Università degli Studi di Pisa

DIPARTIMENTO DI MATEMATICA
Scuola di Dottorato Galileo Galilei

Tesi di Dottorato di Ricerca

Some Analytic Questions in Mathematical Physic Problems

Candidato:
Jmmy Alfonso Mauro
Matricola 299309

Relatore:
Ch.mo Prof. Vladimir Georgiev

Correlatore:
Ch.mo Prof. Paolo Maremonti

Anno Accademico 2008–2009

Introduction

In this thesis we study two different problems of mathematical physics. In the first part of the thesis we discuss some questions related to the partial regularity theory of the Navier-Stokes equations. In particular, we consider the non-stationary Navier-Stokes equations with unit viscosity and zero body force

$$(1)\qquad \begin{aligned} v_t - \Delta v + (v\cdot\nabla)\, v &= -\nabla\pi && \forall\ (x,t) \in \Omega\times(0,T), \\ \nabla\cdot v &= 0 && \forall\ (x,t) \in \Omega\times(0,T), \end{aligned}$$

In our notation $(v\cdot\nabla)\, v = (\nabla v)\, v = v_k \frac{\partial}{\partial x_k} v_h$.
In addition to (1) we require the following initial and boundary conditions

$$(2)\qquad \begin{aligned} v(x,t) &= 0 && \forall\ (x,t) \in \partial\Omega\times(0,T), \\ v(x,0) &= v_0(x) && \forall\ x \in \Omega, \end{aligned}$$

The initial data v_0 should satisfy the compatibility conditions $\nabla\cdot v_0 = 0$ in Ω and $v_0 \cdot \nu_{|\partial\Omega} = 0$, with $\nu(x)$ the outward pointing unit normal vector at $x \in \partial\Omega$, at least in weak form. Moreover, if the domain Ω is unbounded, we also assume the following condition at infinity

$$\lim_{|x|\to\infty} v(x,t) = 0 \qquad \forall\ t \in [0,T)\,.$$

For the Cauchy problem, the existence of weak solutions for initial-boundary value problem (1) – (2) was proved by J. Leray in [41]. In particul, he introduced the first notion of weak solution for the Navier-Stokes system.
In [34], E. Hopf proved the existence of weak solutions for problem (1) – (2), on any smooth enough domain $\Omega \subset R^n$ with $n \geq 2$, which are different to Leray's. Since that time, much effort has been made to establish results on the uniqueness and regularity of weak solutions. Among these results we wish to mention the fundamental works of Serrin [55, 56], which were later improved in [60, 28]. It is also well-known that if the initial data v_0 is smooth enough, problem (1) – (2) has a unique regular solution on $\Omega\times(0,T)$, for some small $T > 0$ (see e.g. [24]).
However uniqueness and regularity questions remain mostly open. In particular, till now, it is not known whether or not a Leray weak solution or a Hopf

weak solution can develop singularities in a finite time, even if the initial data are smooth. The uniqueness problem is strictly related to the regularity one. Indeed, it is well-known that if the solution is smooth enough, then it is unique. Later on, less regular weak solutions were also introduced and they were called *very weak solutions*.

In a series of papers (e.g. see [57, 58]), where he introduced the notion of *suitable* weak solutions and the *generalized energy inequality*, V. Sheffer began to study the partial regularity theory of the Navier-Stokes system. Let us call a point (x,t) *singular* if the velocity v is not essentially bounded in any neighbourhood of (x,t); the remaining points are called *regular*. By a partial regularity theorem, we mean an estimate for the Hausdorff dimension of the set $\mathcal{S}$ of singular points. In [15], L. Caffarelli, R. Kohn, and L. Nirenberg proved a local partial regularity theorem for suitable weak solutions. Improving a previous result of Sheffer, they showed that, for any such weak solution, the associated singular set $\mathcal{S}$ satisfies $\mathscr{P}^1(\mathcal{S})=0$, where $\mathscr{P}^1$ denotes a measure on $\mathbb{R}^3_x\times\mathbb{R}_t$ analogous to one-dimensional Hausdorff measure $\mathscr{H}^1$, but defined using parabolic cylinders instead of Euclidean balls.

So, with fixed initial datum $v_0\in J(\Omega)$, we can determine weak solutions of problem $(1)-(2)$ which enjoy different regularity properties. Since a uniqueness theorem is unknown, as far as we know, *a priori* they must be considered distinct. Let us denote the class of the so called *very weak solutions* by VW, the class of *Hopf weak solutions* by H, the class of *Leray weak solutions* by L and the class of *Caffarelli-Kohn-Nirenberg suitable weak solutions* by CKN. Taking in account their properties, it's possible to establish the following chain of inclusions

$$CKN\subseteq L\subseteq H\subseteq VW\,.$$

The aim of this part of our thesis is twofold:

i) we study some regularity questions about weak solutions in H, L, CKN and we prove that if a Hopf weak solution v of problem $(1)-(2)$ also satisfies $\nabla v\in L^{\bar p}\big(0,T;L^{\bar r}(\Omega)\big)$, for some pair $(\bar p,\bar r)$ such that

$$\text{(3a)}\qquad \frac{3}{\bar r}+\frac{1}{\bar p}\le 2\quad\text{and}\quad \frac{6}{\bar r}+\frac{5}{\bar p}\le 5\qquad \text{if}\quad n=3\,,$$

$$\text{(3b)}\qquad \frac{4}{\bar r}+\frac{3}{\bar p}\le 3\quad\text{and}\quad \bar r\ge 2\qquad \text{if}\quad n=4\,,$$

then, $v\in CKN$. This result assures $H\equiv CKN$, provided that the previous extra-condition is satisfied.
Such an approach was followed by Serrin for the uniqueness of Hopf weak solutions (see Theorem 6 in [56] and Theorem 4.2 in [24]). Nevertheless, later on the extra-condition set by Serrin proved itself to be also enough for the regularity of weak solutions. On the contrary, our request is weaker

than his and, as far as we know, it is unknown whether it may also assure the uniqueness or the regularity of the weak solution;

ii) for n=3, we prove that a) if a Hopf weak solution v of the Cauchy problem also satisfies $v \in L^p\big(0,T;L^q(\mathbb{R}^3)\big)$ for some pair (p,q) such that

$$\frac{3}{q}+\frac{2}{p}>1\,,\quad \frac{3}{q}+\frac{1}{p}\le 1 \quad\text{and}\quad \frac{1}{q}+\frac{1}{p}\le\frac{1}{2} \tag{4}$$

then, for every bounded domain $\widetilde{\Omega}\subset\mathbb{R}^3$, $\mathscr{P}^k\big(\mathcal{S}\cap(\widetilde{\Omega}\times(0,T))\big)=0$, with $k=p\big(\frac{3}{q}+\frac{2}{p}-1\big)$; b) if a Hopf weak solution v of problem (1)−(2) also satisfies $\nabla v\in L^{\bar p}\big(0,T;L^{\bar r}(\Omega)\big)$ for some pair $(\bar p,\bar r)$ such that

$$\frac{3}{\bar r}+\frac{2}{\bar p}>2\,,\quad \frac{3}{\bar r}+\frac{1}{\bar p}\le 2 \quad\text{and}\quad \frac{6}{\bar r}+\frac{5}{\bar p}\le 5 \tag{5}$$

then, for every bounded domain $\widetilde{\Omega}\subseteq\Omega$, $\mathscr{P}^k\big(\mathcal{S}\cap(\widetilde{\Omega}\times(0,T))\big)=0$, with $k=\bar p\big(\frac{3}{\bar r}+\frac{2}{\bar p}-2\big)$. Particularly, if $\frac{3}{q}+\frac{1}{p}<1$ in the first case or $\frac{3}{\bar r}+\frac{1}{\bar p}<2$ in the second one, then, in both cases $k<1$: as far as we know, the best partial regularity result is $\mathscr{P}^1(\mathcal{S})=0$, proved in [15].

To better realize the result which are obtained in this part of the thesis, we proceed in the following way: for each result we give the status of the art and, then, we indicate our contribute.

We start this part of the thesis from some results concerning a pressure field associated to a Hopf weak solution. As far as we know, if $\Omega\subseteq\mathbb{R}^n$ is a bounded or an exterior domain, in [29, Theorem 3.1] it was proved that, to a Hopf weak solution, with initial data v_0 in $B^{2-\frac{2}{p}}_{q,p}(\Omega)\cap J^q(\Omega)$ satisfying a suitable compatibility condition, we can associate a pressure field π with $\nabla\pi\in L^p\big(0,\infty;L^q(\Omega)\big)$ for every pair (p,q) such that $\frac{n}{q}+\frac{2}{p}=n+1$ and $1<q<\frac{n}{n-1}$. In this thesis, with the only assumption $v_0\in J(\Omega)$, we obtain the same result $\nabla\pi\in L^p\big(\varepsilon,\infty;L^q(\Omega)\big)$, for every $\varepsilon\in(0,T)$ and, if $\varepsilon=0$, for $\frac{2(n-1)}{2n-3}<q<\frac{n}{n-1}$; if $1<q\le\frac{2(n-1)}{2n-3}$, we have $\nabla\pi\in\bigcap_{\varepsilon\,:\,0<\varepsilon<1}L^{p-\varepsilon}\big(0,T;L^q(\Omega)\big)$ with $\frac{1}{q}+\frac{2}{p}=\frac{5}{2}$. Moreover, for the pressure π we get $\pi\in\bigcap_{\varepsilon\,:\,0<\varepsilon<1}L^{\frac{4}{3}-\varepsilon}\big(0,T;L^{\frac{2n}{2n-3}}(\Omega')\big)$ and, if $n\ge 4$, $\pi\in L^p\big(0,T;L^{\tilde q}(\Omega')\big)$ [1] for every pair of exponents $(p,\tilde q)$ such that $\frac{n}{\tilde q}+\frac{2}{p}=n$ and $\frac{2n}{2n-3}<\tilde q\le\frac{2n(n-1)}{(2n-1)(n-2)}$. We point out that these last summability properties of π cannot be deduced by the previous properties of $\nabla\pi$, using Sobolev embedding theorem. For the Cauchy problem we obtain $\pi\in L^p\big(0,\infty;L^{\tilde q}(\mathbb{R}^n)\big)$, for every pair $(p,\tilde q)$ such that $\frac{n}{\tilde q}+\frac{2}{p}=n$ and $1<\tilde q\le\frac{n}{n-2}$.

Now we consider the existence of a suitable weak solution on a bounded or exterior domain $\Omega\subset\mathbb{R}^3$. As far as we know, in [58], [15], [46], [61], [65] it was shown that there exists a suitable weak solution (v,π) in $\Omega\times(0,\infty)$ such that, for every $T\in(0,\infty)$,

[1] We denote by $\Omega'\subset\Omega$ an arbitrary bounded domain such that $\operatorname{dist}\big(\Omega\setminus\Omega',\,\partial\Omega\big)>0$, if Ω is an exterior domain, while $\Omega'\equiv\Omega$ if Ω is a bounded domain.

- $\pi \in L^{\frac{5}{3}}\big(\Omega \times (0,T)\big)$, if $\Omega \equiv \mathbb{R}^3$ and $v_0 \in J(\Omega)$ ([58], [15]);
- $\nabla\pi \in L^{\frac{5}{4}}\big(\Omega \times (0,T)\big)$, if $\Omega \subset \mathbb{R}^3$ is a bounded domain and $v_0 \in W^{\frac{2}{5},\frac{5}{4}}(\Omega) \cap J(\Omega)$ ([15]);
- $\nabla\pi \in L^{\frac{5}{4}}\big(\Omega \times (0,T)\big)$, if $\Omega \subset \mathbb{R}^3$ is an exterior domain and $v_0 \in J^{\frac{9}{10},\frac{5}{4}}(\Omega)$ [2] ([46]);
- $\pi \in L^{\frac{5}{3}}\big(\Omega \times (0,T)\big)$, if $\Omega \subset \mathbb{R}^3$ is a bounded or exterior domain and v_0 is in $W^{\frac{4}{5}+\epsilon,\frac{15}{14}}(\Omega) \cap J^{\frac{15}{14}}(\Omega) \cap J(\Omega)$ for some $\epsilon > 0$ ([61]);
- $\pi \in L^{\frac{5}{4}}_{\mathrm{loc}}\big(\Omega \times (0,\infty)\big)$ if $\Omega \subset \mathbb{R}^3$ is a bounded or exterior domain, or a half-space and $v_0 \in J(\Omega)$ ([65]).

As a consequence of the previous estimate for the pressure field π associated to a Hopf weak solution, we are able to prove that, if $\Omega \subset \mathbb{R}^3$ is a bounded or an exterior domain, for each $v_0 \in J(\Omega)$ there exists a suitable weak solution (v,π) of problem (1)–(2), in $\Omega \times (0,\infty)$, with initial data v_0, such that, for every $T \in (0,\infty)$,

- $\nabla\pi \in \bigcap\limits_{\varepsilon\,:\,0<\varepsilon<T} L^p\big(\varepsilon,T;L^q(\Omega)\big)$, for every pair (p,q) such that $\frac{3}{q}+\frac{2}{p}=4$ and $1<q<\frac{3}{2}$;
- $\nabla\pi \in L^p\big(0,T;L^q(\Omega)\big)$, for every pair (p,q) such that $\frac{3}{q}+\frac{2}{p}=4$ and $\frac{4}{3}<q<\frac{3}{2}$;
- $\nabla\pi \in \bigcap\limits_{\varepsilon\,:\,0<\varepsilon<1} L^{p-\varepsilon}\big(0,T;L^q(\Omega)\big)$, for every pair (p,q) such that $\frac{1}{q}+\frac{2}{p}=\frac{5}{2}$ and $1<q\le\frac{4}{3}$;
- $\pi \in \bigcap\limits_{\varepsilon\,:\,0<\varepsilon<1} L^{\frac{4}{3}-\varepsilon}\big(0,T;L^{\tilde{2}}(\Omega')\big)$.

As already mentioned, one of the aims of this part of the thesis is to prove that the class of Hopf weak solutions H and the class of suitable weak solutions CKN coincide, provided that a suitable extra-condition is satisfied. As far as we know, such a question was solved in [65], assuming that a Hopf weak solution v belongs to $L^p\big(0,T;L^q(\Omega)\big)$ for some pair (p,q) such that

$$
\begin{aligned}
&p,q\in(1,\infty)\,, && \frac{3}{q}+\frac{1}{p}\le 1 \quad \text{and} \quad \frac{2}{q}+\frac{2}{p}\le 1 && \text{if } \ n=3\,,\\
&p,q\in(1,\infty)\,, && \frac{2}{q}+\frac{2}{p}\le 1 \quad \text{and} \quad q\ge 4 && \text{if } \ n\ge 4\,.
\end{aligned}
$$

In this thesis, we improve this result, proving that a Hopf weak solution v of problem (1)–(2) is a suitable weak one too, assuming the alternative condition $\nabla v \in L^{\bar{p}}\big(0,T;L^{\bar{r}}(\Omega)\big)$, for some pair $(\bar{p},\bar{r})$ satisfying (3). Moreover, we can prove

[2] $J^{\frac{9}{10},\frac{5}{4}}(\Omega)$ is the completion of $\mathscr{C}_0(\Omega)$ in $W^{\frac{9}{10},\frac{5}{4}}(\Omega)$.

that the suitable weak solution v satisfies the energy equality. We point out that, according to what is known at present, the extra-condition $\nabla v \in L^{\bar p}\big(0,T;L^{\bar r}(\Omega)\big)$ doesn't imply that v satisfies any Taniuchi type extra condition.

We also consider the question of Hausdorff dimension of the possible singular set $\mathcal{S}$ for a Hopf weak solution which satisfies a suitable extra-condition. It is well-known that for a suitable weak solution $\mathscr{P}^1(\mathcal{S}) = 0$ (cf. [15]). As far as we know, there are no contributions that have improved this result. In this thesis, we prove that if v is a Hopf weak solution of problem (1)–(2) such that $v \in L^p\big(0,T;L^q(\mathbb{R}^3)\big)$ for some pair (p,q) satisfying condition (4) (respectively $\nabla v \in L^{\bar p}\big(0,T;L^{\bar r}(\Omega)\big)$ for some pair $(\bar p,\bar r)$ such that satisfying condition (5)), then $\mathscr{P}^k(\mathcal{S}) = 0$ with $k = p\big(\frac{3}{q}+\frac{2}{p}-1\big)$ (respectively with $k = \bar p\big(\frac{3}{\bar r}+\frac{2}{\bar p}-2\big)$). As a consequance, we obtain that the projection $\widetilde{\mathcal{S}}_t$ of $\widetilde{\mathcal{S}}$ onto the t-axis has $\frac{k}{2}$-dimensional Hausdorff measure zero (which is, for the first case, the same result in Theorem 5-i of [28]).

Finally, we partially introduce a study of regularity of suitable weak solutions by means the theory of Morrey-Campanato spaces. We prove that if v is a suitable weak solution of the Cauchy problem which belongs to $L^p\big(0,T;L^q(\mathbb{R}^3)\big)$ for some pair (p,q) such that $p,q \in [3,\infty]$ and $\frac{n}{q}+\frac{2}{p} = \lambda$ for some $1 < \lambda < \dfrac{n+1}{3}$, then, there holds two estimates, from which we can deduce $\nabla v \in L^{2,k}\big(\Omega \times (\varepsilon,T)\big)$, with $k = \frac{n+1-3\lambda}{n+2}$, and $v \in L^\infty\big(\varepsilon,T;L^{2,h}(\Omega)\big)$, with $h = \frac{n+1-3\lambda}{n}$, for an arbitrary bounded domain $\Omega \subset \mathbb{R}^n$ which satisfies the cone condition and for each $\varepsilon \in (0,T)$. These estimates are only the starting point for the approach of regularity results in Morrey-Campanato spaces. In a later paper, we make a deeper analysis of the problem.

In the second part of the thesis we study the asymptotic stability of solitary waves solutions for the Maxwell-Schrödinger system.

A simplified model of the classical M-S (to be written in the general form) system can be considered, assuming the electrostatic case; so, we obtain the following system of Hartree-Fock equations

(6a) $$i\partial_t\psi = -\Delta\psi + e^2\left(\varphi - \frac{Z}{|x|}\right)\psi\,,$$

(6b) $$-4\pi|\psi|^2 - \Delta\varphi = 0.$$

where $\psi(x,t)$ is the wave function of a charged non-relativistic particle of charge $e > 0$ and Z is the number of protons in the nuclei of the atoms which generates the Coulomb potential. Moreover, the wave function $\psi(x,t)$ must satisfy the following charge conservation law

$$\int_{\mathbb{R}^3} |\psi(x,t)|^2\,dx = N\,,$$

where N has to be understood as the number of the electrons.

Solving equation (6b) with respect to φ and substituting this one into (6a), we obtain the following equation

$$i\partial_t\psi(x,t) = -\frac{1}{2}\Delta\psi(x,t) + \left(e^2 \int_{\mathbb{R}^3} \frac{|\psi(y,t)|^2}{|x-y|}\, dy + V(x)\right)\psi(x,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+\,, \tag{7}$$

where

$$V(x) = -\frac{e^2 Z}{|x|}, \tag{8}$$

is the external Coulomb potential.

The charge conservation law can be also written as

$$\int_{\mathbb{R}^3} |\psi(x,t)|^2\, dx = \int_{\mathbb{R}^3} |\psi(x,0)|^2\, dx.$$

In absence of external nuclei we have $Z = 0$ in (8) and, then, we obtain the classical Hartree equation

$$i\partial_t\psi(x,t) = -\frac{1}{2}\Delta\psi(x,t) + e^2\,\psi(x,t) \int_{\mathbb{R}^3} \frac{|\psi(y,t)|^2}{|x-y|}\, dy\,, \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+\,, \tag{9}$$

There is a large amount of literature on the asymptotic behavior for large time of solutions to the Cauchy problem for equation (9); in [94] sharp L^p estimates of the time decay rate of the solutions to (9) are obtained, moreover the authors study also the scattering problem for the same equation.

In [82, 112], the following more general case is studied

$$i\partial_t\psi(x,t) = -\frac{1}{2}\Delta\psi(x,t) + e^2\,\psi(x,t) \int_{\mathbb{R}^3} \frac{|\psi(y,t)|^a}{|x-y|}\, dy\,, \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+\,, \tag{10}$$

where $a = \frac{8}{3}$ corresponds to the *critical mass exponent*.

Studying the problem of local and global existence of solutions to nonlinear Schrödinger equations, one is primarily interested in the scaling symmetry of the equation, under the transformation

$$\psi_\lambda(t,x) = \frac{1}{\lambda^b}\,\psi\Big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\Big), \qquad \lambda > 0, \tag{11}$$

for some constant b depending on the equation. On the other hand, after the above scaling, the L^p-norm (in the space variables) becomes

$$\|\psi_\lambda\|_{L^p(\mathbb{R}^3)} = \frac{1}{\lambda^{b-\frac{3}{p}}}\|\psi\|_{L^p(\mathbb{R}^3)}.$$

A simple calculation shows that, for $b = \frac{3}{2}$, the scaling transformation (11) leaves equation (10), with $a = 8/3$, unperturbed and, in addition, it preserves the L^2-norm.

The main difficult in Hartree equation is the fact that the nonlinearity in equation (9) is subcritical from the point of view of large time asymptotic behavior of solutions, since the equation is not rescaling invariant for $a = 2 < 8/3$.

When there is some nucleus which generates the Coulomb potential, i.e. $Z \neq 0$, it is well-known that there exist solitary type solutions to the Hartree equation (7) (cf. [81], [85]).
Dispersive solutions to NLS tend to spread out in space, although they conserve their L^2 norm. On the contrary, solitary waves have localized spatial profiles that are constant in time. To understand the asymptotic dynamics of general solutions, it is essential to study the interaction between solitary waves and dispersive waves.

For NLS with solitary waves, there are three types of results:

– *control of solutions in a finite time interval*, which does not allow sufficient time for the excited state interaction to make a difference, and construction of all-time solutions with specified asymptotic behaviors (i.e. scattering solutions);

– *orbital stability of solitary waves*; a solution stays close to the family of nonlinear bound states if it is initially close. This is usually proved by energy arguments (cf. [81]);

– *asymptotic stability of solitary waves*; in such a case, one must assume that the spectrum of the linearized operator enjoys suitable spectral properties. Furthermore, the initial data are typically assumed to be localized, so that the dispersive component has fast local decay. Even under restrictive spectral assumptions, only perturbation problems can be treated for large solitary waves, while more general results can be obtained for small solitary waves.

Since orbital stability of solitary waves to Hartree type equation (7) is well-known (cf. [81, Theorem III.2]), our aim is to study the asymptotic stability of solitary wave solutions to equation (7). In this part of the thesis we introduce the linearization of equation (7) around the (positive) ground state standing wave. That lead us to study the dispersive properties of solutions to a certain linear system with variable coefficients that appears naturally after the linearization. Suitably rewriting the system, we can reduce it to a time-independent operator plus a decaying term: so, we obtain a not self-adjoint matrix Schrödinger operator (i.e. a 2×2 not symmetric matrix of operator), for which we can use the dispersive estimates proved in [88]. One of the main difficulties arises from

the fact that in our case the solitary waves don't have small amplitudes. Small amplitudes allow to express the matrix as a scalar Schrödinger operator plus a "small" term. In our case, the fact that the amplitudes of the ground states are not small implies that we cannot proceed in such a way.
Such dispersive estimates of the linear problem are the starting point to prove the asymptotic stability of solitary waves to equation (7).

If we consider system (12)

$$\frac{1}{2}\Delta\chi^* + \omega^* \ \chi^* \ = e^2\left(q(|\chi^*|^2) - \frac{Z}{|x|}\right)\chi^*, \qquad x \in \mathbb{R}^3, \tag{12a}$$

$$\int_{\mathbb{R}^3} |\chi^*|^2 \ = Z\,, \tag{12b}$$

our main stability result is the following Theorem 3.1, which is only announced in our thesis and will be developed in a later paper.

Theorem 0.1. *Let* $\omega^*, \chi^*(x)$ *be the solution to* (12)*. Taken an initial data for* $\psi(x,t)$ *of the type*

$$\psi(x,0) = \varepsilon P_{\chi_{\omega(0)}} u_0 + e^{i\theta_0}\chi_{\omega(0)}, \tag{13}$$

where $u_0(x)$ *is a smooth rapidly decreasing function and* $\varepsilon > 0$ *is a sufficiently small number. Then the Cauchy problem for* (7) *has a global solution*

$$\psi(t,x) \in C(\mathbb{R}_t; H^s)\,,$$

so that

$$\psi(x,t) = P_{\chi^*}(x,t) + e^{i\theta(t)}\chi^*(x) + r(x,t)\,. \tag{14}$$

The remainder $r(x,t)$ *satisfies*

$$\lim_{t\to\infty} \|r(t)\|_{H^1} = 0$$

while $P_{\chi^*}(x,t)$ *satisfies the dispersive estimate*

$$\|P_{\chi^*}(t)\|_{L^3} \le \frac{C}{t^a}, \qquad \text{for } t > 1 \tag{15}$$

for some $0 < a < 1/2$.

The author would like to express his deep gratitude to Prof. Vladimir Georgiev for useful discussions, permanent encouragement and support during the preparation of this thesis.

Contents