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On Virtual Mass of Water contained in a Rectangular Tank whose Side-Walls are Vibrating — II

(Received Dec. 18, 1959)

Fumiki KITO*

Abstract

The author has presented, in the previous paper, theoretical formulae which give the value of virtual mass of water, with regard to vibration of side-walls of a rectangular water tank, full of water. The theoretical formulae were obtained by calculation of kinetic energy of water, which is in vibration induced by the vibration of side-walls. This was an approximate calculation, and not a rigorous one. So, there arose the question about the degree of approximation of the given formulae. In the present paper, this question is examined by taking into account higher modes of vibration of side-walls, and using the minimum principle of Lord Rayleigh. From the result of calculation, we see that the formulae for virtual mass of water in the previous paper are sufficiently accurate for practical use, at least for water tanks having usual proportions.

The formulae for virtual mass, for the case in which the water is only partially filled, is also given here.

I. Introduction

Let us consider a rectangular tank which is filled up with water. When side-walls of the tank vibrate, the water contained in the tank makes also a vibratory motion. Owing to this fact, there arises the effect called the virtual mass of water, upon the natural frequency of vibration of rectangular flat plates which constitute the side-walls of the tank. In the author's previous paper,[1] the formulae which give theoretical value of this virtual mass were presented.

These formulae were not, however, constructed on the ground of a rigorous theory with respect to combined vibration of elastic plate and fluid mass, but were result of an approximate calculation based on evaluation of kinetic energy of vibratory motion of water. Consequently, there arose the question as to whether these formulae are accurate enough for practical uses. In this second report, the author intends to give the result of his examination of the degree of approximation, by means of the method called *Lord Rayleigh's minimum principle.* According to the

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1) F. Kito; On Virtual Mass of Water contained in a Rectangular Tank whose Side-Walls are Vibrating, This Proceedings, Vol. 11, No. 40. (1958)

result of this examination approximate formulae given in the previous paper are seen to be accurate enough for practical purposes, at least for those water tanks having usual dimensions.

Also, in the previous paper, the water was assumed to be completely or almost completely filled up in the tank. Since then, there arose a request for formulae of same kind, for the case in which the water is only partially filled in the tank. The author has made the formulae for virtual mass for this case of partially-filled tank, which is also reported herin.

As in the previous paper, the water is assumed to be an incompressible ideal fluid, and the vibration to be of infinitesimally small amplitude. Also, we use the following notations:——

ϕ = velocity potential of vibratory motion of water, w = transverse displacement of rectangular plate, A = vibration amplitude of ditto, ω = angular frequency of vibration, L = length of the rectangular water tank; H = its height, B = its breadth, ρ_m = density of material composing the rectangular plate, ρ_w = density of water, $W=\omega A$ (amplitude of transverse velocity of the rectangular plate); T_m = kinetic energy of the vibrating plate, for one sheet of the panel $H\times B$, T_w = kinetic energy of water, being the value referred to one panel of side-wall (only one face of which is in contact with water), $V=LBH$ = whole volume of the tank, U = strain energy of rectangular elastic plate, for one panel of dimension $H\times B$, h = depth of water, when water is partially filled, $\alpha=h/H$, $m=\pi/L$, $s=\pi/H$. We take rectangular coordinate axes (o, x, y, z) as sketched in Fig. 1. In the previous paper, the four specified cases (A), (B), (C) and (D), were examined. In the present report, however, the two cases (A) and (C), which is chosen from among the abovementioned four cases, will be taken up, and the degree of accuracy of the approximate formulae will be here examined.

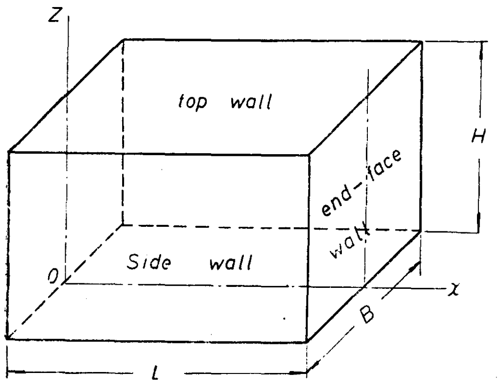

Fig. 1. Sketch of a rectangular water tank

Thus, formerly, the displacement of the rectangular plate was taken to be (tentatively) given by:——

$$w_1=A_1 \sin mx \sin sz \sin \omega t \tag{1}$$

while in the present paper, supplementary term representing the modification of wave-form will be added to Eq. (1). And, it will be examined how much this modification (or correction) term should amount to.

II. *Case A*. Rectangular Tank completely filled up with Water (two Side-Walls making in-phase Vibration)

The transverse velocity of rectangular plate, corresponding to displacement as given by Eq. (1), is

$$\frac{dw_1}{dt} = W_1 \sin mx \sin sz \cos \omega t \qquad (2)$$

where $W_1 = \omega A_1$. Induced by this motion of side-walls, the water contained in the tank will also vibrate. This vibrational motion of water is expressed by a velocity potential ϕ_1, which satisfy *Laplace's equation* together with suitable boundary conditions. As was shown in the previous paper, value of ϕ_1 corresponding to the case (A) is given by

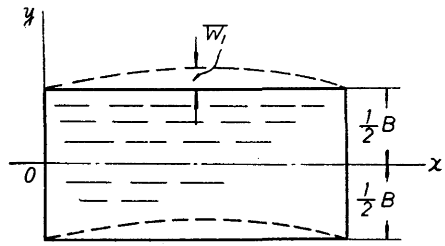

Fig. 2. in-phase vibration of side-walls

$$\phi_0 = B_{oo} f_{oo}(y) + \sum\sum B_{ij} f_{ij}(y) \cos(mix) \cos(sjz) \cos \omega t \qquad (3)$$

where we have put,

$$f_{oo}(y) = y, \qquad f_{ij}(y) = \sinh(n_{ij} y), \qquad B_{oo} = \frac{4}{\pi^2} W_1 \cos \omega t,$$

$$B_{ij} = \frac{4\varepsilon}{\pi^2} \cdot \frac{1}{(i^2-1)(j^2-1)} \cdot \frac{W_1 \cos \omega t}{n_{ij} \cosh(n_{ij} B/2)} \qquad n_{ij} = [(mi)^2 + (sj)^2]^{1/2}$$

The sign $\sum\sum$ of double summation means the double summation for $i = 0, 2, 4, \cdots\cdots$; $j = 0, 2, 4, \cdots\cdots$ ($i = j = 0$ being excepted). When either one of i or j is equal to 0 we take $\varepsilon = 2$, while when both i and j are not equal to 0 we take $\varepsilon = 4$.

Further, we take a modification (or correction) term into consideration. Here, at first, we shall take the modification term of the form

$$\frac{dw_2}{dt} = W_2 \sin 3mx \sin 3sz \cos \omega t \qquad (4)$$

for transverse velocity of the rectangular plate, as a trial case. (see Fig. 3) The value of velocity potential ϕ_2, which corresponds to the vibratory motion of water contained in the tank, induced by the vibration of side-walls as given by Eq. (4), could be obtained by the same method of calculation as shown in the previous paper. The value thus obtained is as follows: —

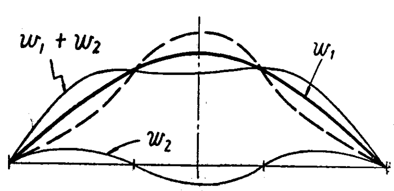

Fig. 3. displacements w_1 and w_2

$$\phi_2 = C_{oo} f_{oo}(y) + \sum\sum C_{ij} f_{ij}(y) \cos(mix) \cos(siz) \cos \omega t \qquad (5)$$

where we put

$$C_{oo} = \frac{4}{9\pi^2} W_2 \cos \omega t$$

$$C_{ij}=\frac{36\varepsilon}{\pi^2}\cdot\frac{1}{(i^2-9)(j^2-9)}\cdot\frac{W_2\cos\omega t}{n_{ij}\cosh(n_{ij}B/2)}$$

the meaning of the sign of dovble summation, and the value of ε, in Eq. (5) are the same as in Eq. (3). When we consider a mode of vibration which consists of superposition of two vibration as expressed by Eqs. (2) and (4), the transverse velocity of the side-wall will be $(dw_1/dt)+(dw_2/dt)$, while the velocity potential of the corresponding vibratory motion of water will be given by $\phi=\phi_1+\phi_2$. The value of kinetic energy T_w of water for this combined vibratory motion $\phi_1+\phi_2$ will be given by the surface integral of the form,

$$T_w=\frac{1}{2}\rho_w\int\int\left[\phi\frac{\partial\phi}{\partial n}\right]dS$$

where $\partial/\partial n$ denotes the differentiation in dircction of outwardly drawn normal to the surface, the integration being to extend to whole boundary surface of the water region. Taking account of the fact that only side-walls are vibrating, the value of T_w can be put into the following form:——

$$T_w=T_{11}+T_{12}+T_{21}+T_{22} \tag{6}$$

where

$$\left.\begin{aligned}T_{11}&=\frac{1}{2}\rho_w\int\int\phi_1\frac{dw_1}{dt}dS\\T_{12}&=\frac{1}{2}\rho_w\int\int\phi_1\frac{dw_2}{dt}dS\\T_{21}&=\frac{1}{2}\rho_w\int\int\phi_2\frac{dw_1}{dt}dS\\T_{22}&=\frac{1}{2}\rho_w\int\int\phi_2\frac{dw_2}{dt}dS\end{aligned}\right\} \tag{6a}$$

It is to be understood that the value (6) is the kinetic energy of water to be attributed to each one panel of side-wall. The integration in Eq. (6a) is, therefore, to extend to one panel of side-wall. Actual values of T_{11}, ect., are found to be as follows:——

$$T_{11}=\frac{1}{2}\rho_w(W_1\cos\omega t)^2V\cdot$$

$$\left[\frac{8}{\pi^4}+\sum\sum\frac{16\varepsilon}{\pi^4}\left\{\frac{1}{(i^2-1)(j^2-1)}\right\}^2\cdot\frac{\tanh(n_{ij}B/2)}{(n_{ij}B)}\right] \tag{7}$$

$$T_{12}=T_{21}=\frac{1}{2}\rho_w W_1W_2(\cos\omega t)^2V\cdot$$

$$\left[\frac{8}{9\pi^4}+\sum\sum\left\{\frac{144}{(i^2-9)(j^2-9)(i^2-1)(j^2-1)}\right\}\frac{\varepsilon}{\pi^4}\frac{\tanh(n_{ij}B/2)}{(n_{ij}B)}\right] \tag{8}$$

$$T_{22}=\frac{1}{2}\rho_w(W_2\cos\omega t)^2V\cdot$$

$$\left[\frac{8}{81\pi^4}+\sum\sum\left\{\frac{36}{(i^2-9)(j^2-9)}\right\}^2\frac{\varepsilon}{\pi^4}\frac{\tanh(n_{ij}B/2)}{(n_{ij}B)}\right] \tag{9}$$

III. *Case C.* Rectangular Tank almost completely filled with Water (two Side-Walls making Opposite-Phase Vibrations)

In this case, we take at first that side-walls are vibrating as expressed by Eq. (2). The corresponding value of velocity potential ϕ_1 was seen to be

$$\phi_1 = \sum\sum B_{ij} f_{ij}(y) \cos(mix) \cos(sjz) \cos \omega t \tag{10}$$

where we put

$$f_{ij}(y) = \cosh(n_{ij}y), \qquad B_{oo} = 0,$$

$$B_{ij} = \frac{2\varepsilon}{\pi^2} \cdot \frac{1}{(i^2-1)(j^2-1)} \cdot \frac{W_1 \cos \omega t}{n_{ij} \sinh(n_{ij}B/2)}$$

The sign $\sum\sum$ of double summation is to be taken for $i=0, 2, 4, \cdots\cdots$; $j=1/2, 3/2, 5/2, \cdots\cdots$. ε is to be taken equal to 2 when $i=0$, but equal to 4 when $i \neq 0$.

Next, we take a modification term. In this case (C), the boundary conditions at top- and bottom surfaces are different. In view of this fact, we put (Fig. 4)

$$\frac{dw_2}{dt} = W_2 \sin mx \sin 2sz \cos \omega t \tag{11}$$

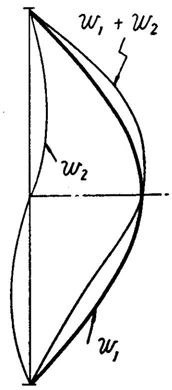

Fig. 4. displacements w_1 and w_2

The corresponding value of velocity potential ϕ_2 can be obtained by the similar method as above, and is found to be

$$\phi_2 = \sum\sum C_{ij} f_{ij}(y) \cos(mix) \cos(sjz) \cos \omega t \tag{12}$$

where we put

$$C_{ij} = \frac{4\varepsilon}{\pi^2} \left\{ \frac{1}{(i^2-1)(j^2-4)} \right\} \frac{W_2 \cos \omega t}{n_{ij} \sinh(n_{ij}B/2)}$$

Also, we take $i=0, 2, 4, \cdots\cdots$; $j=1/2, 3/2, 5/2, \cdots\cdots$; and $\varepsilon=2$ for $i=0$, $\varepsilon=4$ for $i \neq 0$.

Estimating the value of kinetic energy of water T_w by the consideration as before, we have (alloted to each one panel of rectangular plate);

$$T_{11} = \frac{1}{2} \rho_w (W_1 \cos \omega t)^2 \frac{4}{\pi^4} V \cdot$$

$$\sum\sum \left\{ \frac{1}{(i^2-1)(j^2-1)} \right\}^2 \varepsilon \frac{\coth(n_{ij}B/2)}{(n_{ij}B)} \tag{13}$$

$$T_{12} = T_{21} = \frac{1}{2} \rho_w (W_1 W_2 \cos^2 \omega t) \frac{4}{\pi^4} V \cdot$$

$$\sum\sum \left\{ \frac{2}{(i^2-1)(j^2-4)} \right\} \left\{ \frac{1}{(i^2-1)(j^2-1)} \right\} \varepsilon \frac{\coth(n_{ij}B/2)}{(n_{ij}B)} \tag{14}$$

$$T_{22} = \frac{1}{2} \rho_w (W_2 \cos \omega t)^2 \frac{4}{\pi^4} V \cdot$$

$$\sum\sum \left\{ \frac{2}{(i^2-1)(j^2-4)} \right\}^2 \varepsilon \frac{\coth(n_{ij}B/2)}{(n_{ij}B)} \tag{15}$$

IV. Examination of Accuracy of Approximate Formulae for Natural Frequency of Vibration of Side-Walls of rectangular Water-Tank

Ler us first, take up the case (A) considered in section II. The strain energy U of an elastic flat plate, which is in vibration, is given by

$$U=\frac{D}{2}\int\int\left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)^2-2(1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial z^2}-\left(\frac{\partial^2 w}{\partial x\partial z}\right)^2\right\}\right]dxdz \tag{16}$$

where we put D=flexural rigidity of flat plate$=Eh_m^3/[12(1-\nu^2)]$, h_m=thickness of the plate, ν=Poisson's ratio of the material composing the plate, E=Young's modulus of the ditto, w=transverse displacement of the flat plate. For the case (A) the value of transverse displacement w is assumed to be

$$w=w_1+w_2=[A_1\sin mx\sin sz+A_2\sin 3mx\sin 3sz]\sin\omega t$$

($W_1=\omega A_1$, $W_2=\omega A_2$) Putting this value of displacement w into Eq. (16) we obtain

$$U=\frac{1}{2}D(m^2+s^2)\frac{LH}{4}[A_1^2+aA_2^2]\sin^2\omega t$$

The value of a for the present case (A) is found to be equal to 81. The kinetic energy T_m of plate proper is,

$$T_m=\frac{1}{2}\rho_m h_m\int\int\left(\frac{\partial^2 w}{\partial t^2}\right)^2 dxdz=\frac{1}{2}\rho_m h_m\frac{LH}{4}\omega^2[A_1^2+A_2^2]\cos^2\omega t \tag{18}$$

Thus, when the rectangular plate is vibrating, together with the water contained in the tank, total kinetic energy will be the sum of Eqs. (6) and (18), viz., $T_s=T_w+T_m$, while the potential (strain) energy is U. Taking the mean value over one period ($t=\pi/\omega$) of oscillation, we may write;

$$U=C_U(A_1^2+aA_2^2)$$
$$T_s=\omega^2C_T(K_1A_1^2+FA_1A_2+K_2A_2^2) \tag{19}$$

where C_U, C_T, K_1, K_2 and F are constants. Equating these values U and T_s (each being timely mean values), and putting $A_2/A_1=\lambda$ for shortness, we have,

$$\omega^2=\frac{C_U}{C_T}=\frac{1+a\lambda^2}{K_1+F\lambda+K_2\lambda^2} \tag{20}$$

Now, if the mode of vibration was correctly given, the expression (20) would give just exact value for angular frequency ω of free vibration.

But, since the expression for $w=w_1+w_2$ was chosen rather arbitrarily, the expression (20) will give only an approximate value. This being so, we determine the value of $\lambda(=A_2/A_1)$ so as to make the value of expression (20) a minimum, according to the *principle of Lord Rayleigh*[2]. This requires that λ must be a root λ_m of the quadratic equation

$$F+2(K_2-aK_1)\lambda-aF\lambda^2=0 \tag{21}$$

2) G. Temple. W. G. Bickley; *Rayleigh's Principle and its Applications to Engineering*, Oxford Univ. Press, 1933.

The value ω_m which is obtained by putting the value λ_m in place of λ into the expression (20), will give closer approximation to the natural frequency of vibration, than that given in the previous report. And the amount of λ_m will give a measure to check the degree of approximation to it.

Similar treatment can also be made for the case (C) of section III. For this case, we have

$$w=w_1+w_2=[A_1 \sin mx \sin sz+A_2 \sin mx \cdot \sin 2sz] \sin \omega t$$

The value a in the equation (17) is seen to be, $a=(m^2+4s^2)^2/(m^2+s^2)^2$. The expression (18) is left unchanged. Using this value of a, together with the expression for $T_w=T_{11}+T_{12}+T_{21}+T_{22}$ as given in section III, we can obtain the values of constants K_1, K_2 and F, and hence also the value of root λ_m by Eq. (21).

Furthermore, we can make similar estimation for other mode of vibration of the rectangular plate, such as, for example,

$$w=w_1+w_2=[A_1 \sin mx \sin sz+A_2 \sin 3mx \sin 2sz] \sin \omega t$$

V. Numerical Example

In order to illustrate numerically the result of study given in previhus section, let us take the case of a rectangular tank having the dimensions $L=5$ m, $H=3$ m, $B=3$ m,which was also quoted in the previous paper.

Case (*A*) For this case, the values of the coefficient which is defined by

$$T_{11}=\frac{1}{2}\rho_w(W_1 \cos \omega t)^2\, VM_{11}$$

$$T_{12}=\frac{1}{2}\rho_w W_1 W_2 \cos^2 \omega t\, VM_{12}$$

$$T_{21}=\frac{1}{2}\rho_w W_1 W_2 \cos^2 \omega t\, VM_{21}$$

$$T_{22}=\frac{1}{2}\rho_w(W_2 \cos \omega t)^2\, VM_{22}$$

are found, by numerical calculation of Eqs. (7) (8) and (9), to be $M_{11}=0.0985$, $M_{12}=M_{21}=0.00367$, $M_{22}=0.0282$. This means that virtual masses of vibration are 4.40, 0.165 and 1.269 (expressed in terms of equivalent values of tons) respectively. Assuming that the vibrational mass of the plate itself is 0.30 (in ton), we infer that $K_1 : F : K_2=4.70 : 0.330 : 1.57$. Inserting this proportional value, and also the value $a=81$ into Eq. (21), we find that $\lambda_m=0.00326$, which is very small in comparison to unity.

Case C For this case, by numerical calculation we obtain the values $M_{11}=0.16$, $M_{12} = M_{21} = 0.0438$, $M_{22} = 0.0675$. The vibrational masses corresponding to these are, 7.20, 1.97 and 3.04 (in equivalent values of tons). Therefore, $K_1 : F : K_2=7.50 : 3.94 : 3.34$. Also we have

$$a=\left(\frac{1}{25}+\frac{4}{9}\right)^2\Big/\left(\frac{1}{25}+\frac{1}{9}\right)^2=10.25.$$

Putting these values into Eq. (21), we obtain the value $\lambda_m=0.0268$, which means that the ratio of amplitudes A_2/A_1 is only 2.7%. The correction to natural frequency corresponding to this value of λ_m is obtained by taking the ratio of natural frequencies (26) for $\lambda=0$ and $\lambda=\lambda_m$ respectively. We see that the correction does not exceed 1%.

In the above discussion, two cases among the four were taken up, and the wave form of the plate displacement was rather arbitrarily assigned. So that we may have no right to insist that the discussion about the correction term was done completely. But, from the result of the above discussion we may safely infer that the correction for natural frequency will not exceed 2 or 3%, and that the formulae given in the previous report may be said to be accurate enough for practical use.

VI. Rectangular Tank in which the Water is only partially filled

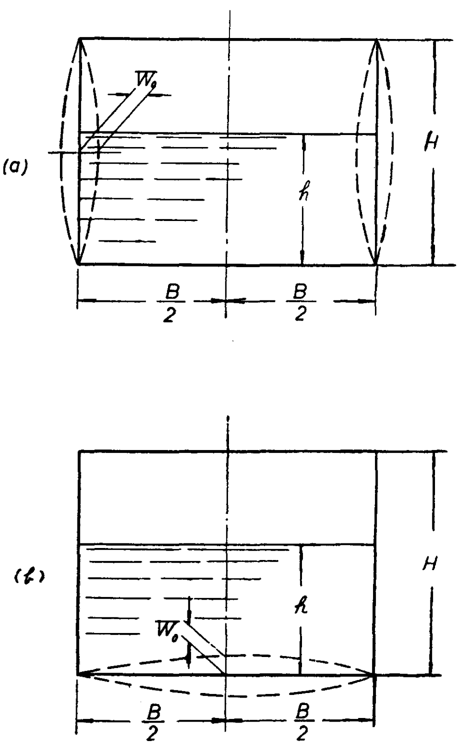

Fig. 5. rectangular tank in which the water is partially filled up

For the case in which water is not filled up to the top ($z=H$) but only partially filled to the level $z=h$ (Fig. 5), we made similar calculation as given in the previous report, and obtained the value of virtual mass of water corresponding to the vibration of side-walls of the tank. The result of calculation is as follows. Namely, the value of kinetic energy T_w of the whole water region, alloted to each one panel (whose one face is in contact with water) of side-wall, is found to be

$$T_w=\frac{1}{2}\rho_w\,[W_0\cos\omega t]^2V_hM \qquad (22)$$

And, the virtual mass (of vibration) M_{wv} is given by $M_{wv}=\rho_wV_hM$. $V_h=LBH$ is the volume of water contained in the tank. The value of coefficient M, for the case of in-phase vibration of side-walls (Fig. 5(a)) is,

$$M=\frac{4}{\pi^4}\sum\sum\left[\frac{\alpha+(-1)^{i+1}j\sin(\alpha\pi)}{(i^2-1)(j^2-\alpha^2)}\right]^2\cdot\frac{\varepsilon\tanh(BN_{ij}/2)}{BN_{ij}} \qquad (23)$$

where we put $\alpha=h/H$, and

$$BN_{ij}=\pi[(iB/L)^2+(jB/h)^2]^{1/2}$$

also, the double summation is to extend to $i=0, 2, 4,\cdots\cdots$; $j=\sigma+\frac{1}{2}$; $\sigma=0, 1, 2,\cdots\cdots$, and also, $\varepsilon=4$ when $i\neq 0$ but $\varepsilon=2$ when $i=0$. As to the case of opposite-phase vibration of side-walls, we have to write coth instead of tanh in the above expression (23). It is to be understood, as in the previous report, that we assume the value of ω^2H/g to be very large in comparison with unity, and that the Eq. (23) gives approximate value.

Lastly, for the case in which the bottom plate is vibrating, as shown in Fig. 5 (b),

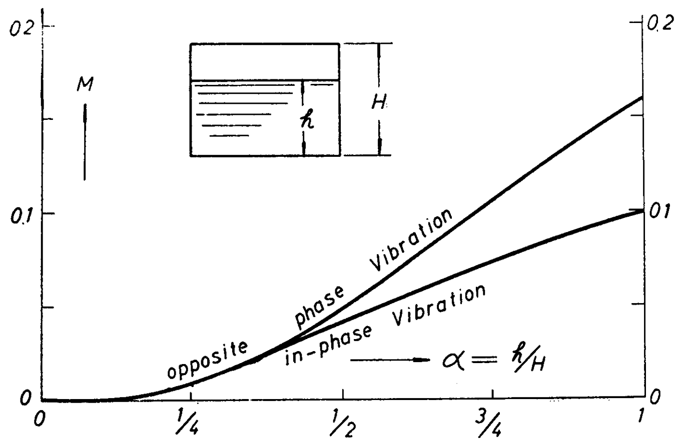

Fig. 6. Value of coefficient M for virtual mass, for a rectangular tank partially filled with water

we have made similar calculation and found that the value of coefficient M is as follows:——

$$M=\frac{16}{\pi^4}\sum\sum\varepsilon\frac{\nu_{ij}}{\xi_{ij}}\left[\frac{1}{(i^2-1)(j^2-1)}\right]^2 \tag{24}$$

where we put

$$\xi_{ij}=\pi[(ih/L)^2+(jh/B)^2]^{1/2}$$

$$\nu_{ij}=[k\tanh\xi_{ij}-\xi_{ij}]/[k-\xi_{ij}\tanh\xi_{ij}]$$

$k=\omega^2h/g$, and the double summation is to extend to $i=0, 2, 4.\cdots\cdots$; $j=0, 2, 4,\cdots\cdots$; also, when $i\neq 0$, $j\neq 0$ we have $\varepsilon=4$, but when $j=0$, $i\neq 0$ or when $j\neq 0$, $i=0$ we have $\varepsilon=2$, the value of ε for $i=0$, $j=0$ being equal to unity. It is to be noted that, when $i=0$, $j=0$ we have $\nu_{ij}/\xi_{ij}=1$.

Since the right hand side of Eq. (24) contains ω, which is not yet known, we must resort to some kind of method of successive approximation, in order to obtain the numerical value of M. But, in actual circumstances, there are many cases wherein the value of k is known to be very large in comparison with unity. In that case, we can take approximately that, $\nu_{ij}=\tanh \xi_{ij}$ and so the left hand side of Eq. (24) is almost independent of ω.