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DYNAMICS OF A GAS BUBBLE IN OSCILLATING PRESSURE FIELDS

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ABSTRACT

The behaviour of a gas bubble in a viscous liquid under oscillating pressure is analyzed. The extention of the theory from the inviscid case to the viscous case is easily done with a linearlized theory, though not mathematically rigorous. Two asymptotic time solutions are shown in the cases of uniform interior at low frequencies and nonuniform interior at high frequencies. For the uniform interior, the results of calculation show that for the glycerine the effect of viscosity is unable to be neglected and becomes maximum at frequencies $\omega=10^5\sim10^6$ [Hz] for the initial bubble radius $R_0=10^{-4}$ [cm]. For water and mercury the viscous effect is negligibly small. And for less than $\omega=10^4$ [Hz] the ratio of the amplitude of perturbed radius to perturbed pressure is constant and its value is nearly $1/(3+2W)$, where W is the Webber number.

Nomenclature

$A=k_1T_\infty/\sqrt{D_1}$
$a=1+W$
$B=k_1T_\infty/R_0$
B_0=universal gas constant
$C=\alpha\delta$
C_v=specific heat at constant volume
c_0=speed of sound
D=coefficient of thermal diffusion
$\vec{D}$=deformation tensor
$E=P_0R_0(1+W)$
k=thermal conductivity
M=total mass of gas
n=number of moles
p=pressure
R=radius of bubble
r=spherical coordinate

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s = Laplace variable
T = temperature
t = time
U = bubble-wall velocity
$\vec{v}$ = velocity vector
W = Webber number
x = dimensionless radius of bubble
$\alpha = P_0/\rho_1 R_0^2$
$\beta = MC_v T_\infty / 4\pi R_0^2)$
$\delta = 4\mu/P_0$
θ = dimensionless temperature
μ = shear viscosity
ρ = density
ϕ = velocity potential
ω = frequency
$\omega_1^2 = \alpha(3+2W)$

Subscripts

$\cdot$ = time derivative
$\circ$ = initial state
∞ = infinity
in = inside
out = outside
1 = in the liquid
2 = in the gas
$\wedge$ = Laplace transform

1. Introduction

The oscillatory behaviour of gas bubbles in a viscous fluid is of a interest in connection with chemical engineering, biological systems, and transmission of sound waves in oil. In particular, the quasi-steady-state motion will be considered in this paper. Lord RAYLEIGH (1917) treated the problem of the free oscillations of a gas bubble in one of the earliest papers, taking no account of heat conduction and viscous effects. Recently, Plesset and HSIEH (1960) discussed the gas bubble dynamics in oscillating pressure fields with the effect of heat conduction, but they disregarded the effect of viscosity. The effect of viscosity automatically vanishes in the equations of motion when the motion is assumed to be spherically symmetric and the fluid to be incompressible. Hence the resultant of viscous stresses per unit volume at any point internal to the fluid vanishes. But the expression for stresses themselves on the free spherical surface of the gas bubble has the viscous term as shown by PORITSKY (1952). Although in almost all cases this viscous effect appearing only in the boundary surface may be disregarded, there are sometimes the cases when one must take this viscous effect into consideration. In the first place we discuss how this viscous effect appears in asymptotic time solutions of the equations of motion of gas bubbles. Next, by investigating whether the interior side of gas bubbles is uniform or not, we try to extend the theory of uniform interior at low frequencies to nonuiform interior at high frequencies.

2. Assumptions

The following assumptions will be made in this theory:

(a) The fluid outside the gas bubble is incompressible and Newtonian.
(b) The bubble consists of a compressible ideal gas.
(c) The bubble is so small that it holds spherical symmetry in motion.
(d) Physical quantities of the liquid and gas, such as coefficients of viscosity and thermal diffusion, perfect gas constant, specific heat at constant volume, and surface tension, do not depend on the frequencies of oscillating pressure.
(e) The acceleration of gravity may be neglected.
(f) There is no diffusion of gas through the bubble wall, and no heat source in the field.
(g) Since the perturbed pressure is small, the equations governing the field may be linearized.
(h) The liquid and gas are at rest at time $t<0$ and at time $t=0$ disturbance is suddenly given through a small oscillating pressure applied at infinity.
(i) The mutual distances among the bubbles are so long compared with the bubble radius that the interaction may be neglected.
(j) There are no boundaries at a finite distance from the center of the bubble.

3. Basic Equations

3.1 Equations of Motion for Viscous Liquid

We shall take a spherical coordinate system r, θ, and φ whose origin is at the center of a bubble. The equation of continuity is given by

$$\frac{1}{\rho}\frac{d\rho}{dt}+\operatorname{div}\vec{v}=0, \tag{1}$$

where ρ is the density of the liquid and $\vec{v}$ the velocity vector. Since from the assumption (a) $\rho=$const. for the viscous liquid outside the bubble,

$$\operatorname{div}\vec{v}=0. \tag{2}$$

Since $v_\theta=v_\phi=0$ owing to the assumption (c) of spherical symmetry, Eq. (2) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)=0 \tag{3}$$

Immediately from Eq. (3) we obtain

$$v_r=\frac{A}{r^2}. \tag{4}$$

Here A is a function of R and $\dot{R}$, where R is the radius of the bubble and $\dot{R}=dR/dt$. Since the boundary condition is given by

$$v_r=\dot{R}=U \quad \text{at} \quad r=R, \tag{5}$$

Eq. (4) becomes

$$v_r = UR^2/r^2 \tag{6}$$

Here U denotes the bubble-wall velocity.

Next we consider the equation of motion. From the assumption (c), the flow outside the spherical bubble is irrotational, so we have

$$\text{curl}\,\vec{v} = 0. \tag{7}$$

This means that $\vec{v}$ must be expressed as the gradient of a scalar potential ϕ, which is called the velocity potential.

The BERNOULLI equation for the potential flow of an incompressible fluid is the first integral of the Navier-Stokes equation and is given by

$$\frac{p}{\rho} + \frac{1}{2}(\nabla\phi)^2 - \frac{\partial\phi}{\partial t} = K(t), \tag{8}$$

where P is the pressure and $K(t)$ is a function of time only. If P_∞ is taken as the pressure in the liquid at infinity, Eq. (8) can be rewritten, as using $\phi = R^2U/r$,

$$\frac{P}{\rho} + \frac{R^4\dot{R}^2}{2r^4} - \frac{1}{r}(2R\dot{R}^2 + R^2\ddot{R}) = \frac{P_\infty(t)}{\rho}. \tag{9}$$

In particular, at the bubble wall where $r = R$, the above equation becomes

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{P_{out} - P_\infty}{\rho}, \tag{10}$$

where P_{out} is the pressure in the liquid just outside the bubble: for $\varepsilon > 0$,

$$P_{out} = \lim_{\varepsilon \to 0} P(R + \varepsilon). \tag{11}$$

Finally, the equation of energy derived from the first law of thermodynamics is given by

$$\frac{\partial T}{\partial t} + \vec{v} \cdot \text{grad}\, T = D\nabla^2 T + \frac{2\mu}{\rho C_v}\, \vec{\vec{D}} : \vec{\vec{D}}, \tag{12}$$

where T is the temperature of the liquid, C_v the specific heat at constant volume, k the thermal conductivity, μ the shear viscosity, $\vec{\vec{D}}$ the rate of deformation tensor, and D the coefficient of thermal diffusion defined by $k/(\rho C_v)$. For the inviscid case the second term on the right-hand side of Eq. (12) disappears. Eqs. (10) and (12) become important for later use.

3.2 Equations of Motion for a Gas Bubble

The gas inside the bubble may be assumed to be a perfect gas, whose compressibility of the gas plays an important role in comparison with that of the liquid outside the bubble. Hence $\text{div}\,\vec{v} \neq 0$ for the gas although $\text{curl}\,\vec{v} = 0$ still remains valid for the spherically symmetric bubble. The conservation of mass in the gas region is expressed by

$$\frac{\partial\rho}{\partial t} + \rho\,\text{div}\,\vec{v} + \vec{v} \cdot \text{grad}\,\rho = 0, \qquad (r \leqq R). \tag{13}$$

With the neglect of viscous effect, the motion in the interior region is governed by Euler's equation:

$$\frac{\partial \vec{v}}{\partial t}+\vec{v}\cdot\mathrm{grad}\,\vec{v}=-\frac{1}{\rho}\,\mathrm{grad}\,P \qquad (r\leqq R) \tag{14}$$

and the energy equation takes the following form

$$\frac{\partial T}{\partial t}+\vec{v}\cdot\mathrm{grad}\,T=D\nabla^2 T-\frac{P}{\rho C_v}\,\mathrm{div}\,\vec{v}, \qquad (r\leqq R). \tag{15}$$

The last term in Eq. (15) represents the reversible part of the stress power due to compressibility.

Furthermore, for the determination of motion of the gas we need the equation of state which is satisfied locally only because of the nonuniformity of the interior, which is now given by

$$P(r,t)=B\rho(r,t)T(r,t), \qquad (r\leqq R) \tag{16}$$

where B is the ratio of the perfect gas canstant to the gram molecular weight of the gas. Thus Eqs. (13) through (16) are the basic equations for the motion of the gas bubble with the nonuniform interior. While, if the interior of the gas bubble is assumed to be uniform, then the temperature T, the pressure P and the density ρ are uniform throughout the bubble. In this case, we have in place of Eq. (16)

$$P[R(t)]R^3(t)=NT[R(t)] \tag{17}$$

at the bubble wall, where N is a constant and is actually equal to $(3/4\pi)nB_0$. Here n is the number of moles of gas in the bubble and B_0 is the universal gas constant.

4. Boundary Conditions

The requirement of continuity of temperature gives at the bubble wall

$$T_1[R(t)]=T_2[R(t)] \tag{18}$$

where the suffixes 1 and 2 mean the state of the liquid outside the bubble and that of the gas inside the bubble, respectively. Similarly the requirement of continuity of stress gives [for example, see Knapp, 1970]

$$P_{\mathrm{in}}=P_{\mathrm{out}}+\frac{2\sigma}{R}+4\mu\frac{\dot{R}}{R}, \tag{19}$$

where σ is the surface tension and P_{in} is the pressure of the gas just inside the bubble wall: for $\varepsilon>0$,

$$P_{\mathrm{in}}=\lim_{\varepsilon\to 0}P(R-\varepsilon).$$

We should emphasize that the last term in the continuity equation of stress (19) represents the viscous effect, although the viscous terms automatically disappear in the equation of motion because of incompressibility and irrotationality. The relative importance of the viscous to the surface tension terms depends on the liquid in accordance with the ratio $2\mu\dot{R}/\sigma$, and the effects of surface tension and viscosity are negligible if $P_{\text{out}} \gg \dfrac{2\sigma}{R} + 4\mu\dfrac{\dot{R}}{R}$. But in general when collapse proceeds, and when a bubble becomes very small, we must take the effects of both viscosity and surface tension into consideration as shown by Poritsky [1952]. Another boundary condition is concerned with the heat equation (12). The requirement of continuity of heat flux gives

$$k_1\frac{\partial T_1}{\partial r} = k_2\frac{\partial T_2}{\partial r}, \qquad \text{at} \quad r = R \tag{20}$$

The last condition requiring continuity of the particle velocity at the bubble wall is

$$v_{r1}(t) = \dot{R}(t) = v_{r2}(t). \tag{21}$$

And all physical quantities must be finite at $r=0$ and remain finite as $r \to \infty$. Since the temperature $T(r, t)$ at infinity is taken to be constant,

$$T_1(\infty, t) = T_\infty \tag{22}$$

where T_∞ is the fixed temperature at a distance from the bubble. On the other hand, the initial conditions are that the system is in equilibrium for $t<0$ and a perturbation of P_∞ sets the system into motion. Therefore, for $t<0$:

$$R(t) = R_0, \tag{23}$$

$$\dot{R}(t) = \ddot{R}(t) = 0, \tag{24}$$

$$P_\infty(t) = P_0, \tag{25}$$

and

$$T_1(r, t) = T_2(r, t) = T_\infty, \tag{26}$$

where R_0 is the equilibrium radius of the bubble and P_0 is the equilibrium pressure in the liquid. If the disturbance is small, then the governing equations of the system may be linearized. Since all the subsequent mathematical operations become linear, the physically significant solution is obtained by taking the real part of the solution with complex quantities. Therefore, if the disturbance is caused by a small oscillating pressure at infinity, then the pressure may be expressed as

$$P_\infty(t) = P_0[1 + \varepsilon(t)] = P_0[1 + \varepsilon_0 e^{i\omega t}], \qquad t>0, \tag{27}$$

where ε_0 is a positive constant much smaller than unity.

5. Case A. Bubble Dynamics with Uniform Interior

The condition of uniformity inside the gas bubble makes the present problem simple, because we need the knowledge of the physical quantities for the gas such as temperature, pressure and so on, only at the bubble wall. The boundary condition for the heat flux, Eq. (20), at the bubble wall can be rewritten as, after integration with the aid of Eq. (15),

$$MC_v\frac{dT_2}{dt}=4\pi R^2k_1\left(\frac{\partial T_1}{\partial r}\right)_{r=R}-4\pi R^2P_2(R)\dot{R}(t),\quad \text{at}\quad r=R \tag{28}$$

where M is the total mass of gas which is assumed to be constant.

5.1 Linearized System with Dimensionless Parameters

The linearization procedure is carried out with respect to the equilibrium configuration and is based on the smallness of ε_0 in comparison with unity. We now define the perturbed quantities by

$$x=(R-R_0)/R_0,$$

$$\theta_1=(T_1-T_\infty)/T_\infty,$$

$$\theta_2=(T_2-T_\infty)/T_\infty,$$

and

$$\frac{p}{a}=[P_2-P_2(R_0)]/P_2(R_0)$$

where x, θ_1, θ_2 and p are of the same order as ε_0, viz. small compared with unity. And the constant a is given by

$$a=1+W$$

where

$$W=2\sigma/P_0R_0,$$

W is called the Webber number.

The linearization of the dynamic equation (10) with the boundary condition (19) gives

$$\ddot{x}+\alpha\delta\dot{x}-\alpha Wx=\alpha(p-\varepsilon) \tag{29}$$

where

$$\alpha=\frac{P_0}{\rho_1R_0{}^2}\qquad\text{and}\qquad\delta=\frac{4\mu}{P_0}.$$

Similarly the linearization of the energy equation (12) gives

$$\frac{\partial\theta_1}{\partial t}=D_1\nabla^2\theta_1. \tag{30}$$

This equation is the ordinary heat equation without convection. The linearization of the gas equation of state, Eq. (17), gives

$$\frac{p}{a}=\theta_2-3x \tag{31}$$

and the boundary condition, Eq. (28), becomes

$$\beta\dot{\theta}_2=-aP_0R_0\dot{x}+k_1T_\infty\left(\frac{\partial\theta_1}{\partial r}\right)_{r=R_0}, \tag{32}$$

since $P_2(R_0)-P_0=\frac{2\sigma}{R_0}$ in a steady state, where

$$\beta=MC_vT_\infty/4\pi R_0^2.$$

We have in a similar way

$$x=\dot{x}=\ddot{x}=\theta_1=\theta_2=0, \qquad \text{for} \quad t<0 \tag{33}$$

and

$$\theta_1(\infty,t)=0, \qquad \text{for all} \quad t. \tag{34}$$

The effect of viscosity expressed by δ appears only on the dynamic equation (29). The rest are the same as those for the liquid without viscosity [PLESSET and HSIEH, 1960].

5.2 The Formal Solution

Here we have four unknowns; p, x, θ_1 and θ_2, and four equations; Eqs. (29), (30), (31) and (32). We shall introduce the Laplace transform of these unknowns as follows

$$\hat{p}=L(p)=\int_0^\infty pe^{-st}dt$$

and similarly,

$$\hat{x}=L(x)\,;\quad \hat{\theta}_1=L(\theta_1)\,;\quad \hat{\theta}_2=L(\theta_2),$$

furthermore, since $\varepsilon=\varepsilon_0e^{i\omega t}$,

$$\hat{\varepsilon}=L(\varepsilon)=\varepsilon_0/(s-i\omega).$$

With the initial conditions given by Eq. (33), Eq. (29) becomes, after the elimination of p with the help of Eq. (31),

$$[s^2+\alpha\delta s+\alpha(3+2W)]\hat{x}=\alpha[(1+W)\hat{\theta}_2-\hat{\varepsilon}]. \tag{35}$$

The equation (30) gives for the spherical symmetry

$$s(r\hat{\theta}_1)=D_1\frac{d^2}{dr^2}(r\hat{\theta}_1) \tag{36}$$

with $\hat{\theta}_1(r=\infty)=0$ from Eq. (34). Since $\theta_2(t)=\theta_1(R,t)$, we have $\hat{\theta}_2=\hat{\theta}_1(R)$. Hence the Laplace transform of the boundary condition given by Eq. (32) becomes

$$s\beta\hat{\theta}_1=-aP_0R_0s\hat{x}+k_1T_\infty\left(\frac{d}{dr}\hat{\theta}_1\right)_{r=R}, \qquad \text{at} \quad r=R. \tag{37}$$

The solution for Eq. (36) is readily found to be

$$\frac{\hat{\theta}_1}{\hat{\varepsilon}}=\frac{\alpha Es}{P(\sqrt{s})}\frac{R_0}{r}\exp\left[-(r-R_0)(s/D_1)^{\frac{1}{2}}\right], \tag{38}$$

where P is a polynomial defined by

$$P(u)=(\beta u^2+Au+B)(u^4+Cu^2+\omega_1^2)+\alpha E(1+W)u^2. \tag{39}$$

The constants which have been introduced are defined as follows:

$$\omega_1^2=\alpha(3+2W);$$

$$A=k_1T_\infty/\sqrt{D_1};$$

$$B=k_1T_\infty/R_0;$$

$$C=\alpha\delta; \quad \text{(effect of viscosity)}$$

$$E=P_0R_0(1+W).$$

And we now find from Eqs. (37) through (39) that

$$\frac{\hat{x}}{\hat{\varepsilon}}=-\frac{\alpha(\beta s+A\sqrt{s}+B)}{P(\sqrt{s})} \tag{40}$$

Since $\hat{\varepsilon}=\varepsilon_0/(s-i\omega)$, the formal solutions to the problem are obtained by the inversion of Eqs. (38) and (40). If the roots of $P(u)=0$ are $-a_j$ $(j=1,2,\cdots,6)$ and $a_7=\omega^{1/2}e^{i\pi/4}$ and $a_8=-\omega^{1/2}e^{i\pi/4}=\omega^{1/2}e^{-i3\pi/4}$, then the inversions are given by

$$\theta_1(r,t)=-\frac{R_0}{r}\sum_{j=1}^{8}a_jb_j\,\text{Erfc}\,[(r-R_0)/2\sqrt{D_1t}+a_j\sqrt{t}\,]$$
$$\cdot\exp\,[a_j(r-R_0)/\sqrt{D_1}+a_j^2t] \tag{41}$$

and

$$x(t)=-\sum_{j=1}^{8}a_jc_j\,\text{Erfc}\,(a_j\sqrt{t})\exp\,(a_j^2t), \tag{42}$$

where b_j and c_j are given by the partial fractians

$$\frac{\alpha E\varepsilon_0 s}{(s-i\omega)P(\sqrt{s})}=\sum_{j=1}^{8}\frac{b_j}{\sqrt{s}+a_j},$$

$$\hat{x}(s)=\sum_{j=1}^{8}\frac{c_j}{\sqrt{s}+a_j}.$$

Here the complementary error function is defined by

$$\mathrm{Erfc}\,(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty \exp\,(-y^2)dy.$$

The unknowns θ_2 and p are also given by similar forms:

$$\theta_2(r,t)=-\sum_{j=1}^{8}a_jb_j\,\mathrm{Erfc}\,(a_j\sqrt{t})\exp\,(a_j^2t) \tag{43}$$

and

$$\begin{aligned}p(t)&=(1+W)(\theta_2-3x)\\&=(1+W)\sum_{j=1}^{8}a_j(3c_j-b_j)\,\mathrm{Erfc}(a_j\sqrt{t})\exp\,(a_j^2t).\end{aligned} \tag{44}$$

The effect of viscosity appears only on the coefficient C in the polynomial $P(u)$. If we set C to be zero, then the formal solutions expressed by Eqs. (41) through (44) are the same as those for the inviscid liquid outside the bubble. But this polynomial $P(u)$ is essential for the behaviour of the gas bubble.

5.3 Asymptotic Time behaviour

The formal solutions expressed by Eqs. (41) through (44) are not convenient to visualize the behaviour of the physical quantities in the oscillating field because the complementary error function has complex arguments and the roots $-a_j$ of $P(u)=0$ are found only by numerical method when specific values for the physical constants are used. But the asymptotic expressions obtained by the application of the method of steepest descent enable to avoid these difficulties and give the steady-state solutions of significance for the thermodynamic relations. As mentioned above, the behaviour of the roots $-a_j$ is very important to examine the asymptotic behaviour of the solutions. For the inviscid case ($\delta=0$, and hence $C=0$), the six roots $-a_j$ of the algebraic equation $P(u)=0$ expressed by Eq. (39), as is well known, lie in the sector

$$|\arg\,(-a_j)|>\pi/4, \qquad \text{or} \qquad |\arg a_j|<3\pi/4$$

by the principle of the argument in the theory of complex analytic functions. But for the viscous case, it is very difficult to do the same discussion as done by PLESSET and HSIEH (1960) because when $u=ve^{i\pi/4}$ the polynomial $P(u)$ becomes

$$P(ve^{i\pi/4})=M(v)+iN(v)$$

where $M(v)$ and $N(v)$ are the real and the imaginary parts of $P(u)$, respectively:

$$M(v)=-\left(\frac{A}{\sqrt{2}}v+B\right)(v^4-\omega_1^2)-C\left(\beta v+\frac{A}{\sqrt{2}}\right)v^2,$$

$$N(v)=\left(\beta v^2+\frac{A}{\sqrt{2}}v\right)(-v^4+\omega_1^2)+C\left(\frac{A}{\sqrt{2}}v+B\right)v^2+\alpha E(1+W)v^2.$$

Here if $C=0$, then $P(ve^{i\pi/4})=0$ has no positive real root, that is to say, $M(v)=0$ and $N(v)=0$ have no common positive real root. This discussion is valid for the small C, i.e., for slightly viscous liquids. We may also say that except the very special physical values, $M(v)=0$ and $N(v)=0$ have no common real root. Therefore we may use the principle of the argument for the viscous case as well as for the inviscid case. After the direct inversion integral of the transformed solution and the change of the contour of integration by use of CAUCHY's residue theorem, the asymptotic expressions follow with the aid of WATSON's lemma (CARRIER et al. 1966) for large t

$$\theta_1(r,t)=\frac{\alpha\varepsilon_0 E\omega}{P(\omega^{1/2}e^{i\pi/4})}\,\frac{R_0}{r}\exp\left[-(r-R_0)/(i\omega/D_1)^{\frac{1}{2}}\right]\cdot\exp\left[i(\omega t+\pi/2)\right]+0(t^{-2})\,; \tag{45}$$

$$x(t)=-\frac{\alpha\varepsilon_0(\beta\omega+A\omega^{1/2}e^{-i\pi/4}-iB]}{P(\omega^{1/2}e^{i\pi/4})}\cdot\exp\left[i(\omega t+\pi/2\right]+0(t^{-1})\,; \tag{46}$$

$$\theta_2(t)=\frac{\alpha\varepsilon_0 E\omega}{P(\omega^{1/2}e^{i\pi/4})}\exp\left[i(\omega t+\pi/2)\right]+0(t^{-2})\,; \tag{47}$$

and

$$p(t)=\frac{\alpha\varepsilon_0(1+W)[(E+3\beta)\omega+3A\omega^{1/2}e^{-i\pi/4}-3iB]}{P(\omega^{1/2}e^{i\pi/4})}\cdot\exp\left[i(\omega t+\pi/2)\right]+0(t^{-2}). \tag{48}$$

These asymptotic expressions are similar to those for the inviscid case only except the form of the polynomial $P(u)$. If C in the polynomial $P(u)$ goes to zero, all of Eqs. (45) through (48) become asymptotic expressions for the inviscid case.

5.4 Examples

As examples for the asymptotic time behaviour with viscosity, in the first place we shall study the dimensionless amplitude of the perturbed radius in the expression of Eq. (46). Now, let us assume that the gas in the bubble is air with the initial temperature 20°C. And as the liquids outside the bubble we adopt water as the most general liquid, glycerine as one of the high viscous liquids and mercury as the liquid with the large surface tension. These fluids have the physical quantities as shown in Table 1 (JSME, 1966).

Table 1. Physical quantities of liquids and air at 20°C.

		air	water	glycerine	mercury	unit
density	ρ	0.00116	0.998	1.264	13.546	g/cm³
thermal conductivity	k	0.000257	0.00594	0.002849	0.0858	joule/cm. s°C
thermal diffusivity	D	0.219	0.00142	0.000944	0.0456	cm²/s
specific heat at constant volume	C	0.718				joule/g°C
surface tension	σ		0.0739	0.0637	0.481	g/cm
shear viscosity	μ	0.000181	0.01	14.896	0.0155	g/cm. s

Here we shall try to examine how the effects of viscosity and surface tension appear in the basic equations. Since Eq. (29) becomes, with the derivative $\dot{x}=dx/dt^*$ with respect to the dimensionless time $t^*=\omega t$,

$$\frac{\omega^2}{\alpha}\ddot{x}+\delta\omega\dot{x}-Wx=p-\varepsilon, \tag{49}$$

the dimensionless numbers such as W, $V=\delta\omega$ and ω^2/α show respectively the effects of surface tension, viscosity and frequency in the oscillating field. Hence these dimensionless numbers are peculiarly suitable for describing the effects on the dimensionless amplitude of the perturbed radius x. And from Eq. (46) we also have

$$|x|=\frac{|\alpha\varepsilon_0[\beta\omega+A\omega^{1/2}e^{-i\pi/4}-iB]|}{|P(\omega^{1/2}e^{i\pi/4})|}. \tag{50}$$

These equations (49) and (50) become important later in the discussion of the effects of viscosity and surface tension.

Next we consider two models in order to compare the effects of viscosity and surface tension with that of surface tension, which are respectively named VW and W models. In Figs. (a) the solide lines are for the VW model and the broken lines are for the W model. For the very small bubble considered here the surface tension usually plays a more important role in motion than the viscosity, so we need not consider the models with the effect of viscosity alone. Next we shall give some explanations for each figure.

Fig. a-1 shows for water how the ratio of amplitude of the radius $|x|$ to the input perturbed pressure ε_0 depends on the initial radius. Generally speaking, the larger the initial radius, the smaller the ratio $|x|/\varepsilon_0$. Since the VW model which contains the effect of viscosity differs little from the W model, we can not distinguish between them in this scale.

Fig. a-2 shows the similar results, namely, the change of $|x|/\varepsilon_0$ when the liquid outside the bubble is glycerine. In this case the effect of viscosity appears considerable at the high frequencies and at the small initial radii.

Fig. a-3 shows the ratio $|x|/\varepsilon_0$ with a constant initial radius changes according

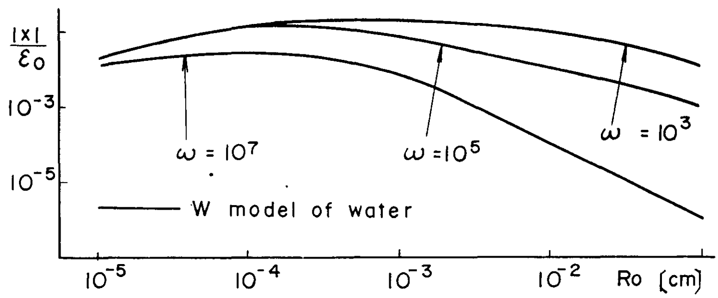

Fig. a-1. The ratio of amplitude of the radius $|x|$ to the input perturbed pressure ε_0 for water.

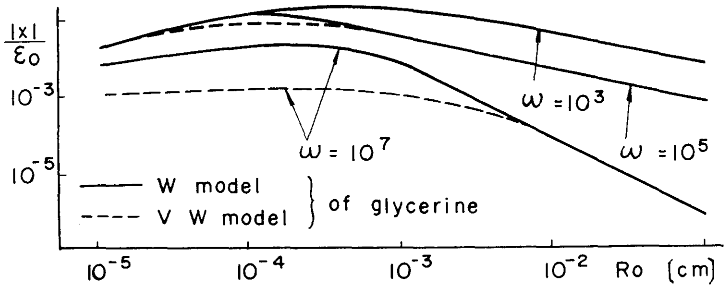

Fig. a-2. The ratio of amplitude of the radius $|x|$ to the input perturbed pressure ε_0 for glycerine.

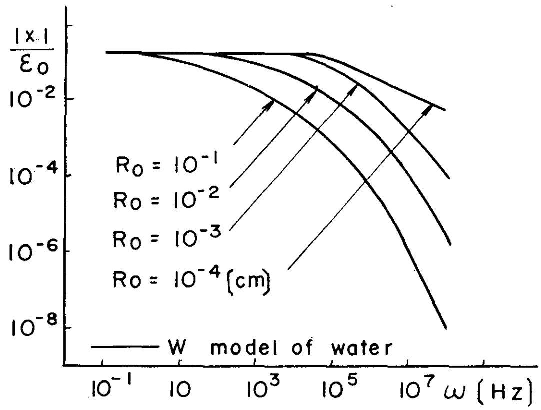

Fig. a-3. The dependence of the ratio $|x|/\varepsilon_0$ on the frequencies.

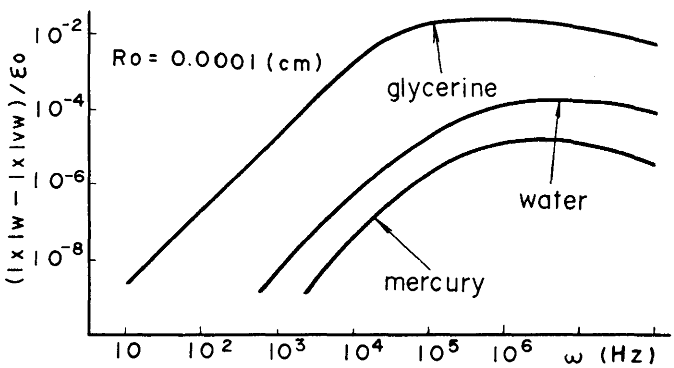

Fig. a-4. The effect of viscosity for the fluctuation of the bubble wall.

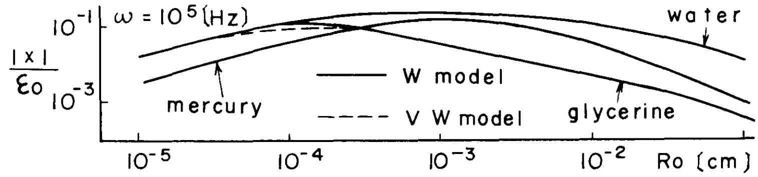

Fig. a-5. The ratios $|x|/\varepsilon_0$ for three liquids at the constant frequency $\omega=10^5$ [Hz].

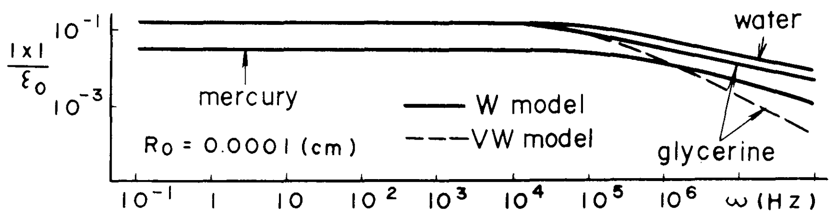

Fig. a-6. The ratios $|x|/\varepsilon_0$ for three liquids at the constant initial radius $R_0=0.0001$ [cm].

to frequencies for water. For less than $\omega=10^2$ [Hz] the ratio $|x|/\varepsilon_0$ is a constant value (0.3) independent of the initial radii. And we have the smaller ratio $|x|/\varepsilon_0$ at higher frequencies for the larger initial radii.

Fig. a-4 shows how the effect of viscosity appears for the different liquids surrounding the bubble at the initial radius $R_0=0.0001$ [cm]. There is the maximum effect of viscosity near the frequency $\omega=10^5\sim10^6$ [Hz] independent of kind of the liquid used. This fact can be explained by Eqs. (49) and (50).

Fig. a-5 shows how the ratio $|x|/\varepsilon_0$ changes for the initial radii and the different liquids outside the bubble at the constant frequency $\omega=10^5$ [Hz]. In comparison with the curves for mercury and glycerine, the ratio $|x|/\varepsilon_0$ for mercury of which surface tension is large is smaller than that for glycerine in the range of the small initial radii. But the ratio $|x|/\varepsilon_0$ for glycerine with large viscosity is smaller than that for mercury in the range of the large radii.

Fig. a-6 shows how the ratio $|x|/\varepsilon_0$ changes according to the frequencies for the three different liquids outside the bubble at the initial radius $R_0=0.0001$ [cm]. The ratio $|x|/\varepsilon_0$ becomes a constant value for less than the frequencies 10^4 [Hz]. The constant value can be obtained in the limit $\omega\rightarrow0$ from Eq. (50) to be $1/(3+2W)$. And its value is 0.17 for water, 0.18 for glycerine, and 0.046 for mercury. But it has a tendency that the ratio $|x|/\varepsilon_0$ becomes small for high frequencies and especially the effect of viscosity is remarkable for glycerine.

Assumption of Uniform Interior

At the high frequency the assumption that the interior of the bubble is uniform seems not to be correct. In order that the interior of the bubble is uniform, for example; the propagation velocity of acoustic pressure although it is quite fast, must be larger than R_0/τ, where τ is the characteristic period of vibration. If the greater part of the change of pressure on the surface of the bubble caused by the vibration can not reaches at the center of the bubble within the characteristic period, the assumption of uniformity is not valid. Therefore, we must examine the validity of this theory based on the uniformity. And we can estimate the range of radius from the uniformity assumption of $c_0\tau \gg R_0$, where c_0 is the speed of sound. Therefore, we shall obtain the following inequality

$$2\pi c_0 \gg \omega R_0.$$

Since the speed of sound is 340 m/sec at 15°C ($2\pi c_0 = 2.14\times10^5$ cm/sec), we may say that the uniformity assumption is a good assumption in the range of $R_0 < 10^{-1}$ [cm] at $\omega = 10^4$ [Hz]. If $\omega R_0 \gg 2\pi c_0$, the interior state of the bubble is a function of space and time. Then we must analyze the interior state of the bubble more precisely.

6. Case B. Bubble Dynamics with Nonuniform Interior

If the change of state is caused in the range of $\omega R_0 \gg 2\pi c_0$, the requirment is no longer imposed that conditions are uniform throughout the bubble. The basic equations for the formulation of this problem are Eqs. (13) through (16) for the bubble of a perfect gas. And the basic equations for the surrounding liquid and the boundary conditions are the same ones as for the uniform case.

6.1 Linearized System with Dimensionless Parameters

The linearization procedure is carried out in the same way as before with the following definitions;

$$x = (R - R_0)/R_0,$$

$$\theta_1 = (T_1 - T_\infty)/T_\infty,$$

$$\theta_2 = (T_2 - T_\infty)/T_\infty,$$

$$\eta = (\rho_2 - \rho_0)/\rho_0,$$

$$\varepsilon(t) = (P_\infty(t) - P_0)/P_0,$$

and

$$p + W = (P_2 - P_0)/P_0,$$

where x, θ_1, θ_2, η and p are small quantities compared with one and are of the same order as ε.

Linearization of the governing equations of the system yields the following equations upon neglect of second-order terms of ε with the same dimensionless parameters as before. For the interior side of the gas bubble, the equation of continuity, Eq. (13), becomes

$$\frac{\partial \eta}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_{r2}) = 0, \tag{51}$$

where v_{r2} is the r-component of the velocity of the perfect gas. And also the equation of motion, Eq. (14), becomes

$$\frac{\partial v_{r2}}{\partial t} = -\frac{P_0}{\rho_0}\frac{\partial p}{\partial r}. \tag{52}$$

The linearized local equation of energy, Eq. (15), is

$$\frac{1}{D_2}\frac{\partial \theta_2}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{\partial \theta_2}{\partial r} - \frac{P_0 a v_{r2}}{k_2 T_\infty}\right)\right], \tag{53}$$

while the equation of state, Eq. (16), becomes

$$p/a = \eta + \theta_2, \tag{54}$$

and the equilibrium condition in the bubble is given by

$$aP_0 = B\rho_0 T_\infty.$$

On the other hand, for the liquid surrounding the bubble, the linearized form of the energy equation, Eq. (12), is

$$\frac{1}{D_1}\frac{\partial}{\partial t}(r\theta_1) = \frac{\partial^2}{\partial r^2}(r\theta_1). \tag{55}$$

Finally the dynamic equation (10) of the boundary of the bubble becomes with the aid of Eq. (19)

$$\ddot{x} + \alpha\delta\dot{x} - \alpha W x = \alpha[p(R_0, t) - \varepsilon]. \tag{56}$$

Furthermore, we will find that the boundary conditions [Eqs. (18), (20), and (21)] expressing the continuity of the temperature, the heat flux, and the rate of displacement become

$$\theta_1(R_0, t) = \theta_2(R_0, t), \tag{57}$$

$$k_1\left(\frac{\partial \theta_1}{\partial r}\right)_{R_0} = k_2\left(\frac{\partial \theta_2}{\partial r}\right)_{R_0}, \tag{58}$$

and

$$v_{r1}(R_0, t) = R_0\dot{x} = v_{r2}(R_0, t) \tag{59}$$

respectively. Here the boundary R is approximated by the initial radius R_0 in order to linearize the equation of the system.

6.2 The Asymptotic Solutions

As in the previous case, we define the Laplace-transformed functions as follows:

$$\hat{x} = L(x);\quad \hat{\theta}_1 = L(\theta_1);\quad \hat{\theta}_2 = L(\theta_2);\quad \hat{p} = L(p);$$

$$\hat{\eta} = L(\eta);\quad \hat{v} = L(v_{r2});\quad \hat{\varepsilon} = L(\varepsilon).$$

After the Laplace-transformation of Eqs. (51) through (56) with the initial conditions of the unperturbed state;

$$x(r,0)=\dot{x}(r,0)=\theta_1(r,0)=\theta_2(r,0)=v_{r2}(r,0)=\eta(r,0)=p(r,0)=0,$$

we have the following equations;

$$s\hat{\eta}+\frac{1}{r^2}\frac{d}{dr}(r^2\hat{v})=0, \qquad \text{for} \quad r\leqq R_0 \tag{60}$$

$$\rho_0 s\hat{v}=-P_0\frac{d\hat{p}}{dr}, \qquad \text{for} \quad r\leqq R_0 \tag{61}$$

$$\frac{s}{D_2}\hat{\theta}_2=\frac{1}{r}\frac{d^2}{dr^2}(r\hat{\theta}_2)-\frac{P_0 a}{k_2 T_\infty}\frac{1}{r^2}\frac{d}{dr}(r^2\hat{v}), \qquad \text{for} \quad r\leqq R_0 \tag{62}$$

$$\hat{p}/a=\hat{\eta}+\hat{\theta}_2, \qquad \text{for} \quad r\leqq R_0 \tag{63}$$

$$\frac{d^2}{dr^2}(r\hat{\theta}_1)=\frac{s}{D_1}r\hat{\theta}_1, \qquad \text{for} \quad r\geqq R_0 \tag{64}$$

$$(s^2+\alpha\delta s-\alpha W)\hat{x}=\alpha[\hat{p}(R_0)-\hat{\varepsilon}], \qquad \text{for} \quad r=R_0. \tag{65}$$

We also have the following boundary conditions at $r=R$:

$$\hat{\theta}_1(R_0)=\hat{\theta}_2(R_0);$$

$$k_1\left(\frac{d\hat{\theta}_1}{dr}\right)_{R_0}=k_2\left(\frac{d\hat{\theta}_2}{dr}\right)_{R_0};$$

$$\hat{v}(R)=R_0 s\hat{x}.$$

Then we shall solve the system of these equations in order to find the asymptotic solutions. Eliminating $\hat{\theta}_2$ and $\hat{v}$ in Eq. (62) in virtue of Eqs. (61), (63), and (60) yields the following equation with respect to $\hat{p}$;

$$\frac{d^4}{dr^4}(r\hat{p})-F(s)\frac{d^2}{dr^2}(r\hat{p})+G(s)(r\hat{p})=0,$$

where

$$F(s)=\frac{\rho_0 s^2}{P_0 a}+\left(\frac{p_0 a}{k_2 T_\infty}+\frac{1}{D_2}\right)s, \qquad G(s)=\frac{\rho_0 s^3}{P_0 a D_2}.$$

This equation may be written in the form

$$\left(\frac{d^2}{dr^2}-\lambda_1^2\right)\left(\frac{d^2}{dr^2}-\lambda_2{}^2\right)(r\hat{p})=0,$$

where

$$\lambda_{1,2}^2(s)=\frac{1}{2}\{F^2(s)\pm\sqrt{F^2(s)-4G(s)}\}.$$

The general solution for $\hat{p}$ may be now written as

$$\hat{p}=\frac{1}{r}[A_1 \cosh \lambda_1 r+A_2 \sinh \lambda_1 r+A_3 \cosh \lambda_2 r+A_4 \sinh \lambda_2 r],$$

since $\hat{p}$ is to finite at $r=0$, $A_1=-A_3$. Equations (60) and (61) give

$$\hat{\eta}=\frac{P_0}{\rho_0 s^2}\frac{1}{r}\frac{d^2}{dr^2}(r\hat{p}).$$

Since $\hat{\eta}$ is also finite at $r=0$, and since in general $\lambda_1=\lambda_2$, we also have $A_1=-A_3=0$. Thus

$$\hat{p}=\frac{1}{r}[A_2 \sinh \lambda_1 r+A_4 \sinh \lambda_2 r]\,; \tag{66}$$

$$\hat{\eta}=\frac{P_0}{\rho_0 s^2}\frac{1}{r}[\lambda_1^2 A_2 \sinh \lambda_1 r+\lambda_2^2 A_4 \sinh \lambda_2 r]. \tag{67}$$

From Eqs. (61), (63), and (64) we obtain the rest solutions for $\hat{v}$, $\hat{\theta}_2$, and $\hat{\theta}_1$, respectively;

$$\hat{v}=-\frac{P_0}{\rho_0 s}\left[A_2\left(\frac{\lambda_1 \cosh \lambda_1 r}{r}-\frac{\sinh \lambda_1 r}{r^2}\right)+A_4\left(\frac{\lambda_2 \cosh \lambda_2 r}{r}-\frac{\sinh (\lambda_2 r)}{r^2}\right)\right], \tag{68}$$

$$\hat{\theta}_2=\frac{1}{r}\left[A_2\left(\frac{1}{a}-\frac{P_0\lambda_1^2}{\rho_0 s_2}\right)\sinh \lambda_1 r+A_4\left(\frac{1}{a}-\frac{P_0\lambda_2^2}{\rho_0 s^2}\right)\sinh \lambda_2 r\right], \tag{69}$$

$$\hat{\theta}_1=\frac{A_0}{r}=\exp[-(s/D_1)^{1/2}r], \tag{70}$$

where the requirement that $\hat{\theta}_1$ remains finite as $r\to\infty$ has been imposed and it is easily verified that $v\to 0$ as $r\to\infty$. It should be noticed that these solutions expressed by Eqs. (66) through (70) are the same ones obtained by PLESSET and HSIEH for the inviscid case but for the constants A_0, A_2, and A_4 which are functions of the parameter s. And these constants are determined by the equation of motion of the bubble wall [Eq. (65)], which has the effect of viscosity, together with the boundary conditions at $r=R$. After some calculations we find

$$A_2(s)=\frac{1}{\Delta(s)}\frac{\alpha\hat{\varepsilon}sR_0}{Q(s)}R(s,\lambda_2)S(s,\lambda_2),$$

$$A_4(s)=-\frac{1}{\Delta(s)}\frac{\alpha\hat{\varepsilon}sR_0}{Q(s)}R(s,\lambda_1)S(s,\lambda_1),$$

where

$$\Delta(s)=\Delta_1(\lambda_1,\lambda_2)-\Delta_1(\lambda_2,\lambda_1),$$

$$Q(s)=s^2+\alpha\delta s-\alpha W,$$

$$R(s,\lambda_i)=\frac{1}{a}-\frac{P_0\lambda_i^2}{\rho_0 s^2}, \qquad (i=1,2)$$

$$S(s,\lambda_i)=k_2(\lambda_i \cosh \lambda_i R_0-\sinh \lambda_i R_0/R_0)+k_1[\sqrt{(s/D_1)}+1/R_0]\sinh \lambda_i R_0,$$

and

$$\Delta_1(\lambda_1, \lambda_2)=\left[\frac{P_0\lambda_1}{\rho_0 R_0 s}\cosh\lambda_1 R_0-\left(\frac{P_0}{\rho_0 R_0^2 s}-\frac{\alpha s}{Q(s)}\right)\sinh\lambda_1 R_0\right]\cdot S(s, \lambda_2)R(s, \lambda_2).$$

Here the polynomial $Q(s)$ has the effect of viscosity. And, in principle, we can determine the quantities of interest such as p, η, v, θ_2, or θ_1 by the inverse Laplace transformation. For example,

$$p(r, t)=\frac{1}{2\pi i}\int_{Br}\hat{p}(r, s)e^{st}ds$$

$$=\frac{1}{2\pi i}\int_{Br}\frac{1}{r}[A_2(s)\sinh\lambda_1 r+A_4(s)\sinh\lambda_2 r]e^{st}ds \tag{71}$$

where the path of integration $[Br]$ is the Bromwich path. Only the asymptotic behaviour for large t will be considered here. After the similar discussion to one done by PLESSET and HSIEH, the following asymptotic solutions are obtained for the viscous case:

$$p(r, t)\simeq\frac{1}{r}[A_2(i\omega)\sinh\lambda_1(i\omega)r+A_4(i\omega)\cdot\sinh\lambda_2(i\omega)r]e^{i\omega t}, \tag{72}$$

$$\theta_2(r, t)\simeq\frac{1}{r}[A_2(i\omega)R(i\omega, \lambda_1(i\omega))\sinh\lambda_1(i\omega)r$$

$$+A_4(i\omega)R(i\omega, \lambda_2(i\omega))\sinh\lambda_2(i\omega)r]e^{i\omega t}, \tag{73}$$

$$x(r, t)\simeq\frac{P_0}{\rho_0 R_0\omega^2}\left\{A_2(i\omega)\left[\frac{\lambda_1(i\omega)\cosh\lambda_1(i\omega)R_0}{R_0}-\frac{\sinh\lambda_1(i\omega)R_0}{R_0^2}\right]\right.$$

$$\left.+A_4(i\omega)\left[\frac{\lambda_2(i\omega)\cosh\lambda_2(i\omega)R_0}{R_0}-\frac{\sinh\lambda_2(i\omega)R_0}{R_0^2}\right]\right\}e^{i\omega t}, \tag{74}$$

where the quantities $A_2(i\omega)$, $A_4(i\omega)$, $\lambda_1(i\omega)$, and $\lambda_2(i\omega)$ are obtained from the expression given in the foregoing by replacing the argument s by $i\omega$. For these high frequencies we can also easily calculate the pressure and temperature in the bubble, and the radius of the bubble wall in the same way as in the case A.

7. Conclusions

A new analysis of viscous effect on the behaviour of a gas bubble in a viscous liquid under oscillating pressure is proposed in this paper. The effect of viscosity appears only in the characteristic polynomial. Formal solutions expressed by Eqs. (41) through (44) and asymptotic time behaviours expressed by Eqs. (45) through (48) are the same as in the inviscid case except for $P(u)$. For the uniform interior, the results of calculation show that for glycerine the effect of viscosity is unable to be neglected and that for water and mercury the viscous effect is negligibly small. The assumption of uniformity is no valid at high frequencies. Asymptotic solutions for nonuniformity inside the gas bubble are shown by Eqs. (72), (73) and (74).

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