Qualitative and Quantitative Properties of Solutions of Ordinary Differential Equations

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Certification

This is to certify that the thesis entitled "**Qualitative and Quantitative Properties of Solutions of Ordinary Differential Equations**" that is been submitted by Mr. B. S. Ogundare for the award of the degree of Doctor of Philosophy at the University of Fort Hare, Alice, is a record of bonafide research work carried out by him under our supervision and guidance. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any Degree or Diploma. Parts of the thesis have appeared in peer reviewed Journals and have been adequately quoted in the bibliography.

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Dedicated to

The Almighty God and my entire family.

Abstract

This thesis is concerned with the qualitative and quantitative properties of solutions of certain classes of ordinary differential equations (ODEs); in particular linear boundary value problems of second order ODE's and non-linear ODEs of order at most four. The Lyapunov's second method of special functions called Lyapunov functions are employed extensively in this thesis. We construct suitable complete Lyapunov functions to discuss the qualitative properties of solutions to certain classes of non-linear ordinary differential equations considered. Though there is no unique way of constructing Lyapunov functions, We adopt Cartwright's method to construct complete Lyapunov functions that are required in this thesis. Sufficient conditions were established to discuss the qualitative properties such as boundedness, convergence, periodicity and stability of the classes of equations of our focus. Another aspect of this thesis is on the quantitative properties of solutions.

New scheme based on interpolation and collocation is derived for solving initial value problem of ODEs. This scheme is derived from the general method of deriving the spline functions. Also by exploiting the Trigonometric identity property of the Chebyshev polynomials, We develop a new scheme for approximating the solutions of two-point boundary value problems.

These schemes are user-friendly, easy to develop algorithm (computer program) and execute. They compare favorably with known standard methods used in solving the classes of problems they were derived for.

Declaration

The work in this thesis is based on research carried out at the Department of Pure and Applied Mathematics, University of Fort Hare, Alice, South Africa. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Preface

Differential equations are essential tools in scientific modeling of physical problems which find relevance in almost every sphere of human endeavor from Agricultural sciences, Engineering, Medical science, Physical sciences to Social sciences. Two broad streams are distinguished in the development of the subject of differential equations. These are: an endeavor to obtain a definite or one of the definite types, either in closed form, which are rarely possible or else by some process of approximation. This is referred to as the Quantitative Theory; and an endeavor to abandon all attempts to reach an exact or approximate solution but strives to obtain information about the whole class of solution. This is called the Qualitative Theory.

This thesis is concerned with the qualitative and quantitative properties of solutions of certain classes of ordinary differential equations (ODEs); in particular linear boundary value problems of second order ODE's and non-linear ODEs of order at most four.

The Lyapunov's second method of special functions called Lyapunov functions is employed extensively in this thesis. We construct suitable complete Lyapunov functions to discuss the qualitative properties of solutions of certain classes of non-linear ordinary differential equations considered. Though there is no unique way of constructing Lyapunov functions, we adopt Cartwright's method to construct complete Lyapunov functions that are required in this thesis. Sufficient conditions were established to discuss the qualitative properties such as boundedness, convergence, periodicity and stability of the classes of equations of our focus.

Another important aspect of this thesis is on the quantitative properties of solutions.

New scheme based on interpolation and collocation is derived for solving initial value problem of ODEs. This scheme is derived from the general method of deriving the spline functions. Also, by exploiting the Trigonometric identity property of the Chebyshev polynomial, we developed a new scheme for approximating solutions of two-point boundary value problems of linear differential equations.
These schemes are user-friendly, easy to develop an algorithm and execute. They compete favorably with known standard methods used in solving the classes of problems they were derived for.

In Chapter 1 we give background information as it relates to the qualitative and quantitative properties of solutions of ODE's. Also our research objectives, literature review as well as the outline of research findings feature in this chapter.

Chapter 2 gives some basic definition on the qualitative properties of solution of ODEs followed by the definition and properties of Lyapunov function (the tool employed for the qualitative properties of solution). Basic Theorems involving the use of Lyapunov functions, as well as the procedure of constructing suitable complete Lyapunov function for the differential equations of orders at most four, also featured in this Chapter. In the second section, we give an overview of the interpolation and approximations in quantitative properties of solutions as well as the characteristics of interpolation methods. The spline method and their properties as well as the interpolation spectral methods.

Chapter 3 gives criteria for global asymptotic stability, boundedness and existence of periodic solutions to certain non-linear non-autonomous differential equation of the second order with less restriction on the non-linear terms.

The global asymptotic stability, boundedness as well as the ultimate boundedness of solutions of a general third order non-linear differential equation are investigated with the use of complete Lyapunov function in Chapter 4.

Chapter 5 gives sufficient conditions for the existence of a stable (globally asymptotically stable), bounded and uniform ultimate bounded solution to a certain fourth order non-linear differential equation using a single complete Lyapunov function without the use of a signum function or any stringent condition on the non-linear terms is given.

The convergence of solutions is a very important and desirable quality in the qualitative studies. In chapter 6 of this thesis, sufficient criteria for the existence of convergence of solutions for a certain class of fourth order non-linear differential equations using the Lyapunov's second method are given.

Our results on the quantitative studies of solutions of ODEs were presented in chapter 7. In the first part of this chapter, New scheme for solving initial value problems of ordinary differential equations is derived. Starting from the general method of deriving the spline function, the scheme is developed based on interpolation and collocation. On the second part, an accurate 'Spectral' method referred to as the pseudo-pseudo-spectral method to approximate the solutions of two-point boundary value problems of linear ordinary differential equations was presented. Exploiting the Trigonometric identity property of the Chebyshev polynomial, we were able to obtain approximate solution which competes favorably with solutions obtained with standard and well known Spectral methods.
The conclusion and suggestions for further studies on the subject of this thesis is presented in the last chapter.

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Chapter 1

INTRODUCTION

1.1 Background

Calculus was the greatest achievement of the seventeenth century and out of it stemmed major branches of mathematics: Differential Equations, Infinite Series, Differential Geometry, the Calculus of Variations, Complex Analysis and many others ([75]).
The history of Ordinary Differential Equations (ODEs) goes all the way back to the XVII century when two great scientists Isaac Newton and Gottfried Leibniz introduced calculus which came to place from the concept of functions ([75]).

Differential equations are essential tools in scientific modeling of physical problems which found their relevance in almost every sphere of human endeavor from Agricultural Sciences, Engineering, Medical Science, Physical Sciences to Social Sciences. Among the earlier work on differential equations, the works of Euler and Lagrange stand out. They first worked on the theory of small oscillations and consequently also, the theory of linear system of ordinary differential equations.

In the development of the subject of differential equations, one may distinguish two broad distinct streams namely:

- An endeavor to obtain a definite or one of the definite types, either in closed form, which are rarely possible or else by some process of approximation. This we shall refer to as the Quantitative Theory.

- An endeavor to abandon all attempts to reach an exact or approximate solution, one strives to obtain information about the whole class of solution. This we shall call the Qualitative Theory.

There are lots of aspects to learn about the solutions to any given differential equation without solving it. For example, consider the equation $\dot{x} = f(x)$; we can find the equilibrium points by finding the zeros $\tilde{x}$ of f. The stability can often be determined by examining the eigenvalues of the derivatives; when this fails, we can still determine the stability by looking at the effects of the non-linear terms. Some of the qualitative questions that can be asked about the solutions are:

1. are there any periodic solution?

2. are the solutions stable?
   and

3. how does the system respond to parameter changes?

The first person to carry out a major investigation in the line of the qualitative theory and hence the development of the "qualitative theory of differential equations" was Henry Poincaré (see [22]). This qualitative theory is now the most actively developing area of the theory of differential equations, with most important applications in diverse areas such as Engineering, Economics, Physical and Biological sciences.
It is well known that mathematical formulations of many physical problems often result in differential equations that are non-linear. In many cases, it is possible to replace a non-linear differential equation with a related linear differential equation that approximates the actual non-linear equation close enough to give useful results. Often, such linearization is not possible or feasible; when it is not, the original non-linear equation must be tackled. Much has been done on the theory and method of dealing with the linear differential equations in Mathematics but just little of

general nature is known about non-linear differential equations.

By non-linear differential equations, we are referring to equations where the unknown functions and their derivatives occur strongly coupled with at least one of the terms of the expressions. In general, the study of non-linear differential equations is restricted to a variety of special cases and the method of solution usually involves one or more of a limited number of different methods. There are several important differences between linear differential equations and non-linear differential equations for instance, for the linear ordinary differential equations, it is possible to derive a closed-form expressions for the solutions of the equations whereas this is not possible in general for the non-linear differential equations. As a consequence, it is desirable to be able to make predictions about the behavior (qualitative analysis) of non-linear ordinary differential equations even in the absence of the closed-form expressions for the solution of the equations.

The analysis of non-linear ordinary differential equations makes use of a wide variety of approaches and mathematical tools than does the analysis of linear differential equations. The main reason for this variety is that no tool or methodology in non-linear differential equations analysis is universally applicable to handle them in a fruitful manner.
Close to half a century now, great efforts have been devoted to the study of qualitative theory of non-linear differential equations, to be precise higher orders non-linear differential equations. During these periods, new methods and outstanding results have appeared. These were extensively summarized in the monograph of [113]. The major directions which must be emphasized in this context, consist in the investigation of solution of non-linear differential equations involving boundedness, stability, periodicity and convergence of solutions.

Some of the techniques used in the investigation of these qualitative properties of solutions include the Lyapunov's Second Method which involves the construction of a suitable positive definite function whose derivative is negative definite. The frequency domain method is another method employed in the investigation. This method involves the study of location of the characteristic polynomial roots in the complex plane. We can also mention the topological degree method which demand the verification of continuity properties of a certain operator and the proof of existence of a particular *a-priori* bound. Each of these methods has its limitations, for instance, the limitation of the Lyapunov's Second Method is on the non-unique way of constructing suitable Lyapunov function; the frequency domain method though overcomes the problem of constructing Lyapunov's functions, it is narrower in scope than the Lyapunov's Second Method (see e.g. ([116]). The Topological Degree Methods on the other hand are mainly used in proving existence of periodic solutions.

This thesis, in one way, is concerned with the following qualitative properties of solutions:

- boundedness,
- convergence,
- periodicity,
- stability;

for differential equations of the **second**, **third** and **fourth** order with different combinations of non-linear terms.
The following classes of equations are considered;

$$\ddot{x}(t) + a(t)g(\dot{x}) + b(t)h(x) = p(t; x, \dot{x}) \tag{1.1.1}$$

$$\dddot{x} + \varphi(x, \dot{x})\ddot{x} + f(x, \dot{x}) = p(t; x, \dot{x}, \ddot{x}), \tag{1.1.2}$$

$$x^{(iv)} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = p(t), \tag{1.1.3}$$

$$x^{(iv)} + a\dddot{x} + f(x, \dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t), \tag{1.1.4}$$

where the functions f, g, h and p are all continuous in their respective arguments. Equations of the form (1.1.1)-(1.1.4) are not only of theoretical importance but also of practical importance. For example, Equations (1.1.1) and (1.1.2) play an important role in the phase locked loop model realized by T.V. Systems. See for instance [8] and [9]; Equations (1.1.3) and (1.1.4) with various combinations of the non-linear and forcing terms can be applied in the modeling for automatic control in Television systems realized by means of R-C filters, see for instance [7]. These and their fifth order counterparts also have applications in some three-loop-electric-circuit problems and control theory (see [114])

Definition 1.1.1

Let

$$\dot{x} = f(t, x(t)) \tag{1.1.5}$$

The solution of Equation (1.1.5) is said to ***converge*** *if given any two solutions* $x_1(t)$ *and* $x_2(t)$ *of the Equation (1.1.5)* $x_2(t) - x_1(t) \to 0$ *as* $t \to \infty$.

Definition 1.1.2

The solution $x(t)$ *of Equation (1.1.5) is said to be* ***periodic*** *if* $x(t) = x(t+T)$ *for* $T > 0, -\infty < t < \infty$ *for all* t. T *is called the period of* x.

Definition 1.1.3

The solution $x(t)$ *of the Equation (1.1.5) with* $f(t,0) = 0$ *is said to be* ***stable*** *if for each* $\epsilon > 0$ *and* $t_0 = 0$, *there exists* $\delta > 0$ *such that* $\|x_0\| < \delta$ *and* $t \geq t_0$ *imply* $\|x(t, t_0, x_0)\| < \epsilon$.

Definition 1.1.4

The solution $x(t)$ *of Equation (1.1.5) is said to be* ***bounded*** *if there exists a* $\beta > 0$ *and there exists a constant* $M (M > 0)$ *such that* $\|x(t, t_0, x_0)\| < M$ *whenever* $\|x_0\| < \beta, t \geq t_0$.

For further expositions on the above definitions, see for instance [24], [30], [56], [57] and [135].

Stability is one of the central properties in System and Control Theory Engineering. From a practical point of view, one of the most important properties that

a system must satisfy is that it has to be stable, otherwise the system is useless and potentially chaotic. The theory of stability has got rich result and could be widely used in concrete problems of the real world. In some cases, a system may be stable or asymptotically stable in theory but it is actually unstable in practice because the stable domain or the domain of attraction is not large enough to allow desired deviation to cancel out. On the other hand, the desired state of a system sometimes may be mathematically unstable and yet the system may oscillate sufficiently near this desired state that its performance is acceptable; that is, it is stable in practice ([77] and [78]). Extreme stability, that is when the difference of each pair of solutions tends to zero as time increases infinitely (convergence of solutions) is also of practical importance.
The convergence property of systems that are stable is important both theoretically and in applications, since small perturbations from the equilibrium point imply that the trajectory will return to it when time goes to infinity.

Our method of approach to study the qualitative properties of this thesis will be the Lyapunov's Second Method. Over the years vast outflow of research and publications has resulted from the use of the Lyapunov's Second Method (or direct method) of stability analysis.
This work stems from the appearance of the original work of Lyapunov (or Liapunov as it appears in some literature) in 1892, more than a century ago, but only in the half century has this concept been appreciated to the point where workers in the area of stability of dynamical systems and automatic control are aware of its application.
The application of the Lyapunov method lies in constructing a scalar function (say V) and its derivatives such that they possess certain properties. When these properties of V and $\dot{V}$ are shown to be satisfied, the stability behavior of the system is known (see [54]).
Because of the difficulties surrounding the construction of suitable Lyapunov functions to study non-linear systems, numerous techniques have been proposed in the literature. These methods are summarized by [113]. To be specific, the construc-

tion of suitable Lyapunov function shall rest and depend solely on the approach of Cartwright (see [28]). See for example other papers where this approach was used: [2-5], [28], [33]-[35], [41]-[45] and [123]-[127].

However, solution of differential equations is of great importance to the Engineers who will want to know what the solution is before concerning themselves with the behavior of the solution. Unfortunately, not all differential equations can be solved analytically. To overcome this problem a search for an approximate solution is sought. This gives rise to numerical methods for solving ODEs.
The pioneering and most famous of the methods are the Euler Forward and Backward methods named after Leonhard Euler ([75]). Most significant contributions were given by Adams and Bashfort (1883) who developed linear multi step methods (LMMs), Runge (1895), Kutta (1901) and Heun (1900) came up with the most celebrated method which is called the Runge-Kutta (RK) method. These methods fall within the class of step-by-step initiatives.

The use of special functions to approximate solution of ODEs also came into the picture but these found their ways much in the Partial Differential Equations, methods like the Weighted Residual method and the Spectral methods. There are many other numerical methods for solving ODEs that have been developed. Some overviews of these can be found in [13], [21], [27], [39], [58], [69], [70], [71], [72], [79]-[80], [81], [82], [92], [108] and [109].
Suffice to note that there is no single numerical method to solve all forms of ODEs. Each of the methods developed so far are restrictive to the classes of equations they were developed for. Also, their method of derivation, implementation and accuracy remains the subject of continuous improvements.

The solution or the behavior of solution of these ODEs when it exists is of great importance. Efforts are made by researchers to come up with accurate methods for solving them or provide reliable information on the behavior of such solutions, more so since all physical problems that are modeled into ODEs may not be the same though they may seem to look alike, yet, the behavior of their solution may not be the same.

1.2 Research Aims and Objectives

The following are the proposed objectives of this thesis:

- To establish sufficient criteria for the qualitative properties of solution of the following classes of differential equations;

$$\ddot{x} + f(t; x, \dot{x}) + g(t; x) = p(t; x, \dot{x}) \tag{1.2.1}$$

$$\dddot{x} + f(x, \dot{x})\ddot{x} + g(x, \dot{x}) + h(x) = p(t; x, \dot{x}, \ddot{x}), \tag{1.2.2}$$

$$x^{(iv)} + f(x, \dot{x}, \ddot{x}, \dddot{x}) + g(x, \dot{x}, \ddot{x}) + h(x, \dot{x}) + l(x) = p(t; x, \dot{x}, \ddot{x}, \dddot{x}), \tag{1.2.3}$$

  Where the functions f, g, h, l and p are all continuous in their respective arguments.

- To find suitable numerical technique(s)/scheme(s) to solve initial value problems (**IVPs**) of ODEs in general.

- To find accurate approximation technique for the solution of two-point boundary value problems (**BVPs**) of ODEs in general.

1.3 Literature Review

The study of qualitative behavior of solutions of differential equations started in the latter part of the nineteenth century and became a subject of intense research since 1940. Most investigations in this direction are of local character. The behavior of solution is studied in a sufficiently small neighborhood of a given solution, e.g in a neighborhood of stationary point or of a periodic solution. The solution becomes different if the investigation is made in the large. In this case, the examined system and a certain domain are given and one has to study all the solutions which are situated in this domain or to find all solutions of a given family which are situated in the same domain.

The first direct reference as far as we know toward this approach is the work of Poincaré ((1879) see [21]). Ever since this work appeared, there has been an intensified interest among researchers to explore its richness. There is a substantial amount of literature dealing with numerous qualitative behavior of solutions of differential equations. These have been summarized in the monographs of [37], [56-57], [59], [76], [113], [115] and [116].

Lyapunov (1892) proposed a fundamental method for studying the problem of stability by constructing functions known as Lyapunov functions in the modern parlance. This function is often represented as $V(t, x)$ defined in some region or the whole state phase that contains the unperturbed solution $x = 0$ for all $t > 0$ and which together with its derivative $\dot{V}(t, x)$ satisfy some sign definiteness. The following definitions of stability were given by Lyapunov.

Definition 1.3.1 [Lyapunov (1892)]
Consider the system

$$\dot{X} = f(t; X) \tag{1.3.1}$$

where X *denotes an* n*-dimensional vector and* $f(t; X)(f : I \times \Re^n \to \Re^n, I = [0, \infty))$ *is continuous. Let* $X(t; X_0, t_0)$ *be a solution of the Equation (1.3.1) through* (X_0, t_0)*, then the trivial solution* $X(X_0, t_0) = 0$ *of the system (1.3.1) is said to be* ***stable*** *at* $t = t_0$*, provided that for arbitrary positive* $\epsilon > 0$*, there is a* $\delta = \delta(\epsilon, t_0)$ *such that whenever* $\|X_0\| < \delta$*, the inequality* $\|X(t; X_0, t_0)\| < \epsilon$ *is satisfied for all* $t \geq t_0$*.*

Definition 1.3.2[Lyapunov (1892)]
The trivial solution $X(t; X_0, t_0)$ *of the system (1.3.1) is said to be* ***asymptotically stable*** *if it is stable, and for each* $t_0 \geq 0$*, there is an* $\eta > 0$ *such that* $\|X_0\| < \eta$ *implies* $\|X(t; X_0, t_0)\| \to 0$ *as* $t \to \infty$*. If in addition all solutions tend to zero, then the trivial solution is* ***asymptotically stable in the large****.*

Lyapunov further gave a result on stability as follows:

Lyapunov's Theorem on Stability [Lyapunov (1892)]

Suppose there is a function V which is positive definite along every trajectory of (1.3.1), and is such that the total derivative $\dot{V}$ is semi definite of opposite sign (or identically zero) along the trajectory of (1.3.1). Then the perturbed motion is stable. If a function V exists with these properties and admits an infinitely small upper bound, and if $\dot{V}$ is definite (with sign opposite of V), it can be shown further that every perturbed trajectory which is sufficiently close to the unperturbed motion $X = 0$ approaches the latter asymptotically.

As a direct consequence of this result, the Lyapunov's theorem on instability is thus deduced from the unchanging sign of V and its derivative. (i.e. if V and its derivative are of the same sign we have what is known as Lyapunov's theorem on instability.)

The existence of a Lyapunov function V that satisfies condition of the Lyapunov's theorem on stability and asymptotic stability has been studied by a number of researchers. Massera [90-91] proved that if $f(t, X)$ in the system (1.3.1) of perturbed motion is periodic in t and continuously differentiable, then there exists a continuously differentiable Lyapunov function in a neighborhood of the asymptotically stable unperturbed motion $X(t; t_0, X_0) = 0$ while for the case in which $f(t, X)$ is continuously differentiable, Malkin (see [76] pg.18) gave necessary and sufficient conditions for existence of a continuously differentiable Lyapunov function $V(t, X)$ in some neighborhood of an asymptotically stable unperturbed trajectory.

Barbashin and Krasovskii (see [113]) gave conditions that ensure the existence of a Lyapunov function $V(t, X)$ throughout phase space $-\infty < t < \infty$ for global stability.

LaSalle [84] introduced an invariance principle and discussed the asymptotic behavior of solutions of an autonomous ordinary differential equation by a Lyapunov function for which its derivative is non positive definite. The invariance principle of LaSalle was extended by [60] to autonomous functional differential equations with finite delay.

Many research results on higher order non-linear differential equations have been obtained using the Lyapunov theorem and its generalizations. Among the earlier work in this category are those of [86], [29] and [141] on convergence of solutions for some second order non-linear differential equations. They made use of a Lyapunov function V constructed to measure the distance between solutions and showed that this distance must approach zero for large enough t. Specifically, they considered the general equation of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t). \tag{1.3.2}$$

[86] considered the case $g(x) = x$. *Cartwright and Littlewood* [29] showed that if g is twice differentiable and satisfies $g(0) = 0$ and if, further, both f and g are strictly positive, then all ultimately bounded solutions of Equation (1.3.2) converge provided that $|g''(x)|$ is sufficiently small. Swick [120] introduced a new dimension in his paper where he made use of the invariance principle due to LaSalle's [83] to study convergence of solutions of non-linear differential equations. LaSalle's theorem enables one to conclude asymptotic stability of an equilibrium point even when -$\dot{V}(t, x)$ is not locally positive definite. However, it applies only to autonomous or periodic systems. In continuation of his earlier investigation, in [118], Swick removed the requirement of boundedness of solutions and thereby improved known results on convergence for second order non-linear differential equations.

Further developments on convergence of solutions in this direction were on the third and fourth order non-linear differential equations. *Tejumola* [127], *Swick* [122] and *Ezeilo* [41] considered some third and fourth order equations and found on them conditions that made solutions of such equations to converge. Of particular interest is the work of Ezeilo [41], where he considered the equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}) \tag{1.3.3}$$

for special values of incrementary ratio $y^{-1}(h(x + y) - h(x))$. Results on this work have been generalized in [5] to fourth order non-linear differential equations.

A new technique (the intrinsic method) was proposed in [31]-[32] where the author constructed new Lyapunov functions for fourth order non-linear differential equations so that they are less restrictive than those presented earlier in the literature. Indeed the intrinsic method is novel in constructing Lyapunov functions for non-linear differential equations.
Let us remark here that in most of these results on third and fourth order, only in few cases were complete Lyapunov function employed. Also only few of the works were able to construct Lyapunov functions of their own. The reason for this is because of difficulty encountered in constructing suitable complete Lyapunov functions in higher orders.

Boundedness and stability properties of solutions of the form of Equations (1.1.2), (1.1.3) and (1.1.4) have received a considerable attention (see [113]). One of the pioneer study on the third order equation was the work of Barbashin [18], where a general third order was considered and conditions for global stability of solutions were established. The result of Barbashin could not handle some of the special cases of the equation as displayed in [42]-[43], [102] and [112]. Qian [112] re-visited the problem of Barbashin and established new results for global stability of solutions of third order equations. Other works on boundedness of third and fourth order equations include the following [10], [23], [28], [34]-[36], [42-45], [62-63], [100], [102], [119], [128]-[129], [130]-[133] and [134].
All these considered various equations with various combination of non-linear terms and all gave results that either generalize or improve the existing ones.

On the quantitative point of view, consider the general nth order Ordinary Differential Equation

$$x^{(n)} = f(t; x, \dot{x}, \dots, x^{(n-1)}) \tag{1.3.4}$$

Equation (1.3.4) can be reduced to a system of n coupled-first order equations given as

$$\dot{X} = f(t; X) \tag{1.3.5}$$

Where $X \in \Re^n, \dot{x} = \dfrac{dx}{dt}, x^{(n)} = \dfrac{d^n x}{dt^n}$.

The numerical solution of Equation (1.3.4) has received lots of attention and it is still receiving such an attention due to the fact that many physical (Engineering, Medical, Financial, Population Dynamics and Biological Sciences) problems formulated into mathematical equation result in the above type.
The solution is generated in a step-by-step fashion by a formula which is regarded as discrete replacement of Equation (1.3.4) (see [13], [69], [70], [71], [79]-[80]).
In the class of methods available in solving the problem numerically, the most celebrated methods are the single-step and the multi-steps methods. In a single step, an information at just one point is enough to advance the solution to the next point while for the multi steps (as the name suggests), information at more than one of the previous points is required to advance the solution to the next point.
The Euler's method (the pioneering method), which is the oldest method, and the Runge-Kutta methods fall into the class of the single-step methods while the Adams methods are in the class of the multi step method.([13], [69], [70], [71], [79]-[80]).
The Adams method is divided into two, namely the Adams-Bashforth (explicit) and Adams-Moulton (implicit). These two methods combined can be used as a predictor-corrector method. This class of methods has been proved to be one of the most efficient methods to solve certain class of IVP (non-stiff).
In the literature, the derivation of the Adams method has been extensively dealt with using the interpolatory polynomial for the discretized problem. For the derivation of linear multi steps method through interpolation and collocation (see [13], [69], [79]-[80]). In [103] the authors used the collocation method to derive a new class of the Adams-Bashforth schemes for ODE while in [104] the authors also used the collocation method for deriving a continuous multi steps method. *Lie & Norsett* [88] discussed the super convergence properties of the collocation methods.

The fundamental problem of approximation of a function by interpolation on an interval paved the way for the spectral methods which are found to be successful for the numerical solution of ordinary and partial differential equations. Spectral representations of analytic studies of differential equations have been in use since the days of Fourier (1822 see [75]). Their applications to Numerical solution of ordinary differential equations refer, at least to the time of Lanczos [81]. Summary of survey of some applications is given in [52].

Some present spectral methods can also be traced back to the "method of weighted residuals" of Finlayson and Scriven [48]. Spectral methods can be viewed as an extreme development of the class of discretization scheme for differential equations known as the method of weighted residuals (MWR) (see [48]). In MWR, the use of approximating functions (called trial functions) is central. These functions are used as basis functions for a truncated series expansion of the solution. Another function called the test function (also known as the weight functions) is used to ensure that the differential equation is satisfied as close as possible by the truncated series expansion. Among the spectral schemes, the three most commonly used are the Tau, Galerkin and collocation (also called pseudo-spectral) methods.

What distinguishes between these methods is the choice of the test functions employed. Galerkin and Tau method are implemented in terms of the expansion coefficients (see [39]), whereas collocation methods are implemented in terms of physical space values of unknown function. Over the past two decades, spectral methods with their current forms appeared as attractive methods in most applications. Some more details on spectral methods could be seen in [53], [72], [105]-[107].

The basic idea of spectral methods to solve differential equations is to expand the solution function as a finite series of very smooth basis functions a_k, as given below

$$y(x) = \sum_{k=0}^{N} a_k \phi_k(x); \quad a \leq x \leq b \tag{1.3.6}$$

where ϕ represents Chebyshev or Legendre polynomials (see [38] for more on Chebyshev polynomials). If $y \in C^{\infty}[a, b]$, the error produced by the approximation approaches zero with exponential rate (see [27]) as N becomes too large (tends to infinity). This phenomenon is referred to as 'spectral accuracy' (see [52]). The accuracy of the derivative obtained by direct term-by-term differentiation of such truncated expansion naturally deteriorates (see [27]), but for low-order derivatives and sufficiently high-order truncations this deterioration is negligible, compared to the restrictions in accuracy introduced by typical difference approximations.

Babolian and Hosseini [15] and Babolian, Bromilow, England & Savari [14] focused on differential equations in which one of the coefficient functions or solution functions is not analytic on the interval of definition. Weak aspect of spectral methods in solving this kind of problems were studied in [14] and [15] and they came up with modifications to the spectral method which proved to be more efficient when compared with existing ones.

1.4 Outline of the Thesis

In the following chapter, the first section starts with some basic definition on the qualitative properties of solution of ODEs followed by the definition and properties of Lyapunov function the major tool employed for such qualitative properties of solution. Basic theorems involving the use of the Lyapunov functions, as well as the procedure of constructing suitable complete Lyapunov functions for the differential equations of orders two, three and four which are different from the ones found in the literature (see Section 1.3), also featured in this chapter.
In the second section of the same chapter, we give an overview of the interpolation and approximations in quantitative properties of solution. This is followed by the characteristics of interpolation methods. The spline method and their properties as well as the interpolation spectral methods.
Chapters Three through Six are published chapters and they appear in that format containing their Abstracts, Introductions and References.

In chapter Three, criteria for global asymptotic stability, boundedness and existence of periodic solutions to certain non-linear non-autonomous differential equation of the second order with less restriction on the non-linear terms, are established.

The global asymptotic stability, boundedness as well as the ultimate boundedness of solutions of a general third order non-linear differential equation is investigated with the use of complete Lyapunov function in Chapter Four.

In chapter Five, sufficient conditions for the existence of a stable (globally asymptotically stable), bounded and uniform ultimate bounded solution to a certain fourth order non-linear differential equation using a single complete Lyapunov function without the use of a signum function or any stringent condition on the non-linear terms is given.

The convergence of solutions is a very important and desirable quality in the qualitative studies. In chapter Six of this thesis, sufficient criteria for the existence of convergence of solutions for a certain class of fourth order non-linear differential equations using the Lyapunov's second method are given.

Our results on the quantitative studies of solutions of ODEs were presented in chapter Seven. The derivation of a new Scheme for solving initial value problem of ordinary differential equation from the general method of deriving the spline function featured in the first part of this chapter, the scheme is developed based on interpolation and collocation. The second part of the chapter presents an accurate 'Spectral' method referred to as the pseudo-pseudo-spectral method to approximate the solutions of two-point boundary value problems linear ordinary differential equations. Exploiting the Trigonometric identity property of the Chebyshev polynomial, we were able to obtain approximate solution which competes favorably with solutions obtained with standard and well known Spectral methods.

The last chapter of this thesis contains the conclusion and suggestions for further studies.

Numbering:
Equations, Theorems and Lemmas are numbered according to the Chapter, section and subsection. e.g Equation (a.b.c.d), Theorem a.b.c.d, Lemma a.b.c.d and Definition a.b.c.d. where a refers to the Chapter, b to the Section, c to the Subsection and d the Number (counter).
Notations:

The following shall be our notations in this thesis;

- $\dot{x}, \ddot{x}$ and $\dddot{x}$ shall mean the first, second and third derivative of the variable x with respect to an independent variable (in this thesis our independent variable is t) respectively.
- $x^{(iv)}$ denotes the fourth derivative of the variable x with respect to t.
- $\dot{V}(t, X)|_{(a.b.c)}$ shall mean the time derivative of the function $V(t, X)$ with respect to the system (a.b.c).

Chapter 2

Ordinary Differential Equations

In this Chapter, we give an overview of the tools used in the discussion of the qualitative and quantitative properties of solutions considered in this thesis. First, we give some definitions related to the qualitative studies, the Lyapunovs' function and its properties, basic theorems on the use of Lyapunov function for qualitative studies as well as details on construction of suitable Lyapunov function for differential equations of the second, third and fourth orders.
Secondly, we give an overview of Interpolation and approximations, the spline function and its properties as well the spectral methods.

2.1 Qualitative Properties

2.1.1 Basic Definitions

Consider a system of differential equations

$$\dot{X} = f(t, X) \tag{2.1.1.1}$$

where X is an n-vector and $f(t, X)$ is an n-vector function which is defined on a region $\Omega \subset I \times \Re^n$ (where I is an interval, a subset of $\Re$) and continuous in (t_0, X_0) so that for each (t_0, X_0) there is a solution $X(t; t_0, X_0)$ satisfying

$$X(t_0; t_0, X_0) = X_0 \tag{2.1.1.2}$$

and

$$X(t; t_0, X_0) = X \tag{2.1.1.3}$$

Let f be smooth enough to guarantee the existence of a solution i.e. f is Lipschitz and continuous. Smoothness of f guarantees the existence of a unique solution for Equation (2.1.1.1). Let this be Equation (2.1.1.3).

Suppose that C is a class of solutions of Equation (2.1.1.1) and $X_0(t)$ is an element of C, then by setting $X = Y + X_0(t)$ together with the continuity of $\dot{X} = f(t, X)$, Equation (2.1.1.1) becomes

$$\dot{Y} = f(t, Y + X_0(t)) - f(t, X_0(t)). \tag{2.1.1.4}$$

Let $G(t, Y) = f(t, Y + X_0(t)) - f(t, X_0(t))$ then $G(t, 0) = 0$. The zero solution $Y(t) \equiv 0$ of Equation (2.1.1.4) corresponds to $X_0(t)$.

We shall stress at this juncture that discussing the stability, boundedness and periodicity of zero solution of the Equation (2.1.1.4) is equivalent to discussing the stability, boundedness and periodicity of $X_0(t)$ of the Equation (2.1.1.1). For this reason, we can assume that $f(t, 0) \equiv 0$ and the following definitions will hold for solutions $X_0(t) \equiv 0$ of the Equation (2.1.1.1).

Definition 2.1.1.1

The zero solution of Equation (2.1.1.1) is STABLE (S), if given $\epsilon > 0$ and $t_0 \in I$, there exists a $\delta(t_0, \epsilon) > 0$, such that whenever

$$|X_0| < \delta(t_0, \epsilon), \quad |X(t; t_0, X_0)| < \epsilon \quad for\ all\ t \geq t_0.$$

Definition 2.1.1.2

The zero solution of Equation (2.1.1.1) is UNIFORMLY STABLE (US), if it is stable and the δ in the definition (2.1.1.1) above is independent of t_0.

Definition 2.1.1.3

The zero solution of Equation (2.1.1.1) is ASYMPTOCALLY STABLE (AS), if it is stable and in addition, there exists an $\alpha \in [t_1, t_2], t_0 \leq t_1 \leq t_2 \leq t$ such that if $X_0 < \delta(t_0, \alpha)$, we have

$$|X(t; t_0, X_0)| \to 0 \ \text{ as } t \to \infty.$$

Definition 2.1.1.4

The zero solution of Equation (2.1.1.1) is UNIFORMLY ASYMPTOCALLY STABLE (UAS), if it is uniformly stable and if there is a $\delta > 0$ *and* $T(\epsilon)$*, such that whenever* $|X_0| < \delta$*, we have*

$$|X(t; t_0, X_0)| < \epsilon \text{ for all } t \geq t_0 + T(\epsilon).$$

Remark 2.1.1.5

It can be seen from the definitions that for any solution to be uniformly stable, asymptotically stable or uniformly asymptotically stable, it has to be stable.

Definition 2.1.1.6

A solution $X(t; t_0, X_0)$ *of Equation (2.1.1.1) is BOUNDED if there exists a* $\beta > 0$*, such that* $|X(t, t_0, X_0)| < \beta$ *for all* $t \geq t_0$*, where* β *may depend on each solution.*

Definition 2.1.1.7

The solution of Equation (2.1.1.1) is EQUI-BOUNDED (EB) if, for any $\alpha > 0$ *and* $t_0 \in I$*, there exists a* $\beta(t_0, \alpha) > 0$ *such that if* $X_0 \in S_\alpha$*, where* $S_\alpha = \{x \in \Re^n : \|x\| < \alpha\}$*, then* $|X(t, t_0, X_0)| < \beta(t_0, \alpha)$ *for all* $t \geq t_0$*, where* α *is the length of interval.*

Definition 2.1.1.8

The solution of Equation (2.1.1.1) is UNIFORMLY BOUNDED if, for any $\alpha > 0$ *and* $t_0 \in I$*, there exists a* $\beta(\alpha) > 0$ *such that if* $X_0 \in S_\alpha$*, then* $|X(t; t_0, X_0)| < \beta(\alpha)$ *for all* $t \geq t_0$*, where* α *is as defined in Definition 2.1.1.7*

Definition 2.1.1.9

The solution of Equation (2.1.1.1) is ULTIMATELY-BOUNDED (UB) for bound M*, if there exists an* $M > 0$ *and for every solution* $X(t, t_0, X_0)$ *of (2.1.1.1), there* *exists a* $T = T(\alpha, X)$, *such that*

$$|X(t, t_0, X_0)| < M$$

for all $t \geq t_0 + T$.

Definition 2.1.1.10

The solution of Equation (2.1.1.1) is UNIFORMLY ULTIMATELY BOUNDED (UUB) for bound M, *if there exists* $M > 0$ *and if for any* $\alpha > 0$ *and* $t_0 \in I$ *there exists a* $T(\alpha) > 0$ *such that* $X_0 \in S_\alpha$ *implies that*

$$|X(t; t_0, X_0)| < M$$

for all $t \geq t_0 + T(\alpha)$.

Remark 2.1.1.11

If M in Definition 2.1.1.9 depends on t_0 and α i.e $M(t_0, \alpha)$ for all t, then the solution of Equation (2.1.1.1) is *EQUI-ULTIMATELY BOUNDED.*

Definition 2.1.1.12

A solution $X(t)$ *of Equation (2.1.1.1) is PERIODIC if for some* $\omega > 0$,

$$X(t + \omega) = X(t),$$

ω *is called the period of* X.

2.1.2 Lyapunov Function and Properties

Lyapunov (1892) dealt with stability by two distinct methods; these are the first and second methods. The first method pre-supposes an explicit solution known and this is applicable to some restricted but important cases. As against this, the second method, which is also called the *Direct* method, is of great generality and power and, above all, does not require the knowledge of the solutions themselves.

The application of the Lyapunov method lies in constructing a scalar function (say V) and its derivatives such that they possess certain properties. When these properties of V and $\dot{V}$ are shown, the stability behavior of the system is known.

The direct method is via a special function called the **Lyapunov** function which we define now.

Definition 2.1.2.1

A Lyapunov function V *defined as* $V : I \times \Re^n \to \Re$, *is a real function of real variables* $X(X \in \Re^n), t$ *with the conditions that* $t \geq T$ *and* $|x_i| < H$. T *and* H *are real constants of which* T *can be supposed to be as large as we wish and* H *as small as we wish but not zero having the following properties:*

(i) Continuity: V(t,X) is continuous and single valued under the condition $t \geq T$ *and* $|x_i| < H$ *and* $V(t,0) \equiv 0$;

(ii) $V(t,X)$ *is positive definite;*

and

(iii) $\dot{V} = \dfrac{\partial V}{\partial x_1}\dot{x}_1 + \dfrac{\partial V}{\partial x_2}\dot{x}_2 + \ldots + \dfrac{\partial V}{\partial x_n}\dot{x}_n + \dfrac{\partial V}{\partial t}$, *representing the total derivative with respect to* t *is negative definite.*

Definition 2.1.2.2: (A Complete Lyapunov function)

A Lyapunov function V *defined as* $V : I \times \Re^n \to \Re$ *is said to be COMPLETE if for* $X \in \Re^n$,

(i) $V(t,X) \geq 0$,

(ii) $V(t,X) = 0$, *if and only if* $X = 0$ *and*

(iii) $\dot{V}|_{(2.1.1)}(t,X) \leq -c\,|X|$ *where c is any positive constant and* $|X|$ *given by*

$$|X| = \left(\sum_{i=1}^{n} (x_i^2)\right)^{\frac{1}{2}} \to \infty$$

It is INCOMPLETE if (iii) is not satisfied.

When the above properties of V and $\dot{V}$ are shown, the qualitative behavior of the system can be discussed. The difficulty, however, arises when the necessary conditions cannot be exhibited; for then no conclusion can be drawn especially about the stability. Each problem is a new challenge, for the functions must be shaped anew for each given systems or class of systems.

The proper choice of V depends to an extent upon the experience, ingenuity, and often, good fortune of the analyst.

2.1.3 Some Basic Theorems

In this section, we shall give without proof some standard theorems on stability and boundedness of Lyapunov second method. The proofs of these theorems can be found in the books by Lyapunov [89] (translated and edited by A. T. Fuller) and Yoshizawa [140].

Let

$$\dot{X} = f(t, X) \tag{2.1.3.1}$$

where $f : I \times \Re^n \to \Re^n$ is a continuous n-vector function.

The Right Hand Side (RHS) of the Equation (2.1.3.1) can be written in the following ways:

$$f(t, X) = A(t)X \tag{2.1.3.2}$$

and

$$f(t, X) = A(t)X + P(t) \tag{2.1.3.3}$$

where $A(t)$ is an $n \times n$ matrix of unknown coefficients, $P : \Re \to \Re^n$ is a continuous function. The Equation (2.1.3.2) is the homogeneous equation while Equation (2.1.3.3) is its non-homogeneous counterpart.

Suppose that $f(t, 0) \equiv 0$ for all t, then the following theorem is true.

Theorem 2.1.3.1: (Lyapunov theorem (Lyapunov (1892)))

If the differential equations of undisturbed motion (the steady state of a system before pertubations are introduced) are such that it is possible to find a definite function V, of which the derivative $\dot{V}$ is a function of fixed sign, which is opposite to that of V or reduces identically to zero, the undisturbed motion is STABLE.

The following theorems are the various simplification of the Theorem 2.1.3.1

Theorem 2.1.3.2

Assume that there exists a function $V(t, X)$ defined for $t \geq 0, |X| < \delta_0$ (δ_0 is a positive constant) continuous with the following properties:

(i) $V(t, 0) \equiv 0$,

(ii) $V(t, X) \geq a(|X|)$,

where $a(r)$ *is continuous monotonically increasing and* $a(0) = 0$,

(iii) $\dot{V}(t, X)|_{(2.1.3.1)} \leq 0$,

then the solution $X(t) \equiv 0$ *(zero solution) of Equation (2.1.3.1) is STABLE.*

Theorem 2.1.3.3

Suppose conditions (i) and (iii) of Theorem 2.1.3.2 hold, and if we replace condition (ii) with

$(iv) a(|X|) \leq V(t, X) \leq b(|X|)$,

$a(r)$ *and* $b(r)$ *being continuous monotone increasing functions and* $a(0) = b(0) = 0$, *then the zero solution of Equation (2.1.3.1) is UNIFORMLY STABLE (US).*

Theorem 2.1.3.4

Under the assumptions of the Theorem 2.1.3.2, if

(v) $\dot{V}(t, X) \leq -c(|X|)$,

where $c(r)$ *is continuous on* $[0, \delta_0]$ *and positive definite, and if* $f(t, X)$ *is bounded, then the zero solution of Equation (2.1.3.1) is ASYMPTOTICALLY STABLE (AS).*

Theorem 2.1.3.5

Under the same assumptions of Theorem 2.1.3.3 with condition (v) of Theorem 2.1.3.4, then the zero solution of Equation (2.1.3.1) is UNIFORMLY ASYMPTOTICALLY STABLE (UAS).

Theorem 2.1.3.6

If $\dot{V}(t, X) \leq -cV(t, X)$, *where* $c > 0$ *is a constant under the same assumptions as in Theorem 2.1.3.3, then the zero solution of Equation (2.1.3.1) is also UNIFORMLY ASYMPTOTICALLY STABLE (UAS).*

Theorems 2.1.3.1 - 2.1.3.6 are Theorems on stability of solutions in the sense of Lyapunov with the use of Lyapunov functions.

The following theorems are on the boundedness of solution in the sense of Lyapunov with the Lyapunov function.

Theorem 2.1.3.7

Suppose there exists a Lyapunov function $V(t, X)$ defined on $I \times \Re^n$ which satisfies the following conditions:

(i) $a(|X|) \leq V(t, X)$, where $a(r)$ is continuous, monotone increasing function and $a(0) = 0$,

(ii) $\dot{V}(t, X) \leq 0$,

then the solutions of Equation (2.1.3.1) are BOUNDED

Theorem 2.1.3.8

Suppose that there exists a Lyapunov function $V(t, X)$ defined on $0 \leq t \leq R, |X| \geq R$ (where R may be large) which satisfies

(i) $a(|X|) \leq V(t, X) \leq b(|X|)$,

where $a(r)$ and $b(r)$ are continuous monotone increasing functions, and

(ii) $\dot{V}(t, X) \leq 0$,

then the solutions of Equation (2.1.3.1) are UNIFORMLY BOUNDED (UB).

Theorem 2.1.3.9

Under the assumptions of Theorem 2.1.3.8, if $\dot{V}(t, X) \leq -c(|X|)$, where $c(r)$ is positive and continuous, then the solutions of Equation (2.1.3.1) are UNIFORM ULTIMATELY BOUNDED (UUB).

2.1.4 Construction of Lyapunov Functions

The major difficulty in applying the second method on Lyapunov to the analysis of qualitative properties of solutions of nonlinear systems is the lack of a straightforward procedure for finding appropriate Lyapunov functions. The construction of Lyapunov function is an art, but like any other art, there are guidelines to follow. For a stable system, there may exist a large or even infinite number of suitable Lyapunov functions. There are generally many methods that have been proposed in the literature for constructing Lyapunov functions. We shall mention the following methods for their novelty.

1. *Krasovskii's Method* [76]

   The method of Krasovskii gives sufficient condition for asymptotic stability of nonlinear systems. Krasovskii's approach assumes the Lyapunov function to be a Hermitian form or quadratic form. We would like to point out here that such assumption of Hermitian form or quadratic form is unnecessarily restrictive simply because a Hermitian form or a quadratic form may not exist for a given system (see [98]).

2. *Schultz-Gibson's Variable Method* [51]

   In order to meet the stability criteria set forth by Lyapunov, a scalar function V and its time derivative must be found. Since the state variables are implicit functions of time, then:

   $$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + ... + \frac{\partial V}{\partial x_n}\dot{x}_n$$

   $$\frac{dV}{dt} = \nabla V \dot{X}$$

   where ∇ is the Gradient Operator.

   V can also be found from the gradient of V by a path integration through state space.

   This path integration will be independent of the path if :

   $$\frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i}$$

   which for the three dimensional case reduces to the vector identity:

   $$\nabla \times [\nabla V] = 0$$

   The gradient of V has the form:

   $$\nabla [V] = \alpha X$$

   (where α is an $n \times n$ matrix of unknown coefficients)

   The Schultz-Gibson's variable method uses a systematic approach based on the fact that if a particular Lyapunov function exists which is capable of proving asymptotic stability of a given non-linear system, then a gradient of this Lyapunov function also exists. In fact, for Euclidean spaces, this method is just another way of looking at the related Lyapunov theory of autonomous system. However, the classical Lyapunov function theory must be modified for systems in Banach space since the most important examples of such systems do not have all their trajectories differentiable, but only dense subset of their trajectories have these properties (see [97] Chapter 15). While the method is straightforward, the process is long and arduous.

3. *Intrinsic Method* [31-32]

   The intrinsic method is used to derive suitable Lyapunov functions for a general class of non-linear systems expressed in state variables as n first order non-linear equations. This method, which applies the integration by parts procedure, derives a Lyapunov function directly from the differential equation under study. For this, the integration is along trajectories and the limits for the integral with respect to time are from zero to t. The derivatives of the Lyapunov function V and its derivative $\dot{V}$ are based on the equation

$$V + \int_0^t -\dot{V}\,d\tau = 0 \qquad (2.1.4.1)$$

   and do not require the gradient of the scalar function V to be obtained.

In constructing our Lyapunov functions, the properties of the Lyapunov functions were taken strictly into consideration and we follow the ideas of LaSalle and Lefschez [85] where a quadratic form was assumed to initiate the construction.

We shall sketch the construction of the Lyapunov functions used in discussing the qualitative properties of the equations considered in this research namely the second, third and fourth order non-linear equation.

For the *second* order equation, we consider the general linear equation of the form

$$\ddot{x} + a\dot{x} + bx = p(t; x, \dot{x}) \qquad (2.1.4.2)$$

with corresponding system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ay - bx + p(t;x,y) \end{aligned} \quad (2.1.4.3)$$

where a and b are positive constants.

To construct suitable complete Lyapunov function for the above system, we assume a quadratic form of the form

$$2V = Ax^2 + By^2 + 2Cxy \quad (2.1.4.4)$$

where $A, B,$ and C are positive constants to be determined.

Differentiating Equation (2.1.4.4) with respect to t using the equivalent system (2.1.4.3) we have

$$\dot{V}|_{(2.1.4.3)} = Ax\dot{x} + By\dot{y} + C(x\dot{y} + \dot{x}y)$$

$$= Axy + By(-ay - bx + p(t;x,y)) + Cy^2 + Cx(-ay - bx + p(t;x,y)) + (Cx + By)p(t;x,y)$$

$$= -Cbx^2 - (Ba - C)y^2 - (Bb + Ca - A)xy + (Cx + By)p(t;x,y)$$

to make the $\dot{V}$ negative definite, we adapt the method of Cartwright [28] by equating the coefficients of mixed variables to zero and the co-efficients of x^2 and y^2 to any positive constant (say δ),i.e

$$Bb + Ca - A = 0, \quad (i)$$

$$Ba - C = \delta, \quad (ii)$$

and

$$Cb = \delta. \quad (iii)$$

Solving these equations for A, B and C, we have that

$$A = \frac{\delta}{ab}\left\{a^2 + b(b+1)\right\}, \quad B = \frac{\delta}{ab}(b+1) \text{ and } C = \frac{\delta}{b}$$

The required Lyapunov function is obtained by substituting for the constants A, B and C in (2.1.4.4) which gives

$$2V = \frac{\delta}{ab}\left\{a^2 + b^2(b+1)\right\}x^2 + \frac{\delta}{ab}(b+1)\,y^2 + 2\frac{\delta}{b}xy \quad (2.1.4.5)$$

re-arranging the above gives

$$2V = \frac{\delta}{ab}\left(y + ax\right)^2 + \frac{\delta}{a}\left(b+1\right)x^2 + \frac{\delta}{a}y^2 \tag{2.1.4.6}$$

Clearly, the above function is positive definite and its derivative negative definite.
For the non-linear counterpart of Equation (2.1.4.2) given as

$$\ddot{x}(t) + g(\dot{x}) + h(x) = p(t; x, \dot{x}) \tag{2.1.4.7}$$

The procedure of construction of suitable Lyapunov function is as explained above except that here we need to find conditions on the non-linear terms to complete the construction.
For a third order equation say

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = p(t; x, \dot{x}, \ddot{x}) \tag{2.1.4.8}$$

with a corresponding system given as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -az - by - cx + p(t; x, \dot{x}, \ddot{x}) \end{aligned} \tag{2.1.4.9}$$

where a, b and c are all positive constants.
The required quadratic form in this case is given as

$$2V = Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz \tag{2.1.4.10}$$

where the A, B, C, D, E and F are constants to be determined.
Differentiating Equation (2.1.4.10) with respect to the system (2.1.4.9) we have

$$\begin{aligned} &\dot{V} = Axy + Byz + Cz(-az - by - cx + p(t; x, \dot{x}, \ddot{x})) \\ &+Dy^2 + Dxz + Eyz + Ex(-az - by - cx + p(t; x, \dot{x}, \ddot{x})) \\ &+Fz^2 + Fz(-az - by - cx + p(t; x, \dot{x}, \ddot{x})) \end{aligned} \tag{2.1.4.11}$$

re-arranging Equation (2.1.4.11) we have

$$\begin{aligned} &\dot{V} = -Ecx^2 - (Fb - D)y^2 - (Ca - F)z^2 - (Eb + Fb - A)xy \\ &-(Cc + Ea - D)xz - (Cb + Fa - B)yz + (Ex + Fy + Cz)p(t; x, \dot{x}, \ddot{x}) \end{aligned} \tag{2.1.4.12}$$

To solve for the constants, we equate the coefficients of the mixed variables to zero and the coefficients of x^2, y^2 and z^2 to any positive constant (say δ), this leads to the following system of equation to solve

$$Ex = \delta \qquad (i)$$
$$Fb - D = \delta \qquad (ii)$$
$$Ca - F = \delta \qquad (iii)$$
$$Eb + Fc - A = 0 \qquad (iv)$$
$$Cc + Ea - D = 0 \qquad (v)$$
and
$$Cb + Fa - B = 0 \qquad (vi)$$

Solving the system we have

$$A = \frac{\delta}{(ab-c)c}\left\{(ab-c) + (a^2+c^2+c)\right\}$$
$$B = \frac{\delta}{(ab-c)c}\left\{b(a+bc+c) + a(a^2+c^2+c)\right\}$$
$$C = \frac{\delta}{(ab-c)c}(a+bc+c)$$
$$D = \frac{\delta}{(ab-c)c}\{(a+bc+c) + a(ab-c)\}$$
$$E = \frac{\delta}{c}$$
$$F = \frac{\delta}{(ab-c)c}\left(a^2+c^2+c\right)$$

these values of the constants guarantee the positive definiteness of V and negative definiteness of its derivative.

For the non-linear equation equation,

$$\dddot{x} + \varphi(x,\dot{x})\ddot{x} + f(x,\dot{x}) = p(t;x,\dot{x},\ddot{x}), \qquad (2.1.4.13)$$

the procedure of construction of suitable Lyapunov function is as explained above except that here we need to find conditions on the non-linear terms to complete the construction.

For the fourth order equation

$$x^{(iv)} + a\dddot{x} + b\ddot{x} + c\dot{x} + dx = p(t;x,\dot{x},\ddot{x},\dddot{x}) \qquad (2.1.4.14)$$

with an equivalent system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= z\\ \dot{z} &= w\\ \dot{w} &= -aw - bz - cy - dx + p(t; x, \dot{x}, \ddot{x}, \dddot{x})\end{aligned} \tag{2.1.4.15}$$

with a, b, c and d are positive constants.

A quadratic form defined as

$$\begin{aligned}2V &= Ax^2 + By^2 + Cz^2 + Dw^2 + 2Exy + 2Fxz + 2Gxw\\ &+2Hyz + 2Iyw + 2Jzw\end{aligned} \tag{2.1.4.16}$$

will be employed for the construction with A, B, C, D, E, F, H, I and J been the constants to be determined.

Differentiating Equation (2.1.4.16) with respect to the system (2.1.4.15) we have

$$\begin{aligned}\dot{V} &= Axy + Byz + Czw + Dw\dot{w} + E[\dot{x}y + x\dot{y}] + F[\dot{x}z + x\dot{z}]\\ &+G[\dot{x}w + x\dot{w}] + H[\dot{y}z + y\dot{z}] + I[\dot{y}w + y\dot{w}] + J[\dot{z}w + z\dot{w}]\end{aligned} \tag{2.1.4.17}$$

Re-arranging, we have

$$\begin{aligned}\dot{V} &= -Gdx^2 - [Ic - E]y^2 - [Jb - H]z^2 - [Da - J]w^2 - [Gc + Id - A]xy\\ &-[Gb + Jd - E]xz - [Dd + Ga - F]xw - [Ib + Jc - F - B]yz\\ &-[Dc + Ia - G - H]yw - [Db + Ja - I - C]zw\\ &+[Gx + Iy + Jz + Dw]p(t; x, \dot{x}, \ddot{x}, \dddot{x})\end{aligned} \tag{2.1.4.18}$$

Equating the coefficients of x^2, y^2, z^2 and w^2 to a positive constant (say $\delta > 0$):

$$Gd = \delta, \tag{i}$$

$$Ic - E = \delta, \tag{ii}$$

$$Jb - H = \delta, \tag{iii}$$

$$Da - J = \delta, \tag{iv}$$

and the coefficients of mixed variables are set to zero. i.e

$$Gc + Id - A = 0, \tag{v}$$

$$Gb + Jd - E = 0, \quad (vi)$$

$$Dd + Ga - F = 0, \quad (vii)$$

$$Ib + Jc - F - B - 0, \quad (viii)$$

$$Ia + Dc - G - H = 0, \quad (ix)$$

and

$$Ja + Da - I - C = 0. \quad (x)$$

Solving these equations we have

$$A = \frac{\delta}{cd\Delta}\left\{\Delta(c^2+d^2) + d\left[\frac{\Delta}{bd}(b^2+d^2) + d[a^2d(b+d) + c^2d(b+1) + ab(ab-c)]\right]\right\}$$
$$B = \frac{\delta}{abc\Delta}\left\{\Delta[a(a^2+b^2-c)] + \frac{b\Delta}{d}(a-c)(a^2+b^2)\right.$$
$$\left. d\left[a^2d(b+d) + c^2d(b+1) + ab(ab-c)\right]\left(ab^2 + \tfrac{abc^2}{d} - bc\right)\right\}$$
$$C = \frac{\delta}{abc\Delta}\left\{c(a^2+b^2+b) + ab\left[b^2cd[a^2d(b+d) + c^2d(b+1) + ab(ab-c)]a^2d(b^2+d^2)\right.\right.$$
$$\left.\left. - (ab-c)c - 1\right]\right\}$$
$$D = \frac{\delta}{a\Delta}\left\{a^2d(b+d) + c^2d(b+1) + ab(ab-c) + \Delta b + \Delta\right\}$$
$$E = \frac{\delta}{\Delta}\left\{\frac{\Delta}{bd}(b^2+d^2) + d\left[a^2d(b+d) + c^2d(b+1) + ab(ab-c)\right]\right\} \quad (2.1.4.19a)$$

$$F = \frac{\delta}{a\Delta}\left\{\frac{\Delta}{d}(a^2+b^2) + d\left[a^2d(b+d) + c^2d(b+1) + ab(ab-c) + \frac{\Delta}{b}\right]\right\}$$
$$G = \frac{\delta}{d}$$
$$H = \frac{b\delta}{\Delta}\left[a^2d(b+d) + c^2d(b+1) + ab(ab-c)\right]$$
$$I = \frac{\delta}{c\Delta}\left\{\frac{\Delta}{bd}(b^2+d^2) + d[a^2d(b+d) + c^2d(b+1) + ab(ab-c)] + \Delta\right\}$$
$$J = \frac{\delta}{\Delta}\left\{a^2d(b+d) + c^2d(b+1) + ab(ab-c) + \frac{\Delta}{b}\right\} \quad (2.1.4.19b)$$

where $\Delta = [(ab-c)c - a^2d]bd$.

The value of the constants defined above guaranteed the positive definiteness of V and negative definiteness of its derivative.

For the non-linear equations

$$x^{(iv)} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = p(t), \quad (2.1.4.20)$$

and

$$x^{(iv)} + a\dddot{x} + f(x, \dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t), \quad (2.1.4.21)$$

suitable conditions can be established for the non-linear terms to construct suitable Lyapunov function to discuss their qualitative properties.

Remark 2.1.4.1: We should remark at this point that from the various Lyapunov functions constructed, the Routh-Hurwitz criteria for the stability of solutions of the equations are clearly seen. i.e. For the second order $a > 0$ and $b > 0$, for the third order $c > 0$ and $ab > c$ and for the fourth order $a > 0, b > 0, c > 0, d > 0$ and $\Delta > 0$. The Lyapunov functions constructed above were adapted for the results reported in this research.

2.2 Quantitative Properties

2.2.1 Interpolation and Approximation

In approximation theory, the challenge faced is how to replace a complicated function (say f) from a large space $\boldsymbol{F}$ by a simple and yet close-by (or "good" in some sense) function p from a small subset $\boldsymbol{P} \subset \boldsymbol{F}$. The literature about approximation is very rich (see [40] and [93]). Usually $\boldsymbol{F}$ is a Banach space, so the distance between $\boldsymbol{p}$ and $\boldsymbol{f}$ is by means of norm. Functions from $\boldsymbol{P}$ are called *approximation functions*.

The approximation functions depend on a set of parameters $\{c_i\}_{i=0}^{n}$.

For example, for $\boldsymbol{p}$ being finite dimensional, $\boldsymbol{p} \in \boldsymbol{P}$ can be represented as

$$\boldsymbol{p}(x) = \sum_{i=0}^{n} \{c_i p_i(x)\} \quad (2.2.1.1)$$

where $\{p_i\}_{i=0}^{n}$ form *basis functions* in $\boldsymbol{P}$.

There are many ways of choosing the space $\mathbf{P}$, these include the use of *ordinary polynomials*, *trigonometric polynomials*, *exponential functions* and *rational functions*. It should however be stated here that the latter two are non-linear and so they are not of special interest in this thesis.

The type of approximation depends on the way how the parameters are obtained. One of the most important is *interpolation*. In the general case for a function

$f : \Re^N \rightarrow \Re^M$, we can define a set of pairs

$$\Omega := \{(x_k, f_k) | x_k \in S \subset \Re^N, f_k := f(x_k) \in \Re^M, k = 0, 1, \ldots, N\}, \tag{2.2.1.2}$$

and the following condition

$$\boldsymbol{p}(x_k) = f_k, \tag{2.2.1.3}$$

We then say that $\boldsymbol{p}$ interpolates f at $x_0, \ldots, x_N$, i.e. Equations (2.2.1.2)-(2.2.1.3) represent the interpolation problem. Here $\mathbf{S}$ is the set $\{x_k\}_{k=0}^N$ of interpolation nodes, i.e. the points where the functional values are known, Γ is the interpolation domain, and $\hat{\Gamma}$ the range (or the co-domain) of f. We can formulate the interpolation problem in this way by providing an answer to the question: How to find a "good" representative of a function that is not known explicitly, but only at some points of the domain of interest.

The interpolation domain Γ and the set of interpolation nodes S play an essential role in the interpolation problem settings. We distinguish between interpolation on regularly spaced data, where the distribution of points satisfy some particular condition(s), and the interpolation on scattered (irregularly) spaced data where $\mathbf{S}$ is any subset of Γ.

Interpolation can be local or global depending on the support of the interpolation function. If all nodes are used for determining all the parameters we have a global approach, which means that any parameter or data perturbation will affect the solution throughout the whole interpolation domain. On the other hand, if the same perturbation does not influence the interpolation functional values outside some sub domain of Γ, the method is considered to be local. Related to this, it is useful to mention that there is also an interesting class of the interpolation functions where the basis functions have local support only (equal to zero outside the sub domain). In general, these methods are global, but few of them are local.

Spline Interpolation functions are typical representatives of such functions.

For interpolation by splines, also called the *piece-wise* functions, one typically needs to discretized the domain, i.e. to generate a grid which covers Γ. The grid is defined by the set S and the choice of the basic elements, say $\Gamma_m \in \Gamma, m = 1, \ldots, N_m$.

The spectral methods are classified into two main classes, the interpolating methods and the non interpolating methods. The Collocation method or the Pseudo spectral method is interpolating in nature while the Galerkin and the Tau method are non-interpolating.

Interpolation methods are well developed to date. The practical applications are numerous and the supporting theory has been essential for developing whole classes of different methods in numerical analysis. Examples can be found in numerical integration and differentiation, numerical methods for solving ODEs and PDEs, etc. In this thesis on the quantitative properties of solution of ODEs our main goal is to employ the spline interpolation as well as the collocation or pseudo spectral method to approximate solutions of ODEs.

2.2.2 Characteristic of Interpolation Methods

Evaluating and comparing characteristics of different interpolation methods are somewhat subjective. However, some attempts were made for obtaining the list of the most important characteristics of interpolation methods. In Franke [50] the special case of the two-dimensional (scattered) data interpolation is considered by performing a test over 32 different methods. The results are evaluated by a set of characteristics, given by the following list:

- *Accuracy*: Accuracy is expressed by the *interpolation error*, say r which is a measure of difference between the interpolation function and the exact values of the interpolating function f. It can be defined point-wise via the error function

  $$r(x) := |||f(x) - \boldsymbol{p}(x)||| \tag{2.2.2.1}$$

  or as a scalar

  $$r := |||f - \boldsymbol{p}||| \tag{2.2.2.2}$$

  where $||.||$ is some suitable norm. Usually the error depends on the space $\boldsymbol{P}$ and the location of the interpolation points. If the interpolation points are the vertices of a grid of simplices, then *the order of accuracy* is related to the maximum diameter of these simplices.

- *Visual Aspect*: Appearance of the interpolation function on Γ is of importance only in low dimensional spaces (N=1,2,3). Visual aspects are often in a close relationship with accuracy, especially at moderate accuracies.

- *Sensitivity to Parameters*: It is desirable that a method is stable with respect to perturbations of parameters and that the solution value is not highly dependent on the sampled function. In principle, as mentioned, local interpolation methods have an advantage, but it does not mean that all global methods will behave badly in general.

- *Computational Costs*: Computational efforts depends on a chosen method. Some methods can be extremely expensive in some applications and has to be avoided, even if all other characteristics are good.

2.2.3 Polynomial Interpolation

One of the foundations of the approximation theory is the Theorem of Weierstrass and its modification, (see [69] and [109]).

Theorem 2.2.3.1:

Let $f \in C[a,b]$. For every $\epsilon > 0$, there exists a polynomial $p(x)$ such that

$$|f(x) - p(x)| < \epsilon, \forall x \in [a,b] \qquad (2.2.3.1)$$

From Equation (2.2.3.1) it follows that one can always find a polynomial that is arbitrary close to a given function on some finite interval. This means that the approximation error is bounded and can be reduced by the choice of adequate polynomial.

The above Theorem is often used in conjunction with the following uniqueness theorem.

Theorem 2.2.3.2:
Let $\{x_k\}_{k=0}^n$ be a set of distinct points, and let $\{y_k\}_{k=0}^n$ be an arbitrary set of points in $\Re$. Then, there exists a unique polynomial of degree n which takes the given values at $n+1$ points.

The proof of this Theorem are numerous and often constructive. One of the most famous ones is related to the well-known Lagrange interpolation method. For this method, we have the following error bound (see [69], [70] and [93]).

Theorem 2.2.3.3:
Suppose $f \in C^{n+1}[a,b]$. Let the interpolation nodes satisfy $a \leq x_0 < x_1 \ldots < x_n \leq b$. The polynomial $p \in P_n$ interpolates f at $\{x_k\}_{k=0}^n$ and $w(x) := \Pi_{k=0}^n (x - x_k)$. Then there exists $\zeta \in [a,b]$ such that the error function $r(x)$ satisfies

$$r(x) := f(x) - p(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!} w(x) \tag{2.2.3.2}$$

Taking $\Gamma = [a,b]$ and using the L_p-norm in (2.2.3.2), one trivially obtained

$$r := |f - p|_{L_p(\Gamma)} \leq \frac{1}{(n+1)!} \left| f^{(n+1)} \right|_{L_p(\Gamma)} |w|_{L_p(\Gamma)} . \tag{2.2.3.3}$$

An interesting special case is that of equidistant points where $x_{k+1} = x_k + \Delta x$ for all k. Then we have

$$|f - p|_{L_\infty(\Gamma)} \leq \frac{\Delta x^{n+1}}{(4(n+1))!} \left| f^{(n+1)} \right|_{L_\infty} (\Gamma). \qquad (2.2.3.4).$$

2.2.4 Piece-wise Interpolation

The main disadvantage of global interpolation is that the interpolation error is related to higher derivatives of the interpolated function f. For example, the condition $f \in C^{n+1}[a,b]$ met in the Theorem 2.2.3.3 can be too strict. This is one of the most important reasons why often another approach is used. Discretizing the interpolation domain and interpolating locally, i.e. on small subsets of $\{x_k\}_{k=0}^n$, the overall accuracy may be significantly improved even if the applied (local) interpolation method is of low order. Interpolation functions obtained on this principle are piece-wise interpolation functions or splines. We will mention some of the most important (most frequently used) piece-wise interpolation methods.

1. *Nearest-neighbor method:* By far the easiest way to interpolate interpolation pairs $\{(x_k, f_k)\}_{k=0}^n$, by piece-wise constants. The method is of $\mathcal{O}(\Delta x)$.

2. *Piece-wise linear interpolation:* An improvement of the previous method made by constructing a linear function between two consecutive nodes. The accuracy of this method is $\mathcal{O}(\Delta x^2)$.

3. *Piece-wise cubic interpolation:* By increasing the order of piece-wise polynomial, one can obtain further improvements in the characteristics of the interpolating method. If the polynomial is of the third order, we have piece-wise cubic interpolation with an accuracy of $\mathcal{O}(\Delta x^4)$.

4. *cubic spline interpolation:* This is probably the most popular choice of obtaining piece-wise interpolation function, which is necessarily differentiable at the interpolation nodes. This provides the interpolation function to be smooth on the entire domain $\Gamma = [a, b]$. This method is also of $\mathcal{O}(\Delta x^4)$.

2.2.5 Interpolating Spectral Method

Spectral methods are quantitative (numerical) methods for solving differential equations in which the dependent variables are expanded as series of orthogonal basis functions. They are divided into two main classes, viz: The Interpolating methods and the Non-interpolating methods.

By interpolating methods, we mean the methods that associate grid of points with each basis set. They are also referred to as the collocation or pseudo spectral method. The pseudo spectral method demands that the differential equation to be approximated be exactly satisfied at a set of points known as collocation method. The Tau method and the Galerkin method are also spectral methods which have been proved to be effective in approximating solution of PDEs but they are both non interpolating spectral methods.

Chapter 3

BOUNDEDNESS, PERIODICITY AND STABILITY OF SOLUTIONS TO $\ddot{x}(t) + a(t)g(\dot{x}) + b(t)h(x) = p(t; x, \dot{x})$

[1]

Abstract:

In this paper, we give criteria for global asymptotic stability, boundedness and existence of periodic solutions to the nonlinear non-autonomous differential equation of the second order with less restriction on the nonlinear terms. This result improves on the existing ones in the literature.

Key words and Phrases: boundedness, Lyapunov function, nonlinear non-autonomous second order non-linear differential equations, periodic solution, stability, asymptotic stability.

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3.1 Introduction

In this paper, we consider the second order nonlinear non-autonomous differential equation

$$\ddot{x} + a(t)g(\dot{x}) + b(t)h(x) = p(t; x, \dot{x}), \tag{3.1.1}$$

where a, b, g, h and p are continuous in the respective argument displayed explicitly. In addition, g and h are such that existence, uniqueness and continuous dependence on the initial condition are guaranteed.

The equation of the above class have received attention of researchers (see [3, 8-9, 11, 20-24]) in fact a lot was done on these class of equations especially the autonomous case (i.e. when $a = b = 1$ or constant independent of t cf [1, 2, 7, 12, 18] and references contained in them).

In some of these works, the use of the second method of Lyapunov played a prominent role in discussing the stability and boundedness. Almost in all these works (for the non-autonomous case) the condition that g and h are differentiable with the function a being decreasing (i.e. $\dot{a}(t) \leq 0$) is quite noticeable.

The motivation for this work are the works of [24] , [16] and [5]. Even though quite old the authors [24] discussed interestingly these properties alongside with the oscillatory nature of the solutions via the Lyapunov second method as well as the oscillatory property of the function a.

In [16], the author also discussed the global asymptotic stability of the trivial solutions of non-autonomous system with an application to a second order equation also via the Lyapunov second method.

In [5], the authors developed a theory whereby all these properties (boundedness, stability and periodicity) been investigated differently could be discussed in a unified way by just one major theorem still on the Lyapunov second method.

We shall in this study adapt [5] to discuss the qualitative properties of solutions of Equation (3.1.1) and give sufficient conditions on the nonlinear terms g and h as well as on the functions a and b that will guarantee the existence of a unique solution which is bounded together with its' derivative on the real line, globally stable and periodic.

Let

$$\dot{X} = f(t; X) \tag{3.1.2}$$

where $X \in \Re^n$, be a system of n coupled-first order equations.

We shall give the following definition for the sake of completeness.

Definition 3.1.1:

A Lyapunov function V defined as $V : I \times \Re^n \to \Re$ is said to be COMPLETE if for $X \in \Re^n$

(i) $V(t, X) \geq 0$

(ii) $V(t, X) = 0$, if and only if $X = 0$

and

(iii) $\dot{V}\mid_{(3.1.2)}(t, X) \leq -c\,|X|$where c is any positive constant and $|X|$ given by

$|X| = (\sum_{i=1}^{n}(x_i^2))^{\frac{1}{2}}$ *such that*

$|X| \to \infty$ *as* $X \to \infty$

Definition 3.1.2:

A Lyapunov function V defined as $V : I \times \Re^n \to \Re$ is said to be INCOMPLETE if for $X \in \Re^n$ (i) and (ii) of the above definition is satisfied and in addition

(iii) $\dot{V}(t, X)\mid_{(3.1.2)} \leq -c\,|X|_{(*)}$ where c is any positive constant and $|X|_{(*)}$ given by

$|X|_{(*)} = (\sum_{i=1}^{<n}(x_i^2))^{\frac{1}{2}}$ *such that*

$|X|_{(*)} \to \infty$ *as* $X \to \infty$.

Equation (3.1.1) can be put in a system form as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -a(t)g(y) - b(t)h(x) + p(t; x, y) \end{aligned} \tag{3.1.3}$$

For expository reasons we will like to give as part of our definition the result of the following:

Generalized Theorems: (Burton et al) In an attempt to discuss the unified theory of periodicity of dissipative ordinary differential equations, Burton et al [5] considered the general differential equation

$$\dot{X} = f(t, X). \tag{3.1.4}$$

When Equation (3.1.4) is linear, it is written as

$$\dot{X} = A(t) + P(t), \tag{3.1.5}$$

with the homogeneous systems

$$\dot{X} = A(t)X, \tag{3.1.6}$$

where $A(t)$ is an $n \times n$ matrix of unknown coefficients, $P : \Re \to \Re^n$ is a continuous function.

The use of Lyapunov functions which led to the formulation of the following scheme was employed:

i) if $f(t, 0) = 0$, and if there is a function $V : [0, \infty) \times \Re^n \to \Re$ such that

$$W_1(|X|) \leq V(t, X) \leq W_2(|X|)$$

and

$$\dot{V}(t, X)|_{(3.1.2)} \leq -W_3(|X|),$$

where $W_i(i = 1, 2, 3)$ are strictly increasing continuous function defined as $W_i : [0, \infty) \to [0, \infty)$ with $W(s) > 0$ and $W(0) = 0$ as wedges. Then the solutions of Equation (3.1.4) is uniformly asymptotically stable (UAS).

ii) if there is a function $V : [0, \infty) \times \Re^3 \to \Re$ such that

$$W_1(|X|) \leq V(t, X) \leq W_2(|X|)$$

and

$$\dot{V}(t, X)|_{(3.1.2)} \leq -W_3(|X|) + M(M > 0),$$

then the solutions of Equation (3.1.4) are ultimately bounded (UB) and uniformly ultimately bounded (UBB).

iii) if the solution of Equation (3.1.4) and Equation (3.1.5) are unique, UB and UUB, then Equation (3.1.4) has a periodic solution.

iv) if the zero solutions of Equation (3.1.6) is uniformly asymptotically stable (UAS), then Equation (3.1.5) has a globally stable periodic solution.

We shall now state without proof, Theorems of Burton et. al [5]

Theorem A [5]:*If f is Lipschitz in X and periodic in t with period T and if the solutions are uniformly bounded and uniformly ultimately bounded for any given bound (say) B, then Equation (3.1.5) has a $T-$ periodic solution.*

Theorem B [5]: *Let the following conditions hold*

a) $f(t+T,X) = f(t,X)$ for all t and some $T > 0$,

b) all solutions of Equation (3.1.4) are bounded,

c) each solution of Equation (3.1.4) is equi-asymptotically stable,

d) the zero solution of the homogeneous system corresponding to Equation (3.1.4) is uniformly asymptotically stable (UAS).

Then Equation (3.1.4) has a globally stable T-periodic solution.

3.2 Formulation of Results

The main result of this paper is given below as

Theorem 3.2.1:*Let g, and h be continuous and also periodic with period ω together with the following conditions:*

(i) $H_0 = \frac{h(x)-h(0)}{x} \leq \alpha \in I_0, x \neq 0$ and $h(0) = 0$,

(ii) $G_0 = \frac{g(y)-g(0)}{y} \leq \beta, y \neq 0$ and $g(0) = 0$,

(ii) $a(t), b(t)$ continuous with $0 < a_0 < a \leq a(t) \leq a_1, 0 < b_0 < b \leq b(t) \leq b_1$

and

(iv) $|p(t;x,y)| \leq M \quad (M \quad constant)$.

Then Equation (3.1.1) has a globally stable ω-periodic solution.

Notations: Throughout this paper $K, K_0, K_1, \dots K_{12}$ will denote finite positive constants. $K_i's$ are not necessarily the same for each time they occur, but each $K_i, i = 1, 2...$ retains its identity throughout.

3.3 The Function V(t;x,y)

We shall use besides Equation (3.1.1) the function $V(t;x,y)$ defined below to prove the main theorem of this paper.

Let

$$2V(t;x,y) = \frac{\delta}{a\alpha\beta}H(t)\left\{(\alpha ab+\beta^2)x^2+\frac{1}{a}y^2+2\beta xy\right\} \tag{3.3.1}$$

where $H(t)=\exp\left(-\int_0^t a(s)ds\right)$ where $a,b,\alpha,\beta,\delta>0$, and for all x,y.

Lemma 3.3.1 *Subject to the assumptions of Theorem 3.2.1 there exist positive constants $K_i = K_i(a,b,\alpha,\beta,\delta), i=1,2$ such that*

$$K_1(x^2+y^2) \le V(t;x,y) \le K_2(x^2+y^2). \tag{3.3.2}$$

Proof: From the function V above it is clear that $V(t;0,0)\equiv 0$.

By rearranging Equation (3.3.1) we have

$$2V(t;x,y) = \frac{\delta}{a\alpha\beta}H(t)\left\{\alpha abx^2+\beta^2(x+\frac{1}{\beta}y)^2+\frac{1-a\beta^2}{a}y^2\right\} \tag{3.3.3}$$

$$2V(t;x,y) \ge \frac{\delta}{a\alpha\beta}H(t)\left\{\alpha abx^2+\frac{1-a\beta^2}{a}y^2\right\} \tag{3.3.4}$$

$$\ge K_1(x^2+y^2) \tag{3.3.5}$$

where

$$K_1=\frac{\delta}{a\alpha\beta}\cdot\min\left\{\alpha ab,\frac{1-a\beta^2}{a}\right\}$$

Therefore,

$$2V(t;x,y)\ge K_1(x^2+y^2).$$

Also from Equation (3.3.1), by using the inequality $xy\le\frac{1}{2}(x^2+y^2)$ we have

$$2V(t;x,y) \le \frac{\delta}{a\alpha\beta}H(t)\left\{(\alpha ab+\beta^2)x^2+\frac{1}{a}y^2+\beta(x^2+y^2)\right\} \tag{3.3.6}$$

Hence,

$$2V\le K_2(x^2+y^2). \tag{3.3.7}$$

where

$$K_2=\frac{\delta}{a\alpha\beta}\cdot\max\left\{(\alpha ab+\beta(\beta+1)),(\frac{1+a\beta}{a})\right\}$$

From inequalities (3.3.5) and (3.3.7), we have

$$K_1(x^2 + y^2) \le V(t; x, y) \le K_2(x^2 + y^2). \tag{3.3.8}$$

This proves Lemma 3.3.1.

Lemma 3.3.2: *Subject to the assumptions of Theorem 3.2.1 there exist positive constants $K_j = K_j(a, b, \alpha, \beta, \delta)(j = 3, 4)$ such that for any solution (x, y) of system (3.1.3),*

$$\dot{V}\mid_{(3.1.3)} \equiv \frac{d}{dt}V\mid_{(3.1.3)}(t; x, y) \le -K_3(x^2 + y^2) + K_4(|x| + |y|)\,|p(t; x, y)|\,. \tag{3.3.9}$$

Proof: From Equation (3.1.1) and Equation (3.1.3) we have,

$$\dot{V}\mid_{(3.1.3)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}.$$

$$= -H(t)R(x, y) + H(t)\left(\frac{\partial V}{\partial x}y + \frac{\partial V}{\partial y}(-ag(y) - bh(x) + p(t))\right)$$

where $R(x, y) = \left\{(\alpha ab + \beta^2)x^2 + \frac{1}{a}y^2 + 2\beta xy\right\}$

$$\dot{V}(t; x, y) \le -\frac{\delta}{a\alpha\beta}H(t)\left\{R(x, y) + K_2(x^2 + y^2) - K_*(|x| + |y|)p(t; x, y)\right\}, \tag{3.3.10}$$

where $K_* = \max(b\alpha, a\beta)$ and K_2 is as defined in the Equation (3.3.8)

By the definition of $H(t)$ we have that inequality (3.3.10) reduces to

$$\dot{V}(t; x, y) \le -K_3(x^2 + y^2) + K_*(|x| + |y|)p(t; x, y), \tag{3.3.11}$$

with $K_3 = 2k_2$

Inequality (3.3.11) can also be simplified and given as

$$\dot{V}(t; x, y) \le -K_3(x^2 + y^2) + K_4(x^2 + y^2)^{\frac{1}{2}}p(t; x, y), \tag{3.3.12}$$

with $K_4 = \sqrt{2}k_*$

This completes the proof of Lemma 3.3.2.

3.4 Proof of the main results

We shall now give the proof of the main theorem stated in Section 2 of this paper.

Proof of Theorem 3.2.1: From Lemmas 3.3.1 and 3.3.2 it had been established that the function $V(t;x,y)$ is a Lyapunov function for the system (3.1.3). Hence, the trivial solution of system (3.1.3) is asymptotically stable.

From the inequality (3.3.12),

$$\dot{V}(t;x,y) \leq -K_3(x^2+y^2) + K_4(x^2+y^2)^{\frac{1}{2}}p(t;x,y),$$

also from inequality (3.3.5), we have

$$(x^2+y^2)^{\frac{1}{2}} \leq \left(\frac{2V}{K_1}\right)^{\frac{1}{2}}.$$

Thus inequality (3.3.12) becomes

$$\frac{dV}{dt} \leq -K_6V + K_7V^{\frac{1}{2}}|p(t)| \tag{3.4.1}$$

It should be noted that $K_3(x^2+y^2) = K_3 \cdot \frac{V}{K_1}$ and

$$\frac{dV}{dt} \leq -K_6V + K_7V^{\frac{1}{2}}|p(t)| \tag{3.4.2}$$

where $K_6 = \frac{K_3}{K_2}$ and $K_7 = \frac{K_5}{K_2^{\frac{1}{2}}}$.

This implies that

$$\dot{V} \leq -K_6V + K_7V^{\frac{1}{2}}|p(t)|$$

and this can also expressed as

$$\dot{V} \leq -2K_8V + K_7V^{\frac{1}{2}}|p(t)| \tag{3.4.3}$$

with $K_8 = \frac{1}{2}K_6$.

Therefore

$$\dot{V} + K_8V \leq -K_8V + K_7V^{\frac{1}{2}}|p(t)| \tag{3.4.4}$$

$$\leq K_7V^{\frac{1}{2}}\left\{|p(t)| - K_9V^{\frac{1}{2}}\right\}, \tag{3.4.5}$$

where $K_9 = \frac{K_8}{K_7}$.

Thus inequality (3.4.5) becomes

$$\leq K_7V^{\frac{1}{2}}V^* \tag{3.4.6}$$

where

$$V^* = |p(t)| - K_9V^{\frac{1}{2}} \tag{3.4.7}$$

$$\leq V^{\frac{1}{2}}|p(t)|$$

$$\leq |p(t)|. \quad (3.4.8)$$

When $|p(t)| \leq K_9 V^{\frac{1}{2}}$,

$$V^* \leq 0 \quad (3.4.9)$$

and when $|p(t)| \geq K_9 V^{\frac{1}{2}}$,

$$V^* \leq |p(t)| \cdot \frac{1}{K_9}. \quad (3.4.10)$$

Substituting inequality (3.4.9) into (3.4.5), we have,

$$\dot{V} + K_8 V \leq K_{10} V^{\frac{1}{2}} |p(t)|$$

where

$$K_{10} = \frac{K_7}{K_9}.$$

This implies that

$$V^{-\frac{1}{2}}\dot{V} + K_8 V^{\frac{1}{2}} \leq K_{10}|p(t)|. \quad (3.4.11)$$

Multiplying both sides of (3.4.11) by $e^{\frac{1}{2}K_8 t}$ we have,

$$e^{\frac{1}{2}K_8 t}\left\{V^{-\frac{1}{2}}\dot{V} + K_8 V^{\frac{1}{2}}\right\} \leq e^{\frac{1}{2}K_8 t} K_{10}|p(t)| \quad (3.4.12)$$

i.e

$$2\frac{d}{dt}\left\{V^{\frac{1}{2}} e^{\frac{1}{2}K_8 t}\right\} \leq e^{\frac{1}{2}K_8 t} K_{10}|p(t)|. \quad (3.4.13)$$

Integrating both sides of inequality (3.4.13) from t_0 to t, gives

$$\left\{V^{\frac{1}{2}} e^{\frac{1}{2}K_8 \gamma}\right\}_{t0}^{t} \leq \int_{t0}^{t} \frac{1}{2} e^{\frac{1}{2}K_8 \tau} K_{10}|p(\tau) d\tau| \quad (3.4.14)$$

which implies that

$$\left\{V^{\frac{1}{2}}(t)\right\} e^{\frac{1}{2}K_8 t} \leq V^{\frac{1}{2}}(t_0) e^{\frac{1}{2}K_8 t_0} + \frac{1}{2}K_{10}\int_{t0}^{t} |p(\tau)| e^{\frac{1}{2}K_8 \tau} d\tau,$$

or

$$V^{\frac{1}{2}}(t) \leq e^{-\frac{1}{2}K_8 t}\left\{V^{\frac{1}{2}}(t_0) e^{\frac{1}{2}K_8 t_0} + \frac{1}{2}K_{10}\int_{t0}^{t} |p(\tau)| e^{\frac{1}{2}K_8 \tau} d\tau\right\}.$$

Using inequalities (3.3.5) and (3.3.7) we have

$$K_1(x^2(t) + \dot{x}^2(t)) \leq e^{-\frac{1}{2}K_8 t}\left\{K_2(x^2(t_0) + \dot{x}^2(t_0)) e^{\frac{1}{2}K_8 t_0} + \frac{1}{2}K_{10}\int_{t0}^{t} |p(\tau)| e^{\frac{1}{2}K_8 \tau} d\tau\right\}^2 \quad (3.4.15)$$

for all $t \geq t_0$ Thus,

$$x^2(t)+\dot{x}^2(t) \leq \frac{1}{K_1}\left\{e^{-\frac{1}{2}K_8 t}\left\{K_2(x^2(t_0)+\dot{x}^2(t_0))e^{\frac{1}{2}K_8 t_0}+\frac{1}{2}K_{10}\int_{t0}^{t}|p(\tau)|\,e^{\frac{1}{2}K_8\tau}d\tau\right\}^2\right\}$$

$$\leq \left\{e^{-\frac{1}{2}K_8 t}\left\{A_1+A_2\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_8\tau}d\tau\right\}^2\right\}. \qquad (3.4.16)$$

By substituting $K_8 = \mu$ in the inequality (3.4.16), we have

$$x^2(t)+\dot{x}^2(t) \leq e^{-\frac{1}{2}\mu t}\left\{A_1+A_2\int_{t_0}^{t}|p(\tau)|\,e^{\frac{1}{2}\mu\tau}d\tau\right\}^2. \qquad (3.4.17)$$

Hence, the completion of the proof.

Remark: From the proof of the theorem the following can be pointed out as the direct consequence of the theorem.

Corollary 3.4.1: *If $p(t;x,y) = 0$, inequality (3.4.17) reduces to*

$$x^2(t)+\dot{x}^2(t) \leq e^{-\frac{1}{2}\mu t}A_1,$$

and as $t \longrightarrow \infty$, $x^2(t)+\dot{x}^2(t) \longrightarrow 0$ which implies that the trivial solution of the system (3.1.3) or better say Equation (3.1.1) is globally asymptotically stable.

Corollary 3.4.2 : *If $p(t;x,y) \leq (|x|+|y|)\phi(t)$ where $\phi(t)$ is a non-negative and continuous function of t and satisfies $\int_0^t \phi(s)ds \leq M < \infty$ and M, a positive constant.*

Then, there exists a constant K_0 which depends on M, K_1, K_2 and t_0 such that every solution $x(t)$ of Equation (3.1.1) satisfies

$$|x(t)| \leq K_0, \quad |\dot{x}(t)| \leq K_0$$

for sufficiently large t.

Below is the sketch of the proof of the corollary 3.4.2

From the inequality (3.3.12) we have that

$$\dot{V} \leq K_4(|x|+|y|)^2\phi(t) \qquad (3.4.18)$$

By using the inequality $|x|\,|y| \leq \frac{1}{2}(x^2+y^2)$ on inequality (3.4.18), we have

$$\dot{V} \leq K_{11}(x^2+y^2)\phi(t) \tag{3.4.19}$$

where $K_{11} = 2K_4$

From the inequalities (3.3.5) and (3.4.19) we have,

$$\dot{V} \leq K_{11}V\phi(t). \tag{3.4.20}$$

Integrating equation (3.4.20) from 0 to t, we obtain

$$V(t) - V(0) \leq K_{12}\int_0^t V(s)\phi(s)ds. \tag{3.4.21}$$

where $K_{12} = \frac{K_{11}}{K_1} = \frac{3K_4}{K_1}$

The inequality (3.4.21) now becomes,

$$V(t) \leq V(0) + K_{12}\int_0^t V(s)\phi(s)ds \tag{3.4.22}$$

By Grownwall-Bellman inequality (3.4.22) yields

$$V(t) \leq V(0)e^{\left(K_{12}\int_0^t \phi(s)ds\right)}. \tag{3.4.23}$$

This proves the corollary.

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Chapter 4

ON THE BOUNDEDNESS AND THE STABILITY OF SOLUTION OF THIRD ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

[1]

Abstract:

In this paper we investigate the global asymptotic stability, boundedness as well as the ultimate boundedness of solutions of a general third order nonlinear differential equation by the use of complete Lyapunov function.

Key words and Phrases: complete Lyapunov function, global asymptotic stability, third order non-linear differential equations.

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4.1 Introduction

The concept of stability as well as the boundedness of solution is a very important one in the theory and applications of differential equations. It is also established so far that the most effective method to study these concepts (especially stability) for non-linear differential equations is the Lyapunov second method.
Consider the equation

$$\dddot{x} + \varphi(x, \dot{x})\ddot{x} + f(x, \dot{x}) = p(t; x, \dot{x}, \ddot{x}), \tag{4.1.1}$$

where φ, f and p are continuous and depend on the arguments displayed explicitly. In addition, they are such that existence, uniqueness and continuous dependence on initial condition are guaranteed.
Boundedness and stability properties of solutions for various form of the equation (4.1.1) had received a considerable amount of attention. Many of these are summarized in [9]. In [6-7], 2 variants or classes of Equation (4.1.1) were considered. Also in [8], the author re-visited the problem of Barbashin [2] where the equation above was considered. Barbashin [2], came up with interesting result on the equation since the equation considered was a general third order nonlinear differential equation. His results could not handle some of the special cases (or variants) of the equation as we have in [4-8]. In an attempt to have result that could handle and accommodate almost all the classes (and variants) of the equation (4.1.1), Qian [8], came up with results which simplified the theory of Barbashin and thereby making the result applicable to wide class or form of the equation (4.1.1).
Our aim in this paper is to further give simplification to the theorem of Barbashin [2] and Quian [8] by extending results in [6] and [7] to discuss the boundedness and ultimately boundedness of the solution of Equation (4.1.1) on a real line.
As in [6] and [7] Equation (4.1.1) is better handled as a system of three-coupled first order equations by letting;

$$\begin{aligned}
\dot{x} &= y \\
\dot{y} &= z \\
\dot{z} &= -\varphi(x, y)z - f(x, y) + p(t; x, \dot{x}, \ddot{x})
\end{aligned} \tag{4.1.2}$$

In this study, we shall use a single complete Lyapunov function to achieve our result. We shall for expository reasons give the following definitions:

Definition 4.1.1: *Let*

$$\dot{X} = f(t, X) \tag{4.1.3}$$

be a system of n-first order differential equations, a Lyapunov function V *defined as* $V : I \times \Re^n \to \Re$ *is said to be COMPLETE if for* $X \in \Re^n$
(i) $V(t, X) \geq 0$
(ii) $V(t, X) = 0$, *if and only if* $X = 0$ *and*
(iii) $\dot{V}|_{4.1.2}(t, X) \leq -c|X|$ *where c is any positive constant and* $|X|$ *given by*
$|X| = \left(\sum_{i=1}^{n}(x_i^2)\right)^{\frac{1}{2}}$ *such that* $|X| \to \infty$ *as* $X \to \infty$

Definition 4.1.2: *A Lyapunov function* V *defined as* $V : I \times \Re^n \to \Re$ *is said to be INCOMPLETE if for* $X \in \Re^n$ *(i) and (ii) of the above definition is satisfied and in addition*
(iii) $\dot{V}(t, X)|_{4.1.2} \leq -c|X|_*$ *where c is any positive constant and* $|X|_*$ *given by*
$|X|_* = \left(\sum_{i=1}^{j}(x_i^2)\right)^{\frac{1}{2}}$, *where by* j $(i \leq j < n)$ *we mean that not all the variables (otherwise called the trajectories) are necessarily involved such that* $|X|_* \to \infty$ *as* $X \to \infty$.
The particular case according to this work is when $n = 3$.

4.2 Formulation of Results

We will consider Equation (4.1.1) in two major ways and have the following theorems to prove.
Case 1: When $p(t, x, \dot{x}, \ddot{x}) \equiv 0$ This may be considered as the homogeneous case.
Theorem 4.2.1: *Let* f, *and* φ *be continuous and let* $I_0 = [\delta, J]$ *where* $J = \beta\kappa\epsilon(1 - \epsilon)$, δ, β, κ *and* ϵ *are positive constants. In addition, let the following conditions hold:*
(i) $f_x = \frac{f(x,y) - f(0,y)}{x} \in I_0 = \alpha, x \neq 0$,

(ii) $f_y = \frac{f(x,y)-f(x,0)}{y} \in I_0 = \beta, y \neq 0,$

(iii) $f(0,y) = f(x,0) = 0$ *and*

(iv)$|\varphi(x,y)| \leq \kappa.$

Then the trivial solution of Equation (4.1.1) is globally asymptotically stable.

Case 2: When $p \neq 0$ The non-homogeneous case

Theorem 4.2.2: *Suppose the conditions of the Theorem 4.2.1 are satisfied, and in addition* $|p(t;x,\dot{x},\ddot{x})| \leq A$, *(A is a positive constant) then there exists a constant* μ $(0 < \mu < \infty)$, *depending only on* β, δ *and* κ *such that every solution of Equation (4.1.1) satisfies*

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) \leq e^{-\frac{1}{2}\mu t}\left\{A_1 + A_2\int_{t_0}^{t}|P(\tau)|\,e^{\frac{1}{2}\mu\tau}d\tau\right\}^2$$

for all $t \geq t_0$, *where the constant* $A_1 > 0$, *depends on* β, δ, κ *as well as on* $t_0, x(t_0), \dot{x}(t_0)$ *and* $\ddot{x}(t_0)$; *and the constant* $A_2 > 0$ *depends on* β, δ *and* κ *only.*

Theorem 4.2.3: *Following the assumptions of Theorem 4.2.2 and taking* $|p(t;x,\dot{x},\ddot{x})| = (|x|+|y|+|z|)\phi(t)$, *where* $\phi(t)$ *is a non-negative and continuous function of* t *and satisfies* $\int_0^t \phi(s)ds \leq M < \infty$ *and* M, *a positive constant.*

Then, there exists a constant K_0 *which depends on* M, K_1, K_2 *and* t_0 *such that every solution* $x(t)$ *of Equation (4.1.1) satisfies*

$$|x(t)| \leq K_0, \quad |\dot{x(t)}| \leq K_0, \quad |\ddot{x(t)}| \leq K_0$$

for sufficiently large t.

Notations: Throughout this paper $K, K_0, \ldots K_{11}$ will denote finite positive constants whose magnitudes depend only on the functions ϕ, f and P as well as constants a, κ, β, Δ and δ but are independent of solutions of Equation (4.1.1). $K_i's$ are not necessarily the same for each time they occur, but each $K_i, i = 1, 2...$ retains its identity throughout.

4.3 The Function $V(x, y, z)$

The main tool in the proof of the theorems is the function $V = V(x, y, z)$ which we obtained below after some lengthy algebraic computations

$$\begin{aligned} 2V &= \tfrac{a\delta}{\Delta}\left\{[\beta^2(1-\epsilon)^2]x^2 + \{(1-\epsilon)[\kappa^2 - \beta(1-\epsilon)] + \beta\}\, y^2 + z^2 \right. \\ &\quad \left. + 2\kappa\beta(1-\epsilon)^2 xy + 2(1-\epsilon)^2\beta xz + 2\kappa(1-\epsilon)yz\right\} \end{aligned} \tag{4.3.1}$$

where $a, \beta, \epsilon, \Delta, \kappa$ and δ are all positive for all x, y, z. with $\delta > 1$ and $\Delta = \alpha\beta(\delta-1)(1-\epsilon)^2$.

The following lemma are to prove that $V(x, y, z)$ is indeed a Lyapunov function.

Lemma 4.3.1 *Subject to the assumptions of Theorem 4.2.1 there exist positive constants* $K_i = K_i(a, \beta, \epsilon, \Delta, \kappa, \delta), i = 1, 2$ *such that*

$$K_1(x^2 + y^2 + z^2) \le V(x, y, z) \le K_2(x^2 + y^2 + z^2). \tag{4.3.2}$$

Proof: Clearly, $V(0, 0, 0) = 0$.

By rearranging Equation (4.3.1) we have

$$\begin{aligned} 2V &= \tfrac{a\delta}{\Delta}\left\{[\beta(1-\epsilon)x + \kappa(1-\epsilon)y + z]^2 + \beta^2(1-\epsilon)^2x^2 + \epsilon[(1-\epsilon)\kappa + \beta\epsilon]y^2 \right. \\ &\quad \left. - \epsilon\beta(1-\epsilon)xz\right\}, \end{aligned} \tag{4.3.3}$$

$$\begin{aligned} 2V &= \tfrac{a\delta}{\Delta}\left\{[\beta(1-\epsilon)x + \kappa(1-\epsilon)y + z]^2 + \beta^2\epsilon(1-\epsilon)^2x^2 - \beta\epsilon(1-\epsilon)(x + \tfrac{1}{2}z)^2 \right. \\ &\quad \left. + \epsilon[\kappa(1-\epsilon) + \beta\epsilon]y^2 + \beta\tfrac{\epsilon^2(1-\epsilon)}{4}z^2\right\}, \end{aligned} \tag{4.3.4}$$

and from Equation (4.3.4) we obtain

$$2V \ge \tfrac{a\delta}{\Delta}\left\{\beta^2\epsilon(1-\epsilon)^2x^2 + \epsilon[\kappa(1-\epsilon) + \beta\epsilon]y^2 + \beta\tfrac{\epsilon^2(1-\epsilon)}{4}z^2\right\} \tag{4.3.5}$$

$$\ge K_1(x^2 + y^2 + z^2), \tag{4.3.6}$$

where

$$K_1 = \frac{a\delta}{2\Delta} \cdot min\left\{\beta^2\epsilon(1-\epsilon)^2, \epsilon[\kappa(1-\epsilon) + \beta\epsilon], \beta\frac{\epsilon^2(1-\epsilon)}{4}\right\}.$$

Using the inequality,

$$xy \le \frac{1}{2}(x^2 + y^2),$$

Equation (4.3.1) becomes,

$$\begin{aligned} 2V &\le \tfrac{a\delta}{\Delta}\left\{[\beta^2(1-\epsilon)^2]x^2 + \{(1-\epsilon)[\kappa^2 - \beta(1-\epsilon)] + \beta\}\, y^2 + z^2 \right. \\ &\quad \left. + \kappa\beta(1-\epsilon)^2(x^2 + y^2) + (1-\epsilon)^2\beta(x^2 + z^2) + \kappa(1-\epsilon)(y^2 + z^2)\right\}. \end{aligned} \tag{4.3.7}$$

$$\begin{aligned} 2V &\leq \tfrac{a\delta}{\Delta}\left\{\beta^2\epsilon^2(1+\beta+\kappa)x^2 + (1-\epsilon)\left\{\kappa(\kappa+1)+\beta(1-\epsilon)(\kappa-1)\right\}y^2\right. \\ &\left. +\left\{1+(1-\epsilon)[\kappa+\beta(1-\epsilon)]\right\}z^2\right\}, \end{aligned} \tag{4.3.8}$$

which reduces to

$$V \leq K_2(x^2 + y^2 + z^2), \tag{4.3.9}$$

with

$$\begin{aligned} K_2 &= \tfrac{\delta}{2\Delta}\cdot max\left\{\beta^2\epsilon^2(1+\beta+\kappa), (1-\epsilon)\left\{\kappa(\kappa+1)+\beta(1-\epsilon)(\kappa-1)\right\}\right. \\ &\left.\left\{1+(1-\epsilon)[\kappa+\beta(1-\epsilon)]\right\}\right\}. \end{aligned}$$

Combining the inequalities (4.3.6) and (4.3.9) we have

$$K_1(x^2 + y^2 + z^2) \leq V(x, y, z) \leq K_2(x^2 + y^2 + z^2). \tag{4.3.10}$$

which proves the Lemma 4.3.1.

Lemma 4.3.2: Suppose that the conditions of Theorem 2.1 hold, then there are positive constants $K_3 = K_3(a, \Delta, \delta)$ such that for any solution (x, y, z) of system (4.1.2),

$$\dot{V}|_{(4.1.2)} \equiv \tfrac{d}{dt}V|_{(4.1.2)}(x, y, z) \leq -K_3(x^2 + y^2 + z^2). \tag{4.3.11}$$

Proof: From Equations (4.1.1) and the system (4.1.2) we have,

$$\dot{V}|_{(4.1.2)} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z}.$$

$$= \tfrac{\partial V}{\partial x}y + \tfrac{\partial V}{\partial y}z + \tfrac{\partial V}{\partial z}(-\varphi(x, y)z - f(x, y)),$$

which gives

$$\begin{aligned} \dot{V} &= \tfrac{a\delta}{\Delta}\left\{[\beta^2(1-\epsilon)^2]xy + \left\{(1-\epsilon)[\kappa^2-\beta(1-\epsilon)]+\beta\right\}yz + z - \varphi(x,y)z - f(x,y)\right) \\ &+\kappa\beta(1-\epsilon)^2[y^2+xz] + (1-\epsilon)^2\beta[yz + x(\varphi(x,y)z - f(x,y)))]+ \\ &\kappa(1-\epsilon)[z^2 + y(\varphi(x,y)z - f(x,y)))]\right\}. \end{aligned} \tag{4.3.12}$$

Then by the conditions on $f(x, y)$, i.e $\frac{f(x,y)-f(0,y)}{x} = f_x$, and $\frac{f(x,y)-f(x,0)}{y} = f_y$, and after much simplification we have

$$\dot{V} = -\frac{a\delta}{\Delta}\left\{x^2 + y^2 + z^2\right\}. \tag{4.3.13}$$

Let $K_3 \leq \frac{a\delta}{\Delta}$

Then

$$\dot{V} \leq -K_3(x^2 + y^2 + z^2).$$

This completes the proof of the Lemma 4.3.2.

Lemma 4.3.3: *Suppose that the conditions of Theorem 4.2.2 hold, then there are positive constants* $K_j = K_j(a, \beta, \epsilon, \kappa, \Delta, \delta)(j = 4, 5)$ *such that for any solution* (x, y, z) *of system* (4.1.2),

$$\dot{V}|_{(4.1.2)} \equiv \frac{d}{dt}V|_{(4.1.2)}(x, y, z) \leq -K_4(x^2 + y^2 + z^2) + K_5(|x| + |y| + |z|)\left|p(t; x, \dot{x}, \ddot{x})\right|. \tag{4.3.14}$$

Proof: Following the same arguments as in Lemma 4.3.2 but this time with $p \neq 0$ set $p(t; x, y, z) = P(t)$ we have that

$$\begin{aligned}\dot{V} &= \tfrac{a\delta}{\Delta}\{[\beta^2(1-\epsilon)^2]xy + \{(1-\epsilon)[\kappa^2 - \beta(1-\epsilon)] + \beta\}yz + z - \varphi(x, y)z - f(x, y) \\ &+P(t) + \kappa\beta(1-\epsilon)^2[y^2 + xz] + (1-\epsilon)^2\beta[yz + x(\varphi(x, y)z - f(x, y) + P(t))] \\ &+\kappa(1-\epsilon)[z^2 + y(\varphi(x, y)z - f(x, y) + P(t))]\}.\end{aligned} \tag{4.3.15}$$

Also by the conditions on $f(x, y)$ and $\varphi(x, y)$

$$\dot{V} = -\frac{a\delta}{\Delta}\left\{x^2 + y^2 + z^2 - (1 - \epsilon^2\beta)x + \kappa(1-\epsilon)y + z)P(t)\right\}. \tag{4.3.16}$$

$$\leq -\frac{a\delta}{\Delta}\left\{x^2 + y^2 + z^2 - K_4(|x| + |y| + |z|)P(t)\right\}, \tag{4.3.17}$$

where $K_4 = max((1-\epsilon)^2\beta, \kappa(1-\epsilon), 1)$

$$\leq -K_3(x^2 + y^2 + z^2) + K_5(|x| + |y| + |z|)\left|P(t)\right|, \tag{4.3.18}$$

where $K_5 = \frac{K_4 a\delta}{\Delta}$.

Since

$$(|x| + |y| + |z|) \leq \sqrt{3}(x^2 + y^2 + z^2)^{\frac{1}{2}},$$

the inequality (4.3.19) becomes

$$\frac{dV}{dt} \leq -K_4(x^2 + y^2 + z^2) + K_6(x^2 + y^2 + z^2)^{\frac{1}{2}}\left|P(t)\right|, \tag{4.3.20}$$

where $K_6 = \sqrt{3}K_5$ and $K_4 = K_3$.

This completes the proof of the Lemma 4.3.3.

From the proofs of the lemmas it is established that the function $V(x, y, z)$ is a Lyapunov function.

4.4 Proof of the main results

We shall give the proofs of the Theorems stated in Section 2 of this paper.

Proof of Theorem 4.2.1: From the proof of the Lemma 4.3.1 and Lemma 4.3.2 it is established that the trivial solution of Equation (4.1.1) is globally asymptotically stable. i.e every solution $(x(t), \dot{x}(t), \ddot{x}(t))$ of the system (4.1.2) satisfies $x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t) \longrightarrow 0$ as $t \longrightarrow \infty$

Proof of Theorem 4.2.2: Indeed from the inequality (4.3.20),

$$\frac{dV}{dt} \leq -K_3(x^2+y^2+z^2) + K_6(x^2+y^2+z^2)^{\frac{1}{2}} \left|P(t)\right|,$$

and also from the inequality (4.3.6), we have

$$(x^2+y^2+z^2)^{\frac{1}{2}} \leq \left(\frac{2V}{K_1}\right)^{\frac{1}{2}}.$$

Thus the inequality (4.3.20) becomes

$$\frac{dV}{dt} \leq -K_7 V + K_8 V^{\frac{1}{2}} \left|P(t)\right|. \tag{4.4.1}$$

We note that

$K_4(x^2+y^2+z^2) = K_4 \cdot \frac{V}{K_1}$

and

$$\frac{dV}{dt} \leq -K_7 V + K_8 V^{\frac{1}{2}} \left|P(t)\right| \tag{4.4.2}$$

where $K_7 = \frac{K_4}{K_2}$ and $K_8 = \frac{K_6}{K_2^{\frac{1}{2}}}$.

These imply that

$$\dot{V} \leq -K_7 V + K_8 V^{\frac{1}{2}} \left|P(t)\right|$$

and this can be written as

$$\dot{V} \leq -2K_9 V + K_8 V^{\frac{1}{2}} \left|P(t)\right|, \tag{4.4.3}$$

where $K_9 = \frac{1}{2} K_7$.

Therefore

$$\dot{V} + K_9 V \leq -K_9 V + K_8 V^{\frac{1}{2}} \left|P(t)\right| \tag{4.4.4}$$

$$\leq K_8 V^{\frac{1}{2}} \left\{ |P(t)| - K_{10} V^{\frac{1}{2}} \right\}, \tag{4.4.5}$$

where $K_{10} = \frac{K_9}{K_8}$.

Thus the inequality (4.4.5) becomes

$$\dot{V} + K_9 V \leq K_8 V^{\frac{1}{2}} V^* \tag{4.4.6}$$

where

$$V^* = |P(t)| - K_{10} V^{\frac{1}{2}} \tag{4.4.7}$$

$$\leq V^{\frac{1}{2}} |P(t)|$$

$$\leq |P(t)|. \tag{4.4.8}$$

When $|P(t)| \leq K_{10} V^{\frac{1}{2}}$,

$$V^* \leq 0 \tag{4.4.9}$$

and when $|P(t)| \geq K_{10} V^{\frac{1}{2}}$,

$$V^* \leq |P(t)| \cdot \frac{1}{K_{10}}. \tag{4.4.10}$$

On substituting the inequality (4.4.9) into the inequality (4.4.5), we have,

$$\dot{V} + K_9 V \leq K_{11} V^{\frac{1}{2}} |P(t)|$$

where

$$K_{11} = \frac{K_8}{K_{10}}.$$

This implies that

$$V^{-\frac{1}{2}} \dot{V} + K_9 V^{\frac{1}{2}} \leq K_{11} |P(t)|. \tag{4.4.11}$$

Multiplying both sides of the inequality (4.4.11) by $e^{\frac{1}{2}K_9 t}$ we have,

$$e^{\frac{1}{2}K_9 t} \left\{ V^{-\frac{1}{2}} \dot{V} + K_9 V^{\frac{1}{2}} \right\} \leq e^{\frac{1}{2}K_9 t} K_{11} |P(t)| \tag{4.4.12}$$

i.e

$$2\frac{d}{dt} \left\{ V^{\frac{1}{2}} e^{\frac{1}{2}K_9 t} \right\} \leq e^{\frac{1}{2}K_9 t} K_{11} |P(t)|. \tag{4.4.13}$$

Integrating both sides of inequality (4.4.13) from t_0 to t, gives

$$\left\{ V^{\frac{1}{2}} e^{\frac{1}{2}K_9 \gamma} \right\}_{t0}^{t} \leq \int_{t0}^{t} \frac{1}{2} e^{\frac{1}{2}K_9 \tau} K_{11} |P(\tau) d\tau| \tag{4.4.14}$$

which implies that

$$\left\{V^{\frac{1}{2}}(t)\right\}e^{\frac{1}{2}K_9t} \leq V^{\frac{1}{2}}(t_0)e^{\frac{1}{2}K_9t_0} + \frac{1}{2}K_{11}\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_9\tau}d\tau,$$

or

$$V^{\frac{1}{2}}(t) \leq e^{-\frac{1}{2}K_9t}\left\{V^{\frac{1}{2}}(t_0)e^{\frac{1}{2}K_9t_0} + \frac{1}{2}K_{11}\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_9\tau}d\tau\right\}.$$

Using inequalities (4.3.9) and (4.3.10) we have

$$K_1(x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)) \leq e^{-\frac{1}{2}K_9t}\left\{K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))e^{\frac{1}{2}K_9t_0} + \frac{1}{2}K_{11}\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_9\tau}d\tau\right\}^2 \tag{4.4.15}$$

for all $t \geq t_0$.

Thus,

$$\begin{aligned} x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t) &\leq \tfrac{1}{K_1}\left\{e^{-\frac{1}{2}K_9t}\left\{K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))e^{\frac{1}{2}K_9t_0}\right.\right. \\ &\qquad \left.\left.+\tfrac{1}{2}K_{11}\textstyle\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_9\tau}d\tau\right\}^2\right\} \\ &\leq \left\{e^{-\frac{1}{2}K_9t}\left\{A_1 + A_2\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_9\tau}d\tau\right\}^2\right\} \end{aligned} \tag{4.4.16}$$

where A_1 and A_2 are constants depending on $\{K_1, K_2, (x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))\}$ and $\{K_1, K_{11}\}$ respectively.

By substituting $K_9 = \mu$ in the inequality (4.4.16), we have

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t) \leq \left\{e^{-\frac{1}{2}\mu t}\left\{A_1 + A_2\int_{t0}^{t}|P(\tau)|\,e^{\frac{1}{2}\mu\tau}d\tau\right\}^2\right\},$$

which completes the proof.

Proof of Theorem 4.2.3: From the function V defined above and the conditions of Theorem 4.2.3, the conclusion of Lemma 4.3.1 can be obtained, as

$$V \geq K_1\left(x^2+y^2+z^2\right), \tag{4.4.17}$$

and since $P \neq 0$ we can revise the conclusion of Lemma 4.3.2, i.e,

$$\dot{V} \leq -K_4(x^2+y^2+z^2) + K_5(|x|+|y|+|z|)\,|P(t)|\,,$$

and we obtain

$$\dot{V} \leq K_5(|x|+|y|+|z|)^2 r(t). \tag{4.4.18}$$

By using the inequality $|x|\,|y| \leq \frac{1}{2}(x^2+y^2)$, on inequality (4.4.18), we have

$$\dot{V} \leq K_{11}(x^2+y^2+z^2)r(t), \tag{4.4.19}$$

where $K_{11} = 3K_5$.

From inequalities (4.4.17) and (4.4.19) we have,

$$\dot{V} \leq K_{11}Vr(t). \tag{4.4.20}$$

Integrating inequality (4.4.20) from 0 to t, we obtain

$$V(t) - V(0) \leq K_{12}\int_0^t V(s)r(s)ds. \tag{4.4.21}$$

where $K_{12} = \frac{K_{11}}{K_1} = \frac{3K_5}{K_1}$

Using the condition on $p(t;x,y,z)$ as stated in the Theorem 4.2.3 we have

$$V(t) \leq V(0) + K_{11}\int_0^t V(s)r(s)ds. \tag{4.4.22}$$

By Grownwall-Bellman inequality, the inequality (4.4.22) yields,

$$V(t) \leq V(0)e^{\left(K_{12}\int_0^t r(s)ds\right)}. \tag{4.4.23}$$

This completes the proof of Theorem 4.2.3.

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Chapter 5

BOUNDEDNESS AND STABILITY PROPERTIES OF SOLUTION TO CERTAIN FOURTH ORDER NON-LINEAR DIFFERENTIAL EQUATION

[1]

Abstract:

We give sufficient conditions for the existence of a stable (globally asymptotically stable), bounded and uniform ultimate bounded solution to a certain fourth order non-linear differential equation using a single complete Lyapunov function without the use of a signum function or any stringent condition on the nonlinear terms.The results include and improve some existing results in literature.

Keywords: Boundedness, complete Lyapunov function, fourth order non-linear differential equations, uniform ultimate boundedness, stability.

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5.1 Introduction

In this paper, we study the fourth order nonlinear differential equation

$$x^{(iv)} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = p(t), \tag{5.1.1}$$

where a and b are positive constants, the functions g, h and p are continuous in the respective argument displayed explicitly. Also dot means the derivative of the variable with respect to t.

Studies on the qualitative properties (boundedness, stability and periodicity) of solutions for higher order nonlinear differential equations have received considerable attention from several scholars who have obtained interesting results. Some of these results have been summarized in [11].

In [1], the authors employed the frequency domain method to investigate the boundedness of this class of equation.

In [9], the Cauchy formula for the particular solution of non-homogeneous linear differential equation was employed to achieve the results on boundedness of solution.

Other articles in this connection include Ezeilo [5-6], Harrow [7-8], Tiryaki and Tunc [13-14], Tunc [15-18], Tunc and Tiryaki [19] where the second method of Lyapunov was used. All these results in one way or the other generalize some results on third order nonlinear equations (see [2, 4, 10 and 12]).

In [18], the author gave criteria for the asymptotic stability and boundedness of solutions of certain class of the equation above by the use of an incomplete Lyapunov function (Yoshizawa [20]) and a stringent condition was placed on the nonlinear terms g and h which is necessary for these functions not only to be continuous but also be differentiable.

In continuation with the study in [9], we will consider Equation (5.1.1) with an equivalent system of equations

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -aw - bz - g(y) - h(x) + p(t), \end{aligned} \tag{5.1.2}$$

this time with the focus on the boundedness and stability properties of the solutions.

Since it has been established that the Lyapunov second method still remains one the most effective method to study these properties of solutions we shall in this paper give criteria for the existence of a unique solution to Equation (5.1.1) which is stable (globally asymptotically stable) and bounded (uniformly ultimately bounded) on the real line. This we shall achieve by the use of a single complete Lyapunov function without the use of any signum function and less restriction on the nonlinear terms g and h other than being continuous.

Even though there is no unique way of constructing a Lyapunov function, we shall adapt Cartwright's approach [3] for the construction of the Lyapunov function used in this work. In order to reach our main results, we shall first give some important basic definitions for the general non-autonomous differential equation. We consider the system

$$\dot{x} = f(t, x) \tag{5.1.3}$$

where $f \in C[I \times S_\rho], I = [0, \infty), t \geq 0$, and $S_\rho = \{x \in \Re^n : \|x\| < \rho\}$. Assume that f is smooth enough to ensure the existence and uniqueness of solutions of Equation (5.1.3) through every point $(t_0, x_0) \in J \times S_\rho$. Also, let $f(t, 0) = 0$ so that the system (5.1.3) admits the zero solution $x \equiv 0$.

Definition 5.1.1 [20]: *The solution $x(t) \equiv 0$ of Equation (5.1.1) is stable if for any $\epsilon > 0$ and any $t_0 \in I$ there exists a $\delta(t_0, \epsilon) < 0$ such that if $x_0 \in S_{\delta(t_0,\epsilon)}$ then $x(t; t_0, x_0) \in S_\epsilon$ for all $t \geq t_0$.*

Definition 5.1.2 [20]: *The solution $x(t) \equiv 0$ of Equation (5.1.1) is asymptotically stable in the whole (globally asymptotically stable) if it is stable and every solution of the Equation (5.1.1) tends to zero as $t \longrightarrow \infty$.*

Definition 5.1.3 [20]: *The solution $x(t) \equiv 0$ of Equation (5.1.1) is uniformly asymptotically stable if it is stable and there exists a $\delta(t_0) > 0$ such that $\|x(t; t_0, x_0)\| \longrightarrow 0$ as $t \longrightarrow \infty$ for all $x_0 \in S_{\delta_0}$.*

We shall also give the following definitions in our context:

Definition 5.1.4: *A Lyapunov function V defined as $V : I \times \Re^n \to \Re$ is said to be complete if for $X \in \Re^n$,*

(i) $V(t, X) \geq 0$

(ii) $V(t, X) = 0$, *if and only if* $X = 0$

and

(iii) $\dot{V}|_{5.1.3}(t, X) \leq -c\,|X|$ *where c is any positive constant and $|X|$ given by*

$|X| = \left(\sum_{i=1}^{n}(x_i^2)\right)^{\frac{1}{2}}$ *such that* $|X| \to \infty$ *as* $X \to \infty$.

Definition 5.1.5: *A Lyapunov function* V *defined as* $V : I \times \Re^n \to \Re$ *is said to be incomplete if for* $X \in \Re^n$, *conditions (i) and (ii) of Definition 5.1.4 is satisfied, and in addition*

(iii) $\dot{V}(t, X)|_{5.1.3} \leq -c\,|X|_*$ *where* c *is any positive constant and* $|X|_*$ *is given as*

$|X|_* = \left(\sum_{i=1}^{<n} x^2\right)^{\frac{1}{2}}$ *such that* $|X|_* \to \infty$ *as* $X \to \infty$.

5.2 Main Results

The following are the main results in this paper.

Case 1. $p \equiv 0$.

Theorem 5.2.1: *Let the functions* g *and* h *be continuous. Furthermore let the following conditions hold:*

(i) $H_0 = \frac{h(x)-h(0)}{x} \leq d \in I_0$, $x \neq 0$ *with* $I_0 = [\delta, \Delta]$, $d, \delta, \Delta > 0$, *and* I_0 *is the Routh Hurwitz interval.*

(ii) $G_0 = \frac{g(y)-g(0)}{y} \leq c \in I_0$, $y \neq 0, c > 0$,

(iii) $h(0) = g(0) = 0$.

Then the trivial solution of Equation (5.1.1) is globally asymptotically stable.

Case 2. $p \neq 0$

Theorem 5.2.2: *Let* p *be continuous and suppose the following conditions are satisfied:*

(i) Conditions(i)-(iii) of Theorem 5.2.1 hold; and

(ii)$|p(t)| \leq M$ *(constant)for all* $t \geq 0$,

then there exists a constant μ,$(0 < \mu < \infty)$ *depending only on* a, b, c, d,*and* δ *such that every solution of Equation (5.1.1) satisfies*

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) \leq e^{-\frac{1}{2}\mu t}\left\{A_1 + A_2\int_{t_0}^{t}|p(\tau)|\,e^{\frac{1}{2}\mu\tau}d\tau\right\}^2$$

for all $t \geq t_0$, *where the constant* $A_1 > 0$, *depends on* a, b, c, d, δ *as well as on* $t_0, x(t_0), \dot{x}(t_0), \ddot{x}(t_0), \dddot{x}(t_0)$; *and the constant* $A_2 > 0$ *depends on* a, b, c, d *and* δ.

We now consider the case when $p(t)$ in Equation (5.1.1) is replaced with $p(t, x, \dot{x}, \ddot{x}, \dddot{x})$.

Theorem 5.2.3: *Following the assumptions of Theorem 5.2.2 and condition (ii) replaced with $\left|p(t, x, \dot{x}, \ddot{x}, \dddot{x})\right| = (|x| + |y| + |z| + |w|)\, r(t)$, where $r(t)$ is a non negative and continuous function of t, and satisfies $\int_0^t r(s)ds \leq M < \infty$ and M, a positive constant. Then there exists a constant K_0 which depends on M, K_1, K_2 and t_0 such that every solution $x(t)$ of Equation (5.1.1) satisfies*

$$|x(t)| \leq K_0, \;\; |\dot{x}| \leq K_0, \;\; |\ddot{x}| \leq K_0, \;\; \left|\dddot{x}\,(t)\right| \leq K_0$$

for all sufficiently large t.

Remark: We wish to remark here that while the Theorem 5.2.1 is on the global asymptotic stability of the trivial solution, the Theorems 5.2.2 and 5.2.3 are dealing with the boundedness and ultimate boundedness of the solutions respectively.

The trivial solution of the corresponding linear equation to Equation (5.1.1) given as

$$x^{(iv)} + a\,\dddot{x} + b\ddot{x} + c\dot{x} + dx = p(t)$$

is asymptotically stable if the Routh-hurwitz condition $(ab - c) > 0, (ab - c)c - a^2 d > 0$ hold.

Notations: For the rest of the article $K, K_0, K_1, \ldots K_{14}$ stand for finite positive constants whose magnitudes depend only on the functions g, h and p as well as constants a, b, c, d and δ but are independent of solutions of Equation (5.1.1). $K_i's$ are not necessarily the same for each time they occur, but each $K_i, i = 1, 2...$ retains its identity throughout.

5.3 Preliminary Results

We shall use as a tool to prove our main results besides Equation (5.1.1), a function $V(x, y, z, w)$ defined by

$$2V(x, y, z, w) = Ax^2 + By^2 + Cz^2 + Dw^2 + 2Exy + 2Fxz + 2Gxw + 2Hyz + 2Iyw + 2Jzw, \tag{5.3.1}$$

where

$$A = \tfrac{a\delta}{\Delta}\left\{(b+d)(c^2+d^2)[d(1-ad)-c] + d^3[a(b^2+d^2)+L]\right\},$$
$$B = \tfrac{\delta}{\Delta}\left\{dL(abd+c) + a(b^2+d^2)[b(d-c)+cd] + [d(1-ad)-c][ad(b^2+c^2)\right.$$
$$\left. -cd^2(b+1)+a^2bc]\right\},$$
$$C = \tfrac{\delta}{\Delta}\left\{a(b^2+d^2)[d(1-ad+a^2c+d)-c] + d[c(a^2+b^2)-ab][d(1-ad)-c] + dL(a^2c+d)\right\},$$
$$D = \tfrac{cd\delta}{\Delta}\left\{L + ab^2 + (d-c) + ab[(1-ad)-c]\right\},$$
$$(5.3.2a)$$

$$E = \tfrac{ac\delta}{\Delta}\left\{d^2L + (b^2+d^2)(d-c)\right\},$$
$$F = \tfrac{cd\delta}{bd\Delta}\left\{d^2L + ad^2(b^2+d^2) + [b(a^2+d^2)+d^2][ab^2d^2[d(1-ad)-c]]\right\},$$
$$G = \tfrac{abc[d(1-ad-c)]\delta}{\Delta},$$
$$H = \tfrac{abcd\delta}{\Delta}\left\{a(b^2+d^2)+L\right\},$$
$$I = \tfrac{a\delta}{\Delta}\left\{d^2L + bd[d(1-ad)-c] + (b^2+d^2)(d-c)\right\}, \qquad (5.3.2b)$$
$$J = \tfrac{acd\delta}{\Delta}\left\{ab^2 + d - c + L\right\},$$
$$\Delta = abcd[d(1-ad)-c],$$
$$L = b[ad + c[c(b+1)-c]],$$

with a, b, c, d positive and $[d(1-ad)-c] > 0$.

Lemma 5.3.1 *Subject to the assumptions of Theorem 5.2.1 there exist positive constants* $K_i = K_i(a, b, c, d, \delta), i = 1, 2$ *such that*

$$K_1(x^2+y^2+z^2+w^2) \le V(x,y,z,w) \le K_2(x^2+y^2+z^2+w^2). \qquad (5.3.3)$$

Proof: Clearly $V(0,0,0,0) \equiv 0$.

By rearranging Equation (5.3.1) we have

$$2V(x,y,z,w) = \left(\tfrac{\delta}{\Delta}\right)\left\{a[d(1-ad)]\{b(cx+dy+w)^2 + d^2(y+b^3d^2x)^2 + b^2d(y+a^2bdx)^2\right.$$
$$\left. +acd(z+\tfrac{b^2d^3}{a}x)^2\right\} + dL\left\{(z+acx)^2 + ac^2(z+\tfrac{1}{a}w)^2 + c(y+\tfrac{ad}{c}w)^2 + ad^2(x+\tfrac{c}{d}y)^2\right.$$
$$\left. +abd(y+\tfrac{c}{d}z)^2\right\} + ad(b^2+d^2)\left\{ad^2(x+\tfrac{c(d-c)}{ad^3}y)^2 + a^2c(z+\tfrac{d}{a}x)^2 + \tfrac{c}{a(b^2+d^2)}(w+az)^2\right.$$
$$\left. +b(d-c)(y+\tfrac{w}{b})^2 + c(y+abz)^2\right\} + \left\{[d(1-ad)-c](ad(c^2+d^2)+abd^2)\right.$$
$$\left. -\tfrac{cd^3}{a}(b^2+d^2) - b^4cd^3 - a^5b^4d^3 - ab^6d^4 - a^2c^2d^2L\right\}x^2$$
$$+\left\{[d(1-ad)-c][ad(b^2+c^2) - cd^2(b+1) + a^2bc - abd^2] - ac^2dL - \tfrac{c^2(d-c)^2}{d^3}\right\}y^2$$
$$+\left\{ad^2(b^2+d^2) + d(b^2c-ab)[d(1-ad)-c] - a^3b^2cd(b^2+d^2) - abc^2L - a^2cd[ab^2+(d-c)]\right\}z^2$$
$$+\left\{L - ab[d(1-ad)-c] - \tfrac{a}{b}(b^2+d^2)(d-c) - \tfrac{a^2d^3}{c} - cdL\right\}w^2\ \},$$
$$(5.3.4)$$

from which we obtain,

$$\begin{aligned}2V(x,y,z,w) &\geq \left(\tfrac{\delta}{\Delta}\right)\{\{[d(1-ad)-c](ad(c^2+d^2)+abd^2)-\tfrac{cd^3}{a}(b^2+d^2)-b^4cd^3-a^5b^4d^3\\ &-ab^6d^4-a^2c^2d^2L\}x^2+\{[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)\\ &+a^2bc-abd^2]-ac^2dL-\tfrac{c^2(d-c)^2}{d^3}\}y^2+\{ad^2(b^2+d^2)+d(b^2c-ab)[d(1-ad)-c]\\ &-a^3b^2cd(b^2+d^2)-abc^2L-a^2cd[ab^2+(d-c)]\}z^2+\{L-ab[d(1-ad)-c]\\ &-\tfrac{a}{b}(b^2+d^2)(d-c)-\tfrac{a^2d^3}{c}-cdL\}w^2\}\end{aligned} \tag{5.3.5}$$

$$\geq K_1(x^2+y^2+z^2+w^2),$$

where

$$\begin{aligned}K_1 &= \tfrac{\delta}{\Delta}\cdot\min\{\,|[d(1-ad)-c](ad(c^2+d^2)+abd^2)-\tfrac{cd^3}{a}(b^2+d^2)-b^4cd^3-a^5b^4d^3\\ &-ab^6d^4-a^2c^2d^2L|,\\ &|[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc-abd^2]-ac^2dL-\tfrac{c^2(d-c)^2}{d^3}|,\\ &|ad^2(b^2+d^2)+d(b^2c-ab)[d(1-ad)-c]-a^3b^2cd(b^2+d^2)-abc^2L-a^2cd[ab^2+(d-c)]|,\\ &|L-ab[d(1-ad)-c]-\tfrac{a}{b}(b^2+d^2)(d-c)-\tfrac{a^2d^3}{c}-cdL|\}\end{aligned}$$

Therefore,

$$2V(x,y,z,w)\geq K_1(x^2+y^2+z^2+w^2). \tag{5.3.6}$$

By using the inequality $xy\leq\frac{1}{2}(x^2+y^2)$, on Equation (5.3.1), we have

$$2V(x,y,z,w)\leq\left(\frac{\delta}{\Delta}\right)\{[A+E+F+G]x^2+[B+E+H+I]y^2+[C+F+H+J]z^2+[D+G+I+J]w^2\}$$

$$\leq K_2(x^2+y^2+z^2+w^2), \tag{5.3.7}$$

where

$$K_2=\left(\frac{\delta}{\Delta}\right)\max\{[A+E+F+G],[B+E+H+I],[C+F+H+J],[D+G+I+J]\}>0.$$

From inequalities (5.3.6) and (5.3.7), we have

$$K_1(x^2+y^2+z^2+w^2)\leq V(x,y,z,w)\leq K_2(x^2+y^2+z^2+w^2). \tag{5.3.8}$$

This proves the Lemma 5.3.1.

Lemma 5.3.2: *Subject to the assumptions of Theorem 5.2.1 and in addition let the condition (ii) of the Theorem 5.2.2 be also satisfied. Then there are positive constants $K_j=K_j(a,b,c,d,\delta)(j=3,4)$ such that for any solution (x,y,z,w) of system (5.1.3),*

$$\dot{V}|_{(5.1.3)}\equiv\tfrac{d}{dt}V|_{(5.1.3)}(x,y,z,w)\leq-K_3(x^2+y^2+z^2+w^2)+K_4(|x|+|y|+|z|+|w|)\,|p(t)|\,. \tag{5.3.9}$$

Proof: From Equations (5.1.1) and (5.1.3), we have

$$\dot{V}|_{(5.1.3)} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} + \frac{\partial V}{\partial w}\dot{w}$$

$$= \tfrac{\partial V}{\partial x}y + \tfrac{\partial V}{\partial y}z + \tfrac{\partial V}{\partial z}w + \tfrac{\partial V}{\partial z}(-aw - bz - g(y) - h(x) + p(t)). \tag{5.3.10}$$

After some simplifications we have,

$$\begin{aligned}\dot{V} &= \left(\tfrac{\delta}{\Delta}\right)\{-Gh(x)x - Ig(y)y - [Jb - H]z^2 - [Da - J]w^2 \\ &-Gg(y)x - Ih(x)y - [Gb - E]xz - Jh(x)z - [Ga - F]xw - Dh(x)w - [Ib - F - B]yz \\ &-Jg(y)z - [Ia - G - H]yw - Dg(y)w - [Db + Ja - I - C]zw \\ &+Ey^2 + Axy + p(t)[Gx + Iy + Jz + Dw]\}.\end{aligned} \tag{5.3.11}$$

Using the conditions on $h(x)$ and $g(y)$, the inequality (5.3.11) becomes

$$\begin{aligned}\dot{V} &\leq \left(\tfrac{\delta}{\Delta}\right)\{-Gdx^2 - [Ic - E]y^2 - [Jb - H]z^2 - [Da - J]w^2 - [Gc + Id - A]xy - [Gb + Jd - E]xz- \\ &-[Ga + Dd - F]xw - [Ib + Jc - F - B]yz - -[Ia + Dc - G - H]yw \\ &-[Db + Ja - I - C]zw[h(0) + g(0) + p(t)][Gx + Iy + Jz + Dw]\}\end{aligned} \tag{5.3.12}$$

and this is equivalent to

$$\dot{V} \leq \left(\tfrac{\delta}{\Delta}\right)\{-K_3(x^2 + y^2 + z^2 + w^2) + [h(0) + g(0) + p(t)][Gx + Iy + Jz + Dw]\} \tag{5.3.13}$$

where $K_3 = \max\{Gd, [Ic - E], [Jb - H], [Da - J]\}$.

Inequality (5.3.13) further reduces to

$$\dot{V} \leq \left(\tfrac{\delta}{\Delta}\right)\{-K_3(x^2 + y^2 + z^2 + w^2) + K_4(|x| + |y| + |z| + |w|)p(t)\} \tag{5.3.14}$$

with $K_4 = \max\{D, G, I, J\}$.

Therefore

$$\dot{V} \leq -K_5(x^2 + y^2 + z^2 + w^2) + K_6(|x| + |y| + |z| + |w|)p(t) \tag{5.3.15}$$

where $K_5 = \left(\tfrac{\delta}{\Delta}\right)K_3$ and $K_6 = \left(\tfrac{\delta}{\Delta}\right)K_4$.

Since

$$(|x| + |y| + |z| + |w|) \leq 2(x^2 + y^2 + z^2 + w^2)^{\frac{1}{2}},$$

the inequality (5.3.15) becomes

$$\frac{dV}{dt} \leq -K_5(x^2 + y^2 + z^2 + w^2) + K_7(x^2 + y^2 + z^2 + w^2)^{\frac{1}{2}}|p(t)|, \tag{5.3.16}$$

where $K_7 = 2K_6$.

This completes the proof of Lemma 5.3.2.

5.4 Proof of the main results

We shall now give the proof of the main results.

Proof of Theorem 5.2.1

From the proof of the Lemmas 5.3.1 and 5.3.2 it is established that the trivial solution of Equation (5.1.1) is globally asymptotically stable. i.e every solution $(x(t), \dot{x}(t), \ddot{x}(t), \dddot{x}\ (t))$ of the system (5.1.2) satisfies $x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}\ (t) \longrightarrow 0$ as $t \longrightarrow \infty$.

Proof of Theorem 5.2.2: Indeed from the inequality (5.3.16),

$$\frac{dV}{dt} \leq -K_5(x^2 + y^2 + z^2 + w^2) + K_7(x^2 + y^2 + z^2 + w^2)^{\frac{1}{2}} \left| p(t) \right|,$$

and also from the inequality (5.3.6), we have

$$(x^2 + y^2 + z^2 + w^2)^{\frac{1}{2}} \leq \left(\frac{2V}{K_1} \right)^{\frac{1}{2}}.$$

Thus the inequality (5.3.16) becomes

$$\frac{dV}{dt} \leq -K_8 V + K_9 V^{\frac{1}{2}} \left| p(t) \right|. \tag{5.4.1}$$

We note that

$K_5(x^2 + y^2 + z^2 + w^2) = K_5 \cdot \frac{V}{K_1}$

and

$$\frac{dV}{dt} \leq -K_8 V + K_9 V^{\frac{1}{2}} \left| p(t) \right| \tag{5.4.2}$$

where $K_8 = \frac{K_6}{K_2}$ and $K_9 = \frac{K_7}{K_2^{\frac{1}{2}}}$.

These imply that

$$\dot{V} \leq -K_8 V + K_9 V^{\frac{1}{2}} \left| p(t) \right|$$

and this can be written as

$$\dot{V} \leq -2K_{10} V + K_9 V^{\frac{1}{2}} \left| p(t) \right|, \tag{5.4.3}$$

where $K_{10} = \frac{1}{2} K_8$.

Therefore

$$\dot{V} + K_{10} V \leq -K_{10} V + K_9 V^{\frac{1}{2}} \left| p(t) \right| \tag{5.4.4}$$

$$\leq K_9 V^{\frac{1}{2}} \left\{ \left| p(t) \right| - K_{11} V^{\frac{1}{2}} \right\}, \tag{5.4.5}$$

where $K_{11} = \frac{K_{10}}{K_9}$.

Thus the inequality (5.4.5) becomes

$$\dot{V} + K_{10} V \leq K_9 V^{\frac{1}{2}} V^* \tag{5.4.6}$$

where

$$V^* = |p(t)| - K_{11}V^{\frac{1}{2}} \tag{5.4.7}$$

$$\leq V^{\frac{1}{2}}\,|p(t)|$$

$$\leq |p(t)|\,. \tag{5.4.8}$$

When $|p(t)| \leq K_{11}V^{\frac{1}{2}}$,

$$V^* \leq 0, \tag{5.4.9}$$

and when $|p(t)| \geq K_{11}V^{\frac{1}{2}}$,

$$V^* \leq |p(t)| \cdot \frac{1}{K_{11}}. \tag{5.4.10}$$

On substituting the inequality (5.4.9) into the inequality (5.4.5), we have,

$$\dot{V} + K_{10}V \leq K_{12}V^{\frac{1}{2}}\,|p(t)|$$

where

$$K_{12} = \frac{K_9}{K_{11}}.$$

This implies that

$$V^{-\frac{1}{2}}\dot{V} + K_{10}V^{\frac{1}{2}} \leq K_{12}\,|p(t)|\,. \tag{5.4.11}$$

Multiplying both sides of the inequality (5.4.11) by $e^{\frac{1}{2}K_{10}t}$ we have,

$$e^{\frac{1}{2}K_{10}t}\left\{V^{-\frac{1}{2}}\dot{V} + K_{10}V^{\frac{1}{2}}\right\} \leq e^{\frac{1}{2}K_{10}t}K_{12}\,|p(t)| \tag{5.4.12}$$

i.e

$$2\frac{d}{dt}\left\{V^{\frac{1}{2}}e^{\frac{1}{2}K_{10}t}\right\} \leq e^{\frac{1}{2}K_{10}t}K_{12}\,|p(t)|\,. \tag{5.4.13}$$

Integrating both sides of (5.4.13) from t_0 to t, gives

$$\left\{V^{\frac{1}{2}}e^{\frac{1}{2}K_{10}\gamma}\right\}_{t0}^{t} \leq \int_{t0}^{t}\frac{1}{2}e^{\frac{1}{2}K_9\tau}K_{12}\,|p(\tau)d\tau| \tag{5.4.14}$$

which implies that

$$\left\{V^{\frac{1}{2}}(t)\right\}e^{\frac{1}{2}K_{10}t} \leq V^{\frac{1}{2}}(t_0)e^{\frac{1}{2}K_{10}t_0} + \frac{1}{2}K_{12}\int_{t0}^{t}|2(\tau)|\,e^{\frac{1}{2}K_{10}\tau}d\tau,$$

or

$$V^{\frac{1}{2}}(t) \leq e^{-\frac{1}{2}K_{10}t}\left\{V^{\frac{1}{2}}(t_0)e^{\frac{1}{2}K_{10}t_0} + \frac{1}{2}K_{12}\int_{t0}^{t}|p(\tau)|\,e^{\frac{1}{2}K_{10}\tau}d\tau\right\}.$$

Using inequalities (5.3.6) and (5.3.7) we have

$$\begin{aligned} K_1(x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}\,(t)) \quad &\leq e^{-\frac{1}{2}K_{10}t}\Big\{K_2(x^2(t_0) + \dot{x}^2(t_0) + \ddot{x}^2(t_0) + \dddot{x}\,(t_0))e^{\frac{1}{2}K_{10}t_0} \\ &+ \tfrac{1}{2}K_{12}\textstyle\int_{t0}^{t}|p(\tau)|\,e^{\frac{1}{2}K_{10}\tau}d\tau\Big\}^2 \end{aligned} \tag{5.4.15}$$

for all $t \geq t_0$.

Thus,

$$\begin{aligned} x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)+\dddot{x}\,(t) &\leq \tfrac{1}{K_1}\left\{e^{-\frac{1}{2}K_{10}t}\left\{K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0)+\dddot{x}\,(t_0))e^{\frac{1}{2}K_{10}t_0}\right.\right. \\ &\quad \left.\left.+\tfrac{1}{2}K_{12}\int_{t0}^{t}|p(\tau)|\,e^{\frac{1}{2}K_{10}\tau}d\tau\right\}^2\right\} \\ &\leq \left\{e^{-\frac{1}{2}K_{10}t}\left\{A_1+A_2\int_{t0}^{t}|p(\tau)|\,e^{\frac{1}{2}K_{10}\tau}d\tau\right\}^2\right\}, \end{aligned} \tag{5.4.16}$$

where A_1 and A_2 are constants depending on $\{K_1, K_2, \ldots K_{12}$ and $(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))+\dddot{x}\,(t_0)\}$.

By substituting $K_{10}=\mu$ in the inequality (5.4.16), we have

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)+\dddot{x}\,(t) \leq \left\{e^{-\frac{1}{2}\mu t}\left\{A_1+A_2\int_{t0}^{t}|p(\tau)|\,e^{\frac{1}{2}\mu\tau}d\tau\right\}^2\right\},$$

which completes the proof.

Proof of Theorem 5.2.3: From the function V defined above and the conditions of Theorem 5.2.3, the conclusion of Lemma 5.3.1 can be obtained, as

$$V \geq K_1\left(x^2+y^2+z^2+w^2\right), \tag{5.4.17}$$

and since $p \neq 0$ we can revise the conclusion of Lemma 5.3.2, i.e,

$$\dot{V} \leq -K_5(x^2+y^2+z^2+w^2)+K_6(|x|+|y|+|z|+|w|)\,|p(t)|\,,$$

and we obtain by using the condition on $p(t;x,y,z,w)$ as stated in the Theorem 5.2.3 that

$$\dot{V} \leq K_6(|x|+|y|+|z|+|w|)^2 r(t). \tag{5.4.18}$$

By using the inequality $|x|\,|y| \leq \frac{1}{2}(x^2+y^2)$, on inequality (5.4.18), we have

$$\dot{V} \leq K_{13}(x^2+y^2+z^2+w^2)r(t), \tag{5.4.19}$$

where $K_{13}=4K_6$.

From inequalities (5.4.17) and (5.4.19) we have,

$$\dot{V} \leq K_{13}Vr(t). \tag{5.4.20}$$

Integrating inequality (5.4.20) from 0 to t, we obtain

$$V(t)-V(0) \leq K_{14}\int_0^t V(s)r(s)ds. \tag{5.4.21}$$

where $K_{14} = \frac{K_{13}}{K_1} = \frac{4K_6}{K_1}$.

Thus,

$$V(t) \leq V(0) + K_{14} \int_0^t V(s)r(s)ds. \qquad (5.4.22)$$

By applying the Grownwall-Reid-Bellman theorem on the inequality, (5.4.22) yields,

$$V(t) \leq V(0)e^{\left(K_{14} \int_0^t r(s)ds\right)}. \qquad (5.4.23)$$

This completes the proof of Theorem 5.2.3.

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Chapter 6

CONVERGENCE OF SOLUTIONS OF CERTAIN FOURTH ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

[1]

Abstract

We give sufficient criteria for the existence of convergence of solutions for a certain class of fourth order nonlinear differential equation using the Lyapunov's second method. A complete Lyapunov function is employed in this work which makes the results to include and improve some existing results in literature.

Key words and Phrases: Complete Lyapunov function, convergence of solution, fourth order nonlinear differential equations.

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6.1 Introduction

In this paper we shall consider the fourth order differential equation

$$x^{(iv)} + a\,\dddot{x} + f(x,\dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t), \tag{6.1.1}$$

where $a > 0$, the functions f, g, h, p are continuous in the respective arguments displayed explicitly, $\dot{x} = \dfrac{dx}{dt}$, $\ddot{x} = \dfrac{d^2x}{dt^2}$, $\dddot{x} = \dfrac{d^3x}{dt^3}$ and $x^{(iv)} = \dfrac{d^4x}{dt^4}$. The conditions on f, g, h and p are such that the existence of solutions of Equation (6.1.1) corresponding to any preassigned initial solutions are guaranteed.

Solutions of the Equation of the form of Equation (6.1.1) have been investigated by several researchers on the account of boundedness, stability and global asymptotic stability (see for instance [5-6], [7], [9], [13-16] and [17]). Some results on these can be found in [10]. Out of the numerous works on this class of equation only a few were devoted to the convergence of the solutions. (see e.g [1] and [3]).

By convergence of solutions we mean, given any two solutions $x_1(t)$ and $x_2(t)$ of Equation (6.1.1), $x_2(t) - x_1(t) \to 0$, $\dot{x_2}(t) - \dot{x_1}(t) \to 0$, $\ddot{x_2}(t) - \ddot{x_1}(t) \to 0$ and $\dddot{x}_2\,(t) - \dddot{x}_1\,(t) \longrightarrow 0$ as $t \to \infty$.

In [2], [8], [11-12], certain classes of third order nonlinear differential equations were investigated and their solutions were proved to converge under certain conditions. In [11], the author considered the equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}),$$

and established that the boundedness of both $p(t)$ and $\int p(\tau)d\tau$ together with the differentiability of the function h guaranteed the convergence of the solutions of the considered equation. This result was improved upon in [12] when the stringent conditions placed on the function h in [11] was dispensed with.

Similarly in [8], the author established that the solutions of the considered equation converged without much restrictions on the nonlinear terms that were involved.

In [1], the author considered Equation (6.1.1) with $g(\dot{x}) = c\dot{x}$ $(c > 0)$, and further with the assumption that h was not necessarily differentiable but satisfied an incrementary ratio $\eta^{-1}\,(h(x+\xi) - h(\xi))\,\eta \neq 0$, which lies in a closed sub interval I_0 of the Routh-Hurwitz interval $(0, \frac{(ab-c)c}{a^2})$, where $I_0 \equiv [\Delta_0, \frac{k(ab-c)c}{a^2}]$ [1 & 3].

The author in [3] considered the equation (6.1.1) with $f(x, \dot{x}) = b$ and criteria for the existence of convergent solutions were established whereas in [1] he considered Equation

(6.1.1) with $f(x, \dot{x}) = b$ and $g(\dot{x}) = c$. The work in [3] extends [1] from equation with one nonlinearity to the one having two nonlinearities which makes it an extension of [1] as well as an extension of [11] to an analogous fourth order equations.

In all these studies Lyapunov's second method has been the main tool of investigation. In the literature, the incomplete Lyapunov functions are frequent and used by a quite appreciable number of researchers due to the nature of construction and simplicity. The works with the complete Lyapunov functions are not as frequent as the ones with incomplete Lyapunov function.

In this present work, we shall extend the work in [8] to Equation (6.1.1). With a suitable complete Lyapunov function and less stringent assumptions on the nonlinear terms f, g, h and p, we shall show that the solutions of Equation (6.1.1) converge.

This work is organized in this order, the main result is presented in Section Two as formulation of results. Section Three deals with the tools needed to proof the main result. The proof of the main theorem is presented is Section Four.

6.2 Formulation of Results

The following is the main result.

Theorem 6.2.1. *Suppose $x_1(t)$ and $x_2(t)$ are two solutions of Equation (6.1.1), suppose further that for arbitrary $\xi, \eta(\eta \neq 0)$*

(i) $\frac{h(\xi+\eta)-h(\xi)}{\eta} \in I_0, \quad \eta \neq 0;$

(ii) $\frac{g(\xi+\eta)-g(\xi)}{\eta} \neq 0;$

(iii) $h(0) = g(0) = 0;$

(iv) $|f(x, y)| \leq b;$

(v) $|p(t)| \leq \Lambda, \qquad (\Lambda \quad \text{constant})$

then there exists a positive constant K_5 such that

(v) $S(t_2) \leq S(t_1)e^{-K_5(t_2-t_1)} \quad \text{for} \quad t_2 \geq t_1,$

where

$$S(t) = \left\{[x_2(t) - x_1(t)]^2 + [\dot{x}_2(t) - \dot{x}_1(t)]^2 + [\ddot{x}_2(t) - \ddot{x}_1(t)]^2 + [\dddot{x}_2\,(t) - \dddot{x}_1\,(t)]^2\right\}.$$

Furthermore all solutions of Equation (6.1.1) converge.

We have the following corollaries as the consequences of the Theorem 6.2.1 when $x_1(t) = 0$ and $t_1 = 0$.

Corollary 6.2.2: *Suppose $p = 0$ in Equation (6.1.1) and suppose further that the conditions of the Theorem hold, then the trivial solution of Equation (6.1.1) is exponentially stable in the large.*

Corollary 6.2.3: *Suppose also that the conditions of the corollary 6.2.2 hold for arbitrary $\eta(\eta \neq 0)$ and $\xi = 0$, then there exists a constant K_0 such that every solution $x(t)$ of Equation (6.1.1) satisfies*

$$|x(t)| \leq K_0, \qquad |\dot{x}(t)| \leq K_0, \qquad |\ddot{x}(t)| \leq K_0, \qquad |\dddot{x}_2(t)| \leq K_0$$

Remark: The corresponding linear equation to (6.1.1) given as

$$x^{(iv)} + a\dddot{x} + b\ddot{x} + c\dot{x} + dx = p(t), \tag{$*$}$$

$d > 0$ and constants b, c (with $h(x) = dx, f(x, \dot{x}) = b, g(\dot{x}) = c\dot{x}$) and $p(t) = 0$ in Equation (6.1.1)) is known to have convergent solutions if the Routh Hurwitz conditions/criteria $ab - c > 0, (ab - c)c - a^2d > 0$ hold [1 & 3].

Notations: Throughout this paper K_3, K_4 and K_5 will denote finite positive constants whose magnitudes depend only on the constants a, b, c, d, δ and Δ but are independent of solutions of Equation (6.1.1). $K_i's$ are not necessarily the same for each time they occur, but each $K_i, i = 1, 2, \ldots 5$ retains its identity throughout.

6.3 Preliminary Results

On setting $\dot{x} = y, \dot{y} = z, \dot{z} = w$, Equation (6.1.1) can be replaced by an equivalent system

$$\begin{aligned}
\dot{x} &= y; \\
\dot{y} &= z; \\
\dot{z} &= w; \\
\dot{w} &= -aw - f(x, y)z - g(y) - h(x) + p(t).
\end{aligned} \tag{6.3.1}$$

Following Cartwright [4] and Reissig et al [10] a possible Lyapunov function is a quadratic function in the variables for which the co-efficients are suitably chosen. In this regard, we shall assume a Lyapunov function of the form

$$2V(x, y, z, w) = Ax^2 + By^2 + Cz^2 + Dw^2 + 2Exy + 2Fxz + 2Ixw + 2Jyz + 2Myw + 2Nzw, \tag{6.3.2}$$

Our investigation rest mainly on the properties of the function

$$W(t) \equiv V(x_2(t)-x_1(t), y_2(t)-y_1(t), z_2(t)-z_1(t), w_2(t)-w_1(t))$$

with $V(x(t), y(t), z(t), w(t))$ written as $V(x, y, z, w)$ where

$$
\begin{aligned}
A &= \tfrac{a\delta}{\Delta}\left\{(b+d)(c^2+d^2)[d(1-ad)-c]+d^3[a(b^2+d^2)+L]\right\};\\
B &= \tfrac{\delta}{\Delta}\left\{dL(abd+c)+a(b^2+d^2)[b(d-c)+cd]+[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc]\right\};\\
C &= \tfrac{\delta}{\Delta}\left\{a(b^2+d^2)[d(1-ad+a^2c+d)-c]+d[c(a^2+b^2)-ab][d(1-ad)-c]+dL(a^2c+d)\right\};\\
D &= \tfrac{cd\delta}{\Delta}\left\{L+ab^2+(d-c)+ab[(1-ad)-c]\right\};\\
E &= \tfrac{ac\delta}{\Delta}\left\{d^2L+(b^2+d^2)(d-c)\right\};\\
F &= \tfrac{cd\delta}{bd\Delta}\left\{d^2L+ad^2(b^2+d^2)+[b(a^2+d^2)+d^2][ab^2d^2[d(1-ad)-c]]\right\};\\
I &= \tfrac{abc[d(1-ad-c)]\delta}{\Delta};\\
J &= \tfrac{abcd\delta}{\Delta}\left\{a(b^2+d^2)+L\right\};\\
M &= \tfrac{a\delta}{\Delta}\left\{d^2L+bd[d(1-ad)-c]+(b^2+d^2)(d-c)\right\};\\
N &= \tfrac{acd\delta}{\Delta}\left\{ab^2+d-c+L\right\};\\
\Delta &= abcd[d(1-ad)-c];\\
L &= b[ad+c[c(b+1)-c]],
\end{aligned}
$$

with a, b, c, d positive and $[d(1-ad)-c] > 0$ were obtained after solving the equations that arose when constructing the Lyapunov function (see section 2.1.4 of this thesis).

Thus, W is equivalent to $V(x, y, z, w)$ with x, y, z, w replaced with $x_2-x_1, y_2-y_1, z_2-z_1$ and w_2-w_1 respectively.

Now define W as

$$
\begin{aligned}
2W(x_2-x_1, y_2-y_1, z_2-z_1, w_2-w_1) &= A(x_2-x_1)^2+B(y_2-y_1)^2+C(z_2-z_1)^2+D(w_2-w_1)^2\\
&+2E(x_2-x_1)(y_2-y_1)+2F(x_2-x_1)(z_2-z_1)\\
&+2I(x_2-x_1)(w_2-w_1)+2J(y_2-y_1)(z_2-z_1)\\
&+2M(y_2-y_1)(w_2-w_1)+2N(z_2-z_1)(w_2-w_1)
\end{aligned}
\tag{6.3.3}
$$

We shall prove the following.

Lemma 6.3.1 *Suppose W is defined as in Equation (6.3.3) and $W(0,0,0,0) = 0$, then there exist constants K_1* and K$_2$ *such that the inequalities*

$$K_1((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2+(w_1-w_2)^2) \le W \le K_2((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2+(w_1-w_2)^2)$$

hold.

Proof of Lemma 6.3.1: Clearly $W(0,0,0,0) \equiv 0$

By rearranging Equation (6.3.3) we have

$$\begin{aligned}2W(x_2-x_1,y_2-y_1,z_2-z_1,w_2-w_1) &= \left(\tfrac{\delta}{\Delta}\right)\Big\{a[d(1-ad)]\{b[c(x_2-x_1)+d(y_2-y_1)+(w_2-w_1)]^2\\
&+d^2[(y_2-y_1)+b^3d^2(x_2-x_1)]^2+b^2d[(y_2-y_1)+a^2bd(x_2-x_1)]^2\\
&+acd[(z_2-z_1)+\tfrac{b^2d^3}{a}(x_2-x_1)]^2\Big\}+dL\left\{[(z_2-z_1)+ac(x_2-x_1)]^2\right.\\
&+ac^2[(z_2-z_1)+\tfrac{1}{a}(w_2-w_1)]^2+c[(y_2-y_1)+\tfrac{ad}{c}(w_2-w_1)]^2\\
&\left.+ad^2[(x_2-x_1)+\tfrac{c}{d}(y_2-y_1)]^2+abd[(y_2-y_1)+\tfrac{c}{d}(z_2-z_1)]^2\right\}\\
&+ad(b^2+d^2)\Big\{ad^2[(x_2-x_1)+\tfrac{c(d-c)}{ad^3}(y_2-y_1)]^2+a^2c[(z_2-z_1)\\
&+\tfrac{d}{a}(x_2-x_1)]^2+\tfrac{c}{a(b^2+d^2)}[(w_2-w_1)+a(z_2-z_1)]^2+b(d-c)[(y_2-y_1)+\tfrac{(w_2-w_1)}{b}]^2\\
&+c[(y_2-y_1)+ab(z_2-z_1)]^2\Big\}+\Big\{[d(1-ad)-c](ad(c^2+d^2)+abd^2)-\tfrac{cd^3}{a}(b^2+d^2)\\
&-b^4cd^3-a^5b^4d^3-ab^6d^4-a^2c^2d^2L\Big\}(x_2-x_1)^2+\Big\{[d(1-ad)-c][ad(b^2+c^2)\\
&-cd^2(b+1)+a^2bc-abd^2]-ac^2dL-\tfrac{c^2(d-c)^2}{d^3}\Big\}(y_2-y_1)^2+\Big\{ad^2(b^2+d^2)\\
&+d(b^2c-ab)[d(1-ad)-c]-a^3b^2cd(b^2+d^2)-abc^2L-a^2cd[ab^2+(d-c)]\Big\}(z_2-z_1)^2\\
&+\Big\{L-ab[d(1-ad)-c]-\tfrac{a}{b}(b^2+d^2)(d-c)-\tfrac{a^2d^3}{c}-cdL\Big\}(w_2-w_1)^2\ \Big\},\end{aligned} \tag{6.3.4}$$

from which we obtain,

$$\begin{aligned}2W(x_2-x_1,y_2-y_1,z_2-z_1,w_2-w_1) &\geq \left(\tfrac{\delta}{\Delta}\right)\{\{[d(1-ad)-c](ad(c^2+d^2)+abd^2)-\tfrac{cd^3}{a}(b^2+d^2)-b^4cd^3\\
&-a^5b^4d^3-ab^6d^4-a^2c^2d^2L\}(x_2-x_1)^2\\
&+\{[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc-abd^2]-ac^2dL\\
&-\tfrac{c^2(d-c)^2}{d^3}\}(y_2-y_1)^2+\{ad^2(b^2+d^2)+d(b^2c-ab)[d(1-ad)-c]\\
&-a^3b^2cd(b^2+d^2)-abc^2L-a^2cd[ab^2+(d-c)]\}(z_2-z_1)^2\\
&+\{L-ab[d(1-ad)-c]\\
&-\tfrac{a}{b}(b^2+d^2)(d-c)-\tfrac{a^2d^3}{c}-cdL\}(w_2-w_1)^2\}\end{aligned} \tag{6.3.5}$$

$$\geq K_1((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(w_2-w_1)^2),$$

where

$$\begin{aligned}K_1 &= \tfrac{\delta}{\Delta}\cdot\min\{\ |[d(1-ad)-c](ad(c^2+d^2)+abd^2)-\tfrac{cd^3}{a}(b^2+d^2)-b^4cd^3\\
&-a^5b^4d^3-ab^6d^4-a^2c^2d^2L|,\\
&|[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc-abd^2]-ac^2dL-\tfrac{c^2(d-c)^2}{d^3}|,\\
&|ad^2(b^2+d^2)+d(b^2c-ab)[d(1-ad)-c]-a^3b^2cd(b^2+d^2)-abc^2L-a^2cd[ab^2+(d-c)]|,\\
&|L-ab[d(1-ad)-c]-\tfrac{a}{b}(b^2+d^2)(d-c)-\tfrac{a^2d^3}{c}-cdL|\}.\end{aligned}$$

Therefore,

$$2W(x_2-x_1,y_2-y_1,z_2-z_1,w_2-w_1)\geq K_1((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(w_2-w_1)^2). \tag{6.3.6}$$

By using the inequality $xy \leq \frac{1}{2}(x^2+y^2)$, on Equation (6.3.2), we have

$$\begin{aligned}2W(x_2-x_1,y_2-y_1,z_2-z_1,w_2-w_1) &\leq \left(\tfrac{\delta}{\Delta}\right)\{[A+E+F+I](x_2-x_1)^2\\
&+[B+E+J+M](y_2-y_1)^2\\
&+[C+F+J+N](z_2-z_1)^2+[D+I+M+N](w_2-w_1)^2\}\end{aligned}$$

$$\leq K_2((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(w_2-w_1)^2), \tag{6.3.7}$$

where

$$K_2=\left(\frac{\delta}{\Delta}\right)\max\{[A+E+F+I],[B+E+J+M],[C+F+J+N],[D+I+M+N]\}>0.$$

From inequalities (6.3.6) and (6.3.7), we have

$$K_1((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(w_2-w_1)^2) \quad \leq W \leq K_2((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(w_2-w_1)^2). \tag{6.3.8}$$

This proves the Lemma 6.3.1.

Lemma 6.3.2 *Suppose* $(x_1(t),y_1(t),z_1(t),w_1(t))$ *and* $(x_2(t),y_2(t),z_2(t),w_2(t))$ *are any* 2 *distinct solutions of the system (6.3.1) such that*

$$H(x_1,x_2)=\frac{h(x_1(t))-h(x_2(t))}{x_1(t)-x_2(t)}\in I_0 \text{ and } G(y_1,y_2)=\frac{g(y_1(t))-g(y_2(t))}{y_1(t)-y_2(t)}\neq 0$$

for all $t>0$, $(0<t<\infty)$, where I_0 carries its usual meaning as $I_0=[\delta,\Delta]$ then the function

$$W=V(x_1-x_2,y_1-y_2,z_1-z_2,w_1-w_2)$$

satisfies

$$\dot W\leq -K_3W$$

for some $K_3>0$

Proof of Lemma 6.3.2: Differentiating W with respect to t using the system (6.3.1) we obtain after some simplifications

$$\begin{aligned}\dot W &= \left(\tfrac{\delta}{\Delta}\right)\{-Ih(x_1(t)-x_2(t))(x_1-x_2)-Mg(y_1(t)-y_2(t))(y_1-y_2)-[Nb-J](z_1-z_2)^2\\ &-[Da-N](w_1-w_2)^2-Ig(y_1(t)-y_2(t))(x_1-x_2)\\ &-Mh(x_1(t)-x_2(t))(y_1-y_2)-[Ib-E](x_1-x_2)(z_1-z_2)\\ &-Nh(x_1(t)-x_2(t))(z_1-z_2)-[Ia-F](x_1-x_2)(w_1-w_2)-Dh(x_1(t)-x_2(t))(w_1-w_2)\\ &-[Mb-F-B](y_1-y_2)(z_1-z_2)-Ng(y_1(t)-y_2(t))(z_1-z_2)-[Ma-I-J](y_1-y_2)(w_1-w_2)\\ &-Dg(y_1(t)-y_2(t))(w_1-w_2)-[Db+Na-M-C](z_1-z_2)(w_1-w_2)+E(y_1-y_2)^2\\ &+A(x_1-x_2)(y_1-y_2)+p(t)[I(x_1-x_2)+M(y_1-y_2)+N(z_1-z_2)+D(w_1-w_2)]\}.\end{aligned} \tag{6.3.9}$$

Using the conditions on $h(x_1-x_2)$ and $g(y_1-y_2)$, Equation (6.3.9) becomes

$$\begin{aligned}\dot W &\leq \left(\tfrac{\delta}{\Delta}\right)\{-IH(x_1,x_2)(x_1-x_2)^2-MG(y_1,y_2)(y_1-y_2)^2-[Nb-J](z_1-z_2)^2-[Da-N](w_1-w_2)^2\\ &-IG(y_1,y_2)(x_1-x_2)(y_1-y_2)-MH(x_1,x_2)(x_1-x_2)(y_1-y_2)-[Ib-E](x_1-x_2)(z_1-z_2)\\ &-NH(x_1,x_2)(x_1-x_2)(z_1-z_2)-NG(y_1,y_2)(y_1-y_2)(z_1-z_2)-[Mb-F-B](y_1-y_2)(z_1-z_2)\\ &-[Ia-F](x_1-x_2)(w_1-w_2)-DH(x_1,x_2)(x_1-x_2)(w_1-w_2)-[Ma-I-J](y_1-y_2)(w_1-w_2)\\ &-DG(y_1,y_2)(y_1-y_2)(w_1-w_2)-[Db+Na-M-C](z_1-z_2)(w_1-w_2)+E(y_1-y_2)^2\\ &+A(x_1-x_2)(y_1-y_2)+p(t)[I(x_1-x_2)+M(y_1-y_2)+N(z_1-z_2)+D(w_1-w_2)]\}\end{aligned} \tag{6.3.10}$$

This can be written as

$$\dot{W} \leq -\frac{\delta}{\Delta} W, \tag{6.3.11}$$

where

$$W = \{W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 + W_{10} + W_{11} + W_{12} - W_{13}\},$$

with

$$W_1 = \alpha_1 H(x_1, x_2)(x_1 - x_2)^2 + \beta_1 MG(y_1, y_2)(y_1 - y_2)^2 + \gamma_1(z_1 - z_2)^2 + \eta_1(w_1 - w_2)^2;$$
$$W_2 = \alpha_2 H(x_1, x_2)(x_1 - x_2)^2 + IG(y_1, y_2)(x_1 - x_2)(y_1 - y_2) + \beta_2 MG(y_1, y_2)(y_1 - y_2)^2;$$
$$W_3 = \alpha_3 H(x_1, x_2)(x_1 - x_2)^2 + MH(x_1, x_2)(x_1 - x_2)(y_1 - y_2) + \beta_3 MG(y_1, y_2)(y_1 - y_2)^2;$$
$$W_4 = \alpha_4 H(x_1, x_2)(x_1 - x_2)^2 + [Ib - E](x_1 - x_2)(z_1 - z_2) + \gamma_2(z_1 - z_2)^2;$$
$$W_5 = \alpha_5 H(x_1, x_2)(x_1 - x_2)^2 + NH(x_1, x_2)(x_1 - x_2)(z_1 - z_2) + \gamma_3(z_1 - z_2)^2;$$
$$W_6 = \alpha_6 H(x_1, x_2)(x_1 - x_2)^2 + [Ia - F](x_1 - x_2)(w_1 - w_2) + \eta_2(w_1 - w_2)^2;$$
$$W_7 = \alpha_7 H(x_1, x_2)(x_1 - x_2)^2 + DH(x_1, x_2)(x_1 - x_2)(w_1 - w_2) + \eta_3(w_1 - w_2)^2;$$
$$W_8 = \beta_4 MG(y_1, y_2)(y_1 - y_2)^2 + [Mb - F - B](y_1 - y_2)(z_1 - z_2) + \gamma_4(z_1 - z_2)^2;$$
$$W_9 = \beta_5 MG(y_1, y_2)(y_1 - y_2)^2 + NG(y_1, y_2)(y_1 - y_2)(z_1 - z_2) + \gamma_5 MG(y_1, y_2)(y_1 - y_2)^2;$$
$$W_{10} = \beta_6 MG(y_1, y_2)(y_1 - y_2)^2 + [Ma - I - J](y_1 - y_2)(w_1 - w_2) + \eta_4(w_1 - w_2)^2;$$
$$W_{11} = \beta_7 MG(y_1, y_2)(y_1 - y_2)^2 + DG(y_1, y_2)(y_1 - y_2)(w_1 - w_2) + \eta_5(w_1 - w_2)^2;$$
$$W_{12} = \gamma_6(z_1 - z_2)^2 + [Db + Na - M - C](z_1 - z_2)(w_1 - w_2) + \eta_6(w_1 - w_2)^2;$$
$$W_{13} = [I(x_1 - x_2) + M(y_1 - y_2) + N(z_1 - z_2) + D(w_1 - w_2)]p(t),$$

and

$$\sum_{i=1}^{7} \alpha_i = 1, \ \sum_{i=1}^{7} \beta_i = 1, \ \sum_{i=1}^{6} \gamma_i = 1 \text{ and } \sum_{\mathrm{i}=1}^{6} \eta_{\mathrm{i}} = 1$$

$W_2, \ W_3, \ \ldots, W_{12}$ are quadratic forms in the variables involved. Since for any quadratic form $AX^2 + BX + C$ to be positive, $B^2 \leq 4AC$. With this property, W_i's $i = 2, 3 \ldots 12$ are positive if

$$\max\left\{\frac{(Ib - E)^2}{\alpha_4\gamma_2}, \frac{(Ia - F)^2}{4\alpha_6\eta_2}\right\} \leq H \leq \min\left\{\frac{4\alpha_5\gamma_3}{N^2}, \frac{4\alpha_7\eta_3}{D^2}\right\} \tag{a}$$

and

$$\max\left\{\frac{(Mb - F - B)^2}{M\beta_4\gamma_4}, \frac{(Ma - I - J)^2}{4M\beta_6\gamma_4}\right\} \leq G \leq \min\left\{\frac{4M\beta_5\gamma_5}{N^2}, \frac{4M\beta_7\eta_5}{D^2}\right\} \tag{b}$$

Moreover, with suitable choice of δ, (small enough), we can always have

$$W_{13} \geq \delta\left\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (w_2 - w_1)^2\right\}^{\frac{1}{2}}$$

With these conditions we have that

$$W \geq W_1$$

and

$$W_1 \leq K_3\left\{((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (w_1 - w_2)^2)\right\}, \tag{6.3.12}$$

with $K_3 = \max\{\alpha_1 H(x_1, x_2), \beta_1 MG(y_1, y_2), \gamma_1, \eta_1\}$

Then from inequality (6.3.11), we could have a K_4 such that

$$\dot{W} \quad \leq \left(\frac{\delta}{\Delta}\right)\left\{-K_4((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2+(w_1-w_2)^2)\right\}. \tag{6.3.13}$$

or

$$\dot{W} \leq -K_5 W, \tag{6.3.14}$$

with $K_5 = \frac{\delta}{\Delta} K_4$.

This completes the proof of the Lemma 6.3.2.

Since $x_1(t)$ and $x_2(t)$ are solutions to be considered, we want to establish that the two solutions converge. Next is to establish that the solutions $x_1(t)$ and $x_2(t)$ converge.

6.4 Proof of the main result

We shall now give the proof of the main results.

Proof of Theorem 6.2.1

Indeed from the inequality (6.3.14),

$$\frac{dW}{dt} \leq -K_5 W$$

On integration from t_1 to t_2, we have that

$$\ln\left(\frac{W(t_2)}{W(t_1)}\right) \leq -K_5(t_2 - t_1)$$

and

$$\frac{W(t_2)}{W(t_1)} \leq e^{-K_5(t_2-t_1)}.$$

Therefore

$$W(t_2) \leq W(t_1)e^{-K_5(t_2-t_1)}. \tag{6.4.1}$$

From the inequality (6.3.12), it follows that

$$W_1 \leq K_3 S,$$

where S is as defined in the Theorem 6.2.1. From the Lemma 6.3.1 we have that

$$W(t_1) \leq K_2((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2+(w_1-w_2)^2) = K_2 S(t_1)$$
$$W(t_2) \leq K_2((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2+(w_1-w_2)^2) = K_2 S(t_2)$$

using this in the inequality (6.4.1), we have

$$S(t_2) \leq S(t_1)e^{-K_5(t_2-t_1)} \tag{6.4.2}$$

for $t_2 \geq t_1$.

As $t \longrightarrow \infty$,we have from the inequality (6.4.1) that

$$\dot{W} \leq 0.$$

Also from the inequality (6.4.2),

$$S(t_2) \longrightarrow 0 \quad \text{as} \quad \mathrm{t}_2 \longrightarrow \infty.$$

This implies that

$$x_2(t) - x_1(t) \longrightarrow 0, \qquad \dot{x}_2(t) - \dot{x}_1(t) \longrightarrow 0, \qquad \ddot{x}_2(t) - \ddot{x}_1(t) \longrightarrow 0 \qquad \text{and} \quad \dddot{x}_1(t) - \dddot{x}_2(t) \longrightarrow 0.$$

Hence the completion of the proof of the Theorem 6.2.1.

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Chapter 7

NUMERICAL SOLUTIONS OF INITIAL VALUE PROBLEMS (IVPs) AND LINEAR INITIAL BOUNDARY VALUE PROBLEMS (LIBVPs)

7.1 Introduction

In this Chapter, we present numerical schemes derived to approximate solution of first order IVPs (which may be generalized to system of first orders ODEs) and solution of LIBVPs. In Sections 7.2 and 7.3, we give the description and derivation of the Pseudo-Spline scheme for approximating solution of IVPs. In Section 7.4, the description and derivation of the numerical scheme for approximating the LIBVPs which we referred to as the pseudo-pseudo spectral scheme is also presented. All numerical experiments of this chapter were carried out using MATLAB 7.6 software 2008a edition and the results presented as appendix.

7.2 Pseudo-Spline Schemes

Consider the initial value problem (IVP)

$$\begin{aligned} y'(t) &= f(t, y(t)) \\ y(t_0) &= y_0 \end{aligned} \tag{7.2.1}$$

where $a \leq t \leq b; a = t_0 < t_1 < t_2 < \ldots < t_{N-1} = b$, $N = \dfrac{(b-a)}{h}$, $N = 0, 1, \ldots, N-1$ and $h = t_{n+1} - t_n$ is called the step length. The conditions on the function $f(t, y(t))$ are such that existence and uniqueness of solution is guaranteed, i.e. f is Lipschitz and continuous. The solution is generated in a step-by-step fashion by a formula which is regarded as discrete replacement of the equation (7.2.1).

In the class of methods available in solving the problem numerically, the most celebrated methods are the single-step and the multi-steps methods. In a single-step method an information at just one point is enough to advance the solution to the next point while for the multi-steps (as the name suggests), information at more than one previous points will be required to advance the solution to the next point.

7.2.1 Piecewise-Interpolation

One of the methods of deriving the multi steps method is by polynomial interpolation for a set of discrete point; however, polynomial interpolation for a set of (N+1)points $\{t_k, y_k\}$ is frequently unsatisfactory because the interpolation error is related to higher derivatives of the interpolated function. To circumvent this, we discretized the interpolation domain and interpolate locally. The overall accuracy may be significantly improved even if the interpolation polynomial is of low order. Interpolation functions obtained on this principle are piece-wise interpolation functions or splines. We define a spline function as follows:

Definition: A function $S(t)$ is called a *spline* of degree k if

(i). the domain of S is the interval $[a, b]$

(ii). $S,\ S',\ S'', \ldots, S^{(k-1)}$ are all continuous on $[a, b]$.

(iii). there are points t_i (called knots) such that $a = t_1 < t_2 < \ldots < t_n = b$ and such that S is a polynomial of degree k on each sub-interval $[t_i, t_{i+1}]$, $i = 1, \ldots, n-1$. subject to the interpolating conditions.

(iv). $S(t_i) = y(t_i) \qquad \forall t \in [t_i, t_{i+1}] \quad i = 1, \ldots n-1.$

(v). $S_r^{(j)}(t_i) = S_{r+1}^{(j)}(t_i);\quad j = 1, \ldots, k-1, \quad r = 1, \ldots, n-1, \quad i = 2, \ldots, n-1.$

Condition (iv) is the collocation while (v) is the continuity condition, only on interior knots.

We shall now use the piece-wise linear and cubic interpolation spline functions to derive our methods. Adams methods are recoverable from our methods.

7.3 Derivation of the scheme

7.3.1 Pseudo Quadratic spline function

Let $S(t)$ be the desired function, the linear Lagrange interpolation formula gives the following representation for $S'(t)$ at the given points t_{n-1} and t_n, for all $t \in [t_{n-1}, t_n]$, as

$$\frac{S'(t) - S'(t_{n-1})}{(t - t_{n-1})} = \frac{S'(t_n) - S'(t)}{(t_n - t)} \tag{7.3.1.1}$$

Simplifying (7.3.1.1) we have

$$S'(t) = \frac{1}{(t_n - t_{n-1})} \{(t - t_{n-1})S'(t_n) + (t_n - t)S'(t_{n-1})\} \tag{7.3.1.2}$$

Integrating (7.3.1.2),

$$S(t) = \frac{1}{2(t_n - t_{n-1})} \left\{ \frac{1}{2}(t - t_{n-1})^2 S'(t_n) - \frac{1}{2}(t_n - t)^2 S'(t_{n-1}) + A \right\} \tag{7.3.1.3}$$

where A is the constant of integration to be determined. Since $S(t)$ interpolates the function f at $t = t_n$, it implies that $S(t_n) = f(t_n, y(t_n))$.

Thus for $t = t_{n-1}$;

$$A = S(t_{n-1}) + \frac{1}{2(t_n - t_{n-1})}(t_n - t_{n-1})^2 S'(t_{n-1}) \tag{7.3.1.4}$$

Substitute (7.3.1.4) into (7.3.1.3) we have,

$$\begin{aligned} S(t) = S(t_{n-1}) + \frac{1}{2(t_n - t_{n-1})}(t_n - t_{n-1})^2 S'(t_{n-1}) + \\ \frac{1}{2(t_n - t_{n-1})} \left\{ \frac{1}{2}(t - t_{n-1})^2 S'(t_n) - \frac{1}{2}(t_n - t)^2 S'(t_{n-1}) \right\} \end{aligned} \tag{7.3.1.5}$$

which on simplification yields

$$S(t) = S(t_{n-1}) + \frac{S'(t_{n-1})}{2(t_n - t_{n-1})} \left\{ (t_n - t_{n-1})^2 - (t_n - t)^2 \right\} + \frac{S'(t_n)}{2(t_n - t_{n-1})}(t - t_{n-1})^2 \tag{7.3.1.6}$$

If in (7.3.1.3) we evaluate $S(t)$ at $t = t_n$,

$$A = S(t_n) - \frac{1}{2(t_n - t_{n-1})}(t_n - t_{n-1})^2 S'(t_n) \tag*{(7.3.1.4)'}$$

if (7.3.1.4)' is substituted into (7.3.1.3) we have

$$S(t) = S(t_n) - \frac{S'(t_n)}{2(t_n - t_{n-1})} \left\{ (t_n - t_{n-1})^2 - (t - t_{n-1})^2 \right\} - \frac{S'(t_{n-1})}{2(t_n - t_{n-1})}(t_n - t)^2 \tag{7.3.1.7}$$

Collocating (7.3.1.6) and (7.3.1.7) at $t = t_{n+1}$ and using the property that $S(t) \approx y(t)$ and that $h = t_n - t_{n-1}$ we have the

$$y_{n+1} = y_{n-1} + 2hf_n \tag{7.3.1.8}$$

and

$$y_{n+1} = y_n + \frac{h}{2}\{3f_n - f_{n-1}\} \tag{7.3.1.9}$$

If we also collocate (7.3.1.6) at $t = t_n$ and simplify we have

$$y_n = y_{n-1} + \frac{h}{2}\{f_n + f_{n-1}\} \tag{7.3.1.10}$$

Equations (7.3.1.8) and (7.3.1.9) correspond to the mid-point rule and the Adams-Bashforth of second order while (7.3.1.10) is an implicit method (the Implicit Trapezoidal Method).
Various multi steps of the Adams forms can be derived from the equations (7.3.1.6) and (7.3.1.7) at different collocation points (say $t = t_{n+2}, t_{n+3}, \ldots$).

7.3.2 The Local Truncation Error

Assume that $y \in C^3[a,b]$ for all x in $a \leq x \leq b$. Due to a standard approach by Lambert [68] we have been able to show that the local truncation errors associated with these numerical algorithms can be expressed respectively as

$$e_{7.3.1.8} = \frac{1}{3}h^2 y'''(\zeta), \qquad \zeta \in (x_{n-1}, x_{n+1})$$

$$e_{7.3.1.9} = \frac{5}{12}h^2 y'''(\zeta), \qquad \zeta \in (x_{n-1}, x_{n+1})$$

$$e_{7.3.1.10} = -\frac{1}{12}h^2 y'''(\zeta), \qquad \zeta \in (x_{n-1}, x_n)$$

Using well known analysis in Herinci [69] and Lambert [81], it can be shown that these methods are all consistent and zero stable. Consistency and zero stability are necessary and sufficient conditions for the convergence of methods of this kind, hence the three numerical schemes are convergent with errors of order $\mathcal{O}(h^2)$.

7.3.3 Pseudo Cubic spline function

Since we are considering a piecewise cubic spline, its second derivative is piecewise linear on $[t_{n-1}, t_n]$, then the linear Lagrange interpolation formula gives the representation for $S''(t)$ at the given points t_{n-1} and t_n as,

$$\frac{S''(t) - S''(t_{n-1})}{(t - t_{n-1})} = \frac{S''(t_n) - S''(t)}{(t_n - t)} \qquad (7.3.3.1).$$

Simplifying (7.3.3.1) gives

$$S''(t) = \frac{1}{(t_n - t_{n-1})}\{(t - t_{n-1})S''(t_n) + (t_n - t)S''(t_{n-1})\} \tag{7.3.3.2}$$

Integrating equation (7.3.3.2) twice we have,

$$S(t) = \frac{1}{(t_n - t_{n-1})}\left\{\frac{S''(t_n}{6}(t - t_{n-1})^3 + \frac{S''(t_{n-1}}{6}(t_n - t)^3\right\} + A(t_n - t) + B(t - t_{n-1}) \tag{7.3.3.3}$$

where A and B are constants. To determine these constants, (7.3.3.3) is collocated at two points say $t = t_{n-1}$ and $t = t_n$, this yield

$$S(t_{n-1}) = \frac{S''(t_{n-1})}{6}(t_n - t_{n-1})^2 + A(t_n - t_{n-1}) \qquad (7.3.3.4)$$

and

$$S(t_n) = \frac{S''(t_n)}{6}(t_n - t_{n-1})^2 + B(t_n - t_{n-1}) \qquad (7.3.3.5).$$

From (7.3.3.4) and (7.3.3.5) we have that

$$A = \frac{1}{(t_n - t_{n-1})}\left\{S(t_{n-1}) - \frac{S''(t_{n-1})}{6}(t_n - t_{n-1})^2\right\}$$

and

$$B = \frac{1}{(t_n - t_{n-1})}\left\{S(t_n) - \frac{S''(t_n)}{6}(t_n - t_{n-1})^2\right\}$$

Substitute for A and B in (7.3.3.3), we have

$$\begin{aligned} S(t) \;&= \frac{1}{(t_n - t_{n-1})}\left\{\frac{S''(t_n)}{6}(t - t_{n-1})^3 + \frac{S''(t_{n-1}}{6}(t_n - t)^3\right\} \\ &+\frac{1}{(t_n - t_{n-1})}\left\{S(t_{n-1}) - \frac{S''(t_{n-1})}{6}(t_n - t_{n-1})^2\right\}(t_n - t) \\ &+\frac{1}{(t_n - t_{n-1})}\left\{S(t_n) - \frac{S''(t_n)}{6}(t_n - t_{n-1})^2\right\}(t - t_{n-1}) \end{aligned} \qquad (7.3.3.6)$$

Collocating (7.3.3.6) at $t = t_{n+1}$ yields

$$S(t_{n+1}) = 2S(t_n) - S(t_{n-1}) + h^2 S''(t_n) \qquad (7.3.3.7)$$

By collocation property, we have

$$y_{n+1} = 2y_n - y_{n-1} + h^2 y''_n \qquad (7.3.3.8)$$

and using (7.2.1),we have that the coefficient of h^2 in the equation (7.3.3.8) can be replaced by

$$y'' = f_t(t_n, y(t_n)) + f_y f_t(t_n, y(t_n)) \qquad (7.3.3.9)$$

where here f_t and $f_y$are the first partial derivatives of $f(t, y(t))$ with respect to t and y respectively. Using the approximation relations, $f_t \approx \frac{f_{n+1} - f_{n-1}}{2h}$ and $f_y \approx \frac{f_{n+1} - f_{n-1}}{2h}$ we simplify (7.3.3.7) to give

$$y_{n+1} = 2y_n - y_{n-1} + \frac{h}{2}\{(1 + f_n)(f_{n+1} - f_{n-1})\}$$

$$y_{n+1} = 2y_n - y_{n-1} + \frac{h}{2}\{(f_{n+1} - f_{n-1}) + f_n f_{n+1} - f_n f_{n-1}\} \qquad (7.3.3.10)$$

Neglecting the nonlinear part in (7.3.3.10), equation (7.3.3.10) becomes

$$y_{n+1} = 2y_n - y_{n-1} + \frac{h}{2}\{f_{n+1} - f_{n-1}\} \qquad (7.3.3.11)$$

which is an implicit 2-step method.

The local truncation error associated with (7.3.3.11)as outlined for the schemes (7.3.3.8)-(7.3.3.10) can be shown to be $-\frac{1}{12}h^3$. The scheme was observed to be consistent but to our surprise the method is not zero stable according to [69] and [81] yet it gives a convergent solution of maximum error of order $\mathcal{O}(h^3)$.

7.3.4 Results

In summary, the schemes derived are given below:

$$y_{n+1} = y_{n-1} + 2hf_n, \tag{7.3.4.1}$$

$$y_{n+1} = y_n + \frac{h}{2}\{3f_n - f_{n-1}\}, \tag{7.3.4.2}$$

$$y_n = y_{n-1} + \frac{h}{2}\{f_n + f_{n-1}\}, \tag{7.3.4.3}$$

and

$$y_{n+1} = 2y_n - y_{n-1} + \frac{h}{2}\{f_{n+1} - f_{n-1}\} \tag{7.3.4.4}$$

7.3.5 Numerical Examples

We shall consider the following problems;

$$1. \qquad y' = -\frac{y}{2(t+1)}, \quad y(0) = 1, \quad t \in [0,1]$$

The exact solution is given as

$$y(t) = \frac{1}{\sqrt{1+t}}$$

$$2. \quad y' = y - t^2 + 1, \quad y(0) = 0.5 \;\; t \in [0,1]$$

The exact solution is given as

$$y(t) = (1+t)^2 - 0.5e^{(t)}$$

The methods described by equations (7.3.4.1) and (7.3.4.2) are respectively represented as Method A and Method B, while Methods C, D, E and F are the combinations of equations (7.3.4.1) with (7.3.4.3), (7.3.4.2) with (7.3.4.3), (7.3.4.1) with (7.3.4.4) and (7.3.4.2) with (7.3.4.4) as predictor-corrector methods respectively.

Table B1 and B2 gives the maximum error of the Methods A and B for the examples with $h = 0.1$ as presented in the Appendix B.

As a means of comparison, the numerical solution generated by these methods are compared with the third order method of Omolehin et al [103] and this is given in Table B3.

From the tables of results displayed it could be seen that one of our methods which is of order 2 performs better than the third order method of Omolehin et al [103].

When the explicit methods of this work are combined to form a predictor-corrector method, the results as seen in the Tables B4 and B5 reveal that these methods give a better accuracy.

7.4 Pseudo-pseudo-spectral Method

In this section we introduce the pseudo-pseudo-spectral method.

Consider the following differential equation

$$Ly = \sum_{i=0}^{m} f_{m-i}(x) D^i y = f(x), \quad x \in [a, b], \tag{7.4.1}$$

$$Ty = K, \tag{7.4.2}$$

where $f_i, \quad i = 0, 1, \ldots m, \quad f$, are known functions of x, D^i is the order of differentiation with respect to the independent variable x, T is a linear functional of rank N and $K \in \Re^m$.

Here (7.4.2) can either be initial, boundary or mixed conditions. To solve the above class of equations using the spectral method is to expand the solution function y, in (7.4.1) and (7.4.2) as a finite series of very smooth functions in the form below

$$y^N(x) = \sum_{k=0}^{N} a_k T_k(x) \tag{7.4.3}$$

where, $\{T_k(x)\}_0^\infty$ is the sequence of Chebyshev polynomials of the first kind. Replacing y by y^N in (7.4.2) the residual is defined as

$$r^N(x) = Ly^N - f \tag{7.4.4}$$

The main target and objective in spectral method is to minimize $r^N(x)$ as much as possible with regard to (7.4.2). The implementation of the spectral methods lead to a system of linear equations with $N+1$ equations in $N+1$ unknowns $a_0, a_1, \ldots, a_N$.

In this section, we present a variation (pseudo) of one of the three spectral methods called collocation (also known as pseudo-spectral) method. We call this method a pseudo-pseudo-spectral method. Also, we use both the Tau and the pseudo-spectral methods for numerical solution of second order linear differential equations to compare the result with pseudo-pseudo-spectral method. We need to state here that this discussion can be extended to the general problem of the form (7.4.1) and (7.4.2).

Consider the following differential equation;

$$\begin{aligned} &P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = S(x), \quad x \in [-1, 1] \\ &y(-1) = \alpha, \quad y(1) = \beta \end{aligned} \tag{7.4.5}$$

With the pseudo-pseudo-spectral method, we suppose that the approximate solution of the equation (7.4.5) is given by

$$y^N(x) = {\sum_{k=0}^{N}}{}' a_k T_k(x) \tag{7.4.3a}$$

instead of (7.4.3) for an arbitrary natural number N, where $\underline{a} = (a_0, a_1, \ldots, a_N)^T \in \Re^{N+1}$ is the constant coefficients vector and $\{T_k(x)\}_0^\infty$ is the sequence of Chebyshev polynomials of the first kind. The prime denotes that the first term in the expansion is halved.

In this method, as against the use of a function $V(x)$ as in the standard Tau method and the Pseudo-spectral method (see [14],[15],[49]), we instead of using the Chebyshev polynomial as a polynomial we exploit the trigonometric property of Chebyshev function.

Let

$$\overline{y} = \sum_{k=0}^{N}{}' a_k T_k(x) \tag{7.4.6}$$

be the approximate solution for the equation (7.4.5), as a solution it must satisfy the equation. Recall the definition of a Chebyshev polynomial,

$$T_k(x) = \cos(k \arccos x)$$

let

$$\theta = \arccos x, \qquad \Rightarrow x = \cos\theta$$

then

$$T_k(x) \equiv T_k(\theta) = \cos k\theta$$

Using the identity defined above, (7.4.6) becomes

$$\overline{y} = \sum_{k=0}^{N}{}' a_k \cos k\theta \tag{7.4.7}$$

The first and second derivatives of (7.4.7) are respectively given as

$$\overline{y'} = \sum_{k=0}^{N}{}' a_k \left(\frac{k \sin k\theta}{\sin\theta}\right) \tag{7.4.8}$$

$$\overline{y''} = \sum_{k=0}^{N}{}' a_k \left(\frac{k \sin k\theta \cos\theta - k^2 \cos k\theta \sin\theta}{\sin^3\theta}\right) \tag{7.4.9}$$

Substituting (7.4.7)-(7.4.9) into the equation (7.4.5) with the functions P, Q, R and S expressed in terms of θ we have

$$\overline{P}(\theta) \sum_{k=0}^{N}{}' a_k \left(\frac{k \sin k\theta \cos\theta - k^2 \cos k\theta \sin\theta}{\sin^3\theta}\right) + \overline{Q}(\theta) \sum_{k=0}^{N}{}' a_k \left(\frac{k \sin k\theta}{\sin\theta}\right) + \overline{R}(\theta) \sum_{k=0}^{N}{}' a_k \cos k\theta = \overline{S}(\theta)$$

$$\theta \in [-\pi, \pi], \quad \overline{y}(-\pi) = \alpha, \quad \overline{y}(\pi) = \beta \tag{7.4.10}$$

Simplifying equation (7.4.10), we have

$$\sum_{k=0}^{N}{}' a_k \phi_k(\theta) = \overline{S}(\theta) \tag{7.4.11}$$

with

$$\phi_k(\theta) = \overline{P}(\theta) \left(\frac{k \sin k\theta \cos\theta - k^2 \cos k\theta \sin\theta}{\sin^3\theta}\right) + \overline{Q}(\theta) \left(\frac{k \sin k\theta}{\sin\theta}\right) + \overline{R}(\theta) \cos k\theta \tag{7.4.12}$$

If we impose the associated conditions on (7.4.11), we have

$$\overline{y}(-1) = \alpha \Rightarrow \sum_{k=0}^{N}{}' a_k T_k(-1) = \sum_{k=0}^{N}{}' a_k (-1)^k = \alpha$$

$$\overline{y}(-1) = \alpha \Rightarrow \sum_{k=0}^{N}{}' a_k T_k(1) = \sum_{k=0}^{N}{}' a_k = \beta$$

So

$$\begin{bmatrix} \frac{1}{2} & -1 & 1 & \dots (-1)^N \\ \frac{1}{2} & 1 & 1 & \dots \quad 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ . \\ . \\ . \\ a_N \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{7.4.13}$$

Relation (7.4.13) form a system with two equations and $N+1$ unknowns, to construct the remaining $N-1$ equations we Collocate (7.4.11) at the zeros of $T_{N-1}(x)$, which are the interior points between -1 and 1 and are given as $\theta_k = \frac{(2k-1)\pi}{N-1}, \quad k = 1, \dots, N-1$, which is in great variance to the Tau Method, the Galerkin method and the Pseudo-spectral method.

The system obtained here solves for the coefficients.

7.4.1 Numerical Experiments

In this section, we consider some ordinary differential equations with Tau Method, Pseudo-spectral method and the Pseudo-pseudo-spectral method and discuss the results.

As notations, we represent the approximations with the Tau method, Pseudo-spectral method and the Pseudo-pseudo-spectral method as $\overline{y}_t$, $\overline{y}_{ps}$ and $\overline{y}_{pps}$ respectively. The tables of results for this experiments are given in Appendix C

The following problems were considered:

Problem 7.4.1.1: Consider the differential equation

$$y''(x) + xy'(x) + y = xcos(x), \quad y(-1) = \sin(-1), \quad y(1) = \sin(1)$$

with the exact solution $y(x) = \sin(x)$. The problem is taken from [14]. The problem was solved with Runge-Kutta of different orders a maximum error of 3.0×10^{-1} were recorded. It was also solved with Tau method and the method described in this paper with $N = 5, 8, 16$. The maximum error produced for these two methods for the various N is given in the table C.1. This table shows the power of spectral methods over Runge-Kutta.

Problem 7.4.1.2: Consider

$$y''(x) + \frac{1}{x}y'(x) = \left(\frac{8}{8-x^2}\right)^2, \quad x \in (0,1), \quad y(1) = 0, \quad y'(0) = 0$$

with the exact solution $2\ln\left(\frac{7}{8-x^2}\right)$.

This problem was taken from [11]. It was solved by extrapolation method with maximum error of 10^{-8}. It was solved in [14] by Tau method for different values of N. Here we solved it by the pseudo-pseudo-spectral method for different values of N as in the Tau method of [14], the maximum error for the two methods is given in table C.2.

Problem 7.4.1.3: Consider

$$y''(x) + |x|y'(x) + \sqrt[3]{\left(x^2 - \tfrac{1}{4}\right)^2}\,y(x) = \left(1 + |x| + \sqrt[3]{\left(x^2 - \tfrac{1}{4}\right)^2}\right)\exp(x), \quad x \in [-1,1],$$
$$y(-1) = \exp(-1), \quad y(1) = \exp(1)$$

with the exact solution $y(x) = \exp(x)$.

This problem was chosen from [15]. It was solved with the method described in this article and the error produced for various N is given in the table C.3 with the maximum error produced when the problem was solved by the Tau method and Pseudo-spectral method of [14].

Problem 7.4.1.4: Consider

$$y''(x) + |x|y'(x) + y(x) = |6x| + |x^3| + 3x^3, \quad x \in [-1,1], \quad y(-1) = y(1) = 1$$

with the exact solution $y(x) = |x^3|$. The problem was taken from [14].

The problem has a non analytic solution function which makes accompany error indispensable.

We apply our method to the problem, the error produced by the method as well as the error produced when it was solved with the Tau method and Pseudo-spectral method of [14] is presented in table C.4.

Problem 7.4.1.5: Consider

$$y''(x) + \exp(\frac{1}{x})y'(x) = y(x) = 6x + x^3 + 3x^2\exp(\frac{1}{x}), \quad x \in [-1,1], \quad y(-1) = -1, \quad y(1) = 1$$

with the exact solution $y(x) = x^3$. This problem was chosen from [14].

When the problem is solved using the Tau method and the pseudo-spectral method in [14], the methods failed and a modified pseudo-spectral(*mps*) method which was the subject of the article was used to solve the problem and the maximum error produced in [14] for the problem is given in the table C.5 with the error produced by the method of this article. This method performs better than the modified method of [14].

Problem 7.4.1.6: Consider the differential equation;

$$y''(x) - \frac{1}{x}y'(x) + \frac{1}{x}y(x) = |x|, \quad y(-1) = -1, \quad y(1) = 1$$

with the exact solution $y(x) = x|x|$. This problem was also from [14].

We tested our method on this problem with different values of N, the results are given in table C.6.

Problem 7.4.1.7: Consider

$$y''(x) + \frac{1}{x}y'(x) + y(x) = \frac{1}{x} + |x|, \quad y(-1) = y(1) = 1$$

with the exact solution $y(x) = |x|$. The problem was chosen from [14].

The results for various values of N are given in table C.7.

Chapter 8

CONCLUSION

In this thesis, as set out in the objectives of the research, we investigated the qualitative and quantitative properties of solutions of certain classes of ordinary differential equations. In the qualitative properties, the Lyapunov second method was used to investigate the qualitative behavior of classes of second order, third order and fourth order nonlinear differential equations.

Variants of the tool employed in these studies had been employed extensively by researchers to study the qualitative properties of solutions of these classes of differential equations.

In most of the studies except for the second order differential equation where it is more convenient to construct a complete Lyapunov function (though not unique), researchers have been (for higher orders) constructing incomplete Lyapunov functions and often make them complete by the use of signum functions.

In this thesis, all the Lyapunov functions used are complete Lyapunov functions which help us to come up with sufficient conditions for the discussion of the qualitative behavior of the classes of equations considered.

We also used the Lyapunov function to discuss the behavior of solutions of non-autonomous equations.

The second method of Lyapunov (the use of Lyapunov functions) remains one of the most effective method to discuss the concepts of stability and boundedness. We have in this thesis used this method to discuss in a unified way the stability, boundedness and periodicity of solutions as well as the convergence of such solutions.

Though,there is a lot of difficulty in constructing complete Lyapunov functions, in this thesis, we had constructed Lyapunov functions different from the ones that are already constructed by any other researcher in the field. At this point it should be stressed that there is no unique way of constructing Lyapunov functions and hence is the reason why there are quite a number of Lyapunov functions in the literature used in discussing the same class of equation and each come up with criteria for discussing the properties.

On the quantitative properties of solution, a new scheme, the pseudo spline method for approximating solution of ordinary differential equations of the first order were derived based on interpolation and collocation through the general method for deriving the spline functions. The scheme derived was compared with methods derived via some standard well known techniques of the same order and were found to be better. The scheme is easier to derive and more user friendly.

This method can handle initial value problems of differential equations. On the use of interpolating spectral method, the pseudo-pseudo-spectral method as we call it is seen to be efficient and competes favorably with other well-known standard methods like the Tau method, Galerkin Method and the Pseudo-spectral (collocation) methods.

One major advantage with this method is that it does not require a tedious means of evaluating the unknown coefficients of the approximating function as in other spectral methods. The method is easy to program and require moderately less of computer time to evaluate.

It is also seen to be suitable for any class of linear differential equations with or without analytical solutions. It is applicable to solve mainly the Initial Boundary Value Problems of ODEs which results from complex systems or PDEs.

Open Problems: The following are the open problems for further research:

- Better and easier ways to construct a complete Lyapunov function to handle nonlinear differential equations of orders higher than 5.
- Due to high growth in the use of machine to solve complex problems, need to develop schemes based on the one derived in this thesis to develop softwares for the people in engineering where accuracy, speed as well as cost matter to most.
- If possible a Lyapunov-like numerical scheme to handle solution of differential equations irrespective of order, type and class.

The identified problems are subjects to be considered in the near future.

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Appendix A

Bellman-Reid- Grownwall Lemma

Let I denote an interval of the real line of the form $[0, \infty$, $[a, b]$ or $[a, b)$ with $a < b$.
Let α, β and u be real valued functions defined on I. Assume that the β and u are continuous and that the negative part of α is integrable on every closed and bounded subinterval of I.

- If β is non-negative and if u satisfies the integral inequality

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds, \quad \forall t \in I \tag{A.1}$$

  then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(r)dr\right)ds, \quad t \in I \tag{A.2}$$

- If in addition the function α is constant then

$$u(t) \leq \alpha \exp\left(\int_a^t \beta(s)\right)ds, \quad t \in I \tag{A.3}$$

Proof: Let

$$v(s) = \exp\left(-\int_a^s \beta(r)dr\right)\int_a^s \beta(r)u(r)dr \quad s \in I \tag{A.4}$$

Differentiating (A.4), we have

$$v'(s) = \left(u(s) - \int_a^s \beta(r)u(r)dr\right)\beta(s)\exp\left(-\int_a^s \beta(r)dr\right), s \in I \tag{A.5}$$

from (A.1) we have that

$$\left(u(s) - \int_a^s \beta(r)u(r)dr\right) \leq \alpha(s) \tag{A.6}$$

using (A.6) in (A.5), we have

$$v'(s) \leq \alpha(a)\beta(s)\exp\left(-\int_a^s \beta(r)dr\right) \tag{A.7}$$

Integrating (A.7) from a to t

$$v(t) - v(a) \leq \int_a^t \alpha(a)\beta(s)\exp\left(-\int_a^s \beta(r)dr\right)ds \tag{A.8}$$

from (A.4), $v(a) = 0$ this reduces (A.8) to

$$v(t) \leq \int_a^t \alpha(a)\beta(s)\exp\left(-\int_a^s \beta(r)dr\right)ds \tag{A.9}$$

Also from (A.4), we have

$$\int_a^t \beta(s)u(s)ds = \exp\left(\int_a^s \beta(r)dr\right)v(t)$$

$$\leq \int_a^t \alpha(s)\beta(s)\exp\left(\int_a^t \beta(r)dr - \int_a^s \beta(r)dr\right)ds \tag{A.10}$$

but

$$\int_a^t \beta(r)dr - \int_a^s \beta(r)dr = \int_s^t \beta(r)dr$$

using (A.10) in (A.1) we have the result (A.2).

If the function α is constant then

$$u(t) \leq \alpha + \left(-\alpha\exp(\int_s^t \beta(r)dr)\right)\Big|_{s=a}^{s=t}$$

$$= \alpha\exp\left(\int_s^t \beta(r)dr\right)$$

Appendix B

Tables of Results for the Pseudo-Spline Scheme

Table B.1: Error of y(t) for Example 1 (h = 0.1)

t	Method **A**	Method **B**
0.3	1.8428727e-004	2.2810119e-004
0.4	2.4052639e-004	5.1440716e-004
0.5	4.5748153e-004	8.5876660e-004
0.6	5.8078449e-004	1.2697656e-003
0.7	8.4856274e-004	1.7575791e-003
0.8	1.0543354e-003	2.3338030e-003
0.9	1.3952231e-003	3.0116219e-003
1.0	1.7044890e-003	3.8060145e-003

Table B.2: Error of y(t) for Example 2 (h = 0.1)

t	Method **A**	Method **B**
0.3	4.5065362e-004	5.8284717e-004
0.4	2.9443804e-004	9.7450032e-004
0.5	6.7867750e-004	1.2528296e-003
0.6	4.3923840e-004	1.4525075e-003
0.7	8.0112923e-004	1.5966312e-003
0.8	5.1017146e-004	1.7007588e-003
0.9	8.6895411e-004	1.7756254e-003
1.0	5.4196872e-004	1.8287918e-003

Table B.3: Error of y(t) for Example 1 (h = 0.1) for methods A, B and [9]

t	Method **A**	Method **B**	**[9]**
0.4	2.9443804e-004	9.7450032e-004	4.2530000e-004
0.5	6.7867750e-004	1.2528296e-003	7.1200000e-004
0.6	4.3923840e-004	1.4525075e-003	1.0556000e-003
0.7	8.0112923e-004	1.5966312e-003	1.4620000e-003
1.0	5.4196872e-004	1.8287918e-003	3.1397000e-003

Table B.4: Error of y(t) for Example 1 (h = 0.1)

t	Method **C**	Method **D**	Method **E**	Method **F**
0.3	3.9207500e-005	3.7016804e-005	2.6716341e-005	2.3962309e-005
0.4	8.4890939e-005	8.1906006e-005	7.1444753e-005	6.3614004e-005
0.5	1.3983292e-004	1.3583079e-004	1.2883332e-004	1.1388709e-004
0.6	2.0539203e-004	2.0019529e-004	1.9547040e-004	1.7156573e-004
0.7	2.8319791e-004	2.7659929e-004	2.6911307e-004	2.3454770e-004
0.8	3.7510163e-004	3.6686362e-004	3.4824665e-004	3.0142337e-004
0.9	4.8320835e-004	4.7305897e-004	4.3182385e-004	3.7122645e-004
1.0	6.0991014e-004	5.9753835e-004	5.1910500e-004	4.4328173e-004

Table B.5: Error of y(t) for Example 2 (h = 0.1)

t	Method **C**	Method **D**	Method **E**	Method **F**
0.3	1.0547721e-004	1.0823124e-004	4.6064141e-006	6.7971098e-006
0.4	1.8156813e-004	1.8399233e-004	1.4349767e-005	2.1497021e-005
0.5	2.3569468e-004	2.3785106e-004	2.9862905e-005	4.5378172e-005
0.6	2.7460440e-004	2.7656997e-004	5.1892600e-005	7.9915113e-005
0.7	3.0275940e-004	3.0458348e-004	8.1313344e-005	1.2680491e-004
0.8	3.2316603e-004	3.2488108e-004	1.1914183e-004	1.8799699e-004
0.9	3.3789855e-004	3.3952672e-004	1.6655518e-004	2.6572679e-004
1.0	3.4841822e-004	3.4997514e-004	2.2491025e-004	3.6255372e-004

Appendix C

Tables of results for the Pseudo-spectral Method

Table C.1:

N	$\|y(x) - \overline{y}_t(x)\|_\infty$	$\left\|y(x) - \overline{y}_{pps}(x)\right\|_\infty$
5	2.11×10^{-5}	1.98×10^{-5}
8	5.71×10^{-8}	4.56×10^{-8}
16	1.11×10^{-16}	5.55×10^{-17}

Table C.2:

N	$\|y(x) - \overline{y}_t(x)\|_\infty$	$\left\|y(x) - \overline{y}_{ps}(x)\right\|_\infty$	$\left\|y(x) - \overline{y}_{pps}(x)\right\|_\infty$
5	5×10^{-5}		2.09×10^{-5}
15	2×10^{-6}		3.33×10^{-16}
16	8×10^{-7}		1.67×10^{-16}
18		4×10^{-19}	1.67×10^{-16}
30	5×10^{-7}		
95	8×10^{-8}		

Table C.3:

N	$\|y(x) - \overline{y}_t(x)\|_\infty$	$\left\|y(x) - \overline{y}_{ps}(x)\right\|_\infty$	$\left\|y(x) - \overline{y}_{pps}(x)\right\|_\infty$
8	3.13×10^{-6}	3.24×10^{-8}	3.20×10^{-8}
11	6.40×10^{-8}	2.52×10^{-12}	5.14×10^{-12}
16	3.92×10^{-8}	3.50×10^{-18}	6.66×10^{-16}

Table C.4:

N	$\|y(x) - \overline{y}_t(x)\|_\infty$	$\left\|y(x) - \overline{y}_{ps}(x)\right\|_\infty$	$\left\|y(x) - \overline{y}_{pps}(x)\right\|_\infty$
8	8.98×10^{-2}	1.21×10^{-1}	1.20×10^{-1}
15	1.54×10^{-2}	1.76×10^{-2}	1.38×10^{-2}
20	1.68×10^{-2}	1.92×10^{-2}	1.50×10^{-2}

Table C.5:

N	$\|y(x)-\overline{y}_t(x)\|_\infty$	$\|y(x)-\overline{y}_{ps}(x)\|_\infty$	$\|y(x)-\overline{y}_{mps}(x)\|_\infty$	$\|y(x)-\overline{y}_{pps}(x)\|_\infty$
5			2.22×10^{-15}	9.09×10^{-16}
8			1.63×10^{-15}	
9				2.53×10^{-16}
12			1.38×10^{-15}	
17				1.25×10^{-16}

Table C.6:

N	$\|y(x)-\overline{y}_t(x)\|_\infty$	$\|y(x)-\overline{y}_{ps}(x)\|_\infty$	$\|y(x)-\overline{y}_{pps}(x)\|_\infty$
5	8.31×10^{-2}	7.64×10^{-2}	1.07×10^{-1}
8	8.75×10^{-1}	8.86×10^{-1}	8.73×10^{-1}
9	1.54×10^{-2}	3.97×10^{-2}	4.86×10^{-2}
17	1.12×10^{-2}	2.05×10^{-2}	2.17×10^{-2}

Table C.7:

N	$\|y(x)-\overline{y}_t(x)\|_\infty$	$\|y(x)-\overline{y}_{ps}(x)\|_\infty$	$\|y(x)-\overline{y}_{pps}(x)\|_\infty$
5	3.26×10^{-1}	5.99×10^{-1}	1.84×10^{-1}
8	2.42×10^{-1}	9.59×10^{-1}	2.10×10^{-2}
16	2.05×10^{-1}	5.27×10^{-1}	7.15×10^{-2}

Appendix D

List of Publications

Below are the details of the publication produced in the course of this research:

1. B. S. Ogundare & G. E. Okecha. (2007): Boundedness, Periodicity and Stability of Solutions to $\ddot{x}(t) + a(t)g(\dot{x}) + b(t)h(x) = p(t; x, \dot{x})$, Math. Sci. Res. J.11(5)2007, 432-443.

2. Ogundare, B. S. & Okecha, G. E. (2007): Convergence of Solutions of Certain Fourth Order Nonlinear Differential Equations. IJMMS Vol. 2007, Article ID 12536, 13 pages.

3. B. S. Ogundare & G. E. Okecha. (2008): On the Boundedness and the Stability of Solution of Third Order non-Linear Differential Equations. Ann. of. Diff. Eqs. 24(1) 2008, 1-8.

4. B. S. Ogundare & G. E. Okecha (2008): On the Boundedness and the Stability Results for the Solutions of Certain Fourth Order Non-Linear Differential Equations , Nonlinear Studies, Vol. 15, No. 1, 2008, 61-70.

5. Ogundare, B. S. & Okecha, G. E. (2008): A Pseudo Spline Methods for solving Initial Value Problem of Ordinary Differential Equation. Journal of Mathematics and Statistics (JMSS) 4(2) 2008, 117-121.

6. Ogundare, B. S. (2009): On the Pseudo-Spectral Method of Solving Linear Ordinary Differential Equations. Journal of Mathematics and Statistics (JMSS) 5(2) 2009, 136-140.