A PEROV TYPE THEOREM FOR CYCLIC CONTRACTIONS AND APPLICATIONS TO SYSTEMS OF INTEGRAL EQUATIONS

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Abstract. In this paper we will prove a fixed point theorem of Perov type for cyclic contractions on complete generalized metric spaces. Then, as an application, we will study the existence, uniqueness and approximation of the solution for a system of integral equations.

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1. PRELIMINARIES

We begin the considerations with some notions and results which will be useful further in this paper.

Let \((X,d)\) be a metric space. We denote:

\[
P(X) := \{ Y \subseteq X \mid Y \text{ is nonempty}\};
\]

\[
P_{cl}(X) := \{ Y \in P(X) \mid Y \text{ is closed}\};
\]

If \(T : Y \subseteq X \to X\) is a single-valued operator, then the symbol

\[
F_T := \{ x \in Y \mid x \in Tx\}
\]

denotes the fixed point set of \(T\).

Definition 1. A matrix \(S \in M_p(\mathbb{R}_+)\) is called a matrix convergent to zero if \(S^k \to 0\) as \(k \to +\infty\).

Theorem 1 ([5], [4]). Let \(S \in M_p(\mathbb{R}_+)\). The following statements are equivalent:

(i) \(S\) is a matrix convergent to zero;

(ii) \(S^k x \to 0\) as \(k \to +\infty\), \(\forall x \in \mathbb{R}^p\);

(iii) \(I_p - S\) is non-singular and

\[
(I_p - S)^{-1} = I_p + S + S^2 + \ldots
\]

(1.1)

(iv) \(I_p - S\) is non-singular and \((I_p - S)^{-1}\) has nonnegative elements;

(v) \(\lambda \in \mathbb{C}\), \(\det(S - \lambda I_p) = 0\) imply \(|\lambda| < 1\).
The matrices convergent to zero were used by A.I. Perov [2] to generalize the contraction principle in the case of metric spaces with a vector-valued distance.

**Definition 2 ([5]).** Let \((X, d)\) be a metric space with \(d : X \times X \to \mathbb{R}_+^p\) a vector-valued distance and \(T : X \to X\). The operator \(T\) is called an \(S\)-contraction if there exists a matrix \(S \in \mathcal{M}_p(\mathbb{R}_+)\) such that:

(i) \(S\) is a matrix convergent to zero;
(ii) \(d(T(x), T(y)) \leq S d(x, y), \forall x, y \in X\).

**Theorem 2 (Perov, [2]).** Let \((X, d)\) be a complete metric space with \(d : X \times X \to \mathbb{R}_+^p\) a vector-valued distance and \(T : X \to X\) be an \(S\)-contraction. Then:

(i) \(T\) has a unique fixed point \(x^* \in X\);
(ii) \(T^k x \xrightarrow{d} x^*\) as \(k \to +\infty\), for all \(x \in X\);
(iii) \(d(T^k x, x^*) \leq S^k(I_p - S)^{-1}d(x, Tx),\) for all \(x \in X\) and \(k \in \mathbb{N}\);
(iv) \(d(x, x^*) \leq (I_p - S)^{-1}d(x, Tx),\) for all \(x \in X\).

Another consistent generalization of the contraction principle was given by Kirk, Srinivasan and Veeramani, using the concept of cyclic operator.

**Theorem 3 ([1]).** Let \(\{A_i\}_{i=1}^m\) be nonempty subsets of a complete metric space, and suppose \(T : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i\) satisfies the following conditions (where \(A_m+1 = A_1\)):

1. \(TA_i \subseteq A_{i+1}\), for \(1 \leq i \leq m\);
2. \(\exists k \in (0, 1)\) such that \(d(Tx, Ty) \leq k d(x, y), \forall x \in A_i, y \in A_{i+1},\) for \(1 \leq i \leq m\).

Then \(T\) has a unique fixed point.

This theorem suggested the introduction of the following

**Definition 3 ([3]).** Let \(X\) be a nonempty set, \(m\) a positive integer and \(T : X \to X\) an operator. By definition, \(\bigcup_{i=1}^m A_i\) is a cyclic representation of \(X\) with respect to \(T\) if:

(i) \(X = \bigcup_{i=1}^m A_i\), with \(A_i \in \mathcal{P}(X),\) for \(1 \leq i \leq m\);
(ii) \(TA_i \subseteq A_{i+1}\), for \(1 \leq i \leq m\), where \(A_{m+1} = A_1\).

2. **Main results**

**Definition 4.** Let \((X, d)\) be a metric space with \(d : X \times X \to \mathbb{R}_+^p\) a vector-valued distance, \(A_1, \ldots, A_m \in \mathcal{P}_{cl}(X)\) and \(T : X \to X\) be an operator. If:
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(i) \( \bigcup_{i=1}^{m} A_i \) is a cyclic representation of \( X \) with respect to \( T \);

(ii) there exists a matrix \( S \in \mathcal{M}_p(\mathbb{R}_+) \) convergent to zero such that

\[ d(Tx, Ty) \leq S \cdot d(x, y), \quad \text{for any } x \in A_i, \quad y \in A_{i+1}, \text{ where } A_{m+1} = A_1, \]

then, by definition, we say that \( T \) is a cyclic \( S \)-contraction.

**Theorem 4.** Let \( (X, d) \) be a complete metric space with \( d : X \times X \to \mathbb{R}_+^p \) a vector-valued distance, \( A_1, A_2, \ldots, A_m \in P_{cl}(X) \). If \( T : X \to X \) is a cyclic \( S \)-contraction then the following statements hold:

1. \( T \) has a unique fixed point \( x^* \in \bigcap_{i=1}^{m} A_i \) and the Picard iteration \( \{x_n\}_{n \geq 0} \)

\[ x_n = T x_{n-1}, \quad n \geq 1, \]

converges to \( x^* \) for any starting point \( x_0 \in X \);

2. the following estimates hold:

\[ d(x_n, x^*) \leq S^n (I_p - S)^{-1} d(x_0, x_1), \quad n \geq 1; \quad (2.1) \]

\[ d(x_n, x^*) \leq (I_p - S)^{-1} d(x_n, x_{n+1}), \quad n \geq 1; \quad (2.2) \]

3. for any \( x \in X \),

\[ d(x, x^*) \leq (I_p - S)^{-1} d(x, Tx). \quad (2.3) \]

**Proof.** (1)

\[ d(x_n, x_{n+1}) = d(T x_n - T x_{n-1}, x_n) \leq S d(x_{n-1}, x_n) \]

\[ \leq \ldots \leq S^n d(x_0, x_1) \]

For \( k \geq 1 \) we have

\[ d(x_n, x_{n+k}) \leq S^n d(x_0, x_1) + S^{n+1} d(x_0, x_1) + \ldots + S^{n+k-1} d(x_0, x_1) \]

\[ = S^n (I_p + S + S^2 + \ldots + S^{k-1}) d(x_0, x_1) \]

\[ \leq S^n (I_p + S + S^2 + \ldots) d(x_0, x_1) \to 0 \text{ as } n \to \infty, \quad (2.4) \]

which means that \( \{x_n\}_{n \geq 0} \) is a Cauchy sequence.

\( (X, d) \) is a complete metric space, so the sequence \( \{x_n\}_{n \geq 0} \) is convergent to a \( q \in X \).

The sequence \( \{x_n\}_{n \geq 0} \) has an infinite number of terms in each \( A_i, i = 1, m \), so from each \( A_i \) one we can extract a subsequence of \( \{x_n\}_{n \geq 0} \) which converges to \( q = \lim_{n \to \infty} x_n \).

Because \( A_i \) are closed, \( q \in \bigcap_{i=1}^{m} A_i \), so \( \bigcap_{i=1}^{m} A_i \neq \emptyset \).
Let be the restriction \( T \big|_{\bigcap_{i=1}^{m} A_i} : \bigcap_{i=1}^{m} A_i \to \bigcap_{i=1}^{m} A_i \).

\( \bigcap_{i=1}^{m} A_i \) is also complete. Applying Perov’s theorem, \( T \big|_{\bigcap_{i=1}^{m} A_i} \) has a unique fixed point, which can be obtained by means of the Picard iteration starting from any initial point. It remains to prove that the Picard iteration converges to \( x^* \), for any initial guess \( x \in X \).

\[
d(x_{n+1}, x^*) = d(Tx_n, Tx^*) \leq Sd(x_n, x^*) \\
\leq \ldots \leq S^nd(x_0, x^*) \to 0 \text{ as } n \to \infty.
\]

(2) \( d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+k-1}, x_{n+k}) \)
\[
\leq d(x_n, x_{n+1}) + Sd(x_n, x_{n+1}) + \ldots + S^{k-1}d(x_n, x_{n+1}) \\
= (I_p + S + \ldots + S^{k-1})d(x_n, x_{n+1}), \text{ for any } n \in \mathbb{N}, \ k \geq 1.
\]

Using the statement (iii) from Theorem 1, by letting \( k \to \infty \) in (2.4) and (2.5) we obtain the estimates (2.1) and (2.2).

(3) Let \( x \in X \). For \( n = 0, x_0 := x \), the a posteriori estimate (2.2) becomes

\[
d(x, x^*) \leq (I_p - S)^{-1}d(x, Tx).
\]

\[\square\]

Theorem 5. (Data dependence theorem) Let \( T : X \to X \) be as in Theorem 4 with \( F_T = \{x^*_T\} \). Let \( U : X \to X \) be an operator such that:

(i) \( U \) has at least one fixed point \( x^*_U \);

(ii) there exists \( \eta > 0 \) such that

\[
d(Tx, Ux) \leq \eta, \text{ for any } x \in X.
\]

Then \( d(x^*_T, x^*_U) \leq \eta(I_p - S)^{-1} \).

Proof. By letting \( x := x^*_U \) in the inequality (2.3), we have

\[
d(x^*_U, x^*_T) \leq (I_p - S)^{-1}d(x^*_U, Tx^*_U) = (I_p - S)^{-1}d(Ux^*_U, Tx^*_U) \\
\leq (I_p - S)^{-1}\eta.
\]

\[\square\]

Theorem 6. Let \( T : X \to X \) be as in Theorem 4. Then the fixed point problem for \( T \) is well posed, that is, assuming there exist \( z_n \in X, n \in \mathbb{N} \) such that \( d(z_n, Tz_n) \to 0, \text{ as } n \to \infty \), this implies that \( z_n \to x^* \), as \( n \to \infty \), where \( F_T = \{x^*\} \).
Proof. By letting \( x := z_n \) in the inequality (2.3), we have
\[
d(z_n, x^*) \leq (I_p - S)^{-1} d(z_n, T z_n), \quad n \in \mathbb{N}
\]
and letting \( n \to \infty \) we obtain \( d(z_n, x^*) \to 0, n \to \infty \). □

3. AN APPLICATION TO A SYSTEM OF INTEGRAL EQUATIONS

We apply the results given by Theorem 2.1 to study the existence and the uniqueness of solutions of the following system of integral equations:

\[
\begin{cases}
  x_1(t) = \int_a^b G_1(t, s) f_1(s, x_1(s), x_2(s)) ds, & t \in [a, b] \\
  x_2(t) = \int_a^b G_2(t, s) f_2(s, x_1(s), x_2(s)) ds
\end{cases}
\]

where \( a, b \in \mathbb{R}, a < b, \)

\( G_1, G_2 \in C([a, b] \times [a, b], [0, \infty]), \)

\( f_1, f_2 \in C([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}). \)

Theorem 7. We suppose that:

(i) there exist \( \alpha_k, \beta_k \in C([a, b], \mathbb{R}), m_k, M_k \in \mathbb{R} \) with \( m_k \leq \alpha_k(t) \leq \beta_k(t) \leq M_k \), for any \( t \in [a, b] \), such that

\[
\begin{cases}
  \alpha_k(t) \leq \int_a^b G_k(t, s) f_k(s, \beta_1(s), \beta_2(s)) ds \\
  \beta_k(t) \geq \int_a^b G_k(t, s) f_k(s, \alpha_1(s), \alpha_2(s)) ds
\end{cases} \quad \text{for } k \in \{1, 2\}
\]

(ii) there exist \( a_1, b_1, a_2, b_2 \in \mathbb{R}_+ \), such that

\[
|f_1(s, u_1, u_2) - f_1(s, v_1, v_2)| \leq a_1|u_1 - v_1| + a_2|u_2 - v_2|,
|f_2(s, u_1, u_2) - f_2(s, v_1, v_2)| \leq b_1|u_1 - v_1| + b_2|u_2 - v_2|,
\]

for any \( s \in [a, b] \) and \( u_k, v_k \in \mathbb{R} \), with

\[
\begin{cases}
  u_k \leq M_k \\
  v_k \geq m_k
\end{cases} \quad \text{or} \quad \begin{cases}
  u_k \geq m_k \\
  v_k \leq M_k
\end{cases} \quad \text{for } k \in \{1, 2\};
\]

(iii) \( \sup_{t \in [a, b]} \int_a^b G_k(t, s) ds \leq 1 \) for \( k \in \{1, 2\}; \)

(iv) \( f_k \) is decreasing in each of the last two variables, that is,

\[
u_1, u_2, v_1, v_2 \in \mathbb{R}, \quad u_1 \leq v_1, \quad u_2 \leq v_2 \Rightarrow f_k(s, u, v) \geq f_k(s, u_2, v_2),
\]

for any \( s \in [a, b], \) and \( k \in \{1, 2\}; \)
(v) the matrix $S = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ converges to zero.

Then the system (3.1) has a unique solution $x^* = (x_1^*, x_2^*) \in C([a, b], \mathbb{R}^2)$, with $\alpha_k \leq x_k^* \leq \beta_k$, for $k \in \{1, 2\}$.

This solution can be obtained by the successive approximations method, starting at any element $x^0 \in C([a, b], \mathbb{R}^2)$. Moreover, if $x^n$ is the $n$th successive approximation, then we have the following estimation:

$$
\|x^* - x^n\| \leq S^n (I_2 - S)^{-1} \|x^0 - x^1\|.
$$

where

$$
\|x\| = \begin{pmatrix} |x_1|_{\infty} \\ |x_2|_{\infty} \end{pmatrix} \quad \text{and} \quad |x|_{\infty} = \max_{t \in [a, b]} |x(t)|.
$$

**Proof.** Let us denote

$$
X := (C([a, b], \mathbb{R}), \cdot, |\cdot|_{\infty}), \quad Z = X \times X
$$

(3.4)

$$
\|\cdot\| : Z \to \mathbb{R}^2, \quad \|x\| = \|(x_1, x_2)\| = \begin{pmatrix} |x_1|_{\infty} \\ |x_2|_{\infty} \end{pmatrix},
$$

where $|x_k|_{\infty} = \max_{t \in [a, b]} |x_k(t)|$ is the Chebyshev norm.

Then $(Z, \|\cdot\|)$ is a generalized Banach space.

We consider the following closed subsets of $X$:

$A_1 = \{(x_1, x_2) \in Z \mid x_k \leq \beta_k, \ k \in \{1, 2\}\}$,

$A_2 = \{(x_1, x_2) \in Z \mid x_k \geq \alpha_k, \ k \in \{1, 2\}\}$,

and the operator $T : Z \to Z$,

$$(x_1, x_2) = x \mapsto Tx = (T_1 x, T_2 x),$$

$$
T_k x(t) := \int_a^b G_k(t, s) f_k(s, x_1(s), x_2(s)) ds, \ \text{for} \ k \in \{1, 2\}.
$$

(3.5)

The system (3.1) is equivalent with the equation $Tx = x$.

We will prove that $A_1 \cup A_2$ is a cyclic representation of $Z$ with respect to $T$.

Let $x = (x_1, x_2) \in A_1 \Rightarrow x_k(s) \leq \beta_k(s), \ \forall s \in [a, b]$, for $k \in \{1, 2\}$.

Using the monotonicity of $f_k$, we have

$$
G_k(t, s) f_k(s, x_1(s), x_2(s)) \geq G_k(t, s) f_k(s, \beta_1(s), \beta_2(s)), \ \text{for} \ k \in \{1, 2\}
$$

and from (i), by integration,

$$
\int_a^b G_k(t, s) f_k(s, x_1(s), x_2(s)) ds \geq \alpha_k(t),
$$

which means that

$$
T_k x(t) \geq \alpha_k(t), \ \forall t \in [a, b], \ \text{for} \ k \in \{1, 2\} \Rightarrow Tx \in A_2.
$$
So \( TA_1 \subseteq A_2 \). In a similar way we have \( TA_2 \subseteq A_1 \).

Using the conditions (ii) and (iii) we have

\[
\|T_k x(t) - T_k y(t)\| \leq \int_a^b G_k(t,s) \| f_k(s, x_1(s), x_2(s)) - f_k(s, y_1(s), y_2(s)) \| ds \\
\leq \int_a^b G_k(t,s) (a_k |x_1(s) - y_1(s)| + b_k |x_2(s) - y_2(s)|) ds \\
\leq \int_a^b G_k(t,s) (a_k |x_1 - y_1|_\infty + b_k |x_2 - y_2|_\infty) \\
\leq a_k |x_1 - y_1|_\infty + b_k |x_2 - y_2|_\infty, \quad \forall \ t \in [a,b] \\
\Rightarrow |T_k x - T_k y|_\infty \leq a_k |x_1 - y_1|_\infty + b_k |x_2 - y_2|_\infty \\
\Rightarrow \left( \frac{|T_1 x - T_1 y|_\infty}{|T_2 x - T_2 y|_\infty} \right) \leq S \left( \frac{|x_1 - y_1|_\infty}{|x_2 - y_2|_\infty} \right),
\]

so we have

\[
\|T x - T y\| \leq S \|x - y\|, \text{ for any } (x, y) \in A_1 \times A_2,
\]

and by the condition (v) it results that the operator \( T \) is a cyclic \( S \)-contraction.

All the conditions of Theorem 4 are satisfied, so \( T \) has a unique fixed point

\[ x^* = (x_1^*, x_2^*) \in A_1 \cap A_2, \text{ with } \alpha_k \leq x_k^* \leq \beta_k, \text{ for } k \in \{1, 2\}. \]

This finishes the proof. \( \square \)

Further on, we will study the continuous dependence phenomenon for the system (3.1).

We consider the perturbed system of integral equations

\[
\begin{cases}
  y_1(t) = \int_a^b H_1(t,s)g_1(s, y_1(s), y_2(s)) ds \\
  y_2(t) = \int_a^b H_2(t,s)g_2(s, y_1(s), y_2(s)) ds
\end{cases}
\quad \text{(3.6)}
\]

where

\[ H_1, H_2 \in C([a,b] \times [a,b], [0, \infty)), \quad g_1, g_2 \in C([a,b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}). \]

**Theorem 8.** We suppose that the conditions of Theorem 7 are satisfied and we denote by \( x^* \) the unique solution of the system of integral equations (3.1).

If \( y^* \in C([a,b], \mathbb{R}^2) \) is a solution of the perturbed system of integral equations (3.6), and

\[
\sup_{t \in [a,b]} \int_a^b H_k(t,s) ds \leq 1,
\]

then

\[
\|y^* - x^*\| \leq S \|y - x\|, \text{ for any } (y, x) \in A_1 \times A_2,
\]

and by the condition (v) it results that the operator \( T \) is a cyclic \( S \)-contraction.
then we have the following estimation:

\[ \|x^*-y^*\|_{\mathbb{R}^2} \leq (I_2 - S)^{-1}(\eta + \tau), \]  

(3.7)

where \( \eta = (\eta_1, \eta_2) \), \( \tau = (\tau_1, \tau_2) \) and

\[
\begin{align*}
\eta_k &= \sup\{|f_k(s,u,v)| \mid s \in [a,b], u, v \in \mathbb{R}\}, \\
\tau_k &= \sup\{|g_k(s,u,v)| \mid s \in [a,b], u, v \in \mathbb{R}\}, \quad \text{for } k \in \{1,2\}.
\end{align*}
\]

**Proof.** We consider the operator \( T : Z \to Z \) attached to the system (3.1), defined by the relation (3.5).

Let \( U : Z \to Z \) be an operator attached to the perturbed system (3.6) and defined by the relation:

\[
y_1(t) = y \mapsto Uy = (U_1y, U_2y),
\]

\[
U_ky(t) := \int_a^b H_k(t,s)g_k(s,y_1(s),y_2(s))ds, \text{ for } k \in \{1,2\}.
\]

We have

\[
|T_kx(t) - U_kx(t)| \leq \int_a^b G_k(t,s)|f_k(s,x_1(s),x_2(s))|ds + \int_a^b H_k(t,s)|g_k(s,x_1(s),x_2(s))|ds
\]

\[
\leq \eta_k \int_a^b G_k(t,s)ds + \tau_k \int_a^b H_k(t,s)ds
\]

\[
\leq \eta_k + \tau_k, \quad \forall t \in [a,b], \quad \text{for } k \in \{1,2\}
\]

\[
\Rightarrow \|T_kx - U_kx\|_{\infty} \leq \eta_k + \tau_k
\]

\[
\Rightarrow \|T_kx - U_kx\| \leq \eta + \tau, \quad \forall x \in Z.
\]

The conditions of Theorem 6 are satisfied, so estimation (3.7) is proved.

\[\square\]

**Remark 1.** A similar approach can be achieved for a system of Volterra type integral equations using, instead of the supremum norm, the Bielecki type norm approach.

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