$$

where $z_1, \ldots, z_n$ are the standard coordinates on $\mathbb{C}^n$, then a SU(n)-structure on M is determined by the following data

- an almost symplectic structure;
- a κ-calibrated almost complex structure J;
- a nowhere vanishing $\varepsilon \in \Lambda^{n,0}_J M$ satisfying $\varepsilon \wedge \overline{\varepsilon} = c_n \frac{\kappa^n}{n!}$, where $c_n = 2^n (-1)^{\frac{n^2+n}{2}} i^n$.

In fact, if (κ, J, ε) are given, then the SU(n)-structure is defined by

$$\mathcal{Q} = \{u \in \mathcal{L}(M) \mid u^*(\kappa_0) = \kappa \;, uJu^{-1} = J \;, u * (\varepsilon_0) = \varepsilon\} \,.$$

On other hand, if $\mathcal{Q}$ is a SU(n)-structure on M, then it defines a triple (κ, J, ε) in an obvious way. In the sequel we will refer to a SU(n)-structure on a n-dimensional manifold as to a triple (κ, J, ε) satisfying the properties stated above and we will call SU(n)-*manifold* the quadruple $(M, \kappa, J, \varepsilon)$.

Since $\mathrm{SU}(n) \subset \mathrm{O}(n, \mathbb{R})$, then a SU($n$)-structure (κ, J, ε) induces a Riemann metric g_J on M. This metric is simply defined by formula (2.9). Let ∇ be the Levi-Civita connection of g_J. Then, according to section 2.4, the SU(n)-structure (κ, J, ε) is integrable if and only if it satisfies

$$\nabla\kappa = 0 \,, \quad \nabla J = 0 \,, \quad \nabla\varepsilon = 0 \,, \tag{3.1}$$

i.e. if and only if it is a Calabi-Yau structure (or equivalently if and only if the Holonomy group of g_J is included in SU(n)). Furthermore it can be seen that equations (3.1) are equivalent to

$$\Delta\kappa = 0 \,, \quad \Delta\varepsilon = 0 \,.$$

Moreover we have the following two lemmas.

Lemma 3.1. *Let (M, J) be an almost complex manifold. Assume that there exists a closed nowhere vanishing $\varepsilon \in \Lambda^{n,0}_J M$; then J is integrable.*

Proof. Let $\alpha \in \Lambda^{0,1}_J M$. Since ε is closed we have

$$d(\varepsilon \wedge \alpha) = (-1)^n \,\varepsilon \wedge d\alpha = (-1)^n \,\varepsilon \wedge (\partial_J \alpha + \overline{\partial}_J \alpha) \,,$$

which forces A_J to vanish. Consequently J has to be integrable. □

Lemma 3.2. *A* $\mathrm{SU}(n)$*-structure* (κ, J, ε) *is integrable if and only if the forms* κ*,* ε *are closed.*

Proof. Let g_J be the J-Hermitian metric induced by (κ, J). The $\mathrm{SU}(n)$-structure (κ, J, ε) is integrable if and only if

$$\nabla\kappa = 0\,, \quad \nabla J = 0\,, \quad \nabla\varepsilon = 0\,.$$

Hence, if (κ, J, ε) is integrable, we immediately get

$$d\kappa = 0\,, \quad d\varepsilon = 0\,.$$

Vice versa assume $d\kappa = 0$, $d\varepsilon = 0$; then, by lemma 3.1, it follows $N_J = 0$. Consequently the pair (κ, J) defines a Kähler structure on M and therefore

$$\nabla\kappa = 0\,, \quad \nabla J = 0\,.$$

Finally we observe that the condition $\varepsilon \wedge \overline{\varepsilon} = c_n \frac{\kappa^n}{n!}$ implies $\nabla\varepsilon = 0$. □

3.1.1 Maslov class of Lagrangian submanifolds

Let (M, κ) be a symplectic manifold and let $p\colon L \hookrightarrow M$ be a submanifold; if for any $x \in L$ the vector space $p_*(T_xL)$ is a Lagrangian subspace of (T_xM, κ_x), then L is said to be a *Lagrangian submanifold* of (M, κ). For any $x \in M$ let us denote by $\Lambda_x(M)$ the set of Lagrangian subspaces of (T_xM, κ_x); then $\Lambda(M) := \bigcup_{x\in M} \Lambda_x(M)$ is a fibre bundle over M with standard fibre $\mathrm{U}(n)/\mathrm{O}(n)$. Note that $p\colon L \hookrightarrow M$ is a Lagrangian submanifold if and only if the Gauss map $G\colon x \mapsto T_xL$ is a section of $p^*(\Lambda(M))$.

Let consider now a symplectic manifold (M, κ) with vanishing first Chern class and fix an almost complex structure $J \in \mathcal{C}_\kappa(M)$. Then the couple (κ, J) defines a $\mathrm{U}(n)$-structure $\mathrm{U}(M)$ on M and, since $c_1(M, \kappa) = 0$, there exists a complex volume form $\varepsilon \in \Lambda^{n,0}_J M$. Such as ε induces a smooth map

$$\det\colon \mathrm{U}(M) \to S^1$$

defined by the relation

$$u^*(\varepsilon_0) = \det(u)\,\varepsilon\,.$$

Consequently we can define the map

$$\phi\colon \Lambda(M) \to S^1\,,$$

given by $\phi = \det^2$. We have

Definition 3.3. Let
$$\vartheta := \frac{1}{2\pi i}\,\frac{dz}{z}$$
be the standard volume form of S^1 and let
$$\mu_L := (\phi \circ G)^*\vartheta\,.$$
Then μ_L is called the *Maslov form* of L with respect to (κ, J, ε) and its class in
$$H^1(L,\mathbb{Z})/p^*(H^1(M,\mathbb{Z}))$$
is said to be the *Maslov index* of L.

Note that the Maslov index of L does not depend form the choice of (J, ε). In [28] the authors prove the following

Lemma 3.4. *Let $\widetilde{\nabla}$ be the Chern connection of (κ, J) and assume that there exists a complex volume form $\varepsilon \in \Lambda^{n,0}_J M$ satisfying*
$$\widetilde{\nabla}\,\varepsilon = 0\,.$$
Then
$$\phi^*(\vartheta) = \frac{1}{\pi i}\,\mathrm{tr}\,\omega_J\,, \tag{3.2}$$
where ω_J denotes the connection 1-form of $\widetilde{\nabla}$.

Some computations imply the following (see [28] again)

Proposition 3.5. *Assume that $\widetilde{\nabla}\varepsilon = 0$ and let*
$$\widetilde{H} := -\sum_{j=1}^{n}(J\nabla_{e_j}Je_j)^N\,,$$
be the complex mean curvature vector *of L, where $\{e_1,\dots,e_n\}$ is a g_J-orthonormal frame of L and $(\,\cdot\,)^N$ denotes the normal component with respect to L. Then the Maslov index of L is represented by $\iota_{\widetilde{H}}\kappa$, i.e.*
$$\mu_L = [\iota_{\widetilde{H}}\kappa] \in H^1(L,\mathbb{Z})/p^*(H^1(M,\mathbb{Z}))\,.$$
Finally we have

Theorem 3.6. *If $\widetilde{\nabla}\varepsilon = 0$ and there exists $\theta \in S^1$ such that*
$$p^*(e^{i\theta}\,\varepsilon) = 0\,,$$
then
$$\mu_L = 0\,.$$
Proof. See [28]. □

3.1.2 Generalized Calabi-Yau structures

The previous results on the Maslov class of Lagrangian submanifolds suggests to consider the following definition

Definition 3.7. [28] A $\mathrm{SU}(n)$-structure (κ, J, ε) on M is said to be a *generalized Calabi-Yau structure* if

- κ is a symplectic form;
- J is a κ-calibrated almost complex structure;
- ε is a nowhere vanishing form in $\Lambda_J^{n,0}M$ satisfying

$$\widetilde{\nabla}\varepsilon = 0\,.$$

The quadruple $(M, \kappa, J, \varepsilon)$ is said to a *generalized Calabi-Yau manifold* (GCY).

Note that in this case the Chern connection $\widetilde{\nabla}$ is a connection compatible with the $\mathrm{SU}(n)$-reduction $\mathcal{Q}$ induced by (κ, J, ε). Furthermore the torsion of $\widetilde{\nabla}$ coincides with the intrinsic torsion of $\mathcal{Q}$ and it is determined by the Nijenhuis tensor of J. Moreover, since $\widetilde{\nabla}^{0,1} = \overline{\partial}_J$ and $\varepsilon \wedge \overline{\varepsilon} = c_n\,\frac{\kappa^n}{n!}$, we have that

$$\widetilde{\nabla}\varepsilon = 0 \iff \overline{\partial}_J\varepsilon = 0\,.$$

Therefore condition $\widetilde{\nabla}\varepsilon = 0$ can be replaced by

$$\overline{\partial}_J\varepsilon = 0\,.$$

Finally we remark that the existence of such a ε implies that the Ricci tensor of $\widetilde{\nabla}$ vanishes (see [28], again).

Example 3.8. On $\mathbb{C}^3$ with coordinates z_1, z_2, z_3 let us consider the following product $*$ defined by

$$(z_1, z_2, z_3) * (w_1, w_2, w_3) = (z_1 + w_1, e^{-w_1}z_2 + w_2, e^{w_1}z_3 + w_3)\,.$$

Then $(\mathbb{C}^3, *)$ is a solvable non-nilpotent Lie group admitting a cocompact lattice Γ (see e.g. [52]).
Let

$$\phi_1 = dz_1\,, \quad \phi_2 = e^{z_1}dz_2\,, \quad \phi_3 = e^{-z_1}dz_3\,.$$

Then $\{\phi_1, \phi_2, \phi_3\}$ define a complex coframe on M that is holomorphic with respect to the complex structure induced by $\mathbb{C}^3$. Set

$$\phi_r = \alpha_r + i\alpha_{r+3}\,, \quad r = 1, 2, 3\,.$$

Then a direct computation gives

$$\begin{cases} d\alpha_1 = d\alpha_4 = 0 \\ d\alpha_2 = \alpha_1 \wedge \alpha_2 - \alpha_4 \wedge \alpha_5 \\ d\alpha_3 = -\alpha_1 \wedge \alpha_3 + \alpha_4 \wedge \alpha_6 \\ d\alpha_5 = \alpha_1 \wedge \alpha_5 - \alpha_2 \wedge \alpha_4 \\ d\alpha_6 = -\alpha_1 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 \,. \end{cases}$$

Let

$$\kappa = \alpha_1 \wedge \alpha_4 + \alpha_3 \wedge \alpha_5 + \alpha_6 \wedge \alpha_2$$

and let J be the almost complex structure defined by relations

$$\begin{aligned} &J(\xi_1) = \xi_4 \,, \quad &&J(\xi_3) = \xi_5 \,, \quad &&J(\xi_6) = \xi_2 \\ &J(\xi_4) = -\xi_1 \,, \quad &&J(\xi_5) = -\xi_3 \,, \quad &&J(\xi_2) = -\xi_6 \,. \end{aligned}$$

Then $d\kappa = 0$ and J is a κ-calibrated non-integrable almost complex structure on M. Set

$$\varepsilon = (\alpha_1 + i\alpha_4) \wedge (\alpha_3 + i\alpha_5) \wedge (\alpha_6 + i\alpha_2) \,.$$

We easily get

$$\begin{cases} \overline{\partial}_J \varepsilon = 0 \\ \varepsilon \wedge \overline{\varepsilon} = -i\kappa^3 \,. \end{cases}$$

Hence $(M, \kappa, J, \varepsilon)$ is a GCY manifold.
In [29] the authors prove that M does not admit any Kähler structure and that κ satisfies the HLC condition.

If $(M, \kappa, J, \varepsilon)$ is a generalized Calabi-Yau manifold, then $c_1(M, \kappa) = 0$. A central problem in the study of this class of manifolds is to establish if the vice versa is true:

Problem: *Does any symplectic manifold (M, κ) with $c_1(M, \kappa) = 0$ admit a structure of generalized Calabi-Yau manifold ?*

This problem is strictly related with the following

Problem: *Given a symplectic manifold (M, κ), describe the moduli space of the κ-calibrated complex structures admitting a generalized Calabi-Yau structure.*

The last problem will be study in the next section where we will compute the tangent space to the moduli space of such structures.

3.1.3 Admissible complex structures

Let (M, κ) be a $2n$-dimensional symplectic manifold. An almost complex structure $J \in \mathcal{C}_\kappa(M)$ is said to be *κ-admissible* if there exists $\varepsilon \in \Lambda_J^{n,0}M$ such that $(M, \kappa, J, \varepsilon)$ is a generalized Calabi-Yau manifold.

Now we describe an example of an almost complex structure calibrated by a symplectic form on a compact nilmanifold which is not admissible.

Example 3.9. Let G be the Lie group

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \; : \; x, y, t \in \mathbb{R} \right\}$$

and let $\Gamma \subset G$ be the cocompact lattice of the matrices with integral entries. Then KT$= G/\Gamma$ is called the *Kodaira-Thurston* manifold.
Let $M = \mathbb{T}^3 \times$KT, where $\mathbb{T}^3$ is the 3-dimensional standard torus. We can identify M with a quotient of $\mathbb{R}^6$, where the class of an arbitrary point $(x_1, \ldots, x_6)$ is given by

$$\begin{aligned} &[(x_1, x_2, x_3, x_4, x_5, x_6)] = \\ &[(x_1 + m_1, x_2 + m_2, x_3 + m_3, x_4 + m_4, x_5 + m_5, x_6 + m_4 x_5 + m_6)]\,, \end{aligned}$$

and $(m_1, m_2, m_3, m_4, m_5, m_6) \in \mathbb{Z}^6$. The 1-forms

$$\begin{aligned} &\alpha_1 = dx_1\,, \quad \alpha_2 = dx_2\,, \quad \alpha_3 = dx_3\,, \\ &\alpha_4 = dx_4\,, \quad \alpha_5 = dx_5\,, \quad \alpha_6 = dx_6 - x_4 dx_5\,, \end{aligned}$$

define a global coframe on M. We have

$$\begin{aligned} &d\alpha_i = 0\,, \text{ for } i = 1, \ldots, 5\,, \\ &d\alpha_6 = -\alpha_4 \wedge \alpha_5\,. \end{aligned}$$

The 2-form

$$\kappa = \alpha_{12} + \alpha_{34} + \alpha_{56}$$

is a symplectic structure on M. Let J be the complex structure defined on the dual frame of $\{\alpha_1, \ldots, \alpha_6\}$ by the relations

$$\begin{aligned} &J(X_1) := X_2, \qquad J(X_3) := X_4, \qquad J(X_5) := X_6, \\ &J(X_2) := -X_1, \qquad J(X_4) := -X_3, \qquad J(X_6) := -X_5\,. \end{aligned}$$

Then J is a κ-calibrated complex structure on M. Moreover

$$\varepsilon := (\alpha_1 + i\alpha_2) \wedge (\alpha_3 + i\alpha_4) \wedge (\alpha_5 + i\alpha_6)$$

is a nowhere vanishing section of $\Lambda^{3,0}_J M$. We easily get

$$\overline{\partial}_J \varepsilon = -(\alpha_3 - i\alpha_4) \wedge \varepsilon\,.$$

In order to prove that there are not nowhere vanishing (3,0)-forms η on M such that $\overline{\partial}_J \eta = 0$ we set

$$Z_j = \frac{1}{2}(X_j - iJX_j)\,, \quad j = 1, 2, 3$$

and

$$\zeta_j = \alpha_j + iJ\alpha_j\,, \quad j = 1, 2, 3\,.$$

Let $\eta \in \Lambda^{3,0}_J M$, then there exists a function $f = u + iv \in C^\infty(M, \mathbb{C})$ such that $\varepsilon = f\,\eta$. We have

$$\overline{\partial}_J \eta = \overline{\partial}_J (f\varepsilon) = \overline{\partial}_J f \wedge \varepsilon + f \overline{\partial}_J \varepsilon = (\sum_{j=1}^{3} \overline{Z}_j(f)\,\overline{\zeta}_j - f\,\overline{\zeta}_2) \wedge \varepsilon\,.$$

Therefore $\overline{\partial}_J \eta = 0$ if and only if the following systems of PDE's are satisfied:

a. $$\begin{cases} \partial_{x_1} u - \partial_{x_2} v = 0 \\ \partial_{x_2} u + \partial_{x_1} v = 0\,, \end{cases}$$

b. $$\begin{cases} \partial_{x_3} u - \partial_{x_4} v - u = 0 \\ \partial_{x_4} u + \partial_{x_3} v - v = 0\,, \end{cases}$$

c. $$\begin{cases} \partial_{x_6} v - \partial_{x_5} u - x_4 \partial_{x_6} u = 0 \\ \partial_{x_6} u + \partial_{x_5} v + x_4 \partial_{x_6} v = 0\,. \end{cases}$$

Equations a. imply that $f = f(x_3, x_4, x_5, x_6)$. Since f is a function on M, then it is $\mathbb{Z}-$periodic in the variables x_3, x_5, x_6. Set

$$\begin{aligned} u(x_3, x_4, x_5, x_6) &= \sum u_N(x_4)\, e^{2\pi i(n_3 x_3 + n_5 x_5 + n_6 x_6)}\,, \\ v(x_3, x_4, x_5, x_6) &= \sum v_N(x_4)\, e^{2\pi i(n_3 x_3 + n_5 x_5 + n_6 x_6)}\,. \end{aligned}$$

We have

$$\partial_{x_5} u = \sum 2\pi i n_5 u_N(x_4)\, e^{2\pi i(n_3 x_3 + n_5 x_5 + n_6 x_6)}\,. \tag{3.3}$$

The same relations hold for $\partial_{x_6} u$, $\partial_{x_5} v$, $\partial_{x_6} v$. Hence, by plugging (3.3) and the other expressions into equations c., we get

$$\begin{cases} (m_5 + x_4 m_6)\, u_N(x_4) - m_6\, v_N(x_4) = 0 \\ m_6 u_N\,(x_4) + (m_5 + x_4 m_6)\, v_N(x_4) = 0\,, \end{cases}$$

for any $N = (n_3, n_5, n_6) \in \mathbb{Z}^3$. If $(m_5 + x_4 m_6)^2 + m_6^2 \neq 0$, then $u_N(x_4) = v_N(x_4) = 0$. Therefore if f satisfies equations a. and c. then $f = f(x_3, x_4)$. In particular f must be $\mathbb{Z}^2$-periodic. By equations b. we immediately get $f \equiv 0$. Hence the almost complex structure J is not admissible.

Moduli space of admissible almost complex structures

Let (M, κ) be a symplectic manifold with vanishing first Chern class. Let us denote by $\mathcal{AC}_\kappa(M)$ the space of κ-admissible complex structures on M. The Lie group $\mathrm{Sp}_\kappa(M)$ of the diffeomorphisms of M preserving κ acts on $\mathcal{AC}_\kappa(M)$ by

$$(\phi, J) = \phi_* J \phi_*^{-1}\,.$$

Let

$$\mathfrak{M}(\mathcal{AC}_\kappa(M)) = \mathcal{AC}_\kappa(M)/\mathrm{Sp}_\kappa(M)\,,$$

be the relative moduli space.

Definition 3.10. Let J be a κ-calibrated almost complex structure on M; another almost complex structure $\widetilde{J} \in \mathcal{C}_\kappa(M)$ is said to be *close to* J if $\det(I - \widetilde{J}J) \neq 0$.

It is known that the space of κ-calibrated almost complex structures close to a fixed J is parameterized by the symmetric tangent bundle endomorphisms anticommuting with J and having norm less than 1: namely $\widetilde{J}$ is close to J if and only if there exists a unique $L \in \mathrm{End}(TM)$ such that

$$\widetilde{J} = RJR^{-1}\,, \quad LJ = -JL\,, \quad {}^tL = L\,, \quad ||L|| < 1\,,$$

where $R = I + L$ and the transpose and the norm of L are taken with respect to the metric g_J.

In order to describe the behavior of the $\overline{\partial}$ operator for $\widetilde{J}$ close to a fixed J we give the following proposition which is interesting in its own.

Proposition 3.11. *Let $R = I + L$ be an arbitrary isomorphism of TM. Then*

$$RdR^{-1}\gamma = d\gamma + [\tau_L, d]\gamma + \sigma_L\gamma \tag{3.4}$$

for any differential form α on M of positive degree; where:

- τ_L *is the zero order derivation defined on the* r*-forms by*

$$\begin{aligned}\tau_L\gamma(X_1, X_2 \dots, X_r) =&\gamma(LX_1, X_2, \dots, X_n) + \gamma(X_1, LX_2, \dots, X_n) + \\ &\dots + \gamma(X_1, X_2, \dots, LX_n)\,;\end{aligned}$$

- $[\tau_L, d] = \tau_L d - d\tau_L$*;*

- σ_L *is the operator defined on the* 1*-form as*

$$\sigma_L\alpha(X, Y) := \alpha(R^{-1}(N_L(X, Y)))$$

(being $N_L(X, Y) := [LX, LY] - L[LX, Y] - L[X, LY] + L^2[X, Y]$*), and it is extended on the forms of arbitrary degree by the Leibniz rule.*

Proof. Let $\alpha \in \Lambda^1(M)$ and $X, Y \in TM$. We have

$$\begin{aligned}RdR^{-1}\alpha(X, Y) =&dR^{-1}\alpha(RX, RY)\\ =&RX\alpha(Y) - RY\alpha(X) + \alpha(R^{-1}[RX, RY])\\ =&X\alpha(Y) - Y\alpha(X) + \alpha([X, Y]) + LX\alpha(Y) - LY\alpha(X)\\ &+ \alpha(R^{-1}[RX, RY] - [X, Y])\\ =&d\alpha(X, Y) + LX\alpha(Y) - LY\alpha(X) + \alpha(R^{-1}[RX, RY] - [X, Y])\,.\end{aligned}$$

Moreover

$$\begin{aligned}\tau_L d\alpha(X, Y) =&d\alpha(LX, Y) + d\alpha(X, LY)\\ =&LX\alpha(Y) - Y\alpha(LX) + \alpha([LX, Y]) + X\alpha(LY) - LY\alpha(X) +\\ &\alpha([X, LY])\end{aligned}$$

and

$$d\tau_L\alpha(X, Y) = X\alpha(LY) - Y\alpha(LX) + \alpha(L[X, Y])\,.$$

Therefore we obtain

$$\begin{aligned}(RdR^{-1} - [\tau_L, d])\alpha(X, Y) =&d\alpha(X, Y) +\\ &\alpha(R^{-1}[RX, RY] - [LX, Y] - [X, LY] + L[X, Y] - [X, Y])\,.\end{aligned}$$

Furthermore

$$\begin{aligned}&R(R^{-1}[RX, RY] - [LX, Y] - [X, LY] + L[X, Y] - [X, Y]) =\\ &= [RX, RY] - R[LX, Y] - R[X, LY] + RL[X, Y] - R[X, Y] =\\ &= [LX, LY] + [LX, Y] + [X, LY] + [X, Y] - [LX, Y] - L[LX, Y]\\ &\quad - [X, LY] - L[X, LY] + L[X, Y] + L^2[X, Y] - [X, Y] =\\ &= N_L(X, Y)\,,\end{aligned}$$

i.e.

$$R^{-1}[RX,RY]-[LX,Y]-[X,LY]+L[X,Y]-[X,Y])=R^{-1}(N_L(X,Y))$$

Hence we have

$$(RdR^{-1}-[\tau_L,d])\alpha(X,Y)=d\alpha(X,Y)+\alpha(R^{-1}(N_L(X,Y)))$$

which proves the proposition when α is a 1-form. Since the operators on the two sides of formula (3.4) satisfy Leibnitz rule, the proof is complete. $\square$

Now we are ready to give the following

Proposition 3.12. *Let $J,\widetilde{J}$ be closed almost complex structures in $\mathcal{C}_\omega(M)$ and let $\overline{\partial}_J$, $\overline{\partial}_{\widetilde{J}}$ be the $\overline{\partial}$-operators with respect to J, $\widetilde{J}$ respectively. Then*

1. $R\overline{\partial}_{\widetilde{J}}f=\overline{\partial}_Jf+L\partial_Jf\,,$

2. $R\overline{\partial}_{\widetilde{J}}R^{-1}\gamma=\overline{\partial}_J\gamma+[\tau_L,d]^{p,q+1}\gamma+\sigma_L^{p,q+1}\gamma\,,$

where $f\in C^\infty(M,\mathbb{C})$, $\gamma\in\Lambda^{p,q}_{\widetilde{J}}(M)$, $\widetilde{J}=RJR^{-1}$, $R=I+L$ and $[\tau_L,d]^{p,q+1}$, $(\sigma_L)^{p,q+1}$ denote the projection of the bracket $[\tau_L,d]=\tau_Ld-d\tau_L$ and of the operator σ_L on the space $\Lambda^{p,q+1}_J(M)$, respectively.

Proof. 1. Let $f\in C^\infty(M,\mathbb{C})$. We have

$$R\overline{\partial}_{\widetilde{J}}f=R(df)^{\widetilde{0,1}}=(Rdf)^{0,1}=(df+Ldf)^{0,1}=\overline{\partial}_Jf+Ldf^{0,1}=\overline{\partial}_Jf+L\partial_Jf\,,$$

where the subscript $\widetilde{0,1}$ denotes the projection onto $\Lambda^{0,1}_{\widetilde{J}}(M)$.

2. Let $\gamma\in\Lambda^{p,q}_J(M)$. Then we have

$$\begin{aligned}R\overline{\partial}_{\widetilde{J}}R^{-1}\gamma&=R(dR^{-1}\gamma)^{\widetilde{p,q+1}}=(RdR^{-1}\gamma)^{p,q+1}\\&=(d\gamma)^{p,q+1}+[\tau_L,d]^{p,q+1}\gamma+\sigma_L^{p,q+1}\gamma\\&=\overline{\partial}_J\gamma+[\tau_L,d]^{p,q+1}\gamma+\sigma_L^{p,q+1}\gamma\,,\end{aligned}$$

where the subscript $\widetilde{p,q+1}$ denotes the projection onto $\Lambda^{p,q+1}_{\widetilde{J}}(M)$. $\square$

Now we compute the (virtual) tangent space to $\mathcal{AC}_\kappa(M)$ at an arbitrary point $[J]$.

Let $J\in\mathcal{C}_\kappa(M)$ be a κ-admissible almost complex structure on M; then there exists a nowhere vanishing $\varepsilon\in\Lambda^{n,0}_JM$ such that $\overline{\partial}_J\varepsilon=0$.

Let $\widetilde{J}$ be a κ-calibrated almost complex structure close to J, then $\widetilde{J}=RJR^{-1}$

where $R = I + L$, $LJ + JL = 0$, $L = {}^tL$, $\|L\| < 1$. The form $R^{-1}\varepsilon$ is a nowhere vanishing form in $\Lambda^{n,0}_{\tilde{J}}M$. Any other section ε' which trivializes $\Lambda^{n,0}_{\tilde{J}}M$ is a multiple of ε, namely $\varepsilon' = f\varepsilon$, with $f \in C^\infty(M, \mathbb{C})$, $f(p) \neq 0$ for any $p \in M$.
Let $\widetilde{J}$ be κ-admissible, then there exists $f \in C^\infty(M, \mathbb{C})$ such that

$$\overline{\partial}_{\tilde{J}} f R^{-1}\varepsilon = 0\,,$$

where $f \neq 0$.
By formulae of proposition 3.12 we have

$$\begin{aligned}
R\overline{\partial}_{\tilde{J}}(fR^{-1}\varepsilon) =& R(\overline{\partial}_{\tilde{J}} f \wedge R^{-1}\varepsilon + f\overline{\partial}_{\tilde{J}} R^{-1}\varepsilon) \\
=& R(\overline{\partial}_{\tilde{J}} f) \wedge \varepsilon + fR\overline{\partial}_{\tilde{J}} R^{-1}\varepsilon \\
=& \overline{\partial}_J f \wedge \varepsilon + L\partial_J f \wedge \varepsilon + f(\overline{\partial}_J\varepsilon + [\tau_L, d]^{n,1}\varepsilon + (\sigma_L)^{n,1}\varepsilon) \\
=& \overline{\partial}_J f \wedge \varepsilon + L\partial_J f \wedge \varepsilon + f([\tau_L, d]^{n,1}\varepsilon + (\sigma_L)^{n,1}\varepsilon)\,,
\end{aligned}$$

i.e.

$$R\overline{\partial}_{\tilde{J}}(fR^{-1}\varepsilon) = \overline{\partial}_J f \wedge \varepsilon + L\partial_J f \wedge \varepsilon + f[\tau_L, d]^{n,1}\varepsilon + f(\sigma_L)^{n,1}\varepsilon\,. \tag{3.5}$$

Let us consider a smooth curve of κ-admissible almost complex structures J_t close to J, such that $J_0 = J$. For any t there exists $L_t \in \mathrm{End}(TM)$ such that if $R_t = I + L_t$, then $J_t = R_t J R_t^{-1}$, for $L_t J + J L_t = 0$, $L_t = {}^tL_t$, $\|L_t\| < 1$.
Let $\varepsilon \in \Lambda^{n,0}_J M$ be a nowhere vanishing $\overline{\partial}_J$-closed form. In correspondence of any t there exists $f_t \colon M \to \mathbb{C}$, $f_t \neq 0$, such that

$$\overline{\partial}_{J_t} f_t R_t^{-1}\varepsilon = 0\,.$$

Hence by formula (3.5) it has to be

$$\overline{\partial}_J f_t \wedge \varepsilon + L_t\partial_J f_t \wedge \varepsilon + f_t[\tau_{L_t}, d]^{n,1}\varepsilon + f_t(\sigma_{L_t})^{n,1}\varepsilon = 0\,. \tag{3.6}$$

We may assume without loss of generality that

$$\begin{aligned}
f_0 &= 1\,,\\
L_0 &= 0\,.
\end{aligned}$$

The derivative of (3.6) at $t = 0$ is

$$\overline{\partial}_J \dot{f}_0 \wedge \varepsilon + [\tau_L, d]^{n,1}\varepsilon\,, \tag{3.7}$$

where we set $L = \dot{L}_0$.
Let us compute $[\tau_L, d]^{n,1}\varepsilon$. We have

$$\begin{aligned}
(\tau_L d\varepsilon)^{n,1} &= (\tau_L \overline{A}_J \varepsilon)^{n,1}\,,\\
(d\tau_L\varepsilon)^{n,1} &= \partial_J \tau_L \varepsilon\,.
\end{aligned}$$

We may write

$$(\tau_L\overline{A}_J\varepsilon)^{n,1} = \mu_L(\varepsilon) \wedge \varepsilon \tag{3.8}$$

and

$$\partial_J\tau_L\varepsilon = \gamma_L(\varepsilon) \wedge \varepsilon\,, \tag{3.9}$$

for $\mu_L(\varepsilon), \gamma_L(\varepsilon) \in \Lambda^{0,1}_J M$. Therefore (3.7) reduces to

$$\overline{\partial}_J\dot{f}_0 \wedge \varepsilon + \mu_L(\varepsilon) \wedge \varepsilon + \gamma_L(\varepsilon) \wedge \varepsilon = 0\,, \tag{3.10}$$

and it is equivalent to

$$\overline{\partial}_J\dot{f}_0 + \mu_L(\varepsilon) + \gamma_L(\varepsilon) = 0\,. \tag{3.11}$$

The following lemma gives the behavior of μ_L and γ_L when the complex volume form ε changes.

Lemma 3.13. *Let ε, $\varepsilon' \in \Lambda^{n,0}_J M$ be $\overline{\partial}_J$-closed. Let $\{Z_1, \dots, Z_n\}$ be a local $(1,0)$-frame and $\{\zeta_1, \dots, \zeta_n\}$ be the dual frame. Then*

1. $\mu_L(\varepsilon') = \mu_L(\varepsilon)\,,$
2. $\gamma_L(\varepsilon') = \gamma_L(\varepsilon) + \eta(f)\,,$

where $\varepsilon' = f\varepsilon$ and $\eta(f)$ is the $(0,1)$-form defined locally as

$$\eta(f) = -\frac{1}{f}\sum_{k,r=1}^{n} Z_k(f)L_{\overline{r}k}\zeta_{\overline{r}}\,.$$

Proof. By definition we get

$$\begin{aligned}\mu_L(\varepsilon') \wedge \varepsilon' =& (\tau_L\overline{A}_J\varepsilon')^{n,1} = (\tau_L\overline{A}_J f\varepsilon)^{n,1} = f(\tau_L\overline{A}_J\varepsilon)^{n,1}\\ =& f\mu_L(\varepsilon) \wedge \varepsilon = \mu_L(\varepsilon) \wedge \varepsilon'\,.\end{aligned}$$

Therefore 1. is proved.
We have

$$\begin{aligned}\gamma_L(\varepsilon) \wedge \varepsilon' &= \partial_J\tau_L\varepsilon' = \partial_J\tau_L f\varepsilon\\ &= \partial_J f \wedge \tau_L\varepsilon + f\partial_J\tau_L\varepsilon\\ &= \partial_J f \wedge \tau_L\varepsilon + f\gamma_L(\varepsilon)\varepsilon\\ &= \partial_J f \wedge \tau_L\varepsilon + \gamma_L(\varepsilon) \wedge \varepsilon'\,.\end{aligned}$$

Now we express $\partial_J f \wedge \tau_L\varepsilon$ in terms of ε'. With respect to the local (1,0)-frame $\{Z_1, \dots, Z_n\}$ we have

$$\varepsilon = h\,\zeta_1 \wedge \dots \wedge \zeta_n$$

and

$$\partial_J f = \sum_{k=1}^{n} Z_k(f)\zeta_k \,.$$

Now we have

$$\begin{aligned}\tau_L\varepsilon &= hL(\zeta_1)\wedge\cdots\wedge\zeta_n+\cdots+h\zeta_1\wedge\cdots\wedge L(\zeta_n)\\ &= h\sum_{k,r=1}^{n}\{L_{\overline{r}1}\zeta_{\overline{r}}\wedge\cdots\wedge\zeta_n+\cdots+(-1)^{n-1}L_{\overline{r}n}\zeta_{\overline{r}}\wedge\cdots\wedge\zeta_{n-1}\}\,.\end{aligned}$$

Therefore

$$\begin{aligned}\partial_J f\wedge\tau_L\varepsilon &= -\sum_{k,r=1}^{n} Z_k(f)L_{\overline{r}k}\zeta_{\overline{r}}\wedge\varepsilon\\ &= -\frac{1}{f}\sum_{k,r=1}^{n} Z_k(f)L_{\overline{r}k}\zeta_{\overline{r}}\wedge\varepsilon'\,.\end{aligned}$$

Hence

$$\gamma_L(\varepsilon') = \gamma_L(\varepsilon)+\eta(f)\,,$$

i.e. 2 is proved. □

From now on we assume that M is compact. In this case, since any holomorphic map $h\colon M\to\mathbb{C}$ is constant (see lemma 2.15), for any $J\in\mathcal{AC}_\kappa(M)$ there exists a unique $\varepsilon\in\Lambda^{n,0}_J M$ (modulo constants) such that $\overline{\partial}_J\varepsilon=0$. Therefore, in view of the last lemma, the $(0,1)$-forms μ_L and γ_L do not depend on the choice of the volume form ε. Therefore by formula (3.11) a tangent vector to $\mathcal{AC}_\kappa(M)$ at a point J is an endomorphism JL, where $L\in\mathrm{End}(TM)$ anticommutes with J and it is such that the $(0,1)$-form $\mu_L-\gamma_L$ is $\overline{\partial}_J$-exact. Hence

$$T_J\mathcal{AC}_\kappa(M)=\{JL\,|\,L\in\mathrm{End}^{0,1}_J(TM)\,,{}^tL=L\,,\mu_L-\gamma_L \text{ is } \overline{\partial}_J-\text{exact}\}\,.$$

We have proved the following

Proposition 3.14. *Let $J\in\mathcal{AC}_\kappa(M)$; then the tangent space to $\mathcal{AC}_\kappa(M)$ at J is given by*

$$T_J\mathcal{AC}_\kappa(M)=\{JL\,|\,L\in\mathrm{End}^{0,1}_J(TM)\,,{}^tL=L\,,\mu_L-\gamma_L \textit{ is } \overline{\partial}_J-\textit{exact}\}\,.$$

In the last part of this section we are going to compute the tangent space to the Moduli space of the κ-admissible almost complex structures.
Recall that by definition

$$\mathfrak{M}(\mathcal{AC}_\kappa(M))=\mathcal{AC}_\kappa(M)/\mathrm{Sp}_\kappa(M)\,.$$

Let $J \in \mathcal{AC}_\kappa(M)$ and let $\mathcal{O}_J(M)$ be the orbit of J under the action of $\mathrm{Sp}_\kappa(M)$; then

$$T_J\mathfrak{M}(\mathcal{AC}_\kappa(M)) = T_J\mathcal{AC}_\kappa(M)/T_J\mathcal{O}_J(M)\,.$$

Therefore we have to compute $T_J\mathcal{O}_J(M)$. We have

Lemma 3.15. *The tangent space to $\mathcal{O}_J(M)$ at J is given by*

$$T_J\mathcal{O}_J(M) = \{\mathcal{L}_X J \mid X \in TM \text{ and } \mathcal{L}_X\kappa = 0\}\,,$$

where $\mathcal{L}$ denotes the Lie derivative.

Proof. Let $\alpha(t)$ be a curve in $\mathcal{O}_J(M)$ such that $\alpha(0) = I$ and let $A = \frac{d}{dt}\alpha(t)_{|t=0}$. Then there exists a smooth curve $\phi_t \in\mathrm{Sp}_\kappa(M)$ such that

$$\phi_0 = I\,, \quad \alpha(t) = \phi_{t*}^{-1}J\phi_{t*}\,.$$

Fix a system of local charts in M $\{x_1,\dots,x_{2n}\}$ and let X be the vector field associated to the 1-parameter group ϕ_t. Then we have

$$\frac{d}{dt}\{\phi_{t*}^{-1}J\phi_{t*}[x](\frac{\partial}{\partial x_i})\}_{|t=0} = \sum_{j,h,r=1}^{2n}\frac{d}{dt}\{\frac{\partial}{\partial x_i}(\phi_t^j)(x)J_{hj}(\phi_t(x))(\phi_{t*}^{-1})_{rh}(\frac{\partial}{\partial x_r})\}_{|t=0} =$$

$$= \sum_{j,h=1}^{2n}\{\frac{\partial}{\partial x_i}(X^j)J_{hj}(x) + X^j\frac{\partial}{\partial x_j}(J_{hi})(x) - J_{ji}\frac{\partial}{\partial x_j}(X^h)(x)\}\frac{\partial}{\partial x_h}\,,$$

i.e.

$$A_{hi} = \sum_{j=1}^{n}(J_{hj}\frac{\partial}{\partial x_i}(X^j) - J_{ji}\frac{\partial}{\partial x_j}(X^h)) + X(J_{hi})\,.$$

Therefore

$$A(Y) = [X, JY] - J[X,Y] = \mathcal{L}_X(Y)$$

for any $Y \in TM$. Now we observe that, since by hypothesis $\phi_t \in\mathrm{Sp}_\kappa(M)$, then

$$0 = \frac{d}{dt}\phi_t^*(\kappa)_{|t=0} = \mathcal{L}_X\kappa\,.$$

□

We can summarize the previous facts in the following

Theorem 3.16. *Let (M,κ) be a compact symplectic manifold of dimension $2n$. Let $\mathcal{AC}_\kappa(M) = \mathcal{AC}_\kappa(M)/\mathrm{Sp}_\kappa(M)$ be the moduli space of κ-admissible almost complex structures on M. Let $J \in \mathcal{AC}_\kappa(M)$; then the tangent space to $\mathfrak{M}(\mathcal{AC}_\kappa(M))$ at $[J]$ is given by*

$$T_{[J]}\mathfrak{M}(\mathcal{AC}_\kappa(M)) = \frac{\{JL \,|\, L \in \mathrm{End}_J^{(0,1)}(TM)\,, {}^tL = L\,, \mu_L - \gamma_L \textit{ is } \overline{\partial}_J - \textit{exact}\}}{\{\mathcal{L}_X J \mid X \in TM \textit{ and } \mathcal{L}_X\kappa = 0\}}\,,$$

where μ_L, γ_L are the $(0,1)$-forms defined by

$$(\tau_L \overline{A}_J \varepsilon)^{n,1} = \mu_L \wedge \varepsilon\,, \quad \partial_J \tau_L \varepsilon = \gamma_L \wedge \varepsilon\,,$$

ε is a nowhere vanishing $\overline{\partial}_J$-closed form in $\Lambda^{n,0}_J M$ and $\mathcal{L}$ denotes the Lie derivative.

Remark 3.17. *If J is an admissible complex structure, then the form μ_L vanishes, since $\overline{A}_J = 0$.*

3.1.4 Admissible complex structures on the Torus

In this section we apply our construction to the torus, computing explicitly the tangent space to $\mathfrak{M}(\mathcal{AC}_\kappa(\mathbb{T}^{2n}))$.

Let $\mathbb{T}^{2n} = \mathbb{C}^n/\mathbb{Z}^{2n}$ be the standard $2n$-dimensional complex torus and let $\{z_1,\dots,z_n\}$ be coordinates on $\mathbb{C}^n$, $z_\alpha = x_\alpha + i x_{\alpha+n}$ for $n = 1,\dots,n$.
Then

$$\kappa_n = \frac{i}{2}\sum_{\alpha=1}^{n} dz_\alpha \wedge d\overline{z}_\alpha\,,$$

$$\varepsilon_n = dz_1 \wedge \cdots \wedge dz_n$$

define a Calabi-Yau structure on $\mathbb{T}^{2n}$. Let $g_n := g_{J_n}$ be the Hermitian metric induced by (J_n,κ_n). The standard complex structure J_n is a κ_n-admissible complex structure on $\mathbb{T}^{2n}$.
Now we want to deform J_n computing the tangent space $T_{J_n}\mathfrak{M}(\mathcal{AC}_{\kappa_n}(M))$ to the moduli space $\mathfrak{M}(\mathcal{AC}_{\kappa_n}(M))$. According to the previous section, given a g_n-symmetric $L \in \mathrm{End}^{(0,1)}_J(TM)$ we have to write down the (0,1)-form γ_L defined by

$$\overline{\partial}_J(\tau_L \varepsilon_n) = \gamma_L \wedge \varepsilon_n\,.$$

Let

$$L = \sum_{s,r=1}^{n}\{L_{\overline{r}s} dz_s \otimes \frac{\partial}{\partial \overline{z}_r} + \overline{L}_{\overline{r}s} d\overline{z}_s \otimes \frac{\partial}{\partial z_r}\}\,,$$

where $\{L_{s\overline{r}}\}$ are $\mathbb{Z}^{2n}$-periodic functions. Then we get

$$\begin{aligned}
\tau_L \varepsilon_n &= L(dz_1) \wedge \cdots \wedge dz_n + \cdots + dz_1 \wedge \cdots \wedge L(dz_n)\\
&= \sum_{r=1}^{n}\{L_{1\overline{r}}\, d\overline{z}_r \wedge \cdots \wedge dz_n\} + \cdots + \sum_{r=1}^{n}\{(-1)^{n-1} L_{n\overline{r}}\, d\overline{z}_r \wedge \cdots \wedge dz_{n-1}\}\\
&= \sum_{r,s=1}^{n} (-1)^{r+1} L_{s\overline{r}}\, d\overline{z}_r \wedge dz_1 \wedge \cdots \wedge \widehat{dz_s} \wedge \cdots \wedge dz_n\,,
\end{aligned}$$

where $\widehat{\ }$ means that the corresponding term is omitted. Therefore we obtain

$$\partial_J(\tau_L\varepsilon) = -\sum_{r,s=1}^{n} \frac{\partial}{\partial z_s} L_{s\bar{r}}\, d\bar{z}_r \wedge \varepsilon\,,$$

i.e.

$$\gamma_L = -\sum_{r,s=1}^{n} \frac{\partial}{\partial z_s} L_{s\bar{r}}\, d\bar{z}_r\,.$$

Then the tangent space to $\mathcal{AC}_\kappa(\mathbb{T}^{2n})$ at J_n is given by

$$T_{J_n}\mathcal{AC}_\kappa(\mathbb{T}^{2n}) = \{J_nL \in \mathrm{End}(T\mathbb{T}^{2n}) \mid L = {}^tL,\ J_nL = -LJ_n \text{ and } \gamma_L \text{ is } \overline{\partial}_{J_n}\text{–exact}\}\,.$$

In order to compute $T_{J_n}\mathfrak{M}(\mathcal{AC}_\kappa(\mathbb{T}^{2n}))$ we have to compute $\mathcal{L}_XJ_n$, for X $\in$End(TM) such that $\mathcal{L}_X\kappa_n = 0$. Let $X = \sum_{r=1}^{2n} a_r \frac{\partial}{\partial x_r}$ be a real vector field on $\mathbb{T}^{2n}$, then

$$\begin{aligned}
\mathcal{L}_X(J_n)\Big(\frac{\partial}{\partial z_s}\Big) &= \sum_{r=1}^{2n} -i\frac{\partial a_r}{\partial z_s}\frac{\partial}{\partial x_r} + \frac{\partial a_r}{\partial z_s}J_n\Big(\frac{\partial}{\partial x_r}\Big)\\
&= \sum_{r=1}^{2n} -i\frac{\partial a_r}{\partial z_s}\Big(\frac{\partial}{\partial x_r} + iJ_n\frac{\partial}{\partial x_r}\Big)\\
&= \sum_{r=1}^{n} -i\frac{\partial a_r}{\partial z_s}\Big(\frac{\partial}{\partial x_r} - i\frac{\partial}{\partial x_{r+n}}\Big) - i\frac{\partial a_{r+n}}{\partial z_s}\Big(\frac{\partial}{\partial x_{r+n}} - i\frac{\partial}{\partial x_r}\Big)\\
&= -\sum_{r=1}^{n} \frac{\partial}{\partial z_s}(ia_r + a_{r+n})\Big(\frac{\partial}{\partial x_r} + i\frac{\partial}{\partial x_{r+n}}\Big)\\
&= -2\sum_{r=1}^{n} \frac{\partial}{\partial z_s}(ia_r + a_{r+n})\frac{\partial}{\partial \bar{z}_r}\,,
\end{aligned}$$

i.e.

$$\mathcal{L}_X(J_n) = -2\sum_{r,s=1}^{n}\Big\{\frac{\partial}{\partial z_s}(ia_r + a_{r+n})dz_s \otimes \frac{\partial}{\partial \bar{z}_r} + \frac{\partial}{\partial \bar{z}_s}(-ia_r + a_{r+n})d\bar{z}_s \otimes \frac{\partial}{\partial z_r}\Big\}\,.$$

Therefore $L = \mathcal{L}_XJ_n$ if and only if

$$L_{\bar{r}s} = 2\frac{\partial}{\partial z_s}(a_r - ia_{r+n})\,, \tag{3.12}$$

for some periodic functions a_r on $\mathbb{R}^{2n}$.

Let consider now $L \in \mathrm{End}(T\mathbb{T}^{2n})$ such that it anticommutes with J_n and let $L_{\bar{r}s}$ be constant functions.

By equation (3.12) there exists $X \in TM$ such that $L = \mathcal{L}_X(J_n)$ if and only if

$$L_{\overline{r}s} = 2\frac{\partial}{\partial z_s}(a_r - ia_{r+n}) = \frac{\partial}{\partial x_s}(a_r - ia_{r+n}) - i\frac{\partial}{\partial x_{s+n}}(a_r - ia_{r+n})$$
$$= \Big(\frac{\partial a_r}{\partial x_s} - \frac{\partial a_{r+n}}{\partial x_{s+n}}\Big) - i\Big(\frac{\partial a_{r+n}}{\partial x_s} + \frac{\partial a_r}{\partial x_{s+n}}\Big).$$

Therefore

$$\begin{cases} \frac{\partial a_r}{\partial x_s} - \frac{\partial a_{r+n}}{\partial x_{s+n}} & = \text{constant} \\ \frac{\partial a_{r+n}}{\partial x_s} + \frac{\partial a_r}{\partial x_{s+n}} & = \text{constant}\,, \end{cases}$$

that imply

$$\frac{\partial^2 a_r}{\partial x_s^2} + \frac{\partial^2 a_r}{\partial x_{s+n}^2} = 0$$

for any $r, s = 1, \ldots, n$.
It follows that the $\{a_r\}$ are harmonic functions on the standard torus $\mathbb{T}^{2n}$ and then they are constant. Therefore any constant g_n-symmetric $L \in \mathrm{End}^{(0,1)}_{J_n}(T\mathbb{T}^{2n})$ defines a non-trivial element of $T_{[J_n]}\mathfrak{M}(\mathcal{AC}_\kappa(M))$. Moreover any constant endomorphisms L_1, L_2 of such type give rise to different elements of $T_{[J_n]}\mathfrak{M}(\mathcal{AC}_\kappa(M))$. Hence J_n is not a rigid structure.

3.2 Six-dimensional generalized Calabi-Yau structures

Since the integrability of a SU(n)-structure forces the induced metric to be Ricci-flat, the Ricci tensor of a SU(n)-manifold $(M, \kappa, J, \varepsilon)$ depends only on the intrinsic torsion of (κ, J, ε).
In this section we write down the Ricci tensor and the scalar curvature of a SU(3)-manifold in terms of torsion forms. Our approach is similar to that one used by Bryant in [15] to compute the Ricci tensor of a G_2-structure. This result has been proved with the aid of MAPLE.

As a direct application of our formulae we have that the scalar curvature of the metric associted to 6-dimensional GCY structure is non-positive and that the Einstein equation forces a special class of 6-dimensional GCY structures to be integrable.

3.2.1 Linear symplectic algebra in dimension 6

In this section we recall same basic facts of Linear algebra in diemension 6.

Let us denote by $\{e_1, \dots, e_6\}$ the standard basis of $\mathbb{R}^6$ and by $\{e^1, \dots, e^6\}$ the dual one. Let

$$\kappa_0 = e^{12} + e^{34} + e^{56}$$

be the standard symplectic structure of $\mathbb{R}^6$. The space of 3-forms on $\mathbb{R}^6$ splits into the following two $\mathrm{Sp}(3, \mathbb{R})$-irreducible vector spaces

$$\begin{aligned} \Lambda^3_0\mathbb{R}^{6*} &= \{\gamma \in \Lambda^3\mathbb{R}^{6*} \mid \gamma \wedge \kappa_0 = 0\}, \\ \Lambda^3_6\mathbb{R}^{6*} &= \{\alpha \wedge \kappa_0 \mid \alpha \in \mathbb{R}^{6*}\}. \end{aligned}$$

The 3-forms lying in $\Lambda^3_0\mathbb{R}^{6*}$ are called *effective* 3-forms. Let

$$\varepsilon_0 = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6),$$

be the standard complex volume form on $\mathbb{C}^3$; then the real part of ε_0

$$\Omega_0 = e^{135} - e^{146} - e^{245} - e^{236},$$

is an effective 3-form. Let consider the action of the Lie group $G = \mathrm{Sp}(3, \mathbb{R}) \times \mathbb{R}_+$ on $\Lambda^3_0\,(\mathbb{R}^6)^*$ given by

$$(\psi, t)\,\alpha = t\,(\psi^{-1})^*(\alpha)$$

and let $\mathcal{O}$ be the orbit of Ω_0 under this action. It is known (see e.g [8, 58]) that the stabilizer of Ω_0 is locally isomorphic to $\mathrm{SU}(3)$. Consequently $\mathcal{O}$ is an open subspace of $\Lambda^3_0\mathbb{R}^{6*}$. If (V, κ) is an arbitrary 6-dimensional symplectic vector space we can fix an isomorphism $\phi\colon V \to \mathbb{R}^6$ satisfying $\phi^*(\kappa_0) = \kappa$ and consider the spaces

$$\Lambda^3_0(V^*) := \phi^*(\Lambda^3_0\mathbb{R}^{6*}), \quad \mathcal{O}(V) := \phi^*(\mathcal{O}).$$

These spaces do not depend form the choice of the symplectomorphism ϕ. The elements of $\Lambda^3_0(V^*)$ are called *κ-effective* 3-forms, while the forms lying in $\mathcal{O}(V)$ are called *κ-positive*.

Let (V, κ) be a 6-dimensional symplectic vector space. Given an effective 3-form Ω let us consider the map $F_\Omega\colon \Lambda^1V^* \to \Lambda^4V^*$ defined by

$$F_\Omega(\alpha) = \Omega \wedge \alpha.$$

Proposition 3.18. *The following facts are equivalent*

1. *Ω is a positive 3-form on V;*

2. *F_Ω is an injective map and κ is negative definite on the image of F_Ω.*

Proof. See [28]. $\square$

Let fix now an arbitrary κ-positive 3-form Ω on V. Since the stabilizer of Ω is isomorphic to SU(3), then it determines a κ-calibrated almost complex structure J_Ω on V. In order to write down an explicit formula for J_Ω

$$P_\Omega \colon \Lambda^1 V^* \to \Lambda^1 V^*$$

defined by

$$P_\Omega \alpha := -\frac{1}{2} \bigstar(\Omega \wedge \bigstar(\Omega \wedge \alpha)) \,.$$

We have

Proposition 3.19. *The endomorphism P_Ω satisfies*

1. $P_\Omega^2 = det(P_\Omega)^{-\frac{1}{6}}\, I$,

2. $\kappa(P_\Omega \alpha,\, \beta) = -\kappa(\alpha,\, P_\Omega \beta)$.

Proof. 1. First we observe that P_Ω is a SU(3)-invariant endomorphism of V^*, since it is built using only Ω and $\bigstar$. Since SU(3) acts irreducibly on V^*, the real version of Schur's lemma assures that $P_\Omega = aI + Jb$, where J is a complex structure on V^* and a, b are real numbers.
Now we claim that P_Ω^2 has a negative eigenvalue. From this claim the conclusion follows. Suppose indeed that there exists $v \neq 0$ such that $P_\Omega^2 v = \lambda\, v$, with $\lambda < 0$. Then

$$2ab\, Jv = (\lambda^2 - a^2 + b^2)\, v \,.$$

If $ab \neq 0$, then J would have a real eigenvalue and this is impossible. On the other hand if $b = 0$ then $P_\Omega^2 = a^2 I$, which is a contradiction with the claim. Hence $P_\Omega = bJ$. To prove the claim we must use an explicit frame $\{e^1, \dots, e^6\}$ of V^* in which κ and Ω takes the standard form and perform the computation e.g. of $P_\Omega^2 e^1$.

2. We have

$$\begin{aligned}
\kappa(P_\Omega \alpha,\, \beta)\frac{\kappa^3}{6} =& -\kappa(\beta,\, P_\Omega \alpha)\frac{\kappa^3}{6} = \frac{1}{2}\beta \wedge \Omega \wedge \bigstar(\Omega \wedge \alpha) = \\
=& -\frac{1}{2}\kappa(\beta \wedge \Omega,\, \alpha \wedge \Omega)\frac{\kappa^3}{6} = -\frac{1}{2}\kappa(\alpha \wedge \Omega,\, \beta \wedge \Omega)\frac{\kappa^3}{6} = \\
=& \kappa(P_\Omega \beta,\, \alpha)\frac{\kappa^3}{6} = -\kappa(\alpha,\, P_\Omega \beta)\frac{\kappa^3}{6} \,.
\end{aligned}$$

□

Therefore if Ω is a κ-positive 3-form, then the endomorphism

$$J_\Omega \colon V \to V$$

dual to $\det(P_\Omega)^{-\frac{1}{6}}\, P_\Omega$ is a complex structure in $\mathcal{C}_\kappa(V)$.

Example 3.20. The form Ω_0 in $\mathbb{R}^6$ induces the standard complex structure and the standard complex volume form on $\mathbb{R}^6$.

An κ-positive 3-form Ω will be said to be *normalized* if $\det P_\Omega = 1$. Now we have

Proposition 3.21. *Let Ω be a normalized κ-positive 3-form on (V, κ) and let J_Ω be the endomorphism dual to P_Ω. Then the form*

$$\varepsilon := \Omega + iJ_\Omega\Omega$$

is of type $(3,0)$ with respect to J_Ω and satisfies

$$\varepsilon \wedge \overline{\varepsilon} = -i\frac{4}{3}\,\kappa^3\,.$$

Vice versa let $J \in \mathcal{C}_\kappa(V)$ and let $\varepsilon \in \Lambda^{3,0}_J V$ such that

$$\varepsilon \wedge \overline{\varepsilon} = -i\frac{4}{3}\,\kappa^3\,,$$

then $\Omega := \mathfrak{Re}\,\varepsilon$ is a normalized κ-positive 3-form on V such that P_Ω is the complex structure dual to J.

It follows that a SU(3)-structure on a 6-dimensional vector space is determined by the following data

- a symplectic structure κ;
- a normalized κ-positive 3-form Ω.

From now on when a SU(3)-structure is given we will denote by (κ, Ω) the structure forms, by J the induced complex structure, by g_J the induced metric, by $*$ the Hodge star operator associated to g_J and by $\bigstar$ be the symplectic star operator of κ. We have the following easy proof

Lemma 3.22. *Let (κ, Ω) be structures forms of a* SU(3)*-structure on V. Then*

1. $\bigstar\kappa = *\kappa = \frac{1}{2}\,\kappa^2$*;*
2. $\Omega \wedge *\Omega = \frac{2}{3}\,\kappa^3$*;*
3. $*\Omega = J\Omega$ *(and consequently $\kappa \wedge J\Omega = 0$).*

Decomposition of the exterior algebra

A SU(3)-structure on a vector space V induces a canonical action on the exterior algebra ΛV^*. Obviously this action is irreducibly on V^* and $\Lambda^5 V^*$, while $\Lambda^2 V^*$ and $\Lambda^3 V^*$ decompose as follows:

$$\Lambda^2 V^* = \Lambda^2_1 V^* \oplus \Lambda^2_6 V^* \oplus \Lambda^2_8 V^* \,,$$
$$\Lambda^3 V^* = \Lambda^3_{\mathfrak{Re}} V^* \oplus \Lambda^3_{\mathfrak{Im}} V^* \oplus \Lambda^3_6 V^* \oplus \Lambda^3_{12} V^* \,,$$

where we set

- $\Lambda^2_1 V^* = \mathbb{R}\,\kappa\,,$
- $\Lambda^2_6 V^* = \{\bigstar(\alpha \wedge \Omega) \,|\, \alpha \in \Lambda^1 V^*\} = \{\phi \in \Lambda^2 V^* \,|\, J\phi = -\phi\}\,,$
- $\Lambda^2_8 V^* = \{\phi \in \Lambda^2 V^* \,|\, \phi \wedge \Omega = 0 \text{ and } \bigstar\phi = -\phi \wedge \kappa\}$
  $= \{\phi \in \Lambda^2 V^* \,|\, J\phi = \phi\,,\, \phi \wedge \kappa^2 = 0\}\,,$

and

- $\Lambda^3_{\mathfrak{Re}} V^* = \mathbb{R}\,\Omega\,,$
- $\Lambda^3_{\mathfrak{Im}} V^* = \mathbb{R}\,J\Omega = \{\gamma \in \Lambda^3 V^* \,|\, \gamma \wedge \kappa = 0\,,\, \gamma \wedge \Omega = c\,\kappa^3,\ c \in \mathbb{R}\}\,,$
- $\Lambda^3_6 V^* = \{\alpha \wedge \kappa \,|\, \alpha \in \Lambda^1 V^*\} = \{\gamma \in \Lambda^3 V^* \,|\, \bigstar\gamma = \gamma\}\,,$
- $\Lambda^3_{12} V^* = \{\gamma \in \Lambda^3 V^* \,|\, \gamma \wedge \kappa = 0\,,\, \gamma \wedge \Omega = 0\,,\, \gamma \wedge J\Omega = 0\}\,.$

Remark 3.23. Now we emphasize some relations which will be useful:

1. If $\phi \in \Lambda^2_6 V^* \oplus \Lambda^2_8 V^*$, then $\bigstar\phi = -\phi \wedge \kappa\,.$
2. If $\gamma \in \Lambda^3_{\mathfrak{Re}} V^* \oplus \Lambda^3_{\mathfrak{Im}} V^* \oplus \Lambda^3_{12} V^*$, then $\bigstar\gamma = -\gamma$ and $\gamma \wedge \kappa = 0$.
3. If α is an arbitrary 1-form, then $J(\alpha \wedge \Omega) = -\alpha \wedge \Omega$, consequently from the definition of J it follows
$$J\Omega \wedge \bigstar(\Omega \wedge \alpha) = -2\,\bigstar\alpha\,.$$
4. If $\beta \in \Lambda^2_8 V^*$ then
$$\begin{aligned} *(\beta \wedge \beta) \wedge \kappa^2 &= \beta \wedge \beta \wedge *\kappa^2 = 2\,\beta \wedge \beta \wedge \kappa \\ &= -2\,\beta \wedge \bigstar\beta = -2|\beta|^2\,\frac{\kappa^3}{6}\,, \end{aligned}$$
so that
$$*(\kappa^2 \wedge *(\beta \wedge \beta)) = -2|\beta|^2. \tag{3.13}$$

We can obtain the decomposition of $\Lambda^4 V^*$ using the duality given by the symplectic star operator.

Moreover we define the projections

$$E_1 \colon \Lambda^2 V^* \to \Lambda^2_8 V^* \,,$$
$$E_2 \colon \Lambda^3 V^* \to \Lambda^3_{12} V^*$$

by

$$E_1(\alpha) = \frac{1}{2}(\alpha + J\alpha) - \frac{1}{18} * ((*(\alpha + J\alpha) + (\alpha + J\alpha) \wedge \kappa) \wedge \kappa)\, \kappa \,, \tag{3.14}$$

$$E_2(\beta) = \beta - \frac{1}{2} * (J\beta \wedge \kappa) \wedge \kappa - \frac{1}{4} * (\beta \wedge J\Omega)\, \Omega - \frac{1}{4} * (\Omega \wedge \beta)\, J\Omega \,. \tag{3.15}$$

Note that E_2 commutes with $*$ since the latter is an automorphism of $\Lambda^3_{12} V^*$. The same is true for J (hence also for $\bigstar$).

The ϵ-identities

As done by Bryant in the G_2-case we introduce the following ϵ-notation, which will be useful in the following. Let (κ_0, Ω_0) be the standard SU(3)-structure on $\mathbb{R}^6$; we write

$$\Omega_0 = \frac{1}{6}\epsilon_{ijk} e^{ijk} \,, \quad *\Omega = \frac{1}{6}\overline{\epsilon}_{ijk} e^{ijk} \,, \quad \kappa_0 = \frac{1}{2}\kappa_{ij} e^{ij} \,.$$

We have the following identities, whose proof is straightforward:

$$\begin{aligned}
&\epsilon_{ipq}\kappa_{pq} = 0 \,, \\
&\kappa_{ip}\kappa_{pj} = -\delta_{ij} \,, \\
&\epsilon_{ijp}\kappa_{pr} = \overline{\epsilon}_{ijr} \,, \\
&\overline{\epsilon}_{ijp}\kappa_{pr} = -\epsilon_{ijr} \,, \\
&\overline{\epsilon}_{ipq}\epsilon_{jpq} = -4\kappa_{ij} \,, \\
&\epsilon_{ipq}\epsilon_{jpq} = 4\delta_{ij} = \overline{\epsilon}_{ipq}\overline{\epsilon}_{jpq} \,, \\
&\overline{\epsilon}_{ijp}\epsilon_{klp} = -\kappa_{ik}\delta_{jl} + \kappa_{jk}\delta_{il} + \kappa_{il}\delta_{jk} - \kappa_{jl}\delta_{ik} \,, \\
&\epsilon_{ijp}\epsilon_{klp} = -\kappa_{ik}\kappa_{jl} + \kappa_{il}\kappa_{jk} + \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} = \overline{\epsilon}_{ijk}\overline{\epsilon}_{ipq} \,.
\end{aligned} \tag{3.16}$$

As first application of formulae (3.16) we have the decomposition

$$\mathfrak{so}(6) = \mathfrak{su}(3) \oplus [\mathbb{R}]_1 \oplus [\mathbb{R}^6]_2, \tag{3.17}$$

of $\mathfrak{so}(3)$ in SU(3)-invariant subspaces, where we use the notation

$$([a]_1)_{ij} = a\kappa_{ij} \,, \qquad ([v]_2)_{ij} = \epsilon_{ijp} v_p \,.$$

Decomposition of symmetric 2-tensors

The 21-dimensional space of symmetric 2-tensor on a vector space V equipped by a SU(3)-structure splits into irreducible $\mathfrak{su}(3)$-modules as follows:

$$S^2V^* = \mathbb{R}g_J \oplus S^2_+ \oplus S^2_- \,,$$

where

$$\begin{aligned} S^2_+ &= \{h \in S^2V^* \,:\, Jh = h,\, \mathrm{tr}_g h = 0\}\,,\\ S^2_- &= \{h \in S^2V^* \,:\, Jh = -h\}\,. \end{aligned}$$

The spaces S^2_+ and S^2_- are isomorphic respectively to $\Lambda^2_8V^*$ and $\Lambda^3_{12}V^*$. In the standard case $(\mathbb{R}^6, \kappa_0, \Omega_0)$ the maps

$$\begin{aligned} \iota\colon S^2_+ &\longrightarrow \Lambda^2_8V^*\,,\\ \gamma\colon S^2_- &\longrightarrow \Lambda^3_{12}V^* \end{aligned}$$

given by

$$\begin{aligned} \iota(h_{ij}e^ie^j) &= h_{ip}\kappa_{pj}e^{ij}\,,\\ \gamma(h_{ij}e^ie^j) &= h_{ip}\varepsilon_{pjk}e^{ijk} \end{aligned}$$

defines $\mathfrak{su}(3)$-isomorphisms.

3.2.2 SU(3)-manifolds

First of all we introduce the following

Definition 3.24. Let (M, κ) be a 6-dimensional almost symplectic manifold. A 3-form Ω on M is called *effective* if $\Omega \wedge \kappa = 0$. If further Ω_x is normalized and κ_x-positive for any $x \in M$, then Ω will be called normalized κ-positive.

By proposition 3.21 a SU(3)-structure on a 6-dimensional manifold M is determined by the choice of:

- an almost symplectic structure κ,
- a normalized κ-positive 3-form Ω.

In fact if Ω is a normalized κ-positive it determines a κ-calibrated almost complex structure J on M such that $\varepsilon = \Omega + iJ\Omega$ is of type $(3,0)$.
Furthermore the spaces of r-forms on M split in $\mathfrak{su}(3)$-modules as follows:

$$\begin{aligned} \Lambda^2M &= \Lambda^2_1M \oplus \Lambda^2_6M \oplus \Lambda^2_8M\,,\\ \Lambda^3M &= \Lambda^3_{Re}M \oplus \Lambda^3_{Im}M \oplus \Lambda^3_6M \oplus \Lambda^{12}_3M\,,\\ \Lambda^4M &= \Lambda^4_1M \oplus \Lambda^4_6M \oplus \Lambda^4_8M\,, \end{aligned}$$

where the meaning of symbols is obvious. Consequently the derivatives of the structure forms decompose as

$$
\begin{aligned}
d\kappa &= \nu_0\,\Omega + \alpha_0\,J\Omega + \nu_1\wedge\kappa + \nu_3\,,\\
d\Omega &= \pi_0\,\kappa^2 + \pi_1\wedge\Omega - \pi_2\wedge\kappa\,,\\
dJ\Omega &= \sigma_0\,\kappa^2 + \sigma_1\wedge\Omega - \sigma_2\wedge\kappa\,,
\end{aligned}
\tag{3.18}
$$

where $\nu_0,\alpha_0,\pi_0,\sigma_0 \in C^\infty(M,\mathbb{R})$, $\nu_1,\pi_1,\sigma_1 \in \Lambda^1 M$, $\pi_2,\sigma_2 \in \Lambda^2_8 M$ and $\nu_3 \in \Lambda^3_{12} M$.

3.2.3 A formula for SU(3)-manifolds

The goal of this section is to prove the following

Lemma 3.25. *The formulae*

$$
*(d\kappa)\wedge\Omega - \frac{1}{2}\kappa^2\wedge *dJ\Omega = 0\,, \tag{3.19}
$$

$$
J\Omega\wedge(*dJ\Omega) - (*d\Omega)\wedge\Omega = 0 \tag{3.20}
$$

hold for any SU(3)*-structure* (κ,Ω).

In order to prove this theorem we need recall some basic facts on G_2-structures.

We consider on $\mathbb{R}^7$ the 3-form

$$
\sigma_0 = \Omega_0 + \kappa_0\wedge e^7
$$

where $\mathbb{R}^7 =< e_1,\dots,e_7 >$ and the standard forms κ_0, Ω_0 are computed with respect to $\{e_1,\dots,e_6\}$. The Lie group G_2 is by definition the stabilizer of σ_0 under the standard action of $\mathrm{GL}(7,\mathbb{R})$ on the vector space $\Lambda^3\mathbb{R}^{7*}$ of 3-forms on $\mathbb{R}^7$. This group preserves the standard metric and the standard orientation of $\mathbb{R}^7$. Since the dimension of G_2 is 14, the orbit of σ_0 under the action of $\mathrm{GL}(7,\mathbb{R})$ is an open subspace of $\Lambda^3\mathbb{R}^{7*}$. Let us denote by $\Lambda^3_+\mathbb{R}^{7*}$ this space and we call its elements *definite 3-forms.*

Let V be an arbitrary 7-dimensional vector space and $\phi\colon V\to\mathbb{R}^7$ be an isomorphism. Let $\Lambda^3_+V^* = \phi^*(\Lambda^3_+\mathbb{R}^{7*})$. Since $\Lambda^3_+\mathbb{R}^{7*}$ consists of a single orbit, $\Lambda^3_+V^*$ does not depend from the choice of $\phi\,\in$End$(V,\mathbb{R}^7)$. The forms $\sigma\in\Lambda^3_+V^*$ will be called *definite* 3-forms. If σ is an effective 3-form on V, then it induces a canonical metric g in the following way: one fixes an isomorphism $\phi\colon V\to\mathbb{R}^7$ satisfying $\phi^*(\sigma_0)=\sigma$ and takes $g=\phi^*(g_0)$.

Let consider now a 7-dimensional manifold N. A G_2-structure on N is determined by the choice of a 3-form σ on M such that

$$
\sigma_x \in \Lambda^3_+T^*_xM\,,
$$

for any $x \in M$. A G_2-structure induces a Riemannian metric g_σ on M. Let us denote by $*$ the Hodge star operator of g_σ. In [14] R. Bryant proves the following

Theorem 3.26. *A form σ defining a G_2-structure satisfies the following formula*

$$*_\sigma\sigma \wedge *_\sigma(d *_\sigma \sigma) + (*_\sigma d\sigma) \wedge \sigma = 0\,. \tag{3.21}$$

Now we can prove lemma 3.25.

Proof. Let (M, κ, Ω) be a SU(3)-manifold, $N = M \times \mathbb{R}$ and

$$p_1 \colon N \to M\,,$$
$$p_2 \colon N \to \mathbb{R}$$

be the standard projections. Identifying κ and Ω with their pull-backs by p_1 and dt with its pull-back by p_2, we get that the 3-form

$$\sigma = \Omega + \kappa \wedge dt\,,$$

defines a G_2-structure on N. A computation gives

$$d\sigma = d\Omega + d\kappa \wedge dt\,, \tag{3.22}$$

$$*_\sigma\sigma = (*\Omega) \wedge dt + *\kappa = (J\Omega \wedge dt + \frac{1}{2}\kappa^2)\,, \tag{3.23}$$

$$d*_\sigma\sigma = (dJ\Omega) \wedge dt + (d\kappa) \wedge \kappa\,, \tag{3.24}$$

$$*_\sigma d\sigma = *d\Omega \wedge dt - *d\kappa\,. \tag{3.25}$$

Furthermore we have

$$\begin{aligned} *_\sigma\sigma \wedge *_\sigma(d\sigma) + (*_\sigma d\sigma) \wedge \sigma =& J\Omega \wedge (*dJ\Omega) \wedge dt + \frac{1}{2}\kappa^2 \wedge *(d\kappa \wedge \kappa) \wedge dt \\ &+ \frac{1}{2}\kappa^2 \wedge *dJ\Omega - (*d\Omega) \wedge \Omega \wedge dt \\ &- *(d\kappa) \wedge \Omega - *(d\kappa) \wedge \kappa \wedge dt\,. \end{aligned}$$

Therefore equation (3.21) implies

- $*(d\kappa) \wedge \Omega = \frac{1}{2}\kappa^2 \wedge *dJ\Omega$,
- $J\Omega \wedge (*dJ\Omega) + \frac{1}{2}\kappa^2 \wedge *(d\kappa \wedge \kappa) - (*d\Omega) \wedge \Omega - *(d\kappa) \wedge \kappa = 0\,.$

Equation (3.19) is proved. In order to show that equation (3.20) holds we need to prove the following identity

$$\frac{1}{2}\kappa^2 \wedge *(d\kappa \wedge \kappa) = *(d\kappa) \wedge \kappa\,. \tag{3.26}$$

The decomposition of 3-forms on M implies

$$\frac{1}{2}\kappa^2 \wedge *(d\kappa \wedge \kappa) = \frac{1}{2}\kappa^2 \wedge *(\nu_1 \wedge \kappa^2) = (\bigstar\kappa) \wedge *(\nu_1 \wedge \kappa^2) \tag{3.27}$$

and

$$*(d\kappa) \wedge \kappa = *(\nu_1 \wedge \kappa) \wedge \kappa\,, \tag{3.28}$$

where $\nu_1 \wedge \kappa \in \Lambda^3_6 M = \{\gamma \in \Lambda^3 M \,|\, \bigstar\gamma = \gamma\}$. Now we apply equation (2.6) taking $\zeta = *(\nu_1 \wedge \kappa^2)$ and $\gamma = 1 \in \Lambda^0 M$. We have

$$(\bigstar\kappa) \wedge *(\nu_1 \wedge \kappa^2) = \bigstar(*(\nu_1 \wedge \kappa^2)) = *J(*(\nu_1 \wedge \kappa^2)) = -J\nu_1 \wedge \kappa^2\,. \tag{3.29}$$

Moreover, since $\nu_1 \in \Lambda^3_6 M$, we have

$$*(\nu_1 \wedge \kappa) \wedge \kappa = -J\nu_1 \wedge \kappa^2\,. \tag{3.30}$$

Equation (3.29) together with equation (3.30) imply (3.26), so that equation (3.20) is proved. □

3.2.4 Torsion forms

Let (M, κ, Ω) be a SU(3)-manifold. Then, with notation of section 3.2.2, we have the following

Proposition 3.27. *These relations hold:*

1. $\pi_0 = \frac{2}{3}\alpha_0\,,$
2. $\sigma_0 = -\frac{2}{3}\nu_0\,,$
3. $\sigma_1 = J\pi_1\,.$

Proof. 1. The relation $\Omega \wedge \kappa = 0$ implies

$$\begin{aligned} 0 = d(\Omega \wedge \kappa) &= d\Omega \wedge \kappa - \Omega \wedge d\kappa \\ &= \pi_0\kappa^3 - \pi_2 \wedge \kappa - \alpha_0\Omega \wedge J\Omega - \Omega \wedge \nu_3 \\ &= (\pi_0 - \frac{2}{3}\alpha_0)\kappa^3 \end{aligned}$$

and the claim follows.
2. Analogous to 1 starting from $\kappa \wedge J\Omega = 0$.

3. This formula is a consequence of the formula (3.20) together with the definition of J; in fact we have

$$\begin{aligned}
0 &= (*d\Omega) \wedge \Omega + J\Omega \wedge *dJ\Omega \\
&= *(\pi_1 \wedge \Omega) \wedge \Omega + J\Omega \wedge *(\sigma_1 \wedge \Omega) \\
&= -J(\bigstar(\pi_1 \wedge J\Omega) \wedge \Omega) - J(\Omega \wedge \bigstar(\sigma_1 \wedge \Omega)) \\
&= J(-2J\bigstar\pi_1) - J(2J\bigstar\sigma_1) \\
&= -2J\bigstar\pi_1 + 2\bigstar\sigma_1\,.
\end{aligned}$$

□

Definition 3.28. The forms $\{\pi_0, \sigma_0, \pi_1, \nu_1, , \sigma_2, \nu_3\}$ are called the *torsion forms* of the SU(3)-structure.

We immediately get that a SU(3)-structure is integrable if and only if all of the torsion forms vanish identically.

Now we characterizes two special SU(3)-structures in terms of torsion forms

- **Half-flat structures:** We recall that a SU(3)-structure (κ, Ω) is said to be *half-flat* if the pair (κ, Ω) satisfies

$$d\kappa \wedge \kappa = 0\,, \quad d\Omega = 0\,.$$

This definition has been introduced by Chiossi-Salamon in [18]. A hypersurface of a G_2-manifold inherits a Half-flat structure in a natural way. Furthermore has been proved that any analytic half-flat manifold can be realized has a hypersurface of a G_2-manifold (but Bryant showed that this fact is not always true if the Half-flat manifold is not analytic!).

Let (κ, Ω) be a half-flat structure. By the hypothesis $d\Omega = 0$ we get

$$\pi_i = 0\,, \quad i = 0, 1, 2\,,$$

so that

$$d\kappa = -\frac{3}{2}\sigma_0\,\Omega \wedge \kappa + \nu_1 \wedge \kappa + \nu_3\,.$$

On other hand the hypothesis $d\kappa \wedge \kappa = 0$ implies

$$0 = d\kappa \wedge \kappa = \nu_1 \wedge \kappa^2$$

which forces ν_1 to vanish.

- **Six-dimensional GCY structures:** Let $(M, \kappa, J, \varepsilon)$ be a 6-dimensional GCY manifold. The equation $d\kappa = 0$ implies

$$\pi_0 = \sigma_0 = 0\,, \quad \nu_1 = 0\,, \quad \nu_3 = 0\,.$$

Therefore $d\Omega$ and $dJ\Omega$ reduce to

$$\begin{aligned} d\Omega &= \pi_1 \wedge \Omega - \pi_2 \wedge \kappa\,, \\ dJ\Omega &= J\pi_1 \wedge \Omega - \sigma_2 \wedge \kappa\,. \end{aligned}$$

Since the complex volume form ε associated to (κ, Ω) is of type (3,0), $\overline{\partial}_J \varepsilon$ is the $(3,1)$-part (hence the J anti-invariant part) of $d\varepsilon$. Thus we have

$$\overline{\partial}_J \varepsilon = \frac{1}{2}(d\varepsilon - Jd\varepsilon)$$

and

$$\begin{aligned} \overline{\partial}_J \varepsilon =& \frac{1}{2}(d\varepsilon - Jd\varepsilon) \\ =& \frac{1}{2}(d\Omega + idJ\Omega - Jd\Omega - iJdJ\Omega) \\ =& \frac{1}{2}\{d\Omega - Jd\Omega + i(dJ\Omega - JdJ\Omega)\} \\ =& \frac{1}{2}\{\pi_1 \wedge \Omega - J(\pi_1 \wedge \Omega) + i(J\pi_1 \wedge \Omega - J(J\pi_1 \wedge \Omega))\} \\ =& \pi_1 \wedge \Omega + i\, J\pi_1 \wedge \Omega. \end{aligned}$$

Hence, by proposition 3.18, equation $\overline{\partial}_J \varepsilon = 0$ is equivalent to $\pi_1 = 0$. It follows that 6-dimensional GCY structures can be defined as SU(3)-structures satisfying

$$\pi_0 = \sigma_0 = 0\,, \quad \nu_1 = \pi_1 = 0\,, \quad \nu_3 = 0\,.$$

3.2.5 The Ricci tensor of a SU(3)-manifold in terms of torsion forms

Fix a SU(3)-reduction $\mathcal{Q}$ of the linear frame bundle $\mathcal{L}(M)$, given by the pair (κ, Ω). Then $\mathcal{Q}$ is a subbundle of the principal SO(6)-bundle $p \colon \mathcal{F} \to M$ of the normal frames of the metric g associated to the pair (κ, Ω). Consider on the bundle $\mathcal{F}$ the tautological $\mathbb{R}^6$-valued 1-form ω defined by $\omega[u](v) = u(p_*[u]v)$ for every $u \in \mathcal{F}$ and $v \in T_u\mathcal{F}$. On $\mathcal{F}$ we have also the Levi-Civita connection 1-form ψ taking values in $\mathfrak{so}(6)$. Using the canonical basis $\{e_1, \ldots, e_6\}$ of $\mathbb{R}^6$ we will regard ω as a vector of $\mathbb{R}$-valued 1-forms on $\mathcal{F}$

$$\omega = \omega_1 e_1 + \cdots + \omega_6 e_6$$

and ψ as a skew-symmetric matrix of 1-forms, i.e. $\psi = (\psi_{ij})$. With these notations the first structure equation relating ω and ψ

$$d\omega = -\psi \wedge \omega \,, \tag{3.31}$$

becomes $d\omega_i = -\psi_{ij} \wedge \omega_j$. Note that equation (3.31) simply means that ψ is torsion-free.
The curvature of ψ is by definition the $\mathfrak{so}(6)$-valued 2-form $\Psi = d\psi + \psi \wedge \psi$. In index notation

$$\Psi_{ij} = d\psi_{ij} + \psi_{ik} \wedge \psi_{kj} = \frac{1}{2} R_{ijkl}\, \omega_k \wedge \omega_l \,.$$

We consider the pull-backs of ψ and ω to $\mathcal{Q}$ and denote them by the same symbols for the sake of brevity. The intrinsic torsion of the SU(3)-structure measures the failing of ψ to take values in $\mathfrak{su}(3)$. More precisely, according to the splitting $\mathfrak{so}(6) = \mathfrak{su}(3) \oplus [\mathbb{R}]_1 \oplus [\mathbb{R}^6]_2$, we decompose ψ as follows

$$\psi = \theta + [\mu]_1 + [\tau]_2 \,.$$

Thus θ is a connection 1-form on $\mathcal{Q}$ which in general is not torsion-free.
As before we shall regard τ as a vector of 1-forms $\tau = \tau_i e_i$. Furthermore we can write

$$\tau_i = T_{ij}\, \omega_j \quad \text{and} \quad \mu = M_i\, \omega_i \,, \tag{3.32}$$

where T_{ij} and M_i are smooth functions. The fact that ψ is torsion-free implies

$$d\omega_i = -\theta_{ij} \wedge \omega_j - \epsilon_{ijk}\, \tau_k \wedge \omega_j - \kappa_{ij}\, \mu \wedge \omega_j \,. \tag{3.33}$$

We have the following

Lemma 3.29. *These identities hold:*

1. $\theta \wedge [\mu]_1 + [\mu]_1 \wedge \theta = 0$;
2. $[\tau]_2 \wedge [\mu]_1 - [\mu]_1 \wedge [\tau]_2 = 0$;
3. $\theta \wedge [\tau]_2 + [\tau]_2 \wedge \theta = [\theta \wedge \tau]_2$;
4. $[\tau]_2 \wedge [\mu]_1 + [[\mu]_1 \wedge \tau]_2 = 0$.

Proof. The proof is a straightforward application of ϵ-identities. To see how things work, we prove the first one. Since θ takes values in $\mathfrak{su}(3)$ we have

$$\epsilon_{pkl}\, \theta_{kl} = \epsilon_{klp}\, \theta_{kl} = 0 \,.$$

So

$$\bar{\epsilon}_{ijp}\epsilon_{klp}\,\theta_{kl} = 0$$

for every $i, j = 1, \ldots, 6$. Then applying the ϵ-identities we get

$$\begin{aligned} 0 =& \bar{\epsilon}_{ijp}\epsilon_{klp}\,\theta_{kl} \\ =& (-\kappa_{ik}\delta_{jl} + \kappa_{jk}\delta_{il} + \kappa_{il}\delta_{jk} - \kappa_{jl}\delta_{ik})\,\theta_{kl} \\ =& 2\kappa_{jk}\,\theta_{ki} - 2\kappa_{ik}\,\theta_{kj}\,, \end{aligned}$$

i.e.

$$\kappa_{jk}\,\theta_{ki} = \kappa_{ik}\,\theta_{kj}\,.$$

Consequently

$$\theta_{ik} \wedge \kappa_{kj}\,\mu + \kappa_{ik}\,\mu \wedge \theta_{kj} = 0\,,$$

i.e.

$$\theta \wedge [\mu]_1 + [\mu]_1 \wedge \theta = 0\,.$$

□

Now we can introduce the following quantities

$$D\theta = d\theta + \theta \wedge \theta + [\tau]_2 \wedge [\tau]_2 - \frac{2}{3}[\kappa_{ij}\,\tau_i \wedge \tau_j]_1\,, \tag{3.34}$$

$$D\tau = d\tau + \theta \wedge \tau - 2\,[\mu]_1 \wedge \tau\,, \tag{3.35}$$

$$D\mu = d\mu + \frac{2}{3}\kappa_{ij}\,\tau_i \wedge \tau_j\,. \tag{3.36}$$

With this definition $D\theta$ takes values in $\mathfrak{su}(3)$. Moreover by lemma 3.29 we get

$$\begin{aligned} \Psi =& d(\theta + [\tau]_2 + [\mu]_1) + (\theta + [\tau]_2 + [\mu]_1) \wedge (\theta + [\tau]_2 + [\mu]_1) \\ =& D\theta + [D\tau]_2 + [D\mu]_1\,. \end{aligned}$$

Using the ω-frame we shall write

$$D\theta_{ij} = \frac{1}{2}S_{ijkl}\,\omega_k \wedge \omega_l\,, \tag{3.37}$$

$$D\tau_i = \frac{1}{2}T_{ijk}\,\omega_j \wedge \omega_k\,, \tag{3.38}$$

$$D\mu = \frac{1}{2}N_{kl}\,\omega_k \wedge \omega_l\,. \tag{3.39}$$

By the definition of the curvature form we have

$$R_{ijkl} = S_{ijkl} + \epsilon_{ijp}T_{pkl} + \kappa_{ij}N_{kl}\,.$$

In this notation the first Bianchi identity

$$\Psi \wedge \omega = 0\,,$$

has the indicial expression

$$\begin{aligned} &S_{ijkl} + S_{iljk} + S_{iklj} + \\ &+ \epsilon_{ijp} T_{pkl} + \epsilon_{ilp} T_{pjk} + \epsilon_{ikp} T_{plj} + \kappa_{ij} N_{kl} + \kappa_{il} N_{jk} + \kappa_{ik} N_{lj} = 0 \end{aligned} \tag{3.40}$$

Let $Ric_{ij} = R_{ikkj}$ and $s = Ric_{kk}$ be respectively the Ricci tensor and the scalar curvature of (M, g). Starting from equation (3.40) we can derive the following

Theorem 3.30. *In the previous notation we have*

$$Ric_{ij} = 2\epsilon_{ipq} T_{pqj} - 3\kappa_{ip} N_{pj} \,,$$

$$s = 2\epsilon_{kpq} T_{pqk} - 3\kappa_{kp} N_{pk} \,.$$

Denote by π the projection $\pi\colon \mathcal{Q} \to M$. In terms of the ω-frame the pull-backs of the structure forms take their standard expression, i.e.

$$\begin{aligned} \pi^*(\Omega) &= \frac{1}{6}\epsilon_{ijk}\, \omega_i \wedge \omega_j \wedge \omega_k \,, \\ \pi^*(J\Omega) &= \frac{1}{6}\bar{\epsilon}_{ijk}\, \omega_i \wedge \omega_j \wedge \omega_k \,, \\ \pi^*(\kappa) &= \frac{1}{2}\kappa_{ij}\, \omega_i \wedge \omega_j \,. \end{aligned}$$

Taking into account formula (3.33) and ϵ-identities, we immediately get

Proposition 3.31. *The derivatives of the structure forms are*

$$\begin{aligned} d\pi^*(\Omega) &= \frac{1}{2}(-\kappa_{ja}\kappa_{kb} + \kappa_{jb}\kappa_{ka})\, \tau_b \wedge \omega_a \wedge \omega_j \wedge \omega_k - 3\,\mu \wedge \pi^*(J\Omega) \,, \\ d\pi^*(J\Omega) &= (\tau_j \wedge \omega_j) \wedge \pi^*(\kappa) - 3\,\mu \wedge \pi^*(\Omega) \,, \\ d\pi^*(\kappa) &= \bar{\epsilon}_{lrj}\, \tau_l \wedge \omega_r \wedge \omega_j \,. \end{aligned}$$

A direct computation gives the following formulae

$$\begin{aligned} \pi^*(\pi_0) &= \frac{2}{3} T_{ii} \,, \\ \pi^*(\pi_1) &= \epsilon_{ijk} T_{ij}\, \omega_k + 3\kappa_{ik} M_i\, \omega_k \,, \\ \pi^*(\pi_2) &= \frac{1}{2}\bar{\epsilon}_{sra}\epsilon_{aij} T_{sr}\, \omega_i \wedge \omega_j - 2\kappa_{ia} T_{aj}\, \omega_i \wedge \omega_j + \frac{2}{3} T_{ii}\, \pi^*(\kappa) \,, \\ \pi^*(\sigma_0) &= \frac{2}{3}\kappa_{ij}\, T_{ij} \,, \\ \pi^*(\sigma_2) &= \frac{1}{2}\epsilon_{rsa}\epsilon_{aij} T_{rs}\, \omega_i \wedge \omega_j - 2T_{ij}\omega_i \wedge \omega_j + \frac{2}{3}\kappa_{ij} T_{ij}\, \pi^*(\kappa) \,, \\ \pi^*(\nu_1) &= \epsilon_{ijk} T_{ij}\, \omega_k \,, \\ \pi^*(\nu_3) &= \bar{\epsilon}_{aij} T_{ak}\, \omega_i \wedge \omega_j \wedge \omega_k + \frac{1}{6}\kappa_{ab} T_{ab}\epsilon_{ijk}\, \omega_i \wedge \omega_j \wedge \omega_k \\ &\quad - \frac{1}{6} T_{aa}\bar{\epsilon}_{ijk}\, \omega_i \wedge \omega_j \wedge \omega_k - \frac{1}{2} T_{ab}\epsilon_{abi}\kappa_{jk}\, \omega_i \wedge \omega_j \wedge \omega_k \,. \end{aligned}$$

Warning: From now on we identify the torsion forms with their pull-backs to the principal SU(3)-bundle $\mathcal{Q}$.

Combining the previous formulae and (3.33) we are able to prove the following

Theorem 3.32. *In terms of torsion forms the scalar curvature of the metric induced by the* SU(3)*-structure is expressed as*

$$s = \frac{15}{2}\pi_0^2 + \frac{15}{2}\sigma_0^2 + 2d^*\pi_1 + 2d^*\nu_1 - |\nu_1|^2 - \frac{1}{2}|\sigma_2|^2 - \frac{1}{2}|\pi_2|^2 - \frac{1}{2}|\nu_3|^2 + 4\langle \pi_1, \nu_1 \rangle \,. \tag{3.41}$$

Proof. First of all we introduce the 1-forms $S_{ijk}\,\omega_k$, $V_{ik}\,\omega_k$, defined by the relations

$$\begin{aligned} dT_{ij} &= T_{ik}\,\theta_{kj} + T_{kj}\,\theta_{ki} + S_{ijk}\,\omega_k \,, \\ dM_i &= M_k\,\theta_{ki} + V_{ik}\,\omega_k \,. \end{aligned}$$

Using equations (3.35) and (3.36) and the definition of T_{ij}, M_i given in (3.32) we have

$$\begin{aligned} D\tau_i =& dT_{ij} \wedge \omega_j + T_{ij}\,d\omega_j - 2\,\kappa_{ij}\,\mu \wedge \tau_j \\ =& (S_{iba} - T_{ij}T_{qa}\epsilon_{jbq} - T_{ij}\kappa_{jb}M_a - 2\kappa_{ij}M_aT_{jb})\,\omega_a \wedge \omega_b \,, \end{aligned}$$

and

$$\begin{aligned} D\mu =& dM_r \wedge \omega_r + M_r d\omega_r + \frac{2}{3}\kappa_{ij}\,\tau_i \wedge \tau_j \\ =& (V_{ba} - M_r\epsilon_{rbq}T_{qa} - M_r\kappa_{rb}M_a + \frac{2}{3}\kappa_{ij}T_{ia}T_{jb})\,\omega_a \wedge \omega_b \,. \end{aligned}$$

Therefore, taking into account (3.38), (3.39), we obtain

$$\begin{aligned} T_{iab} &= 2(S_{iba} - T_{ij}T_{qa}\epsilon_{jbq} - T_{ij}\kappa_{jb}M_a - 2\kappa_{ij}M_aT_{jb}) \,, \\ N_{ab} &= 2(V_{ba} - M_r\epsilon_{rbq}T_{qa} - M_r\kappa_{rb}M_a + \frac{2}{3}\kappa_{ij}T_{ia}T_{jb}) \,. \end{aligned}$$

It follows that

$$\begin{aligned} \epsilon_{ipq}T_{pqj} &= 2(\epsilon_{ipq}S_{pjq} - \epsilon_{ipq}\epsilon_{rjs}T_{pr}T_{sq} - \epsilon_{ipq}T_{pr}\kappa_{rj}M_q + 2\overline{\epsilon}_{iqr}T_{rj}M_q) \,, \\ \kappa_{ip}N_{pj} &= 2(\kappa_{ip}V_{jp} - \kappa_{ip}\epsilon_{rjq}T_{qp}M_r - \kappa_{ip}\kappa_{rj}M_rM_p + \frac{2}{3}\kappa_{ip}\kappa_{qr}T_{qp}T_{rj}) \end{aligned}$$

and using the ϵ-identities

$$\begin{aligned}\epsilon_{ipq}T_{pqi} =&2(-\epsilon_{ipq}S_{ipq}-\epsilon_{ipq}\epsilon_{ris}T_{pr}T_{sq}-\bar{\epsilon}_{prq}T_{pr}M_q+2\bar{\epsilon}_{qri}T_{ri}M_q)\\ =&2(-\epsilon_{ipq}S_{ipq}-\epsilon_{ipq}\epsilon_{ris}T_{pr}T_{sq}+\bar{\epsilon}_{prq}T_{pr}M_q)\,,\\ \kappa_{ip}N_{pi} =&2(\kappa_{ip}V_{ip}-\kappa_{ip}\epsilon_{riq}T_{qp}M_r-\kappa_{ip}\kappa_{ri}M_rM_p+\frac{2}{3}\kappa_{ip}\kappa_{qr}T_{qp}T_{ri})\\ =&2(\kappa_{ip}V_{ip}+\bar{\epsilon}_{rqp}T_{qp}M_r+\frac{2}{3}\kappa_{ip}\kappa_{qr}T_{qp}T_{ri}+\Sigma_i\,M_i^2)\,.\end{aligned}$$

Then by theorem 3.30 we get

$$\begin{aligned}s =&4(-\epsilon_{ipq}S_{ipq}-\epsilon_{ipq}\epsilon_{ris}T_{pr}T_{sq}+\bar{\epsilon}_{prq}T_{pr}M_q)\\ &-6(\kappa_{ip}V_{ip}+\bar{\epsilon}_{rqp}T_{qp}M_r+\frac{2}{3}\kappa_{ip}\kappa_{qr}T_{qp}T_{ri}+\Sigma_i\,M_i^2)\\ =&-4\epsilon_{ipq}S_{ipq}-4\epsilon_{ipq}\epsilon_{ris}T_{pr}T_{sq}-2\bar{\epsilon}_{prq}T_{pr}M_q\\ &-6\kappa_{ip}V_{ip}-4\kappa_{ip}\kappa_{qr}T_{qp}T_{ri}-6\Sigma_i\,M_i^2\,.\end{aligned}$$

Furthermore a straightforward computation gives the following formulae

$$\begin{aligned}&\pi_0^2=\frac{4}{9}T_{ii}T_{jj}\,,\\ &\sigma_0^2=\frac{4}{9}\kappa_{ij}\kappa_{sr}T_{ij}T_{sr}\,,\\ &|\pi_2|^2=-\frac{4}{3}T_{ii}T_{jj}+4T_{ij}^2-2\epsilon_{sra}\epsilon_{aij}T_{sr}T_{ij}+4\kappa_{ir}\kappa_{js}T_{ij}T_{sr}\,,\\ &|\sigma_2|^2=-2\epsilon_{sra}\epsilon_{aij}T_{sr}T_{ij}-\frac{4}{3}\kappa_{ij}\kappa_{ab}T_{ij}T_{ab}-4T_{ij}T_{ji}+4\Sigma_{ij}T_{ij}^2\,,\\ &|\nu_1|^2=\epsilon_{ijk}\epsilon_{kab}T_{ij}T_{ab}\,,\\ &|\nu_3|^2=2T_{ij}^2+2T_{ij}T_{ji}-2\kappa_{jr}\kappa_{is}T_{ij}T_{rs}-2\kappa_{ir}\kappa_{js}T_{ij}T_{rs}\,,\\ &d^*\pi_1=-\epsilon_{sra}\epsilon_{aij}T_{sr}T_{ij}+4\bar{\epsilon}_{ijk}T_{ij}M_k-\epsilon_{sra}S_{sra}-3\kappa_{ij}V_{ij}-3\Sigma_i\,M_i^2\,,\\ &d^*\nu_1=-\epsilon_{sra}\epsilon_{aij}T_{sr}T_{ij}+\bar{\epsilon}_{ijk}T_{ij}M_k-\epsilon_{sra}S_{sra}\,,\\ &\langle\pi_1,\nu_1\rangle=\epsilon_{abk}\epsilon_{kij}T_{ab}T_{ij}-3\bar{\epsilon}_{ijk}T_{ij}M_k\,.\end{aligned}$$

Therefore we get

$$\begin{aligned}&\frac{15}{2}\pi_0^2+\frac{15}{2}\sigma_0^2+2d^*\pi_1+2d^*\nu_1-|\nu_1|^2-\frac{1}{2}|\sigma_2|^2-\frac{1}{2}|\pi_2|^2-\frac{1}{2}|\nu_3|^2+4\langle\pi_1,\nu_1\rangle=\\ =&4T_{ii}T_{jj}+4\kappa_{ij}\kappa_{sr}T_{ij}T_{sr}-5\Sigma_{ij}T_{ij}+\epsilon_{sra}\epsilon_{aij}T_{sr}T_{ij}+T_{ij}T_{ji}-2\bar{\epsilon}_{ijk}T_{ij}M_k\\ &-6\kappa_{ij}V_{ij}-6\Sigma_i\,M_i^2+(-\kappa_{ia}\kappa_{jb}+\kappa_{ib}\kappa_{ja})T_{ij}T_{ba}-4\epsilon_{ijk}S_{ijk}=\\ =&4\epsilon_{ipq}S_{ipq}-4\epsilon_{ipq}\epsilon_{ris}T_{pr}T_{sq}-2\bar{\epsilon}_{prq}T_{pr}M_q-6\kappa_{ip}V_{ip}-4\kappa_{ip}\kappa_{qr}T_{qp}T_{ri}-6\Sigma_i\,M_i^2\,,\end{aligned}$$

i.e.

$$s = \frac{15}{2}\pi_0^2 + \frac{15}{2}\sigma_0^2 + 2d^*\pi_1 + 2d^*\nu_1 - |\nu_1|^2 - \frac{1}{2}|\sigma_2|^2 - \frac{1}{2}|\pi_2|^2 - \frac{1}{2}|\nu_3|^2 + 4\langle\pi_1, \nu_1\rangle$$

and the theorem is proved. □

Here we collect some consequences of formula (3.41) when the SU(3)-structure has special features.

1. **Half-flat structures.** The condition $d\kappa \wedge \kappa = 0$ reads in terms of torsion forms as $\nu_1 = 0$. Thus in the half-flat case the scalar curvature takes the form
$$s = \frac{15}{2}\sigma_0^2 - \frac{1}{2}|\sigma_2|^2 - \frac{1}{2}|\nu_3|^2 \,.$$

2. **GCY structures.** The condition $\overline{\partial}_J \epsilon = 0$ reads as $\pi_1 = 0$, so that, taking into account $d\kappa = 0$,
$$s = -\frac{1}{2}|\sigma_2|^2 - \frac{1}{2}|\pi_2|^2 \,.$$

Corollary 3.33. *The scalar curvature of a* 6*-dimensional generalized Calabi-Yau manifold is everywhere non-positive and it vanishes identically if and only if the* SU(3)*-structure has no torsion.*

Now we write the Ricci curvature $\mathrm{Ric}_{ij} = 2\epsilon_{ipq}T_{pqj} - 3\kappa_{ip}N_{pj}$ in terms of the torsion forms using the operators ι and γ defined in section 3.2.1.

Theorem 3.34. *If M is endowed with the* SU(3)*-structure* (κ, Ω) *with torsion forms given by* (3.18) *, then the traceless part of the Ricci tensor of the induced metric is*

$$\mathrm{Ric}_0 = \iota^{-1}(E_1(\phi_1)) + \gamma^{-1}(E_2(\phi_2))\,, \tag{3.42}$$

where

$$\begin{aligned}\phi_1 = &- *(\nu_1 \wedge J\nu_3) + \frac{1}{4} * (\pi_2 \wedge \pi_2) + \frac{1}{4} * (\sigma_2 \wedge \sigma_2) + \\ &+ dJ\pi_1 + \frac{1}{2}\, d^*\nu_3 + \frac{1}{2}\, d^*(\nu_1 \wedge \kappa) - \frac{1}{4}\, d * (\pi_0\, \Omega) + \frac{1}{4}\, d^*(\sigma_0\, \Omega)\,,\end{aligned}$$

$$\begin{aligned}\phi_2 = &- 2\sigma_0\, \nu_3 - 4\,\sigma_2 \wedge \nu_1 - 2\, Jd\pi_2 - 2 \bigstar d\sigma_2 - 4\, d * (\nu_1 \wedge *\Omega) + \\ &- 2d * (J\pi_1 \wedge \Omega) + 2\pi_0\, J\,\nu_3 - 2\, Jd * (\pi_1 \wedge \Omega) - 4\,\pi_2 \wedge J\pi_1 + \\ &+ 4\,\nu_1 \wedge *(J\pi_1 \wedge \Omega) - 2\, J\nu_1 \wedge *(\nu_1 \wedge \Omega) - \frac{1}{2}Q(\nu_3, \nu_3)\,,\end{aligned}$$

E_1 and E_2 are the maps defined by equations (3.14) *and* (3.15) *and Q is the bilinear form $Q\colon \Lambda^3_{12}M \times \Lambda^3_{12}M \to \Lambda^3 M$ defined by*

$$Q(\alpha,\beta) = \epsilon_{ijl}\iota_{e_j}\iota_{e_i}\,\alpha \wedge \iota_{e_l}\beta\,,$$

where $\{e_1,\dots,e_6\}$ is a unitary frame and ι denotes the contraction of forms.

Remark 3.35. The formulae for the scalar curvature and for the traceless part of the Ricci tensor are justified by representation theory. Both s and Ric_0 must be the linear combination of linear terms in $V_2(\mathfrak{su}(3))$ and quadratic terms in $V_1(\mathfrak{su}(3))$. For the scalar curvature the terms must take values in the $V_{0,0}$ copies of V_1 and V_2, while for the Ricci curvature the terms must take values in Λ^2_8 and Λ^3_{12} copies of V_1 and V_2. (For $S^2_0 = \Lambda^2_8 \oplus \Lambda^3_{12}$). So we have to consider:

$$\begin{aligned} S^2(V_1(\mathfrak{su}(3))) =& 11\, V_{0,0} \oplus 13\, V_{1,0} \oplus 17\, V_{1,1} \oplus 12\, V_{2,0} \oplus \\ & \oplus 3\, V_{3,0} \oplus 4\, V_{2,2} \oplus 9\, V_{2,1} \oplus 2\, V_{3,1}\,. \end{aligned}$$

The 11 copies of $V_{0,0}$ are generated by

- π_0^2, σ_0^2, $\pi_0\sigma_0$;
- $|\pi_1|^2$, $|\nu_1|^2$, $<\pi_1,\nu_1>$ and another bilinear expression in π_1, ν_1 which does not appear in formula (3.41);
- $|\sigma_2|^2$, $|\pi_2|^2$, and a bilinear expression in π_2, σ_2 which does not appear;
- $|\nu_3|^2$.

The 17 copies of $V_{1,1}$ are generated by the projections of

- $\pi_0\pi_2$, $\pi_0\sigma_2$, $\sigma_0\sigma_2, \sigma_0\pi_2$;
- 4 bilinear expressions in π_1 and ν_1 which does not appear in formula (3.42);
- $*\pi_1 \wedge J\nu_3$ and 3 more bilinear expressions in π_1 and ν_3;
- $*(\pi_2 \wedge \pi_2)$, $*(\sigma_2 \wedge \sigma_2)$ and 2 more bilinear expressions in π_2 and σ_2;
- a bilinear form in ν_3.

The 12 copies of $V_{2,0}$ are generated by the projections of

- $\pi_0\nu_3$, $\sigma_0\nu_3$;
- $\nu_1 \wedge *(J\pi_1 \wedge \Omega)$, $J\nu_1 \wedge *(\nu_1 \wedge \Omega)$ and other 2 bilinear expressions in π_1, ν_1;

- $\sigma_2 \wedge \nu_1$, $\pi_2 \wedge \nu_1$, $\sigma_2 \wedge \pi_1$, $\pi_2 \wedge \pi_1$;
- two bilinear expressions in σ_2, ν_3 and π_2, ν_3;
- $Q(\nu_3, \nu_3)$.

An analogous discussion can be done for the second order expressions after considering the splitting:

$$V_2(\mathfrak{su}(3)) = 3\,V_{0,0} \oplus 4\,V_{1,0} \oplus 5\,V_{1,1} \oplus 3\,V_{2,1} \oplus 4\,V_{2,0} \oplus V_{3,0} \oplus V_{2,2}\,.$$

3.2.6 Ricci tensor of generalized Calabi-Yau manifolds

Suppose now that the pair (κ, Ω) gives a generalized Calabi-Yau structure on M. In this case all the torsion is encoded by π_2 and σ_2; in fact $d\Omega$ and $dJ\Omega$ reduce to

$$d\Omega = -\pi_2 \wedge \kappa\,, \quad dJ\Omega = -\sigma_2 \wedge \kappa\,.$$

Therefore we get

$$\begin{aligned} 0 &= d^2\Omega = -d\pi_2 \wedge \kappa\,,\\ 0 &= d^2 J\Omega = -d\sigma_2 \wedge \kappa\,, \end{aligned}$$

i.e. $d\pi_2$ and $d\sigma_2$ are effective 3-forms. Since $\pi_2 \in \Lambda^2_8 M$

$$\begin{aligned} 0 = d(\pi_2 \wedge \Omega) &= d\pi_2 \wedge \Omega + \pi_2 \wedge d\Omega\\ &= d\pi_2 \wedge \Omega - \pi_2 \wedge \pi_2 \wedge \kappa\\ &= d\pi_2 \wedge \Omega + \pi_2 \wedge *\pi_2\\ &= d\pi_2 \wedge \Omega + |\pi_2|^2 * 1\,, \end{aligned}$$

i.e.

$$d\pi_2 \wedge \Omega = -|\pi_2|^2 * 1\,.$$

Analogously we get

$$d\sigma_2 \wedge J\Omega = -|\sigma_2|^2 * 1\,.$$

Now we can express the Ricci tensor of a generalized Calabi-Yau manifold in terms of π_2 and σ_2. In this case equation (3.42) reduces to

$$\mathrm{Ric}_0 = \frac{1}{4}\,\iota^{-1}(E_1(*(\pi_2 \wedge \pi_2 + \sigma_2 \wedge \sigma_2))) - 2\,\gamma^{-1}(E_2(Jd\pi_2 + \bigstar d\sigma_2))\,.$$

Since $d\sigma_2$ is effective, $\bigstar d\sigma_2 = -d\sigma_2$. Thus

$$\mathrm{Ric}_0 = \frac{1}{4}\,\iota^{-1}(E_1(*(\pi_2 \wedge \pi_2 + \sigma_2 \wedge \sigma_2))) - 2\,\gamma^{-1}(E_2(Jd\pi_2 - d\sigma_2))\,.$$

By the definitions of E_1 and E_2, using the J-invariance of π_2 and formula (3.13), we have

$$\begin{aligned} E_1(*(\pi_2\wedge\pi_2)) &= *(\pi_2\wedge\pi_2) - \frac{1}{9}*((\pi_2\wedge\pi_2 + *(\pi_2\wedge\pi_2)\wedge\kappa)\wedge\kappa)\kappa \\ &= *(\pi_2\wedge\pi_2) + \frac{1}{9}|\pi_2|^2\kappa - \frac{1}{9}*(*(\pi_2\wedge\pi_2)\wedge\kappa^2)\kappa \\ &= *(\pi_2\wedge\pi_2) + \frac{1}{9}|\pi_2|^2\kappa + \frac{2}{9}|\pi_2|^2\kappa \\ &= *(\pi_2\wedge\pi_2) + \frac{1}{3}|\pi_2|^2\kappa \end{aligned}$$

and

$$\begin{aligned} E_2(d\pi_2) &= d\pi_2 - \frac{1}{2}*(Jd\pi_2\wedge\kappa)\wedge\kappa - \frac{1}{4}*(d\pi_2\wedge J\Omega)\,\Omega + \frac{1}{4}*(d\pi_2\wedge\Omega)\,J\Omega \\ &= d\pi_2 - \frac{1}{4}*(d\pi_2\wedge J\Omega)\,\Omega - \frac{1}{4}|\pi_2|^2\,J\Omega \\ &= d\pi_2 + \frac{1}{4}*(\pi_2\wedge\sigma_2\wedge\kappa)\,\Omega - \frac{1}{4}|\pi_2|^2\,J\Omega\,, \end{aligned}$$

where in the last step we have used

$$0 = d(\pi_2\wedge J\Omega) = d\pi_2\wedge J\Omega + \pi_2\wedge dJ\Omega = d\pi_2\wedge J\Omega - \pi_2\wedge\sigma_2\wedge\kappa\,.$$

In the same way we get

$$E_1(*(\sigma_2\wedge\sigma_2)) = *(\sigma_2\wedge\sigma_2) + \frac{1}{3}|\sigma_2|^2\kappa$$

and

$$E_2(d\sigma_2) = d\sigma_2 + \frac{1}{4}*(\pi_2\wedge\sigma_2\wedge\kappa)\,J\Omega + \frac{1}{4}|\sigma_2|^2\,\Omega\,.$$

Therefore, taking into account that E_2 commutes with J, the traceless Ricci tensor of a special generalized Calabi-Yau manifold is given by

$$\begin{aligned} \mathrm{Ric}_0 =& \frac{1}{4}\,\iota^{-1}(*(\sigma_2\wedge\sigma_2 + \pi_2\wedge\pi_2) + \frac{1}{3}(|\sigma_2|^2 + |\pi_2|^2)\,\kappa) \\ & - 2\gamma^{-1}(Jd\pi_2 - d\sigma_2 + \frac{1}{4}(|\pi_2|^2 - |\sigma_2|^2)\,\Omega)\,. \end{aligned} \tag{3.43}$$

Formula (3.43) implies that the metric induced by a GCY structure (κ,Ω) is Einstein (*i.e.* $\mathrm{Ric}_0 = 0$) if and only if the torsion forms π_2,σ_2 satisfies

$$\begin{cases} \sigma_2\wedge\sigma_2 + \pi_2\wedge\pi_2 + \frac{1}{6}(|\pi_2|^2 + |\sigma_2|^2)\,\kappa\wedge\kappa = 0 \\ Jd\pi_2 - d\sigma_2 + \frac{1}{4}(|\pi_2|^2 - |\sigma_2|^2)\,\Omega = 0\,. \end{cases} \tag{3.44}$$

We have the following

Corollary 3.36. *Let (M,κ,Ω) be a* GCY *manifold and assume $\pi_2 = 0$ (or $\sigma_2 = 0$), then (M,κ,Ω) is Einstein if and only if it is a genuine Calabi-Yau manifold.*

The GCY manifolds having $\pi_2 = 0$ are called *Special generalized Calabi-Yau manifold* and will be to taken into account in the next chapter.
The proof of Corollary 3.36 relies on the following lemma which is interesting in its own.

Lemma 3.37. *Let (V,κ,Ω) be a 6-dimensional symplectic vector space endowed with a normalized κ-positive 3-form. Let $\alpha \in \Lambda^2_8 V^*$ be a non-zero form, then $\alpha\wedge\alpha$ does no belong to the 1-dimensional* SU(3)*-module generated by $\kappa\wedge\kappa$.*

Proof. The key observation here is that $\Lambda^2_8 V^*$ is isomorphic as a SU(3)-representation to the adjoint representation $V_{1,1}$. Since every element in $\mathfrak{su}(3)$ is Ad(SU(3))-conjugated to an element of a fixed Cartan subalgebra of $\mathfrak{su}(3)$, there exists a SU(3)-basis $\{e^1,\dots,e^6\}$ of V^* such that

$$\alpha = \lambda_1\, e^{12} + \lambda_2\, e^{12} - (\lambda_1+\lambda_2)\, e^{56}\,,$$

for some $\lambda_1,\lambda_2 \in \mathbb{R}$. Now suppose that $\alpha\wedge\alpha = q\,\kappa\wedge\kappa$ for some $q\in\mathbb{R}$. Setting to zero the three components of $\alpha\wedge\alpha - q\,\kappa\wedge\kappa$ gives the equations

$$\begin{aligned} &\lambda_1^2 + \lambda_1\lambda_2 + q = 0\,,\\ &\lambda_2^2 + \lambda_1\lambda_2 + q = 0\,,\\ &\lambda_1\lambda_2 - q = 0\,, \end{aligned}$$

which readily imply $q = 0$. □

Proof of corollary 3.36. By lemma 3.37, since $\pi_2 = 0$, the first equation of (3.44) can be satisfied if and only if $|\sigma_2|^2 = 0$. Therefore the Einstein condition forces (κ,Ω) to be a Calabi-Yau structure on M.
The same argument can be used starting with $\sigma_2 = 0$ instead of $\pi_2 = 0$ □

Remark 3.38. In [31] it has been proven (see theorem 1) that a compact Einstein almost Kähler manifold with vanishing first Chern class is actually a Kähler-Einstein manifold. Note that our result holds with no the compactness assumption.

Chapter 4

Special generalized Calabi-Yau manifolds and deformations of Special Lagrangian submanifolds

Let (M, g) be a Riemannian manifold. An *oriented p-plane* ξ on M is a p-dimensional vector subspace ξ of some tangent space T_xM, equipped with an orientation. If ξ is an oriented p-plane on M, then the orientation of ξ and the restriction of the metric g to it induce a natural volume form $\mathrm{Vol}(\xi)$ on ξ. Let ϕ be a p-form on M, then

$$\phi_{|\xi} = a\,\mathrm{Vol}(\xi)$$

for some $a \in \mathbb{R}$. If $a \leq 1$, then we say that $\phi_{|\xi} \leq \mathrm{Vol}(\xi)$. The following definition was introduced by Harvey and Lawson in [37]

Definition 4.1. A p-form ϕ on M is said to be a *calibration* if

1. $d\phi = 0$,
2. $\phi_{|\xi} \leq \mathrm{Vol}(\xi)$ for any oriented p-plane ξ on M.

The triple (M, g, ϕ) is sometimes called a *calibrated manifold.* Let $i\colon L \hookrightarrow (M, g, \phi)$ be an oriented submanifold, then the pull-back of the metric g to L induces a volume form $\mathrm{Vol}(L)$ on L.

Definition 4.2. If
$$i^*(\phi) = \mathrm{Vol}(L)$$
then $i\colon L \hookrightarrow M$ is said to be a *calibrated submanifold* of (M, g, ϕ).

We have the following

Theorem 4.3 (Harvey-Lawson). *Let (M, g, ϕ) be a calibrated manifold and let $i\colon N \hookrightarrow M$ be a compact calibrated submanifold. Then*
$$\mathrm{V}(L) \leq \mathrm{V}(L'),$$
for any compact submanifold $i'\colon L' \hookrightarrow M$ homologous to L, where $V(L)$ demotes the volume of L.

Let $(M, \kappa, J, \varepsilon)$ be a Calabi-Yau manifold, then the form $\mathfrak{Re}\,\varepsilon$ defines a calibration on (M, g_J). An oriented submanifold $p\colon L \hookrightarrow M$ calibrated by $\mathfrak{Re}\,\varepsilon$ is said to be *special Lagrangian*. We have the following easy proof

Lemma 4.4. *Let $(M, \kappa, J, \varepsilon)$ be a Calabi-Yau manifold and let $p\colon L \hookrightarrow M$ be a submanifold. The following facts are equivalent*

1. *there exists an orientation on L making it calibrated by $\mathfrak{Re}\,\varepsilon$;*

2. *$p^*(\kappa) = 0$ and $p^*(\mathfrak{Im}\,\varepsilon) = 0$.*

In [49] McLean proves that the moduli space $\mathfrak{M}(L)$ of special Lagrangian submanifolds near a fixed compact one L is a smooth manifold of dimension equal to the first Betti number of L. Furthermore Hitchin in [41] proves that $\mathfrak{M}(L)$ has a natural embeddings in $(H^1(L, \mathbb{R}) \times H^{n-1}(L, \mathbb{R}), \kappa_L)$ as Lagrangian submanifold, where κ_L denotes the canonic (linear) symplectic structure on $H^1(L, \mathbb{R}) \times H^{n-1}(L, \mathbb{R})$.

In [45] Peng Lu studies the following problem:

Let $(M, \kappa_t, J_t, \varepsilon_t)$, $t \in (-\delta, \delta)$, be a smooth family of Calabi-Yau manifolds and let $p\colon L \hookrightarrow M$ be a compact special Lagrangian submanifold of $(M, \kappa_0, J_0, \varepsilon_0)$. Is possible to find a family $p_t\colon L \hookrightarrow M$ of special Lagrangian submanifolds of $(M, \kappa_t, J_t, \varepsilon_t)$ such that $p_0 = p$?

Note that in general this problem can not be solved. Indeed, if $p_t\colon L \hookrightarrow (M, \kappa_t, J_t, \mathfrak{Im}\,\varepsilon_t)$ is a smooth family of special Lagrangian submanifolds, then the cohomology classes
$$[p_0^*(\kappa_t)] \in H^2(L, \mathbb{R}), \quad [p_0^*(\mathfrak{Im}\,\varepsilon_t)] \in H^3(L, \mathbb{R})$$

vanishes, since $p_0^*(\kappa_t)$ is homotopic to $p_t^*(\kappa_t)$ and $p_0^*(\Im\mathfrak{m}\,\varepsilon_t)$ is homotopic $p_t^*(\Im\mathfrak{m}\,\varepsilon_t)$.

We have the following

Theorem 4.5 ([45, 56])**.** *The Lu Peng problem can be solved if and only if the special Lagrangian submanifold $p\colon L \hookrightarrow M$ satisfies, for t sufficiently small, the equations*

$$p^*(\kappa_t) = 0\,, \quad p^*(\Im\mathfrak{m}\,\varepsilon_t) = 0\,.$$

In next the section we study the Lu Peng problem in a special class of SU(3)-manifolds.

4.1 Special Generalized Calabi-Yau manifolds

Lemma 4.4 allows to generalize the definition of special Lagrangian submanifold to a non-integrable SU(n)-manifold.

Definition 4.6. Let $(M, \kappa, J, \varepsilon)$ be a SU(n)-manifold and let $p\colon L \hookrightarrow M$ be a submanifold. If

$$p^*(\kappa) = 0\,, \quad p^*(\Im\mathfrak{m}\,\varepsilon) = 0\,,$$

then L is said to be a *special Lagrangian submanifold*.

First of all we note that theorem 3.6 implies that if $(M, \kappa, J, \varepsilon)$ is a GCY manifold and $p\colon L \hookrightarrow M$ is a special Lagrangian submanifold, then the Maslov class of L vanishes. Unfortunately in the GCY case the real part of the complex volume form ε is not necessary closed. Consequently $\Re\mathfrak{e}\,\varepsilon$ does not define a calibration on M and special Lagrangian submanifolds are not calibrated submanifolds. This fact suggest to consider a new class of SU(n)-structures constituted by the GCY structures (κ, J, ε) satisfying

$$d\Re\mathfrak{e}\,\varepsilon = 0\,. \tag{4.1}$$

It turns out that, if $n > 3$ and $(M, \kappa, J, \varepsilon)$ is a $2n$-dimensional GCY manifold with complex volume form satisfying equation (4.1), then it is a genuine Calabi-Yau manifold (see [28]). However in dimension 6 this is not still true and there are a lot of examples of GCY structures satisfying equation (4.1) and defined on manifolds which do not admit Kähler structures (see [28], [29] and the examples described in the next section). Moreover it easy to check that if $(M, \kappa, J, \varepsilon)$ is a 6-dimensional symplectic SU(3)-manifold endowed with a complex volume form satisfying equation (4.1), then it is in particular a GCY manifold. This justifies the following

Definition 4.7. A *special generalized Calabi-Yau manifold*, or shortly a SGCY manifold, is a quadruple (M,κ,J,ε) where

- (M,κ) is a 6-dimensional symplectic manifold;
- J is a κ-calibrated almost complex structure on M;
- $\varepsilon\in\Lambda^{3,0}_J M$ satisfies

$$\begin{cases}\varepsilon\wedge\overline{\varepsilon}=-i\frac{4}{3}\kappa^3\\ d\mathfrak{Re}\,\varepsilon=0\,.\end{cases}$$

We immediately get that a submanifold L of a SGCY manifold is special Lagrangian if and only if admits an orientation making it calibrated by $\mathfrak{Re}\,\varepsilon$.

4.1.1 Examples

In this section we give some examples of SGCY manifolds and special Lagrangian submanifolds.

Example 4.8. Let G be the Lie group of matrices of the form

$$A=\begin{pmatrix} e^t & 0 & xe^t & 0 & 0 & y_1\\ 0 & e^{-t} & 0 & xe^{-t} & 0 & y_2\\ 0 & 0 & e^t & 0 & 0 & w_1\\ 0 & 0 & 0 & e^{-t} & 0 & w_2\\ 0 & 0 & 0 & 0 & 1 & t\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

Let

$$\begin{aligned}&\alpha_1=dt,\quad \alpha_2=dx,\quad \alpha_3=e^{-t}dy_1-xe^{-t}dw_1\\ &\alpha_4=e^{t}dy_2-xe^{t}dw_2,\quad \alpha_5=e^{-t}dw_1,\quad \alpha_6=e^{t}dw_2\,.\end{aligned}\tag{4.2}$$

Then $\{\alpha_1,\dots,\alpha_6\}$ is a basis of left-invariant 1-forms. By (4.2) we easily get

$$\begin{cases}d\alpha_1=d\alpha_2=0\\ d\alpha_3=-\alpha_1\wedge\alpha_3-\alpha_2\wedge\alpha_5\\ d\alpha_4=\alpha_1\wedge\alpha_4-\alpha_2\wedge\alpha_6\\ d\alpha_5=-\alpha_1\wedge\alpha_5\\ d\alpha_6=\alpha_1\wedge\alpha_6\,.\end{cases}\tag{4.3}$$

Let $\{\xi_1, \dots \xi_6\}$ be the dual frame of $\{\alpha_1, \dots, \alpha_6\}$; we have

$$
\begin{aligned}
&\xi_1 = \frac{\partial}{\partial t}\,, \quad \xi_2 = \frac{\partial}{\partial x}\,, \quad \xi_3 = e^t \frac{\partial}{\partial y_1}\,, \quad \xi_4 = e^{-t}\frac{\partial}{\partial y_2}\\
&\xi_5 = e^t \frac{\partial}{\partial w_1} + x e^t \frac{\partial}{\partial y_1}\,, \quad \xi_6 = e^{-t}\frac{\partial}{\partial w_2} + x e^{-t}\frac{\partial}{\partial y_2}\,.
\end{aligned}
\tag{4.4}
$$

From (4.4) we obtain

$$
\begin{aligned}
&[\xi_1, \xi_3] = \xi_3, \quad [\xi_1, \xi_4] = -\xi_4, \quad , [\xi_1, \xi_5] = \xi_5\\
&[\xi_1, \xi_6] = -\xi_6, \quad [\xi_2, \xi_5] = \xi_3, \quad [\xi_2, \xi_6] = \xi_4
\end{aligned}
\tag{4.5}
$$

and the other brackets are zero. Therefore G is a non-nilpotent solvable Lie group. By [33] G has a cocompact lattice Γ. Hence

$$
M = G/\Gamma
$$

is a compact solvmanifold of dimension six. Let us denote with $\pi\colon \mathbb{R}^6 \to M$ the natural projection. Define

$$
\kappa = \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5
$$

and

$$
\begin{aligned}
&J(\xi_1) = \xi_2\,, \qquad J(\xi_3) = \xi_6\,, \qquad J(\xi_4) = \xi_5\\
&J(\xi_2) = -\xi_1\,, \quad J(\xi_6) = -\xi_3\,, \quad J(\xi_5) = -\xi_6\,.
\end{aligned}
$$

Then κ is a symplectic form on M and J is a κ-calibrated almost complex structure on M. Set

$$
\varepsilon = i(\alpha_1 + i\alpha_2) \wedge (\alpha_3 + i\alpha_6) \wedge (\alpha_4 + i\alpha_5)\,;
$$

a direct computation shows that (κ, J, ε) is a special generalized Calabi-Yau structure on M. Let consider now the lattice $\Sigma \subset \mathbb{R}^4$ given by

$$
\Lambda := Span_{\mathbb{Z}} \left\{ \begin{pmatrix} -\mu \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \mu \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\mu \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \mu \end{pmatrix} \right\},
$$

where $\mu = \frac{\sqrt{5}-1}{2}$. Let $\mathbb{T}^4$ be the torus

$$
\mathbb{T}^4 = \mathbb{R}^4/\Lambda\,.
$$

For any $p, q \in \mathbb{Z}$ let $\rho(p,q)$ be the transformation of $\mathbb{T}^4$ represented by the matrix

$$\begin{pmatrix} e^{p\lambda} & 0 & qe^{p\lambda} & 0 \\ 0 & e^{-p\lambda} & 0 & qe^{-p\lambda} \\ 0 & 0 & e^{p\lambda} & 0 \\ 0 & 0 & 0 & e^{-p\lambda} \end{pmatrix},$$

where $\lambda = \log \frac{3+\sqrt{5}}{2}$. Then

$$A(p,q)([y_1, y_2, z_1, z_2], (t,x)) = (\rho(p,q)[y_1, y_2, z_1, z_2], (t+p, x+q))$$

is a transformation of $\mathbb{T}^4 \times \mathbb{R}^2$ for any $p, q \in \mathbb{Z}$. Let Θ be the group of such transformations. The manifold M can be identified with

$$\frac{\mathbb{T}^4 \times \mathbb{R}^2}{\Theta} \tag{4.6}$$

(see [33]).
Let consider now the involutive distribution $\mathcal{D}$ generated by $\{\xi_2, \xi_3, \xi_4\}$ and let $p\colon L \hookrightarrow M$ be the leaf through $\pi(0)$.
By (4.4) and the identification (4.6) we get

$$\pi^{-1}(L) = \{x = (x_1, \dots, x_6) \in \mathbb{R}^6 \ \mid\ x_1 = x_5 = x_6 = 0\};$$

hence L is a compact submanifold of M. By a direct computation one can check that

$$\begin{cases} p^*(\kappa) = 0\,, \\ p^*(\Im\mathfrak{m}\,\varepsilon) = 0\,. \end{cases}$$

Hence L is a special Lagrangian submanifold.

Example 4.9. Let $(x_1, \dots, x_6)$ be coordinates on $\mathbb{R}^6$ and let

$$\kappa_3 = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6\,.$$

Let $a = a(x_1)$, $b = b(x_2)$, $c = c(x_3)$ be three smooth functions such that

$$\lambda_1 := b(x_2) - c(x_3), \quad \lambda_2 := -a(x_1) + c(x_3), \quad \lambda_3 = a(x_1) - b(x_2)$$

are $\mathbb{Z}^6$-periodic. Let us consider the κ_3-calibrated complex structure on $\mathbb{R}^6$ defined by

$$\begin{cases} J(\frac{\partial}{\partial x_r}) & = & e^{-\lambda_r}\frac{\partial}{\partial x_{3+r}} \\ J(\frac{\partial}{\partial x_{3+r}}) & = & -e^{\lambda_r}\frac{\partial}{\partial x_r} \end{cases}$$

$r = 1, 2, 3$. Define a (3,0)-form on $\mathbb{R}^6$ by

$$\varepsilon = i(dx_1 + ie^{\lambda_1} dx_4) \wedge (dx_2 + ie^{\lambda_2} dx_5) \wedge (dx_3 + ie^{\lambda_3} dx_6)\,.$$

Then we get

$$\begin{cases} \varepsilon \wedge \overline{\varepsilon} = -i\frac{4}{3}\kappa_3^3 \\ d\mathfrak{Re}\,\varepsilon = 0\,. \end{cases}$$

Since $\lambda_1, \lambda_2, \lambda_3$ are $\mathbb{Z}^6$-periodic, $(\kappa_3, J, \varepsilon)$ defines a special generalized Calabi-Yau structure on the torus $\mathbb{T}^6 = \mathbb{R}^6/\mathbb{Z}^6$. Now consider the three-torus $L = \pi(X)$, where $\pi\colon \mathbb{R}^6 \to \mathbb{T}^6$ is the natural projection and

$$X = \{(x_1, \dots, x_6) \in \mathbb{R}^6 \;\mid\; x_1 = x_2 = x_3 = 0\}.$$

It is immediate to check that L is a special Lagrangian submanifold of $\mathbb{T}^6$.

Remark 4.10. The previous example shows that, if $(M, \kappa_t, J_t, \varepsilon_t)$ is a family of special generalized Calabi-Yau manifolds with holomorphic initial datum $(M, \kappa_0, J_0, \varepsilon_0)$, then $(M, \kappa_t, J_t, \varepsilon_t)$ is not necessary holomorphic Calabi-Yau, for small t. Indeed, with the notation used above, define

$$\begin{cases} J(\frac{\partial}{\partial x_r}) & = & e^{-t\lambda_r}\frac{\partial}{\partial x_{3+r}} \\ J(\frac{\partial}{\partial x_{3+r}}) & = & -e^{t\lambda_r}\frac{\partial}{\partial x_r}\,, \end{cases}$$

for $r = 1, 2, 3$,

$$\kappa_t = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6$$

and

$$\varepsilon_t = i(dx_1 + ie^{t\lambda_1} dx_4) \wedge (dx_3 + ie^{t\lambda_2} dx_5) \wedge (dx_3 + ie^{t\lambda_3} dx_6)\,.$$

Then $(\mathbb{T}^6, \kappa_t, J_t, \varepsilon_t)$ is a special generalized Calabi-Yau manifold for any $t \in \mathbb{R}$, such that $(\mathbb{T}^6, \kappa_0, J_0, \varepsilon_0)$ is the standard holomorphic Calabi-Yau torus and J_t is non integrable for $t \neq 0$ (here we assume that $\lambda_1, \lambda_2, \lambda_3$ are not constant).

Example 4.11. Let consider now the Lie group G of matrices of the form

$$A = \begin{pmatrix} 1 & 0 & x_1 & u_1 & 0 & 0 \\ 0 & 1 & x_2 & u_2 & 0 & 0 \\ 0 & 0 & 1 & y & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where x_1, x_2, u_1, u_2, y, t are real numbers. Let Γ be the subgroup G formed by the matrices having integral entries. Since Γ is a cocompact lattice of G, then $M := G/\Gamma$ is a six-dimensional nilmanifold.
Let consider

$$\xi_1 = \frac{\partial}{\partial y} + x_1 \frac{\partial}{\partial u_1} + x_2 \frac{\partial}{\partial u_2}\,, \quad \xi_2 = \frac{\partial}{\partial x_2}\,,$$
$$\xi_3 = \frac{\partial}{\partial x_1}\,, \quad \xi_4 = \frac{\partial}{\partial t}\,, \quad \xi_5 = \frac{\partial}{\partial u_1}\,, \quad \xi_6 = \frac{\partial}{\partial u_2}\,.$$

Then $\{\xi_1, \dots, \xi_6\}$ is a G-invariant global frame on M.
The respective coframe $\{\alpha_1, \dots, \alpha_6\}$ satisfies

$$\begin{cases} d\alpha_1 = d\alpha_2 = d\alpha_3 = d\alpha_4 = 0 \\ d\alpha_5 = \alpha_1 \wedge \alpha_3 \\ d\alpha_6 = \alpha_1 \wedge \alpha_2\,. \end{cases} \tag{4.7}$$

The special generalized Calabi-Yau structure on M is given by the symplectic form

$$\kappa = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6\,,$$

by the κ-calibrated almost complex structure

$$\begin{array}{lll} J(\xi_1) = \xi_4\,, & J(\xi_2) = \xi_5\,, & J(\xi_3) = \xi_6\,, \\ J(\xi_4) = -\xi_1\,, & J(\xi_5) = -\xi_2\,, & J(\xi_6) = -\xi_3 \end{array}$$

and by the complex volume form

$$\varepsilon = (\alpha_1 + i\alpha_4) \wedge (\alpha_2 + i\alpha_5) \wedge (\alpha_3 + i\alpha_6)\,.$$

By a direct computation we get

$$\mathfrak{Re}\,\varepsilon = \alpha_{123} - \alpha_{345} + \alpha_{246} - \alpha_{156}\,,$$
$$\mathfrak{Im}\,\varepsilon = \alpha_{234} - \alpha_{135} + \alpha_{126} - \alpha_{456};\,.$$

Let

$$X = \{A \in G \mid y = x_2 = u_2 = 0\}$$

and

$$L = \pi(X)\,,$$

$\pi\colon G \to M$ being the canonical projection. Then L is a special Lagrangian torus embedded in $(M, \kappa, J, \varepsilon)$.

Now we give an example of a compact 6-dimensional complex manifold admitting generalized Calabi-Yau structures, but which can not admit any special generalized Calabi-Yau structure.

Example 4.12. Let

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \; : \; z_1, z_2, z_3 \in \mathbb{C} \right\}$$

be the complex Heisenberg group and let $\Gamma \subset G$ be the subgroup with integral entries. Then $M = G/\Gamma$ is the Iwasawa manifold. It is known that M is symplectic, but it has no Kähler structures (see [34]).
Let $z_r = x_r + ix_{r+3}$, $r = 1, 2, 3$, and set

$$\begin{aligned} &\alpha_1 = dx_1, \quad \alpha_2 = dx_3 - x_1 dx_2 + x_4 dx_5, \quad \alpha_3 = dx_5 \\ &\alpha_4 = dx_4, \quad \alpha_5 = dx_2, \quad \alpha_6 = dx_6 - x_4 dx_2 - x_1 dx_5 \,, \end{aligned}$$

then $\{\alpha_1, \dots, \alpha_6\}$ are G-invariant, so that $\{\alpha_1, \dots, \alpha_6\}$ is a global coframe on M. We immediately get

$$\begin{cases} d\alpha_1 = d\alpha_3 = d\alpha_4 = d\alpha_5 = 0 \\ d\alpha_2 = -\alpha_1 \wedge \alpha_5 - \alpha_3 \wedge \alpha_4 \\ d\alpha_6 = -\alpha_4 \wedge \alpha_5 - \alpha_1 \wedge \alpha_3 \,. \end{cases}$$

Let $\{\xi_1, \dots \xi_6\}$ be the dual frame of $\{\alpha_1, \dots, \alpha_6\}$, then

$$\begin{cases} J(\xi_r) = \xi_{r+3} & r = 1, 2, 3 \,, \\ J(\xi_{3+r}) = -\xi_r & r = 1, 2, 3 \,, \end{cases}$$

defines a complex structure on M calibrated by the symplectic form

$$\kappa = \alpha_{14} + \alpha_{25} + \alpha_{36} \,.$$

Let $\varepsilon = (\alpha_1 + i\alpha_4) \wedge (\alpha_2 + i\alpha_5) \wedge (\alpha_3 + i\alpha_6)$, then a direct computation gives

$$\begin{cases} \varepsilon \wedge \overline{\varepsilon} = -\frac{4}{3} i \kappa^3 \\ \overline{\partial}_J \varepsilon = 0 \,, \end{cases}$$

i.e. (κ, J, ε) is a generalized Calabi-Yau structure on M.
Now we prove that there are no nowhere vanishing (3,0)-forms η on M such that

$$d \mathfrak{Re}\, \eta = 0 \,.$$

In particular (M, J) does not admit any special generalized Calabi-Yau structure. In order to show this let $\eta \in \Lambda^{3,0}_J M$; then there exists $f \in C^\infty(M, \mathbb{C})$ such that

$$\eta = f \varepsilon \,.$$

Let $f = u + iv$ and set

$$du = \sum_{i=1}^{n} u_i \alpha_i\,, \quad dv = \sum_{i}^{n} v_i \alpha_i\,.$$

A direct computation shows that

$$\begin{aligned} d\mathfrak{Re}\,\eta =& (u_6 + v_3)\alpha_{3456} + (v_2 + u_5)\alpha_{2456} + (-u_1 + v_4)\alpha_{1345} + \\ & + (-u_6 - v_3)\alpha_{1236}(-u_3 + v_6)\alpha_{2346} + (-u_5 - v_2)\alpha_{1235} + \\ & + (u_1 - v_4)\alpha_{1246} + (u_3 - v_6)\alpha_{1356} + v\,\alpha_{1245} + \\ & - v\,\alpha_{1346} + (u + u_4 + v_1)\alpha_{1456} + (u - u_4 - v_1)\alpha_{1234} + \\ & + (-u_2 + v_5)\alpha_{2345} + (u_2 - v_5)\alpha_{1256}\,. \end{aligned}$$

Hence $d\mathfrak{Re}\,\eta = 0$ if and only if $u = v = 0$.

4.1.2 The Lu Peng problem in SGCY manifolds

In this section we study the Lu Peng problem in SGCY manifolds.
We have the following

Theorem 4.13. *Let $(M, \kappa_t, J_t, \varepsilon_t)$ be a family of* SGCY *manifolds and let $p_0\colon L \hookrightarrow M$ be a compact special Lagrangian submanifold of $(M, \kappa_0, J_0, \varepsilon_0)$. Assume that :*

- *the cohomology classes*

$$[p_0^*(\kappa_t)] \in H^2(L, \mathbb{R})\,, \quad [(\exp_V)^* \mathfrak{Im}\,\varepsilon_t)] \in H^3(L, \mathbb{R})$$

  vanishes for any t and any vector field V normal to L, where $\exp_V\colon L \to M$ *is the smooth map*

$$\exp_V(x) := \exp_x(V(x));$$

- *for any vector field V normal to L, ε_0 satisfies*

$$p_0^*(\iota_V d\mathfrak{Im}\,\varepsilon_0) = 0\,;$$

then there exists $\delta > 0$ and a family $p_t\colon L \hookrightarrow M$ of special Lagrangian submanifolds of $(M, J_t, \kappa_t, \varepsilon_t)$, for $t \in (-\delta, \delta)$, that extends $p_0\colon L \hookrightarrow M$.

Before the proof we need the following preliminar

Lemma 4.14. *Let (V, κ) be a symplectic vector space and let $p\colon W \hookrightarrow V$ be a Lagrangian subspace. Then*

1. *$\tau\colon V/W \to W^*$ defined as $\tau([v]) = p^*(\iota_v \kappa)$ is an isomorphism;*
2. *let J be a κ-calibrated complex structure on V and let $\varepsilon \in \Lambda^{n,0}_J V^*$ satisfying*
$$p^*(\mathfrak{Im}\,\varepsilon) = 0\,, \quad \varepsilon \wedge \overline{\varepsilon} = c_n \frac{\kappa^n}{n!}\,.$$
Then $\theta\colon V/W \to \Lambda^{n-1}(W^*)$ defined as $\theta([v]) := p^*(\iota_v \mathfrak{Im}\,\varepsilon)$ is an isomorphism. Moreover for any $v \in V$, we have
$$\theta([v]) = - * \tau([v])\,,$$

where $*$ is computed with respect to $p^*(g_J(\cdot,\cdot)) := p^*(\kappa(\cdot, J\cdot))$ and the volume form $\mathrm{Vol}(W) := p^*(\mathfrak{Re}\,\epsilon)$.

Proof. See [49], page 722. □

Now are ready to prove theorem 4.13.

Proof of theorem 4.13. Let
$$F\colon (-\sigma,\sigma) \times C^{(1,\alpha)}(N(L)) \to C^{(0,\alpha)}(\Lambda^2 L) \bigoplus C^{(0,\alpha)}(\Lambda^3 L)$$
be defined by
$$F(t,V) = ((\exp_V)^* \kappa_t, (\exp_V)^* \mathfrak{Im}\,\varepsilon_t)\,.$$
Observe that for any fixed t the form $(\exp_V)^*\kappa_t$ is homotopic to $p_0^*(\kappa_t)$. Therefore our hypothesis imply that
$$F\big((-\sigma,\sigma) \times C^{(1,\alpha)}(N(L))\big) \subset d\big(C^{(1,\alpha)}(\Lambda^1 L\big) \bigoplus d\big(C^{(1,\alpha)}(\Lambda^2 L\big)\,.$$
A direct computation gives that the differential DF of the map F at the point $[(0,0)]$ is
$$DF[(0,0)](0,V) = (p_0^*(d(\iota_V \kappa_0)), p_0^*(d(\iota_V \mathfrak{Im}\,\varepsilon_0) + \iota_V d(\mathfrak{Im}\,\varepsilon_0)))\,.$$
Then, by our assumptions, we get
$$DF[(0,0)](0,V) = (p_0^*(d(\iota_V \kappa_0)), p_0^*(d(\iota_V \mathfrak{Im}\,\varepsilon_0)))\,. \tag{4.8}$$
In view of lemma 4.14, we get
$$-p_0^*(\iota_V \mathfrak{Im}\,\varepsilon_0) = *p_0^*(\iota_V \kappa_0)\,,$$
where $*$ is computed with respect to the metric $p_0^*(g_J)$ and the volume form $p_0^*(\mathfrak{Re}\,\varepsilon)$. Consequently we obtain
$$DF[(0,0)](0,V) = (d(p_0^*(\iota_V \kappa_0)), -d * (p_0^*(\iota_V \kappa_0)))\,.$$

By the Hodge decomposition of $\Lambda^1 L$ it follows that

$$DF[(0,0)]\colon \mathbb{R}\times C^{(1,\alpha)}(N(L))\to d\big(C^{(1,\alpha)}(\Lambda^1 L)\bigoplus d\big(C^{(1,\alpha)}(\Lambda^2 L)\big).$$

is surjective and

$$\ker DF[(0,0)]|_{\{0\}\times C^{(1,\alpha)}(N(L))}\cong \mathcal{H}^1(L)\,, \tag{4.9}$$

where $\mathcal{H}^1(L)$ denotes the space of $C^{(1,\alpha)}$ harmonic 1-forms on L.
Let A be the space of normal vector fields V in $C^{(1,\alpha)}(N(L))$, identified by the isomorphism $\sharp_{\kappa_0}\colon C^{(1,\alpha)}(N(L))\to C^{(1,\alpha)}(\Lambda^1 L)$ to one-forms on L belonging to

$$d(C^{(2,\alpha)}(L))\bigoplus d^*(C^{(2,\alpha)}(\Lambda^2 L)$$

and let

$$\hat{F}=F|_{(-\sigma,\sigma)\times A}\,.$$

Then, (4.9) and the Hodge decomposition theorem imply that

$$D\hat{F}|_{\{0\}\times A}\colon A\to d(C^{(2,\alpha)}L)\bigoplus d^*(C^{(2,\alpha)}(\Lambda^2 L))$$

is an isomorphism.
Therefore we can apply the implicit function theorem to $\hat{F}$ and find, for small t, a solution V of the equation

$$\begin{cases} V(0)=0\\ \hat{F}(t,V(t))=0\,. \end{cases}$$

Taking the derivative of $\hat{F}(t,V(t))=0$ with respect to t, we get that $\dot{V}(t)$ is solution of an elliptic equation. Consequently $V(t)$ is a smooth vector field for any t. Then

$$p_t(x):=\exp_x V(t,x)$$

is a family of special Lagrangian submanifolds of M that extends L. $\square$

4.2 Four-dimensional generalized Calabi-Yau manifolds

In this section we study a generalization of the Calabi-Yau structure to dimension 4.
First of all we consider the following proposition, essentially due to Conti and Salamon (see [22]), which gives a characterization of SU(2)-structures on 4-manifolds in terms of torsion forms

Proposition 4.15 ([22]). SU(2)-*structures on a* 4-*manifold* M *are in one-to-one correspondence with the triple of* 2-*forms* $(\omega_1, \omega_2, \omega_3)$ *satisfying*

$$\omega_i \wedge \omega_j = \delta_{ij}\, v \quad \textit{for } i = 1, 2, 3$$

for some 4-*form* $v \neq 0$, *and*

$$\iota_X \omega_1 = \iota_Y \omega_2 \Longrightarrow \omega_3(X, Y) \geq 0\,.$$

In particular any ω_i is a symplectic structure on M. The triple $(\omega_1, \omega_2, \omega_3)$ induces a triple of almost complex structures an M which we describe in the following

Lemma 4.16. *Let*

$$P_r\colon \Lambda^1 M \to \Lambda^1 M\,,\ \textit{for } r = 1, 2, 3\,,$$

be the $C^\infty(M)$-*linear endomorphisms defined by*

$$\begin{aligned} P_1(\phi) &= \bigstar_1(\omega_3 \wedge \bigstar_1(\omega_2 \wedge \phi))\,,\\ P_2(\phi) &= \bigstar_2(\omega_1 \wedge \bigstar_2(\omega_3 \wedge \phi))\,,\\ P_3(\phi) &= \bigstar_3(\omega_2 \wedge \bigstar_3(\omega_1 \wedge \phi))\,, \end{aligned}$$

where $\bigstar_r$ *is the symplectic star operator induced by* ω_r. *Denote by* J_r *the endomorphism dual to* P_r *with respect to the duality induced by* ω_r. *Then* J_r *is a* ω_r-*calibrated almost complex structure on* M. *Moreover these almost complex structures satisfy the following "quaternionic-like" identities*

$$J_r J_s = -J_s J_r\,, \qquad \textit{for } r, s = 1, 2, 3$$

and

$$J_1 J_2 = J_3\,.$$

Moreover the triple $(\omega_1, \omega_2, \omega_3)$ induces a Riemannian metric g on M by the following formulae

$$g(X, Y) = \omega_1(X, J_1 Y) = \omega_2(X, J_2 Y) = \omega_3(X, J_3 Y)$$

for every $X, Y \in TM$ and a triple of complex volume forms $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ by the realtions

$$\begin{aligned} \varepsilon_1 &= \omega_2 + iJ_1\omega_3\,,\\ \varepsilon_2 &= \omega_1 - iJ_2\omega_3\,,\\ \varepsilon_3 &= \omega_1 + iJ_3\omega_2\,. \end{aligned}$$

In this way one has $\varepsilon_r \in \Lambda^{2,0}_{J_r} M$.

Equivalently an SU(2)-structure on a 4-manifold M is completely determined by a triple (κ, J, ε), where

- κ is a symplectic form,
- J is a κ-calibrated almost complex structure on M,
- ε is a non-vanishing $(2,0)$-form satisfying

$$\varepsilon \wedge \overline{\varepsilon} = 2\,\kappa^2\,,$$

in fact in this case one takes

$$\omega_1 = \kappa\,, \quad \omega_2 = \mathfrak{Re}\,\varepsilon\,, \quad \omega_3 = \mathfrak{Im}\,\varepsilon\,.$$

Now we consider the symplectic case:
Let (M,κ) be a four-dimensional symplectic manifold and let J be a κ-calibrated almost complex structure on M. Let ε be a nowhere vanishing $(2,0)$-form on M satisfying

$$\varepsilon \wedge \overline{\varepsilon} = 2\,\kappa^2\,.$$

Then conditions

$$\begin{cases} \widetilde{\nabla}\varepsilon = 0 \\ d\mathfrak{Re}\,\varepsilon = 0 \end{cases}$$

imply

$$d\mathfrak{Im}\,\varepsilon = 0\,.$$

Indeed, if $\widetilde{\nabla}\varepsilon = 0$, then $\overline{\partial}_J\varepsilon = 0$ and

$$d\mathfrak{Re}\,\varepsilon = 0 \Longrightarrow \overline{\partial}_J\varepsilon + \partial_J\overline{\varepsilon} + \overline{A}_J\varepsilon + A_J\overline{\varepsilon} = \overline{A}_J\varepsilon + A_J\overline{\varepsilon} = 0\,.$$

Since $\overline{A}_J\varepsilon \in \Lambda_J^{1,2}M$ and $A_J\overline{\varepsilon} \in \Lambda_J^{2,1}M$, we get $d\varepsilon = 0$ which implies that J is integrable. In dimension four we adopt the following definition

Definition 4.17. Let M be a (compact) four-manifold. A *generalized Calabi-Yau structure* on M is a triple (κ, J, ε), where

- κ is a symplectic form,
- J is a κ-calibrated almost complex structure on M,
- ε is a non-vanishing $(2,0)$-form satisfying

$$\begin{cases} \varepsilon \wedge \overline{\varepsilon} = 2\,\kappa^2 \\ d\mathfrak{Re}\,\varepsilon = 0\,. \end{cases}$$

In an obvious way we have the definition of special Lagrangian submanifold also in the four dimensional case.

Example 4.18. We recall the construction of the Kodaira-Thurston manifold. Let G be the Lie subgroup of $\mathrm{GL}(5,\mathbb{R})$ whose matrices have the following form

$$A = \begin{pmatrix} 1 & x & z & 0 & 0 \\ 0 & 1 & y & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z, t \in \mathbb{R}$. Let Γ be the subgroup of G of matrices with integers entries. Since Γ is a cocompact lattice in G, we get that

$$M = G/\Gamma$$

is a compact manifold. M is called the *Kodaira-Thurston manifold*.
Let $\{\xi_1, \ldots, \xi_4\}$ be the global frame of M given by

$$\xi_1 = \frac{\partial}{\partial x}\,, \quad \xi_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\,, \quad \xi_3 = \frac{\partial}{\partial z}\,, \quad \xi_4 = \frac{\partial}{\partial t}\,.$$

We easily get

$$[\xi_1, \xi_2] = \xi_3$$

and the other brackets are zero. The dual frame of $\{\xi_1, \ldots, \xi_4\}$ is given by

$$\alpha_1 = dx\,, \quad \alpha_2 = dy\,, \quad \alpha_3 = dz - x dy\,, \quad \alpha_4 = dt\,.$$

We have

$$d\alpha_1 = d\alpha_2 = d\alpha_4 = 0\,, \quad d\alpha_3 = -\alpha_1 \wedge \alpha_2\,.$$

The generalized Calabi-Yau structure on M is given by the forms

$$\begin{cases} \kappa = \alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_4 \\ \varepsilon = i(\alpha_1 + i\alpha_3) \wedge (\alpha_2 + i\alpha_4) \end{cases}$$

and by the almost complex structure

$$\begin{aligned} J(\xi_1) &= \xi_3\,, & J(\xi_2) &= \xi_4\,, \\ J(\xi_3) &= -\xi_1\,, & J(\xi_4) &= -\xi_2\,. \end{aligned}$$

We immediately get

$$\begin{aligned} \mathfrak{Im}\,\varepsilon &= \alpha_1 \wedge \alpha_2 - \alpha_3 \wedge \alpha_4\,, \\ \mathfrak{Re}\,\varepsilon &= \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_4\,. \end{aligned}$$

Hence

$$d\kappa = 0\,,$$
$$d\mathfrak{Re}\,\varepsilon = 0\,.$$

Let $X \subset G$ be the set

$$X = \{A \in G \mid x = t = 0\}$$

and

$$L = \pi(X),$$

where $\pi : G \to M$ is the natural projection. Hence L is a compact manifold embedded in M. Moreover the tangent bundle to L is generated by $\{\xi_2, \xi_3\}$; so we get

$$p^*(\kappa) = 0\,,$$
$$p^*(\mathfrak{Im}\,\varepsilon) = 0\,.$$

Hence L is a special Lagrangian torus.

The following lemma gives a topological obstruction for the existence of generalized Calabi-Yau structures on compact 4-manifolds.

Lemma 4.19. *Let M be a 4-dimensional compact manifold admitting a special generalized Calabi-Yau structure, then*

$$\dim(H^2(M, \mathbb{R})) \geq 2\,.$$

Proof. Let (κ, J, ε) be a generalized Calabi-Yau structure on M and let $\omega_2 = \mathfrak{Re}\,\varepsilon$, $\omega_3 = \mathfrak{Im}\,\varepsilon$. First of all we observe that ω_2 is a symplectic form on M and consequently it cannot be exact. Furthermore if $a[\kappa] + b[\omega_2] = 0$ for some $a, b \in \mathbb{R}$, then

$$a\kappa + \omega_2 = d\alpha\,,$$

for some $\alpha \in \Lambda^1 M$ and this last equation together with $\omega_2 \wedge \kappa = 0$ readily implies $b\kappa^2 = d(\alpha \wedge \kappa)$, which forces b to vanish. Hence κ and ω_2 induce $\mathbb{R}$-linear independent classes in $H^2(M, \mathbb{R})$ and $\dim(H^2(M, \mathbb{R})) \geq 2$. □

Chapter 5

SU(n)-structures on contact manifolds

In this chapter we give a generalization of Calabi-Yau structure to the context of contact manifolds. A *contact Calabi-Yau manifold* is a $2n+1$-dimensional manifold M endowed with a contact form α, a $d\alpha$-calibrated complex structure J on the contact distribution $\xi = \ker\alpha$ and a closed basic complex volume form ε. As in the Calabi-Yau case, if a contact Calabi-Yau structure $(M, \alpha, J, \varepsilon)$ is given, the real part of ε is a calibration on M. An oriented submanifold is said to be special Legendrian if it is calibrated by $\mathfrak{Re}\,\varepsilon$. In section 5.2.1 we prove that the moduli space of special Legendrian submanifolds near a fixed compact one is always a smooth 1-dimensional manifold. Hence this case is quite different from the Calabi-Yau one where the dimension of the moduli space of special Lagrangian submanifolds near a compact one depends form the first Betti number of the base point.
In section 5.2.2 we study the Lu Peng problem for special Legendrian submanifolds and in section 5.4 we classify invariant contact Calabi-Yau structures on 5-dimensional nilmanifolds. In the last section we generalize to the codimension r.

5.1 SU(n)-structures on $2n+1$-manifolds

Let M be a $2n{+}1$-dimensional manifold and let $\alpha \in \Lambda^1 M$ be a nowhere vanishing 1-form. Assume that there exists a 2-form κ on M satisfying

$$\alpha \wedge \kappa^n \neq 0$$

and let $\xi = \ker\alpha$. Then the couple (ξ, κ) defines a symplectic vector bundle on M. Let us denote by $\mathcal{C}_\kappa(\xi)$ the space of the complex structures on the vector bundle ξ calibrated by κ. Let $J \in \mathcal{C}_\kappa(\xi)$ and let R_α be the Reeb vector field of the couple (α, κ), i.e. the unique vector field on TM satisfying

$$\alpha(R_\alpha) = 1\,, \quad \iota_{R_\alpha}\kappa = 0\,.$$

We recall the following

Definition 5.1. A complex p-form γ on M is said to be *transverse* if

$$\iota_{R_\alpha}\gamma = 0\,.$$

If further

$$\iota_{R_\alpha}d\gamma = 0$$

γ is said to be *basic.*

Let us denote by $\Lambda^p_0 M$ the set of the tansverse p-form on M and by $\Lambda^p_B M$ the space of basic p-form. Note that d takes basic forms in basic forms. Furthermore one can define the basic cohomology groups $H^r_B(M)$ of (M, α) as the cohomology groups of the complex $(\Lambda_B M, d)$. Extending the complex structure J on TM as zero on R_α, we have

$$J(\Lambda^p_0 M) \subset \Lambda^p_0 M\,.$$

Consequently $\Lambda^p_0 M \otimes \mathbb{C}$ splits as

$$\Lambda^p_0 M \otimes \mathbb{C} = \bigoplus_{r+s=p} \Lambda^{r,s}_J \xi\,.$$

Proposition 5.2. *Let M be a $2n+1$-dimensional manifold. A* SU(n)*-structure on M is determined by the following data*

- *a nowhere vanishing* 1*-form α on M;*
- *a* 2*-form κ satisfying $\alpha \wedge \kappa^n \neq 0$;*
- *a complex structure $J \in \mathcal{C}_\kappa(\xi)$ (where $\xi = \ker\alpha$);*
- *a nowhere vanishing $\varepsilon \in \Lambda^{n,0}_J \xi$ satisfying $\varepsilon \wedge \overline{\varepsilon} = c_n \frac{\kappa^n}{n!}$.*

Since $\mathrm{SU}(n) \subset \mathrm{O}(2n+1, \mathbb{R})$, a SU($n$)-structure on M induces a Riemannian metric g. This metric can be described in terms of $(\alpha, \kappa, J, \varepsilon)$ by

$$g = g_J + \alpha \otimes \alpha\,,$$

where

$$g_J(\cdot,\cdot) := \kappa(\cdot, J\cdot) \tag{5.1}$$

is the metric on ξ induced by the pair (κ, J). Note that a SU(n)-structure is integrable if and only if

$$\nabla\alpha = 0\,,\quad \nabla\kappa = 0\,,\quad \nabla J = 0\,,\quad \nabla\varepsilon = 0\,.$$

Moreover we have that $\kappa \in \Lambda^{1,1}_J\xi$ and consequently

$$\varepsilon \wedge \kappa = 0\,.$$

Any contact SU(n)-structure on M induces a SU$(n+1)$-structure $(\widetilde{\kappa}, \widetilde{J}, \widetilde{\varepsilon})$ on the cone

$$V = M \times \mathbb{R}^+$$

by taking

$$\widetilde{\kappa} = tdt \wedge \alpha + t^2\,\kappa\,,\quad \widetilde{J} = \begin{cases} J \text{ on } \xi \\ \widetilde{J}(\partial_t) = \frac{1}{t}R_\alpha \end{cases}\,,\quad \widetilde{\varepsilon} = t^n\,\varepsilon \wedge (t\alpha + idt)\,.$$

From the definitions of $\widetilde{\kappa}$ and $\widetilde{\varepsilon}$ we immediately get

$$\widetilde{\kappa}^{n+1} = \lambda\,\widetilde{\varepsilon} \wedge \overline{\widetilde{\varepsilon}}\,,$$

where λ is a complex constant. A direct computation gives

$$d\widetilde{\kappa} = 0 \iff \kappa = \frac{1}{2}d\alpha\,.$$

We recall that $\alpha \in \Lambda^1 M$ is a *contact form* if

$$\alpha \wedge (d\alpha)^n \neq 0\,.$$

In this case the distribution $\xi = \ker\alpha$ is said to be a *contact structure*. A SU(n)-structure $(\alpha, \kappa, J, \varepsilon)$ is said to be *contact* if

$$\kappa = \frac{1}{2}d\alpha\,.$$

Note that if $(M, \alpha, J, \varepsilon)$ is a contact SU(n)-manifold, then $\widetilde{\kappa}$ is a symplectic structure on V. We can introduce the following

Definition 5.3. A contact SU(n)-structure (α, J, ε) is said to be *contact Calabi-Yau* if the complex volume form ε is closed. In this case the quadruple $(M, \alpha, J, \varepsilon)$ is said to be a *contact Calabi-Yau manifold.*

Note that if (α, J, ε) is a contact Calabi-Yau structure, then ε is a basic form.

Example 5.4. Consider $\mathbb{R}^{2n+1}$ endowed with Euclidean standard coordinates $\{x_1, \ldots, x_n, y_1, \ldots, y_n, t\}$. Let

$$\alpha_0 = 2dz - 2\sum_{i=1}^{n} y_i \, dx_i \, .$$

be the *standard contact form* on $\mathbb{R}^{2n+1}$ and let $\xi_0 = \ker \alpha_0$. Then ξ_0 is spanned by

$$\{y_1 \, \partial_t + \partial_{x_1}, \, \ldots, \, y_n \, \partial_t + \partial_{x_n}, \, \partial_{y_1}, \, \ldots, \, \partial_{y_n}\} \, .$$

For simplicity, set $V_i = y_i \, \partial_t + \partial_{x_i}$, $W_j = \partial_{y_j}$, $i, j = 1, \ldots, n$ and

$$\begin{cases} J_0(V_r) = W_r \\ J_0(W_r) = -V_r \end{cases} \qquad r = 1, \ldots n \, .$$

Then J_0 defines a complex structure in $\mathcal{C}_{\kappa_0}(\xi_0)$, where $\kappa_0 = \frac{1}{2} d\alpha_0$. Since the space of transverse 1-forms is spanned by $\{dx_1, \, \ldots, \, dx_n, \, dy_1, \ldots, \, dy_n\}$, then the complex valued form

$$\varepsilon_0 := (dx_1 + i dy_1) \wedge \cdots \wedge (dx_n + i dy_n)$$

is of type $(n, 0)$ with respect to J_0 and it satisfies

$$\begin{cases} \varepsilon_0 \wedge \bar{\varepsilon}_0 = c_n \, \kappa_0^n \\ d\varepsilon_0 = 0 \, . \end{cases}$$

Therefore $(\mathbb{R}^{2n+1}, \alpha_0, J_0, \varepsilon_0)$ is a contact Calabi-Yau manifold.

The following will describe a compact contact Calabi-Yau manifold.

Example 5.5. Let

$$H(3) := \left\{ A = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

be the 3-dimensional Heisenberg group and let $M = H(3)/\Gamma$, where Γ denotes the subgroup of $H(3)$ given by the matrices with integral entries. The 1-forms $\alpha_1 = dx$, $\alpha_2 = dy$, $\alpha_3 = x \, dy - dz$ are $H(3)$-invariant and therefore they define a global coframe on M. Then $\alpha = 2\alpha_3$ is a contact form whose contact distribution ξ is spanned by $V = \partial_x$, $W = \partial_y + x \, \partial_z$. Again

$$\begin{cases} J(V) = W \\ J(W) = -V \end{cases}$$

defines a κ-calibrated complex structure on ξ and $\varepsilon = \alpha_1 + i\,\alpha_2$ is a $(1,0)$-form on ξ such that $(M, \alpha_3, J, \varepsilon)$ is a contact Calabi-Yau manifold.

The last example gives an invariant contact Calabi-Yau structure on a nilmanifold. It can be generalized to the dimension $2n+1$ in this way: let $\mathfrak{g}$ be the Lie algebra spanned by $\{X_1, \ldots, X_{2n+1}\}$ with

$$[X_{2k-1}, X_{2k}] = -X_{2n+1}$$

for $k = 1, \ldots, n$ and the other brackets are zero. Then $\mathfrak{g}$ is a $2n+1$-dimensional nilpotent Lie algebra with rational constant structures and, by Malcev theorem, it follows that if G is the simply connected Lie group with Lie algebra $\mathfrak{g}$, then G has a compact quotient. Let $\{\alpha_1, \ldots, \alpha_{2n+1}\}$ be the dual basis of $\{X_1, \ldots, X_{2n+1}\}$. Then we immediately get

$$d\alpha_1 = 0\,, \ldots, \, d\alpha_{2n} = 0\,, \quad d\alpha_{2n+1} = \sum_{k=1}^{n} \alpha_{2k-1} \wedge \alpha_{2k}\,.$$

Hence the contact form

$$\alpha = \frac{1}{2}\,\alpha_3\,,$$

the complex structure on ξ determined by the realtions

$$\begin{cases} J(X_{2k-1}) = X_{2k} \\ J(X_{2k}) = -X_{2k-1} \end{cases}$$

for $k = 1, \ldots, n$ and the complex form

$$\varepsilon = (\alpha_1 + i\,\alpha_2) \wedge \cdots \wedge (\alpha_{2n-1} + i\,\alpha_{2n})$$

define a contact Calabi-Yau structure on any compact nilmanifold associated with $\mathfrak{g}$.

The following proposition gives simple topological obstructions in order that a compact $2n+1$-dimensional manifold M carries a contact Calabi-Yau structure.

Proposition 5.6. *Let M be a $2n+1$-dimensional compact manifold. Assume that M admits a contact Calabi-Yau structure; then*

1. *if n is* even*, then $b_{n+1}(M) \neq 0$;*

2. *if n is* odd*, then*

$$\begin{cases} b_n(M) \geq 2 \\ b_{n+1}(M) \geq 2\,, \end{cases}$$

where $b_j(M)$ denotes the j^{th} Betti number of M.

Proof. Let (α, J, ε) be a contact Calabi-Yau structure on M and let $\xi = \ker\alpha$. Set $\Omega = \Re\mathfrak{e}\,\varepsilon$; then, since $\varepsilon \in \Lambda^{n,0}_J\xi$, we have $\varepsilon = \Omega + i\,J\Omega$. In view of the assumption $d\varepsilon = 0$, we obtain $d\Omega = dJ\Omega = 0$ and since $d\alpha \in \Lambda^{1,1}_J\xi$ it follows that

$$\Omega \wedge d\alpha = J\Omega \wedge d\alpha = 0\,.$$

Hence

$$d(\Omega \wedge \alpha) = d(J\Omega \wedge \alpha) = 0\,.$$

Furthermore we have

$$\varepsilon \wedge \overline{\varepsilon} = \Omega \wedge \Omega + J\Omega \wedge J\Omega \qquad \text{if } n \text{ is even;}$$

$$\varepsilon \wedge \overline{\varepsilon} = -2i\,\Omega \wedge J\Omega \qquad \text{if } n \text{ is odd.}$$

1. If n is even, then $\alpha \wedge (\Omega \wedge \Omega + J\Omega \wedge J\Omega)$ is a volume form on M. Assume that the cohomology classes $[\Omega \wedge \alpha]$, $[J\Omega \wedge \alpha]$ vanish; then there exist $\beta, \gamma \in \Lambda^n M$ such that

$$\alpha \wedge \Omega = d\beta\,, \quad \alpha \wedge J\Omega = d\gamma\,.$$

By Stokes theorem we have

$$\begin{aligned} 0 \neq \int_M \alpha \wedge \Omega \wedge \Omega + \alpha \wedge J\Omega \wedge J\Omega &= \int_M d\beta \wedge \Omega + d\gamma \wedge J\Omega \\ &= \int_M d(\beta \wedge \Omega) + d(\gamma \wedge J\Omega) = 0\,, \end{aligned}$$

which is absurd. Therefore one of $[\Omega \wedge \alpha]$, $[J\Omega \wedge \alpha]$ does not vanish. Consequently $b_{n+1}(M) \neq 0$.

2. Let n be odd. We prove that the cohomology classes $[\Omega]$ and $[J\Omega]$ are $\mathbb{R}$-independent. Assume that there exist $a, b \in \mathbb{R}$ such that $a[\Omega] + b[J\Omega] = 0$, $(a, b) \neq (0, 0)$. Then there exists $\beta \in \Lambda^{n-1}M$ such that

$$a\,\Omega + b\,J\Omega = d\beta\,.$$

We may assume that $a = 1$, so that $\Omega = d\beta - bJ\Omega$. Stokes theorem implies

$$0 \neq \int_M \alpha \wedge \Omega \wedge J\Omega = \int_M \alpha \wedge d\beta \wedge J\Omega = -\int_M d(\alpha \wedge \beta \wedge J\Omega) = 0$$

which is a contradiction. Hence $b_n(M) \geq 2$. With the same argument, it is possible to prove that $b_{n+1}(M) \geq 2$ by showing that $[\Omega \wedge \alpha]$ and $[J\Omega \wedge \alpha]$ are $\mathbb{R}$-independent in $H^{n+1}(M)$.

□

The following is an immediate consequence of proposition 5.6.

Corollary 5.7. *A 3-dimensional compact manifold M admitting contact Calabi-Yau structure has $b_1(M) \geq 2$. In particular, there are no compact 3-dimensional simply connected contact Calabi-Yau manifolds.*
Moreover, the $2n+1$-dimensional sphere has no contact Calabi-Yau structures.

5.1.1 Sasakian manifolds and contact Calabi-Yau manifolds

Let (M, ξ) be a contact manifold. Let α be a 1-form defining ξ, $\kappa = 1/2\, d\alpha$ and $J \in \mathcal{C}_\kappa(\xi)$. The pair (α, ξ) induces an almost Kähler structure $(\widetilde{\kappa}, \widetilde{J})$ on the cone $V = M \times \mathbb{R}^+$, by taking

$$\widetilde{\kappa} = t dt \wedge \alpha + t^2\, \kappa\,, \quad \widetilde{J} = \begin{cases} J \text{ on } \xi \\ \widetilde{J}(\partial_t) = \frac{1}{t} R_\alpha\,. \end{cases}$$

Furthermore, as in the almost complex case, it is defined the Nijenhuis tensor of J by

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X,Y]$$

for $X, Y \in TM$, where J is extended in TM by $J(R_\alpha) = 0$. We recall the following

Theorem 5.8. *The following facts are equivalent*

1. *the almost Kähler structure $(\widetilde{\kappa}, \widetilde{J})$ is integrable;*

2. $N_J = -d\alpha \otimes R_\alpha$.

Now we can recall the following

Definition 5.9. A *Sasakian structure* on M is a pair (α, J), where

- α is a contact form,
- $J \in \mathcal{C}_\kappa(\xi)$ satisfies $N_J = -d\alpha \otimes R_\alpha$, being $\xi = \ker \alpha$.

The triple (M, α, J) is called a *Sasakian manifold*.

It is known that if (M,α,J) is a Sasakian manifold, then R_α is a Killing vector field of the metric $g=g_J+\alpha\otimes\alpha$ (where as usual g_J is the transversal metric induced by (κ,J)) and

$$\nabla_X R_\alpha = JX\,,$$

for any $X\in\xi$, where ∇ denotes the Levi Civita connection of g. Moreover in this case J takes basic forms in basic forms and consequently the basic cohomology groups of (M,α) split as

$$H^p_B(M)=\bigoplus_{r+s=p} H^{r,s}_B(M)\,.$$

One easily get that $d\alpha$ defines a non zero class in $H^{1,1}_B(M)$.
A Sasakian structure (α,J) induces a natural connection ∇^ξ on ξ given by the relations

$$\nabla^\xi_X Y=\begin{cases}(\nabla_X Y)^\xi & \text{if } X\in\xi\\ [R_\alpha,Y] & \text{if } X=R_\alpha\,,\end{cases}$$

where the subscript ξ denotes the projection onto ξ. One easily gets

$$\nabla^\xi_X J=0\,,\quad \nabla^\xi_X g_J=0\,,\quad \nabla^\xi_X d\alpha=0\,,\quad \nabla^\xi_X Y-\nabla^\xi_Y X=[X,Y]^\xi$$

for any $X,Y\in TM$. Consequently we have

$$\mathrm{Hol}(\nabla^\xi)\subseteq \mathrm{U}(n).$$

Moreover it is defined the *transverse Ricci tensor* Ric^T as the Ricci tensor associated to ∇^ξ, i.e.

$$\mathrm{Ric}^T(X,Y)=\sum_{i=1}^{2n} g(\nabla^\xi_X\nabla^\xi_{e_i}e_i-\nabla^\xi_{e_i}\nabla^\xi_X e_i-\nabla^\xi_{[X,e_i]}e_i,Y)$$

for any $X,Y\in\xi$, where $\{e_1,\dots,e_{2n}\}$ is an arbitrary orthonormal frame of ξ. It is known that Ric^T satisfies

$$\mathrm{Ric}^T(X,Y)=\mathrm{Ric}(X,Y)+2\,g(X,Y)$$

for any $X,Y\in\xi$, where Ric denotes the Ricci tensor of the Riemannian metric $g=g_J+\alpha\otimes\alpha$. Let us denote by ρ^T the Ricci form of Ric^T, i.e.

$$\rho^T(X,Y)=\mathrm{Ric}^T(JX,Y)=\mathrm{Ric}(JX,Y)+2\,\kappa(X,Y)$$

for any $X,Y\in\xi$. We recall that ρ^T is a closed form (see e.g. [32]); it is often called the *transverse Ricci form* of (α,J).

Definition 5.10. The basic cohomology class

$$c_1^B(M) = \frac{1}{2\pi}[\rho^T] \in H_B^{1,1}(M)$$

is called the *first basic Chern class* of (M, α, J) and if it vanishes (M, α, J) is said to be *null-Sasakian*.

Now we have

Proposition 5.11. *Let (M, α, J) be a $2n+1$-dimensional Sasakian manifold. The following facts are equivalent*

1. $\mathrm{Hol}^0(\nabla^\xi) \subseteq \mathrm{SU}(n)$
2. $\mathrm{Ric}^T = 0$.

Proof. The connection ∇^ξ induces a connection ∇^K on $\Lambda_J^{n,0}\xi$ which has $\mathrm{Hol}(\nabla^K) \subseteq \mathrm{U}(1)$. Since $\mathrm{Hol}^0(\nabla^K)$ and $\mathrm{Hol}^0(\nabla^\xi)$ are related by

$$\mathrm{Hol}^0(\nabla^K) = \det(\mathrm{Hol}^0(\nabla^\xi))\,,$$

where det is the map induced by the determinant $\mathrm{U}(n) \to \mathrm{U}(1)$, $\mathrm{Hol}^0(\nabla^\xi) \subseteq \mathrm{SU}(n)$ if and only if $\mathrm{Hol}^0(\nabla^K) = \{1\}$ and in this case ∇^K is flat. It can be showed that the curvature form of ∇^K coincides with the transverse Ricci form of (α, J). Hence $\mathrm{Hol}^0(\nabla^\xi) \subseteq \mathrm{SU}(n)$ if and only if $\mathrm{Ric}^T = 0$. □

A Sasakian structure (α, J) is said to be *α-Einstein* if the Riemannian metric g induced by (α, J) satisfies

$$\mathrm{Ric}(g) = \lambda g + \nu\alpha \otimes \alpha\,,$$

where (λ, ν) is a pair of constants. This class of metrics was introduced by Okumura in [55]. Moreover it is known that for a generic $2n+1$-dimensional Sasakian structure the transverse Ricci tensor satisfies

$$\mathrm{Ric}(R_\alpha, X) = 2n\,\alpha(X)\,,$$

for any $X \in TM$ (see e.g. [55]). Therefore if a Sasakian structure has

$$\mathrm{Ric}^T = 0\,,$$

then it is α-Einstein and the Ricci tensor reduces to

$$\mathrm{Ric} = -2g + (2n+2)\,\alpha \otimes \alpha\,. \tag{5.2}$$

Let consider now a contact Calabi-Yau case. We have the following

Lemma 5.12. *Let $(M, \alpha, J, \varepsilon)$ be a contact Calabi-Yau manifold. Then (α, J) is a Sasakian structure on M.*

Proof. Let $\widetilde{\kappa}, \widetilde{J}$ be the almost Kähler structure induced by (α, J) on the cone $V = M \times \mathbb{R}^+$ and let

$$\psi = \varepsilon \wedge (\alpha + i1/t\, dt) \in \Lambda^{n+1,0}_{\widetilde{J}} M\,.$$

The form ψ is nowhere vanishing on V and

$$d\psi = d(\varepsilon \wedge (\alpha + i1/t\, dt)) = \varepsilon \wedge d\alpha = 0\,.$$

Therefore ψ is a closed complex volume form on V. Consequently $\widetilde{J}$ is integrable and (M, α, J) is a Sasakian manifold. □

Moreover we have that if $(M, \alpha, J, \varepsilon)$ is contact Calabi-Yau, then

$$\mathrm{Hol}(\nabla^{\xi}) \subset \mathrm{SU}(n)$$

and, by proposition 5.11, the transverse Ricci tensor of (α, J) vanishes. Summarizing we have the following

Proposition 5.13. *Let $(M, \alpha, J, \varepsilon)$ be a contact Calabi-Yau manifold, then (M, α, J) is a null-Sasakian manifold with vanishing transverse Ricci tensor. Consequently M is α-Einstein and has scalar curvature equal to $-2n$.*

Therefore contact Calabi-Yau manifolds can be considered a special class of null-Sasakian manifolds.

5.2 Special Legendrian submanifolds

Let us consider on $\mathbb{R}^{2n+1}$ the standard basis $\{e_1, \ldots, e_{2n+1}\}$ and let $V < \mathbb{R}^{2n+1}$ be the subspace spanned by $\{e_1, \ldots, e_{2n}\}$. Let J_0 be the endomorphism which coincides with the standard complex structure on V and fixes e_{2n+1}. Let

$$\alpha_0 := e^{2n+1}\,, \quad \kappa_0 := \sum_{i=1}^{n} e^{2i-1} \wedge e^{2i}\,, \quad \varepsilon_0 := \bigwedge_{i=1}^{n} (e^{2i-1} + i\, e^{2i})\,.$$

Then, with notation of last section, $\varepsilon_0 \in \Lambda^{n,0}_{J_0} V$. The Lie group SU($n$) can be viewed as the set of transformations in $\mathrm{GL}(2n+1, \mathbb{R})$ fixing $(\alpha_0, \kappa_0, J_0, \varepsilon_0)$. Let $\mathcal{G}(n)$ be the set of the n-dimensional subspaces of $\mathbb{R}^{2n+1}$. We have the following easy-proof

Proposition 5.14. *The form $\mathfrak{Re}\,\varepsilon_0$ is a calibration on $\mathbb{R}^{2n+1}$ and $W \in \mathcal{G}(n)$ is $\mathfrak{Re}\,\varepsilon_0$-calibrated if and only if*

- $W < V$;
- $i^*(\kappa_0) = 0\,, \quad i^*(\mathfrak{Im}\,\varepsilon_0 = 0)$;

where $i\colon W \hookrightarrow \mathbb{R}^{2n+1}$ is the natural embedding.

We recall the following

Definition 5.15. Let (M,ξ) be a contact $2n+1$-dimensional manifold and let α be the 1-form defining ξ. A submanifold $p\colon L \hookrightarrow M$ is said to be a *Legendrian submanifold* if

1. $p^*(\alpha) = 0$;
2. $\dim L = n$.

Let consider now a $2n+1$-dimensional contact Calabi-Yau manifold (M,α,J,ε). The n-form $\Omega = \mathfrak{Re}\,\varepsilon$ is a calibration on M with respect to the metric g induced by (α,J). Furthermore, by proposition 5.14, a submanifold $p\colon L \hookrightarrow M$ satisfies the conditions

$$p^*(\alpha) = 0\,, \quad p^*(\mathfrak{Im}\,\varepsilon) = 0$$

if and only if there exists an orientation making it calibrated by $\mathfrak{Re}\,\varepsilon$. Hence we can give the following

Definition 5.16. Let (M,α,J,ε) be a contact Calabi-Yau manifold. A Legendrian submanifold $p\colon L \hookrightarrow M$ is said to be *special Legendrian* if it satisfies the equations

$$p^*(\alpha) = 0\,, \quad p^*(\mathfrak{Im}\,\varepsilon) = 0\,.$$

Example 5.17. Let $(M = H(3)/\Gamma, \alpha, J, \varepsilon)$ be the contact manifold of the example 5.5; then the submanifold

$$L := \left\{ [A] \in M \mid A = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \simeq S^1$$

is a compact special Legendrian submanifold.

5.2.1 Deformation of Special Legendrian submanifolds

Let $(M, \alpha, J, \varepsilon)$ be a contact Calabi-Yau manifold and let $p\colon L \hookrightarrow M$ be a special Legendrian submanifold. Another special Legendrian submanifold $p_1\colon L \hookrightarrow M$ is said to be a *deformation* of $p_0\colon L \hookrightarrow M$ if there exist a smooth map $F\colon L \times [0,1] \to M$ such that

1. $F_t\colon L \hookrightarrow M$ is a special Legendrian submanifold for any $t \in [0,1]$;

2. $F_0 = p$, $F_1 = p_1$.

Let

$$\begin{aligned}\mathfrak{M}(L) := \{&special\ Legendrian\ submanifolds\ of\ (M, \alpha, J, \varepsilon)\\ &which\ are\ deformations\ of\ p\colon L \hookrightarrow M\}/\sim,\end{aligned}$$

be the moduli space of special Legendrian submanifolds near $p\colon L \hookrightarrow M$, where two embeddings are considered equivalent if they differ by a diffeomorphism of L. We have the following

Theorem 5.18. *Assume that L is compact. Then $\mathfrak{M}(L)$ is a 1-dimensional smooth manifold.*

Proof. Let $\mathcal{N}(L)$ be the normal bundle to L. Then

$$\mathcal{N}(L) = < R_\alpha > \oplus J(p_*(TL)),$$

where R_α is the Reeb vector field of α. Let Z be a vector filed normal to L and let $\exp_Z\colon L \to M$ be defined as

$$\exp_Z(x) := \exp_x(Z(x)),$$

for any $x \in L$. Let U be a neighborhood of 0 in $C^{2,\alpha}(< R_\alpha >) \oplus C^{1,\alpha}(J(p_*(TL)))$ and let

$$F\colon U \to C^{1,\alpha}(\Lambda^1 L) \oplus C^{0,\alpha}(\Lambda^n L),$$

be defined as

$$F(Z) = (\exp_Z^*(\alpha), 2\exp_Z^*(\Im\mathfrak{m}\,\varepsilon)).$$

We obviously have

$$Z \in F^{-1}(0,0) \cap C^\infty(\mathcal{N}(L)) \iff \exp_Z(L) \text{ is a special Legendrian submanifold.}$$

Note that since $\exp_Z$ and p are homotopic via $\exp_{tZ}$, we have

$$[\exp_Z^*(\Im\mathfrak{m}\,\varepsilon)] = [p^*(\Im\mathfrak{m}\,\varepsilon)] = 0.$$

Therefore

$$F\colon U \to C^{1,\alpha}(\Lambda^1 L) \oplus dC^{1,\alpha}(\Lambda^{n-1}L)\,.$$

Let us compute the differential of the map F.

$$F_*[0](Z) = \frac{d}{dt}(\exp^*_{t\,Z}(\alpha), 2\exp^*_{t\,Z}(\Im\mathfrak{m}\,\varepsilon)) = (p^*(\mathcal{L}_Z\alpha), 2p^*(\mathcal{L}_Z\Im\mathfrak{m}\,\varepsilon))\,,$$

where $\mathcal{L}$ denotes the Lie derivative. We may write $Z = JX + f\,R_\alpha$; then applying Cartan formula we obtain

$$\begin{aligned} F_*[0](Z) &= (p^*(\mathcal{L}_Z\alpha), 2p^*(\mathcal{L}_Z\Im\mathfrak{m}\,\varepsilon))\\ &= (p^*(d\iota_Z\alpha + \iota_Z d\alpha), 2p^*(d\iota_Z\Im\mathfrak{m}\varepsilon))\\ &= (p^*(d\iota_{JX+f\,R_\alpha}\alpha + \iota_{JX+f\,R_\alpha}d\alpha), 2p^*(d\iota_{JX+f\,R_\alpha}\Im\mathfrak{m}\,\varepsilon))\\ &= (p^*(d\iota_{fR_\alpha}\alpha + \iota_{JX}d\alpha), 2p^*(d\iota_{JX}\Im\mathfrak{m}\,\varepsilon))\\ &= (p^*(df + \iota_{JX}d\alpha), 2dp^*(\iota_{JX}\Im\mathfrak{m}\,\varepsilon))\,. \end{aligned}$$

Applying lemma 4.14, we get

$$2p^*(\iota_{JX}\Im\mathfrak{m}\,\varepsilon) = -*p^*(\iota_{JX}d\alpha)\,,$$

where $*$ is the Hodge star operator with respect to the metric $p^*(g_J)$ and the volume form $p^*(\Re\mathfrak{e}\,\varepsilon)$. Consequently we obtain

$$F_*[0](Z) = (d(f\circ p) + p^*(\iota_{JX}d\alpha), -d*p^*(\iota_{JX}d\alpha))\,. \tag{5.3}$$

The next step consists to show that $F_*[0]$ is a surjective operator. Let $(\eta, d\gamma) \in C^{1,\alpha}(\Lambda^1 L) \oplus dC^{1,\alpha}(\Lambda^n(L))$. By the Hodge decomposition theorem we may assume

$$d\gamma = -d*du \text{ with } u \in C^{3,\alpha}(L)$$

and we have

$$\eta = dv + d^*\beta + h(\eta)$$

where $v \in C^{2,\alpha}L$, $\beta \in C^{2,\alpha}(\Lambda^2 L)$ and $h(\eta)$ denotes the harmonic component of η. Then we have

$$\begin{aligned} (\eta, d\gamma) =&(du - du + dv + d^*\beta + h(\eta), -d*du)\\ &(dv - du + du + d^*\beta + h(\eta), -d*(du + d^*\beta + h(\eta))\,. \end{aligned}$$

We can find $f \in C^{2,\alpha}(p(L))$ and $X \in C^{1,\alpha}(p_*(TL))$ such that

$$\begin{aligned} &f\circ p = v - u\\ &p^*(\iota_{JX}d\alpha) = du + d^*\beta + h(\eta)\,. \end{aligned}$$

Hence

$$(\eta, d\gamma) = (d(f \circ p) + p^*(\iota_{JX} d\alpha), -d * p^*(\iota_{JX} d\alpha))$$

and $F_*[0]$ is surjective. Therefore $(0,0)$ is a regular value of F.
Now we compute $\ker F_*[0]$. Formula (5.3) implies that $Z \in \ker F_*[0]$ if and only if

$$d(f \circ p) + p^*(\iota_{JX} d\alpha) = 0 \tag{5.4}$$

$$d^* p^*(\iota_{JX} d\alpha) = 0\,. \tag{5.5}$$

By applying d^* to both sides of (5.4) and taking into account (5.5) we get

$$0 = d^* d(f \circ p) + d^* p^*(\iota_{JX} d\alpha) = d^* d(f \circ p)\,,$$

i.e.

$$\Delta(f \circ p) = 0\,.$$

Since L is compact f is constant. Hence (5.4) reduces to

$$p^*(\iota_{JX} d\alpha) = 0\,. \tag{5.6}$$

The map

$$\Theta \colon p_*(TL) \to \Lambda^1(L)$$

defined by

$$\Theta(X) = p^*(\iota_{JX} d\alpha)$$

is an isomorphism; hence equation (5.6) implies $X = 0$. Therefore $Z = W + f\, R_\alpha$ belongs to $\ker F_*[0]$ if and only if

$$\begin{cases} W = 0 \\ f = \text{constant}\,. \end{cases}$$

It follows that $\ker F_*[0]$ $=\mathrm{Span}_{\mathbb{R}}(R_\alpha) \subset C^\infty(\mathcal{N}(L))$. The implicit function theorem between Banach spaces implies that the Moduli space $\mathfrak{M}(L)$ is a 1-dimensional smooth manifold. □

Remark 5.19. Note that the dimension of $\mathfrak{M}(L)$ does not depend on that one of L. This is quite different from the Calabi-Yau case, where the dimension of the Moduli space of deformations of special Lagrangian submanifolds near a fixed compact L is equal to the first Betti number of L. This difference can be explained in the following way: the deformations parametrized by curves tangent to the contact structure are trivial, while those one along the Reeb vector field R_α parameterize the Moduli space.

5.2.2 The Lu Peng problem in contact Calabi-Yau manifolds

In this section we study the Lu Peng problem for special Legendrian submanifolds. We have the following

Theorem 5.20. *Let $(M, \alpha_t, J_t, \varepsilon_t)_{t\in(-\delta,\delta)}$ be a smooth family of contact Calabi-Yau manifolds. Let $p\colon L \hookrightarrow M$ be a compact special Legendrian submanifold of $(M, \alpha_0, J_0, \varepsilon_0)$. There exists, for small t, a family of compact special Legendrian submanifolds $p\colon L \hookrightarrow (M, \alpha_t, J_t, \varepsilon_t)$ such that $p_0 = p$ if and only if the condition*

$$[p^*(\Im\mathfrak{m}\,\varepsilon_t)] = 0 \tag{5.7}$$

holds for t small enough.

Proof. The condition (5.8) is necessary. Indeed if we can extend L, then $\Im\mathfrak{m}\,\varepsilon_t$ is a closed form such that $p_t^*(\Im\mathfrak{m}\,\varepsilon_t) = 0$. Since p_t is homotopic to p_0 we have

$$[p_0^*(\Im\mathfrak{m}\,\varepsilon_t)] = [p_t^*(\Im\mathfrak{m}\,\varepsilon_t)] = 0\,.$$

In order to prove that condition (5.8) is sufficient, we can consider the map

$$G\colon (-\sigma, \sigma) \times C^{(1,\alpha)}(J(p_*TL)) \to C^{0,\alpha}(\Lambda^2 L) \oplus C^{(0,\alpha)}(\Lambda^n L)$$

defined by

$$G(t, Z) = (\exp_Z^*(d\alpha_t), 2\exp_Z^*(\Im\mathfrak{m}\,\varepsilon_t))\,.$$

By our assumption it follows that

$$\mathrm{Im}(G) \subset dC^{1,\alpha}(\Lambda^1 L) \oplus dC^{(1,\alpha)}(\Lambda^{n-1} L)$$

Let $X \in p_*(TL)$; a direct computation gives

$$\begin{aligned} G_*[(0,0)](0, JX) =& (dp^*(\iota_{JX} d\alpha_0), 2dp^*(\iota_{JX}\Im\mathfrak{m}\,\varepsilon)) \\ =& (dp^*(\iota_{JX} d\alpha_0), -d * p^*(\iota_{JX} d\alpha_0))\,, \end{aligned}$$

where $*$ is the Hodge operator of the metric $p^*(g_J)$ with respect to the volume form $p^*(\Re\mathfrak{e}\,\varepsilon)$ (here we have applied again formula lemma 4.14). It follows that $G_*[(0,0)](0,\cdot)$ is surjective and that

$$\ker G_*[(0,0)]_{\{0\}\times C^{1,\alpha}(p_*(J(TL)))} \equiv \mathcal{H}^1(L)\,.$$

Let

$$A = \{X \in C^{1,\alpha}(p_*(TL)) \text{ s.t. } p^*(\iota_{JX} d\alpha) \in dC^{1,\alpha}(L) \oplus d^*C^{1,\alpha}(\Lambda^2 L)\}$$

and

$$\hat{G} = G_{|(-\delta,\delta)\times A}\,.$$

Then by the Hodge decomposition of $\Lambda(L)$ it follows that

$$G_*[(0,0)]_{\{0\}\times A}\colon A \to dC^{1,\alpha}(L) \oplus d^* C^{1,\alpha}(\Lambda^2 L)$$

is an isomorphism. Again by the implicit function theorem and the there exists a local solution of the equation

$$\begin{cases} \psi(0) = 0\,, \\ \hat{G}(t,\psi(t)) = 0\,. \end{cases}$$

Taking into account the derivative of the equation $\hat{G}(t,\psi(t)) = 0$ with respect to t, we get that $\dot{\psi}(t)$ is solution of an elliptic equation for any t. It follows that $\psi(t)$ is a curve of smooth vector fields. The extension of $p\colon L \hookrightarrow M$ is obtained by considering

$$p_t := \exp_{\psi(t)}\,.$$

□

5.3 Interplay between Calabi-Yau and contact Calabi-Yau structures

The interplay between Calabi-Yau and contact Calabi-Yau manifolds can be summarized with the following table

(M,κ,J,ε) Calabi-Yau	(M,α,J,ε) Contact Calabi-Yau
$\mathrm{Hol}(\nabla)\subset \mathrm{SU}(n)$	$\mathrm{Hol}(\nabla^\xi)\subset \mathrm{SU}(n)$
Ric$= 0$	$\mathrm{Ric}^T = 0$
$c_1(M)$=0	$c_1^B(M) = 0$
$\mathfrak{Re}\,\varepsilon$ is a calibration on M	$\mathfrak{Re}\,\varepsilon$ is a calibration on M
a submanifold $p\colon L \hookrightarrow M$ is calibrated by $\mathfrak{Re}\,\varepsilon$ if and only if $p^*(\kappa) = p^*(\mathfrak{Im}\,\varepsilon) = 0$	a submanifold $p\colon L \hookrightarrow M$ is calibrated by $\mathfrak{Re}\,\varepsilon$ if and only if $p^*(\alpha) = p^*(\mathfrak{Im}\,\varepsilon) = 0$
the moduli space of special Lagrangian submanifolds closed to a compact one L is a smooth of dimension $b_1(L)$	the moduli space of special Lagendrian submanifolds closed to a compact one L is a a 1-dimensional manifold

Where in the left side ∇ denotes the Levi-Civita connection of the Riemannian metric g_J associated to (κ, J) and Ric is the Ricci tensor of g_J.

5.4 The 5-dimensional nilpotent case

In this section we study invariant contact Calabi-Yau structure on 5-dimensional nilmanifolds. We will prove that a compact 5-dimensional nilmanifold carrying an invariant Calabi-Yau structure is covered by a Lie group whose algebra is isomorphic to

$$\mathfrak{g} = (0,0,0,0,12+34)\,,$$

just described in section 2. Notation $\mathfrak{g} = (0,0,0,0,12+34)$ means that there exists a basis $\{\alpha_1,\ldots,\alpha_5\}$ of the dual space of the Lie algebra $\mathfrak{g}$ such that

$$d\alpha_1 = d\alpha_2 = d\alpha_3 = d\alpha_4 = 0\,,\quad d\alpha_5 = \alpha_1\wedge\alpha_2 + \alpha_3\wedge\alpha_4\,.$$

First of all we note that 5-dimensional contact Calabi-Yau are in particular Hypo. Recall that an *Hypo structure* on a 5-dimensional manifold is the datum of $(\alpha,\omega_1,\omega_2,\omega_3)$, where $\alpha\in\Lambda^1(M)$ and $\omega_i\in\Lambda^2(M)$ and

1. $\omega_i\wedge\omega_j = \delta_{ij}\,v$, for some $v\in\Lambda^4(M)$ satisfying $v\wedge\alpha\neq 0$;
2. $\iota_X\omega_1 = \iota_Y\omega_2 \iff \omega_3(X,Y)\geqslant 0$:
3. $d\omega_1 = 0$, $d(\omega_2\wedge\alpha) = 0$, $d(\omega_3\wedge\alpha)=0$.

These structures have been introduced and studied by D. Conti and S. Salamon in [22].
Let (M,α,J,ε) be a contact Calabi-Yau manifold of dimension 5. Then

$$\alpha\,,\quad \omega_1 = \frac{1}{2}d\alpha\,,\quad \omega_2 = \mathfrak{Re}\,\varepsilon\,,\quad \omega_3 = \mathfrak{Im}\,\varepsilon\,,$$

define an Hypo structure on M.
The following lemma, whose proof is immediate, will be useful in the sequel

Lemma 5.21. *Let $M = G/\Gamma$ be a manifold of dimension 5. If M admits an invariant contact form, then the Lie algebra of G is isomorphic to one of the following models*

- $(0,0,12,13,14+23)\,;$
- $(0,0,0,12,13+24)\,;$
- $(0,0,0,0,12+34)\,.$

Let $\mathfrak{g}$ be a non-trivial 5-dimensional nilpotent Lie algebra and denote by $V = \mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. There exists a filtration of V

$$V^1 \subset V^2 \subset V^3 \subset V^4 \subset V^5 = V\,,$$

with $dV^i \subset \Lambda^2 V^{i-1}$ and $\dim_{\mathbb{R}} V^i = i$. We may chose the filtration V in such a way that $V^2 \subset \ker d \subset V^4$.

Let $(M = G/\Gamma, \alpha, \omega_1, \omega_2, \omega_3)$ be a nilmanifold endowed with a invariant Hypo structure $(\alpha, \omega_1, \omega_2, \omega_3)$

1. Assume that $\alpha \in V^4$. Then we have the following (see [22])

   Theorem 5.22. *If $\alpha \in V^4$, then $\mathfrak{g}$ is either* $(0,0,0,0,12)$, $(0,0,0,12,13)$, *or* $(0,0,12,13,14)$.

   In particular if $(M, \alpha, J, \varepsilon)$ is contact Calabi-Yau, then $\alpha \in V^4$.

2. Assume that $\alpha \notin V^4$. We have the following two theorems (see [22] again)

   Lemma 5.23. *If $\alpha \notin V^4$ and all ω_i are closed, then α is orthogonal to V^4.*

   Theorem 5.24. *If α is orthogonal to V^4, then $\mathfrak{g}$ is one of*

   $$(0,0,0,0,12)\,, \quad (0,0,0,0,12+34)\,.$$

Let $(M, \alpha, J, \varepsilon)$ be a contact Calabi-Yau manifold of dimension 5 endowed by an invariant contact Calabi-Yau structure, then by 1. α does not belong to V^4. By lemma 5.23 α is orthogonal to V^4 and by theorem 5.24 $\mathfrak{g} = (0,0,0,0,12+34)$. Hence we have proved the following

Theorem 5.25. *Let $M = G/\Gamma$ be a nilmanifold of dimension* 5 *admitting an invariant contact Calabi-Yau structure. Then $\mathfrak{g}$ is isomorphic to*

$$(0,0,0,0,12+34)\,.$$

5.5 Calabi-Yau manifolds of codimension r.

In this section we extend the definition of contact Calabi-Yau manifold to codimension r proving the analogous of theorem 5.20.
Let us consider the following

Definition 5.26. Let M be a $2n+r$-dimensional manifold. A *r-contact structure* on M is the datum $\mathcal{D} = \{\alpha_1, \dots, \alpha_r\}$, where $\alpha_i \in \Lambda^1 M$, such that

- $d\alpha_1 = d\alpha_2 = \cdots = d\alpha_r$;
- $\alpha_1 \wedge \cdots \wedge \alpha_r \wedge (d\alpha_1)^n \neq 0$.

Note that if $\mathcal{D} = \{\alpha_1, \dots, \alpha_r\}$ is a r-contact structure and $\xi := \bigcap \ker \alpha_i$, then $(\xi, d\alpha_1)$ is a symplectic vector bundle on M and there exists a unique set of vector fields $\{R_1, \dots, R_r\}$ satisfying

$$\alpha_i(R_j) = \delta_{ij}\,, \quad \iota_{R_i} d\alpha_i = 0 \text{ for any } i, j = 1, \dots, r\,.$$

Let us denote by $\mathcal{C}_\kappa(\xi)$ the set of complex structures on ξ calibrated by the symplectic form $\kappa = \frac{1}{2} d\alpha_1$ and by $\Lambda^r_0 M$ the set of r-form γ on M satisfying

$$\iota_{R_i}\gamma = 0 \text{ for any } i = 1, \dots, r.$$

Since $J(\Lambda^r_0 M) \subset \Lambda^r_0 M$ we have a natural splitting of $\Lambda^r_0 M \otimes \mathbb{C}$ in

$$\Lambda^r_0 M \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{p,q}_J \xi\,.$$

If $J \in \mathfrak{C}_\kappa \xi$ is given, we extend it in TM by defining

$$J(R_i) = R_i\,.$$

We can give the following

Definition 5.27. A r-contact Calabi-Yau manifold is the datum of $(M, \mathcal{D}, J, \epsilon)$, where

- M is a $2n+r$-dimensional manifold;
- $\mathcal{D} = \{\alpha_1, \dots, \alpha_r\}$ is a r-contact structure;
- $J \in \mathfrak{C}_\kappa \xi$
- $\varepsilon \in \Lambda^{n,0}_J(\xi)$ satisfies
$$\begin{cases} \varepsilon \wedge \overline{\varepsilon} = c_n\, \kappa^n \\ d\varepsilon = 0\,. \end{cases}$$

Example 5.28. Let $M = H(3)/\Gamma \times S^1$ be the Kodaira-Thurston manifold, where $H(3)$ is the 3-dimensional Heisenberg group and Γ is the lattice of $H(3)$ of matrices with integers entries. Let

$$\begin{aligned} \alpha_1 &= 2dz + 2xdy\,, \\ \alpha_2 &= 2dz + 2xdy + dt\,. \end{aligned}$$

One easily get

$$d\alpha_1 = d\alpha_2 = 2dx \wedge dy$$

and that $\mathcal{D} = \{\alpha_1, \alpha_2\}$ is a 2-contact structure on M. Note that $\xi = \ker \alpha_1 \cap \ker \alpha_2$ is spanned by $\{X_1 = \partial_x, X_2 = x\partial_y - \partial_z\}$. Moreover the Reeb fields of $\mathcal{D}$ are

$$\begin{aligned} R_1 &= \frac{1}{2}\partial_z - \frac{1}{2}\partial_t\,, \\ R_2 &= \frac{1}{2}\partial_t\,. \end{aligned}$$

Therefore $\Lambda^1_0 M$ is generated by $\{dx, dy\}$. Let $J \in \mathrm{End}(\xi)$ be the complex structure given by

$$J(X_1) = X_2\,, \quad J(X_2) = -X_1$$

and let $\varepsilon \in \Lambda^{2,0}_J \xi$ be the form

$$\varepsilon = dx + iJdy\,.$$

Then $(M, \mathcal{D}, J, \varepsilon)$ is a 2-contact Calabi-Yau structure.

As in the contact Calabi-Yau case if $(M, \mathcal{D}, J, \varepsilon)$ is a r-contact Calabi-Yau manifold, then the n-form $\Omega = \mathfrak{Re}\,\varepsilon$ is a calibration on M. Moreover a n-dimensional submanifold $p\colon L \hookrightarrow M$ admits an orientation making it calibrated by Ω if and only if

$$\begin{aligned} &p^*(\alpha_i) = 0 \text{ for any } \alpha_i \in \mathcal{D}\,, \\ &p^*(\mathfrak{Im}\,\varepsilon) = 0\,. \end{aligned}$$

A submanifold satisfying these equations will be called *special Legendrian.*

Example 5.29. Let $(M, \mathcal{D}, J, \varepsilon)$ be the 2-contact Calabi-Yau structure described in example 5.28. Then

$$L := \left\{ [A] \in H(3)/\Gamma \,|\, A = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, x \in \mathbb{R} \right\} \times \{q\} \simeq S^1$$

is a Special Legendrian submanifold for any $q \in S^1$.

The proof of next theorem is very similar to that one of theorem 5.20 and it is left to the reader

Theorem 5.30. *Let $(M, \mathcal{D}_t, J_t, \varepsilon_t)_{t\in(-\delta,\delta)}$ be a smooth family of r-contact Calabi-Yau manifolds. Let $p\colon L \hookrightarrow M$ be a compact special Legendrian submanifold of $(M, \mathcal{D}_0, J_0, \varepsilon_0)$. Then there exists, for small t, a family of compact special Legendrian submanifolds $p_t\colon L \hookrightarrow (M, \mathcal{D}_t, J_t, \varepsilon_t)$ extending $p\colon L \hookrightarrow M$ if and only if the condition*

$$[p^*(\Im\mathfrak{m}\, \varepsilon_t)] = 0 \tag{5.8}$$

holds for t small enough.

Bibliography

[1] Abbena E.: An Example of an Almost Kähler Manifold which is not Kählerian, *Boll.U.M.I.* (6) **3**-A (1984), pp. 383–392.

[2] Agricola I., Friedrich Th.: On the holonomy of connections with skew-symmetric torsion. *Math. Ann.* **328** (2004), pp. 711–748.

[3] Agricola, I., Friedrich, Th., Nagy, P.-A., Puhle C.: On the Ricci tensor in the common sector of type II string theory, *Classical Quantum Gravity* **22** (2005), pp. 2569–2577.

[4] Alexandrov B., Friedrich Th., Schoemann, N.: Almost Hermitian 6-manifolds revisited, *J. Geom. Phys.* **53** (2005), pp. 1–30.

[5] Apostolov V., Draghici T.: The curvature and the integrability of almost-Kähler manifolds: a survey, *Symplectic and contact topology: interactions and perspectives* (Toronto, ON/Montreal, QC, 2001), 25-53, Fields Inst. Commun., 35, Amer. Math. Soc., Providence, RI, 2003.

[6] Apostolov V., Salamon S.: Kähler reduction of metrics with holonomy G_2, *Comm. Math. Phys.* **246** (2004), pp. 43–61.

[7] Audin M., Lafontaine J.: *Holomorphic curves in symplectic geometry.* Progress in Mathematics, 117. Birkhäuser Verlag, Basel, 1994. xii+328 pp.

[8] Banos B.: On symplectic classification of effective 3-forms and Monge-Ampère equations, *Differential Geom. Appl.* **19** (2003), pp. 147–166.

[9] Bedulli L.: *Tre-varietà di Calabi-Yau generalizzate*, PhD thesis, Università di Firenze (2004).

[10] Bedulli L., Vezzoni L.: The Ricci Tensor of SU(3)-Manifolds, to appear in *J. Geom. Phys.* (2006).

[11] Boyer C., Galicki K.: 3-Sasakian manifolds, *Surveys in differential geometry: essays on Einstein manifolds*, Surv. Differ. Geom., VI, Int. Press, Boston, MA (1999), pp. 123–184.

[12] Boyer C. P., Galicki K., Matzeu P.: On Eta-Einstein Sasakian Geometry, *Commun. Math. Phys.* **262** (2006), pp. 177–208.

[13] Boyer C. P., Galicki K., Nakamaye M.: On the geometry of Sasakian-Einstein 5-manifolds, *Math. Ann.* **325** n. 3 (2003), pp. 485-524.

[14] Bryant R.: Metric with exceptional holonomy, *Ann. of Math.* (2) **126** (1987), pp. 525–576.

[15] Bryant R.: Some remarks on G_2-structures, `e-print: math.DG/0305124`.

[16] Brylinski J.L: A Differential Complex for Poisson Manifolds, *J. Diff. Geom.* **28** (1988) pp. 93-114.

[17] Cabrera F. M., Swann A.: Curvature of (special) almost Hermitian manifolds, `e-print: math.DG/0501062`.

[18] Chiossi S., Salamon S.: The intrinsic torsion of SU(3) and G_2 structures. *Differential geometry, Valencia*, 2001, 115–133, World Sci. Publishing, River Edge, NJ, 2002.

[19] Cleyton R., Ivanov S.: *On the geometry of closed G_2-structure*, `e-print: math.DG/0306362`.

[20] Chiossi S. G., Swann A.: G_2*-structures with torsion from half-integrable nilmanifolds*, J. Geom. Phys. **54** (2005), pp. 262–285.

[21] Conti D., Tomassini A.: *Special Symplectic Six-Manifolds*, `e-print: math.DG/0601002`

[22] Conti D., Salamon S.: Generalized Killing spinors in dimension 5, `e-print: math.DG/0508375`, to appear in *Trans. Amer. Math. Soc.*.

[23] de Bartolomeis P.: Geometric Structures on Moduli Spaces of Special Lagrangian Submanifolds, *Ann. di Mat. Pura ed Applicata*, IV, Vol. CLXXIX, (2001), pp. 361–382.

[24] de Bartolomeis P.: GBV Algebras, Formality Theorems, and Frobenius Manifolds, *Seminari di Geometria Algebrica 1998-1999, Scuola Normale Superiore, Pisa*, pp. 161–178.

[25] de Bartolomeis P.: *Symplectic and Holomorphic Theory in Kähler Geometry*, XIII Escola de Geometria Diferencial Istituto Matemática e Estatística Universidade de São Paulo, (2004)

[26] de Bartolomeis, P.: $\mathbb{Z}_2$ and $\mathbb{Z}$-deformation theory for holomorphic and symplectic manifolds. Complex, contact and symmetric manifolds, 75–103, *Progr. Math.*, 234, Birkhäuser Boston, Boston, MA, (2005).

[27] de Bartolomeis P., Tomassini A.: On Formality of Some Symplectic Manifolds, *Inter. Math. Res. Notic.* **24** (2001) pp. 1287–1314.

[28] de Bartolomeis P., Tomassini A.: On the Maslov Index of Lagrangian Submanifolds of Generalized Calabi-Yau Manifolds, to appear in *Int. J. of Math* **17** (8) (2006), pp. 921–947.

[29] de Bartolomeis P., Tomassini A.: On solvable Generalized Calabi-Yau Manifolds, to appear in *Ann. Inst. Fourier.*

[30] Deligne P., Griffiths P., Morgan J., Sullivan D.: Real Homotopy Theory of Kähler Manifolds, *Inventiones Math.* **29** (1975), pp. 245–274.

[31] Draghici T.: Symplectic obstructions to the existence of ω-compatible Einstein metrics, `preprint` (2003).

[32] El Kacimi-Alaoui A.: Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications, *Compositio Math.* **73**, n. 1 (1990), pp. 57–106.

[33] Fernàndez M., de León M., Saralegui M.: A six dimensional Compact Symplectic Solvmanifold without Kähler Structures, *Osaka J. Math* **33** (1996) pp. 19–34.

[34] Fernàndez M., Gray A.: Compact Symplectic Solvmanifolds not admitting Complex Structures, *Geometriae Dedicata* **34** (1990) pp. 295–299.

[35] Gompf R.E., Mrowka T.S., Irreducible 4-manifolds Need not be Complex, *Ann. of Math.* (2) **138** (1993) pp. 61-111.

[36] Gross M., Huybrechts D., Joyce D.: *Calabi-Yau manifolds and related geometries*, Lectures from the Summer School held in Nordfjordeid, June 2001. Universitext. Springer-Verlag, Berlin, 2003. viii+239 pp.

[37] Harvey R., Lawson H. Blaine, Jr.: Calibrated geometries, *Acta Math.* **148** (1982) pp. 47–157.

[38] Hasegawa K.: A note on compact solvmanifolds with Kähler structures, `e-print CV/0406227` (2004).

[39] Hitchin N.: Generalized Calabi-Yau Manifolds, *Quart. J. Math.* **54** (2003), pp. 281–308.

[40] Hitchin N.: Stable forms and special metrics, *Global differential geometry: the mathematical legacy of Alfred Gray* (Bilbao, 2000), 70–89, Contemp. Math., 288, Amer. Math. Soc., Providence, RI, 2001.

[41] Hitchin N.J.: The moduli space of special Lagrangian submanifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **25** (1997), no. 3-4, pp. 503–515 (1998). N. J. Hitchin, The Moduli Space of Special Lagrangian Submanifolds , dg-ga/9711002

[42] Hopf E.: Elementare Bemerkungen ueber die Loesung parzieller Differentialgleichungen zweiter Ordnung von elliptischen Typus, *Sitzungber. Preuss. Akad. Wiss. phys. math. Kl.* 19 (1927), pp. 147–152.

[43] Joyce, Dominic D.: *Compact manifolds with special holonomy*, Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. xii+436 pp.

[44] Kodaira K.: *Complex manifolds and deformation of complex structures*, translated from the 1981 Japanese original by Kazuo Akao. Reprint of the 1986 English edition. Classics in Mathematics. Springer-Verlag, Berlin, 2005. x+465 pp.

[45] Lu P.: Kähler-Einstein metrics on Kummer threefold and special Lagrangian tori, *Comm. Anal. Geom.* **7** (1999), no. 4, pp. 787–806.

[46] Malcev A.: On a class of homogeneous spaces, *Amer. Math. Soc. Translation* no.42 (1951).

[47] Mathieu O.: Harmonic cohomology classes of symplectic manifolds, *Comm. Math. Helvetici* **70** (1995) pp. 1–9.

[48] McDuff D., Salamon D.: *Introduction to Symplectic Topology*, Oxford Mathematical Monographs, Oxford University Press, New York (1995).

[49] McLean R. L.: Deformations of Calibrated Geometries, *Comm. Anal. Geom.* **6**, no.4, (1998) pp. 705–747.

[50] Mostow G.D.: Factor spaces of solvable groups, *Ann. of Math.* (2) **60** (1954) pp. 1–27.

[51] Mostow G.D.: Representative Functions on Discrete groups and Solvable Arithmetic Subgroups, *American J. of Math.* **92** (1970) pp. 1–32.

[52] Nakamura I.: Complex parallelisable manifolds and their small deformations, *J. of Diff. Geom.* **10** (1975) pp. 85–112.

[53] Newlander A., Nirenberg L.: Complex anallytic coordinates in almost complex maqnifold, *Ann. of Math.* **65** (1957), pp. 391–404.

[54] Nomizu K.: On the cohomology of compact homogeneous spaces of nilpotent Lie groups, *Ann. Math.* 59 (1954), pp. 531–538.

[55] Okumura M.: Some remarks on space with a certain contact structure. Tôhoku Math J. (2), **14** (1962), pp. 135–145.

[56] Paoletti R.: On families of Lagrangian submanifolds, *Manuscripta Math.* 107 (2002), no. 2, pp. 145–150.

[57] Salamon S.: Riemannian geometry and holonomy groups, *Pitman Research Notes in Mathematics Series*, 201. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989. viii+201 pp.

[58] Sato M., Kimura T.: *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. 65, 1977, pp. 1–155.

[59] Silva A.: Spazi omogenei Kähleriani di gruppi di Lie complessi risolubili, *Boll. U.M.I.* **2 A** (1983) pp. 203–210.

[60] Strominger A., Yau S-T, Zaslow E.: *Mirror Symmetry is T-Duality* , Nucl. Phys. B 479, No. 1-2, 1996, pp.243–25

[61] Thurston W., *Some Example of Symplectic Manifolds*, Proc. A.M.S. 5 (1976), pp. 467–468.

[62] Tomassini A., Vezzoni L.: Admissible Complex Structures and Moduli Space, `e-print` (2006)

[63] Tomassini A., Vezzoni L.: Contact Calabi-Yau Manifolds and Special Legendrian Submanifolds, to appear in *Osaka J. Math.*.

[64] Tomassini A., Vezzoni L.: Special Lagrangian Submanifolds in Generalized Calabi-Yau Manifolds, preprint n. 411, *Dipartimento di Matematica Università di Parma* (2005).

[65] Tralle A. Kedra J.: Compact completely solvable Kähler solvmanifolds are tori, *Int. Math. Res. Not.* **15** (1997) pp. 727–732.

[66] Tralle A. Oprea J.: *Symplectic Manifolds with no Kähler Structure*, Lecture Notes in Mathematics, **1661**, Springer-Verlag, Berlin, 1997.

[67] Vezzoni L.: A Generalization of the Normal Holomorphic Frames in Symplectic Manifolds, *Boll. U.M.I.* **9**-B (2006), pp. 723–732.

[68] Vezzoni L.: On the Hermitian Curvature of a Symplectic Manifold, *Adv. Geom.* **7** (2007), no. 2, pp. 207–214.

[69] C. Viterbo: Intersection de Sous-Varétés Lagrangiennes, Functionelles d"action et Indice des Systèmes Hamiltoniens, *Bull. Soc. Math. France* **115** (1987), pp. 361–390.

[70] Yan D.: Hodge Structure on Symplectic Manifolds, *Adv. in Math.* **120** (1996), pp. 143-154.

[71] Yau S.T.: On Calabi's Conjecture and some new results in algebraic geometry, *Proc. Nat. U.S.A.* **74** (1977), pp. 1789–1799.

[72] Yau S.T.: On the Ricci curvature of a Compact Kähler manifold and the complex Monge-Ampère Equation I, *Comm. Pure Appl. Math.* **31** (1978), pp. 339–411.