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# Fibonacci and Lucas series with elliptic functions

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# FIBONACCI AND LUCAS SERIES WITH ELLIPTIC FUNCTIONS

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## ABSTRACT

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By TuAnh Gia Nguyen

The sequences $\{F_n\}$ and $\{L_n\}$ are well known as the Fibonacci and Lucas sequences respectively. In this thesis, we present in detail several methods to evaluate certain types of series involving reciprocals of Fibonacci and Lucas sequences, named Fibonacci and Lucas series. Some new results are achieved and introduced.

In addition, we study some interesting properties of the Jacobi elliptic functions $sn(z)$, $cn(z)$, and $dn(z)$. We also discuss some properties of the Lambert series. These properties are essential for evaluation of Fibonacci and Lucas series.

Moreover, we will evaluate series involving the reciprocals of the Horadam sequence $\{W_n\}$ and its special cases including Fibonacci, Lucas, Pell, and Fermat series.

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# INTRODUCTION

In 1202, the Italian mathematician Leonardo Fibonacci, also known as Leonardo of Pisano, introduced the rabbit problem. He provided the general solution to the problem using the sequence $\{F_n\}$ defined recursively by

$F_n = F_{n-2} + F_{n-1},\ F_1 = 1,\ F_2 = 1,\ n \geq 3$.

Changing the initial values give the Lucas sequence, $\{L_n\}$,

$L_n = L_{n-2} + L_{n-1},\ L_1 = 1,\ L_2 = 3,\ n \geq 3$ named after E. Lucas (1842-1891).

There are many research papers that discuss these numbers. This thesis presents in detail the evaluation of the series involving combinations and reciprocals of the Fibonacci and Lucas numbers and their generalizations.

Most of the articles referred to in this thesis can be found in a journal, "The Fibonacci Quarterly", which was established by professor Verner E. Hoggatt, Jr. of San Jose State University in February, 1963.

In Chapter I, we discuss the very first four problems, which appeared in the *Fibonacci Quarterly* in 1963, involving the series of reciprocals and infinite products of Fibonacci numbers. Then, based on the idea of telescoping sum, we find several formulas related to the series of reciprocal of the Fibonacci and Lucas numbers. Specially, the author introduced a new theorem that evaluates recursively the sum of the series $\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+k}}$, where k is a positive integer. Also, the author provided a program that calculates the sums of series $\sum_{n=1}^{k} \frac{1}{F_n}$ and $\sum_{n=1}^{k} \frac{1}{L_n}$, for any designed positive integer k, accurately to a hundred decimal places.

In Chapter II, we study the Jacobi elliptic functions *sn*(z), *cn*(z), and *dn*(z) as well as their properties. We prove a difficult theorem that provides the Fourier series of the Jacobi elliptic function *sn*(z). Then, based on this theorem and the Fourier series expansions of *sn*(z), *cn*(z), and *dn*(z), we evaluate the sums of several series involving the reciprocals of the Fibonacci and Lucas sequences.

In Chapter III, we introduce the Horadam sequence $\{W_n\}$ which contains Fibonacci, Lucas, Pell, and Fermat sequences as special cases. Then, we use some properties of the Lambert series and Jacobi elliptic functions to evaluate the sums of infinite series involving combinations of $\{W_n\}$. As special cases of these results, we obtain additional identities involving the Fibonacci and Lucas sequences.

# CHAPTER I

# SPECIAL SERIES INVOLVING FIBONACCI AND LUCAS NUMBERS

## 1.1 The Binet Forms for The Fibonacci and Lucas Numbers

The sequence $\{F_n\}$ is recursively defined by $F_n = F_{n-2} + F_{n-1},\ F_1 = 1,\ F_2 = 1,\ n \geq 3$ is well known as the Fibonacci sequence. With the same recurrence relation, choosing different initial values, we have the Lucas sequence $\{L_n\}$ defined by

$L_n = L_{n-2} + L_{n-1},\ L_1 = 1,\ L_2 = 3,\ n \geq 3$. The two sequences are related by the formula

$L_n = F_{n-1} + F_{n+1},\ n \geq 1$. More relations between the two sequences can be found in Hoggatt [12].

The quadratic equation

$$(1.1) \qquad \boxed{x^2 - x - 1 = 0}$$

has two roots $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$. These two roots play an important role in studying the Fibonacci and Lucas sequences. The number $\alpha$ is called the Golden Ratio. It is clear that

$$(1.2) \qquad \boxed{\alpha + \beta = 1,\ \alpha - \beta = \sqrt{5},\ \alpha\beta = -1,\ \alpha^2 = \alpha + 1,\ \textit{and}\ \beta^2 = \beta + 1}.$$

The equation (1.1) is called the Fibonacci quadratic equation. The Binet form for the Fibonacci numbers, named after the French mathematician Jacques-Phillipe-Marie Binet (1786-1856) is given by

$$(1.3) \qquad \boxed{F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}},\ n = 1, 2, 3, \ldots .$$

Proof: We will prove (1.3) by using strong mathematical induction on n.

When n = 1, it is obvious that $F_1 = 1 = \dfrac{\alpha - \beta}{\alpha - \beta}$.

Hence, $F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ is true for n = 1.

Let k is an arbitrary positive integer. Suppose $F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ for any positive integer n from 1 to k. Then, we must show that equation also holds when n = k + 1.

Thus, by the inductive hypothesis, we have $F_k = \dfrac{\alpha^k - \beta^k}{\alpha - \beta}$, and $F_{k-1} = \dfrac{\alpha^{k-1} - \beta^{k-1}}{\alpha - \beta}$.

Hence it follows that $F_{k+1} = F_k + F_{k-1}$

$$\begin{aligned}
&= \frac{\alpha^k - \beta^k}{\alpha - \beta} + \frac{\alpha^{k-1} - \beta^{k-1}}{\alpha - \beta} \\
&= \frac{\alpha^{k-1}(\alpha + 1) - \beta^{k-1}(\beta + 1)}{\alpha - \beta} \\
&= \frac{\alpha^{k-1}\alpha^2 - \beta^{k-1}\beta^2}{\alpha - \beta} \quad \text{by (1.2)} \\
&= \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}.
\end{aligned}$$

Thus, by strong mathematical induction, $F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ is true for all positive integer n.

□

However, in 2004, Sury [15] provided a new proof of (1.3) based on polynomial identities.

The Binet form for the Lucas numbers is given by

(1.4) $\boxed{L_n = \alpha^n + \beta^n}$, n = 1, 2, 3, ... .

Proof: We also prove (1.4) by using strong mathematical induction on n.

When n = 1, it is clear that $L_1 = 1 = \alpha + \beta$ by (1.2).

Hence, $L_n = \alpha^n + \beta^n$ is true for n = 1.

We now suppose that $L_n = \alpha^n + \beta^n$ is true for any integer n from 1 to k, where k is an arbitrary positive integer. Then, we will show the above equation also holds when n = k + 1.

Thus, we have

$$\begin{aligned}
L_{k+1} &= L_k + L_{k-1} && \text{by the definition of Lucas number} \\
&= \alpha^k + \beta^k + \alpha^{k-1} + \beta^{k-1} && \text{by the inductive hypothesis} \\
&= \alpha^{k-1}(\alpha + 1) + \beta^{k-1}(\beta + 1) \\
&= \alpha^{k-1}\alpha^2 + \beta^{k-1}\beta^2 && \text{by (1.2)} \\
&= \alpha^{k+1} + \beta^{k+1}.
\end{aligned}$$

Hence, by strong mathematical induction, $L_n = \alpha^n + \beta^n$ is true for all positive integer n.

□

## 1.2 The First Four Problems

In the year of 1963, professor Verner E. Hoggatt, Jr. of San Jose State University started the journal, namely, "The Fibonacci Quarterly." The first issue was published in February 1963, with Verner E. Hoggatt, Jr. as the editor and Brother Alfred Brousseau as the managing editor. The first four problems involving series of the reciprocals and infinite products of the Fibonacci numbers appeared in this journal.

**1.2.1** Problem 1: It was proposed by R. L. Graham, Bell Telephone Laboratories, Murray Hill, New Jersey in the Fibonacci Quarterly in April 1963. Show that

$$(1.5) \qquad \boxed{\sum_{n=1}^{\infty}\frac{1}{F_n}=3+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{F_nF_{n+1}F_{n+2}}}.$$

A solution by the famous author Leonard Carlitz of Duke University, Durham, N. C. was published in the Fibonacci Quarterly in December 1963 . We have

$$\sum_{n=2}^{\infty}\frac{1}{F_n}-\sum_{n=2}^{\infty}\frac{F_n}{F_{n-1}F_{n+1}}=\sum_{n=2}^{\infty}\left(\frac{1}{F_n}-\frac{F_n}{F_{n-1}F_{n+1}}\right)$$
$$=\sum_{n=2}^{\infty}\frac{F_{n-1}F_{n+1}-F_n^2}{F_{n-1}F_nF_{n+1}}.$$

Using $F_{n-1}F_{n+1}-F_n^2=(-1)^n$ (see Hoggatt [12], page 57, ($I_{13}$)) on the right hand side of above equation, it follows that

$$\sum_{n=2}^{\infty}\frac{1}{F_n}-\sum_{n=2}^{\infty}\frac{F_n}{F_{n-1}F_{n+1}}=\sum_{n=2}^{\infty}\frac{(-1)^n}{F_{n-1}F_nF_{n+1}}. \quad (*)$$

On the other hand, we also have

$$\sum_{n=2}^{\infty}\frac{F_n}{F_{n-1}F_{n+1}}=\sum_{n=2}^{\infty}\frac{F_{n+1}-F_{n-1}}{F_{n-1}F_{n+1}}$$
$$=\sum_{n=2}^{\infty}\left(\frac{1}{F_{n-1}}-\frac{1}{F_{n+1}}\right)$$
$$=\left(\frac{1}{F_1}-\frac{1}{F_3}\right)+\left(\frac{1}{F_2}-\frac{1}{F_4}\right)+\left(\frac{1}{F_3}-\frac{1}{F_5}\right)+\ldots$$
$$=\frac{1}{F_1}+\frac{1}{F_2}=2.$$

Hence (*) can be rewritten as $\sum_{n=2}^{\infty}\frac{1}{F_n}-2=\sum_{n=2}^{\infty}\frac{(-1)^n}{F_{n-1}F_nF_{n+1}}$, or $$\sum_{n=1}^{\infty}\frac{1}{F_n}-\frac{1}{F_1}-2=\sum_{n=2}^{\infty}\frac{(-1)^n}{F_{n-1}F_nF_{n+1}},$$

or $$\sum_{n=1}^{\infty}\frac{1}{F_n}=3+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{F_nF_{n+1}F_{n+2}}.$$

□

**1.2.2** Problem 2: It was proposed by R. L. Graham, Bell Telephone Laboratories, Murray Hill, New Jersey in the Fibonacci Quarterly in April 1963. Show that

$$\boxed{\sum_{n=2}^{\infty}\frac{1}{F_{n-1}F_{n+1}}=1}. \tag{1.6}$$

A solution by Francis Parker, University of Alaska, was published in the Fibonacci Quarterly in December 1963.

We have $$\frac{1}{F_{n-1}F_{n+1}} = \frac{F_n}{F_{n-1}F_nF_{n+1}}$$

$$= \frac{F_{n+1}-F_{n-1}}{F_{n-1}F_nF_{n+1}}$$

$$= \frac{1}{F_{n-1}F_n}-\frac{1}{F_nF_{n+1}}.$$

Hence $$\sum_{n=2}^{\infty}\frac{1}{F_{n-1}F_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{1}{F_{n-1}F_n}-\frac{1}{F_nF_{n+1}}\right)$$

$$=\left(\frac{1}{1\cdot1}-\frac{1}{1\cdot2}\right)+\left(\frac{1}{1\cdot2}-\frac{1}{2\cdot3}\right)+\left(\frac{1}{2\cdot3}-\frac{1}{3\cdot5}\right)+\dots$$

$$=1.$$

□

**1.2.3** Problem 3: It was proposed by L. Carlitz, Duke University, Durham, N.C. in the Fibonacci Quarterly in October 1963. Show that

(1.7) $$\boxed{\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+2}^2F_{n+3}}+\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}^2F_{n+3}}=\frac{1}{2}}.$$

A solution by John H. Avila, University of Maryland, College, Park Maryland, was published in the Fibonacci Quarterly in February 1964.

Let $a = F_n,\ b = F_{n+1},\ c = F_{n+2},\ and\ d = F_{n+3}.$

Then, a + b = c, b + c = d, and the left hand side of the desired formula is

$$\sum_{n=1}^{\infty}\left(\frac{1}{ac^2d}+\frac{1}{ab^2d}\right) = \sum_{n=1}^{\infty}\left(\frac{b}{abc^2d}+\frac{c}{ab^2cd}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{c-a}{abc^2d}+\frac{d-b}{ab^2cd}\right) \quad \text{since a + b = c and b + c = d}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{abcd}-\frac{1}{bc^2d}+\frac{1}{ab^2c}-\frac{1}{abcd}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{ab^2c}-\frac{1}{bc^2d}\right).$$

The last sum is the infinite telescoping series, which is

$$\left(\frac{1}{F_1F_2^2F_3}-\frac{1}{F_2F_3^2F_4}\right)+\left(\frac{1}{F_2F_3^2F_3}-\frac{1}{F_3F_4^2F_5}\right)+\dots = \frac{1}{F_1F_2^2F_3}=\frac{1}{2}.$$

□

**1.2.4** Problem 4: It was proposed by S. L. Basin, Sylvania Electronic Systems, Mt. View, Calif. in the Fibonacci Quarterly in October 1963.

Prove the identities:

(1.8) $$\boxed{F_{n+1} = \prod_{i=1}^{n}\left(1+\frac{F_{i-1}}{F_i}\right)},$$

(1.9) $$\boxed{\frac{F_{n+1}}{F_n} = 1+\sum_{i=2}^{n}\frac{(-1)^i}{F_iF_{i-1}}},$$

(1.10) $$\boxed{\frac{1+\sqrt{5}}{2} = 1+\sum_{i=2}^{\infty}\frac{(-1)^i}{F_iF_{i-1}}}.$$

A solution by J. L. Brown, Jr., Pennsylvania State University, State College, Pa., was published in the Fibonacci Quarterly in February 1964. We have

$$F_{n+1} = \frac{F_{n+1}\times F_n\times F_{n-1}\times\ldots\times F_2}{F_n\times F_{n-1}\times\ldots\times F_2\times F_1} = \prod_{i=1}^{n}\left(\frac{F_{i+1}}{F_i}\right) = \prod_{i=1}^{n}\left(\frac{F_i+F_{i-1}}{F_i}\right) = \prod_{i=1}^{n}\left(1+\frac{F_{i-1}}{F_i}\right).$$

This proves identity (1.8).

□

To prove identity (1.9), we start with the quotient on the left hand side.

$$\frac{F_{n+1}}{F_n} = \left(\frac{F_{n+1}}{F_n}-\frac{F_n}{F_{n-1}}\right)+\left(\frac{F_n}{F_{n-1}}-\frac{F_{n-1}}{F_{n-2}}\right)+\ldots+\left(\frac{F_3}{F_2}-\frac{F_2}{F_1}\right)+1$$

$$= 1+\sum_{i=2}^{n}\left(\frac{F_{i+1}}{F_i}-\frac{F_i}{F_{i-1}}\right)$$

$$= 1+\sum_{i=2}^{n}\frac{F_{i+1}F_{i-1}-F_i^2}{F_iF_{i-1}}$$

$$= 1+\sum_{i=2}^{n}\frac{(-1)^i}{F_iF_{i-1}},$$ where we used the well-known identity $F_{i+1}F_{i-1}-F_i^2 = (-1)^i$.

□

Identity (1.10) comes from the limit of (1.9) as $n\to\infty$. As the limit of (1.9) gives

$\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = 1+\sum_{i=2}^{\infty}\frac{(-1)^i}{F_iF_{i-1}}$, we use the Binet form to evaluate the left hand side.

$$\text{LHS} = \lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \lim_{n\to\infty}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha^n-\beta^n}$$

$$= \lim_{n\to\infty}\frac{\alpha^{n+1}\left[1-\left(\frac{\beta}{\alpha}\right)^{n+1}\right]}{\alpha^n\left[1-\left(\frac{\beta}{\alpha}\right)^n\right]}$$

$$= \lim_{n\to\infty}\alpha \text{ since } \frac{\beta}{\alpha}<1$$

$$= \alpha = \frac{1+\sqrt{5}}{2}.$$

□

### 1.3 Telescoping Series Involving Fibonacci and Lucas numbers

The series involving reciprocals of various Fibonacci and Lucas numbers will be evaluated by using the idea of telescoping series. A telescoping series will have all intermediate terms cancel each other in pairs. Thus, a telescoping series always equals to the sum of the first few terms for the infinite case, and additional last few terms for the finite case. For example, let us evaluate the finite telescoping sum $\sum_{k=1}^{n}\frac{1}{k(k+1)}$.

$$\sum_{k=1}^{n}\frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

$$= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right).$$

Since intermediate terms add up to zero in pairs, only the first and the last term are left.

Thus, we have $\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$.

Here are some well-known finite telescoping identities that we will use later in this section:

(1.11) $$\boxed{\sum_{k=1}^{n} (a_k - a_{k+1}) = a_1 - a_{n+1}},$$

(1.12) $$\boxed{\sum_{k=1}^{n} (a_k - a_{k+2}) = (a_1 + a_2) - (a_{n+1} + a_{n+2})},$$

(1.13) $$\boxed{\sum_{k=1}^{n} (a_k - a_{k+3}) = (a_1 + a_2 + a_3) - (a_{n+1} + a_{n+2} + a_{n+3})},$$

(1.14) $$\boxed{\sum_{k=1}^{n} (a_k - a_{k+4}) = (a_1 + a_2 + a_3 + a_4) - (a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4})}.$$

<u>Theorem 1.3.1</u> Let $F_n$ be a Fibonacci number. Then

(1.15) $$\boxed{\sum_{n=1}^{\infty} \frac{F_n}{F_{n+1} F_{n+2}} = 1}.$$

<u>Proof</u>: We start by observing that $\frac{F_n}{F_{n+1} F_{n+2}} = \frac{F_{n+2} - F_{n+1}}{F_{n+1} F_{n+2}} = \frac{1}{F_{n+1}} - \frac{1}{F_{n+2}}$.

Now, consider the sequence of partial sums $S_n = \sum_{k=1}^{n} \frac{F_k}{F_{k+1} F_{k+2}}$.

Then, it follows that $S_n = \sum_{k=1}^{n} \left( \frac{1}{F_{k+1}} - \frac{1}{F_{k+2}} \right)$

$= \sum_{k=1}^{n} (a_k - a_{k+1})$, where $a_k = \frac{1}{F_{k+1}}$, $a_{k+1} = \frac{1}{F_{k+2}}$

$$= a_1 - a_{n+1} \text{ by } (1.11)$$

$$= \frac{1}{F_2} - \frac{1}{F_{n+2}}$$

$$= 1 - \frac{1}{F_{n+2}} \text{ since } F_2 = 1.$$

We now take the limit of $S_n$ as $n \to \infty$ to yield (1.15) since $\lim_{n\to\infty} \frac{1}{F_{n+2}} = 0$.

□

<u>Theorem 1.3.2</u> Let $F_n$ be a Fibonacci number. Then

$$(1.16) \qquad \boxed{\sum_{n=1}^{\infty} \frac{F_{n+1}}{F_n F_{n+3}} = \frac{5}{4}}.$$

<u>Proof</u>: By using the definition of the Fibonacci numbers, we obtain

$$\frac{F_{n+1}}{F_n F_{n+3}} = \frac{F_{n+3} - F_{n+2}}{F_n F_{n+3}}$$

$$= \frac{F_{n+3} - (F_n + F_{n+1})}{F_n F_{n+3}}$$

$$= \left(\frac{1}{F_n} - \frac{1}{F_{n+3}}\right) - \frac{F_{n+1}}{F_n F_{n+3}}.$$

Then, it is easy to see that $\frac{F_{n+1}}{F_n F_{n+3}} = \frac{1}{2}\left(\frac{1}{F_n} - \frac{1}{F_{n+3}}\right)$.

We now consider the finite sum $S_n = \sum_{k=1}^{n} \frac{F_{k+1}}{F_k F_{k+3}}$.

It follows that $S_n = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{F_k} - \frac{1}{F_{k+3}}\right)$

$$= \frac{1}{2}\sum_{k=1}^{n}(a_k - a_{k+3}) \quad \text{where} \quad a_k = \frac{1}{F_k},\ a_{k+3} = \frac{1}{F_{k+3}}$$

$$= \frac{1}{2}\left[(a_1 + a_2 + a_3) - (a_{n+1} + a_{n+2} + a_{n+3})\right] \quad \text{by (1.13)}$$

$$= \frac{1}{2}\left[\left(\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3}\right) - \left(\frac{1}{F_{n+1}} + \frac{1}{F_{n+2}} + \frac{1}{F_{n+3}}\right)\right].$$

Therefore,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1}{2}\left[\left(\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3}\right) - \left(\frac{1}{F_{n+1}} + \frac{1}{F_{n+2}} + \frac{1}{F_{n+3}}\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3}\right)$$

$$= \frac{5}{4} \quad \text{since} \quad F_1 = F_2 = 1,\ F_3 = 2.$$

□

Theorem 1.3.3 Let $F_n$, $L_n$ be the Fibonacci and Lucas numbers. Then

$$(1.17) \qquad \boxed{\sum_{n=1}^{\infty}\frac{L_{n+2}}{F_n F_{n+4}} = \frac{17}{6}}.$$

Proof: We shall use the identity $L_{n+2} = F_{n+1} + F_{n+3}$ (see Hoggatt [12], page 56, ($I_8$)) to prove the theorem.

$$\frac{L_{n+2}}{F_n F_{n+4}} = \frac{F_{n+1} + F_{n+3}}{F_n F_{n+4}}$$

$$= \frac{F_{n+1} + (F_n - F_n) + F_{n+3}}{F_n F_{n+4}}$$

$$= \frac{F_{n+2} + F_{n+3} - F_n}{F_n F_{n+4}} \quad \text{since} \quad F_{n+2} = F_n + F_{n+1}$$

$$= \frac{F_{n+4} - F_n}{F_n F_{n+4}} \quad \text{since} \quad F_{n+4} = F_{n+2} + F_{n+3}$$

$$= \frac{1}{F_n} - \frac{1}{F_{n+4}}.$$

We now consider the finite sum $S_n = \sum_{k=1}^{n} \frac{L_{k+2}}{F_k F_{k+4}}$ together with the above result.

It follows that

$$S_n = \sum_{k=1}^{n} \left( \frac{1}{F_k} - \frac{1}{F_{k+4}} \right)$$
$$= \sum_{k=1}^{n} (a_k - a_{k+4}), \text{ where } a_k = \frac{1}{F_k},\ a_{k+4} = \frac{1}{F_{k+4}}$$
$$= (a_1 + a_2 + a_3 + a_4) - (a_{n+1} + a_{n+2} + a_{n+3} + a_{k+4}) \text{ by (1.14)}$$
$$= \left( \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} \right) - \left( \frac{1}{F_{n+1}} + \frac{1}{F_{n+2}} + \frac{1}{F_{n+3}} + \frac{1}{F_{n+4}} \right).$$

Then, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[ \left( \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} \right) - \left( \frac{1}{F_{n+1}} + \frac{1}{F_{n+2}} + \frac{1}{F_{n+3}} + \frac{1}{F_{n+4}} \right) \right]$

$$= \left( \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} \right) = \frac{17}{16} \quad \text{since } F_1 = F_2 = 1,\ F_3 = 2, F_4 = 3.$$

□

Theorem 1.3.4 Let $F_n$ be a Fibonacci number. Then

(1.18) $$\boxed{\sum_{n=3}^{\infty} \frac{F_{2n}}{F_{n+2}^2 F_{n-2}^2} = \frac{85}{108}}.$$

Proof: We use the identity $F_{n+2}^2 - F_{n-2}^2 = 3F_{2n}$ (see Hoggatt [12] page 59, ($I_{28}$)) in proving this theorem. Applying the identity for the numerator of the rational expression, we get

$$\frac{F_{2n}}{F_{n+2}^2 F_{n-2}^2} = \frac{\frac{1}{3}\left(F_{n+2}^2 - F_{n-2}^2\right)}{F_{n+2}^2 F_{n-2}^2} = \frac{1}{3}\left[ \frac{1}{F_{n-2}^2} - \frac{1}{F_{n+2}^2} \right].$$

Now, let us consider the finite sum $S_n = \sum_{k=3}^{n} \frac{F_{2k}}{F_{k+2}^2 F_{k-2}^2}$.

$$S_n = \sum_{k=3}^{n} \frac{1}{3}\left[\frac{1}{F_{k-2}^2} - \frac{1}{F_{k+2}^2}\right]$$

$$= \frac{1}{3}\sum_{k=3}^{n}\left(a_{k-2} - a_{k+2}\right), \text{ where } a_{k-2} = \frac{1}{F_{k-2}^2},\ a_{k+2} = \frac{1}{F_{k+2}^2}$$

$$= \frac{1}{3}\left[\left(a_1 + a_2 + a_3 + a_4\right) - \left(a_{n-1} + a_n + a_{n+1} + a_{n+2}\right)\right] \text{ by (1.14)}$$

$$= \frac{1}{3}\left[\left(\frac{1}{F_1^2} + \frac{1}{F_2^2} + \frac{1}{F_3^2} + \frac{1}{F_4^2}\right) - \left(\frac{1}{F_{n-1}^2} + \frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2}\right)\right].$$

Taking the limit of $S_n$ as $n \to \infty$ will yield the result

$$\lim_{n\to\infty} S_n = \frac{1}{3}\left(\frac{1}{F_1^2} + \frac{1}{F_2^2} + \frac{1}{F_3^2} + \frac{1}{F_4^2}\right) = \frac{85}{108} \text{ since } F_1 = F_2 = 1,\ F_3 = 2, F_4 = 3.$$

□

Lemma 1.3.1 Let α be the Golden Ratio $\frac{1+\sqrt{5}}{2}$. Then for any positive integer k,

(1.19) $$\boxed{\lim_{n\to\infty} \frac{F_{n+k}}{F_n} = \alpha^k}.$$

Proof: We will prove the lemma by using the Binet form.

From (1.3), we have
$$\frac{F_{n+k}}{F_n} = \frac{\alpha^{n+k} - \beta^{n+k}}{\alpha^n - \beta^n}$$

$$= \frac{\alpha^{n+k}\left[1 - \left(\frac{\beta}{\alpha}\right)^{n+k}\right]}{\alpha^n\left[1 - \left(\frac{\beta}{\alpha}\right)^n\right]}$$

$$= \frac{\alpha^k \left[1-\left(\frac{\beta}{\alpha}\right)^{n+k}\right]}{\left[1-\left(\frac{\beta}{\alpha}\right)^{n}\right]}.$$

Since $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, and $\frac{\beta}{\alpha} < 1$, it implies that $\lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n = \lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^{n+k} = 0$.

Hence, we obtain $\lim_{n\to\infty}\frac{F_{n+k}}{F_n} = \alpha^k$.

□

<u>Lemma 1.3.2</u> Let $F_n$, $L_n$ be the Fibonacci and Lucas numbers. Then

(1.20) $\boxed{F_nL_n - F_{n+1}L_{n-1} = (-1)^n}$, $n \geq 1$.

<u>Proof</u>: We prove (1.20) by using mathematical induction on n.

When n = 1, $F_nL_n - F_{n+1}L_{n-1} = F_1L_1 - F_2L_0 = (1)(1) - (1)(2) = -1 = (-1)^1$.

Thus, (1.20) is true for n = 1.

Suppose that (1.20) is true when n = k, where k is any arbitrary integer greater or equal to 1; we will show it also holds when n = k + 1.

$$\begin{aligned} F_{k+1}L_{k+1} - F_{k+2}L_k &= F_{k+1}\left(L_k + L_{k-1}\right) - L_k\left(F_k + F_{k+1}\right) \\ &= F_{k+1}L_k + F_{k+1}L_{k-1} - L_kF_k - L_kF_{k+1} \\ &= -\left(F_kL_k - F_{k+1}L_{k-1}\right) \\ &= -(-1)^k \quad \text{by the inductive hypothesis} \\ &= (-1)^{k+1}. \end{aligned}$$

Therefore, by mathematical induction, (1.20) is true for every positive integer n.

□

Theorem 1.3.5 Let $L_n$ be a Lucas number. Then

(1.21) $$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{L_{n-1}L_n}=\frac{\sqrt{5}}{10}}.$$

Proof: We prove theorem 1.3.5 using (1.19) and (1.20).

Observe that

$$\begin{aligned}\frac{(-1)^{n-1}}{L_{n-1}L_n} &= -\frac{(-1)^n}{L_{n-1}L_n}\\ &= -\frac{F_nL_n - F_{n+1}L_{n-1}}{L_{n-1}L_n} \quad \text{by (1.20)}\\ &= -\left(\frac{F_n}{L_{n-1}}-\frac{F_{n+1}}{L_n}\right).\end{aligned}$$

Consider the finite sum $S_n=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{L_{k-1}L_k}$.

$$\begin{aligned}S_n &= -\sum_{k=1}^{n}\left(\frac{F_k}{L_{k-1}}-\frac{F_{k+1}}{L_k}\right)\\ &= -\sum_{k=1}^{n}(a_k-a_{k+1}),\ \text{where } a_k=\frac{F_k}{L_{k-1}},\ a_{k+1}=\frac{F_{k+1}}{L_k}\\ &= -(a_1-a_{n+1}) \ \text{by (1.11)}\\ &= -\left(\frac{F_1}{L_0}-\frac{F_{n+1}}{L_n}\right)\\ &= -\frac{1}{2}+\frac{F_{n+1}}{L_n} \ \text{since } L_0=2 \text{ and } F_1=1.\end{aligned}$$

By using the identity $L_{n+2}=F_{n+1}+F_{n+3}$ (see Hoggatt [12], page 56, ($I_8$)) we obtain

$$S_n=-\frac{1}{2}+\frac{F_{n+1}}{F_{n-1}+F_{n+1}}=-\frac{1}{2}+\frac{\frac{F_{n+1}}{F_{n-1}}}{1+\frac{F_{n+1}}{F_{n-1}}}.$$

So, $\lim_{n\to\infty} S_n = -\frac{1}{2} + \lim_{n\to\infty}\left(\frac{\frac{F_{n+1}}{F_{n-1}}}{1+\frac{F_{n+1}}{F_{n-1}}}\right)$.

Since $\lim_{n\to\infty}\frac{F_{n+1}}{F_{n-1}} = \alpha^2$ by (1.19), it implies that $\lim_{n\to\infty} S_n = -\frac{1}{2}+\frac{\alpha^2}{1+\alpha^2}$.

Substituting $\alpha = \frac{1+\sqrt{5}}{2}$ leads to $\lim_{n\to\infty} S_n = \frac{\sqrt{5}}{10}$.

□

The following lemma is needed for the next theorem.

Lemma 1.3.3 If $F_n$ be a Fibonacci number, then

(1.22) $$\boxed{F_{n-1}F_{n+3} - F_nF_{n+2} = 2(-1)^n}, \ n \geq 1.$$

Proof: This lemma can be easily proved by mathematical induction on n.

□

Theorem 1.3.6 Let $F_n$ be a Fibonacci number. Then

(1.23) $$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{F_nF_{n+3}} = \frac{6\alpha-9}{4}, \ where\ \alpha = \frac{1+\sqrt{5}}{2}}.$$

Proof: Observe that

$$\begin{aligned}\frac{(-1)^{n+1}}{F_nF_{n+3}} &= -\frac{(-1)^n}{F_nF_{n+3}} \\ &= -\frac{1}{2}\left(\frac{F_{n-1}F_{n+3} - F_nF_{n+2}}{F_nF_{n+3}}\right) \quad \text{by (1.22)} \\ &= -\frac{1}{2}\left(\frac{F_{n-1}}{F_n} - \frac{F_{n+2}}{F_{n+3}}\right).\end{aligned}$$

Consider the finite sum $S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{F_k F_{k+3}}$ and follow the observation above,

$$S_n = \sum_{k=1}^{n} -\frac{1}{2}\left(\frac{F_{k-1}}{F_k} - \frac{F_{k+2}}{F_{k+3}}\right)$$

$$= -\frac{1}{2}\sum_{k=1}^{n}(a_k - a_{k+3}), \text{ where } a_k = \frac{F_{k-1}}{F_k},\ a_{k+3} = \frac{F_{k+2}}{F_{k+3}}$$

$$= -\frac{1}{2}\left[(a_1 + a_2 + a_3) - (a_{n+1} + a_{n+2} + a_{n+3})\right] \text{ by (1.13)}$$

$$= -\frac{1}{2}\left[\left(\frac{0}{1} + \frac{1}{1} + \frac{1}{2}\right) - \left(\frac{F_n}{F_{n+1}} + \frac{F_{n+1}}{F_{n+2}} + \frac{F_{n+2}}{F_{n+3}}\right)\right]$$

$$= -\frac{1}{2}\left[\frac{3}{2} - \left(\frac{1}{\frac{F_{n+1}}{F_n}} + \frac{1}{\frac{F_{n+2}}{F_{n+1}}} + \frac{1}{\frac{F_{n+3}}{F_{n+2}}}\right)\right]$$

$$= -\frac{1}{2}\left[\frac{3}{2} - \left(\frac{1}{\alpha} + \frac{1}{\alpha} + \frac{1}{\alpha}\right)\right] \text{ by (1.19) for k = 1}$$

$$= -\frac{1}{2}\left[\frac{3}{2} - \frac{3}{\alpha}\right]$$

$$= \frac{6 - 3\alpha}{4\alpha}$$

$$= \frac{6 - 3\left(\frac{1+\sqrt{5}}{2}\right)}{4\left(\frac{1+\sqrt{5}}{2}\right)} \text{ since } \alpha = \frac{1+\sqrt{5}}{2}$$

$$= \frac{3\sqrt{5} - 6}{4}$$

$$= \frac{6\left(\frac{1+\sqrt{5}}{2}\right) - 9}{4}$$

$$= \frac{6\alpha - 9}{4}.$$

□

Theorem 1.3.7 Let $F_n$ be a Fibonacci number. Then

(1.24) $$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{2n-1}F_{2n+3}} = \frac{1}{6}}.$$

Proof: Observe that

$$\begin{aligned} 3F_{2n+1} &= F_{2n+1} + F_{2n+1} + F_{2n+1} \\ &= F_{2n+1} + \underbrace{F_{2n+1} + (F_{2n}}_{F_{2n+2}} + F_{2n-1}) \\ &= \underbrace{F_{2n+1} + F_{2n+2}}_{F_{2n+3}} + F_{2n-1} \\ &= F_{2n+3} + F_{2n-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{(-1)^{n-1}}{F_{2n-1}F_{2n+3}} &= \frac{(-1)^{n-1} F_{2n+1}}{F_{2n-1}F_{2n+1}F_{2n+3}} \\ &= \frac{1}{3} \frac{(-1)^{n-1} (F_{2n+3} + F_{2n-1})}{F_{2n-1}F_{2n+1}F_{2n+3}} \\ &= \frac{(-1)^{n-1}}{3} \left[ \frac{1}{F_{2n-1}F_{2n+1}} + \frac{1}{F_{2n+1}F_{2n+3}} \right]. \end{aligned}$$

Consider the finite sum $S_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{F_{2k-1}F_{2k+3}}$ with the result above. Then we have

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{3} \left[ \frac{1}{F_{2k-1}F_{2k+1}} + \frac{1}{F_{2k+1}F_{2k+3}} \right] \\ &= \frac{1}{3}\left[ \left( \frac{1}{F_1F_3} + \frac{1}{F_3F_5} \right) - \left( \frac{1}{F_3F_5} + \frac{1}{F_5F_7} \right) + \left( \frac{1}{F_5F_7} + \frac{1}{F_7F_9} \right) - \ldots + (-1)^{n-1} \left( \frac{1}{F_{2n-1}F_{2n+1}} + \frac{1}{F_{2n+1}F_{2n+3}} \right) \right] \\ &= \frac{1}{3} \left[ \frac{1}{F_1F_3} + (-1)^{n-1} \cdot \frac{1}{F_{2n+1}F_{2n+3}} \right]. \end{aligned}$$

Taking the limit both sides of the above equation as $n \to \infty$, we get $\lim_{n\to\infty} S_n = \frac{1}{3F_1F_3} = \frac{1}{6}$.

□

Lemma 1.3.4 If $F_n$ and $L_n$ be Fibonacci and Lucas numbers, then

(1.25) $$\boxed{\lim_{n\to\infty}\frac{L_n}{F_n}=\sqrt{5}}.$$

Proof: Use the identity $L_n = F_{n-1} + F_{n+1}$ (see Hoggatt [12], page 56, ($I_8$)) we have

$$\begin{aligned}\lim_{n\to\infty}\frac{L_n}{F_n} &= \lim_{n\to\infty}\left(\frac{F_{n-1}+F_{n+1}}{F_n}\right)\\ &= \lim_{n\to\infty}\frac{F_{n-1}}{F_n}+\lim_{n\to\infty}\frac{F_{n+1}}{F_n}\\ &= \frac{1}{\alpha}+\alpha \quad \text{by (1.19) for k = 1}\\ &= -\beta+\alpha \quad \text{since } \alpha\beta=-1\\ &= \sqrt{5}.\end{aligned}$$

□

Theorem 1.3.8 Let $F_n$ and $L_n$ be the Fibonacci and Lucas numbers. Then

(1.26) $$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}F_{2n+2}}{L_n^2L_{n+2}^2}=\frac{8}{45}}.$$

Proof: We will use the identities ($I_{28}$) with k = 2, $(-1)^{n-1} = \frac{1}{2}(F_nL_{n+2} - F_{n+2}L_n)$ and ($I_{38}$) with m = 2n + 2, $L_nF_{n+2} + L_{n+2}F_n = 2F_{2n+2}$ (see Hoggatt [12], page 59) to yield the following result:

$$\begin{aligned}\frac{(-1)^{n-1}F_{2n+2}}{L_n^2L_{n+2}^2} &= \frac{(-1)^{n-1}\cdot\frac{1}{2}(L_nF_{n+2}+L_{n+2}F_n)}{L_n^2L_{n+2}^2} \quad \text{by identity } (I_{38})\\ &= \frac{\frac{1}{2}(F_nL_{n+2}-F_{n+2}L_n)\cdot\frac{1}{2}(L_nF_{n+2}+L_{n+2}F_n)}{L_n^2L_{n+2}^2} \quad \text{by idientiy } (I_{28})\end{aligned}$$

$$=\frac{1}{4}\frac{F_n^2L_{n+2}^2-F_{n+2}^2L_n^2}{L_n^2L_{n+2}^2}$$

$$=\frac{1}{4}\left[\left(\frac{F_n}{L_n}\right)^2-\left(\frac{F_{n+2}}{L_{n+2}}\right)^2\right].$$

Consider the finite sum $S_n = \sum_{k=1}^{n}\frac{(-1)^{k-1}F_{2k+2}}{L_k^2L_{k+2}^2}$.

$$S_n=\frac{1}{4}\sum_{k=1}^{n}\left[\left(\frac{F_k}{L_k}\right)^2-\left(\frac{F_{k+2}}{L_{k+2}}\right)^2\right]$$

$$=\frac{1}{4}\sum_{k=1}^{n}(a_k-a_{k+2}),\ \text{ where } a_k=\frac{F_k^2}{L_k^2},\ a_{k+2}=\frac{F_{k+2}^2}{L_{k+2}^2}$$

$$=\frac{1}{4}\left[(a_1+a_2)-(a_{n+1}+a_{n+2})\right]\ \text{ by (1.12)}$$

$$=\frac{1}{4}\left[\left(\frac{F_1}{L_1}\right)^2+\left(\frac{F_2}{L_2}\right)^2-\left(\frac{F_{n+1}}{L_{n+1}}\right)^2-\left(\frac{F_{n+2}}{L_{n+2}}\right)^2\right].$$

Taking the limit both sides of the equation as $n\to\infty$, we obtain

$$\lim_{n\to\infty}S_n=\frac{1}{4}\left[1+\frac{1}{9}-\frac{1}{5}-\frac{1}{5}\right]=\frac{8}{45}\ \text{ where we used (1.25).}$$

□

<u>Theorem 1.3.9</u> Let $F_n$ and $L_n$ be the Fibonacci and Lucas numbers. Then

$$(1.27)\qquad \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}L_{n+1}}{F_nF_{n+1}F_{n+2}}=1}.$$

<u>Proof</u>: Using the identity $L_{n+1}=F_{n+2}+F_n$ (see Hoggatt [12], page 56, ($I_8$)), we get

$$\frac{(-1)^{n-1}L_{n+1}}{F_nF_{n+1}F_{n+2}}=\frac{(-1)^{n-1}(F_{n+2}+F_n)}{F_nF_{n+1}F_{n+2}}=(-1)^{n-1}\left[\frac{1}{F_nF_{n+1}}+\frac{1}{F_{n+1}F_{n+2}}\right].$$

Thus, $S_n = \sum_{k=1}^{n} \frac{(-1)^{k-1} L_{k+1}}{F_k F_{k+1} F_{k+2}} = \sum_{k=1}^{n} (-1)^{k-1} \left[ \frac{1}{F_k F_{k+1}} + \frac{1}{F_{k+1} F_{k+2}} \right]$.

$S_n$ is a finite telescoping sum; so, $S_n = \frac{1}{F_1 F_2} + (-1)^{n-1} \frac{1}{F_{n+1} F_{n+2}}$ and $\lim_{n \to \infty} S_n = \frac{1}{F_1 F_2} = 1$.

□

<u>Theorem 1.3.10</u> Let $F_n$ and $L_n$ be the Fibonacci and Lucas numbers. Then

$$(1.28) \qquad \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} L_{2n+2}}{F_n F_{n+1} L_{n+1} L_{n+2}} = \frac{1}{3}}.$$

<u>Proof</u>: The following equation (1.29) is easy to prove using mathematical induction on n. So, we omit the proof and use (1.29) in proving theorem 1.3.10.

$$(1.29) \qquad \boxed{F_{n+1} L_{n+2} + F_n L_{n+1} = L_{2n+2}}.$$

We have $\frac{(-1)^{n-1} L_{2n+2}}{F_n F_{n+1} L_{n+1} L_{n+2}} = \frac{(-1)^{n-1} (F_{n+1} L_{n+2} + F_n L_{n+1})}{F_n F_{n+1} L_{n+1} L_{n+2}}$ by (1.29)

$$= (-1)^{n-1} \left[ \frac{1}{F_n L_{n+1}} + \frac{1}{F_{n+1} L_{n+2}} \right].$$

Let $S_n$ be the finite sum $\sum_{k=1}^{n} \frac{(-1)^{k-1} L_{2k+2}}{F_k F_{k+1} L_{k+1} L_{k+2}}$, then

$$S_n = \sum_{k=1}^{n} (-1)^{k-1} \left[ \frac{1}{F_k L_{k+1}} + \frac{1}{F_{k+1} L_{k+2}} \right]$$

$$= \frac{1}{F_1 L_2} + (-1)^{n-1} \frac{1}{F_{n+1} L_{n+2}} \quad \text{since } S_n \text{ is a telescoping series.}$$

Hence, $\lim_{n \to \infty} S_n = \frac{1}{F_1 L_2} = \frac{1}{3}$.

□

In conclusion, this section sketches some ideas about the applications of the telescoping method to evaluate some series involving the reciprocals of the Fibonacci and Lucas numbers. More identities that were evaluated by using the telescoping method can be found in Brother Alfred Brousseau [5].

### 1.4 Recursive Relations of Special Series

In this section, we will investigate recursive relations of some series involving reciprocals of the Fibonacci numbers having the types,

(1.30) $$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+k_1} F_{n+k_2} \ldots F_{n+k_r}}, \text{ where } k_1, k_2, \ldots, k_r \text{ are positive integers.}$$

#### 1.4.1 Definition of series of k^th degree:

If the series (1.30) has k Fibonacci numbers in its denominator, then it is called a "series of the k^th degree."

#### 1.4.2 Convergence of series of Fibonacci reciprocals

We start by showing the series $\sum_{n=1}^{\infty} \frac{1}{F_n}$ is convergent. This will help us to conclude that all series having the form of (1.30) are convergent since their terms are less than or equal to the terms of the series $\sum_{n=1}^{\infty} \frac{1}{F_n}$. By the Binet form (1.3), we have

$$\lim_{n\to\infty} \frac{\frac{1}{F_{n+1}}}{\frac{1}{F_n}} = \lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \frac{1}{\alpha} \text{ by (1.19)}$$

$$= \frac{2}{1+\sqrt{5}} < 1 \text{ since } \alpha = \frac{1+\sqrt{5}}{2}.$$

Therefore, by the ratio test theorem, the series $\sum_{n=1}^{\infty}\frac{1}{F_n}$ is absolutely convergent.

**1.4.3** The recursive relation of the second degree series:

This section is the original work of the author. We will define a recursion to evaluate the second degree series of Fibonacci reciprocals of the form:

(1.31) $$\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+k}}.$$

In order to do that, we need to prove the following lemmas and theorems.

Lemma 1.4.1 For any positive integers n and k,

(1.32) $$\boxed{F_{n+k} = F_k F_{n+1} + F_{k-1} F_n}.$$

Proof: We prove (1.32) by using strong mathematical induction on n where k is fixed.

When n = 1, it is easy to see that

$$\begin{aligned} F_{n+k} &= F_{1+k} \\ &= F_k + F_{k-1} \\ &= (1)F_k + (1)F_{k-1} \\ &= F_k F_2 + F_{k-1} F_1 \quad \text{since } F_2 = 1,\ F_1 = 1. \end{aligned}$$

Hence, $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$ is true for n =1.

Suppose that $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$ is true for all integer n ≤ p, where p is an arbitrary integer. We will show that the above equation still holds at n = p + 1.

We have

$$\begin{aligned} F_{p+1+k} &= F_{p+k} + F_{p-1+k} \quad \text{by definition} \\ &= \left(F_k F_{p+1} + F_{k-1} F_p\right) + \left(F_k F_p + F_{k-1} F_{p-1}\right) \quad \text{by inductive hypothesis} \end{aligned}$$

$$= F_k\left(F_{p+1} + F_p\right) + F_{k-1}\left(F_p + F_{p-1}\right)$$
$$= F_kF_{p+2} + F_{k-1}F_{p+1}.$$

Therefore, by strong mathematical induction, $F_{n+k} = F_kF_{n+1} + F_{k-1}F_n$ is true for all positive integer n. Similar induction on k will yield the result.

□

Lemma 1.4.2 For any positive integers n and k,

(1.33) $$\boxed{F_{n-1}F_{n+k} - F_nF_{n+k-1} = (-1)^n F_k}.$$

Again, we will prove (1.33) by using mathematical induction on n.

When n = 1, it is easy to see that

$$F_{n-1}F_{n+k} - F_nF_{n+k-1} = F_0F_{1+k} - F_1F_k = 0 \cdot F_{k+1} - 1 \cdot F_k = -F_k.$$

Hence, $F_{n-1}F_{n+k} - F_nF_{n+k-1} = (-1)^n F_k$ is true when n = 1.

Now, suppose that n ≥ 1 be an integer such that $F_{n-1}F_{n+k} - F_nF_{n+k-1} = (-1)^n F_k$. We will show that the equation also holds for n + 1.

$$\begin{aligned}
F_nF_{n+1+k} - F_{n+1}F_{n+k} &= F_n\left(F_{n+k} + F_{n+k-1}\right) - F_{n+k}\left(F_n + F_{n-1}\right) \\
&= F_nF_{n+k} + F_nF_{n+k-1} - F_nF_{n+k} - F_{n-1}F_{n+k} \\
&= -\left(F_{n-1}F_{n+k} - F_nF_{n+k-1}\right) \\
&= -(-1)^n F_k \text{ by inductive hypothesis} \\
&= (-1)^{n+1} F_k.
\end{aligned}$$

Therefore, by mathematical induction, $F_{n-1}F_{n+k} - F_nF_{n+k-1} = (-1)^n F_k$ is true for all positive integer n. Similar induction on k will yield the result.

□

Lemma 1.4.3: Let $F_n$ be a Fibonacci number. Then

$$(1.34) \qquad \boxed{\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1}.$$

Proof: By replacing n in (1.6) by n + 1, we will obtain (1.34)

□

Theorem 1.4.1: Let $F_n$ be a Fibonacci number. Then for any positive integers n and k,

$$(1.35) \qquad \boxed{\sum_{n=1}^{\infty} \left( \frac{F_k}{F_n F_{n+k}} - \frac{F_{k+2}}{F_n F_{n+k+2}} \right) = (-1)^k \left( \sum_{m=1}^{k} \frac{1}{F_m F_{m+2}} - 1 \right)}.$$

Proof: Replacing k by (k+2) in (1.32), we have $F_{n+k+2} = F_{k+2}F_{n+1} + F_{k+1}F_n$. Then the summand is given by

$$\begin{aligned}
\frac{F_k}{F_n F_{n+k}} - \frac{F_{k+2}}{F_n F_{n+k+2}} &= \frac{F_k F_{n+k+2} - F_{k+2}F_{n+k}}{F_n F_{n+k} F_{n+k+2}} \\
&= \frac{F_k \left(F_{k+2}F_{n+1} + F_{k+1}F_n\right) - F_{k+2}\left(F_k F_{n+1} + F_{k-1}F_n\right)}{F_n F_{n+k} F_{n+k+2}} \\
&= \frac{F_k F_{k+1} F_n - F_{k+2} F_{k-1} F_n}{F_n F_{n+k} F_{n+k+2}} \\
&= \frac{-F_n \left(F_{k+2} F_{k-1} - F_{k+1} F_k\right)}{F_n F_{n+k} F_{n+k+2}} \\
&= \frac{-\left(F_{k+2} F_{k-1} - F_{k+1} F_k\right)}{F_{n+k} F_{n+k+2}} \\
&= \frac{(-1)^{k+1}}{F_{n+k} F_{n+k+2}} \quad \text{by (1.33) with k = 2.}
\end{aligned}$$

Therefore, we can write $\sum_{n=1}^{\infty} \left[ \frac{F_k}{F_n F_{n+k}} - \frac{F_{k+2}}{F_n F_{n+k+2}} \right] = (-1)^{k+1} \sum_{n=1}^{\infty} \frac{1}{F_{n+k} F_{n+k+2}}$.

On the other hand, we also have

$$\sum_{n=1}^{\infty}\frac{1}{F_{n+k}F_{n+k+2}} = \sum_{m=k+1}^{\infty}\frac{1}{F_m\,F_{m+2}},\ \text{where } m = n + k$$
$$= \sum_{m=1}^{\infty}\frac{1}{F_m\,F_{m+2}} - \sum_{m=1}^{k}\frac{1}{F_m\,F_{m+2}}$$
$$= 1 - \sum_{m=1}^{k}\frac{1}{F_m\,F_{m+2}}\ \text{ since } \sum_{m=1}^{\infty}\frac{1}{F_m\,F_{m+2}} = 1 \text{ by (1.34).}$$

It implies that $\displaystyle\sum_{n=1}^{\infty}\left[\frac{F_k}{F_n\,F_{n+k}} - \frac{F_{k+2}}{F_n\,F_{n+k+2}}\right] = (-1)^{k+1}\left(1 - \sum_{m=1}^{k}\frac{1}{F_m\,F_{m+2}}\right)$

$$= (-1)^k\left(\sum_{m=1}^{k}\frac{1}{F_m\,F_{m+2}} - 1\right).$$

□

Now, we use theorem 1.4.1 to define a recursive relation for the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{F_n\,F_{n+k}}$, $k \geq 1$.

***If k is even***, say k = 2r, where r is an integer and r ≥ 1, then the series becomes

$$\sum_{n=1}^{\infty}\frac{1}{F_n\,F_{n+2r}}.$$

Let us define $T_r = \displaystyle\sum_{n=1}^{\infty}\frac{1}{F_n\,F_{n+2r}}$, r ≥ 1, then $T_1 = \displaystyle\sum_{n=1}^{\infty}\frac{1}{F_n\,F_{n+2}} = 1$ by (1.34) and

$$T_{r+1} = \sum_{n=1}^{\infty}\frac{1}{F_n\,F_{n+2r+2}}.$$

We also define $A_o = 0$ and $A_r = \displaystyle\sum_{m=1}^{2r}\frac{1}{F_m\,F_{m+2}}$ with r ≥ 1. Then, it is easy to see that

(1.36) $$\boxed{A_r = A_{r-1} + \frac{1}{F_{2r-1}\,F_{2r+1}} + \frac{1}{F_{2r}\,F_{2r+2}}},\ \text{where } r \geq 1.$$

With the above definitions, using equation (1.35) with k = 2r, we get

$F_{2r}T_r - F_{2(r+1)}T_{r+1} = (-1)^{2r}(A_r - 1)$, or $F_{2r}T_r - F_{2(r+1)}T_{r+1} = A_r - 1$.

Solving the above equation for $T_{r+1}$, we obtain

(1.37)
$$\boxed{\begin{aligned} &T_{r+1} = \left(\frac{F_{2r}}{F_{2(r+1)}}\right)T_r + \frac{1}{F_{2(r+1)}}(1 - A_r),\ r \geq 1, \\ &T_1 = 1,\ A_0 = 0, \\ &A_r = A_{r-1} + \frac{1}{F_{2r-1}F_{2r+1}} + \frac{1}{F_{2r}F_{2r+2}}. \end{aligned}}$$

Next, we will evaluate some terms of $T_r$.

For r = 1, we have

$$A_1 = A_0 + \frac{1}{F_1F_3} + \frac{1}{F_2F_4} = \frac{5}{6}, \text{ and}$$

$$T_2 = \left(\frac{F_2}{F_4}\right)T_1 + \frac{1}{F_4}\left(1 - \frac{5}{6}\right) = \frac{7}{18}.$$

It means that

(1.38)
$$\boxed{T_2 = \sum_{n=1}^{\infty}\frac{1}{F_nF_{n+4}} = \frac{7}{18}}.$$

For r = 2, we have

$$A_2 = A_1 + \frac{1}{F_3F_5} + \frac{1}{F_4F_6} = \frac{5}{6} + \frac{1}{2\times5} + \frac{1}{3\times8} = \frac{39}{40}, \text{ and}$$

$$T_3 = \left(\frac{F_4}{F_6}\right)T_2 + \frac{1}{F_6}(1 - A_2) = \frac{3}{8}\times\frac{7}{18} + \frac{1}{8}\left(1 - \frac{39}{40}\right) = \frac{143}{960}.$$

This implies that

(1.39)
$$\boxed{T_3 = \sum_{n=1}^{\infty}\frac{1}{F_nF_{n+6}} = \frac{143}{960}}.$$

For r = 3, it follows that

$$A_3 = A_2 + \frac{1}{F_5 F_7} + \frac{1}{F_6 F_8} = \frac{39}{40} + \frac{1}{5\times 13} + \frac{1}{8\times 21} = \frac{272}{273}, \text{ and}$$

$$T_4 = \left(\frac{F_6}{F_8}\right) T_3 + \frac{1}{F_8}\left(1 - A_3\right) = \frac{8}{21}\times\frac{143}{960} + \frac{1}{21}\left(1 - \frac{272}{273}\right) = \frac{4351}{76440}.$$

Hence,

(1.40) $$\boxed{T_4 = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+8}} = \frac{4351}{76440}}.$$

In general, we can recursively calculate exactly the value of any series of the form

$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2r}}$, where r ≥ 1.

***If k is odd***, say k = 2r – 1, where r is an integer and r ≥ 1, then the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2r-1}}.$$

Let us define $P_r = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2r-1}}$, then $P_1 = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}}$, and $P_{r+1} = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2r+1}}$.

We also define $B_r = \sum_{m=1}^{2r-1} \frac{1}{F_m F_{m+2}}$ with r ≥ 1, then $B_1 = \frac{1}{F_1 F_3} = \frac{1}{2}$.

It is easy to see that

(1.41) $$\boxed{B_{r+1} = B_r + \frac{1}{F_{2r} F_{2r+2}} + \frac{1}{F_{2r+1} F_{2r+3}}}.$$

With the above definitions, using equation (1.35), we get

$$F_{2r-1}P_r - F_{2r+1}P_{r+1} = (-1)^{2r-1}(B_r - 1), \text{ or } F_{2r-1}P_r - F_{2r+1}P_{r+1} = -(B_r - 1).$$

Solving this equation for $P_{r+1}$ yields

(1.42)
$$\boxed{\begin{aligned} &P_{r+1} = \left(\frac{F_{2r-1}}{F_{2r+1}}\right)P_r + \frac{1}{F_{2r+1}}(B_r - 1),\ r \geq 1,\\ &B_1 = \frac{1}{2},\ P_1 = \sum_{n=1}^{\infty}\frac{1}{F_n F_{n+1}},\\ &B_{r+1} = B_r + \frac{1}{F_{2r}F_{2r+2}} + \frac{1}{F_{2r+1}F_{2r+3}}. \end{aligned}}$$

Now, we will calculate some particular values of $P_r$ using (1.42).

For r = 1, we have $P_2 = \frac{F_1}{F_3}P_1 + \frac{1}{F_3}(B_1 - 1) = \frac{1}{2}P_1 + \frac{1}{2}\left(\frac{1}{2} - 1\right) = \frac{1}{2}P_1 - \frac{1}{4}$.

It means that

(1.43)
$$\boxed{P_2 = \sum_{n=1}^{\infty}\frac{1}{F_n F_{n+3}} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+1}} - \frac{1}{4}}.$$

For r = 2, we have

$$B_2 = B_1 + \frac{1}{F_2 F_4} + \frac{1}{F_3 F_5} = \frac{1}{2} + \frac{1}{1\times 3} + \frac{1}{2\times 5} = \frac{14}{15},\ \text{and}$$

$$P_3 = \left(\frac{F_3}{F_5}\right)P_2 + \frac{1}{F_5}(B_2 - 1) = \frac{2}{5}\left(\frac{1}{2}P_1 - \frac{1}{4}\right) + \frac{1}{5}\left(\frac{14}{15} - 1\right) = \frac{1}{5}P_1 - \frac{17}{150}.$$

Therefore,

(1.44)
$$\boxed{P_3 = \sum_{n=1}^{\infty}\frac{1}{F_n F_{n+5}} = \frac{1}{5}\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+1}} - \frac{17}{150}}.$$

For r = 3, it follows that

$$B_3 = B_2 + \frac{1}{F_4 F_6} + \frac{1}{F_5 F_7} = \frac{14}{15} + \frac{1}{3\times 8} + \frac{1}{5\times 13} = \frac{103}{104},\ \text{and}$$

$$P_4 = \left(\frac{F_5}{F_7}\right)P_3 + \frac{1}{F_7}(B_3 - 1) = \frac{5}{13}\left(\frac{1}{5}P_1 - \frac{17}{150}\right) + \frac{1}{13}\left(\frac{103}{104} - 1\right) = \frac{1}{13}P_1 - \frac{279}{20250}.$$

Which gives

(1.45) $$\boxed{P_4 = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+7}} = \frac{1}{13}\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} - \frac{279}{20280}}.$$

In general, recursively, we can find the relationship between the series $\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2r-1}}$, where r ≥ 1 and the series $\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}}$. The sum of the series $\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}}$ can be found using Lambert series which will be discussed in chapter 3.

**1.4.4** Alternating series of the second degree:

The alternating series of a second degree has the form $\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+k}}$.

Theorem 1.4.2: Let $F_n$ be a Fibonacci number. Then for any positive integer k,

(1.46) $$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+k}} = \frac{1}{F_k}\left(\sum_{m=1}^{k} \frac{F_{m-1}}{F_m} - \frac{k}{\alpha}\right), \text{ where } \alpha = \frac{1+\sqrt{5}}{2}}.$$

Proof: By using equation (1.33), we have

$$\begin{aligned} \frac{(-1)^n F_k}{F_n F_{n+k}} &= \frac{F_{n-1}F_{n+k} - F_n F_{n+k-1}}{F_n F_{n+k}} \\ &= \frac{F_{n-1}}{F_n} - \frac{F_{n+k-1}}{F_{n+k}}. \end{aligned}$$

Now, we consider the finite sum $S_n = \sum_{m=1}^{n} \frac{(-1)^m F_k}{F_m F_{m+k}}$. Then, it follows that

$$S_n = \sum_{m=1}^{n}\left(\frac{F_{m-1}}{F_m} - \frac{F_{m+k-1}}{F_{m+k}}\right)$$

$$= \sum_{m=1}^{n}\frac{F_{m-1}}{F_m} - \sum_{m=1}^{n}\frac{F_{m+k-1}}{F_{m+k}}$$

$$= \left(\frac{F_0}{F_1} + \ldots + \frac{F_{k-1}}{F_k} + \frac{F_k}{F_{k+1}} + \ldots + \frac{F_{n-1}}{F_n}\right)$$

$$- \left(\frac{F_k}{F_{k+1}} + \ldots + \frac{F_{n-1}}{F_n} + \frac{F_n}{F_{n+1}} + \ldots + \frac{F_{n+k-1}}{F_{n+k}}\right)$$

$$= \left(\frac{F_0}{F_1} + \ldots + \frac{F_{k-1}}{F_k}\right) - \left(\frac{F_n}{F_{n+1}} + \ldots + \frac{F_{n+k-1}}{F_{n+k}}\right)$$ since the the last block of the first parentheses cancel with the first block of the second parentheses.

By (1.19), we have $\lim_{n\to\infty}\frac{F_n}{F_{n+1}} = \lim_{n\to\infty}\frac{F_{n+1}}{F_{n+2}} = \ldots = \lim_{n\to\infty}\frac{F_{n+k-1}}{F_{n+k}} = \frac{1}{\alpha}$.

Thus, $$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\left[\frac{F_0}{F_1} + \ldots + \frac{F_{k-1}}{F_k}\right] - \left[\frac{F_n}{F_{n+1}} + \ldots + \frac{F_{n+k-1}}{F_{n+k}}\right]\right)$$

$$= \left(\lim_{n\to\infty}\sum_{m=1}^{k}\frac{F_{m-1}}{F_m}\right) - \left(\underbrace{\frac{1}{\alpha} + \ldots + \frac{1}{\alpha}}_{k\ times}\right)$$

$$= \left(\sum_{m=1}^{k}\frac{F_{m-1}}{F_m}\right) - \frac{k}{\alpha}.$$

Hence, $\sum_{n=1}^{\infty}\frac{(-1)^n F_k}{F_n F_{n+k}} = \left(\sum_{m=1}^{k}\frac{F_{m-1}}{F_m}\right) - \frac{k}{\alpha}$, or $\sum_{n=1}^{\infty}\frac{(-1)^n}{F_n F_{n+k}} = \frac{1}{F_k}\left[\sum_{m=1}^{k}\frac{F_{m-1}}{F_m} - \frac{k}{\alpha}\right]$.

□

**1.4.5** Series of the third degree:

Series of the third degree have the form $\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+a} F_{n+b}}$ where a and b are positive integers. In this section, we just evaluate some special cases of series of the third degree.

Theorem 1.4.3: Let $F_n$ be a Fibonacci number, then

(1.47) $$\boxed{\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+2} F_{n+3}} = \frac{1}{4}}.$$

Proof: Observe that

$$\begin{aligned}\frac{1}{F_n F_{n+2} F_{n+3}} &= \frac{2F_{n+1}}{2F_n F_{n+1}F_{n+2} F_{n+3}}\\ &= \frac{F_{n+3} - F_n}{2F_n F_{n+1}F_{n+2} F_{n+3}} \quad \text{since } 2F_{n+1} = F_{n+3} - F_n\\ &= \frac{1}{2}\left(\frac{1}{F_n F_{n+1}F_{n+2}} - \frac{1}{F_{n+1}F_{n+2} F_{n+3}}\right).\end{aligned}$$

Let $S_n$ be a finite sum given by $S_n = \sum_{k=1}^{n}\frac{1}{F_k F_{k+2} F_{k+3}}$. Then, we have

$$\begin{aligned}S_n &= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{F_k F_{k+1}F_{k+2}} - \frac{1}{F_{k+1}F_{k+2} F_{k+3}}\right)\\ &= \frac{1}{2}\sum_{k=1}^{n}(a_k - a_{k+1}), \text{ where } a_k = \frac{1}{F_k F_{k+1}F_{k+2}}\\ &= \frac{1}{2}(a_1 - a_{n+1}) \text{ by (1.11)}\\ &= \frac{1}{2}\left(\frac{1}{F_1 F_2 F_3} - \frac{1}{F_{n+1}F_{n+2} F_{n+3}}\right).\end{aligned}$$

Taking limit of $S_n$ as $n \to \infty$, it follows that $\lim_{n\to\infty} S_n = \frac{1}{2}\frac{1}{F_1F_2F_3} = \frac{1}{4}$.

□

Theorem 1.4.4: Let $F_n$ be a Fibonacci number, then

(1.48) $$\boxed{\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+2} F_{n+4}} = \frac{7}{18} - \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+1} F_{n+2}}}.$$

Proof: From (1.32), we obtain $F_{n+3} = F_3F_{n+1} + F_2F_n$ and $F_{n+4} = F_4F_{n+1} + F_3F_n$. Then,

$$\begin{aligned}
\frac{F_3}{F_n F_{n+2} F_{n+3}} - \frac{F_4}{F_n F_{n+2} F_{n+4}} &= \frac{F_3F_{n+4} - F_4F_{n+3}}{F_n F_{n+2} F_{n+3}F_{n+4}} \\
&= \frac{F_3\left(F_4F_{n+1} + F_3F_n\right) - F_4\left(F_3F_{n+1} + F_2F_n\right)}{F_n F_{n+2} F_{n+3}F_{n+4}} \\
&= \frac{\left(F_3^2 - F_4F_2\right)F_n}{F_n F_{n+2} F_{n+3}F_{n+4}} \\
&= \frac{1}{F_{n+2} F_{n+3}F_{n+4}} \text{ since } F_2 = 1,\ F_3 = 2, \text{ and } F_4 = 3.
\end{aligned}$$

Hence, we have $\sum_{n=1}^{\infty}\frac{F_3}{F_n F_{n+2} F_{n+3}} - \sum_{n=1}^{\infty}\frac{F_4}{F_n F_{n+2} F_{n+4}} = \sum_{n=1}^{\infty}\frac{1}{F_{n+2} F_{n+3}F_{n+4}}$.

On the other hand, we also have

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{F_{n+2} F_{n+3}F_{n+4}} &= \sum_{k=3}^{\infty}\frac{1}{F_k F_{k+1}F_{k+2}}, \text{ where } k = n+2 \\
&= \sum_{k=1}^{\infty}\frac{1}{F_k F_{k+1}F_{k+2}} - \frac{1}{F_1 F_2F_3} - \frac{1}{F_2 F_3F_4} \\
&= \sum_{k=1}^{\infty}\frac{1}{F_k F_{k+1}F_{k+2}} - \frac{1}{2} - \frac{1}{6}.
\end{aligned}$$

It implies that $\sum_{n=1}^{\infty}\frac{F_3}{F_n F_{n+2} F_{n+3}} - \sum_{n=1}^{\infty}\frac{F_4}{F_n F_{n+2} F_{n+4}} = \sum_{n=1}^{\infty}\frac{1}{F_n F_{n+1}F_{n+2}} - \frac{2}{3}$.

Using (1.47) and solving the above equation for $\sum_{n=1}^{\infty}\frac{1}{F_n F_{n+2} F_{n+4}}$, we get (1.48).

□

In general, by applying (1.32), we will get appropriate coefficients like $F_3$ and $F_4$ in the proof of theorem 1.4.4 above. Then by using similar steps in the proof above, we can obtain the following result

(1.49) $$\boxed{\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+a} F_{n+b}} = c + d \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} F_{n+2}}}$$, where a, b, c, and d are positive integers.

More information about a, b, c, and d can be found in Brousseau [6].

The approach in this section has a limitation. It cannot evaluate directly to the final result as a formula. It requires going through recursive steps until the final result.

## 1.5 Evaluation of Series Using Programming

### 1.5.1 Finite sum of the reciprocals of Fibonacci numbers:

Here is the program written in Java to evaluate $\sum_{n=1}^{k} \frac{1}{F_n}$ for any positive integer k.

```
import java.text.*;
import java.math.*;
import java.io.*;
public class Fibonacci
{public static void main(String[] args) throws IOException
    {int n;
    char option = 'y';
    BigDecimal a1 = new BigDecimal(0);
    BigDecimal b1 = new BigDecimal(0);
    BigDecimal sum = new BigDecimal(2);
    BigDecimal f1 = new BigDecimal(1);
    BigDecimal f2 = new BigDecimal(1);
```

```
BigDecimal fn = new BigDecimal(0);
BufferedReader console = new BufferedReader(new InputStreamReader(System.in));
System.out.println("\n\t\tSUMMATION OF 1/Fn OF FIBONACCI NUMBER");
do  {f1 = new BigDecimal(1);
    f2 = new BigDecimal(1);
    fn = new BigDecimal(0);
    a1 = new BigDecimal(0);
    b1 = new BigDecimal(0);
    n=0;
    sum = new BigDecimal(2);
    System.out.print("Please enter value n = ");
    n = Integer.parseInt(console.readLine());
    if (n >= 0)
        {for (int i = 0; i <= (n - 3); i++)
            {fn = f1.add(f2);
            String f1copy = f1+ ""; //converts to a string
            f2 = new BigDecimal(f1copy); // string is used to make a new BigDecimal
            String f2copy = fn + "";
            f1 = new BigDecimal(f2copy);
            a1 = new BigDecimal(1);
            String fncopy = fn + "";
            b1 = new BigDecimal(fncopy);
```

```
            BigDecimal c1 = a1.divide(b1, 100, BigDecimal.ROUND_HALF_UP);
            sum = sum.add(c1);}
         System.out.println("\nFn= " + b1);
         System.out.println("Summation of 1/Fn = " + sum + "\n");}
      System.out.print("Would you like to continue? (y or n)>");
      option = (console.readLine()).charAt(0);
      if (option != 'y' && option != 'n')
      System.out.println("I assume you would like to continue");}
   while (option != 'n');}
}
```

Running this program for certain positive integers, we get the following results accurately to one hundred decimal places:

$$\sum_{n=1}^{100} \frac{1}{F_n} = 3.3598856662431775531674434867505621095621676787364058686707$$

1921818252342064903596728340535884336473339,

$$\sum_{n=1}^{200} \frac{1}{F_n} = 3.3598856662431775531720113029189271796888993668031301019180$$

419369233944833935522398279141710598758965,

$$\sum_{n=1}^{400} \frac{1}{F_n} = 3.3598856662431775531720113029189271796889051337319684864955$$

538153251303189966833836062240783148035250,

$$\sum_{n=1}^{1000} \frac{1}{F_n} = 3.3598856662431775531720113029189271796889051337319684864955$$

5381532513031899668338361541621645679008884.

### 1.5.2 Finite sum of the reciprocals of Lucas numbers:

Here is the program written in Java to evaluate $\sum_{n=1}^{k} \frac{1}{L_n}$ for any positive integer k.

```
import java.text.*;

import java.math.*;

import java.io.*;

public class LucasSieries

{    public static void main(String[] args) throws IOException

     {int n;

     char option = 'y';

     BigDecimal a1 = new BigDecimal(0);

     BigDecimal b1 = new BigDecimal(0);

     BigDecimal c1 = new BigDecimal(0);

     BigDecimal sum = new BigDecimal(1);

     BigDecimal L1 = new BigDecimal(1);

     BigDecimal L2 = new BigDecimal(3);

     BigDecimal Ln = new BigDecimal(0);

     BufferedReader console = new BufferedReader(new InputStreamReader(System.in));

     System.out.println("\n\t\tSUMMATION OF 1/Ln OF FIBONACCI NUMBER");

     do
```

```
{L1 = new BigDecimal(1);
L2 = new BigDecimal(3);
Ln = new BigDecimal(0);
a1 = new BigDecimal(0);
b1 = new BigDecimal(0);
sum = new BigDecimal(1);
n=0;
a1 = new BigDecimal(1);
b1 = new BigDecimal(3);
c1 = a1.divide(b1, 100, BigDecimal.ROUND_HALF_UP);
sum = sum.add(c1);
System.out.print("Please enter value n = ");
n = Integer.parseInt(console.readLine());
if (n >= 0)
    {for (int i = 0; i <= (n - 3); i++)
        {Ln = L1.add(L2);
         String L1copy = L2+ ""; //converts to a string
         L1 = new BigDecimal(L1copy); // string makes a new BigDecimal
        String L2copy = Ln + "";
        L2 = new BigDecimal(L2copy);
        a1 = new BigDecimal(1);
        String Lncopy = Ln + "";
```

```
            b1 = new BigDecimal(Lncopy);
            c1 = a1.divide(b1, 100, BigDecimal.ROUND_HALF_UP);
            sum = sum.add(c1);}
          System.out.println("\nLn= " + b1);
        System.out.println("Summation of 1/Ln = " + sum + "\n");}
      System.out.print("Would you like to continue? (y or n):");
      option = (console.readLine()).charAt(0);
      if (option != 'y' && option != 'n')
      System.out.println("I assume you would like to continue");}
   while (option != 'n');}
}
```

Running this program for certain positive integers, we get the following results accurately to one hundred decimal places:

$$\sum_{n=1}^{100} \frac{1}{L_n} = 1.962858173209645782866752339182946142407233607578736017602901410109936486349849291668825856992280 0475,$$

$$\sum_{n=1}^{400} \frac{1}{L_n} = 1.9628581732096457828687951286751835266495930171622194211307152404170616075464603779749310499352245825,$$

$$\sum_{n=1}^{1000} \frac{1}{L_n} = 1.9628581732096457828687951286751835266495930171622194211307152404170616075464603779790418990840346964.$$

# CHAPTER II

# ELLIPTIC FUNCTIONS AND APPLICATIONS

## 2.1 Theory of Elliptic Functions

**2.1.1** Definitions   Let z = u + iv be a complex variable.

Definition 1. An analytic function $f(z)$ is *doubly-periodic* with periods $w_1$ and $w_2$ ($w_1$ and $w_2$ are complex and independent) if

(2.1) $\boxed{f(z+w_1)=f(z),\ f(z+w_2)=f(z)}$ for all z in the domain of $f(z)$.

The existence of doubly-periodic analytic functions was established in 1827 by N.H. Abel and C.G.J. Jacobi (1804 – 1851), independent of each other.

Definition 2 An analytic function $f(z)$ is said to *be elliptic* if

(2,2) $\boxed{\begin{array}{l}(i)\ f(z) \text{ is doubly-periodic,}\\ (ii)\ f(z) \text{ is analytic in the finite plane except for poles.}\end{array}}$

An explicit example of an elliptic function is

$$f(z)=\sum_{m=-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty}(z-m-ni)^{-3}\right].$$

This function has the periods $w_1$ = 1 and $w_2$ = i, and the poles z = m + ni.

The periods of a doubly periodic function $f(z)$ are

(2.3) $\boxed{mw_1+nw_2}$, where m, n = 0, ±1, ±2, ….

These periods form a "mesh" in the z-plane and determine a set of congruent parallelograms, known as *periodic parallelograms* (Figure 1).

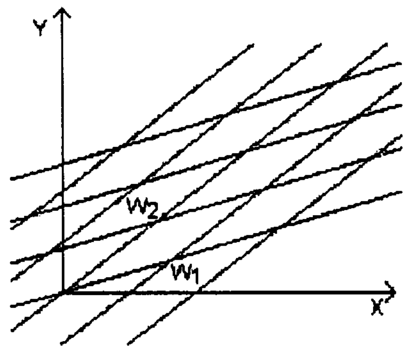


Figure 1

The range of any doubly-periodic function coincides with the set of values which it assumes in a period parallelogram.

Definition 3 The order of an elliptic function is the sum of the orders of the poles of the elliptic function in the period parallelogram.

More information about doubly-periodic functions can be found in Einar Hille [11].

**2.1.2** Introduction of Jacobi's elliptic functions (*sn* z, *cn* z, and *dn* z)

Consider the following integral:

(2.4) $$\boxed{z=\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}}$$ , where $0 < k < 1$ and y is a complex variable.

This integral defines z as a function of k and y where k is called the modulus of the integral. Conversely, y may be considered as a function of z defined by (2.4) except at its singularities; we use a new symbol for this function and write $y = sn\,(z,k)$, or simply

(2.5) $\boxed{y = sn\,z}$ when it is not necessary to emphasize the modulus k.

*sn* z is called an elliptic function.

In addition, Jacobi [14] defined two more elliptic functions as follows:

(2.6) $\boxed{cn\, z = \sqrt{1 - sn^2 z}}$, and

(2.7) $\boxed{dn\, z = \sqrt{1 - k^2 sn^2 z}}$.

**2.1.3** Simple properties of *sn* z, *cn* z, and *dn* z

If we set y = 0 in (2.4), then z = 0. Hence, from (2.5), it follows that *sn* 0 = 0.

Substitute *sn* 0 = 0 into (2.6) and (2.7), we obtain the properties,

(2.8) $\boxed{sn\, 0 = 0,\ cn\, 0 = 1,\ dn\, 0 = 1}$.

If we replace t by – t in (2.4), we will get $z = -\int_0^{-y} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$, or

$$-z = \int_0^{-y} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Then, by definition of *sn* z, it implies that *sn* (-z) = – y.

Since, y = *sn* z by (2.5), it follows that *sn* z = – *sn* (-z). Thus, *sn* z is an odd function.

Since *sn* z = – *sn* (-z), it implies that $sn^2$ z = $sn^2$ (-z). Therefore, it is easy to see that *cn* z, and *dn* z are even functions by (2.6) and (2.7).

Thus, we have the properties

(2.9) $\boxed{sn(-z) = -sn\, z,\ cn(-z) = cn\, z, \text{ and } dn(-z) = dn\, z}$.

Now, differentiating both sides of (2.4) with respect to y, we get

$$\frac{dz}{dy} = \frac{1}{\sqrt{(1-y^2)(1-k^2y^2)}}, \text{ or}$$

$$\frac{dy}{dz} = \sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}$$

$$= \sqrt{1 - sn^2 z}\sqrt{1-k^2 sn^2 z}, \text{ since } y = sn\, z$$

$= cn\,z \cdot dn\,z$ by (2.6) and (2.7).

Hence, we have

$$\text{(2.10)} \qquad \boxed{\frac{d(sn\,z)}{dz} = cn\,z \cdot dn\,z}.$$

Next, we differentiate both sides of (2.6) with respect to z.

$$\begin{aligned}
\frac{d(cn\,z)}{dz} &= \frac{d}{dz}\left(\sqrt{1-sn^2 z}\right) \\
&= \frac{1}{2}\,\frac{\frac{d}{dz}\left(-sn^2 z\right)}{\sqrt{1-sn^2 z}} \\
&= \frac{-sn\,z \cdot cn\,z \cdot dn\,z}{\sqrt{1-sn^2 z}} \quad \text{by (2.10)} \\
&= \frac{-sn\,z \cdot cn\,z \cdot dn\,z}{cn\,z} \quad \text{by (2.6)} \\
&= -sn\,z \cdot dn\,z.
\end{aligned}$$

Thus, we have shown another property,

$$\text{(2.11)} \qquad \boxed{\frac{d(cn\,z)}{dz} = -sn\,z \cdot dn\,z}.$$

Similarly, differentiating both side of (2.7) with respect to z, we get the next property,

$$\text{(2.12)} \qquad \boxed{\frac{d(dn\,z)}{dz} = -k^2 sn\,z \cdot cn\,z}.$$

If we define $K = \int_0^1 \frac{dt}{\sqrt{\left(1-t^2\right)\left(1-k^2t^2\right)}}$, then using (2.5), (2.6), and (2.7), we obtain

$$\text{(2.13)} \qquad \boxed{sn\,K = 1,\ cn\,K = 0,\ \text{and}\ dn\,K = \sqrt{1-k^2}}.$$

The following addition properties are well-known (see Woods [16], p. 372).

(2.14)
$$\begin{aligned} sn(z_1+z_2) &= \frac{sn\,z_1\; cn\,z_2\; dn\,z_2 + sn\,z_2\; cn\,z_1\; dn\,z_1}{1-k^2 sn^2 z_1\; sn^2 z_2}, \\ cn(z_1+z_2) &= \frac{cn\,z_1\; cn\,z_2\; - sn\,z_1\; sn\,z_2\; dn\,z_1\; dn\,z_2}{1-k^2 sn^2 z_1\; sn^2 z_2}, \text{ and} \\ dn(z_1+z_2) &= \frac{dn\,z_1\; dn\,z_2\; -k^2\, sn\,z_1\; sn\,z_2\; cn\,z_1\, cn\,z_2}{1-k^2 sn^2 z_1\; sn^2 z_2}. \end{aligned}$$

### 2.1.4 Periods of Jacobi's elliptic functions

Let t = sin α in (2.4), then it becomes

(2.15)
$$z = \int_0^{\alpha_1} \frac{d\alpha}{\sqrt{(1-k^2\sin^2\alpha)}}, \text{ where } \alpha_1 = \sin^{-1} y \text{ and } 0 < |y| < 1.$$

The angle $\alpha_1$ is called the amplitude of z, written as $\alpha_1$= am z. Then,

(2.16)
$$\begin{aligned} \sin\alpha_1 &= \sin(am\,z) = sn\,z, \\ \cos\alpha_1 &= \cos(am\,z) = cn\,z, \\ \sqrt{1-k^2\sin^2\alpha_1} &= \sqrt{1-k^2\sin^2(am\,z)} = dn\,z. \end{aligned}$$

We now study the effect on z in (2.15) by adding π to $\alpha_1$.

$$\int_0^{\alpha_1+\pi} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \int_0^{\pi} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} + \int_{\pi}^{\pi+\alpha_1} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} + \int_{\pi}^{\pi+\alpha_1} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} \text{ since } \sin\alpha \text{ is symmetric}$$

about x equals $\frac{\pi}{2}$

$$= 2\int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} + \int_0^{\alpha_1} \frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} \text{ by letting } \alpha=(\pi+\beta).$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} + z \text{ by (2.15).}$$

If we put t = sinα in $K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$, then

(2.17) $$\boxed{K = \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}}.$$

Hence, it follows that $$z + 2K = \int_0^{\pi+\alpha_1} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}.$$

Thus, using (2.16) we obtain

$$sn(z+2K) = \sin(\alpha_1 + \pi) = -\sin\alpha_1 = -sn\, z,$$

$$cn(z+2K) = \cos(\alpha_1 + \pi) = -\cos\alpha_1 = -cn\, z,$$

$$dn(z+2K) = \sqrt{1-\sin^2(\alpha_1+\pi)} = \sqrt{1-\sin^2\alpha_1} = dn\, z.$$

Replacing z by z + 2K into these equations above, we have

(2.18) $$\boxed{\begin{aligned} &sn(z+4K) = -sn(z+2K) = sn\, z, \\ &cn(z+4K) = -cn(z+2K) = cn\, z, \text{ and} \\ &dn(z+4K) = dn(z+2K) = dn\, z \end{aligned}}$$

*Thus we have proved that the elliptic functions sn* z *and cn* z *are periodic with period 4K, and dn* z *is periodic with period 2K.*

Now, we define $K'$ by the formula

(2.19) $$\boxed{K' = \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1-k'^2\sin^2\alpha}} = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k'^2s^2)}}} \text{ where } k' = \sqrt{1-k^2},$$

k is the modulus of integral, and $k'$ is the complement modulus of integral.

If we let t = $\frac{1}{\sqrt{1-k'^2s^2}}$, then we have the following results:

$$\frac{dt}{ds} = \frac{k'^2 s}{\left(1-k'^2 s^2\right)\sqrt{1-k'^2 s^2}},$$

$$\text{kt} = \frac{k}{\sqrt{1-k'^2 s^2}}, \text{ or } \quad \sqrt{1-k^2 t^2} = \sqrt{\frac{\left(1-k^2\right)-k'^2 s^2}{1-k'^2 s^2}}$$

$$= \frac{\sqrt{k'^2 - k'^2 s^2}}{\sqrt{1-k'^2 s^2}} \text{ since } k'^2 + k^2 = 1$$

$$= \frac{k'\sqrt{1-s^2}}{\sqrt{1-k'^2 s^2}},$$

and $\sqrt{1-t^2} = -\frac{ik's}{\sqrt{1-k'^2 s^2}}$ (we ignore the positive answer).

Therefore, we have

$$\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2 t^2}} = \frac{\dfrac{k'^2 s\, ds}{\left(1-k'^2 s^2\right)\sqrt{1-k'^2 s^2}}}{\left(\dfrac{-ik's}{\sqrt{1-k'^2 s^2}}\right)\left(\dfrac{k'\sqrt{1-s^2}}{\sqrt{1-k'^2 s^2}}\right)} = \frac{i\, ds}{\sqrt{\left(1-s^2\right)\left(1-k'^2 s^2\right)}}.$$

Since $t = \frac{1}{\sqrt{1-k'^2 s^2}}$, it follows that $\begin{cases} t = 1, \text{when } s = 0 \\ t = \dfrac{1}{\sqrt{1-k'^2}} = \dfrac{1}{k}, \text{when } s = 1. \end{cases}$

Therefore, we have $\int_1^{\frac{1}{k}} \frac{dt}{\sqrt{\left(1-t^2\right)\left(1-k^2 t^2\right)}} = \int_0^1 \frac{i\, ds}{\sqrt{\left(1-s^2\right)\left(1-k'^2 s^2\right)}}$.

Since the integral on the right side is equal to $iK'$ by (2.19), it implies that

(2.20) $$\boxed{K' = -i\int_1^{\frac{1}{k}} \frac{dt}{\sqrt{\left(1-t^2\right)\left(1-k^2 t^2\right)}}}.$$

From (2.17), (2.19), and (2.20), we have

$$K+iK'=\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}+\int_1^{\frac{1}{k}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\int_0^{\frac{1}{k}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Hence, it follows that

(2.21)
$$\boxed{\begin{aligned} &sn(K+iK')=\frac{1}{k} \text{ by (2.5),}\\ &cn(K+iK')=-\frac{ik'}{k} \text{ by (2.6), and}\\ &dn(K+iK')=0 \text{ by (2.7).}\end{aligned}}$$

For the negative sign of $cn(K+iK')$, see Woods [16], pages 337 – 339.

Using the results of (2.21) and (2.14), we obtain

(2.22)
$$\boxed{\begin{aligned} &sn(z+K+iK')=\frac{dn\,z}{k\,cn\,z},\\ &cn(z+K+iK')=-\frac{ik'}{k\,cn\,z}, \text{ and}\\ &dn(z+K+iK')=\frac{ik'\,sn\,z}{cn\,z}.\end{aligned}}$$

Let us derive the first identity of (2.22); the other two will be derived in a similar way.

From (2.14), we have

$$\begin{aligned} sn(z+K+iK') &= \frac{sn\,z\ cn(K+iK')\ dn(K+iK')+sn(K+iK')\ cn\,z\ dn\,z}{1-k^2sn^2z\,sn^2(K+iK')}\\ &= \frac{0+\frac{1}{k}cn\,z\ dn\,z}{1-k^2sn^2z\left(\frac{1}{k^2}\right)} \text{ by}(2.21)\\ &= \frac{cn\,z\ dn\,z}{k(1-sn^2z)}\\ &= \frac{dn\,z}{k\,cn\,z} \text{ since } cn^2z=1-sn^2z. \end{aligned}$$

□

Again, we can use (2.22) and (2.14) to yield

(2.23)
$$\boxed{\begin{aligned} &sn(z+iK') = \frac{1}{k\,sn\,z},\\ &cn(z+iK') = -\frac{i\,dn\,z}{k\,sn\,z}, \text{ and}\\ &dn(z+iK') = -\frac{i\,cn\,z}{sn\,z}.\end{aligned}}$$

We derive the first identity of (2.23); the other two will be derived similarly.

Since $z+iK' = (z+K+iK')+(-K)$, using (2.14), we have

$$\begin{aligned} sn(z+iK') &= sn\left[(z+K+iK')+(-K)\right]\\ &= \frac{sn(z+K+iK')\,cn(-K)\,dn(-K)+sn(-K)\,cn(z+K+iK')\,dn(z+K+iK')}{1-k^2sn^2(z+K+iK')sn^2(-K)}.\end{aligned}$$

We also have $cn(-K) = cnK = 0$ and $sn(-K) = -snK = -1$, by using (2.9) and (2.13).

Then, using (2.22), we get

$$sn(z+iK') = \frac{0+(-1)\left(\dfrac{-ik'}{k\,cn\,z}\right)\left(\dfrac{-ik'sn\,z}{cn\,z}\right)}{1-k^2\left(\dfrac{dn^2 z}{k^2cn^2 z}\right)(-1)^2} = \frac{k'^2 sn\,z}{k\left(dn^2 z - cn^2 z\right)}.$$

Since $cn^2 z = 1-sn^2 z$ from (2.6) and $dn^2 z = 1-k^2sn^2 z$ from (2.7), it follows that

$$\begin{aligned} dn^2 z - cn^2 z &= sn^2 z - k^2 sn^2 z\\ &= (1-k^2)sn^2 z\\ &= k'^2 sn^2 z.\end{aligned}$$

Hence, $sn(z+iK') = \dfrac{k'^2 sn\,z}{k\times k'^2 sn^2 z} = \dfrac{1}{k\,sn\,z}$.

□

If we replace z by $z+iK'$ into (2.23), we get

(2.24) $$\boxed{\begin{array}{l} sn(z+2iK') = sn\,z, \\ cn(z+2iK') = -cn\,z, \text{ and} \\ dn(z+2iK') = -dn\,z. \end{array}}$$

We verify the first identity.

If we replace z by $z+iK'$ into (2.23), we get

$$sn(z+2iK') = \frac{1}{k\,sn(z+iK')} \underset{by(2.23)}{=} \frac{1}{k\dfrac{1}{k\,sn\,z}} = sn\,z.$$

□

Similarly, if we replace z by $z+2iK'$ into (2.24), we obtain

(2.25) $$\boxed{\begin{array}{l} sn(z+4iK') = sn\,z, \\ cn(z+4iK') = cn\,z, \text{ and} \\ dn(z+4iK') = dn\,z. \end{array}}$$

From (2.18), (2.24), and (2.25) we can conclude that:

*The elliptic functions sn z, cn z, and dn z are doubly periodic functions: the function sn z has the periods 4K and $2iK'$; the function cn z has the periods 4K and $4iK'$, and the function dn z has the periods 2K and $4iK'$.*

### **2.1.5** Maclaurin's series expansion

Let $f(z) = sn$ z, then the Maclaurin's series of the function $f(z)$ at 0 is given by

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \ldots.$$

Let's recall (2.8), (2.10), (2.11), and (2.12) as the following:

$sn0 = 0,\ cn0 = 1,\ dn0 = 1,\ (sn\,z)' = cn\,z \cdot dn\,z$ ,

$(cn\,z)' = -sn\,z \cdot dn\,z$, and $(dn\,z)' = -k^2 sn\,z \cdot cn\,z$.

Hence,

$f(z) = sn$ z implies $f(0) = sn\ 0 = 0$,

$f'(z) = (sn\,z)' = cn\,z \cdot dn\,z$ implies $f'(0) = cn\,0 \cdot dn\,0 = 1$ , and

$f''(z) = -sn\,z \cdot dn^2\,z - k^2 sn\,z \cdot cn^2\,z$ implies $f''(0) = 0$.

All the even derivatives contain *sn* z as a factor, therefore,

$f^{(2)}(0) = f^{(4)}(0) = f^{(6)}(0) = f^{(8)}(0) = \ldots = 0$.

Thus, the Maclaurin's series for *sn* z can be written as

(2.26) $$\boxed{sn\,z = z - \left(1+k^2\right)\frac{z^3}{3!} + \left(1+14k^2+k^4\right)\frac{z^5}{5!} - \ldots}.$$

Similarly, we also get the Maclaurin's series for *cn* z and *dn* z as

(2.27) $$\boxed{cn\,z = 1 - \frac{z^2}{2!} + \left(1+4k^2\right)\frac{z^4}{4!} - \left(1+44k^2+16k^4\right)\frac{z^6}{6!} + \ldots},$$

(2.28) $$\boxed{dn\,z = 1 - k^2\frac{z^2}{2!} + k^2\left(4+k^2\right)\frac{z^4}{4!} - k^2\left(16+44k^2+k^4\right)\frac{z^6}{6!} + \ldots}.$$

## 2.2 Fourier Series of Jacobi Elliptic Functions

In this section, z is a complex variable. Our objective is to sketch the proof of the following theorem

(2.29) $$sn\,z = \frac{2\pi\sqrt{q}}{kK}\sum_{m=1}^{\infty}\left(\frac{q^{m-1}}{1-q^{2m-1}}\right)\sin\left[(2m-1)\frac{\pi z}{2K}\right],$$ where $q = e^{-\frac{\pi K'}{K}}$ and $\left|q \cdot e^{\pm i\frac{\pi z}{K}}\right| < 1$.

Since the proof is long and very difficult, we will not go over all the details. For a complete proof, one can see Hancock [10], pages 99 – 261. The reason that we introduce this proof is the Fourier expansion will be used heavily later. Also, the details here will enhance the understanding of the Jacobi elliptic functions.

First, we define the function $\phi(z)$ and the constant q as follows

(2.30) $$\phi(z) = 1 + e^{\frac{i\pi z}{K}}, \text{ and } q = e^{-\frac{\pi K'}{K}}.$$

We use this definition to prove the following lemma.

Lemma 2.2.1

(2.31) $$\phi(z + niK')\,\phi(-z + niK') = 1 + 2q^n \cos\frac{\pi z}{K} + q^{2n}.$$

Proof: By definition, we have

$$\begin{aligned}
\phi(z + niK')\,\phi(-z + niK') &= \left[1 + e^{i\pi\left(\frac{z+niK'}{K}\right)}\right]\left[1 + e^{i\pi\left(\frac{-z+niK'}{K}\right)}\right] \\
&= \left[1 + e^{\left(\frac{i\pi z}{K} - \frac{n\pi K'}{K}\right)}\right]\left[1 + e^{\left(-\frac{i\pi z}{K} - \frac{n\pi K'}{K}\right)}\right] \\
&= \left(1 + q^n e^{\frac{i\pi z}{K}}\right)\left(1 + q^n e^{-\frac{i\pi z}{K}}\right) \\
&= 1 + q^n\left(e^{\frac{i\pi z}{K}} + e^{-\frac{i\pi z}{K}}\right) + q^{2n}
\end{aligned}$$

$$= 1 + 2q^n \cos\frac{\pi z}{K} + q^{2n} \quad \text{since} \quad 2\cos z = e^{-iz} + e^{iz}.$$

□

Now we consider the two functions:

$$\Phi(z) = [\phi(z+iK')\cdot\phi(z+3iK')\phi(z+5iK')\cdots][\phi(-z+iK')\cdot\phi(-z+3iK')\phi(-z+5iK')\cdots],$$

and $\Phi_1(z) = \phi(z)[\phi(z+2iK')\phi(z+4iK')\cdots][\phi(-z+2iK')\phi(-z+4iK')\cdots]$.

Then using (2.31), it is easy to verify the two following identities.

(2.32) $$\boxed{\Phi(z) = \prod_{n=1}^{\infty}\left(1 + 2q^{2n-1}\cos\frac{\pi z}{K} + q^{2(2n-1)}\right)},$$

(2.33) $$\boxed{\Phi_1(z) = \left(1 + e^{\frac{i\pi z}{K}}\right)\prod_{n=1}^{\infty}\left(1 + 2q^{2n}\cos\frac{\pi z}{K} + q^{4n}\right)}.$$

Next, we introduce the four Jacobi's theta functions.

(2.34) $$\boxed{\begin{array}{l}\Theta_1(z) = A\,\Phi(z),\ H_1(z) = Ae^{\frac{i\pi}{2K}\left(\frac{1}{2}iK'-z\right)}\cdot\Phi_1(z),\\ \Theta(z) = \Theta_1(K-z),\ H(z) = H_1(K-z),\\ \text{where A is a constant.}\end{array}}$$

With these definitions, using (2.32), we get

(2.35) $$\boxed{\Theta_1(z) = A\prod_{n=1}^{\infty}\left(1 + 2q^{2n-1}\cos\frac{\pi z}{K} + q^{2n(2n-1)}\right)},$$

(2.36) $$\boxed{\Theta(z) = A\prod_{n=1}^{\infty}\left(1 - 2q^{2n-1}\cos\frac{\pi z}{K} + q^{2n(2n-1)}\right)}$$

$$\text{since } \cos\frac{\pi}{K}(K-z) = \cos\left(\pi - \frac{\pi z}{K}\right) = -\cos\frac{\pi z}{K}.$$

We also have $e^{\frac{i\pi}{2K}\left(\frac{1}{2}iK'-z\right)} = e^{\left(-\frac{\pi K'}{4K} - \frac{i\pi z}{2K}\right)} = \sqrt[4]{q}\cdot e^{-\frac{i\pi z}{2K}}$ since $q = e^{-\frac{\pi K'}{K}}$.

Hence, from (2.34) we can rewrite $H_1(z)$ as follow:

$$\begin{aligned} H_1(z) &= A\cdot\sqrt[4]{q}\cdot e^{-\frac{i\pi z}{2K}}\cdot\Phi_1(z) \\ &= A\cdot\sqrt[4]{q}\cdot e^{-\frac{i\pi z}{2K}}\cdot\left(1+e^{\frac{i\pi z}{K}}\right)\prod_{n=1}^{\infty}\left(1+2q^{2n}\cos\frac{\pi z}{K}+q^{4n}\right) \text{ by (2.33)} \\ &= A\sqrt[4]{q}\left(e^{-\frac{i\pi z}{2K}}+e^{\frac{i\pi z}{2K}}\right)\prod_{n=1}^{\infty}\left(1+2q^{2n}\cos\frac{\pi z}{K}+q^{4n}\right) \end{aligned}$$

Again, using $2\cos z = e^{-iz}+e^{iz}$, we obtain:

(2.37) $$\boxed{H_1(z) = \left(2A\sqrt[4]{q}\cos\frac{\pi z}{2K}\right)\prod_{n=1}^{\infty}\left(1+2q^{2n}\cos\frac{\pi z}{K}+q^{4n}\right)}.$$

Now, using (2.34), (2.37), and the facts that $\cos\frac{\pi}{2K}(K-z)=\cos\left(\frac{\pi}{2}-\frac{\pi z}{2K}\right)=\sin\frac{\pi z}{2K}$,

and $\cos\frac{\pi}{K}(K-z)=\cos\left(\pi-\frac{\pi z}{K}\right)=-\cos\frac{\pi z}{K}$, we can rewrite $H(z)$ as follow:

(2.38) $$\boxed{H(z) = \left(2A\sqrt[4]{q}\sin\frac{\pi z}{2K}\right)\prod_{n=1}^{\infty}\left(1-2q^{2n}\cos\frac{\pi z}{K}+q^{4n}\right)}.$$

<u>Lemma 2.2.2</u>

(2.39) $$\boxed{1-2q^{m}\cos 2z+q^{2m} = \left(1-q^{m}e^{i2z}\right)\left(1-q^{m}e^{-i2z}\right)}.$$

<u>Proof</u>: We have

$$\begin{aligned} \left(1-q^{m}e^{i2z}\right)\left(1-q^{m}e^{-i2z}\right) &= 1-q^{m}\left(e^{i2z}+e^{-i2z}\right)+q^{2m} \\ &= 1-2q^{m}\left(\frac{e^{i2z}+e^{-i2z}}{2}\right)+q^{2m} \\ &= 1-2q^{m}\cos(2z)+q^{2m}. \end{aligned}$$

□

Lemma 2.2.3

(2.40) $$\boxed{\text{Let } z = u + iv, \text{ then } \lim_{v\to\infty}\left|\frac{1}{\sin z}\right| = 0}.$$

Proof: We have

$$\begin{aligned}
|\sin z| &= |\sin(u+iv)| \\
&= \left|\frac{e^{iu-v} - e^{-iu+v}}{2i}\right| \\
&\geq \left|\frac{|e^{iu-v}| - |e^{-iu+v}|}{|2i|}\right| \\
&= \left|\frac{e^{-v} - e^{v}}{2}\right| \\
&= |\sinh v|.
\end{aligned}$$

Hence, it follows that $\frac{1}{|\sin z|} \leq \frac{1}{|\sinh v|} = 0$ as $v \to \infty$.

□

Theorem 2.2.1 Suppose $f(z)$ is a doubly-periodic function of the $2n^{th}$ order with periods $4K$ and $i2K'$ such that $f(z+2K) = -f(z)$, and $f(z)$ has n poles $\alpha_t$ within the period- rectangle RSTU, where R is an arbitrary point $z_0$, S and U are the two points $z_0 + 2K$ and $z_0 + i2K'$. Then

(2.41) $$\boxed{f(z) = \frac{\pi}{2K}\sum_{m=-\infty}^{\infty}\sum_{t=1}^{n}\frac{A_t}{\sin\frac{\pi}{2K}(z - \alpha_t - i2mK')}},$$ where

$A_t$ is the residue of $f(z)$ relative to $\alpha_t$.

Proof: Let LMNP be a rectangle (figure 2) whose vertices L and P are the points $z_o - i2lK'$ and $z_o + i2lK'$, while M and N are the points $(z_o + 2K) - i2lK'$ and $(z_o + 2K) + i2lK'$, where $l$ is an integer.

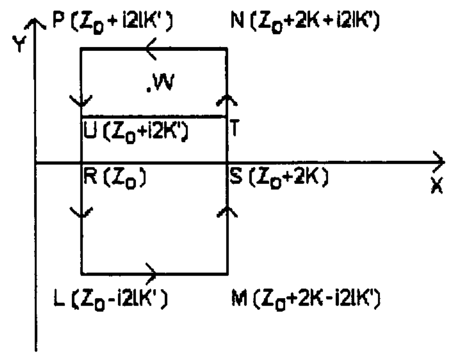


Figure 2

Let w be an arbitrary point within the rectangle LMNP.

Consider the line integral $\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}dz$, where $\gamma$ is taken counterclockwise over the sides of the rectangle LMNP.

Since $$\frac{f(z+2K)}{\sin\frac{\pi}{2K}(z+2K-w)} = \frac{-f(z)}{\sin\left(\pi + \frac{\pi}{2K}(z-w)\right)} = \frac{-f(z)}{-\sin\frac{\pi}{2K}(z-w)} = \frac{f(z)}{\sin\frac{\pi}{2K}(z-w)},$$

it is clear that $\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}$ has the period 2K and its poles are the points z = w and $z = \alpha_t + i2mK'$ where m varies from $-l$ to $l - 1$.

From Cauchy's theorem, it follows that the integral $\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}dz$ equals to the sum of the residues relative to the poles that are located with the rectangle LMNP.

We have

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}dz=\frac{1}{2\pi i}\int_{LM}\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}dz+\frac{1}{2\pi i}\int_{MN}\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}dz$$
$$+\frac{1}{2\pi i}\int_{NP}\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}dz+\frac{1}{2\pi i}\int_{PL}\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}dz$$

Since the integrand has a period 2K, it implies that

$\frac{1}{2\pi i}\int_{MN}\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}dz+\frac{1}{2\pi i}\int_{PL}\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}dz=0$ because these two integrals have opposite direction of each other. On the other hand, we also have

$\frac{1}{2\pi i}\int_{LM}\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}dz+\frac{1}{2\pi i}\int_{NP}\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}dz=0$ because when $l$ becomes large, then $\frac{1}{\sin\frac{\pi}{2K}(z-w)}$ approaches zero by (2.40).

Thus, the sum of the residues equals to zero, which means

$$\text{Res}\left(\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)},w\right)+\sum_{t=1}^{n}\text{Res}\left(\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)},\alpha_t\right)=0.$$

Also, since

$$\text{Res}\left(\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}, w\right) = \lim_{z\to w}\left(\frac{(z-w)f(z)}{\sin\frac{\pi}{2K}(z-w)}\right)$$

$$= \lim_{z\to w}\left(\frac{1}{\frac{\pi}{2K}\cos\frac{\pi}{2K}(z-w)} f(w)\right) \text{ by L'Hospital rule}$$

$$= \frac{2K}{\pi} f(w),$$

it follows that $\sum_{t=1}^{n}\text{Res}\left(\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}, \alpha_t\right) = -\frac{2K}{\pi} f(w)$, or

$$f(w) = -\frac{\pi}{2K}\sum_{t=1}^{n}\text{Res}\left(\frac{f(z)}{\sin\frac{\pi}{2K}(z-w)}, \alpha_t\right) = \frac{\pi}{2K}\sum_{t=1}^{n}\text{Res}\left(\frac{f(z)}{\sin\frac{\pi}{2K}(w-z)}, \alpha_t\right).$$

We know $f(z)$ has n poles $\alpha_t$ within the period-rectangle RSTU. Let $A_t$ be the residue of $f(z)$ relative to $\alpha_t$. Then for all w in the finite complex plane (see Hancock [10]),

we have $f(w) = \frac{\pi}{2K}\sum_{m=-\infty}^{\infty}\sum_{t=1}^{n}\frac{A_t}{\sin\frac{\pi}{2K}\left(w-\alpha_t - i2mK'\right)}$.

Then by replacing w by z, we get (2.41).

□

Corollary

(2.42) $$\boxed{\frac{\Theta(z)}{H(z)} = \frac{\pi}{2K}\frac{\Theta(0)}{H'(0)}\sum_{m=-\infty}^{\infty}\frac{1}{\sin\frac{\pi}{2K}(z-2mK'i)}}$$, where

$\Theta$ (z) and H (z) are the Jacobi theta functions.

Proof From (2.36) and (2.38), we have

$$\frac{\Theta(z)}{H(z)}=\frac{\left(1-2q\cos\frac{\pi z}{K}+q^2\right)\left(1-2q^3\cos\frac{\pi z}{K}+q^6\right)\dots}{\left(2\sqrt[4]{q}\sin\frac{\pi z}{2K}\right)\left(1-2q^2\cos\frac{\pi z}{K}+q^4\right)\left(1-2q^4\cos\frac{\pi z}{K}+q^8\right)\dots}$$

It is clear that

$$\frac{\Theta(z+2K)}{H(z+2K)}=-\frac{\Theta(z)}{H(z)},\ \text{since}\ \cos\frac{\pi}{K}(z+2K)=\cos\frac{\pi z}{K},\ \text{and}\ \sin\frac{\pi}{2K}(z+2K)=-\sin\frac{\pi z}{2K}$$

and $$\frac{\Theta(z+4K)}{H(z+4K)}=\frac{\Theta(z)}{H(z)},\ \text{since}\ \cos\frac{\pi}{K}(z+4K)=\cos\frac{\pi z}{K},\ \text{and}\ \sin\frac{\pi}{2K}(z+4K)=\sin\frac{\pi z}{2K}.$$

Thus, the function $\frac{\Theta(z)}{H(z)}$ satisfies all conditions of the theorem 2.2.1, then by (2.41) we

have $\frac{\Theta(z)}{H(z)}=\frac{\pi}{2K}\sum_{m=-\infty}^{\infty}\sum_{t=1}^{n}\frac{A_t}{\sin\frac{\pi}{2K}\left(z-\alpha_t-i2mK'\right)}$, where $A_t$ is the residue of

$\frac{\Theta(z)}{H(z)}$ relative to $\alpha_t$ .

The zeros of $H(z)$ are the poles of $\frac{\Theta(z)}{H(z)}$, then $\alpha_t$ are the solutions of equation

$H(z)=0$, which is $2\sqrt[4]{q}\sin\frac{\pi z}{2K}\left(1-2q^2\cos\frac{\pi z}{K}+q^4\right)\left(1-2q^4\cos\frac{\pi z}{K}+q^8\right)\dots=0$.

Now, applying (2.39) yields

$$2\sqrt[4]{q}\sin\frac{\pi z}{2K}\left[\left(1-q^2e^{i\frac{\pi z}{K}}\right)\left(1-q^2e^{-i\frac{\pi z}{K}}\right)\right]\left[\left(1-q^4e^{i\frac{\pi z}{K}}\right)\left(1-q^4e^{-i\frac{\pi z}{K}}\right)\right]\ldots=0.$$

Consider equation $1-q^{2m}e^{\frac{\pi z}{K}i}=0$. Using $q=e^{-\frac{K'\pi}{K}}$ as defined in (2.29), we get

$e^{\left(\frac{-2m\pi K'}{K}+\frac{\pi z}{K}i\right)}=1$ or $\frac{-2m\pi K'}{K}+\frac{\pi z}{K}i=2h\pi i,\ h=0,\pm1,\pm2,\ldots$ since $e^z=1$ iff $z=2h\pi i$,

or $z=2hK-2mK'i$.

Thus, the equation $1-q^{2m}e^{\frac{\pi z}{K}i}=0$ has solutions $z=2hK-2mK'i$.

Similarly, the equation $1-q^{2m}e^{-\frac{\pi z}{K}i}=0$ has solutions $z=-2hK+2mK'i$.

From the above two results, we have $\alpha_t=2hK+2mK'i$, where $h,m=0,\pm1,\pm2,\ldots$.

This shows that for each pair values of (h, m), the corresponding $\alpha_t$ lies in distinct rectangle period (Recall theorem 2.2.1, the vertices of a period- rectangle RSTU are located at R which is an arbitrary point $z_0$, S and U are the two points $z_0+2K$ and $z_0+i2K'$.) That means, for each period- rectangle RSTU, $\frac{\Theta(z)}{H(z)}$ has only one pole.

Then, it follows that $$\sum_{t=1}^{n}\frac{A_t}{\sin\frac{\pi}{2K}\left(z-\alpha_t-2mK'i\right)}=\frac{\text{Res of }\frac{\Theta(z)}{H(z)}\text{ at }0}{\sin\frac{\pi}{2K}\left(z-0-2mK'i\right)}.$$

We have $\text{Res}\left(\frac{\Theta(z)}{H(z)},0\right)=\lim_{z\to0}\left((z-0)\frac{\Theta(z)}{H(z)}\right)=\frac{\Theta(0)}{H'(0)}$ by L'Hospital rule.

It implies $\dfrac{\Theta(z)}{H(z)} = \dfrac{\pi}{2K}\sum_{m=-\infty}^{\infty} \dfrac{\dfrac{\Theta(0)}{H'(0)}}{\sin\dfrac{\pi}{2K}(z-2mK'i)}$.

□

Lemma 2.2.4

(2.43) $$\boxed{\frac{H(z)}{\Theta(z)} = \frac{\pi}{2K}\frac{\Theta(0)}{H'(0)}\sum_{m=-\infty}^{\infty} \frac{1}{\sin\frac{\pi}{2K}\left(z-i(2m-1)K'\right)}}.$$

Proof We replace z by $z+K'i$ into (2.42) to obtain

$$\frac{\Theta(z+iK')}{H(z+iK')} = \frac{\pi}{2K}\frac{\Theta(0)}{H'(0)}\sum_{m=-\infty}^{\infty} \frac{1}{\sin\frac{\pi}{2K}(z+K'i-2mK'i)}.$$

On the other hand, Hancock [10], pages 222-223, it is shown that $\dfrac{\Theta(z+iK')}{H(z+iK')} = \dfrac{H(z)}{\Theta(z)}$.

Hence, it follows $\dfrac{H(z)}{\Theta(z)} = \dfrac{\pi}{2K}\dfrac{\Theta(0)}{H'(0)}\sum_{m=-\infty}^{\infty} \dfrac{1}{\sin\dfrac{\pi}{2K}\left(z-i(2m-1)K'\right)}$.

□

Lemma 2.2.5

(2.44) $$\boxed{sn\, z = \frac{\pi}{2kK}\sum_{m=-\infty}^{\infty} \frac{1}{\sin\frac{\pi}{2K}\left(z-i(2m-1)K'\right)}}.$$

Proof Jacobi [14], pages 225-256, proved the following formula, which relates the theta functions, $\Theta(z)$ and $H(z)$, to the elliptic function, *sn* z.

$sn\,z = \frac{1}{\sqrt{k}}\frac{H(z)}{\Theta(z)}$ (also see Hille [11]).

Differentiating both sides of the above equation with respect to z, we obtain

$cn\,z \cdot dn\,z = \frac{1}{\sqrt{k}}\frac{H'(z)\Theta(z) - H(z)\Theta'(z)}{\Theta^2(z)}$ (using (2.10) on the left hand side).

Then, at z = 0, we have $cn\,0 \cdot dn\,0 = \frac{1}{\sqrt{k}}\frac{H'(0)\Theta(0) - H(0)\Theta'(0)}{\Theta^2(0)}$.

Since $cn\ 0 = 1$, $dn\ 0 = 1$, and $H(0) = 0$, it follows that $\frac{\Theta(0)}{H'(0)} = \frac{1}{\sqrt{k}}$.

Thus, $sn\,z = \frac{1}{\sqrt{k}}\frac{H(z)}{\Theta(z)} = \frac{1}{\sqrt{k}}\frac{\pi}{2K}\frac{\Theta(0)}{H'(0)}\sum_{m=-\infty}^{\infty}\frac{1}{\sin\frac{\pi}{2K}\left(z - i(2m-1)K'\right)}$ by (2.43).

Substituting $\frac{\Theta(0)}{H'(0)} = \frac{1}{\sqrt{k}}$, we obtain (2.44).

□

Lemma 2.2.6

(2.45) $$\boxed{\frac{1}{\sin(y+inx)} + \frac{1}{\sin(y-inx)} = \frac{4\sqrt{q^n}\left(q^n+1\right)\sin y}{1-2q^n\cos 2y + q^{2n}}},$$

where $e^x = \sqrt{q}$, and $q = e^{-\frac{\pi K'}{K}}$.

Proof: We have

$$\text{LHS} = \frac{2i}{e^{i(y+inx)} - e^{-i(y+inx)}} + \frac{2i}{e^{i(y-inx)} - e^{-i(y-inx)}}$$

$$= \frac{2i\, e^{nx} e^{iy}}{e^{i2y} - e^{2nx}} + \frac{2i\, e^{nx} e^{iy}}{e^{i2y} e^{2nx} - 1}$$

$$= \frac{2i\sqrt{q^n}\, e^{iy}}{e^{i2y} - q^n} + \frac{2i\sqrt{q^n}\, e^{iy}}{q^n e^{i2y} - 1} \text{ since } e^x = \sqrt{q}$$

$$= \frac{2i\sqrt{q^n}\, e^{-iy}}{1 - q^n e^{-i2y}} - \frac{2i\sqrt{q^n}\, e^{iy}}{1 - q^n e^{i2y}}$$

$$= -2i\sqrt{q^n}\left[\frac{\left(1+q^n\right)\left(e^{iy} - e^{-iy}\right)}{\left(1 - q^n e^{-i2y}\right)\left(1 - q^n e^{i2y}\right)}\right].$$

Using (2.39) and the formula $e^{iy} - e^{-iy} = 2i \sin y$, we obtain (2.45).

□

Theorem 2.2.2

(2.46)
$$sn\, z = \frac{2\pi\sqrt{q}}{kK} \sin\frac{\pi z}{2K} \sum_{m=1}^{\infty} \frac{q^{m-1}\left(1+q^{2m-1}\right)}{1 - 2q^{2m-1}\cos\frac{2\pi z}{2K} + q^{4m-2}}.$$

Proof Let $y = \frac{\pi z}{2K}$, $x = \frac{\pi K'}{2K}$ or $e^x = \sqrt{q}$ since $q = e^{-\frac{\pi K'}{K}}$.

Then, by expanding the right hand side of (2.44), we have

$$\sum_{m=-\infty}^{\infty} \frac{1}{\sin\frac{\pi}{2K}\left(z - i(2m-1)K'\right)} = \sum_{m=-\infty}^{\infty} \frac{1}{\sin\left[y - i(2m-1)x\right]}$$

$$= \ldots + \ldots + \frac{1}{\sin(y+i3x)} + \frac{1}{\sin(y+ix)} + \frac{1}{\sin(y-ix)} + \frac{1}{\sin(y-i3x)} + \ldots$$

$$= \left[\frac{1}{\sin(y+ix)} + \frac{1}{\sin(y-ix)}\right] + \left[\frac{1}{\sin(y+i3x)} + \frac{1}{\sin(y-i3x)}\right] + \ldots$$

$$= \sum_{m=1}^{\infty}\left[\frac{1}{\sin\left[y + i(2m-1)x\right]} + \frac{1}{\sin\left[y - i(2m-1)x\right]}\right]$$

$$= \sum_{m=1}^{\infty} \frac{4\sqrt{q^{2m-1}}\left(q^{2m-1}+1\right)\sin y}{1-2q^{2m-1}\cos 2y+q^{4m-2}}, \text{ by (2.45)}.$$

Then, replace $y = \dfrac{\pi z}{2K}$ to yield (2.46).

□

Theorem 2.2.3

$$(2.47) \qquad \boxed{\sum_{m=1}^{\infty} q^{m-1}\sin(2m-1)y = \frac{(1+q)\sin y}{1-2q\cos 2y+q^2}, \text{ where } \left|q\cdot e^{\pm 2iy}\right|<1}.$$

Proof We have

$$\sum_{m=1}^{\infty} q^{m-1}\sin(2m-1)y = \sum_{m=1}^{\infty} q^{m-1}\left(\frac{e^{i(2m-1)y}-e^{-i(2m-1)y}}{2i}\right)$$
$$= \frac{1}{2i}\left[\sum_{m=1}^{\infty} q^{m-1}e^{i(2m-1)y} - \sum_{m=1}^{\infty} q^{m-1}e^{-i(2m-1)y}\right]. \qquad (*)$$

Let us consider the series $\sum_{m=1}^{\infty} q^{m-1}e^{i(2m-1)y}$. Then, we have

$$\sum_{m=1}^{\infty} q^{m-1}e^{i(2m-1)y} = \sum_{m=1}^{\infty} q^{m-1}w^{2m-1}, \text{ where } w = e^{iy}$$
$$= \frac{1}{qw}\sum_{m=1}^{\infty}\left(qw^2\right)^m$$
$$= \frac{qw^2}{qw}\sum_{m=1}^{\infty}(r)^{m-1}, \text{ where r} = qw^2$$
$$= w\sum_{m=0}^{\infty}(r)^m.$$

Since $|r| = \left|qw^2\right| = \left|q\cdot e^{2iy}\right| < 1$ by the hypothesis, it gives $\sum_{m=0}^{\infty}(r)^m = \dfrac{1}{1-r}$.

Hence, it follows that

$$\sum_{m=1}^{\infty} q^{m-1} e^{i(2m-1)y} = \frac{e^{iy}}{1-qe^{i2y}} \text{ since } w = e^{iy} \text{ and } r = q w^2$$

$$= \frac{e^{iy}\left(1-qe^{-i2y}\right)}{\left(1-qe^{i2y}\right)\left(1-qe^{-i2y}\right)}$$

$$= \frac{e^{iy}\left(1-qe^{-i2y}\right)}{1-2q\cos 2y+q^2} \text{ by (2.39).}$$

Similarly, we also obtain $\sum_{m=1}^{\infty} q^{m-1} e^{-i(2m-1)y} = \frac{e^{-iy}\left(1-qe^{i2y}\right)}{1-2q\cos 2y+q^2}$.

Hence, from $(*)$ it follows that

$$\sum_{m=1}^{\infty} q^{m-1}\sin(2m-1)y = \frac{1}{2i}\left[\frac{e^{iy}\left(1-qe^{-i2y}\right)-e^{-iy}\left(1-qe^{i2y}\right)}{1-2q\cos 2y+q^2}\right]$$

$$= \frac{1}{2i}\left[\frac{\left(e^{iy}-e^{-iy}\right)+q\left(e^{iy}-e^{-iy}\right)}{1-2q\cos 2y+q^2}\right]$$

$$= \frac{\sin y + q\cdot \sin y}{1-2q\cos 2y+q^2}.$$

□

Now, we are ready to prove the theorem mentioned in the beginning of this section 2.2, which is the equation $sn\, z = \frac{2\pi\sqrt{q}}{kK}\sum_{m=1}^{\infty}\left(\frac{q^{m-1}}{1-q^{2m-1}}\right)\sin\left[(2m-1)\frac{\pi z}{2K}\right]$, where $q = e^{-\frac{\pi K'}{K}}$.

<u>Proof</u>: From (2.46) and $y = \frac{\pi z}{2K}$, we have

$$sn\, z = \frac{2\pi\sqrt{q}}{kK}\left[\left(\frac{q^0(1+q)\sin y}{1-2q\cos 2y+q^2}\right)+\left(\frac{q^1\left(1+q^3\right)\sin y}{1-2q^3\cos 2y+q^6}\right)+\ldots\right].$$

From (2.47), we also have

$$\frac{q^0(1+q)\sin y}{1-2q\cos 2y+q^2}=\sum_{m=1}^{\infty} q^0 q^{m-1}\sin(2m-1)y,$$
$$\frac{q^1(1+q^3)\sin y}{1-2q^3\cos 2y+q^6}=\sum_{m=1}^{\infty} q(q^3)^{m-1}\sin(2m-1)y,$$
$$\ldots$$

Hence, it follows that

$$sn\,z=\frac{2\pi\sqrt{q}}{kK}\sum_{m=1}^{\infty}\left[q^0(q)^{m-1}+q(q^3)^{m-1}+q^2(q^5)^{m-1}+\ldots\right]\sin(2m-1)y$$

$$=\frac{2\pi\sqrt{q}}{kK}\sum_{m=1}^{\infty}q^{m-1}\left[\sum_{n=0}^{\infty}\left(q^{2m-1}\right)^n\right]\sin(2m-1)y.$$

Since $|q|=\left|e^{\frac{-\pi K'}{K}}\right|<1$, we get $\sum_{n=0}^{\infty}\left(q^{2m-1}\right)^n=\frac{1}{1-q^{2m-1}}$ by the geometric series.

Thus, $sn\,z=\frac{2\pi\sqrt{q}}{kK}\sum_{m=1}^{\infty}\left(\frac{q^{m-1}}{1-q^{2m-1}}\right)\sin\left[(2m-1)y\right]$, which is the equation (2.29).

□

More expansions of the elliptic functions can be found in Hancock [10] and Jacobi [14]. The restrictions on the expansions can be found in Byrd [8], pp 304-305.

### 2.3 Applications of The Jacobi Elliptic Functions

#### 2.3.1 Relationships between constants K, $K'$ and E, $E'$

Let's recall the definitions of K, $K'$ and E, $E'$ as follows:

$$K=\int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}},\quad K'=\int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-k'^2\sin^2\alpha}},$$

$$E=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\alpha}\;d\alpha,\quad E'=\int_0^{\pi/2}\sqrt{1-k'^2\sin^2\alpha}\;d\alpha,\ \text{where } k^2+k'^2=1.$$

If we let $m=k^2$, then $k'^2=1-m$, and

$$K=K(m),\ K'=K'(m)=K(1-m),$$
$$E=E(m),\ E'=E'(m)=E(1-m).$$

### 2.3.2 Series expansions for the Jacobi elliptic functions

We define functions x and q by

(2.48) $$\boxed{x=x(m)=\frac{\pi K'(m)}{K(m)},\ \text{and}\ \ q=e^{-\frac{\pi K'(m)}{K(m)}}=e^{-x}}.$$

The following equations are the Jacobi expansions of the elliptic function *sn* z, *cn* z, and *dn* z in terms of sine and cosine and related expansions given by Bruckman [7]:

(2.49) $$\boxed{sn\,z=\frac{2\pi}{\sqrt{m}\,K}\sum_{n=0}^{\infty}\frac{q^{n+\frac{1}{2}}}{1-q^{2n+1}}\sin(2n+1)\frac{\pi z}{2K}},$$

(2.50) $$\boxed{cn\,z=\frac{2\pi}{\sqrt{m}\,K}\sum_{n=0}^{\infty}\frac{q^{n+\frac{1}{2}}}{1+q^{2n+1}}\cos(2n+1)\frac{\pi z}{2K}},$$

(2.51) $$\boxed{dn\,z=\frac{\pi}{2K}+\frac{2\pi}{K}\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\cos(2n)\frac{\pi z}{2K}},$$

(2.52) $$\boxed{\left(\frac{K}{\pi}\right)^2 dn^2 z-\frac{KE}{\pi^2}=2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\cos(2n)\frac{\pi z}{2K}},$$

(2.53) $$\boxed{\frac{4}{3}(2-m)\left(\frac{K}{\pi}\right)^2-\frac{4KE}{\pi^2}+\frac{1}{3}=8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}},$$

(2.54) $$\boxed{1-\frac{4KE}{\pi^2}=8\sum_{n=1}^{\infty}\frac{(-1)^n nq^{2n}}{1-q^{2n}}},$$

(2.55) $$\boxed{-\frac{1}{16}\log(1-m) = \sum_{n=1}^{\infty}\frac{nq^{2n-1}}{(2n-1)\left(1-q^{4n-2}\right)}}.$$

### 2.3.3 Series of hyperbolic functions

We apply the results of the expansions of Jacobi elliptic functions to obtain the series of hyperbolic functions.

Lemma 2.3.1

(2.56) $$\boxed{\frac{q^m}{1-q^{2m}} = \frac{1}{2}\operatorname{csch} mx, \text{ and } \frac{q^m}{1+q^{2m}} = \frac{1}{2}\operatorname{sech} mx}, \text{where } q = e^{-x}.$$

Proof: Observe that

$$\begin{aligned}\frac{q^m}{1-q^{2m}} &= \frac{1}{q^{-m}-q^m} \\ &= \frac{1}{e^{mx}-e^{-mx}} \text{ since } q = e^{-x} \\ &= \frac{1}{2\sinh mx} = \frac{1}{2}\operatorname{csch} mx.\end{aligned}$$

Similarly, it is easy to show that $\frac{q^m}{1+q^{2m}} = \frac{1}{2}\operatorname{sech} mx$.

□

Theorem 2.3.1

(2.57) $$\boxed{\sum_{n=1}^{\infty}(-1)^{n-1}\operatorname{csch}\left(n-\frac{1}{2}\right)x = \frac{K\sqrt{m}}{\pi}}, \text{ where } x = \frac{\pi K'}{K}.$$

Proof: We prove (2.57) by substituting z = K in (2.49). That gives

$$sn K = \frac{2\pi}{\sqrt{m}\,K} \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{1-q^{2n+1}} \sin(2n+1)\frac{\pi}{2}$$

$$= \frac{2\pi}{\sqrt{m}\,K} \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{1-q^{2n+1}} (-1)^n$$

$$= \frac{2\pi}{\sqrt{m}\,K} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}(-1)^{n-1}}{1-q^{2n-1}}, \quad \text{replacing n by } n-1$$

$$= \frac{2\pi}{\sqrt{m}\,K} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2} \operatorname{csch}\left(n-\frac{1}{2}\right)x, \quad \text{by (2.56) with } m = n - ½.$$

Using $sn\, K = 1$, we obtain (2.57)

□

<u>Theorem 2.3.2</u>

(2.58) $$\boxed{\sum_{n=-\infty}^{\infty} \operatorname{sech}\left(n-\frac{1}{2}\right)x = 2\sum_{n=1}^{\infty} \operatorname{sech}\left(n-\frac{1}{2}\right)x = \frac{2K\sqrt{m}}{\pi}}, \text{ where } x = \frac{\pi K'}{K}.$$

<u>Proof</u>: From (2.50), we substitute z = 0 to yield

$$cn\, 0 = \frac{2\pi}{\sqrt{m}\,K} \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{1+q^{2n+1}} \cos 0$$

$$= \frac{2\pi}{\sqrt{m}\,K} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}}, \quad \text{replacing n by n - 1}$$

$$= \frac{2\pi}{\sqrt{m}\,K} \sum_{n=1}^{\infty} \frac{1}{2} \operatorname{sech}\left(n-\frac{1}{2}\right)x, \quad \text{by (2.56) with } m = n - ½$$

$$= \frac{\pi}{\sqrt{m}\,K} \sum_{n=1}^{\infty} \operatorname{sech}\left(n-\frac{1}{2}\right)x.$$

Using $cn\ 0 = 1$ and the fact that $\sum_{n=-\infty}^{\infty} \operatorname{sech}\left(n-\frac{1}{2}\right)x = 2\sum_{n=1}^{\infty} \operatorname{sech}\left(n-\frac{1}{2}\right)x$, we get (2.58).

□

The following theorems are proved similarly as the above theorem. Therefore, we only sketch the ideas how to get them.

Using z = 0 in (2.51) and the facts $dn\ 0 = 1$ and $\sum_{n=-\infty}^{\infty} \operatorname{sech} nx = 2\sum_{n=1}^{\infty} \operatorname{sech} nx + 1$, we get

(2.59) $$\boxed{\sum_{n=-\infty}^{\infty} \operatorname{sech} nx = \frac{2K}{\pi}}.$$

Substituting z = K in (2.51), using $dn K = \sqrt{1-m}$ by (2.13) and (2.56), we have

(2.60) $$\boxed{\sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sech} nx = \frac{2K\sqrt{1-m}}{\pi}}.$$

Let z = 0 in (2.52), and with $dn\ 0 = 1$ and (2.56), we obtain

(2.61) $$\boxed{\sum_{n=1}^{\infty} n \operatorname{csch} nx = \frac{K(K-E)}{\pi^2}}.$$

Let z = K in (2.52) and use $dn K = \sqrt{1-m}$. Then by (2.13) and (2.56) we get

(2.62) $$\boxed{\sum_{n=1}^{\infty} (-1)^{n-1} n \operatorname{csch} nx = \frac{KE}{\pi^2} - (1-m)\left(\frac{K}{\pi}\right)^2}.$$

**2.3.4** Applications to infinite series involving Fibonacci and Lucas numbers

In this section we evaluate Fibonacci series using some results from the elliptic functions.

Lemma 2.3.2 Let $F_n$ be a Fibonacci number. Then

(2.63) $$\boxed{F_{2n} = \frac{2}{\sqrt{5}}\sinh(2n\lambda)}, \text{ where } \lambda = \ln \alpha, \text{ and } \alpha = \frac{1+\sqrt{5}}{2}.$$

Proof: From the Binet form, we have

$$F_{2n} = \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \text{ where } \alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}.$$

Since $\alpha\beta = -1$, $\alpha - \beta = \sqrt{5}$, $\lambda = \ln \alpha$, and $\alpha = e^{\lambda}$, it follows that

$$F_{2n} = \frac{e^{2n\lambda} - e^{-2n\lambda}}{\sqrt{5}} = \frac{2}{\sqrt{5}}\sinh(2n\lambda).$$

□

<u>Lemma 2.3.3</u> Let $F_n$ be a Fibonacci number. Then

(2.64) $$\boxed{F_{2n+1} = \frac{2}{\sqrt{5}}\cosh(2n+1)\lambda}.$$

<u>Proof</u>: Using the Binet form, we have

$$\begin{aligned} F_{2n+1} &= \frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta} \\ &= \frac{\alpha^{2n+1} + \alpha^{-(2n+1)}}{\sqrt{5}}, \text{ since } \alpha\beta = -1 \text{ and } \alpha - \beta = \sqrt{5} \\ &= \frac{e^{(2n+1)\lambda} + e^{-(2n+1)\lambda}}{\sqrt{5}} \text{ since } \alpha = e^{\lambda} \\ &= \frac{2}{\sqrt{5}}\cosh(2n+1)\lambda. \end{aligned}$$

□

In the same way, we use the technique above, that is the Binet form and definitions of α and β, to find a hyperbolic expansion for the Lucas numbers. It is easy to show that

(2.65) $$\boxed{L_{2n} = 2\cosh 2n\lambda}, \text{ and}$$

(2.66) $$\boxed{L_{2n+1} = 2\sinh(2n+1)\lambda}.$$

Next, let us consider the equation $\frac{\pi K'(m)}{K(m)} = 2\lambda = \ln(1+\alpha)$.

Then, by definition (2.48), the left hand side of this equation is x, and the right hand side equals to $2\ln\left(\frac{1+\sqrt{5}}{2}\right)$ by definition of α in (2.63). This equation has a unique solution m = μ where 0 < μ < 1, (see Bruckman [7], page 296).

It means $x = \frac{\pi K'(\mu)}{K(\mu)} = 2\lambda = 2\ln\frac{1+\sqrt{5}}{2}$.

Then, if we define $\rho = \frac{K(\mu)}{\pi}$ and $\sigma = \frac{E(\mu)}{\pi}$ (see definitions of K and E in section 2.3.1), this particular value of x will lead to the following theorem.

<u>Theorem 2.3.3</u> Let $L_n$ be a Lucas number. Then

$$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{L_{2n-1}} = \frac{1}{2}\rho\sqrt{\mu}}. \tag{2.67}$$

<u>Proof</u>: Using the hyperbolic form as in (2.66), we get

$$\begin{aligned}\frac{1}{L_{2n-1}} &= \frac{1}{2\sinh(2n-1)\lambda} \\ &= \frac{1}{2\sinh\left(n-\frac{1}{2}\right)x} \text{ since } \lambda = \frac{x}{2} = \ln\frac{1+\sqrt{5}}{2} \\ &= \frac{1}{2}\operatorname{csch}\left(n-\frac{1}{2}\right)x.\end{aligned}$$

Hence, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{L_{2n-1}} = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\operatorname{csch}\left(n-\frac{1}{2}\right)x = \frac{1}{2}\frac{K\sqrt{m}}{\pi}$ by (2.57).

By replacing K / π by ρ and m by μ in the equation above, we obtain (2.67).

□

<u>Theorem 2.3.4</u> Let $F_n$ be a Fibonacci number. Then

(2.68) $$\boxed{\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}}=\frac{1}{2}\rho\sqrt{5\mu}}.$$

Proof: Using (2.64), it is easy to show that $\frac{1}{F_{2n-1}}=\frac{\sqrt{5}}{2}\operatorname{sech}\left(n-\frac{1}{2}\right)x$ with λ = x / 2.

Hence, it follows that $\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}}=\frac{\sqrt{5}}{2}\sum_{n=1}^{\infty}\operatorname{sech}\left(n-\frac{1}{2}\right)x=\frac{\sqrt{5}}{2}\frac{K\sqrt{m}}{\pi}$ by (2.58).

Then, replacing K / π by ρ and m with μ, we get (2.68).

□

Similar to the proofs of theorem 2.3.3 and 2.3.4, we can obtain the following identities:

(2.69) $$\boxed{\sum_{n=1}^{\infty}\frac{1}{L_{2n}}=\frac{1}{2}\rho-\frac{1}{4}},$$

(2.70) $$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{L_{2n}}=\frac{1}{4}-\frac{1}{2}\rho\sqrt{1-\mu}},$$

(2.71) $$\boxed{\sum_{n=1}^{\infty}\frac{1}{F_{2n}}=\frac{\sqrt{5}}{2}\rho(\rho-\sigma)},$$

(2.72) $$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{F_{2n}}=\frac{\sqrt{5}}{2}\left[\rho\sigma-(1-\mu)\sigma^2\right]}.$$

## 2.4 Bruckman's Theorem and Applications

Theorem 2.4.1 If $\frac{2K'(m_1)}{K(m_1)}=\frac{K'(m_2)}{K(m_2)}$, then

$$
(2.73)\qquad
\boxed{
\begin{aligned}
&(a)\quad K(m_1)=\left(1+\sqrt{m_2}\right)K(m_2),\\
&(b)\quad K'(m_1)=\frac{1}{2}\left(1+\sqrt{m_2}\right)K'(m_2),\\
&(c)\quad m_1=1-\left(\frac{1-\sqrt{m_2}}{1+\sqrt{m_2}}\right)^2=\frac{4\sqrt{m_2}}{\left(1+\sqrt{m_2}\right)^2},\\
&(d)\quad E'(m_1)=\frac{E'(m_2)+\sqrt{m_2}K'(m_2)}{1+\sqrt{m_2}},\\
&(e)\quad E(m_1)=\frac{2E(m_2)-(1-m_2)K(m_2)}{1+\sqrt{m_2}}.
\end{aligned}
}
$$

Proof: Part (a) Let $x=\dfrac{\pi K'(m_1)}{K(m_1)}$ and $x'=\dfrac{\pi K'(m_2)}{K(m_2)}$, then by the hypothesis of the theorem, $x=\dfrac{1}{2}x'$. Observe that $\sum_{n=-\infty}^{\infty}\operatorname{sech}(2n)x+\sum_{n=-\infty}^{\infty}\operatorname{sech}(2n-1)x=\sum_{n=-\infty}^{\infty}\operatorname{sech} nx$.

Using $x=\dfrac{1}{2}x'$ in the left hand side of the equation, we get

$$\sum_{n=-\infty}^{\infty}\operatorname{sech} nx'+\sum_{n=-\infty}^{\infty}\operatorname{sech}\left(n-\frac{1}{2}\right)x'=\sum_{n=-\infty}^{\infty}\operatorname{sech} nx.$$

By (2.59), we have $\sum_{n=-\infty}^{\infty}\operatorname{sech} nx'=\dfrac{2K(m_2)}{\pi}$ and $\sum_{n=-\infty}^{\infty}\operatorname{sech} nx=\dfrac{2K(m_1)}{\pi}$.

Also, by (2.58), we have $\sum_{n=-\infty}^{\infty}\operatorname{sech}\left(n-\dfrac{1}{2}\right)x'=\dfrac{2K(m_2)\sqrt{m_2}}{\pi}$.

Thus, we get a relation $\dfrac{2K(m_2)}{\pi}+\dfrac{2K(m_2)\sqrt{m_2}}{\pi}=\dfrac{2K(m_1)}{\pi}$, which implies (a).

Part (b) From (a), we have $\left(1+\sqrt{m_2}\right)=\frac{K(m_1)}{K(m_2)}$, which is $\frac{2K'(m_1)}{K'(m_2)}$ by the hypothesis. It follows that $K'(m_1)=\frac{1}{2}\left(1+\sqrt{m_2}\right)K'(m_2)$, which is (b).

Part (c) By the formula 17.3.29 of Abramowitz and Stegun [1], we have

$K(m)=\frac{2}{1+\sqrt{1-m}}K\left(\frac{\left(1-\sqrt{1-m}\right)^2}{\left(1+\sqrt{1-m}\right)^2}\right)$. Replacing m by $(1-m_2)$ on both sides of the above equation yields $K(1-m_2)=\frac{2}{1+\sqrt{m_2}}K\left(\frac{\left(1-\sqrt{m_2}\right)^2}{\left(1+\sqrt{m_2}\right)^2}\right)$.

Since $K(1-m_2)=K'(m_2)$ by definition 2.3.1, it follows that

$$K'(m_2)=\frac{2}{1+\sqrt{m_2}}K\left(\frac{\left(1-\sqrt{m_2}\right)^2}{\left(1+\sqrt{m_2}\right)^2}\right), \text{ or } \frac{1}{2}\left(1+\sqrt{m_2}\right)K'(m_2)=K\left(\frac{\left(1-\sqrt{m_2}\right)^2}{\left(1+\sqrt{m_2}\right)^2}\right).$$

Since $\frac{1}{2}\left(1+\sqrt{m_2}\right)K'(m_2)=K'(m_1)$ by part (b), it follows

$$K'(m_1)=K\left(\frac{\left(1-\sqrt{m_2}\right)^2}{\left(1+\sqrt{m_2}\right)^2}\right), \text{ or } K(1-m_1)=K\left(\frac{\left(1-\sqrt{m_2}\right)^2}{\left(1+\sqrt{m_2}\right)^2}\right) \text{ by definition 2.3.1.}$$

This implies $1-m_1=\frac{\left(1-\sqrt{m_2}\right)^2}{\left(1+\sqrt{m_2}\right)^2}$. Solving this equation for $m_1$, we obtain (c).

Part (d) From the formula 17.3.30 of Abramowitz and Stegun [1], we have

$$E(m)=\left(1+\sqrt{1-m}\right)E\left(\frac{\left(1-\sqrt{1-m}\right)^2}{\left(1+\sqrt{1-m}\right)^2}\right)-\frac{2\sqrt{1-m}}{1+\sqrt{1-m}}K\left(\frac{\left(1-\sqrt{1-m}\right)^2}{\left(1+\sqrt{1-m}\right)^2}\right).$$

Replacing m by $(1 - m_2)$ in the above equation yields

$$E(1-m_2)=\left(1+\sqrt{m_2}\right)E\left(\frac{\left(1-\sqrt{m_2}\right)^2}{\left(1+\sqrt{m_2}\right)^2}\right)-\frac{2\sqrt{m_2}}{1+\sqrt{m_2}}K\left(\frac{\left(1-\sqrt{m_2}\right)^2}{\left(1+\sqrt{m_2}\right)^2}\right).$$

Since $E(1-m_2)=E'(m_2)$ by by definition 2.3.1 and $\dfrac{\left(1-\sqrt{m_2}\right)^2}{\left(1+\sqrt{m_2}\right)^2}=1-m_1$ from part (c),

it follows that $E'(m_2)=\left(1+\sqrt{m_2}\right)E(1-m_1)-\dfrac{2\sqrt{m_2}}{1+\sqrt{m_2}}K(1-m_1)$

$=\left(1+\sqrt{m_2}\right)E'(m_1)-\dfrac{2\sqrt{m_2}}{1+\sqrt{m_2}}K'(m_1)$ by definition 2.3.1.

Next, we substitute the result of part (b) into the right hand side of the above equation to

get $E'(m_2)=\left(1+\sqrt{m_2}\right)E'(m_1)-\dfrac{2\sqrt{m_2}}{1+\sqrt{m_2}}\left[\dfrac{1}{2}\left(1+\sqrt{m_2}\right)K'(m_2)\right].$

Solving this equation for $E'(m_1)$ yields (d).

Part (e) Recall the well-known Legendre's formula,

$E(m)K'(m)+E'(m)K(m)-K(m)K'(m)=\dfrac{\pi}{2}$ for any parameter m.

Letting m = $m_1$, we get $E(m_1)K'(m_1)+E'(m_1)K(m_1)-K(m_1)K'(m_1)=\dfrac{\pi}{2}$.

Letting m = $m_2$, we get $E(m_2)K'(m_2)+E'(m_2)K(m_2)-K(m_2)K'(m_2)=\dfrac{\pi}{2}$.

Hence, it follows that

$$E(m_1)K'(m_1)+E'(m_1)K(m_1)-K(m_1)K'(m_1)=E(m_2)K'(m_2)+E'(m_2)K(m_2)-K(m_2)K'(m_2)$$

Now, use (a), (b), and (d) in the above equation and solve it for $E(m_1)$ to obtain (e).

□

Now, if we set $m_1 = \mu$ (where $\mu$ satisfies $\frac{\pi K'(\mu)}{K(\mu)} = 2\ln\frac{1+\sqrt{5}}{2}$) and define $\mu''$ by the relation $x' = \frac{\pi K'(\mu'')}{K(\mu'')} = 4\lambda = 4\ln\frac{1+\sqrt{5}}{2}$, then $\mu''$ plays the role of $m_2$ in theorem 2.4.1.

We also define $\rho'' = \frac{K(\mu'')}{\pi}$ and $\sigma'' = \frac{E(\mu'')}{\pi}$. Based on these definitions, we have the three following corollaries.

<u>Corollary 2.4.1</u>

(2.74) $$\boxed{\sqrt{\mu''} = \frac{1-\sqrt{1-\mu}}{1+\sqrt{1-\mu}}},$$

(2.75) $$\boxed{1-\mu'' = \frac{4\sqrt{1-\mu}}{\left(1+\sqrt{1-\mu}\right)^2}}.$$

<u>Proof</u>: Let $m_2 = \mu''$ and $m_1 = \mu$ in (c) of (2.73), we get $\mu = 1-\left(\frac{1-\sqrt{\mu''}}{1+\sqrt{\mu''}}\right)^2$.

Solve this equation for $\sqrt{\mu''}$ to yield (2.74) and solve for $1 - \mu''$ to obtain (2.75).

□

Corollary 2.4.2

(2.76) $$\boxed{\rho'' = \frac{1}{2}\left(1+\sqrt{1-\mu}\right)\rho}.$$

Proof: Using (a) of (2.73) with $m_1 = \mu$ and $m_2 = \mu''$, and dividing both sides of that equation by π we get $\frac{K(\mu)}{\pi} = \left(1+\sqrt{\mu''}\right)\frac{K(\mu'')}{\pi}$. Since $\frac{K(\mu)}{\pi} = \rho$ (definition on page 71) and $\frac{K(\mu'')}{\pi} = \rho''$, it follows that $\rho = \left(1+\sqrt{\mu''}\right)\rho''$. Then solve this equation for $\rho''$ and use $\sqrt{\mu''}$ as in (2.74), we will obtain (2.76).

□

Corollary 2.4.3

(2.77) $$\boxed{\sigma'' = \frac{\sigma + \rho\sqrt{1-\mu}}{1+\sqrt{1-\mu}}}.$$

Proof: Using (e) of (2.73) with $m_1 = \mu$ and $m_2 = \mu''$, and dividing both sides of that equation by π we get $\frac{E(\mu)}{\pi}\left(1+\sqrt{\mu''}\right) = \frac{2E(\mu'')}{\pi} - (1-\mu'')\frac{K(\mu'')}{\pi}$.

Since $\frac{K(\mu)}{\pi} = \rho$, $\frac{K(\mu'')}{\pi} = \rho''$, $\sigma'' = \frac{E(\mu'')}{\pi}$, and $\sigma = \frac{E(\mu)}{\pi}$ it follows that $\sigma\left(1+\sqrt{\mu''}\right) = 2\sigma'' - (1-\mu'')\rho''$. Then, use (2.74) for the expression of $\sqrt{\mu''}$, (2.75) for $(1 - \mu'')$, and (2.76) for $\rho''$. Now, solve that equation for $\sigma''$ to yield (2.77).

□

Now we are ready to evaluate additional Fibonacci and Lucas series.

Theorem 2.4.2 Let $F_n$ be a Fibonacci number. Then

(2.78)
$$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_{4n-2}}=\frac{\sqrt{5}}{4}\left(1-\sqrt{1-\mu}\right)\rho}.$$

Proof: From (2.63), we have

$$\frac{2}{\sqrt{5}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_{4n-2}}=\frac{2}{\sqrt{5}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\frac{2}{\sqrt{5}}\sinh(4n-2)\lambda}$$
$$=\sum_{n=1}^{\infty}(-1)^{n-1}\operatorname{csch}\left(n-\frac{1}{2}\right)x' \text{ since } \lambda=\frac{x'}{4}=\ln\frac{1+\sqrt{5}}{2}.$$

Using (2.57), we also have

$$\sum_{n=1}^{\infty}(-1)^{n-1}\operatorname{csch}\left(n-\frac{1}{2}\right)x=\frac{K(m_2)\sqrt{m_2}}{\pi}=\rho''\sqrt{\mu''} \text{ since } m_2=\mu'' \text{ and } \frac{K(\mu'')}{\pi}=\rho''.$$

Now, using (2.74) and (2.76) for the expressions of $\sqrt{\mu''}$ and $\rho''$, we get

$$\frac{2}{\sqrt{5}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_{4n-2}}=\frac{1}{2}\left(1+\sqrt{1-\mu}\right)\rho\cdot\left(\frac{1-\sqrt{1-\mu}}{1+\sqrt{1-\mu}}\right).$$ Simplifying this equation gives (2.78).

□

Theorem 2.4.3 Let $L_n$ be a Lucas number. Then

(2.79)
$$\boxed{\sum_{n=1}^{\infty}\frac{1}{L_{4n-2}}=\frac{1}{4}\left(1-\sqrt{1-\mu}\right)\rho}.$$

Proof: Using (2.65), we have

$$\sum_{n=1}^{\infty}\frac{1}{L_{4n-2}}=\sum_{n=1}^{\infty}\frac{1}{2\cosh(4n-2)\lambda}$$
$$=\frac{1}{2}\sum_{n=1}^{\infty}\operatorname{sech}\left(n-\frac{1}{2}\right)x' \text{ since } \lambda=\frac{x'}{4}=\ln\frac{1+\sqrt{5}}{2}$$

$$= \frac{K(m_2)\sqrt{m_2}}{2\pi} \text{ by (2.58)}$$

$$= \frac{1}{2}\rho''\sqrt{\mu''} \text{ since } \mu'' = m_2 \text{ and } \rho'' = \frac{K(\mu'')}{\pi}.$$

Now, using (2.74) and (2.76) yields $\sum_{n=1}^{\infty} \frac{1}{L_{4n-2}} = \frac{1}{2} \cdot \frac{1}{2}\left(1+\sqrt{1-\mu}\right)\rho \cdot \left(\frac{1-\sqrt{1-\mu}}{1+\sqrt{1-\mu}}\right)$.

Simplifying this equation, we obtain (2.79).

□

Similar to the proofs of theorems 2.4.2 and 2.4.3, we can derive the following three identities.

(2.80) $$\boxed{\sum_{n=1}^{\infty} \frac{1}{L_{4n}} = \frac{1}{4}\left(1+\sqrt{1-\mu}\right)\rho - \frac{1}{4}},$$

(2.81) $$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{4n}} = \frac{1}{4} - \frac{1}{2}(1-\mu)^{\frac{1}{4}}\rho},$$

(2.82) $$\boxed{\sum_{n=1}^{\infty} \frac{n}{F_{4n}} = \frac{\sqrt{5}}{8}\left[(2-\mu)\rho^2 - 2\rho\sigma\right]}.$$

In general, if we let x = 2λ, 4λ, or 8λ, and so on, and then using the theorem 2.4.1, we can derive the other relationships between three basic constants μ, ρ, and σ. If we continue to use these relations, we can obtain more results on the infinite series for the reciprocals of the Fibonancci and Lucas numbers. More identities can be found in Bruckman [7].

# CHAPTER III

# SERIES INVOLVING THE

# GENERALIZED FIBONACCI AND LUCAS NUMBERS

## 3.1 The Generalized Sequence $\{W_n\}$ and Other Related Sequences

### 3.1.1 Introduction

In this chapter, we evaluate the sum of infinite series of the sequence $\{W_n\}$ introduced by Horadam [13]. The sequence $\{W_n\}$ is a generalization of the Fibonacci, Lucas, Pell, and other sequences. The sequence $\{W_n\}$ which is described below was introduced by Horadam in 1965. More details can be found in Bodas [4].

Consider the sequence of real numbers $\{W_n\}$ defined by the recurrence relation:

(3.1) $$\boxed{W_n = pW_{n-1} - qW_{n-2}},$$ with the initial conditions

(3.2) $$\boxed{W_0 = a,\ W_1 = b},$$ where

$a \geq 0$, $b \geq 1$, $p \geq 1$, $q \neq 0$ are integers with $p^2 - 4q > 0$.

The characteristic equation of the sequence $\{W_n\}$,

(3.3) $$\boxed{x^2 - px + q = 0}$$ has the roots

(3.4) $$\boxed{\alpha = \frac{p + \sqrt{p^2 - 4q}}{2},\ \ \beta = \frac{p - \sqrt{p^2 - 4q}}{2}}.$$

It is obvious that

(3.5) $$\boxed{\alpha > \beta,\ \ \alpha\beta = q,\ \ \alpha + \beta = p,\ \text{ and }\ \alpha - \beta = \sqrt{p^2 - 4q} > 0}.$$

The Binet form for $\{W_n\}$ is given by

(3.6) $\boxed{W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}}$, where

(3.7) $\boxed{A = b - a\beta \text{ and } B = b - a\alpha}$.

**3.1.2** Special cases of $\{W_n\}$

When a = 0, b = 1, p = 1, and q = −1, we have the Fibonacci sequence $\{F_n\}$.

(3.8) $\boxed{F_n = F_{n-1} + F_{n-2}, \text{ where } F_0 = 0 \text{ and } F_1 = 1}$.

From (3.4), (3.5), and (3.6), we have

(3.9) $\boxed{\alpha = \frac{1+\sqrt{5}}{2}, \text{ and } \beta = \frac{1-\sqrt{5}}{2}}$,

(3.10) $\boxed{\alpha\beta = -1, \ \alpha + \beta = 1, \text{ and } \alpha - \beta = \sqrt{5}}$, and

(3.11) $\boxed{F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}}}$.

When a = 2, b = 1, p = 1, and q = −1 in (3.1) and (3.2), we have the Lucas sequence $\{L_n\}$.

(3.12) $\boxed{L_n = L_{n-1} + L_{n-2}, \text{ where } L_0 = 2 \text{ and } L_1 = 1}$, and its Binet form is

(3.13) $\boxed{L_n = \alpha^n + \beta^n}$ where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Similarly, if a = 0 and b = 1, we obtain the generalized Fibonacci sequence $\{U_n\}$,

(3.14) $\boxed{U_n = pU_{n-1} - qU_{n-2}}$, and (3.6) implies that

(3.15) $\boxed{U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}}$.

If a = 2 and b = p, we get the generalized Lucas sequence $\{V_n\}$,

(3.16) $\boxed{V_n = pV_{n-1} - qV_{n-2}}$.

Then by (3.7), we have A = $\sqrt{p^2 - 4q}$ and B = – $\sqrt{p^2 - 4q}$.

Hence, from (3.6) we obtain the Binet form for the sequence { $V_n$ },

(3.17) $\boxed{V_n = \alpha^n + \beta^n}$ where α and β as in (3.4).

If a = 0, b = 1, p = 2, and q = –1, we have the Pell sequence $\{P_n\}$,

(3.18) $\boxed{P_n = 2P_{n-1} + P_{n-2} \text{ where } P_0 = 0 \text{ and } P_1 = 1}$.

If a = 2, b = 2, p = 2, and q = –1, we have the Pell-Lucas sequence $\{Q_n\}$,

(3.19) $\boxed{Q_n = 2Q_{n-1} + Q_{n-2} \text{ where } Q_0 = 2 \text{ and } Q_1 = 2}$.

If a = 0, b = 1, p = 3, and q = 2, we have the Fermat sequence $\{f_n\}$,

(3.20) $\boxed{f_n = 3f_{n-1} - 2f_{n-2} \text{ where } f_0 = 0 \text{ and } f_1 = 1}$.

If a = 2, b = 3, p = 3, and q = 2, we have the Fermat-Lucas sequence $\{g_n\}$,

(3.21) $\boxed{g_n = 3g_{n-1} - 2g_{n-2} \text{ where } g_0 = 2 \text{ and } g_1 = 3}$.

## 3.2 Analyzing the Horadam Series

### 3.2.1 Convergence of series $\sum_{n=1}^{\infty} \frac{1}{W_n}$

By equation (3.6), we have

$$\lim_{n\to\infty}\left|\frac{\frac{1}{W_{n+1}}}{\frac{1}{W_n}}\right| = \lim_{n\to\infty}\left|\frac{W_n}{W_{n+1}}\right|$$

$$= \lim_{n\to\infty} \left| \frac{\dfrac{A\alpha^n - B\beta^n}{\alpha-\beta}}{\dfrac{A\alpha^{n+1} - B\beta^{n+1}}{\alpha-\beta}} \right|$$

$$= \frac{1}{\alpha} \lim_{n\to\infty} \left| \frac{A - B\left(\dfrac{\beta}{\alpha}\right)^n}{A - B\left(\dfrac{\beta}{\alpha}\right)^{n+1}} \right| .$$

Since $\left|\dfrac{\beta}{\alpha}\right| < 1$, it follows that $\lim_{n\to\infty}\left(\dfrac{\beta}{\alpha}\right)^n = \lim_{n\to\infty}\left(\dfrac{\beta}{\alpha}\right)^{n+1} = 0$.

Hence, we obtain

(3.22) $$\boxed{\lim_{n\to\infty} \left| \frac{\dfrac{1}{W_{n+1}}}{\dfrac{1}{W_n}} \right| = \frac{1}{\alpha}} .$$

This result generalizes what we have proved earlier in (1.20).

From (3.4), we note that $2\alpha = p + \sqrt{p^2 - 4q}$, $p \geq 1$ and $q \neq 0$.

If $q < 0$ then $p + \sqrt{p^2 - 4q} > p + p = 2p \geq 2$. It means $2\alpha > 2$, or $\alpha > 1$.

If $q > 0$ then $4q \geq 4$ since q is an integer. By the hypothesis $p^2 - 4q > 0$, it follows that

$p^2 > 4q \geq 4$, or $p > 2$.

Hence, we have $2\alpha = p + \sqrt{p^2 - 4q} > p > 2$ or $\alpha > 1$.

For both of the above cases, we have shown that $\alpha > 1$ for all integer $q \neq 0$.

From (3.22), we get $\lim_{n\to\infty}\left|\dfrac{\frac{1}{W_{n+1}}}{\frac{1}{W_n}}\right| = \dfrac{1}{\alpha} < 1.$

Thus, by the ratio test theorem, we conclude that $\sum_{n=1}^{\infty}\frac{1}{W_n}$ converges absolutely and therefore convergent.

### 3.2.2 Evaluation of the sum of the series $\sum_{n=1}^{\infty}\frac{1}{W_n}$

We desire to evaluate $\sum_{n=1}^{\infty}\frac{1}{W_n}$, but up to now, no one can evaluate exactly the result. In 1969, Brousseau [6], calculated by approximating to 10 decimal place for the case $\sum_{n=1}^{\infty}\frac{1}{F_n}$ and $\sum_{n=1}^{\infty}\frac{1}{L_n}$ as follow: $\sum_{n=1}^{400}\frac{1}{F_n} = 3.3598856662\ldots$

$$\sum_{n=1}^{400}\frac{1}{L_n} = 1.9628581732\ldots .$$

The author of this thesis, in section 1.5 of chapter I, provided a program using the Java language which gives more accurate values of $\sum_{n=1}^{\infty}\frac{1}{F_n}$ and $\sum_{n=1}^{\infty}\frac{1}{L_n}$,

$\sum_{n=1}^{400}\frac{1}{F_n} = 3.35988566624317755317201130291892717968890513373196848649555381532513031899668338360622407831480352 50\ldots,$

$\sum_{n=1}^{400}\frac{1}{L_n} = 1.962858173209645782868795128675183526649593017162219421130715240417061607546460377974931049935224582 5\ldots.$

In 1883, Catalan [9] had divided the series $\sum_{n=1}^{\infty}\frac{1}{F_n}$ into two parts: even terms of $\{F_n\}$, $\sum_{n=1}^{\infty}\frac{1}{F_{2n}}$, and odd terms of $\{F_n\}$, $\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}}$. The series $\sum_{n=1}^{\infty}\frac{1}{F_{2n}}$ can be expressed in term of the Lambert series (to be discussed later) and of Jacobi elliptic functions for $\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}}$.

Therefore, we find that it is appropriate to partition the series $\sum_{n=1}^{\infty}\frac{1}{W_n}$ into two parts, one with odd terms $\{W_{2n-1}\}$, and other with even terms $\{W_{2n}\}$. We will use the Jacobi elliptic functions (in section 3.3) and the Lambert series (in section 3.4) to investigate the series $\sum_{n=1}^{\infty}\frac{1}{W_{2n-1}}$ and $\sum_{n=1}^{\infty}\frac{1}{W_{2n}}$.

### 3.3 Evaluating Special Series Using The Jacobi Elliptic Functions

In chapter II, we discussed the two elliptic integral constants K and $K'$. They are given by $K = K(k) = \int_0^{\frac{\pi}{2}}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}$, and $K' = K'(k') = \int_0^{\frac{\pi}{2}}\frac{d\alpha}{\sqrt{1-k'^2\sin^2\alpha}}$ where $k'^2 + k^2 = 1$. In all the lemmas and theorems in this section will use these two constants. Also, Jacobi's symbol $q$ is here replaced by $r$ to avoid confusion with the use of $q$ in the recurrence relation (3.1).

<u>Lemma 3.3.1</u> Let $r = e^{\frac{-\pi K'}{K}}$. Then

$$\boxed{\sum_{n=1}^{\infty}\frac{4r^n}{1+r^{2n}} = \frac{2K}{\pi} - 1}. \quad (3.23)$$

Proof: From equation (2.51), we have the following identity

$$dn\,z = \frac{\pi}{2K} + \frac{2\pi}{K}\sum_{n=1}^{\infty}\frac{r^n}{1+r^{2n}}\cos(2n)\frac{\pi z}{2K}.$$

Now, let z = 0 in this equation, we get $dn\,0 = \frac{\pi}{2K} + \frac{2\pi}{K}\sum_{n=1}^{\infty}\frac{r^n}{1+r^{2n}}\cos 0$.

Using $dn$ 0 = 1 by (2.8) and solving this equation for $\sum_{n=1}^{\infty}\frac{r^n}{1+r^{2n}}$, we obtain (3.23).

□

Lemma 3.3.2 Let $r = e^{\frac{-\pi K'}{K}}$. Then

(3.24) $$\boxed{\sum_{n=1}^{\infty}\frac{r^{n+\frac{1}{2}}}{1+r^{2n+1}} = \frac{kK}{2\pi}}.$$

Proof: let z = 0 in equation (2.50), we obtain $cn\,0 = \frac{2\pi}{\sqrt{m}\,K}\sum_{n=0}^{\infty}\frac{r^{n+\frac{1}{2}}}{1+r^{2n+1}}\cos 0$.

Since $cn$ 0 = 1 by (2.8) and $\sqrt{m} = k$, solving this equation for $\frac{kK}{\pi}$ yields (3.24).

□

Theorem 3.3.1 Suppose A = B = 1 in (3.6) and q = −1. Then

(3.25) $$\boxed{\sum_{n=1}^{\infty}\frac{1}{W_{2n-1}} = \sqrt{p^2+4}\cdot\frac{\mu K}{2\pi}},$$ where μ is a real number such that

$$r = e^{\frac{-\pi K'(\mu')}{K(\mu)}} = \beta^2,\ \beta = \frac{1}{2}\left(p - \sqrt{p^2+4}\right),\ \text{and}\ \mu^2 + \mu'^2 = 1.$$

Proof: Since A = B = 1, from (3.6), we obtain $W_{2n-1} = \frac{\alpha^{2n-1} - \beta^{2n-1}}{\alpha - \beta}$. Thus,

$$\frac{1}{W_{2n-1}} = (\alpha - \beta)\frac{\beta^{2n-1}}{(\alpha\beta)^{2n-1} - \beta^{4n-2}}$$
$$= (\alpha - \beta)\frac{\beta^{2n-1}}{(-1)^{2n-1} - \beta^{4n-2}} \quad \text{since } \alpha\beta = -1$$
$$= (\alpha - \beta)\frac{\beta \cdot \beta^{2n-2}}{(-1) - \beta^{4n-2}}.$$

Since $\beta = \frac{1}{2}\left(p - \sqrt{p^2 + 4}\right) < 0$. Hence, from the hypothesis, $r = \beta^2$, we get $\sqrt{r} = -\beta$

and $0 < \sqrt{r} < 1$ since $-\beta = \frac{1}{2}\left(\sqrt{q^2 + 4} - p\right) < 1$ when $p \geq 1$. Thus, we can write

$$\sum_{n=1}^{\infty}\frac{1}{W_{2n-1}} = (\alpha - \beta)\sum_{n=1}^{\infty}\frac{\left(-\sqrt{r}\right)\cdot\left(-\sqrt{r}\right)^{2n-2}}{(-1) - \left(-\sqrt{r}\right)^{4n-2}}$$
$$= (\alpha - \beta)\sum_{n=1}^{\infty}\frac{r^{n-1/2}}{1 + r^{2n-1}}$$
$$= (\alpha - \beta)\sum_{n=0}^{\infty}\frac{r^{n+1/2}}{1 + r^{2n+1}} \quad \text{by replacing n with n + 1}$$
$$= (\alpha - \beta)\cdot\frac{1}{2}\frac{\mu K}{\pi} \quad \text{by (3.24) with } k = \mu.$$

Substituting $\alpha - \beta = \sqrt{p^2 + 4}$ in the above equation obtains (3.25).

□

Here are some special cases of theorem 3.3.1.

Since A = B = 1 and q = −1, we can use (3.25) to evaluate series involving the reciprocals of the Fibonacci and Pell numbers, $\{F_{2n-1}\}$ and $\{P_{2n-1}\}$.

If p = 1, then from (3.25), we have

(3.26) $$\boxed{\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}} = \sqrt{5}\cdot\frac{\mu K}{2\pi}}$$, where μ satisfies

$$\beta^2 = \left(\frac{1-\sqrt{5}}{2}\right)^2 = e^{\frac{-\pi K'(\mu')}{K(\mu)}} \quad \text{and } \mu^2 + \mu'^2 = 1.$$

(3.27) $$\boxed{\sum_{n=1}^{\infty}\frac{1}{P_{2n-1}} = \sqrt{2}\cdot\frac{\mu K}{2\pi}}$$ where μ satisfies

$$\beta^2 = \left(1-\sqrt{2}\right)^2 = e^{\frac{-\pi K'(\mu')}{K(\mu)}} \quad \text{and } \mu^2 + \mu'^2 = 1.$$

The results of (3.26) and (3.27) are consistent with those of Bruckman [7].

Theorem 3.3.2 Suppose $A = -B = \alpha - \beta$ and $q = -1$. Then

(3.28) $$\boxed{\sum_{n=1}^{\infty}\frac{1}{W_{2n}} = \frac{1}{4}\left(\frac{2K}{\pi}-1\right)}.$$

Proof: Again, using (3.6), we have

$$\begin{aligned}
\frac{1}{W_{2n}} &= \frac{\alpha-\beta}{A\alpha^{2n} - B\beta^{2n}} \\
&= \frac{\alpha-\beta}{A\left(\alpha^{2n} - \frac{B}{A}\beta^{2n}\right)} \\
&= \frac{A}{A\left(\alpha^{2n} - (-1)\beta^{2n}\right)} \quad \text{since } \alpha-\beta = A \text{ and } \frac{B}{A} = -1 \\
&= \frac{\beta^{2n}}{(\alpha\beta)^{2n} + \beta^{4n}} \\
&= \frac{\beta^{2n}}{1+\beta^{4n}} \quad \text{since } \alpha\beta = q = -1 \text{ by (3.5).}
\end{aligned}$$

Now let k = μ such that $r = e^{\frac{-\pi K'(\mu')}{K(\mu)}} = \beta^2,\ \beta = \frac{1}{2}\left(p - \sqrt{p^2 - 4q}\right)$, then $\sqrt{r}$ = –β since

$\beta = \frac{1}{2}\left(p - \sqrt{p^2 + 4}\right) < 0$. Thus, $\sum_{n=1}^{\infty} \frac{1}{W_{2n}} = \sum_{n=1}^{\infty} \frac{r^n}{1 + r^{2n}}$. Using (3.23) for the right hand side of this equation, we get (3.28)

□

Here are some special cases of theorem 3.3.2

Since A = –B and q = –1, we can use (3.28) to evaluate series involving the reciprocals of Lucas and Pell-Lucas numbers.

If p = 1, then from (3.25), we have

(3.29) $$\boxed{\sum_{n=1}^{\infty} \frac{1}{L_{2n}} = \frac{1}{4}\left(\frac{2K}{\pi} - 1\right)},\ \text{where } K = \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1 - \mu^2 \sin^2 \alpha}} \text{ with}$$

μ satisfies $\beta^2 = \left(\frac{1 - \sqrt{5}}{2}\right)^2 = e^{\frac{-\pi K'(\mu')}{K(\mu)}}$ and $\mu^2 + \mu'^2 = 1$.

(3.30) $$\boxed{\sum_{n=1}^{\infty} \frac{1}{Q_{2n}} = \frac{1}{4}\left(\frac{2K}{\pi} - 1\right)},\ \text{where } K = \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1 - \mu^2 \sin^2 \alpha}} \text{ with}$$

μ satisfies $\beta^2 = \left(1 - \sqrt{2}\right)^2 = e^{\frac{-\pi K'(\mu')}{K(\mu)}}$ and $\mu^2 + \mu'^2 = 1$.

Notice that Bruckman [8] had calculated $\sum_{n=1}^{\infty} \frac{1}{L_{2n}} = 0.56617767\ \ldots$, and E. W. Bowen found $\sum_{n=1}^{\infty} \frac{1}{Q_{2n}} = 0.2017495\ \ldots$ .

More information about the applications of the Jacobi elliptic functions on the reciprocals of $\{W_n\}$ can be found in Horadam [13].

### 3.4 Lambert Series and Applications

The Lambert series is a special case of the series

$$(3.31) \qquad \boxed{\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}}.$$

Specifically, the Lambert series is defined as

$$(3.32) \qquad \boxed{L(x)=\sum_{n=1}^{\infty} \frac{x^n}{1-x^n},\ |x|<1}.$$

It is clear that the Lambert series is convergent by the ratio test as follows:

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{1-x^{n+1}}}{\frac{x^n}{1-x^n}}\right| = \lim_{n\to\infty}|x|\left|\frac{1-x^n}{1-x^{n+1}}\right|$$

$$= \lim_{n\to\infty}|x| \quad \text{since } \lim_{n\to\infty} x^n = \lim_{n\to\infty} x^{n+1} = 0 \text{ when } |x|<1$$

$$= |x| < 1.$$

The generalized Lambert series is defined (see Arista [3]).

$$(3.33) \qquad \boxed{L(a,x)=\sum_{n=1}^{\infty} \frac{ax^n}{1-ax^n},\ |x|<1,\ |ax|<1}.$$

<u>Lemma 3.4.1</u>

$$(3.34) \qquad \boxed{\sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{4n}} = L(x^2)-L(x^4),\ |x|<1}.$$

<u>Proof</u>: Using ratio test, it is easy to show that $\sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{4n}}$ is convergent when $|x|<1$.

Observe that $\frac{x}{1-x}-\frac{x^2}{1-x^2}=\frac{x}{1-x^2}$.

Now, replacing x by $x^{2n}$ in the above equation to get $\frac{x^{2n}}{1-x^{2n}}-\frac{x^{4n}}{1-x^{4n}}=\frac{x^{2n}}{1-x^{4n}}$.

Thus, it follows that $\sum_{n=1}^{\infty}\frac{x^{2n}}{1-x^{4n}}-\sum_{n=1}^{\infty}\frac{x^{4n}}{1-x^{4n}}=\sum_{n=1}^{\infty}\frac{x^{2n}}{1-x^{4n}}$, or $L(x^2)-L(x^4)=\sum_{n=1}^{\infty}\frac{x^{2n}}{1-x^{4n}}$.

□

Lemma 3.4.2

(3.35) $$\boxed{\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1-x^{2n+1}}=L(x)-L(x^2),\ |x|<1}.$$

Proof: Again, $\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1-x^{2n+1}}$ converges for $|x|<1$ by applying the ratio test.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{x^n}{1-x^n}&=\frac{x}{1-x}+\frac{x^2}{1-x^2}+\frac{x^3}{1-x^3}+\frac{x^4}{1-x^4}+\cdots\\&=\left(\frac{x}{1-x}+\frac{x^3}{1-x^3}+\cdots\right)+\left(\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}+\cdots\right)\\&=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1-x^{2n+1}}+\sum_{n=1}^{\infty}\frac{x^{2n}}{1-x^{2n}}.\end{aligned}$$

It implies that $\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1-x^{2n+1}}=\sum_{n=1}^{\infty}\frac{x^n}{1-x^n}-\sum_{n=1}^{\infty}\frac{x^{2n}}{1-x^{2n}}=L(x)-L(x^2)$.

□

Lemma 3.4.3

(3.36) $$\boxed{\sum_{n=1}^{\infty}\frac{x^n}{1+x^n}=L(x)-2L(x^2),\ |x|<1}.$$

Proof: Again, one can verify that $\sum_{n=1}^{\infty}\frac{x^n}{1+x^n}$ converges via the ratio test.

Observe that $\frac{x}{1-x}-\frac{2x^2}{1-x^2}=\frac{x}{1+x}$.

Replacing x by $x^n$ in the above equation obtains $\frac{x^n}{1-x^n}-\frac{2x^{2n}}{1-x^{2n}}=\frac{x^n}{1+x^n}$.

Thus, it follows that $\sum_{n=1}^{\infty}\frac{x^n}{1-x^n}-\sum_{n=1}^{\infty}\frac{2x^{2n}}{1-x^{2n}}=\sum_{n=1}^{\infty}\frac{x^n}{1+x^n}$, or $L(x)-L(x^2)=\sum_{n=1}^{\infty}\frac{x^n}{1+x^n}$.

□

Lemma 3.4.4

(3.37) $$\boxed{\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1+x^{2n+1}}=L(x)-3L(x^2)+2L(x^4),\ |x|<1}.$$

Proof: Again, one can verify that $\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1+x^{2n+1}}$ converges via the ratio test.

Expanding $\sum_{n=1}^{\infty}\frac{x^n}{1+x^n}$, we have

$$\sum_{n=1}^{\infty}\frac{x^n}{1+x^n}=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1+x^{2n+1}}+\sum_{n=1}^{\infty}\frac{x^{2n}}{1+x^{2n}},\text{ or}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1+x^{2n+1}}&=\sum_{n=1}^{\infty}\frac{x^n}{1+x^n}-\sum_{n=1}^{\infty}\frac{x^{2n}}{1+x^{2n}}\\&=\left[L(x)-2L(x^2)\right]-\left[L(x^2)-2L(x^4)\right]\text{ by (3.36)}\\&=L(x)-3L(x^2)+2L(x^4).\end{aligned}$$

□

Lemma 3.4.5

(3.38) $$\boxed{\sum_{n=1}^{\infty}\frac{x^{2n-1}}{1-x^{4n-2}}=L(x)-2L(x^2)+L(x^4),\ |x|<1}.$$

Proof: One can verify that $\sum_{n=1}^{\infty}\frac{x^{2n-1}}{1-x^{4n-1}}$ converges via the ratio test.

Observe that $\frac{x}{1-x}+\frac{x}{1+x}=\frac{2x}{1-x^2}$.

Replacing x by $x^{2n-1}$ in the above equation, we obtain

$$\sum_{n=1}^{\infty}\frac{x^{2n-1}}{1-x^{2n-1}}+\sum_{n=1}^{\infty}\frac{x^{2n-1}}{1+x^{2n-1}}=2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{1-x^{4n-2}}.$$

Since $\sum_{n=1}^{\infty}\frac{x^{2n+1}}{1-x^{2n+1}}=L(x)-L(x^2)$ by (3.35) and

$\sum_{n=1}^{\infty}\frac{x^{2n-1}}{1+x^{2n-1}}=L(x)-3L(x^2)+2L(x^4)$ by (3.37),

it follows that $\sum_{n=1}^{\infty}\frac{x^{2n-1}}{1-x^{4n-2}}=\frac{1}{2}\left[L(x)-L(x^2)\right]+\frac{1}{2}\left[L(x)-3L(x^2)+2L(x^4)\right]$, which

implies (3.38).

□

Now we evaluate the series $\sum_{n=1}^{\infty}\frac{1}{W_{2n-1}}$ and $\sum_{n=1}^{\infty}\frac{1}{W_{2n}}$ in terms of the Lambert series.

Theorem 3.4.1 Suppose A = B = 1 and q = − 1. Then

(3.39) $$\boxed{\sum_{n=1}^{\infty}\frac{1}{W_{2n}}=(\alpha-\beta)\left[L(\beta^2)-L(\beta^4)\right]}.$$

Proof: From (3.6), we have

$$\begin{aligned}\frac{1}{W_{2n}}&=\frac{\alpha-\beta}{A\alpha^{2n}-B\beta^{2n}}\\&=(\alpha-\beta)\frac{\beta^{2n}}{(\alpha\beta)^{2n}-\beta^{4n}}\quad\text{since } A=B=1\end{aligned}$$

$$= (\alpha - \beta)\frac{\beta^{2n}}{1-\beta^{4n}} \text{ since } \alpha\beta = -1.$$

It follows that $\sum_{n=1}^{\infty}\frac{1}{W_{2n}} = (\alpha - \beta)\sum_{n=1}^{\infty}\frac{\beta^{2n}}{1-\beta^{4n}}$.

Since $|\beta| = \left|\frac{1}{2}\left(p - \sqrt{p^2+4}\right)\right| < 1$ when p ≥ 1, apply (3.34) to obtain (3.39).

□

Here are some special cases of theorem 3.4.1

Since A = B = 1 and q = − 1, we can apply (3.39) to the Fibonacci and Pell sequences as follow.

(3.40) $$\boxed{\sum_{n=1}^{\infty}\frac{1}{F_{2n}} = \sqrt{5}\left[L\left(\frac{3-\sqrt{5}}{2}\right) - L\left(\frac{7-3\sqrt{5}}{2}\right)\right]}$$

since $\alpha - \beta = \sqrt{5}$ and $\beta = \frac{1-\sqrt{5}}{2}$ for the Fibonacci numbers.

(3.41) $$\boxed{\sum_{n=1}^{\infty}\frac{1}{P_{2n}} = 2\sqrt{2}\left[L\left(3-2\sqrt{2}\right) - L\left(17-12\sqrt{2}\right)\right]}$$

since $\alpha - \beta = 2\sqrt{2}$ and $\beta = 1-\sqrt{2}$ for the Pell numbers.

<u>Theorem 3.4.2</u> Suppose A = − B = α − β and q = −1. Then

(3.42) $$\boxed{\sum_{n=1}^{\infty}\frac{1}{W_{2n-1}} = -L(\beta) + 2L(\beta^2) - L(\beta^4)}.$$

<u>Proof</u>: From (3.6), we have

$$\frac{1}{W_{2n-1}} = \frac{\alpha - \beta}{A\alpha^{2n-1} - B\beta^{2n-1}}$$

$$= \frac{A}{A\left(\alpha^{2n-1} - \frac{B}{A}\beta^{2n-1}\right)} \text{ since } \alpha - \beta = A$$

$$= \frac{\beta^{2n-1}}{(\alpha\beta)^{2n-1} + \beta^{4n-2}} \text{ since } \frac{B}{A} = -1$$

$$= -\frac{\beta^{2n-1}}{1 - \beta^{4n-2}} \text{ since } \alpha\beta = -1 .$$

It implies that $\sum_{n=1}^{\infty} \frac{1}{W_{2n-1}} = -\sum_{n=1}^{\infty} \frac{\beta^{2n-1}}{1-\beta^{4n-2}} = -\left[L(\beta) - 2L(\beta^2) + L(\beta^4)\right]$ by (3.38).

□

Here are some special cases of theorem 3.4.2

With conditions A = – B = α – β and q = –1, we can apply (3.42) to the Lucas and Pell-Lucas sequences as follow.

(3.43) $$\boxed{\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}} = -L\left(\frac{1-\sqrt{5}}{2}\right) + 2L\left(\frac{3-\sqrt{5}}{2}\right) - L\left(\frac{7-3\sqrt{5}}{2}\right)} \text{ since } \beta = \frac{1-\sqrt{5}}{2} .$$

(3.44) $$\boxed{\sum_{n=1}^{\infty} \frac{1}{Q_{2n-1}} = -L\left(1-\sqrt{2}\right) + 2L\left(3-2\sqrt{2}\right) - L\left(17-12\sqrt{2}\right)} \text{ since } \beta = 1-\sqrt{2} .$$

### 3.5 Further Applications of Lambert Series

In this section, we investigate the series of the type $\sum_{n=1}^{\infty} \frac{1}{W_{rn} W_{r(n+1)}}$, where r is a positive odd integer. We first justify the following two lemmas.

Lemma 3.5.1 Suppose A = B = 1, q = –1, and r is a positive odd integer. Then

(3.45) $$\boxed{2\sum_{n=1}^{\infty} \frac{1}{\alpha^{rn} W_{rn}} = \frac{1}{\alpha^r W_r} + W_r \cdot \sum_{n=1}^{\infty} \frac{1}{W_{rn} W_{r(n+1)}}} .$$

Proof: From (3.6), we have

$$\alpha^r W_{r(n+1)} + W_{rn} = \alpha^r \cdot \frac{A\alpha^{r(n+1)} - B\beta^{r(n+1)}}{\alpha - \beta} + \frac{A\alpha^{rn} - B\beta^{rn}}{\alpha - \beta}$$

$$= \frac{1}{\alpha - \beta}\left[\left(\alpha^{r(n+2)} - \alpha^r \beta^{r(n+1)}\right) + \left(\alpha^{rn} - \beta^{rn}\right)\right] \text{ since } A = B = 1$$

$$= \frac{1}{\alpha - \beta}\left[\left(\alpha^{r(n+2)} - (\alpha\beta)^r \beta^{rn}\right) + \left(\alpha^{rn} - \beta^{rn}\right)\right]$$

$$= \frac{1}{\alpha - \beta}\left[\alpha^{r(n+2)} - (-1)^r \beta^{rn} + \alpha^{rn} - \beta^{rn}\right] \text{ since } \alpha\beta = -1.$$

Since r is a positive odd integer, $-(-1)^r \beta^{rn} - \beta^{rn} = 0$. Thus, we have

$$\alpha^r W_{r(n+1))} + W_{rn} = \frac{1}{\alpha - \beta}\left[\alpha^{r(n+2)} + \alpha^{rn}\right]$$

$$= \frac{\alpha^{r(n+1)}}{\alpha - \beta}\left[\alpha^r + \frac{1}{\alpha^r}\right]$$

$$= \frac{\alpha^{r(n+1)}}{\alpha - \beta}\left[\alpha^r - \beta^r\right] \text{ since } \alpha\beta = -1, \text{ r is odd}$$

$$= \alpha^{r(n+1)} W_r. \qquad (*)$$

Thus, it follows that

$$\frac{1}{\alpha^{rn} W_{rn}} + \frac{1}{\alpha^{r(n+1)} W_{r(n+1)}} = \frac{\alpha^r W_{r(n+1)} + W_{rn}}{\alpha^{r(n+1)} W_{rn} W_{r(n+1)}}$$

$$= \frac{\alpha^{r(n+1)} W_r}{\alpha^{r(n+1)} W_{rn} W_{r(n+1)}} \text{ by } (*)$$

$$= \frac{W_r}{W_{rn} W_{r(n+1)}}.$$

Therefore, it follows that $\sum_{n=1}^{\infty} \frac{1}{\alpha^{rn} W_{rn}} + \sum_{n=1}^{\infty} \frac{1}{\alpha^{r(n+1)} W_{r(n+1)}} = \sum_{n=1}^{\infty} \frac{W_r}{W_{rn} W_{r(n+1)}}.$

Since $\sum_{n=1}^{\infty}\frac{1}{\alpha^{r(n+1)}W_{r(n+1)}}=\sum_{n=1}^{\infty}\frac{1}{\alpha^{rn}W_{rn}}-\frac{1}{W_r\alpha^r}$, it follows that

$$2\sum_{n=1}^{\infty}\frac{1}{\alpha^{rn}W_{rn}}=\frac{1}{\alpha^r W_r}+W_r\cdot\sum_{n=1}^{\infty}\frac{1}{W_{rn}W_{r(n+1)}}.$$

□

Lemma 3.5.2 Suppose A = B = 1, q = –1, and r is a positive odd integer. Then

$$(3.46)\qquad \boxed{\sum_{n=1}^{\infty}\frac{1}{\alpha^{rn}W_{rn}}=(\alpha-\beta)\left[L\left(\beta^{2r}\right)-2L\left(\beta^{4r}\right)+2L\left(\beta^{8r}\right)\right]}.$$

Proof: Using the Binet form for $W_{rn}$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{\alpha^{rn}W_{rn}}&=\sum_{n=1}^{\infty}\frac{\alpha-\beta}{\alpha^{rn}\left(\alpha^{rn}-\beta^{rn}\right)}\\&=(\alpha-\beta)\sum_{n=1}^{\infty}\frac{1}{\alpha^{2rn}-(\alpha\beta)^{rn}}\\&=(\alpha-\beta)\sum_{n=1}^{\infty}\frac{1}{\alpha^{2rn}-(-1)^{rn}}\quad\text{since } \alpha\beta=-1\\&=(\alpha-\beta)\sum_{n=1}^{\infty}\frac{\alpha^{-2rn}}{1-(-1)^{rn}\alpha^{-2rn}}\\&=(\alpha-\beta)\sum_{n=1}^{\infty}\frac{\beta^{2rn}}{1-(-1)^{rn}\beta^{2rn}}\quad\text{since } \beta=-\alpha^{-1}.\end{aligned}$$

Expanding the right hand side yields $\sum_{n=1}^{\infty}\frac{\beta^{2rn}}{1-(-1)^{rn}\beta^{2rn}}=\sum_{n=0}^{\infty}\frac{\left(\beta^{2r}\right)^{2n+1}}{1+\left(\beta^{2r}\right)^{2n+1}}+\sum_{n=1}^{\infty}\frac{\beta^{4rn}}{1-\beta^{4rn}}$.

Using (3.37) with x = $\beta^{2r}$ and the definition (3.32) of the Lambert function with x = $\beta^{4r}$, we get (3.46).

□

Theorem 3.5.1 Suppose A = B = 1, q = −1, and r is a positive odd integer. Then

$$(3.47) \qquad \boxed{\sum_{n=1}^{\infty}\frac{1}{W_{rn}W_{r(n+1)}}=\frac{2(\alpha-\beta)}{W_r}\left[L(\beta^{2r})-2L(\beta^{4r})+2L(\beta^{8r})\right]+\frac{\beta^r}{W_r^2}}.$$

Proof: From (3.45)

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{W_{rn}W_{r(n+1)}}&=\frac{1}{W_r}\left(2\sum_{n=1}^{\infty}\frac{1}{\alpha^{rn}W_{rn}}-\frac{1}{\alpha^r W_r}\right)\\&=\frac{1}{W_r}\left\{2(\alpha-\beta)\left[L(\beta^{2r})-2L(\beta^{4r})+2L(\beta^{8r})\right]-\frac{1}{\alpha^r W_r}\right\} \text{ by (3.46)}\\&=\frac{2(\alpha-\beta)}{W_r}\left[L(\beta^{2r})-2L(\beta^{4r})+2L(\beta^{8r})\right]+\frac{\beta^r}{W_r^2} \quad \text{since}\ \frac{1}{\alpha}=-\beta,\ \text{r is odd.}\end{aligned}$$

□

Here are some special cases of theorem 3.5.1.

With condition A = B = 1, q = −1, and r is a positive odd integer, we can apply (3.47) to the Fibonacci and Pell sequences.

For the Fibonacci sequence, we have $\alpha-\beta=\sqrt{5}$, $\beta=\frac{1-\sqrt{5}}{2}$, and

$$(3.48) \qquad \boxed{\sum_{n=1}^{\infty}\frac{1}{F_{rn}F_{r(n+1)}}=\frac{2\sqrt{5}}{F_r}\left[L(\beta^{2r})-2L(\beta^{4r})+2L(\beta^{8r})\right]+\frac{\beta^r}{F_r^2}}.$$

For the Pell sequence, we have $\alpha-\beta=2\sqrt{2}$ and $\beta=1-\sqrt{2}$, and

$$(3.49) \qquad \boxed{\sum_{n=1}^{\infty}\frac{1}{P_{rn}P_{r(n+1)}}=\frac{4\sqrt{2}}{P_r}\left[L(\beta^{2r})-2L(\beta^{4r})+2L(\beta^{8r})\right]+\frac{\beta^r}{P_r^2}}.$$

Lemma 3.5.3 Suppose A = − B = α − β, q = −1, and r is a positive odd integer. Then

(3.50) $$\boxed{2\sum_{n=1}^{\infty}\frac{1}{\alpha^{rn}W_{rn}}=\frac{1}{\alpha^{r}W_{r}}+(\alpha-\beta)U_{r}\cdot\sum_{n=1}^{\infty}\frac{1}{W_{rn}W_{r(n+1)}}}\text{ where } U_{r}=\frac{\alpha^{r}-\beta^{r}}{\alpha-\beta}.$$

Proof: Since A = – B = α – β, by the Binet form, it is easy to see that $W_{rn}=\alpha^{rn}+\beta^{rn}$.

$$\begin{aligned}
\alpha^{r}W_{r(n+1)}+W_{rn} &= \alpha^{r}\cdot\left(\alpha^{r(n+1)}+\beta^{r(n+1)}\right)+\left(\alpha^{rn}+\beta^{rn}\right)\\
&=\left(\alpha^{r(n+2)}+(\alpha\beta)^{r}\beta^{rn}\right)+\left(\alpha^{rn}+\beta^{rn}\right)\\
&=\alpha^{r(n+2)}+(-1)^{r}\beta^{rn}+\alpha^{rn}+\beta^{rn} \text{ since } \alpha\beta=-1\\
&=\alpha^{r(n+2)}+\alpha^{rn} \quad \text{since r is odd}\\
&=\alpha^{r(n+1)}\left(\alpha^{r}+\frac{1}{\alpha^{r}}\right)\\
&=\alpha^{r(n+1)}\left(\alpha^{r}-\beta^{r}\right) \text{ since } \alpha\beta=-1\text{, r is odd}\\
&=(\alpha-\beta)\frac{\alpha^{r(n+1)}\left(\alpha^{r}-\beta^{r}\right)}{(\alpha-\beta)}\\
&=(\alpha-\beta)\alpha^{r(n+1)}U_{r}. \quad (**)
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
\frac{1}{\alpha^{rn}W_{rn}}+\frac{1}{\alpha^{r(n+1)}W_{r(n+1)}} &= \frac{\alpha^{r}W_{r(n+1)}+W_{rn}}{\alpha^{r(n+1)}W_{rn}W_{r(n+1)}}\\
&=\frac{(\alpha-\beta)\alpha^{r(n+1)}U_{r}}{\alpha^{r(n+1)}W_{rn}W_{r(n+1)}} \text{ by } (**)\\
&=\frac{(\alpha-\beta)U_{r}}{W_{rn}W_{r(n+1)}}.
\end{aligned}$$

Therefore, it follows that $\sum_{n=1}^{\infty}\frac{1}{\alpha^{rn}W_{rn}}+\sum_{n=1}^{\infty}\frac{1}{\alpha^{r(n+1)}W_{r(n+1)}}=\sum_{n=1}^{\infty}\frac{(\alpha-\beta)U_{r}}{W_{rn}W_{r(n+1)}}$.

Since $\sum_{n=1}^{\infty}\frac{1}{\alpha^{r(n+1)}W_{r(n+1)}}=\sum_{n=1}^{\infty}\frac{1}{\alpha^{rn}W_{rn}}-\frac{1}{W_{r}\alpha^{r}}$, it implies that

$$2\sum_{n=1}^{\infty}\frac{1}{\alpha^{rn}W_{rn}}=\frac{1}{\alpha^{r}W_{r}}+\sum_{n=1}^{\infty}\frac{(\alpha-\beta)U_{r}}{W_{rn}W_{r(n+1)}}.$$

□

Lemma 3.5.4 Suppose A = – B = α – β, q = –1, and r is a positive odd integer. Then

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{\alpha^{rn}W_{rn}}=L\left(\beta^{2r}\right)-2L\left(\beta^{8r}\right)}. \tag{3.51}$$

Proof: Using the Binet form for $W_{rn}$ with condition A = – B = α – β, we have

$\sum_{n=1}^{\infty}\frac{1}{\alpha^{rn}W_{rn}}=\sum_{n=1}^{\infty}\frac{1}{\alpha^{rn}\left(\alpha^{rn}+\beta^{rn}\right)}$. Substituting $\alpha=\frac{-1}{\beta}$ and simplifying, we get

$$\sum_{n=1}^{\infty}\frac{1}{\alpha^{rn}W_{rn}}=\sum_{n=1}^{\infty}\frac{\beta^{2m}}{1+(-1)^{m}\beta^{2m}}.$$

Now, expand the right hand side of this equation; we get

$$\sum_{n=1}^{\infty}\frac{\beta^{2rn}}{1-\left(-1\right)^{rn}\beta^{2rn}}=\sum_{n=0}^{\infty}\frac{\left(\beta^{2r}\right)^{2n+1}}{1-\left(\beta^{2r}\right)^{2n+1}}+\sum_{n=1}^{\infty}\frac{\beta^{4rn}}{1+\beta^{4rn}}.$$

Using (3.35) with x = $\beta^{2r}$ and (3.36) with x = $\beta^{4r}$, we get (3.51).

□

Theorem 3.5.2 Suppose A = – B = α – β, q = –1, and r is a positive odd integer. Then

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{W_{rn}W_{r(n+1)}}=\frac{2}{(\alpha-\beta)U_{r}}\left[L\left(\beta^{2r}\right)-2L\left(\beta^{8r}\right)\right]+\frac{\beta^{r}}{(\alpha-\beta)U_{r}W_{r}}}, \text{ where} \tag{3.52}$$

$$U_{r}=\frac{\alpha^{r}-\beta^{r}}{\alpha-\beta},\ W_{r}=\alpha^{r}+\beta^{r}.$$

Proof: From (3.50), we have

$$\sum_{n=1}^{\infty}\frac{1}{W_{rn}W_{r(n+1)}}=\frac{1}{(\alpha-\beta)U_r}\left(2\sum_{n=1}^{\infty}\frac{1}{\alpha^{rn}W_{rn}}-\frac{1}{\alpha^r W_r}\right)$$

$$=\frac{1}{(\alpha-\beta)U_r}\left[2\left(L\left(\beta^{2r}\right)-2L\left(\beta^{8r}\right)\right)-\frac{1}{\alpha^r W_r}\right] \text{ by (3.51)}.$$

□

Replacing $\frac{1}{\alpha}=-\beta$ in the right side of the above equation yields (3.52).

Here are some special cases of theorem 3.5.2.

With condition A = – B = α – β, we can apply (3.52) to yield following results.

For the Lucas sequence, we have $\alpha-\beta=\sqrt{5}$, $\beta=\frac{1-\sqrt{5}}{2}$, $U_r=F_r$, and

(3.53) $$\sum_{n=1}^{\infty}\frac{1}{L_{rn}L_{r(n+1)}}=\frac{2}{F_r\sqrt{5}}\left[L\left(\beta^{2r}\right)-2L\left(\beta^{8r}\right)\right]+\frac{\beta^r}{\sqrt{5}F_rL_r}.$$

For Pell-Lucas sequence, we have $\alpha-\beta=2\sqrt{2}$, $\beta=1-\sqrt{2}$, $U_r=P_r$, and

(3.54) $$\sum_{n=1}^{\infty}\frac{1}{Q_{rn}Q_{r(n+1)}}=\frac{1}{P_r\sqrt{2}}\left[L\left(\beta^{2r}\right)-2L\left(\beta^{8r}\right)\right]+\frac{\beta^r}{2\sqrt{2}P_rQ_r}.$$

If r = 1, then by (3.48) we have (3.55) and (3.56) by (3.53).

(3.55) $$\sum_{n=1}^{\infty}\frac{1}{F_nF_{(n+1)}}=2\sqrt{5}\left[L\left(\frac{3-\sqrt{5}}{2}\right)-2L\left(\frac{7-3\sqrt{5}}{2}\right)+2L\left(\frac{47-21\sqrt{5}}{2}\right)\right]+\frac{1-\sqrt{5}}{2}.$$

(3.56) $$\sum_{n=1}^{\infty}\frac{1}{L_nL_{(n+1)}}=\frac{2}{\sqrt{5}}\left[L\left(\frac{3-\sqrt{5}}{2}\right)-2L\left(\frac{47-21\sqrt{5}}{2}\right)\right]+\frac{1-\sqrt{5}}{2\sqrt{5}}.$$

More information about the applications of the Lambert series can be found in Andre – Jeannin [2].

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