Higher Supergeometry Revisited

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Supergeometry and applications

December 14-15, 2017

Outline *

Motivations and applications

$\mathbb{Z}_2^n$ Berezinian

$\mathbb{Z}_2^n$ Manifolds and $\mathbb{Z}_2^n$ morphisms

$\mathbb{Z}_2^n$ Batchelor-Gawedzki theorem

$\mathbb{Z}_2^n$ integral calculus

Outlook

* Joint with T. Covolo, J. Grabowski, V. Ovsienko, S. Kwok, ...

Motivations and Applications

Higher gradings and modified sign rule

Supergeometry: coordinates

x of degree 0

ξ of degree 1

Higher gradings and modified sign rule

$\mathbb{Z}_2^2$-Supergeometry: coordinates

x of degree $(0,0)$ y of degree $(1,1)$
ξ of degree $(0,1)$ η of degree $(1,0)$

Higher gradings and modified sign rule

$\mathbb{Z}_2^3$-Supergeometry: coordinates

x of degree $(0,0,0)$ y of degree $(0,1,1)$ ...

ξ of degree $(0,0,1)$ η of degree $(0,1,0)$ ...

$$y \cdot \eta = (-1)^{\langle (0,1,1),(0,1,0) \rangle} \eta \cdot y$$

Higher gradings and modified sign rule

$\mathbb{Z}_2^3$-Supergeometry: coordinates

$$x \text{ of degree } (0,0,0) \quad y \text{ of degree } (0,1,1) \quad \ldots$$
$$\xi \text{ of degree } (0,0,1) \quad \eta \text{ of degree } (0,1,0) \quad \ldots$$

$$y \cdot \eta = (-1)^{\langle (0,1,1),(0,1,0) \rangle} \eta \cdot y$$

New features:

Even coordinates may anticommute: $(-1)^{\langle (1,1,0),(1,0,1) \rangle} = -1$

Odd coordinates may commute: $(-1)^{\langle (1,0,0),(0,1,0) \rangle} = +1$

Non-zero degree coordinates may not be nilpotent: $(-1)^{\langle (1,1,0),(1,1,0) \rangle} = +1$

Examples

Physics:

- Parastatistical supersymmetry
- String orbifolds
- Anyons

Algebra:

- Super differential forms $(n = 2)$

$$\alpha \wedge \beta = (-1)^{\deg(\alpha)\deg(\beta)+\mathrm{p}(\alpha)\,\mathrm{p}(\beta)}\beta \wedge \alpha$$

- Quaternions $\mathbb{H}$ $(n = 3)$
- Clifford algebras $C\ell_{p,q}$ $(n = p + q + 1)$

Geometry:

- Superized higher vector bundles, e.g., $TTM, T^*TM \ldots$
- Tangent bundle of a supermanifold

Tangent bundle to a supermanifold

Supermanifold $\mathcal{M}$: (x, ξ)

$(x, \xi, dx, d\xi) \rightsquigarrow (0, 1, 1+0, 1+1)$

+ usual sign rule

$T\mathcal{M}$: supermanifold

$C^\infty(x, d\,\xi)[\xi, d\,x]$

$(x, \xi, dx, d\xi) \rightsquigarrow ((0,0),(0,1),(1,0),(1,1))$

+ new sign rule

$T\mathcal{M}$: $\mathbb{Z}_2^2$-manifold

$C^\infty(x)[[\xi, d\,x, d\,\xi]]$

Coherent differential calculus

$\mathbb{Z}_2^2$-manifold:

$$(0,0),(1,1),(0,1),(1,0)$$

$$\phi:\ \{x,y,\xi,\eta\}\mapsto\{x',y',\xi',\eta'\}$$

$$x'=x+y^2,\quad y'=y,\quad \xi'=\xi,\quad \eta'=\eta$$

$$F(x')=F(x+y^2)=\sum_\alpha \frac{1}{\alpha!}(\partial_{x'}^\alpha F)(x)y^{2\alpha}$$

Local model:

$$(\mathbb{R}^p,\ \mathcal{C}^\infty(U)[[y,\xi,\eta]])$$

$$\begin{cases} x'=\sum_r f_r^{x'}(x)y^{2r}+\sum_r g_r^{x'}(x)y^{2r+1}\xi\eta \\ y'=\sum_r f_r^{y'}(x)y^{2r+1}+\sum_r g_r^{y'}(x)y^{2r}\xi\eta \\ \xi'=\sum_r f_r^{\xi'}(x)y^{2r}\xi+\sum_r g_r^{\xi'}(x)y^{2r+1}\eta \\ \eta'=\sum_r f_r^{\eta'}(x)y^{2r}\eta+\sum_r g_r^{\eta'}(x)y^{2r+1}\xi \end{cases}$$

Key-concept

$$\partial_{(x,y,\xi,\eta)}(x',y',\xi',\eta') = \left(\begin{array}{cc|cc} (0,0) & (1,1) & (0,1) & (1,0) \\ \hline (1,1) & (0,0) & (1,0) & (0,1) \\ \hline (1,0) & (0,1) & (0,0) & (1,1) \\ \hline (0,1) & (1,0) & (1,1) & (0,0) \end{array}\right)$$

Big diagonal blocks: even $\mathbb{Z}_2^n$-degrees

Small diagonal blocks: degree zero

Higher Trace and Berezinian of matrices over a Clifford Algebra

Journal of Geometry and Physics (2012), 62(11), 2294-2319

Berezinian

$\mathcal{A}$: supercommutative algebra

Theorem

$\exists!$ group morphism

$$\mathrm{Ber} : \mathsf{GL}^0(\mathcal{A}) \to (\mathcal{A}^0)^\times$$

such that

- $$\mathrm{Ber}\left(\begin{array}{c|c} \mathsf{A} & \\ \hline & \mathsf{D} \end{array}\right) = \det(\mathsf{A})\,\det^{-1}(\mathsf{D})$$
- $$\mathrm{Ber}\left(\begin{array}{c|c} \mathbb{I} & \mathsf{B} \\ \hline & \mathbb{I} \end{array}\right) = 1 = \mathrm{Ber}\left(\begin{array}{c|c} \mathbb{I} & \\ \hline \mathsf{C} & \mathbb{I} \end{array}\right)$$

It is defined by

$$\mathrm{Ber}\left(\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right) = \det(A - BD^{-1}C)\,\det^{-1}(D)$$

$\mathbb{Z}_2^n$-Berezinian

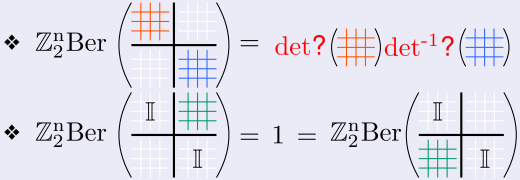
$\mathcal{A}$: $\mathbb{Z}_2^n$-commutative algebra

Theorem

∃! group morphism

$$\mathbb{Z}_2^n\mathrm{Ber} : \mathsf{GL}^0(\mathcal{A}) \to (\mathcal{A}^0)^\times$$

such that



It is defined by

$$\mathbb{Z}_2^n\mathrm{Ber}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = ?$$

$\mathbb{Z}_2^n$-Determinant

$\mathcal{A}$: $(\mathbb{Z}_2^n)_{\text{even}}$-commutative algebra

Theorem

1. $\exists!$ algebra morphism

$$\mathbb{Z}_2^n\det : \mathsf{gl}^0(\mathcal{A}) \to \mathcal{A}^0$$

such that

- $$\mathbb{Z}_2^n\det\begin{pmatrix} \star & & & \\ & \star & & \\ & & \star & \\ & & & \star \end{pmatrix} = \prod \det(\,\star\,)$$
- $$\mathbb{Z}_2^n\det\begin{pmatrix} \mathbb{I} & \star & \star & \star \\ & \mathbb{I} & \star & \star \\ & & \mathbb{I} & \star \\ & & & \mathbb{I} \end{pmatrix} = 1 = \mathbb{Z}_2^n\det\begin{pmatrix} \mathbb{I} & & & \\ \star & \mathbb{I} & & \\ \star & \star & \mathbb{I} & \\ \star & \star & \star & \mathbb{I} \end{pmatrix}$$

2. $\mathbb{Z}_2^n\det(X)$ is linear in the entries of X

Quasideterminants (I.Gelfand and V.Retakh)

$$\mathrm{Ber}\left(\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right) = \det(A - BD^{-1}C)\,\det{}^{-1}(D)$$

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$$\left|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right|_{11} = A - BD^{-1}C$$

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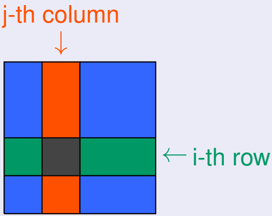
Definition

For a square matrix X with entries in a ring R,

$$|X|_{ij} := x_{ij} - r_i^j (X^{ij})^{-1} c_j^i \quad \in R$$



Examples

- $$X = \begin{pmatrix} x & a & b \\ c & y & d \\ e & f & z \end{pmatrix}$$
  $$|X|_{11} = x - bz^{-1}e - (a - bz^{-1}f)(y - dz^{-1}f)^{-1}(c - dz^{-1}e)$$
- $$X = \left(\begin{array}{cc|c} x & a & b \\ c & y & d \\ \hline e & f & z \end{array}\right)$$
  $$|X|_{11} = \begin{pmatrix} x - bz^{-1}e & a - bz^{-1}f \\ c - dz^{-1}e & y - dz^{-1}f \end{pmatrix}$$

Quasi-determinants are rational functions

$\mathbb{Z}_2^n$-Determinant

$\mathcal{A}$: $(\mathbb{Z}_2^n)_{\text{even}}$-commutative algebra

Theorem

1. $\exists!$ algebra morphism

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UDL decomposition

$$X = UDL$$

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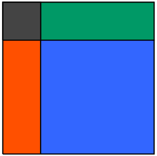
$$D = \begin{pmatrix} |X|_{11} & & & \\ & |X^{1:1}|_{22} & & \\ & & \ddots & \\ & & & X_{qq} \end{pmatrix}$$



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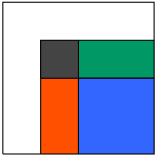

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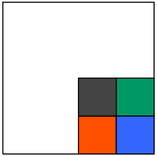

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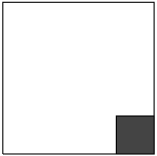

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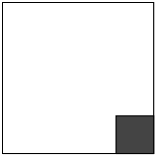

Theorem

$$\mathbb{Z}_2^{\mathrm{n}}\det(X) = \det(|X|_{11})\ \det(|X^{1:1}|_{22})\ \ldots\ \det(X_{rr})$$

$\mathbb{Z}_2^n$-Berezinian

Theorem

∃! group morphism

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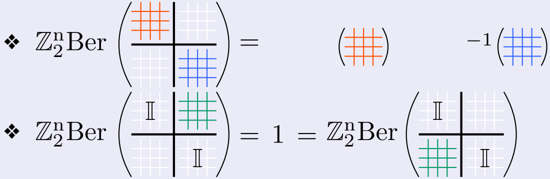

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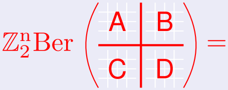

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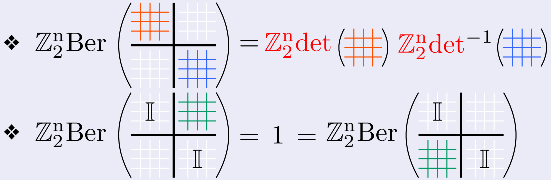

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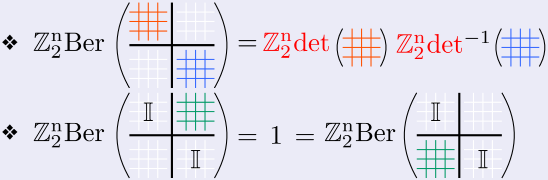

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$$\mathbb{Z}_2^n\mathrm{Ber}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \mathbb{Z}_2^n\mathrm{det}(A - BD^{-1}C)\ \mathbb{Z}_2^n\mathrm{det}^{-1}(D)$$

For $n = 1$, $\mathbb{Z}_2^n\mathrm{Ber} = \mathrm{Ber}$; for $\mathcal{A} = \mathbb{H}$, $\mathbb{Z}_2^n\mathrm{Ber} = |\,\mathrm{Ddet}\,|$; Liouville formula

The category of $\mathbb{Z}_2^n$-supermanifolds

Journal of Mathematical Physics (2016), 57(7)

Functor of points approach

$$P \in \mathbb{C}[\mathbb{C}^n], \quad V = \{z \in \mathbb{C}^n : P(z) = 0\} \in \mathtt{Aff}, \quad \mathbb{C}[V] = \mathbb{C}[\mathbb{C}^n]/(P) \in \mathtt{CA}$$

$$\mathrm{Sol}_P : \mathtt{CA} \ni A \mapsto \mathrm{Sol}_P(A) = \{a \in A^n : P(a) = 0\} \in \mathtt{Set}$$

$$\mathrm{Sol}_P = \mathrm{Hom}_{\mathtt{CA}}(\mathbb{C}[V], -) \in \mathtt{Fun}(\mathtt{CA}, \mathtt{Set})$$

$$\mathrm{Hom}_{\mathtt{Aff}}(-, V) \in \mathtt{Fun}(\mathtt{Aff}^{\mathrm{op}}, \mathtt{Set})$$

$$\underline{\bullet} : \mathtt{C} \ni c \mapsto \underline{c} := \mathrm{Hom}_{\mathtt{C}}(-, c) \in \mathtt{FunSh}(\mathtt{C}^{\mathrm{op}}, \mathtt{Set})$$

$$\mathrm{Lim}\, \underline{c} \simeq \underline{\mathrm{Lim}\, c}$$

$$\mathrm{Colim}\, \underline{c} \to \underline{\mathrm{Colim}\, c}$$

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$$\mathrm{Lim}\, \underline{c} \simeq \underline{\mathrm{Lim}\, c}$$

$$\mathrm{Colim}\, \underline{c} \to \underline{\mathrm{Colim}\, c}$$

Functor of points approach

Representable $\mathtt{Sh}(\mathtt{C})$: trivial spaces

$\mathtt{Sh}(\mathtt{C})$: spaces

Locally representable $\mathtt{Sh}(\mathtt{C})$: varieties or manifolds $\mathtt{Var}(\mathtt{C})$

Examples:

$\mathtt{Var}(\mathtt{Aff})$: schemes

$\mathtt{Var}(\mathbb{Z}_2^n\text{-}\mathtt{Domain})$: $\mathbb{Z}_2^n$-manifolds

Locally ringed space approach

Definition

*A $\mathbb{Z}_2^n$***-manifold** *is a $\mathbb{Z}_2^n$-graded locally ringed space $(M, \mathcal{A}_M)$ that is locally modeled on*

$$(\mathbb{R}^p, \mathcal{C}^\infty_{\mathbb{R}^p}(-)[[\xi^1, \ldots, \xi^q]]) \ ,$$

where the ξ^a are $\mathbb{Z}_2^n$-commutative.

Reconstruction theorem

Proposition

A topological space that is covered by $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$- commutative coordinate systems

$$(x, y, \ldots, \xi, \eta, \ldots)$$

and is endowed with $\mathbb{Z}_2^n$-degree preserving coordinate transformations

$$\varphi_{\beta\alpha} : (x, y, \ldots, \xi, \eta, \ldots) \mapsto (x', y', \ldots, \xi', \eta', \ldots)$$

that satisfy the cocycle condition

$$\varphi_{\gamma\beta}\, \varphi_{\beta\alpha} = \varphi_{\gamma\alpha}\ ,$$

*defines a $\mathbb{Z}_2^n$-**manifold**.*

Nilpotency – Formal series

Invertibility of superfunctions:

$$f \in \mathcal{C}^\infty(U)[\xi^1, \ldots, \xi^q] \text{ invertible} \Leftrightarrow f_0 \in \mathcal{C}^\infty(U) \text{ invertible}$$

Proof: Nilpotency

Nilpotency – Formal series

Invertibility of $\mathbb{Z}_2^n$-functions:

$$f \in \mathcal{C}^\infty(U)[[\xi^1, \ldots, \xi^q]] \text{ invertible} \Leftrightarrow f_0 \in \mathcal{C}^\infty(U) \text{ invertible}$$

Proof: Formal power series

$\mathbb{Z}_2^n$-morphisms

$\mathbb{Z}_2^n$**-morphism**: ringed space morphism $(\psi, \psi^*) : (M, \mathcal{A}) \to (N, \mathcal{B})$

Commutation with base projections:

$$\begin{array}{ccc} \mathcal{B}(V) & \xrightarrow{\psi^*} & \mathcal{A}(\psi^{-1}(V)) \\ \varepsilon_V \downarrow & \circlearrowleft & \downarrow \varepsilon_{\psi^{-1}(V)} \\ \mathcal{C}^\infty_N(V) & \xrightarrow{\psi^*} & \mathcal{C}^\infty_M(\psi^{-1}(V)) \end{array}$$

ψ^* is C^0 with respect to the $\mathcal{J}$-adic topology, $\mathcal{J} = \ker \varepsilon$

$\mathcal{A}$ is Hausdorff-complete for the $\mathcal{J}$-adic topology

Fundamental $\mathbb{Z}_2^n$-Morphism Theorem

Theorem

A $\mathbb{Z}_2^n$-morphism

$$(\psi, \psi^*) : (M, \mathcal{A}_M) \to (V, \mathcal{C}_V^\infty[[\xi^1, \dots, \xi^q]])$$

is completely and uniquely defined by the pullbacks

$$\psi^* x^i \quad \text{and} \quad \psi^* \xi^a$$

of the base coordinates x^i and the formal coordinates ξ^a

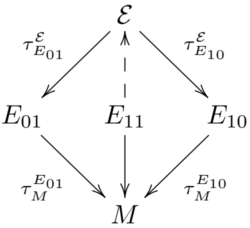
"The result that makes $\mathbb{Z}_2^n$-Geometry a reasonable theory"

SPLITTING THEOREM FOR $\mathbb{Z}_2^n$-MANIFOLDS

JOURNAL OF GEOMETRY AND PHYSICS (2016), 110

Double vector bundles

Definition 1:



$$E_{11} = \ker \tau^{\mathcal{E}}_{E_{01}} \cap \ker \tau^{\mathcal{E}}_{E_{10}}$$

Trivial example:

$$E_{01} \oplus E_{10} \oplus E_{11}$$

$\mathcal{E} \simeq E_{01} \oplus E_{10} \oplus E_{11}$ and $\Gamma(E^*_{01} \otimes E^*_{10} \otimes E_{11})$

Double vector bundles

Definition 2: Pair of commuting ‘Euler’ vector fields on a manifold.

Definition 3: Locally trivial fiber bundle with standard fiber $V_{01} \oplus V_{10} \oplus V_{11}$ s.th.

$$\begin{cases} \xi'^a &= f^a_u(x)\xi^u \\ \eta'^b &= g^b_v(x)\eta^v \\ y'^c &= h^c_w(x)y^w + k^c_{u,v}(x)\xi^u\eta^v \end{cases}$$

Split $\mathbb{Z}_2^n$-manifolds

Vector bundle:

$E \to M$ $\dashrightarrow$ $\Pi E := E[1]$

$$\mathcal{A}(\Pi E) = \Gamma(\bigwedge E^*) = \bigoplus_{k=0}^{r} \Gamma(\odot^k (\Pi E)^*)$$

Supermanifold: $\mathcal{M} = (M, \mathcal{A}(\Pi E))$

Graded vector bundle:

$E = E_{01} \oplus E_{10} \oplus E_{11} \to M$ $\dashrightarrow$ $\Pi E := E_{01}[01] \oplus E_{10}[10] \oplus E_{11}[11]$

$$\mathcal{A}(\Pi E) = \prod_{k \geq 0} \Gamma(\odot^k (\Pi E)^*)$$

$\mathbb{Z}_2^2$-manifold: $\mathcal{M} = (M, \mathcal{A}(\Pi E))$

Non-split $\mathbb{Z}_2^n$-manifold

Double vector bundle: $\mathcal{E}$

$$\begin{cases} \xi'^a & = f^a_u(x)\xi^u \\ \eta'^b & = g^b_v(x)\eta^v \\ y'^c & = h^c_w(x)y^w + k^c_{u,v}(x)\xi^u\eta^v \end{cases}$$

- - - -> Superization, coherence, cocycle condition

$\mathbb{Z}_2^2$**-manifold – NOT canonically split**

$\mathbb{Z}_2^n$-Batchelor-Gawedzki Theorem

Batchelor, Gawedzki, Kirillov and Rudakov

Smooth and real analytic, but not holomorphic

Theorem

Any $\mathbb{Z}_2^n$-manifold $(M, \mathcal{A})$ is non-canonically split, i.e., there exists a non-canonical isomorphism

$$\mathcal{A} \simeq \mathcal{A}(\Pi E) ,$$

where E is a $\mathbb{Z}_2^n \setminus \{0\}$-vector bundle.

Sketch of proof (I)

Step 1:

$$\mathcal{A} \underset{\mathrm{Sh}}{\simeq} \mathcal{A}(\Pi E) = \prod_{k\geq 0} \odot^k \Gamma((\Pi E)^*) \quad ?$$

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$$\mathcal{A} \underset{\mathrm{Sh}}{\simeq} \mathcal{A}(\Pi E) = \prod_{k\geq 0} \odot^k \Gamma((\Pi E)^*) \quad ?$$

$$0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{C}^\infty \to 0$$

$$\mathcal{J}/\mathcal{J}^2$$

Sketch of proof (I)

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$$0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{C}^\infty \to 0$$

$$\mathcal{J}/\mathcal{J}^2 = \Gamma((\Pi E)^*)$$

$$\prod_{k\geq 0} \odot^k (\mathcal{J}/\mathcal{J}^2) \underset{\mathrm{Sh}}{\simeq} \mathcal{A} \ ?$$

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$$\prod_{k\geq 0} \odot^k (\mathcal{J}/\mathcal{J}^2) \underset{\mathrm{Sh}}{\longrightarrow} \mathcal{A} \ ?$$

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$$\mathcal{A} \underset{\mathrm{Sh}}{\simeq} \mathcal{A}(\Pi E) = \prod_{k\geq 0} \odot^k \Gamma((\Pi E)^*)$$

$$0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{C}^\infty \to 0$$

$$\mathcal{J}/\mathcal{J}^2 = \Gamma((\Pi E)^*)$$

$$\prod_{k\geq 0} \odot^k (\mathcal{J}/\mathcal{J}^2) \to \mathcal{A}$$

➙ $\mathcal{J}/\mathcal{J}^2 \to \mathcal{A}$?

Sketch of proof (I)

Step 1:

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➨ $\mathcal{J}/\mathcal{J}^2 \to \mathcal{A}$

➨ $0 \to \mathcal{J}^2 \to \mathcal{J} \to \mathcal{J}/\mathcal{J}^2 \to 0$?

Sketch of proof (I)

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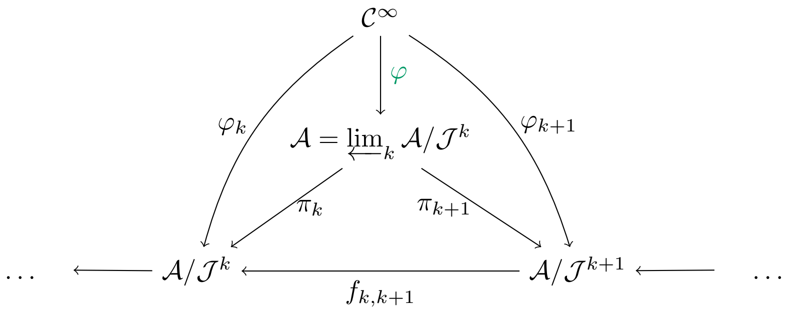
➡ $\mathcal{J}/\mathcal{J}^2 \to \mathcal{A}$

➡ $0 \to \mathcal{J}^2 \to \mathcal{J} \to \mathcal{J}/\mathcal{J}^2 \to 0$

➡ $\mathcal{C}^\infty \xrightarrow{\varphi} \mathcal{A}$?

Sketch of proof (II)

Step 2:



$$\varphi_{k+1,\Omega} : \mathcal{C}^\infty(\Omega) \to \mathcal{A}(\Omega)/\mathcal{J}^{k+1}(\Omega)$$
$$\varphi_{k+1,U} : \mathcal{C}^\infty(U) \to \mathcal{A}(U)/\mathcal{J}^{k+1}(U)$$
$$\varphi_{k+1,U}|_{U\cap V} = \varphi_{k+1,V}|_{U\cap V}$$

Sketch of proof (III)

$$\omega_{k+1,UV} := \varphi_{k+1,U}|_{U\cap V} - \varphi_{k+1,V}|_{U\cap V}$$

$$\omega_{k+1,UV} \in \mathrm{Der}(\mathcal{C}^\infty(U\cap V), \Gamma(U\cap V, \odot^k(\Pi E)^*))$$

Sketch of proof (III)

$$\omega_{k+1,UV} := \varphi_{k+1,U}|_{U\cap V} - \varphi_{k+1,V}|_{U\cap V}$$

$$\omega_{k+1,UV} \in \Gamma(U \cap V, TM \otimes \odot^k(\Pi E)^*)$$

$$\omega_{k+1} \in \check{Z}^1 = \check{B}^1$$

$$\varphi_{k+1,U}|_{U\cap V} - \varphi_{k+1,V}|_{U\cap V} = \eta_{k+1,V}|_{U\cap V} - \eta_{k+1,U}|_{U\cap V}$$

$\varphi_{k+1,U} + \eta_{k+1,U}$ consistent (correction of $\varphi_{k+1,U}$)

Cohomological Approach to the Graded Berezinian

Journal of Noncommutative Geometry, 9 (2015), 543–565

Determinant module

$\mathcal{A}$ a commutative algebra

M a free $\mathcal{A}$-module, rank r, bases $(e_i), (e'_i)$, $e_j = e'_i B^i_j$

$\mathrm{Det}(M) = \bigwedge^r M$: rank 1 $\mathcal{A}$-module with

$$e_1 \wedge \ldots \wedge e_r = e'_1 \wedge \ldots \wedge e'_r \cdot \det(B)$$

$\mathbb{Z}_2^n$-Berezinian module

$\mathcal{A}$ a $\mathbb{Z}_2^n$-commutative algebra

M a free $\mathbb{Z}_2^n$-graded $\mathcal{A}$-module, total rank r, bases $(e_i), (e'_i)$, $e_j = e'_i B^i_j$

Definition

$\mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(M)$ *is a rank* 1 $\mathcal{A}$*-module on which* $B \in \mathsf{GL}^0(\mathcal{A})$ *acts as* $- \cdot \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(B)$

$\mathbb{Z}_2^n$-Berezinian module

$$\mathcal{K} = \odot_{\mathcal{A}} \Pi M \otimes \odot_{\mathcal{A}} M^* \,, \qquad d = \sum_i \Pi e_i \, \varepsilon^i$$

$$H(\mathcal{K}) = H^r(\mathcal{K}) = [\omega] \cdot \mathcal{A}$$

$$\Phi : \; B \in \mathsf{GL}^0(\mathcal{A}) \mapsto \Phi_B \simeq (B, {}^{\mathbb{Z}_2^n\mathrm{t}}B^{-1}) \in \mathrm{Aut}^0(H(\mathcal{K})) \simeq (\mathcal{A}^0)^\times$$

$$\Phi_B = - \cdot \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(B)$$

$$\omega = \omega' \cdot \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(B)$$

ω : algebraic $\mathbb{Z}_2^n$ Berezinian volume

Integration on $\mathbb{Z}_2^n$-manifolds

in progress – ORBilu http://hdl.handle.net/10993/27319

Local Berezinian section

$$(U, X = (x, y, \xi, \eta)) : \quad ((0,0), (1,1), (0,1), (1,0))$$

$$M = \Omega^1(\mathcal{M})(U) : \quad (\mathrm{d}\, x, \mathrm{d}\, y, \mathrm{d}\, \xi, \mathrm{d}\, \eta)$$

$$M^* = T(\mathcal{M})(U) : \quad (\partial_x, \partial_y, \partial_\xi, \partial_\eta)$$

Local Berezinian section

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$$\omega = \mathrm{d}\, x\, \mathrm{d}\, y \otimes \partial_\xi \partial_\eta$$

Local Berezinian section

$$(U, X = (x, y, \xi, \eta)) : \quad ((0,0),(1,1),(0,1),(1,0))$$

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$$M^* = T(\mathcal{M})(U) : \quad (\partial_x, \partial_y, \partial_\xi, \partial_\eta)$$

$$\omega = \mathrm{d}\,x\,\mathrm{d}\,y \otimes \partial_\xi \partial_\eta$$

$$\omega(X) = \omega(X')\ \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial_{X'} X)$$

$$\omega(X)\, f(X) = \omega(X')\ f(X(X'))\ \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial_{X'} X) =: \omega(X')\ f'(X')$$

Global Berezinian section

Definition

A **Berezinian section** *of a $\mathbb{Z}_2^2$-manifold with an oriented base is a family*

$$\omega(X)f(X),\, \omega(X')f'(X'),\ldots\ ,$$

whose components transform according to the rule

$$f'(X') = f(X(X'))\ \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial_{X'}X)$$

$\mathbb{Z}_2^2$-integral I

β: Berezinian section supported in a $\mathbb{Z}_2^2$-domain $X = (x, y, \xi, \eta)$

$$\beta = \omega(X) f(X) = (\mathrm{d}\,x\,\mathrm{d}\,y \otimes \partial_\xi \partial_\eta) \sum_{k=0}^{\infty} \sum_{a=0}^{1} \sum_{b=0}^{1} f_{kab}(x) y^k \xi^a \eta^b$$

Definition

$$\int \beta := \int \mathrm{d}\,x \int \mathrm{d}\,y \;\; \partial_\xi \partial_\eta \; f(x, y, \xi, \eta) =$$

$$\int \mathrm{d}\,x \int \mathrm{d}\,y \; \sum_{k=0}^{\infty} f_{k11}(x) y^k := \int \mathrm{d}\,x \; f_{011}(x) \in \mathbb{R}$$

β: arbitrary Berezinian section $\rightsquigarrow$ partition of unity

$\mathbb{Z}_2^2$-integral II

σ: generalized Berezinian section supported in a $\mathbb{Z}_2^2$-domain $X = (x, y, \xi, \eta)$

$$\sigma = \omega(X)\, L(X) = (\mathrm{d}\,x\,\mathrm{d}\,y \otimes \partial_\xi \partial_\eta) \sum_{k=-N}^{\infty} \sum_{a=0}^{1} \sum_{b=0}^{1} f_{kab}(x) y^k \xi^a \eta^b$$

Definition

$$\int \sigma = \int \mathrm{d}\,x \int \mathrm{d}\,y \sum_{k=-N}^{\infty} f_{k11}(x) y^k :=$$

$$\int \mathrm{d}\,x\, f_{-111}(x) \ \in \mathbb{R}$$

σ: arbitrary generalized Berezinian section $\rightsquigarrow$ partition of unity

Change-of-variables formula

σ: generalized Berezinian section supported in two domains $\mathcal{U}, \mathcal{U}'$ (X, X')

$$\sigma = \omega(X)\, L(X) = \omega(X')\, L'(X')$$

Change-of-variables formula

σ: generalized Berezinian section supported in two domains $\mathcal{U}, \mathcal{U}'$ (X, X')

$$\sigma = \omega(X)\, L(X) = \omega(X')\, L'(X') = \omega(X')\, L(X(X'))\ \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial_{X'} X)$$

Change-of-variables formula

σ: generalized Berezinian section supported in two domains $\mathcal{U}, \mathcal{U}'$ (X, X')

$$\sigma = \omega(X)\, L(X) = \omega(X')\, L'(X') = \omega(X')\, L(X(X'))\; \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial_{X'} X)$$

Theorem

$$\int_{\mathcal{U}} \omega(X)\, L(X) = \int_{\mathcal{U}'} \omega(X')\, L(X(X'))\; \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial_{X'} X)$$

$\mathbb{Z}_2^2$-integral III

Δ: distributional Berezinian section supported in a $\mathbb{Z}_2^2$-domain $X = (x, y, \xi, \eta)$

$$\Delta = \omega(X) \sum_{\ell \leq N} f_\ell(X) \delta^{(\ell)}(y) =$$

$$(\mathrm{d}\, x \, \mathrm{d}\, y \otimes \partial_\xi \partial_\eta) \sum_{\ell \leq N} \left(\sum_{k=0}^{\infty} \sum_{a=0}^{1} \sum_{b=0}^{1} f_{kab;\ell}(x) y^k \xi^a \eta^b \right) \delta^{(\ell)}(y)$$

Definition

$$\int \Delta = \int \mathrm{d}\, x \int \mathrm{d}\, y \sum_{\ell \leq N} \left(\sum_{k=0}^{\infty} f_{k11;\ell}(x) y^k \right) \delta^{(\ell)}(y) =$$

$$\int \mathrm{d}\, x \int \mathrm{d}\, y \sum_{\ell \leq N} (-1)^\ell \partial_y^\ell \left(\sum_{k=0}^{\infty} f_{k11;\ell}(x) y^k \right) \delta(y) = \int \mathrm{d}\, x \sum_{\ell \leq N} (-1)^\ell \ell \, ! f_{\ell 11;\ell}(x) \in \mathbb{R}$$

$$\int \omega(X) \sum_{\ell \leq N} f_\ell(X)\delta^{(\ell)}(y) \stackrel{?}{=} \int \omega(X') \, \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial\,\Phi) \; \Phi^* \sum_{\ell \leq N} f_\ell(X)\delta^{(\ell)}(y)$$

$$\int \omega(X) \sum_{\ell \leq N} f_\ell(X)\delta^{(\ell)}(y) \stackrel{?}{=} \int \omega(X') \, \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial\, \Phi) \; \Phi^* \sum_{\ell \leq N} f_\ell(X)\delta^{(\ell)}(y)$$

$$= \int \omega(X') \; \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial\, (\Phi_2 \circ \Phi_1)) \; (\Phi_2 \circ \Phi_1)^* \sum_{\ell \leq N} f_\ell(X)\delta^{(\ell)}(y)$$

$$\int \omega(X) \sum_{\ell \leq N} f_\ell(X)\delta^{(\ell)}(y) \stackrel{?}{=} \int \omega(X') \, \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial\, \Phi) \; \Phi^* \sum_{\ell \leq N} f_\ell(X)\delta^{(\ell)}(y)$$

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$$= \int \omega(X') \, \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial\, \Phi_2) \, \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial\, \Phi_1) \; (\Phi_2 \circ \Phi_1)^* \sum_{\ell \leq N} f_\ell(X)\delta^{(\ell)}(y)$$

$$\int \omega(X) \sum_{\ell \leq N} f_\ell(X)\delta^{(\ell)}(y) \stackrel{?}{=} \int \omega(X') \, \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial\, \Phi) \; \Phi^* \sum_{\ell \leq N} f_\ell(X)\delta^{(\ell)}(y)$$

$$= \int \omega(X') \; \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial\, (\Phi_2 \circ \Phi_1)) \; (\Phi_2 \circ \Phi_1)^* \sum_{\ell \leq N} f_\ell(X)\delta^{(\ell)}(y)$$

$$= \int \omega(X') \, \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial\, \Phi_2) \, \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial\, \Phi_1) \; (\Phi_2 \circ \Phi_1)^* \sum_{\ell \leq N} f_\ell(X)\delta^{(\ell)}(y)$$

$$= \int \omega(X') \, \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial\, \Phi_2) \, \mathbb{Z}_2^{\mathrm{n}}\mathrm{Ber}(\partial\, \Phi_1) \; \Phi_1^* \; \Phi_2^* \sum_{\ell \leq N} f_\ell(X)\delta^{(\ell)}(y)$$

OUTLOOK

Current and upcoming work

- $\mathbb{Z}_2^n$-differential calculus ✓

  arXiv:1608.00949

- $\mathbb{Z}_2^n$-versions of inverse & implicit function, constant rank, Frobenius ✓

  arXiv:1608.00961

- $\mathbb{Z}_2^n$-integral calculus ✓

  ORBilu: http://hdl.handle.net/10993/27319

- Categorical $\mathbb{Z}_2^n$-Geometry and Molotkov's work ✓

- Functional analytic issues in $\mathbb{Z}_2^n$-Geometry ✓

- Applications in Physics ✓

Thank you for your attention