

PRIME NEAR-RING MODULES AND THEIR
LINKS WITH THE GENERALISED GROUP
NEAR-RING

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PREFACE

In view of the facts that the definition of a ring led to the definition of a near-ring, the definition of a ring module led to the definition of a near-ring module, prime rings resulted in investigations with respect to primeness in near-rings, one is naturally inclined to attempt to define the notion of a group near-ring seeing that the group ring had already been defined and investigated into by, inter alia, Groenewald in [7]. However, in trying to define the group near-ring along the same lines as the group ring was defined, it was found that the resulting multiplication was, in general, not associative in the near-ring case due to the lack of one distributive property.

In 1976, Meldrum [19] achieved success in defining the group near-ring. However, in his definition, only distributively generated near-rings were considered and the distributive generators played a vital role in the construction. In 1989, Le Riche, Meldrum and van der Walt [17], adopted a similar approach to that which led to a successful and fruitful definition of matrix near-rings, and defined the group near-ring in a more general sense. In particular, they defined $R[G]$, the group near-ring of a group G over a near-ring R , as a subnear-ring of $M(R^G)$, the near-ring of all mappings of the group R^G into itself.

More recently, Groenewald and Lee [14], further generalised the definition of $R[G]$ to $R[S : M]$, the generalised semigroup near-ring of a semigroup S over any faithful R -module M . Again, the natural thing to do would be to extend the results obtained for $R[G]$ to $R[S : M]$, and this they achieved with much success.

In this thesis, we define $R[G : M]$, the generalised group near-ring of a group G over any faithful R -module M , as a subnear-ring of all mappings from M^G to M^G . Immediately, we realise that $R[G] \subseteq R[G : M] \subseteq R[S : M]$ and hence all results that apply to $R[S : M]$ must obviously apply to $R[G : M]$. We include some of these results (with modifications in some cases), and we also bring in some additional results. However, our main purpose here was to investigate primeness in $R[G : M]$. This brought us to the realisation that since $R[G : M]$ was constructed over the R -module M , we first needed to investigate primeness in M - an area where very little work has been done. Hence, we decided to follow the following path in this thesis.

In the first chapter, we define (as usual) the basic and essential concepts related to near-rings, near-ring modules, prime near-rings, radicals of near-rings, and to the group near-ring, $R[G]$.

In chapter 2, we let M be a faithful left R -module, where R is a right near-ring, and we proceed to define the notions of prime, semiprime, s -prime and strongly prime in M . As in the case of R , we distinguish between various types of (related, but nonequivalent) primes and semiprimes in M and consequently, based on our definition of an s -prime module, we were also able to identify various types of s -primes in M . Besides the many general results we obtain with respect to prime modules, we also investigate the inter-relationships between primeness in M and primeness in R . However, to successfully achieve a two-way relationship, we are compelled to introduce the idea of a multiplication module. This we do at the end of this chapter.

In chapter 3, we turn our attention to some radical theory with respect to near-ring modules. We define special classes of near-ring modules, and then show that classes of some of the primes defined in chapter 2 turn out to be special radical classes in our sense of the definition.

In the final chapter, we define the generalised group near-ring, $R[G : M]$. We, then, establish some general results on $R[G : M]$, but, in this chapter our main focus is to establish links between primeness in $R[G : M]$ and primeness in the R -module M and/ or the base near-ring R . To this end, we achieved much

success in commuting from $R[G : M]$ to M and/ or R , but the return journey presented may obstacles. However, we conclude this chapter (and this thesis) with a flicker of hope by showing that if R is a near-field and G is an ordered group, then $R[G]$ (ie. $R[G : M]$ with $M = {}_R R$) is 2-prime.

All near-rings in this thesis will be zerosymmetric right near-rings.

Chapter 1

PRELIMINARIES

In this chapter, we present some results (many known) which will be used recurrently in the latter chapters. In some cases, proofs may be included. For most of the results that follow, we refer the reader to either Meldrum [18] or Pilz [21].

1.1 Near-rings

Definition 1 (*Near-ring*)

A set R together with the two operations of addition and multiplication (written as $(R, +, \cdot)$) is called a near-ring if the following conditions are satisfied:

- (a) $(R, +)$ is a group.
- (b) (R, \cdot) is a semigroup ie. R is closed and associative under multiplication.
- (c) At least one of the following two distributive conditions hold:

$$(i) \quad a(b + c) = ab + ac \quad \text{or} \quad (ii) \quad (a + b)c = ac + bc$$

for all $a, b, c \in R$.

Remark 2 (a) If $\mathbf{c}(i)$ holds, then R is called a left near-ring and if $\mathbf{c}(ii)$ holds, then R is called a right near-ring.

- (b) R is called a near-ring with identity if R has multiplicative identity.
- (c) If R is a right near-ring, then it is always true that $0.r = 0$ for all $r \in R$.
If it is also true that $r.0 = 0$ for all $r \in R$, then R is referred to as a zerosymmetric right near-ring.
- (d) If $(R, +)$ is an abelian group, then R is called an abelian near-ring.

From here on, all near-rings will be zerosymmetric right near-rings.

Definition 3 (*R-Subgroup*)

A subset H of R is called a (two sided or invariant) R -subgroup of R if:

- (a) $(H, +)$ is a subgroup of $(R, +)$.
- (b) $RH \subseteq H$.
- (c) $HR \subseteq H$.

If in the above definition, (a) and (b) are satisfied, then H is called a left R -subgroup whereas if (a) and (c) are satisfied, then H is called a right R -subgroup. (If H is a subgroup of R , this will be denoted by $H \leq R$)

Definition 4 (*Normal Subgroup*)

An R -subgroup H of R is called a normal subgroup if for all $r \in R$ and all $h \in H$, we have $r + h - r \in H$.

Theorem 5 If $a \in R$, then Ra is a left R -subgroup of R .

Proof. Let $r_1a, r_2a \in Ra$. Then $r_1a - r_2a = (r_1 - r_2)a \in Ra$. Furthermore, $R(Ra) = (RR)a \subseteq Ra$. So Ra is a left R -subgroup of R . ■

Definition 6 (*Ideal*)

A subset I of R is called a (two sided) ideal of R if

- (a) $(I, +)$ is a normal subgroup of $(R, +)$.
- (b) $IR \subseteq I$.

(c) $r_1(r_2 + i) - r_1r_2 \in I$ for all $r_1, r_2 \in R$ and $i \in I$.

If I satisfies conditions (a) and (b), I is called a right ideal of R while I is called a left ideal of R if (a) and (c) are satisfied. (An ideal I of R will be denoted by $I \triangleleft R$, and for left ideals and right ideals we use the notations \triangleleft_l and \triangleleft_r respectively).

Definition 7 (Essential Ideal)

An ideal I of R is called an essential ideal (denoted by $I \triangleleft \cdot R$) if $I \cap A \neq 0$ for every $0 \neq A \triangleleft R$.

Definition 8 (Annihilator)

Let $I \triangleleft R$. Then we define the left annihilator of I in R as:

$$l(I) = \{r \in R : rI = 0\}.$$

Theorem 9 If I is a left R -subgroup of R , then $l(I) \triangleleft R$.

Definition 10 (Quotient Near-ring)

Let $I \triangleleft R$. Let $\frac{R}{I} = \{r + I : r \in R\}$ be the set of cosets of I in R . Then $(\frac{R}{I}, +, \cdot)$ is called the quotient near-ring of R over I where $+$ and \cdot are defined by:

$$\begin{aligned} (r_1 + I) + (r_2 + I) &= (r_1 + r_2) + I \\ (r_1 + I) \cdot (r_2 + I) &= (r_1 \cdot r_2) + I \end{aligned}$$

for all $r_1, r_2 \in R$.

Definition 11 (Prime Ideals)

Let $P \triangleleft R$. Then P is called:

- (a) 0-prime if for all ideals A, B of R , $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.
- (b) 1-prime if for all left ideals A, B of R , $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.
- (c) 2-prime if for all left R -subgroups A, B of R , $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

- (d) 3-prime if $a, b \in R$ and $aRb \subseteq P$ implies that $a \in P$ or $b \in P$.
- (e) completely prime (c-prime) if $a, b \in R$ and $ab \in P$ implies that $a \in P$ or $b \in P$.

Definition 12 (Semiprime Ideals)

Let $P \triangleleft R$. Then P is called:

- (a) 0-semiprime if for all ideals A of R such that $A^2 \subseteq P$, we have $A \subseteq P$.
- (b) 1-semiprime if for all left ideals A of R such that $A^2 \subseteq P$, we have $A \subseteq P$.
- (c) 2-semiprime if for all left R -subgroups A of R such that $A^2 \subseteq P$, we have $A \subseteq P$.
- (d) 3-semiprime if $a \in R$ and $aRa \subseteq P$ implies that $a \in P$.
- (e) completely semiprime (c-semiprime) if $a \in R$ and $a^2 \in P$ implies that $a \in P$.

Definition 13 If $\nu = 0, 1, 2, 3, c$, then R is called a ν -prime (ν -semiprime) near-ring if $\{0\}$ is a ν -prime (ν -semiprime) ideal of R .

Theorem 14 Let $P \triangleleft R$ and let $\nu = 0, 1, 2, 3, c$. Then P is ν -prime (ν -semiprime) if and only if $\frac{R}{P}$ is a ν -prime (ν -semiprime) near-ring.

Remark 15 Let $P \triangleleft R$. Then the following have already been established in [4]:

- (a) P is c-prime $\implies P$ is 3-prime $\implies P$ is 2-prime $\implies P$ is 1-prime $\implies P$ is 0-prime.
- (b) If R has multiplicative identity, then P is 2-prime $\iff P$ is 3-prime.
- (c) P is c-semiprime $\implies P$ is 3-semiprime $\implies P$ is 2-semiprime $\implies P$ is 1-semiprime $\implies P$ is 0-semiprime.
- (d) If R has multiplicative identity, then P is 2-semiprime $\iff P$ is 3-semiprime.

(e) If $\nu = 0, 1, 2, 3, c$ and P is ν -prime, then P is ν -semiprime.

Definition 16 (*Equiprime Ideals*)

Let $P \triangleleft R$. Then P is said to be equiprime if $a \in R \setminus P$ and $x, y \in R$, then $arx - ary \in P$ for all $r \in R$ implies $x - y \in P$. The near-ring R is called equiprime if $\{0\}$ is an equiprime ideal of R .

Definition 17 (*Nil Ideal*)

An element $r \in R$ is said to be nilpotent if there exists $n \in \mathbb{N}$ such that $r^n = 0$. An ideal A of R is called a nil ideal if every $a \in A$ is nilpotent.

Definition 18 (*Near-field*)

Let $(R, +, \cdot)$ be a near-ring and let $R^* = R \setminus \{0\}$. If (R^*, \cdot) is a group, then $(R, +, \cdot)$ is called a near-field.

1.2 Near-ring Modules

Definition 19 (*R-module, R-submodule*)

Let R be a near-ring and $(M, +)$ be a group.

- (a) Then M is called a left R -module if for all $r_1, r_2 \in R$ and $m \in M$, it follows that $(r_1 + r_2)m = r_1m + r_2m$ and $(r_1r_2)m = r_1(r_2m)$.
- (b) Let $H \leq M$ such that for all $r \in R$ and $h \in H$, we have that $rh \in H$. Then H is called an R -submodule of M . (We shall denote this by $H \leq_R M$).

Remark 20 If R is a near-ring, then an obvious R -module is the group $(R, +)$. We shall denote this particular R -module by ${}_R R$.

Definition 21 An R -module M will be called an abelian module if $(M, +)$ is an abelian group.

Proposition 22 If M is an R -module and $m \in M$, then Rm is an R -submodule of M .

Proof. Similar to the proof of Theorem 5. ■

Definition 23 (*R-ideal*)

An *R-ideal* of M is a normal subgroup P of M such that for all $r \in R, m \in M$ and $n \in P, r(m+n) - rm \in P$. (We shall denote this as $P \triangleleft_R M$).

Note that if R is a zerosymmetric near-ring, then every R -ideal of M is an R -submodule of M . Furthermore, if $M = {}_R R$, then the R -ideals of M are essentially the left ideals of R and the R -submodules of M are the left R -subgroups of R . The following construction from a subset of M plays an important role in the study of structural relationships between R and M .

Definition 24 (*The set $(P : M)_R$*)

Let $P \subseteq M$. Then we define the set, written as $(P : M)_R$, by:

$$(P : M)_R = \{r \in R : rM \subseteq P\}.$$

If no confusion arises, we will simply write $(P : M)$ in place of $(P : M)_R$. Furthermore, for our purposes here, we will write $\tilde{P} = (P : M)_R$.

Theorem 25 *Let M be an R -module and let $P \subseteq M$. Then:*

- (a) *If P is an R -submodule of M , then \tilde{P} is a left R -subgroup of R .*
- (b) *If P is an R -ideal of M , then \tilde{P} is an ideal of R .*

Proof. (a) Let $a, b \in \tilde{P}$. Then $aM \subseteq P$ and $bM \subseteq P$. Since P is an R -submodule, $(a-b)M = aM - bM \subseteq P$ and hence $a-b \in \tilde{P}$. Now let $r \in R$ and $p \in \tilde{P}$. Then $pM \subseteq P$. So $rpM \subseteq rP \subseteq P$ implies that $rp \in \tilde{P}$ whence $R\tilde{P} \subseteq \tilde{P}$. This proves that \tilde{P} is a left R -subgroup of R .

(b) Following the method for (a), we can show that \tilde{P} is a subgroup of R .

Let $r \in R, a \in \tilde{P}$ and $m \in M$. Then $(r+a-r)m = rm + am - rm \in P$ since $rm \in M, am \in P$ and P is a normal subgroup of M . So $r+a-r \in \tilde{P}$ implies that \tilde{P} is a normal subgroup of R .

Now let $r_1, r_2 \in R$ and $a \in \tilde{P}$. Then, if $m \in M$, we have:

$$[r_1(r_2+a) - r_1r_2]m = r_1(r_2m + am) - r_1(r_2m) \in P$$

since $r_1 \in R, r_2m \in M, am \in P$ and P is an R -ideal of M .

Hence $r_1(r_2 + a) - r_1r_2 \in \tilde{P}$.

Finally, let $a \in \tilde{P}$ and $r \in R$. Then $aM \subseteq P$ and we get:

$$(ar)M = a(rM) \subseteq aM \subseteq P$$

So $ar \in \tilde{P}$ implies that $\tilde{P}R \subseteq \tilde{P}$, and the proof is complete. ■

Proposition 26 *Let $P \triangleleft R$. Then $\frac{R}{P}$ is an R -module with scalar multiplication defined by:*

$$r(r_1 + P) = rr_1 + P \text{ where } r, r_1 \in R.$$

Proof. Let $r_1, r_2, r_3 \in R$. Then:

$$\begin{aligned} \text{(a)} \quad & (r_1 + r_2)(r_3 + P) \\ &= [(r_1 + r_2)r_3] + P \\ &= (r_1r_3 + r_2r_3) + P \\ &= (r_1r_3 + P) + (r_2r_3 + P) \\ &= r_1(r_3 + P) + r_2(r_3 + P). \end{aligned}$$

$$\text{(b)} \quad (r_1r_2)(r_3 + P) = (r_1r_2r_3) + P = r_1(r_2r_3 + P). \quad \blacksquare$$

Proposition 27 *Let $P \triangleleft R$. Then:*

(a) *If K is an R -submodule of the R -module $\frac{R}{P}$, then $K = \frac{L}{P}$ for some left R -subgroup L of R containing P .*

(b) *If K is an R -ideal of the R -module $\frac{R}{P}$, then $K = \frac{L}{P}$ for some left ideal L of R containing P .*

Proof. Consider the R -modules, ${}_R R$ and $\frac{R}{P}$. Then, by [18, Lemma 2.24], $\pi: {}_R R \longrightarrow \frac{R}{P}$ defined by $\pi(r) = r + P$ is an R -epimorphism. Hence, by [18, Theorem 2.26], there is a one-to-one correspondence between the R -submodules (R -ideals) of $\frac{R}{P}$ and the R -submodules (R -ideals) of ${}_R R$ containing P . But the R -submodules of ${}_R R$ are the left R -subgroups of R and the R -ideals of ${}_R R$ are the left ideals of R . Hence, since π preserves and reflects inclusions, the results follow. ■

Definition 28 (*Quotient Group*)

Let M be an R -module and $P \triangleleft_R M$. Then $\frac{M}{P} = \{m + P : m \in M\}$ is called the quotient group of M over P .

Theorem 29 Let M be an R -module and $P \triangleleft_R M$. Then the quotient group, $\frac{M}{P}$, is an R -module (called the quotient R -module) with scalar multiplication defined by:

$$r(m + P) = rm + P \text{ where } r \in R \text{ and } m \in M.$$

Proof. Same as the proof of Proposition 26 ■

Definition 30 (*Left Annihilator*)

If $P \subseteq M$, then the left annihilator of P in R is defined by:

$$l(P) = \{r \in R : rP = 0\}.$$

Remark 31 From the above definition, it clearly follows that $l(P) = (0 : P)_R$.

Theorem 32 Let M be an R -module and let $P \subseteq M$. Then:

(a) $l(P)$ is a left ideal of R .

(b) If P is an R -submodule of M , then $l(P)$ is an ideal of R .

Proof. (a) Let $x, y \in l(P)$. Then $xP = yP = 0$. So $(x - y)P = xP - yP = 0$ implies that $x - y \in l(P)$.

Now let $r \in R$ and $x \in l(P)$. Then $xP = 0$ implies:

$$(r + x - r)P = rP + xP - rP = rP - rP = 0$$

Hence $r + x - r \in l(P)$ and we have that $l(P)$ is a normal subgroup of R .

Finally, let $r_1, r_2 \in R$ and $x \in l(P)$. Then $xP = 0$ implies:

$$\begin{aligned} & [r_1(r_2 + x) - r_1r_2]P \\ &= r_1(r_2P + xP) - (r_1r_2)P \\ &= r_1r_2P - r_1r_2P \\ &= 0. \end{aligned}$$

Therefore, $l(P)$ is a left ideal of R .

(b) As in (a), we can show that $l(P)$ is a left ideal of R . To show that $l(P)$ is also a right ideal of R , let $x \in l(P)$ and $r \in R$. Then $(xr)P = x(rP) \subseteq xP = 0$. Hence $xr \in l(P)$ and so we have that $l(P)R \subseteq l(P)$; thus proving that $l(P)$ is an ideal of R . ■

Proposition 33 [21, Proposition 3.14] *Let R be a near-ring and let $I \triangleleft R$.*

Then:

(a) *If M is an $\frac{R}{I}$ -module, then under the scalar multiplication $rm = (r + I)m$, M becomes an R -module with $I \subseteq (0 : M)_R$.*

(b) *If M is an R -module and $I \subseteq (0 : M)_R$, then M is an $\frac{R}{I}$ -module with respect to $(r + I)m = rm$.*

(c) *In both cases, $(0 : M)_{\frac{R}{I}} = \frac{(0 : M)_R}{I}$*

Proof. (a) Let $r_1, r_2 \in R$ and $m \in M$. Then:

$$(i) \quad (r_1 + r_2)m = ((r_1 + r_2) + I)m = ((r_1 + I) + (r_2 + I))m = (r_1 + I)m + (r_2 + I)m = r_1m + r_2m.$$

$$(ii) \quad (r_1r_2)m = (r_1r_2 + I)m = (r_1(r_2 + I))m = r_1((r_2 + I)m) = r_1(r_2m).$$

So M is an R -module.

Furthermore, if $x \in I$, then $x + I = 0 \implies (x + I)m = 0 \implies xm = 0 \implies x \in (0 : M)_R$.

Therefore $I \subseteq (0 : M)_R$.

(b) Reverse the process in (a).

(c) Let $x \in (0 : M)_{\frac{R}{I}}$. Then $x = r + I \in \frac{R}{I}$ for some $r \in R$ such that $xM = 0$. Now $rM = (r + I)M = xM = 0$. Hence $r \in (0 : M)_R$ which implies that $x \in \frac{(0 : M)_R}{I}$. So $(0 : M)_{\frac{R}{I}} \subseteq \frac{(0 : M)_R}{I}$.

On the other hand, let $x \in \frac{(0 : M)_R}{I}$. Then $x = r + I$ for some $r \in (0 : M)_R$. Then $rM = 0$ and we have:

$$xM = (r + I)M = rM + I = I$$

So $xM = 0$ and $x \in \frac{R}{I}$ implies that $x \in (0 : M)_{\frac{R}{I}}$. Hence we have that $\frac{(0 : M)_R}{I} \subseteq (0 : M)_{\frac{R}{I}}$ and the proof is complete. ■

Definition 34 (*Monogenic, Faithful, Tame*)

Let M be an R -module. Then M is called:

- (a) *monogenic* if there exists $m_0 \in M$ (called the generator of M) such that $Rm_0 = M$.
- (b) *faithful* if $r \in R$ and $rM = 0$ implies $r = 0$.
- (c) *tame* if every R -submodule of M is also an R -ideal of M .

Lemma 35 [21, Proposition 3.4] Let M be a monogenic R -module with generator m_0 . Then, if A is a left ideal of R , Am_0 is an R -ideal of M .

Proof. If $a_1m_0 \in Am_0$ and $a_2m_0 \in Am_0$ where $a_1, a_2 \in A$, then:

$$a_1m_0 - a_2m_0 = (a_1 - a_2)m_0 \in Am_0 \text{ since } a_1 - a_2 \in A.$$

So Am_0 is a subgroup of M .

Let $m \in M$. Then, since M is monogenic, $m = rm_0$ for some $r \in R$.

Therefore, if $am_0 \in Am_0$, we get:

$$m + am_0 - m = rm_0 + am_0 - rm_0 = (r + a - r)m_0 \in Am_0 \text{ since } r + a - r \in A.$$

Therefore Am_0 is a normal subgroup of M .

Finally, if $r \in R$, $r_1m_0 = m \in M$ and $am_0 \in Am_0$, then:

$r(r_1m_0 + am_0) - r(r_1m_0) = [r(r_1 + a) - rr_1]m_0 \in Am_0$, since A is a left ideal of R implies that $r(r_1 + a) - rr_1 \in A$.

Hence Am_0 is an R -ideal of M . ■

We conclude this section with some important isomorphisms which will prove to be very useful in the third and fourth chapters.

Definition 36 [18, Definition 2.22] Let M be an R -module and I be an R -ideal of M . Then $\pi : M \longrightarrow \frac{M}{I}$ defined by $\pi(m) = m + I$ is called the natural R -homomorphism associated with I .

Lemma 37 [18, Lemma 2.24] *The natural homomorphism, π , is an R -epimorphism such that $\text{Ker } \pi = I$.*

Theorem 38 [18, Theorem 2.26] *Let M be an R -module and let A and B be R -ideals of M with $A \supseteq B$. Then:*

$$\frac{\frac{M}{B}}{\frac{A}{B}} \cong_R \frac{M}{A}$$

Theorem 39 [18, Theorem 2.28] *Let M be an R -module, P an R -ideal and H an R -submodule of M . Then:*

(a) $P + H$ is an R -submodule of M .

(b) $\frac{P + H}{P} \cong_R \frac{H}{P \cap H}$.

1.3 Radicals of Near-rings

Definition 40 (Closure)

A class \mathcal{R} of near-rings is said to be:

- (a) *homomorphically closed if it is closed under homomorphic images (that is, if $I \in \mathcal{R}$ and J is a homomorphic image of I , then $J \in \mathcal{R}$).*
- (b) *essentially closed if E is an essential ideal of a near-ring R and $E \in \mathcal{R}$ implies that $R \in \mathcal{R}$.*
- (c) *closed under ideals if $I \triangleleft R$ and $R \in \mathcal{R}$ implies that $I \in \mathcal{R}$.*
- (d) *closed under subdirect sums if any subdirect sum of elements from \mathcal{R} also belongs to \mathcal{R} .*

Definition 41 *Let \mathcal{V} denote any class of near-rings. Then:*

- (a) \mathcal{V} is called a universal class if it is closed with respect to homomorphic images and ideals.
- (b) \mathcal{V} is called a variety if it is closed with respect to homomorphic images and subdirect sums.

Definition 42 (*Kurosh-Amitsur Radical class*)

A subclass \mathcal{R} in the variety \mathcal{V} of all near-rings is called a Kurosh-Amitsur radical class (*KA-radical*) if \mathcal{R} satisfies:

(K1) \mathcal{R} is homomorphically closed.

(K2) $\mathcal{R}(N) = \sum\{A \triangleleft N : A \in \mathcal{R}\} \in \mathcal{R}$ for every $N \in \mathcal{V}$.

(K3) $\mathcal{R}\left(\frac{N}{\mathcal{R}(N)}\right) = 0$ for every $N \in \mathcal{V}$.

Let \mathcal{R} be a *KA-radical class* and let the term \mathcal{R} -ideal refer to an ideal belonging to \mathcal{R} . Then for any near-ring $R \in \mathcal{R}$, there exists an \mathcal{R} -ideal I of R which contains every other \mathcal{R} -ideal of R . This ideal is called the radical of the radical class. Many radicals have been defined for near-rings. If R is a near-ring, then some of the various radicals for R are listed below:

(a) The 0-prime radical: $\mathcal{P}_0(R) = \cap\{I \triangleleft R : I \text{ is a 0-prime ideal of } R\}$.

(b) The 2-prime radical: $\mathcal{P}_2(R) = \cap\{I \triangleleft R : I \text{ is a 2-prime ideal of } R\}$.

(c) The 3-prime radical: $\mathcal{P}_3(R) = \cap\{I \triangleleft R : I \text{ is a 3-prime ideal of } R\}$.

(d) The c -prime radical: $\mathcal{P}_c(R) = \cap\{I \triangleleft R : I \text{ is a } c\text{-prime ideal of } R\}$.

(e) The nil radical: $\mathfrak{N}(R) = \sum\{I \triangleleft R : I \text{ is a nil ideal of } R\}$.

Definition 43 (*Hoehnke Radical class*)

Let \mathcal{R} be a subclass in the variety of all near-rings. For $R \in \mathcal{R}$, let ρ be a function that associates R with its radical, $\rho(R)$. Then \mathcal{R} is called a Hoehnke radical class if for every $R \in \mathcal{R}$, ρ satisfies:

(a) $\frac{(\rho(R) + I)}{I} \subseteq \rho\left(\frac{R}{I}\right)$ for all $I \triangleleft R$

(b) $\rho\left(\frac{R}{\rho(R)}\right) = 0$ for all R .

By definition, it is clear that the prime radicals listed above are all Hoehnke radicals.

We will use \mathcal{R}_0 to represent the class of zerosymmetric near-rings.

1.4 Group Near-rings

For the definitions and other basic concepts in this section, we refer you to Le Riche, Meldrum and Van der Walt [17].

We begin with the following construction:

Let R be a near-ring with identity 1 , G be a (multiplicatively written) group with identity e , and let R^G denote the cartesian direct sum of $|G|$ copies of $(R, +)$ indexed by the elements of G . Then $M(R^G)$ is the right near-ring of all mappings of the group R^G into itself. Now, for $r \in R$ and $g \in G$, let $[r, g]$ denote the function in $M(R^G)$ defined by:

$$([r, g](\mu))(h) = r\mu(hg) \text{ for all } \mu \in R^G, h \in G.$$

Definition 44 (*Group Near-ring*)

The subnear-ring of $M(R^G)$, generated by the set $\{[r, g] : r \in R, g \in G\}$, will be called the group near-ring constructed from R and G and will be denoted by $R[G]$.

A powerful method for proofs in $R[G]$ is by using induction on the complexity of an element A of $R[G]$. Hence, we provide the following definitions:

Definition 45 (*Generating Sequence*)

A generating sequence for an element A of $R[G]$ is a finite sequence A_1, A_2, \dots, A_n of elements of $R[G]$ such that $A_n = A$, and for all i , $1 \leq i \leq n$, one of the following three cases applies:

- (a) $A_i = [r, g]$ for some $r \in R, g \in G$.
- (b) $A_i = A_k + A_l$ for some k and l where $1 \leq k, l < i$.
- (c) $A_i = A_k A_l$ for some k and l where $1 \leq k, l < i$.

Definition 46 (*Complexity*)

The length of a generating sequence of minimal length for $A \in R[G]$ will be called the complexity of A and denoted by $c(A)$.

Remark 47 *From the above definitions, it is clear that:*

- (a) $c(A) \geq 1$ for all $A \in R[G]$.
- (b) $c(A) = 1$ if and only if $A = [r, g]$ for some $r \in R, g \in G$.
- (c) If $c(A) > 1$, then $A = B + C$ or $A = BC$ for some $B, C \in R[G]$ with $c(B) < c(A)$ and $c(C) < c(A)$.

Le Riche, Meldrum and Van der Walt [17] associated two important and useful ideals of $R[G]$ with an ideal of R . These are defined next:

Definition 48 *(The ideals \mathcal{A}^* and \mathcal{A}^+)*

Let \mathcal{A} be an ideal of the near-ring R . Then the following are ideals of $R[G]$:

- (a) $\mathcal{A}^* = \{A \in R[G] : A\mu(g) \in \mathcal{A} \text{ for all } \mu \in R^G, g \in G\}$, and
- (b) $\mathcal{A}^+ = \langle \{[a, e] : a \in \mathcal{A}\} \rangle$ (that is, the ideal that is generated by the set $\{[a, e] : a \in \mathcal{A}\}$).

Chapter 2

PRIMENESS IN NEAR-RING MODULES

INTRODUCTION

The notions of prime rings, semiprime rings, s -prime rings and strongly prime rings and their extensions to ring modules have been extensively researched by various authors, some of whom we shall cite in the main content of this chapter. The natural question one would then ask is what about the extension of the definitions of the various types of prime rings to near-rings? In 1970, Holcombe [16] introduced the notion of a prime near-ring. However, due to the lack of one distributive property as well as the fact that addition is in general noncommutative in a near-ring, he was able to define three types of prime near-rings within the class of all prime near-rings. This resulted in the definitions of 0-prime (or prime), 1-prime and 2-prime near-rings. Groenewald [8] further added to these three types by introducing 3-prime and c -prime (completely prime) near-rings. All of these five types were found to be equivalent in the class of associative rings but, in general, different within the class of near-rings.

Similarly, in a natural way, there followed the definition of a semiprime near-

ring (again five distinct types, ν -semiprime where $\nu = 0, 1, 2, 3, c$). In 1964, s -prime near-rings were introduced by van der Walt [22] and in 1988 Groenewald [9] introduced the notion of a strongly prime near-ring.

In particular, we may view the various primes in a near-ring R as primes in the R -module $(R, +)$. Now let R be a right near-ring and let M be any left R -module. In this chapter, we attempt to generalise the various notions of primeness that were defined in R to the module M . Booth and Groenewald [5] have already done this in the case of equiprime near-rings. So, for the purpose of completeness and since many of their results can be extended to the various other notions of primes, we begin this chapter by capturing some of their results on equiprime near-ring modules. Thereafter, we provide various characterizations of the different types of prime modules and show equivalences between these characterizations. This results in the observation that, in general, we cannot distinguish between 0-prime and 1-prime near-ring modules. Thus 1-prime modules were omitted from further investigations. Furthermore, if P is a prime (semiprime, s -prime, strongly prime) R -ideal of the module M , we show that the ideal $\tilde{P} = (P : M)_R$ of R is also prime (semiprime, s -prime, strongly prime). However, in some cases the reverse implication created some difficulties. To overcome these problems, in the last section we introduce the concept of a multiplication module.

In this chapter, R will denote a zerosymmetric right near-ring (without identity, unless specified) and M a left R -module. Furthermore, if no ambiguity is created, we will simply write the ideal $(P : M)_R$ of R as $(P : M)$.

2.1 Equiprime Modules

Based on the definition and characterization of prime modules for rings, Booth and Groenewald [5] provided a similar characterization for equiprime modules of a near-ring. In this short section, we state some of their results. In section 2.2 we will observe that many of their results are analogously prevalent in the

various notions of prime near-ring modules that we shall define.

Definition 49 *Let M be an R -module and P be an R -ideal of M . Then:*

(a) *M is called equiprime if $RM \neq 0$ and the following condition is satisfied:*

If $a \in R$ with $a \notin (0 : M)$ and $m, m' \in M$, then $arm - arm' = 0$ for all $r \in R$ implies that $m - m' = 0$.

(b) *P is an equiprime R -ideal if $a \in R$ with $a \notin (P : M)$ and $m, m' \in M$, then $arm - arm' \in P$ for all $r \in R$ implies that $m - m' \in P$.*

If P is an R -ideal of M , then it is well known that $\tilde{P} = (P : M)$ is an ideal of R . The following proposition shows the relationship between the equiprimeness of P and that of \tilde{P} .

Proposition 50 *If P is an equiprime R -ideal of the R -module M , then \tilde{P} is an equiprime ideal of R .*

Proof. Let $a \in R \setminus \tilde{P}$ and $x, y \in R$ such that $arx - ary \in \tilde{P}$ for all $r \in R$. Then $(arx - ary)M \subseteq P$ for all $r \in R$. Hence $ar(xm) - ar(ym) = (arx - ary)m \in P$ for all $r \in R$ and for all $m \in M$. Since P is equiprime and $a \notin \tilde{P}$, we have that $(x - y)m = xm - ym \in P$ for all $m \in M$. So $(x - y)M \subseteq P$ implies that $x - y \in \tilde{P}$. ■

Corollary 51 *If M is an equiprime R -module, then $(0 : M)$ is an equiprime ideal of R .*

Proposition 52 [5, Proposition 2.1] *Let R be a near-ring and $P \triangleleft R$ with $P \neq R$. Then the following are equivalent:*

- (a) *P is an equiprime ideal of R .*
- (b) *There exists an equiprime R -module M such that $P = (0 : M)$.*

Corollary 53 [5, Corollary 2.2] *If $0 \neq R$ is a near-ring, then R is equiprime if and only if there exists a faithful equiprime R -module M .*

Proposition 54 [5, Lemma 2.3] *Let M be an equiprime R -module and suppose that $0 \neq H$ is an R -submodule of M . Then:*

- (a) $(0 : M)_R = (0 : H)_R$.
- (b) H is an equiprime R -submodule.

Proposition 55 [5, Proposition 2.5] *Let M be an equiprime R -module and let A be an invariant subgroup of R such that $A \not\subseteq (0 : M)$. Then M is an equiprime A -module.*

2.2 Prime Modules

We begin this section with the definition of a ν -prime R -ideal ($\nu = 0, 1, 2, 3, c$) of the R -module M . We, then, proceed by providing equivalent definitions to the initial definition of a ν -prime R -ideal and by investigating various characterisations (including those with respect to annihilators and m -systems) of the ν -prime R -ideals. Furthermore, as was done in the case of prime near-rings, we look at the inter-relationships between the five (essentially four) types of prime R -ideals (modules) defined in this section. Of importance, also, is the link between the primeness of an R -ideal of M to that of an ideal of the base near-ring \tilde{R} . To this end, we show that if $P \triangleleft_R M$ is ν -prime, then so is the ideal \tilde{P} of R . We conclude this section with some hereditary properties concerning the ν -prime R -ideals (modules).

Definition 56 *Let $P \triangleleft_R M$ such that $RM \not\subseteq P$. Then P is called:*

- (a) *0-prime if $AB \subseteq P$ implies $AM \subseteq P$ or $B \subseteq P$ for all ideals, A of R , and all R -ideals, B of M .*
- (b) *1-prime if $AB \subseteq P$ implies $AM \subseteq P$ or $B \subseteq P$ for all left ideals, A of R , and all R -ideals, B of M .*
- (c) *2-prime if $AB \subseteq P$ implies $AM \subseteq P$ or $B \subseteq P$ for all left R -subgroups, A of R , and all R -submodules, B of M .*

- (d) 3-prime if $rRm \subseteq P$ implies that $rM \subseteq P$ or $m \in P$ for all $r \in R$ and $m \in M$.
- (e) completely prime (c-prime) if $rm \in P$ implies that $rM \subseteq P$ or $m \in P$ for all $r \in R$ and $m \in M$.

Definition 57 M is said to be a ν -prime ($\nu = 0, 1, 2, 3, c$) R -module if $RM \neq 0$ and 0 is a ν -prime R -ideal of M .

Note that the definitions of the 0-prime, 1-prime and 2-prime R -ideals (or modules) involve some substructure of R . In the results that follow, we provide equivalent definitions to show that in the case of prime R -ideals, the substructures of R can be reduced to elements of R while, in the case of prime modules, the definitions can be reduced to substructures in M only. However, we first need the following lemma:

Lemma 58 Let $P \triangleleft_R M$ and $B \leq_R M$. Then:

$$B \not\subseteq P \text{ implies that } (P : B) = (P : P + B).$$

Proof. Let $a \in (P : P + B)$. Then $a(P + B) \subseteq P$. Since $B \subseteq P + B$, it follows that $aB \subseteq P$. Hence $a \in (P : B)$ and therefore it follows that $(P : P + B) \subseteq (P : B)$.

On the other hand, let $a \in (P : B)$. We need to show that $a(p + b) \in P$ for all $p \in P$ and $b \in B$. Now $a(p + b) = [a(p + b) - ab] + ab$. Since P is an R -ideal and since $ab \in P$, we have that $[a(p + b) - ab] + ab \in P$, whence it follows that $a(p + b) \in P$. Therefore $a(P + B) \subseteq P$ implies $a \in (P : P + B)$. So $(P : B) \subseteq (P : P + B)$, and the proof is complete. ■

Corollary 59 Let P and B both be R -ideals of M . Then $B \not\subseteq P$ implies that $(P : B) = (P : P + B)$.

Proof. Since R is a zerosymmetric near-ring, the R -ideal B of M is an R -submodule of M . So the proof follows as for the previous lemma. ■

Proposition 60 Let $P \triangleleft_R M$. Then the following are equivalent:

- (a) P is a 2-prime R -ideal.
- (b) For all $a \in R$ and R -submodules B of M such that $aB \subseteq P$, it follows that $aM \subseteq P$ or $B \subseteq P$.
- (c) For all $a \in R$ and $b \in M$ such that $a[b]_R \subseteq P$, it follows that $aM \subseteq P$ or $b \in P$. (Here $[b]_R$ is the R -submodule of M generated by b).
- (d) For all R -submodules N of M such that $P \subset N$, it follows that:

$$(P : M) = (P : N).$$

Proof. (d) \Rightarrow (c) : Let $a \in R$ and let $b \in M$ such that $a[b]_R \subseteq P$. Suppose that $b \notin P$. Then we have the following two possibilities:

(i) $P \subset [b]_R$: Then $a[b]_R \subseteq P$ implies that $a \in (P : [b]_R) = (P : M)$. Hence $aM \subseteq P$.

(ii) There exists $x \in P$ such that $x \notin [b]_R$. In this case, the submodule $P + [b]_R$ strictly contains P and, by the given condition, $(P : M) = (P : P + [b]_R)$. Furthermore, $a[b]_R \subseteq P$ implies that $a \in (P : [b]_R) = (P : P + [b]_R)$ from Lemma 58. So $a \in (P : M)$ and once again we have that $aM \subseteq P$.

(c) \Rightarrow (b) : Let $a \in R$ and let B be an R -submodule of M such that $aB \subseteq P$. Then for all $b \in B$, $a[b]_R \subseteq aB \subseteq P$. So, from (c), we have that $aM \subseteq P$ or $b \in P$ for all $b \in B$. Hence $aM \subseteq P$ or $B \subseteq P$.

(b) \Rightarrow (a) : Let A be a left R -subgroup of R and B be an R -submodule of M such that $AB \subseteq P$. If $B \subseteq P$, then we are done. So suppose that $B \not\subseteq P$. Since $aB \subseteq P$ for all $a \in A$, by the given condition, $aM \subseteq P$. Since $aM \subseteq P$ for all $a \in A$, it follows that $AM \subseteq P$. So P is 2-prime.

(a) \Rightarrow (d) : Let N be an R -submodule of M such that $P \subset N$. If $x \in (P : M)$, then $xM \subseteq P$ implies that $xN \subseteq P$ and hence $x \in (P : N)$. So, clearly, we have $(P : M) \subseteq (P : N)$.

On the other hand, if $y \in (P : N)$, then the left R -subgroup of R generated by y is in $(P : N)$. Since N is an R -submodule of M and P is 2-prime, a routine

computation yields that $yM \subseteq P$ or $N \subseteq P$. Since $P \subset N$, it follows that $yM \subseteq P$ whence $y \in (P : M)$. So $(P : N) \subseteq (P : M)$. ■

In a similar way to Proposition 60 we can construct and prove equivalent definitions for 0-prime and 1-prime R -ideals. These are stated in the following proposition:

Proposition 61 *Let P be an R -ideal of M . Then the following are equivalent:*

- (a) P is a 0-prime (or 1-prime) R -ideal.
- (b) For all $a \in R$ and for all R -ideals B of M such that $aB \subseteq P$, we have that $aM \subseteq P$ or $B \subseteq P$.
- (c) For all $a \in R$ and $b \in M$ such that $a\langle b \rangle_R \subseteq P$, we have that $aM \subseteq P$ or $b \in P$. (Here $\langle b \rangle_R$ is the R -ideal of M generated by b).
- (d) For all R -ideals N of M such that $P \subset N$, we have that $(P : M) = (P : N)$.

In view of Proposition 61, we note that that the definitions of a 0-prime and a 1-prime R -ideal depend on elements of R and the same substructure of M . Thus, in general, we cannot distinguish between 0-prime and 1-prime R -ideals or R -modules. Henceforth, we will therefore restrict our investigations to 0-prime R -ideals (modules) only.

Corollary 62 *An R -module M is:*

- (a) 0-prime if and only if for all nonzero R -ideals N of M , it follows that $(0 : M) = (0 : N)$.
- (b) 2-prime if and only if for all nonzero R -submodules N of M , it follows that $(0 : M) = (0 : N)$

Proof. Follows from part (d) of Proposition 60 and Proposition 61. ■

Proposition 63 *Let M be an R -module and $P \triangleleft_R M$. Then the following are equivalent:*

(a) P is 3-prime and $(P : m) \triangleleft R$ for every $m \in M \setminus P$.

(b) $RM \not\subseteq P$ and $(P : m) = (P : M)$ for every $m \in M \setminus P$

Proof. (a) \Rightarrow (b): Since P is a 3-prime R -ideal, we have $RM \not\subseteq P$. Now let $m \in M \setminus P$ and consider $x \in (P : m)$. Since $(P : m)$ is an ideal of R , we have $xR \subseteq (P : m)$ and therefore $xRm \subseteq P$. Again, since P is 3-prime, we have that $x \in (P : M)$ or $m \in P$. But $m \notin P$. So $x \in (P : M)$ implies that $(P : m) \subseteq (P : M)$. Clearly, $(P : M) \subseteq (P : m)$.

(b) \Rightarrow (a): Let $m \in M \setminus P$. Clearly $(P : m) \triangleleft R$ since $(P : m) = (P : M)$. Let $a \in R$ be such that $aRm \subseteq P$. Since $RM \not\subseteq P$ and $(P : m) = (P : M)$ we have $Rm \not\subseteq P$. So for every $b \in Rm \setminus P$, we have:

$$(P : M) \subseteq (P : Rm) \subseteq (P : b) = (P : M) \text{ whence } (P : Rm) = (P : M).$$

Since $aRm \subseteq P$, we get $a \in (P : Rm) = (P : M)$. So $aM \subseteq P$ implies that P is a 3-prime R -ideal. ■

Proposition 64 Let $P \triangleleft_R M$. Then the following are equivalent:

(a) P is a 3-prime R -ideal.

(b) $RM \not\subseteq P$ and $(P : Rm) = (P : M)$ for every $m \in M \setminus P$

Proof. (a) \Rightarrow (b): Since P is a 3-prime R -ideal, we have $RM \not\subseteq P$. Now let $m \in M \setminus P$ and consider $x \in (P : Rm)$. Then $xRm \subseteq P$. Now, since P is a 3-prime R -ideal and since $m \notin P$, we get $x \in (P : M)$ and consequently $(P : Rm) \subseteq (P : M)$. Clearly, $(P : M) \subseteq (P : Rm)$.

(b) \Rightarrow (a): Let $m \in M \setminus P$ and let $a \in R$ such that $aRm \subseteq P$. Then $a \in (P : Rm) = (P : M)$ implies $aM \subseteq P$ and hence P is a 3-prime R -ideal. ■

Corollary 65 If P is a 3-prime R -ideal of M and $(P : m) \triangleleft R$ for every $m \in M \setminus P$, then $RM \not\subseteq P$ and $(P : m) = (P : Rm)$.

Proposition 66 Let $P \triangleleft_R M$. Then the following are equivalent:

- (a) P is a completely prime R -ideal.
- (b) $RM \not\subseteq P$ and $(P : m) = (P : M)$ for every $m \in M \setminus P$.
- (c) P is 3-prime and $(P : m) \triangleleft R$ for every $m \in M \setminus P$.

Proof. (a) \Rightarrow (b): Since P is a completely prime R -ideal, we have $RM \not\subseteq P$. Let $m \in M \setminus P$ and consider $x \in (P : m)$. Then $xm \in P$. Since P is completely prime and $m \notin P$, we must have that $x \in (P : M)$. Hence $(P : m) \subseteq (P : M)$. The other inclusion is trivial.

(b) \Rightarrow (a): Let $m \in M \setminus P$ and let $a \in R$ such that $am \in P$. So $a \in (P : m)$ implies $a \in (P : M)$ implies $aM \subseteq P$. Hence P is a completely prime R -ideal.

(b) \Leftrightarrow (c): Follows from Proposition 63. ■

Before we state the next proposition, we recall the following from Definition 30 and Theorem 32:

- (a) If $P \subseteq M$, then the left annihilator of P in R defined by

$$l(P) = \{r \in R : rP = 0\}$$

is a left ideal of R .

- (b) If $P \leq_R M$, then $l(P) \triangleleft R$.

Proposition 67 *Let M be a faithful R -module. Then M is 2-prime if and only if for all $0 \neq A \leq_R M$, we have that $l(A) = 0$.*

Proof. Suppose that M is 2-prime and faithful, and let $0 \neq A \leq_R M$. Then, by definition, $l(A).A = 0$. Since M is 2-prime and $A \neq 0$, we have that $l(A)M = 0$. Since M is faithful, $l(A) = 0$.

Conversely, Suppose M is not 2-prime. Then there exists some $B \leq R$ and some $C \leq_R M$ such that $BC = 0$ but $BM \neq 0$ and $C \neq 0$. Since $C \neq 0$, by the given condition, $l(C) = 0$. Furthermore, since $BC = 0$, clearly we must have that $B \subseteq l(C) = 0$ so that $B = 0$ and hence $BM = 0$, which is a contradiction. Thus M is 2-prime. ■

Corollary 68 *If M is a faithful R -module, then M is 0-prime if and only if for all $0 \neq A \triangleleft_R M$, we have that $l(A) = 0$.*

Proof. Note that if $A \triangleleft_R M$, then $l(A) \triangleleft R$. The rest of the proof follows as in Proposition 67 ■

Recall that if R is a near-ring and $X \subseteq R$, then X is called an m -system if for all $a, b \in X$, there exists $a_1 \in \langle a \rangle$ and $b_1 \in \langle b \rangle$ such that $a_1 b_1 \in X$. This definition can be found in [21, Definition 2.78]. In the definition that follows, we extend this definition to near-ring modules. However, certain adjustments together with various classifications were necessary.

Definition 69 *Let $X \subseteq M$ where M is an R -module. Then X is called an:*

(a) m_0 -system if $0 \neq r \in R$ and $0 \neq x \in X$ implies that $r\langle x \rangle_R \cap X \neq \emptyset$.

(Recall that $\langle x \rangle_R$ is the R -ideal of M generated by x).

(b) m_2 -system if $0 \neq r \in R$ and $0 \neq x \in X$ implies that $r[x]_R \cap X \neq \emptyset$.

(Recall that $[x]_R$ is the R -submodule of M generated by x).

(c) m_3 -system if $0 \neq r \in R$ and $0 \neq x \in X$ implies that $rRx \cap X \neq \emptyset$.

(d) m_c -system if $0 \neq r \in R$ and $0 \neq x \in X$ implies that $rx \in X$.

Proposition 70 *Let $\nu = 0, 2, 3, c$ and let $P \triangleleft_R M$. Then P is ν -prime if and only if $M \setminus P$ is an m_ν -system.*

Proof. We prove the case for $\nu = 0$. The others follow similarly.

Suppose that P is 0-prime. Let $0 \neq r \in R$ and $0 \neq x \in M \setminus P$. Suppose that $r\langle x \rangle_R \cap (M \setminus P) = \emptyset$. Then clearly it must follow that $r\langle x \rangle_R \subseteq P$. Since P is 0-prime, we have that $rM \subseteq P$ or $x \in P$. If $rM \subseteq P$, then since $r \in R$ was arbitrary, $RM \subseteq P$ which contradicts the definition of a 0-prime R -ideal. If $x \in P$, then $x \notin M \setminus P$ which is again a contradiction. So $r\langle x \rangle_R \cap (M \setminus P) \neq \emptyset$.

Now suppose that $M \setminus P$ is an m_0 -system. Let $0 \neq r \in R$ and $0 \neq x \in M$ such that $r\langle x \rangle_R \subseteq P$. Suppose that $x \notin P$. Then $0 \neq x \in M \setminus P$. Since $M \setminus P$ is

an m_0 -system, $r\langle x \rangle_R \cap (M \setminus P) \neq \emptyset$. Hence there exists some $a \in r\langle x \rangle_R \cap (M \setminus P)$. So $a \in r\langle x \rangle_R$ and $a \in M \setminus P \Rightarrow a \notin P$. This contradicts that $r\langle x \rangle_R \subseteq P$. Hence $x \in P$ and therefore P is 0-prime. ■

If $A \triangleleft R$, then we know that A is completely prime $\implies A$ is 3-prime $\implies A$ is 2-prime $\implies A$ is 0-prime. Furthermore, if R is a near-ring with identity, then the notions of a 2-prime ideal and a 3-prime ideal become equivalent. We show that similar types of relationships exist among the R -ideals of M .

Proposition 71 *Let $P \triangleleft_R M$. Then P is completely prime $\implies P$ is 3-prime $\implies P$ is 2-prime $\implies P$ is 0-prime.*

Proof. Suppose P is completely prime. Let $a \in R$ and $m \in M$ such that $aRm \subseteq P$. Then, since $a \in R$, we have $a(am) \in P$. Since P is completely prime, we have that $aM \subseteq P$ or $am \in P$. If $aM \subseteq P$, then we are done. If $am \in P$, then again since P is completely prime, it follows that $aM \subseteq P$ or $m \in P$. Hence P is 3-prime.

Suppose that P is 3-prime. Let A be a left R -subgroup of R and B be an R -submodule of M such that $AB \subseteq P$. Let $a \in A$ and $m \in B$. Then $aRm \subseteq ARB \subseteq AB \subseteq P$. Since P is 3-prime, we have that $aM \subseteq P$ or $m \in P$. Since a and m were arbitrary elements of A and B respectively, we conclude that $AM \subseteq P$ or $B \subseteq P$; whence P is 2-prime.

Let P be 2-prime, and let A be an ideal of R and B be an R -ideal of M such that $AB \subseteq P$. Then, clearly, A is a left R -subgroup of R and B is an R -submodule of M . Since P is 2-prime, $AM \subseteq P$ or $B \subseteq P$. So P is 0-prime. ■

Corollary 72 *If M is an R -module, then M is completely prime $\implies M$ is 3-prime $\implies M$ is 2-prime $\implies M$ is 0-prime.*

In general, a 0-prime R -ideal need not be 2-prime and a 2-prime R -ideal need not be 3-prime. To show this, we present the following examples:

Example 73 *Let K be the Klein-4-group, $K = \{0, 1, 2, 3\}$ with multiplication given by: $r \cdot 3 = r$ and $rb = 0$ if $b \in \{0, 1, 2\}$, $r \in K$. We illustrate this in table form:*

×	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	0	2
3	0	0	0	3

Consider the R -module $M = {}_R R$. Then M has no proper R -ideals and $\{0, 2\}$ is a proper R -submodule. Furthermore, $\{0\}$ is a 0-prime R -ideal since R is the only nonzero R -ideal and $R^2 \neq \{0\}$. However, $\{0\}$ is not 2-prime since $2 \cdot \{0, 2\} = \{0\}$ but $2M \neq 0$ and $\{0, 2\} \not\subseteq \{0\}$.

Example 74 Let R be the near-ring defined on $\mathbb{Z}_3 = \{0, 1, 2\}$ by:

×	0	1	2
0	0	0	0
1	0	0	1
2	0	0	2

Let $M = {}_R R$. Then M has no proper R -submodules; hence no proper R -ideals. Since $RM \neq 0$, $\{0\}$ is a 2-prime R -ideal. However, $1R1 = \{0\}$ but $1R=1M \not\subseteq \{0\}$ since $1 \cdot 2 = 1$ and also $1 \notin \{0\}$. Therefore $\{0\}$ is not a 3-prime R -ideal.

Proposition 75 Let R be a near-ring with identity, 1, and let P be an R -ideal of M . Then P is 2-prime if and only if P is 3-prime.

Proof. The fact that if P is 3-prime implies that it is 2-prime has already been proved in Proposition 71 .

Now suppose that P is a 2-prime R -ideal. Let $a \in R$ and $m \in M$ such that $aRm \subseteq P$. Then $RaRm \subseteq RP \subseteq P$. Since Ra is a left R -subgroup of R , Rm is an R -submodule of M and since P is 2-prime, it follows that $RaM \subseteq P$ or $Rm \subseteq P$. In particular, since $1 \in R$, it follows that $1 \cdot aM = aM \subseteq P$ or $1 \cdot m = m \in P$. Therefore P is 3-prime. ■

Proposition 76 If P is an equiprime R -ideal of an R -module M , then P is 3-prime.

Proof. Let $a \in R$ and $m \in M$ such that $aRm \subseteq P$. Suppose $aM \not\subseteq P$. Then $a \notin (P : M)$. Since $aRm \subseteq P$ and R is zerosymmetric, we have that $aRm - aR0 \subseteq P$ or $arm - ar0 \in P$ for all $r \in R$. Since P is equiprime, we have that $m = m - 0 \in P$; thus proving that P is 3-prime. ■

Corollary 77 *If M is an equiprime R -module, then M is 3-prime.*

In associative rings, the notions of 3-prime and equiprime (in fact, all our notions of prime) coincide. However, in general, this is not true in the case of near-rings and, hence, in the case of near-ring modules as the following example demonstrates:

Example 78 *Let R be the near-ring built on the cyclic group $(\mathbb{Z}_5, +)$ with multiplication on R given by the following table:*

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	2	2	2	2
3	0	3	3	3	3
4	0	4	4	4	4

Let $M = {}_R R$. Then clearly M is 3-prime. However, M is not equiprime since, for example:

$$2\{1\}4 - 2\{1\}3 = 0 \text{ but } 4 - 3 \neq 0.$$

If $P \triangleleft_R M$, then we recall that $\tilde{P} = (P : M)$ is an ideal of R . Now, if P is a ν -prime ($\nu = 2, 3, c$) R -ideal then does this imply that \tilde{P} is also ν -prime? We investigate this in the propositions that follow. The case $\nu = 0$ is treated separately.

Proposition 79 *Let P be an R -ideal of M . Then:*

- (a) P is a 2-prime R -ideal of M implies that \tilde{P} is a 2-prime ideal of R .
- (b) P is a 3-prime R -ideal of M implies that \tilde{P} is a 3-prime ideal of R .

(c) P is a completely prime R -ideal of M implies that \tilde{P} is a completely prime ideal of R .

Proof. (a) : Let A, B be left R -subgroups of R such that $AB \subseteq \tilde{P}$. Then $ABM \subseteq P$, so that for all $m \in M$, $ABm \subseteq P$. Since P is a 2-prime R -ideal and Bm is an R -submodule of M , we have that $AM \subseteq P$ or $Bm \subseteq P$ for all $m \in M$. If $AM \subseteq P$, then $A \subseteq (P : M) = \tilde{P}$ and we are done. If $Bm \subseteq P$ for all $m \in M$, then $BM \subseteq P$ and hence $B \subseteq (P : M) = \tilde{P}$. Therefore \tilde{P} is a 2-prime ideal of R .

(b) : Let $x, y \in R$ such that $xRy \subseteq \tilde{P}$. Suppose that $y \notin \tilde{P}$. Then $yM \not\subseteq P$ implies that there exists an $m \in M$ such that $ym \notin P$. Now $xRy \subseteq \tilde{P}$ implies that $xRym \subseteq P$. Since P is a 3-prime R -ideal and $ym \notin P$, we must have that $xM \subseteq P$ ie. $x \in \tilde{P}$. Thus \tilde{P} is a 3-prime ideal of R .

(c) : Let $x, y \in R$ such that $xy \in \tilde{P}$ and suppose that $y \notin \tilde{P}$. Then $yM \not\subseteq P$ implies that there exists some $m \in M$ such that $ym \notin P$. Since $xyM \subseteq P$, $x(ym) \in P$. But P is a completely prime R -ideal of M and $ym \notin P$. Hence $xM \subseteq P$ implies that $x \in \tilde{P}$; whence \tilde{P} is a completely prime ideal of R . ■

Corollary 80 *Let $\nu = 2, 3, c$. If M is a ν -prime R -module, then $(0 : M)$ is a ν -prime ideal of R .*

Proposition 81 *Let M be a faithful R -module and let $\nu = 2, 3, c$. Then, if M is ν -prime, R is a ν -prime near-ring.*

Proof. $\nu = 2$: Let A and B be left R -subgroups of R such that $AB = 0$. Then $(AB)M = 0$ which implies that $AB \subseteq (0 : M)$. Since M is 2-prime, so is $(0 : M)$. Hence it follows that $A \subseteq (0 : M)$ or $B \subseteq (0 : M)$ and thus $AM = 0$ or $BM = 0$. Since M is faithful, we have that $A = 0$ or $B = 0$.

$\nu = 3$: Let $a, b \in R$ such that $aRb = 0$. Then $(aRb)M = 0$ implies that $aRb \subseteq (0 : M)$. Since $(0 : M)$ is 3-prime, $a \in (0 : M)$ or $b \in (0 : M)$ whence $aM = 0$ or $bM = 0$. Once again the faithfulness of M implies that $a = 0$ or $b = 0$, and hence R is a 3-prime near-ring.

$\nu = \mathbf{c}$: Similar to the previous case. ■

That P is a 0-prime R -ideal implies that \tilde{P} is a 0-prime ideal of R , unfortunately, does not follow as naturally as for the 2-prime, 3-prime and c -prime cases. However, with certain restrictions on M , we find that the relationship holds.

Proposition 82 *Let P be a 0-prime R -ideal of a monogenic R -module M . Then \tilde{P} is a 0-prime ideal of R .*

Proof. Let A, B be ideals of R such that $AB \subseteq \tilde{P}$. Suppose that $A \not\subseteq \tilde{P}$ and $B \not\subseteq \tilde{P}$. Then we have that $AM \not\subseteq P$ and $BM \not\subseteq P$. Since M is monogenic, there exists $m_0 \in M$ such that $Rm_0 = M$. Therefore, $BM \not\subseteq P$ implies that $BRm_0 \not\subseteq P$. Furthermore, since B is an ideal of R , $BR \subseteq B$, and hence it follows that $BRm_0 \subseteq Bm_0$. So $Bm_0 \not\subseteq P$. From Lemma 35, Bm_0 is an R -ideal of M such that $Bm_0 \not\subseteq P$. Since we also have that $AM \not\subseteq P$ and that P is a 0-prime R -ideal, it follows that $ABm_0 \not\subseteq P$. So $ABM \not\subseteq P$ implies that $AB \not\subseteq \tilde{P}$ which is a contradiction. Therefore $A \subseteq \tilde{P}$ or $B \subseteq \tilde{P}$ implies that \tilde{P} is a 0-prime ideal of R . ■

Proposition 83 *Let P be a 0-prime R -ideal of a tame R -module M . Then \tilde{P} is a 0-prime ideal of R .*

Proof. Let A, B be ideals of R such that $AB \subseteq \tilde{P}$. If $B \subseteq \tilde{P}$, then we are done. Suppose that $B \not\subseteq \tilde{P}$. Then there exists $m \in M$ such that $Bm \not\subseteq P$. Now Bm is an R -submodule of M and, since M is tame, Bm is also an R -ideal of M . Furthermore, $ABm \subseteq ABM \subseteq P$. Since P is a 0-prime R -ideal and $Bm \not\subseteq P$, it must follow that $AM \subseteq P$. Hence $A \subseteq \tilde{P}$ implies that \tilde{P} is 0-prime. ■

Corollary 84 *If M is a 0-prime monogenic (or tame) R -module, then $(0 : M)$ is a 0-prime ideal of R .*

Proposition 85 *Let M be a faithful, monogenic (or tame) 0-prime R -module. Then R is a 0-prime near-ring.*

Proof. Let A and B be ideals of R such that $AB = 0$. Then $(AB)M = 0$ implies that $AB \subseteq (0 : M)$. Since $(0 : M)$ is a 0-prime ideal of R , we have that $A \subseteq (0 : M)$ or $B \subseteq (0 : M)$ whence $AM = 0$ or $BM = 0$. Since M is faithful, $A = 0$ or $B = 0$ which implies that R is 0-prime. ■

The notions of monogenic and tame R -modules used in the preceding propositions are, in general, unrelated notions. In the example that follows, we show that a tame R -module need not be monogenic. Thereafter we provide an example of an R -module that is neither tame nor monogenic.

Example 86 Let R be the near-ring constructed on $K_4 = \{0, 1, 2, 3\}$ with multiplication on R given by the following table:

\times	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	0	0
3	0	1	0	1

Let $M = {}_R R$. Then the R -submodules of M are $\{0\}, \{0, 1\}, \{0, 2\}$ and M which are exactly the R -ideals of M . Hence M is a tame R -module.

However, for all $m \in M = {}_R R$ we have $Rm \neq M$. Hence M is not monogenic.

Example 87 Let R be the near-ring constructed on $K_4 = \{0, 1, 2, 3\}$ with multiplication on R given by the following table:

\times	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	0	1

Let $M = {}_R R$. Then the R -submodules of M are $\{0\}, \{0, 1\}, \{0, 2\}$ and M . However, $\{0, 2\}$ is not an R -ideal of M ; thus implying that M is not tame. Furthermore, for all $m \in M$, $Rm \subseteq \{0, 1\}$. So $Rm \neq M$ for all $m \in M$ implies that M is not monogenic.

Now suppose that P is an R -ideal of M such that $(P : M)$ is a ν -prime ideal of R for $\nu = 0, 2, 3$ and c . Does this imply that P is a ν -prime R -ideal of M ? We investigate this in a later section when we introduce the notion of a multiplication module. However, at present we have the following:

Lemma 88 *Let $P \triangleleft R$ with $P \neq R$.*

(a) *If P is 0-prime then $\frac{R}{P}$ is a faithful R -module and $P = \left(0 : \frac{R}{P}\right)_R$.*

(b) *If P is 1-prime then there exists a 0-prime R -module M with $P = (0 : M)_R$.*

Proof. (a): $\frac{R}{P}$ is an R -module with the natural operations. If $p \in P$ and $(r + P) \in \frac{R}{P}$, then $p(r + P) = pr + P = P$ and we have $P \subseteq \left(0 : \frac{R}{P}\right)_R$. Now, let $a \in \left(0 : \frac{R}{P}\right)_R$. Hence $aR \subseteq P$ and consequently $\langle a \rangle R \subseteq P$. Since P is a 0-prime ideal we get $\langle a \rangle \subseteq P$ and thus $a \in P$. Hence $P = \left(0 : \frac{R}{P}\right)_R$ and $\frac{R}{P}$ is a faithful R -module.

(b): Let $M = \frac{R}{P}$. Since P is 1-prime, it is also 0-prime. Hence, from (a), M is an R -module with $P = (0 : M)_R$. We need to show that M is 0-prime. So let $A \triangleleft_l R$ and $B \triangleleft_R M$ such that $AB = 0$. Then, by Proposition 27, $B = \frac{L}{P}$ for some left ideal L of R . Hence $AB = 0$ implies that $AL \subseteq P$. Since P is 1-prime, we have $A \subseteq P = (0 : M)_R$ or $L \subseteq P$. So $AM = 0$ or $B = 0$ implies that M is 1-prime and, consequently, 0-prime. ■

Corollary 89 *If R is a 0-prime near-ring, then ${}_R R$ is a faithful R -module.*

Proposition 90 *Let $\nu = 2, 3, c$ and let $P \triangleleft R$ with $P \neq R$. Then there exists a ν -prime R -module M with $(0 : M)_R = P$ if and only if P is a ν -prime ideal of R .*

Proof. If M is a ν -prime R -module, then it follows from Corollary 80 that $P = (0 : M)_R$ is a ν -prime ideal of R .

For the converse, let P be a ν -prime ideal of R . It follows from Lemma 88 that $R \left(\frac{R}{P}\right) \neq 0$ and $\left(0 : \frac{R}{P}\right)_R = P$ in all three cases $\nu = 2, 3, c$. We, now, consider the three cases separately.

For the 2-prime case, let $K \leq R$ and $L \leq_R \frac{R}{P}$ such that $KL = 0$. Then, by Proposition 27, $L = \frac{L_1}{P}$ for some left R -subgroup L_1 of R . Hence we have $K \left(\frac{L_1}{P} \right) = 0$ and thus $KL_1 \subseteq P$. Since P is a 2-prime ideal, it follows that $K \subseteq P = \left(0 : \frac{R}{P} \right)_R$ or $L_1 \subseteq P$ whence $K \left(\frac{R}{P} \right) = 0$ or $L = 0$. Hence $M = \frac{R}{P}$ is a 2-prime R -module.

For the 3-prime case, let $m \in \frac{R}{P}$ and $a \in R$ such that $aRm = 0$.

If $m = m_1 + P$ for some $m_1 \in R$, we get $aR(m_1 + P) = aRm_1 + P = 0$ which implies that $aRm_1 \subseteq P$. If $m_1 \in P$, then $m = 0$ and we are done. Now suppose $m_1 \notin P$. Since P is a 3-prime ideal, we get $a \in P = \left(0 : \frac{R}{P} \right)_R \implies a \left(\frac{R}{P} \right) = 0$. Hence $M = \frac{R}{P}$ is a 3-prime R -module.

For the c -prime case, let $a \in R$ and $m = m_1 + P \in \frac{R}{P}$ for some $m_1 \in R$ such that $am = a(m_1 + P) = am_1 + P = 0$. Then $am_1 \in P$, and since P is a c -prime ideal, it follows that $a \in P = \left(0 : \frac{R}{P} \right)_R$ or $m_1 \in P$ and thus $a \left(\frac{R}{P} \right) = 0$ or $m = m_1 + P = 0$. Hence $M = \frac{R}{P}$ is a c -prime R -module. ■

Corollary 91 *Let $\nu = 2, 3, c$ and let $P \triangleleft R$ with $P \neq R$. Then $\frac{R}{P}$ is a ν -prime near-ring if and only if $\frac{R}{P}$ is a ν -prime R -module.*

Proof. Let $\frac{R}{P}$ be a ν -prime near-ring. Then P is a ν -prime ideal of R . By Proposition 90, it follows that $\frac{R}{P}$ is a ν -prime R -module.

Conversely, if $\frac{R}{P}$ is a ν -prime R -module, then by Corollary 80, $\left(0 : \frac{R}{P} \right)_R$ is a ν -prime ideal of R . But $P = \left(0 : \frac{R}{P} \right)_R$. Hence P is a ν -prime ideal of R and so $\frac{R}{P}$ is a ν -prime near-ring. ■

Corollary 92 *Let $P \triangleleft R$ with $P \neq R$. Then $\frac{R}{P}$ is a 1-prime near-ring implies that $\frac{R}{P}$ is a 0-prime R -module.*

Proof. Follows from Lemma 88. ■

Thus far, we have considered M as an R -module. Now suppose that A is an ideal of R . In what follows, we consider some prime relationships between M as an R -module and M as an A -module. It is quite clear that if M is a completely prime R -module, then M is a completely prime A -module. However, this result does not follow easily in the ν -prime cases for $\nu = 0, 2, 3$. For these cases, certain restrictions were required on the near-ring or its substructures. Hence, before we prove the next proposition we give the definition of an \mathcal{A} -near-ring introduced in [2].

Definition 93 *A near-ring R is called an \mathcal{A} -near-ring if for any ideal A of R and for any ideal B of A there exists $n \in \mathbb{N}$ such that $(\langle B \rangle_R)^n \subseteq B$.*

Proposition 94 *Let R be an \mathcal{A} -near ring. If $A \triangleleft R$ and M is a 2-prime R -module, then M is a 2-prime A -module.*

Proof. Let B be a left A -subgroup of A and N a nonzero A -submodule of M such that $BN = 0$. Since R is an \mathcal{A} -near-ring there exists $n \in \mathbb{N}$ such that

$$(\langle\langle B \rangle_A \rangle_R)^n N \subseteq \langle B \rangle_A N$$

Since $BN = 0$, we have $B \subseteq (0 : N)_A \triangleleft A$. Hence $\langle B \rangle_A \subseteq (0 : N)_A$ implies $\langle B \rangle_A N = 0$. Therefore $(\langle\langle B \rangle_A \rangle_R)^n N = 0$.

Let m be the minimal number such that $(\langle\langle B \rangle_A \rangle_R)^m N = 0$.

If $m = 1$, then $(\langle\langle B \rangle_A \rangle_R)N = 0$. If N is also an R -submodule, then since M is a 2-prime R -module we get $(\langle\langle B \rangle_A \rangle_R)M = 0$ or $N = 0$. If $N = 0$, then we are done. So suppose $N \neq 0$. Hence $(\langle\langle B \rangle_A \rangle_R)M = 0$ and therefore $BM = 0$. If N is not an R -submodule of M , then there exists $t \in N$ such that $Rt \not\subseteq N$. Hence Rt is a nonzero R -submodule of M . Now we have:

$$(\langle\langle B \rangle_A \rangle_R)Rt \subseteq (\langle\langle B \rangle_A \rangle_R)t \subseteq (\langle\langle B \rangle_A \rangle_R)N = 0.$$

Since M is a 2-prime R -module and $Rt \neq 0$, we get $(\langle\langle B \rangle_A \rangle_R)M = 0$ and therefore $BM = 0$.

If $m > 1$, then $(\langle\langle B \rangle_A \rangle_R)^{m-1}N \neq 0$ and therefore there exists an $x \in (\langle\langle B \rangle_A \rangle_R)^{m-2}N \subseteq M$ such that $(\langle\langle B \rangle_A \rangle_R)x \neq 0$.

Now, $(\ll B \gg_A \gg_R)(\ll B \gg_A \gg_R)x = 0$ and, again, since M is a 2-prime R -module we get $(\ll B \gg_A \gg_R)M = 0$. Hence it follows that $BM = 0$ and we are done. ■

Proposition 95 *Let R be a near-ring. If A is an invariant subgroup of R such that $A \not\subseteq (0 : M)_R$ and M is a 3-prime R -module, then M is a 3-prime A -module.*

Proof. Since $A \not\subseteq (0 : M)_R$ we have $AM \neq 0$. Let $a \in A$ and $m \in M$ such that $aAm = 0$. If $m = 0$ then we are done. Suppose $m \neq 0$. If $Am = 0$, then we have $aRm \subseteq Am = 0$ and since M is a 3-prime R -module, it follows that $aM = 0$. So, again, we are done.

Now suppose $Am \neq 0$ and let $0 \neq t = a_1m \in Am$. Now, $aRAM \subseteq aAm = 0$. Hence $aRt = 0$ and since M is a 3-prime R -module we have $aM = 0$. Hence M is a 3-prime A -module. ■

Proposition 96 *If M is a 2-prime (resp. 3-prime) R -module and $0 \neq H \triangleleft_R M$, then H is a 2-prime (resp. 3-prime) R -module.*

Proof. First, let M be a 2-prime R -module, and let A be a left R -subgroup of R and B an R -submodule of H such that $AB = 0$. Since B is also an R -submodule of M we get $AM = 0$ or $B = 0$. Hence $AH = 0$ or $B = 0$ and it follows that H is a 2-prime R -module.

Now, let M be a 3-prime R -module and take $a \in R$ and $h \in H$ such that $aRh = 0$. Since M is 3-prime, we get that $aM = 0$ or $h = 0$. But $aH \subseteq aM = 0$. Hence H is 3-prime. ■

Proposition 97 *If A is any subset of a near-ring R and M is a c -prime R -module, then M is a c -prime A -module.*

Proof. Let $a \in A$ and $m \in M$ such that $am = 0$. Then $a \in R$ and since M is a c -prime R -module, it follows that $aM = 0$ or $m = 0$. ■

Proposition 98 *If M is a c -prime R -module and $0 \neq H \triangleleft_R M$, then H is a c -prime R -module.*

Proof. Let $a \in R$ and $h \in H$ such that $ah = 0$. Since $h \in M$ and M is a c -prime R -module, we have that $aM = 0$ or $h = 0$. So $aH \subseteq aM = 0$ or $h = 0$. Hence H is also c -prime. ■

2.3 Semiprime Modules

As in the case of prime modules, in this section we generalise the concept of semiprimeness from the near-ring R to any R -module M . However, we do not delve as deeply into the theory of semiprime near-ring modules as we did for prime near-ring modules. We present here some basic results analogous to those obtained for semiprime near-rings or for prime near-ring modules.

Definition 99 Let P be an R -ideal of an R -module M such that $RM \not\subseteq P$. Then P is called:

- (a) 0-semiprime if $A^2M \subseteq P$ implies $AM \subseteq P$ for all ideals A of R .
- (b) 1-semiprime if $A^2M \subseteq P$ implies $AM \subseteq P$ for all left ideals A of R .
- (c) 2-semiprime if $A^2M \subseteq P$ implies $AM \subseteq P$ for all R -subgroups A of R .
- (d) 3-semiprime if $aRam \subseteq P$ implies $am \in P$ for all $a \in R$ and $m \in M$.
- (e) completely semiprime (c -semiprime) if $a^2m \in P$ implies $am \in P$ for all $a \in R$ and $m \in M$.

Definition 100 An R -module M is called ν -semiprime ($\nu = 0, 1, 2, 3, c$) if $RM \neq 0$ and 0 is a ν -semiprime R -ideal of M .

In Section 2.2, we included many equivalent definitions for the various types of primes. For example, in Proposition 60, we proved that: " P is a 2-prime R -ideal of an R -module M if and only if for all $a \in R$ and for all submodules B of M such that $aB \subseteq P$, we have that $aM \subseteq P$ or $B \subseteq P$ ". An analogue of this proposition for a 2-semiprime R -ideal would be either:

- (a) P is a 2-semiprime R -ideal of M if and only if for all $a \in R$ and all R -subgroups A of R such that $a(AM) \subseteq P$, we have $aM \subseteq P$; or
- (b) P is a 2-semiprime R -ideal of M if and only if for all $a \in R$ such that $a^2M \subseteq P$, we have $aM \subseteq P$.

The problem with (a) is that AM may only be a subset of M . Furthermore, if we reconstruct (b) for a 0-semiprime R -ideal and a 1-semiprime R -ideal, the results would imply that the three types of semiprime R -ideals are equivalent structures. This we already know is in general not true if we consider $M = {}_R R$. Hence results obtained for prime modules may not follow naturally to semiprime modules as they do in the near-ring case. However, we do have some analogous results.

Proposition 101 *Let P be an R -ideal of M . Then P is c -semiprime $\implies P$ is 3-semiprime $\implies P$ is 2-semiprime $\implies P$ is 1-semiprime $\implies P$ is 0-semiprime.*

Proof. Suppose that P is c -semiprime. Let $a \in R$ and $m \in M$ such that $aRam \subseteq P$. Then $a^4m = a(aa)am \in aRam \subseteq P$. Hence $(a^2)^2m \in P$. Since P is c -semiprime, $a^2m \in P$. Applying this again, we get that $am \in P$, which implies that P is 3-semiprime.

Now suppose that P is 3-semiprime. Let A be a left R -subgroup of R such that $A^2M \subseteq P$. Then $a^2m \in P$ for all $a \in A$ and $m \in M$. Now, $aRam \subseteq ARAM \subseteq A^2M \subseteq P$ for all $a \in A, m \in M$. Since P is 3-semiprime, for all $a \in A$ and $m \in M$ we get $am \in P$. Hence $AM \subseteq P$ and thus it follows that P is 2-semiprime.

The rest of the proof follows as in Proposition 71. ■

Proposition 102 *If R is a near-ring with identity, then $P \triangleleft_R M$ is 2-semiprime if and only if P is 3-semiprime.*

Proof. Suppose that P is 2-semiprime. Let $a \in R$ and $m \in M$ be such that $aRam \subseteq P$. Then $(Ra)^2m = RaRam \subseteq RP \subseteq P$. Since P is 2-semiprime, $Ram \subseteq P$ and hence, since $1 \in R$, we get $am = 1.am \in P$. ■

It is well known that in a near-ring R , a ν -prime ideal is clearly ν -semiprime for $\nu = 0, 1, 2, 3, c$. In an R -module M , the same follows easily for a ν -prime R -ideal of M where $\nu = 2, 3, c$. However, in the 0-prime or the 1-prime case, this result is not trivial, unless we choose M to be a monogenic or a tame R -module.

Proposition 103 *Let $\nu = 2, 3, c$. If P is a ν -prime R -ideal of an R -module M , then P is also ν -semiprime.*

Proof. $\nu = 2$: Let A be a left R -subgroup of R such that $A^2M \subseteq P$. Then $A^2 \subseteq (P : M)$. Now, since P is 2-prime, $(P : M)$ is also 2-prime and hence, being an ideal of R , it is also 2-semiprime. Therefore $A \subseteq (P : M)$ and so $AM \subseteq P$ implies that P is 2-semiprime.

$\nu = 3$: Let $a \in R$ and $m \in M$ such that $aRam \subseteq P$. If $m \in P$, then $am \in P$ and we are done. So suppose $m \notin P$. Then, since P is an R -ideal, $am \notin P$. Since P is a 3-prime R -ideal, by Proposition 64, we have that $(P : Ram) = (P : M)$. Then:

$$aRaRam \subseteq aRam \subseteq P \implies aRa \subseteq (P : Ram) = (P : M).$$

Now, since $(P : M)$ is a 3-prime ideal of R , it is also 3-semiprime. Hence it follows that $a \in (P : M)$. So $aM \subseteq P$ implies $am \in P$ whence P is 3-semiprime.

$\nu = c$: Since P is a c -prime R -ideal of M , $(P : M)$ is a c -prime ideal of R and hence $(P : M)$ is c -semiprime. Now let $a \in R$ and $m \in M$ such that $a^2m \in P$. If $m \in P$, then $am \in P$ and we are done.

Suppose $m \notin P$. Then, by Proposition 66, we have that $(P : m) = (P : M)$. So we have:

$$a^2m \in P \implies a^2 \in (P : m) \implies a^2 \in (P : M).$$

Since $(P : M)$ is c -semiprime, $a \in (P : M)$ and hence $aM \subseteq P \implies am \in P$. This proves that P is a c -semiprime R -ideal. ■

Proposition 104 *Let M be a monogenic R -module with generator m_0 . Then P is a ν -prime ($\nu = 0, 1$) R -ideal of M implies that P is a ν -semiprime ($\nu = 0, 1$) R -ideal of M .*

Proof. Let $A \triangleleft R$ such that $A^2M \subseteq P$. Then $A(AM) \subseteq P$ implies that $A(Am_0) \subseteq P$. By Lemma 35, Am_0 is an R -ideal of M and since P is 0-prime, we have that $AM \subseteq P$ or $Am_0 \subseteq P$. If $AM \subseteq P$, then we are done.

On the other hand, if $Am_0 \subseteq P$, then $AM = ARm_0 \subseteq Am_0 \subseteq P$. So, again $AM \subseteq P$. Hence P is 0-semiprime.

The 1-prime case follows in exactly the same way. ■

Proposition 105 *Let M be a tame R -module. Then P is a 0-prime (or 1-prime) R -ideal of M implies that P is a 0-semiprime (or 1-semiprime) R -ideal of M .*

Proof. Let $A \triangleleft R$ such that $A^2M \subseteq P$. Then $A(AM) \subseteq P$ implies that $A(Am) \subseteq P$ for every $m \in M$. Since M is tame, Am is an R -ideal of M and since P is 0-prime, we have that $AM \subseteq P$ or $Am \subseteq P$. If $AM \subseteq P$, then we are done.

If $Am \subseteq P$, then $Am \subseteq P$ for every $m \in M$. So, again it follows that $AM \subseteq P$. Hence P is 0-semiprime.

Again, the 1-prime case follows in exactly the same way. ■

We have seen already that if P is a 2-prime (3-prime or c -prime) R -ideal, then $\tilde{P} = (P : M)$ is a 2-prime (3-prime or c -prime) ideal of R . Furthermore, if M is a monogenic (or tame) R -module, then \tilde{P} is 0-prime whenever P is 0-prime. We show that similar results also hold in the case of semiprime ideals. Furthermore, for the 0-prime, 1-prime and 2-prime case, we show that their converses also hold.

Proposition 106 *Let P be an R -ideal of the R -module M and let \tilde{P} be the corresponding ideal of R . Then:*

- (a) P is a 0-semiprime R -ideal of M if and only if \tilde{P} is a 0-semiprime ideal of R .
- (b) P is a 1-semiprime R -ideal of M if and only if \tilde{P} is a 1-semiprime ideal of R .

(c) P is a 2-semiprime R -ideal of M if and only if \tilde{P} is a 2-semiprime ideal of R .

Proof. (a) Let $A \triangleleft R$ such that $A^2 \subseteq \tilde{P}$. Then $A^2M \subseteq P$. Since P is 0-semiprime, $AM \subseteq P$. So $A \subseteq \tilde{P}$ implies that \tilde{P} is 0-semiprime.

Conversely, let $A \triangleleft R$ such that $A^2M \subseteq P$. Then $A^2 \subseteq \tilde{P}$. Since \tilde{P} is a 0-semiprime ideal of R , it follows that $A \subseteq \tilde{P}$. Hence $AM \subseteq P$ implies that P is a 0-semiprime R -ideal.

(b) Let A be a left ideal of R such that $A^2 \subseteq \tilde{P}$. The rest of the proof follows as in (a).

(c) Let A be a left R -subgroup of R such that $A^2 \subseteq \tilde{P}$. The rest of the proof follows as in (a). ■

Proposition 107 *Let P be an R -ideal of M . Then:*

(a) P is a 3-semiprime R -ideal of M implies that \tilde{P} is a 3-semiprime ideal of R .

(b) P is a c -semiprime R -ideal of M implies that \tilde{P} is a c -semiprime ideal of R .

Proof. (a) If $a \in R$ such that $aRa \subseteq \tilde{P}$, then $aRaM \subseteq P$ implies that $aRam \subseteq P$ for all $m \in M$. Since P is 3-semiprime, $am \in P$ for all $m \in M$. Hence $aM \subseteq P$ implies that $a \in \tilde{P}$. So \tilde{P} is 3-semiprime.

(b) Let $a \in R$ such that $a^2 \in \tilde{P}$. Then $a^2m \in P$ for all $m \in M$. Since P is c -semiprime, it follows that $am \in P$ for all $m \in M$. Therefore $aM \subseteq P$ and hence $a \in \tilde{P}$. So \tilde{P} is c -semiprime. ■

Whether the converses of the statements in the above proposition hold or not is still an open question. However, we do have the following observation:

Corollary 108 *Let R be a near-ring with identity. Then P is a 3-semiprime R -ideal of M if and only if \tilde{P} is a 3-semiprime ideal of R .*

Proof. Follows from the fact that 2-semiprime and 3-semiprime structures are equivalent in both R and M when R has identity. ■

2.4 s -Prime Modules

In [22], van der Walt defined the notion of an s -system: "A subset S of a near-ring R is called an s -system if S contains a multiplicative system S^* such that for every $s \in S$, we have $\langle s \rangle \cap S^* \neq \emptyset$ ". He then called an ideal A of R an s -prime ideal if $R \setminus A$ is an s -system. In [3], Birkenmeier et al called an ideal A of a near-ring R nilprime if A is 0-prime and $\frac{R}{A}$ has no nonzero nil ideals. They then went on to prove that every s -prime near-ring is a nilprime near-ring. In this section, we generalise the approach adopted by Birkenmeier et al to any R -module M . However, once again we identify various types.

Definition 109 *Let P be an R -ideal of an R -module M such that $RM \neq 0$. Let $\nu = 0, 2, 3$. Then P is called ν - s -prime if:*

- (a) P is ν -prime.
- (b) $\frac{R}{(P : M)_R}$ contains no nonzero nil ideals (ie. for every $A \triangleleft R$ such that $A \not\subseteq (P : M)$, there exists an $a \in A \setminus (P : M)$ such that $a^n M \not\subseteq P$ for all $n \in \mathbb{N}$).

If we transfer the above definition to the module M itself, then we have that M is ν - s -prime if M is ν -prime and $\frac{R}{(0 : M)_R}$ has no nonzero nil ideals (ie. for every $A \triangleleft R$, $A \not\subseteq (0 : M)$, there exists an $a \in A \setminus (0 : M)$ such that $a^n M \neq 0$ for all $n \in \mathbb{N}$).

Furthermore, it is well known that the nil radical of a near-ring R is defined as: $\mathfrak{N}(R) = \sum \{A \triangleleft R : A \text{ is a nil ideal of } R\}$. With this in mind, we restate Definition 109 as follows:

Definition 110 *Let $\nu = 0, 2, 3$ and $RM \neq 0$. Then $P \triangleleft_R M$ is called ν - s -prime if P is ν -prime and $\mathfrak{N} \left(\frac{R}{(P : M)_R} \right)_R = 0$.*

It is clear that every ν -*s*-prime R -ideal (R -module) is ν -prime for $\nu = 0, 2, 3$. Furthermore, in section 2.2, we proved the following results for an R -ideal P of M :

- (a) P is 3-prime $\implies P$ is 2-prime $\implies P$ is 0-prime. (Note here that we provided examples to show that these three types of primes are nonequivalent in general. We can use the same examples to conclude the nonequivalence of the three types of *s*-primes).
- (b) If R has identity, then P is 2-prime $\iff P$ is 3-prime.
- (c) For $\nu = 2, 3$, if P is ν -prime, then $\tilde{P} = (P : M) \triangleleft R$ is ν -prime.
- (d) If P is 0-prime and M is a monogenic (or tame) R -module, then $\tilde{P} \triangleleft R$ is 0-prime.

By applying the above results and Definition 109, we conclude the following string of results with respect to a ν -*s*-prime R -ideal P of an R -module M (or with respect to M itself).

Proposition 111 P (or M) is 3-*s*-prime $\implies P$ (or M) is 2-*s*-prime $\implies P$ (or M) is 0-*s*-prime. Furthermore, if R has identity, then P (or M) is 2-*s*-prime $\iff P$ (or M) is 3-*s*-prime.

Proposition 112 Let $\nu = 2, 3$. Then:

- (a) $P \triangleleft_R M$ is ν -*s*-prime implies $\tilde{P} \triangleleft R$ is also ν -*s*-prime.
- (b) M is a ν -*s*-prime module implies $(0 : M)$ is a ν -*s*-prime ideal of R .

Proposition 113 Let M be a monogenic (or tame) R -module. Then:

- (a) $P \triangleleft_R M$ is 0-*s*-prime implies $\tilde{P} \triangleleft R$ is also 0-*s*-prime.
- (b) M is a 0-*s*-prime R -module implies $(0 : M)$ is a 0-*s*-prime ideal of R .

Proposition 114 Let $\nu = 2, 3$ and let P be a ν -*s*-prime ideal of R such that $P \neq R$. Then there exists a ν -*s*-prime R -module M with $P = (0 : M)$.

Proof. Since P is ν -prime, from Proposition 90, we know that $\frac{R}{P}$ is a nonzero R -module, $\frac{R}{P}$ is ν -prime and $P = \left(0 : \frac{R}{P}\right)_R$. So let $M = \frac{R}{P}$. Since M is ν -prime, the first condition of Definition 109 is satisfied.

For the second condition, since P is a ν - s -prime ideal, it follows from the definition of an s -prime ideal that $\frac{R}{P}$ has no nonzero nil ideals.

But $P = \left(0 : \frac{R}{P}\right)_R$. Hence $\frac{R}{(0 : \frac{R}{P})_R} = \frac{R}{(0 : M)_R}$ has no nonzero nil ideals.

So M is a ν - s -prime R -module. ■

Corollary 115 *If $P \triangleleft R$ with $P \neq R$, then there exists a 2- s -prime (3- s -prime) R -module M with $P = (0 : M)_R$ if and only if P is a 2- s -prime (3- s -prime) ideal.*

Proposition 116 *Let M be a 2- s -prime (3- s -prime) R -module and let P be an R -ideal of M . Then P is a 2- s -prime (3- s -prime) R -module.*

Proof. From Proposition 96, P is a 2-prime (3-prime) R -module. Hence we need only show that $\frac{R}{(0 : P)_R}$ contains no nonzero nil ideals.

Since M is 2- s -prime (3- s -prime), $\frac{R}{(0 : M)_R}$ contains no nonzero nil ideals. But $(0 : M)_R \subseteq (0 : P)_R$ implies that $\frac{R}{(0 : P)_R} \subseteq \frac{R}{(0 : M)_R}$. So $\frac{R}{(0 : P)_R}$ also contains no nonzero nil ideals. ■

Before we prove the next proposition, we state the following lemma from [3, Corollary 12]:

Lemma 117 *If R is an \mathcal{A} -near-ring and $I \triangleleft R$, then $\aleph(I) = I \cap \aleph(R)$.*

Proposition 118 *Let R be an \mathcal{A} -near-ring and let $A \triangleleft R$. If M is a 2- s -prime R -module, then M is a 2- s -prime A -module.*

Proof. It follows from Proposition 94 that M is a 2-prime A -module. So we need only show that $\aleph\left(\frac{A}{(0 : M)_A}\right) = 0$. Since M is 2- s -prime, we know that $\aleph\left(\frac{R}{(0 : M)_R}\right) = 0$. Furthermore, $A \cap (0 : M)_R = (0 : M)_A$. Hence

it follows that $\frac{A}{(0 : M)_A} = \frac{A}{[A \cap (0 : M)_R]} \cong \frac{[A + (0 : M)_R]}{(0 : M)_R} \triangleleft \frac{R}{(0 : M)_R}$.
 Therefore, since $\frac{A}{(0 : M)_A} \triangleleft \frac{R}{(0 : M)_R}$, it follows from the previous lemma that $\aleph \left(\frac{A}{(0 : M)_A} \right) \subseteq \aleph \left(\frac{R}{(0 : M)_R} \right) = 0$ and we are done. ■

Corollary 119 *Let R be an \mathcal{A} -near-ring and $A \triangleleft R$. If M is a 3- s -prime R -module, then M is a 3- s -prime A -module.*

Proof. It follows from Proposition 95 that M is a 3-prime A -module. The rest of the proof follows as for the previous proposition. ■

2.5 Strongly Prime Modules

Strongly prime rings were originally introduced by Handelman and Lawrence [15]. They defined a nonzero module M of a ring R to be strongly prime if:

for all $0 \neq m \in M$, there exists a finite subset F of R (depending on m) such that if $a \in R$ and $aFm = 0$, then $a = 0$.

If R is a near-ring and M is an R -module, the above definition can be used in exactly the same way. (We will refer to this definition as: *HL-strongly prime*).

Beachy [1] introduced another notion of a strongly prime ring module. He defined a nonzero module M of a ring R to be strongly prime if:

*for each $m' \in M$ and $0 \neq m \in M$, there exists a finite subset F of R such that $a \in R$ and $aFm = 0$ implies $am' = 0$. (We will refer to this definition as: *Beachy-strongly prime*).*

Groenewald [10] extended the Handelman-Lawrence definition to near-rings and defined a near-ring R to be right strongly prime if for every $0 \neq a \in R$, there exists a finite subset F of R such that if $r \in R$ and $aFr = 0$, then $r = 0$. Analogously, a near-ring R is defined to be left strongly prime if $r(Fa) = 0$ implies $r = 0$. Furthermore, an ideal P of R is called left strongly prime if $\frac{R}{P}$ is a left strongly prime near-ring. In this section we generalise these ideas to any R -module M .

Definition 120 Let M be an R -module such that $RM \neq 0$. Then:

- (a) M is said to be (left) strongly prime if for all $0 \neq m \in M$, there exists a finite subset $F = \{r_1, r_2, \dots, r_n\} \subseteq R$ (depending on m) such that $a \in R$ and $aFm = 0$ implies $aM = 0$.
- (b) An R -ideal P of M is (left) strongly prime if $RM \not\subseteq P$ and $\frac{M}{P}$ is a (left) strongly prime module. (ie. for all $m \in M \setminus P$, there exists a finite subset F of R such that $a \in R$ and $aFm \subseteq P$ implies that $aM \subseteq P$).

Hereafter we shall refer to left strongly prime simply as strongly prime. Furthermore, if we refer to a module M as being strongly prime we would mean that it is strongly prime in terms of our definition above. It is quite clear (proof can be seen in the proposition that follows) that a module M of a near-ring R is *HL*-strongly prime $\implies M$ is *Beachy*-strongly prime $\implies M$ is strongly prime. To get equivalence amongst the three definitions, we impose some additional conditions. However, we first state the following definition:

Definition 121 An R -module M is said to be cofaithful if there exists a finite subset F of M such that $a \in R$ and $aF = 0$ implies $a = 0$.

Proposition 122 Let M be an R -module of the near-ring R . Then the following are equivalent:

- (a) M is *HL*-strongly prime.
- (b) M is cofaithful and *Beachy*-strongly prime.
- (c) M is faithful and strongly prime.

Proof. (a) \implies (b): If M is *HL*-strongly prime, then for each $0 \neq m \in M$, there exists a finite $F \subseteq R$ such that $a \in R$ and $aFm = 0$ implies $a = 0$. So for each $m' \in M$ it also follows that $am' = 0$ and therefore M is *Beachy*-strongly prime. To show that M is cofaithful, choose $F' = Fm \subseteq M$ and the result follows.

(b) \implies (c): Suppose M is cofaithful and *Beachy*-strongly prime. Since M is cofaithful, it is clearly also faithful and there exists $F' = \{m_1, m_2, \dots, m_t\} \subseteq M$

such that $r \in R$ and $rF' = 0 \implies r = 0$. Let $0 \neq m \in M$. Then, since M is *Beachy*-strongly prime, for each $m_i \in F'$ ($1 \leq i \leq t$) there exists a finite $F_i \subseteq R$ such that $a \in R$ and $aF_i m = 0 \implies am_i = 0$. Now let $F = \cup F_i$ where $i = 1, 2, \dots, t$. Then $aFm = 0 \implies a(\cup F_i)m = 0 \implies aF_i m = 0 \implies am_i = 0$ for all $i = 1, 2, \dots, t$. Thus $aFm = 0 \implies aF' = 0 \implies a = 0$. Hence $aM = 0$ and M is strongly prime.

(c) \implies (a): Since M is strongly prime, for each $0 \neq m \in M$, there exists a finite $F \subseteq R$ such that $a \in R$ and $aFm = 0$ implies $aM = 0$. Since M is faithful, $a = 0$ and so M is *HL*-strongly prime. ■

Proposition 123 *If M is a strongly prime R -module, then M is 3-prime.*

Proof. Let $a \in R$ and $m \in M$ such that $aRm = 0$. Suppose $m \neq 0$. Since M is strongly prime, there exists a finite subset F of R such that $aFm \subseteq aRm = 0$ implies that $aM = 0$. Hence M is 3-prime. ■

Proposition 124 *Let M be a strongly prime R -module. Then for every nonzero R -submodule S of M , there exists a finite subset $F = \{s_1, s_2, \dots, s_n\} \subseteq S$ such that $a \in R$ and $aF = 0$ implies $aM = 0$.*

Proof. Let $0 \neq S \leq_R M$ and $0 \neq m \in S$. Since M is left strongly prime, there exists a finite subset $F = \{r_1, r_2, \dots, r_n\} \subseteq R$ such that $a \in R$ and $aFm = 0$ implies that $aM = 0$. Let $F_1 = Fm = \{r_1m, r_2m, \dots, r_nm\}$. Then $F_1 \subseteq S$ since S is an R -submodule of M . Furthermore, $aF_1 = 0 \implies aFm = 0$ and hence it follows that $aM = 0$. ■

Proposition 125 *Let M be an R -module such that for every $0 \neq m \in M$ there exists an $r \in R$ such that $rm \neq 0$. If for every nonzero R -submodule S of M , there exists a finite subset $F = \{s_1, s_2, \dots, s_n\} \subseteq S$ such that $a \in R$ and $aF = 0$ implies $aM = 0$, then M is strongly prime.*

Proof. Let $0 \neq m \in M$. Since Rm is a nonzero R -submodule of M , there exists a finite subset $F = \{s_1m, s_2m, \dots, s_nm\} \subseteq Rm$ such that $a \in R$ and

$aF = 0$ imply $aM = 0$. Let $F_1 = \{s_1, s_2, \dots, s_n\} \subseteq R$. Then it follows that $aF_1m = 0 \implies aF = 0 \implies aM = 0$. Hence M is strongly prime. ■

Corollary 126 *If R is a near-ring with identity then the R -module M is strongly prime if and only if for every nonzero R -submodule S of M , there exists a finite subset $F = \{s_1, s_2, \dots, s_n\} \subseteq S$ such that $aF = 0$ implies $aM = 0$.*

Proof. Let $0 \neq m \in M$. Since R has identity, $1.m = m \neq 0$. So the proof follows from the previous two propositions. ■

Proposition 127 *Let M be a HL-strongly prime R -module. Then for every nonzero R -submodule S of M , there exists a finite subset $F = \{s_1, s_2, \dots, s_n\} \subseteq S$ such that $a \in R$ and $aF = 0$ implies $a = 0$.*

Proof. Follows by a similar argument used in the proof of Proposition 124. ■

Proposition 128 *Let M be an R -module such that for every $0 \neq m \in M$ there exists an $r \in R$ such that $rm \neq 0$. If for every nonzero R -submodule S of M , there exists a finite subset $F = \{s_1, s_2, \dots, s_n\} \subseteq S$ such that $a \in R$ and $aF = 0$ implies $a = 0$, then M is HL-strongly prime.*

Proof. Follows by a similar argument used in the proof of Proposition 125. ■

Corollary 129 *If R is a near-ring with identity then the R -module M is HL-strongly prime if and only if for every nonzero R -submodule S of M , there exists a finite subset $F = \{s_1, s_2, \dots, s_n\} \subseteq S$ such that $aF = 0$ implies $a = 0$.*

Proposition 130 *If R is a near-ring with identity and M is an R -module with no nonzero, proper R -submodules then M is Beachy-strongly prime.*

Proof. Let $m \in M$ and $0 \neq m_1 \in M$. Since R has an identity element, we have that Rm_1 is a nonzero R -submodule of M . Hence, from our assumption, we have $Rm_1 = M$. So there exists an $r \in R$ such that $m = rm_1$. If we let $F = \{r\}$ and $aFm_1 = 0$, then $am = arm_1 = 0$. Thus M is Beachy-strongly prime. ■

Example 131 Let F be a field, $R = F \times F$ and ${}_R M = 0 \times F$. Then ${}_R M$ has no nonzero, proper R -submodules. So, from Proposition 130, it follows that ${}_R M$ is Beachy-strongly prime. Since ${}_R M$ is clearly not faithful, it follows from Proposition 122, that ${}_R M$ is not HL -strongly prime.

Although, in general, a 3-prime R -module is not strongly prime, we have the following observation:

Proposition 132 Let M be an R -module. Then the following are equivalent:

- (a) M is strongly prime.
- (b) M is 3-prime and for every nonzero submodule S of M , there exists a finite subset $F = \{s_1, s_2, \dots, s_n\} \subseteq S$ such that $a \in R$ and $aF = 0$ implies $aM = 0$.

Proof. (a) \Rightarrow (b): Follows from Propositions 123 and 124.

(b) \Rightarrow (a): Let $0 \neq m \in M$. Then $Rm \neq 0$ (for if $Rm = 0$, then $aRm = 0$ for all $a \in R$ implies that $aM = 0$ for all $a \in R$ and hence $RM = 0$. Since M is 3-prime, this is not possible). Hence there exists an $r \in R$ such that $rm \neq 0$. So the result follows from Proposition 125. ■

Proposition 133 If P is a c -prime R -ideal of M , then P is a strongly prime R -ideal.

Proof. Let $m \in M \setminus P$. Since P is c -prime, $Rm \not\subseteq P$ and $(P : m) = (P : M)$. Furthermore, if $Rm \subseteq P$ then $R \subseteq (P : m) = (P : M)$ implies $RM \subseteq P$ is a contradiction. Hence $Rm \not\subseteq P$.

Let $a \in R$ such that $am \notin P$ and let $F = \{a\}$. If $r \in R$ such that $rFm \subseteq P$, we get $r \in (P : Fm) = (P : am)$. But $am \notin P$ implies that $(P : am) = (P : M)$. Hence $r \in (P : M)$ implies $rM \subseteq P$ whence P is strongly prime. ■

Corollary 134 If P is a 3-prime R -ideal of M and $(P : m) \triangleleft R$ for every $m \in M \setminus P$, then P is a strongly prime R -ideal.

Proof. Since P is a 3-prime R -ideal and $(P : m) \triangleleft R$ for every $m \in M \setminus P$, it follows from Proposition 66, that P is a c -prime R -ideal of M . So the result follows. ■

Since any 3-prime R -module is also 0-prime, clearly it follows that any strongly prime R -module is also 0-prime. However, a 0-prime R -module is, in general, not strongly prime - not even in the case of a zerosymmetric near-ring. We demonstrate this in the following example taken from Groenewald [10]:

Example 135 Let R be the near-ring with addition and multiplication defined as in the following tables:

+	0	a	b	c	x	y	×	0	a	b	c	x	y
	0	0	a	b	c	x	y	0	0	0	0	0	0
	a	a	0	x	y	c	b	a	0	0	a	a	0
	b	b	x	0	y	a	c	b	0	0	c	b	0
	c	c	y	x	0	b	a	c	0	0	b	c	0
	x	x	b	c	a	y	0	x	0	0	x	x	0
	y	y	c	a	b	0	x	y	0	0	y	y	0

Then R is a zerosymmetric near-ring. Consider the R -module $M = {}_R R$. Then M has no proper R -ideals. Furthermore, $\{0\}$ is a 0-prime R -ideal of M since $R^2 \neq \{0\}$. Hence M is a 0-prime R -module. However, M is not strongly prime since for any subset F of R , $aFx = 0$ but $a \neq 0$.

Proposition 136 If P is a strongly prime R -ideal of M , then there exists a finite subset F of R such that $(P : Fm) = (P : M)$ for all $m \in M \setminus P$.

Proof. Let $m \in M \setminus P$. Since P is strongly prime, there exists a finite subset F of R such that $a \in R$ and $aFm \subseteq P \implies aM \subseteq P$. Hence $r \in (P : Fm) \implies rFm \subseteq P \implies rM \subseteq P \implies r \in (P : M)$, so that we have $(P : Fm) \subseteq (P : M)$. The other inclusion is obvious. ■

Nicholson and Watters [20] defined the notion of an fm -system for a subset X of a ring module M based on the Handelman-Lawrence definition of a strongly

prime ring. We adopt their idea and provide the following definition of an F_m -system for a near-ring module M :

Definition 137 *Let R be a near-ring. A nonempty subset X of an R -module M is called an F_m -system if for each $m \in X$, there exists a finite subset F of R such that $aFm \cap X \neq \emptyset$ for all $a \notin (0 : M)$.*

Proposition 138 *An R -ideal P of M is strongly prime if and only if $M \setminus P$ is an F_m -system.*

Proof. Suppose that $P \triangleleft_R M$ is strongly prime. Then $\frac{M}{P} \neq 0$ (for, if $\frac{M}{P} = 0$, then $M \subseteq P$ implies that $RM \subseteq P$ contradicts the definition of a strongly prime R -ideal). Hence $M \setminus P \neq \emptyset$. Let $m \in M \setminus P$. Then, since P is strongly prime, there exists a finite subset F of R such that $a \in R$ and $aFm \subseteq P$ implies that $aM \subseteq P$. Suppose there exists $r \notin (P : M)$ such that $rFm \cap (M \setminus P) = \emptyset$. Then clearly this implies that $rFm \subseteq P$ but $rM \not\subseteq P$, which contradicts that P is a strongly prime R -ideal of M . Hence $rFm \cap (M \setminus P) \neq \emptyset$ for all $r \in (P : M)$ and so $M \setminus P$ is an F_m -system.

Now assume that $M \setminus P$ is an F_m -system. Let $m \in M \setminus P$. Then there exists a finite subset F of R such that $aFm \cap (M \setminus P) \neq \emptyset$ for all $a \notin (P : M)$. Let $r \in R$ be such that $rFm \subseteq P$. If $rM \not\subseteq P$, then $r \notin (P : M)$ implies that $rFm \cap (M \setminus P) \neq \emptyset$. So there exists an $x \in M$ such that $x \in rFm$ and $x \in M \setminus P$ and hence $x \in rFm$ but $x \notin P$. This contradicts that $rFm \subseteq P$. So $rM \subseteq P$ whence P is a strongly prime R -ideal of M . ■

Proposition 139 *If P is a strongly prime R -ideal of M , then $\tilde{P} = (P : M)$ is a strongly prime ideal of R .*

Proof. Let $a \in R \setminus (P : M)$. Then $aM \not\subseteq P$ implies that there exists $m \in M$ such that $am \notin P$. So $am \neq 0$. Since P is strongly prime in M , for this $0 \neq am \in M$ there exists a finite subset F of R such that if $b \in R$ and $bFam \subseteq P$, then $bM \subseteq P$. Now, if $bFa \subseteq (P : M)$, then it follows that $bFaM \subseteq P \implies bFam \subseteq P \implies bM \subseteq P \implies b \in (P : M)$. Hence, by definition, $(P : M)$ is a strongly prime ideal of R . ■

Corollary 140 *If M is a strongly prime R -module, then $(0 : M)$ is a strongly prime ideal of R .*

Proof. Replace P by the 0-ideal in the previous proof. ■

Proposition 141 *If M is a faithful R -module such that M is strongly prime, then R is a strongly prime near-ring.*

Proof. Let $0 \neq a \in R$. Since M is strongly prime, $(0 : M)$ is a strongly prime ideal of R . Hence there exists a finite subset F of R such that $r \in R$ and $rFa \subseteq (0 : M)$ implies that $r \in (0 : M)$. So $rM = 0$ and, since M is faithful, we have that $r = 0$.

So $rFa = 0 \implies rFa \subseteq (0 : M) \implies r = 0$. Therefore, R is a strongly prime near-ring. ■

Proposition 142 *Let $P \triangleleft R$ with $P \neq R$. Then P is a strongly prime ideal of R if and only if there exists a strongly prime R -module M with $P = (0 : M)_R$.*

Proof. Let P be a strongly prime ideal of R with $P \neq R$. We know that $\frac{R}{P}$ is an R -module with natural operations. Furthermore, since P is strongly prime, P is 3-prime and hence 0-prime. So by Lemma 88, we have that $P = \left(0 : \frac{R}{P}\right)_R$. We show that $\frac{R}{P}$ is a strongly prime R -module. Let $a + P \in \frac{R}{P}$ where $a \notin P$. Since P is strongly prime, there exists a finite subset F of R such that $r \in R$ and $rFa \subseteq P$ implies that $r \in P$. Now for this F , we get: $rF(a+P) = rFa+P \subseteq P$ since $rFa \subseteq P$. Hence $rF(a+P) \subseteq P$ implies that $r \in P$ and so $\frac{R}{P}$ is strongly prime. We let $M = \frac{R}{P}$.

For the converse, if M is a strongly prime R -module then $(0 : M)_R$ is a strongly prime ideal of R . Since $P = (0 : M)_R$, the result follows. ■

Corollary 143 *Let $P \triangleleft R$ with $P \neq R$. Then $\frac{R}{P}$ is a strongly prime near-ring if and only if $\frac{R}{P}$ is a strongly prime R -module.*

Proof. If $\frac{R}{P}$ is a strongly prime near-ring, then P is a strongly prime ideal of R . Hence, by Proposition 142, $M = \frac{R}{P}$ is a strongly prime R -module.

On the other hand, $\frac{R}{P}$ is a strongly prime R -module implies that $\left(0 : \frac{R}{P}\right)_R$ is a strongly prime ideal of R . But, by Proposition 142, $P = \left(0 : \frac{R}{P}\right)_R$. Hence P is a strongly prime ideal of R and it follows that $\frac{R}{P}$ is a strongly prime near-ring. ■

Proposition 144 *Let $A \triangleleft R$, $A \neq (0 : M)$ and M be a strongly prime R -module. Then M is a strongly prime A -module.*

Proof. Let $0 \neq m \in M$. Since M is a strongly prime R -module, there exist a finite subset F of R such that $a \in A \subseteq R$ and $aFm = 0$ implies that $aM = 0$. We consider the R -submodule, Am of M .

If $Am = 0$, then for every $a \in A$, we have $aAm \subseteq Am = 0$. Since M is a strongly prime R -module, we know that M is also a 3-prime R -module and hence, from Proposition 95, M is a 3-prime A -module. Therefore, since $m \neq 0$, it must follow that $aM = 0$ for all $a \in A$ and consequently $AM = 0$ which contradicts $A \neq (0 : M)$. So $Am \neq 0$.

Since $Am \neq 0$, there exists $b \in A$ such that $bm \neq 0$. Hence there exists a finite subset, $F' \subseteq R$, such that $a \in A \subseteq R$ and $aF'(bm) = 0$ implies $aM = 0$. If we let $F_1 = F'b$, then since $A \triangleleft R$ we have $F_1 \subseteq A$ and hence we get $aF_1m = a(F'b)m = aF'(bm) = 0$ which implies $aM = 0$. Therefore M is a strongly prime A -module. ■

Proposition 145 *If M is a strongly prime R -module and $H \triangleleft_R M$, then H is a strongly prime R -module.*

Proof. Let $a \in R$ and $0 \neq h \in H$. Then, since $0 \neq h \in M$ and M is strongly prime, there exists a finite subset F of R such that $aFh = 0$ implies that $aM = 0$. Hence $aFh = 0$ implies $aH \subseteq aM = 0$ and, therefore, H is strongly prime. ■

2.6 Multiplication modules

In the previous sections, we defined various types of prime R -ideals (modules) and we were easily able to prove that if an R -ideal P of M satisfied a certain prime condition, then so did the corresponding ideal $\widetilde{P} = (P : M)$ of R . However the converse relation turned out to be problematic in many situations, especially since it is difficult to construct an R -ideal of M by starting with an ideal of R . To overcome this problem, we now introduce the notion of a multiplication module.

Definition 146 *Let M be an R -module. Then:*

- (a) $C \subseteq M$ is called a multiplication set if $\widetilde{C}M = C$.
- (b) $m \in M$ is called a multiplication element if the singleton set $\{m\}$ is a multiplication set.

Note that (b) above translates to $\widetilde{\{m\}}M = m$. For future applications we will simply write $\widetilde{\{m\}}$ as \widetilde{m} .

Definition 147 *Let M be an R -module. Then:*

- (a) M is called a 0-multiplication module if every R -ideal of M is multiplication set.
- (b) M is called a 2-multiplication module if every R -submodule of M is multiplication set.
- (c) M is called a c -multiplication module if every $m \in M$ is a multiplication element.

The three types of multiplication modules defined above are, in general, nonequivalent. We demonstrate the existence of ν -multiplication ($\nu = 0, 2, c$) modules and their nonequivalence with the following examples:

Example 148 *Let R be the near-ring constructed on $K_4 = \{0, 1, 2, 3\}$ with multiplication on R given by the following table:*

\times	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	1	2	2
3	0	1	2	3

Let $M = {}_R R$. Then the R -ideals of M are $\{0\}, \{0, 1\}$ and M . Clearly, $\widetilde{\{0\}}M = \{0\}$ and $\widetilde{M}M = M$. Furthermore, $\widetilde{\{0, 1\}} = \{0, 1\}$ implies that $\widetilde{\{0, 1\}}M = \{0, 1\}$. Hence M is a 0-multiplication module. However, $\{0, 2\}$ is an R -submodule of M such that $\widetilde{\{0, 2\}}M = \{0\}M = \{0\} \neq \{0, 2\}$. So M is not a 2-multiplication module.

Example 149 Let R be the near-ring constructed on $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ with multiplication given by the following table:

\times	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	0	1	1
2	0	2	2	0	2	2
3	0	3	3	0	3	3
4	0	4	4	0	4	4
5	0	5	5	0	5	5

Let $M = {}_R R$. Then the R -submodules of M are $\{0\}, \{0, 3\}$ and M with $\widetilde{\{0\}}M = \{0\}, \widetilde{\{0, 3\}}M = \{0, 3\}$ and $\widetilde{M}M = M$. So M is a 2-multiplication module. However, for any $m \neq 0$ we have $\widetilde{\{m\}}M \neq \{m\}$ which implies that M is not a c -multiplication module.

Example 150 Let R be the near-ring constructed on $K_4 = \{0, 1, 2, 3\}$ with multiplication on R given by the following table:

\times	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

Let $M = {}_R R$. Then for all $m = 0, 1, 2, 3$ we have $\tilde{m}M = mM = m$. Hence M is a c -multiplication module.

Proposition 151 *Let M be an R -module. Then M is a c -multiplication module $\implies M$ is a 2-multiplication module $\implies M$ is a 0-multiplication module.*

Proof. Suppose that M is a c -multiplication module. Let A be an R -submodule of M . Since M is a c -multiplication module, for each $a \in A$, we have $\tilde{a}M = a$. Hence $\tilde{A}M = A$, thus proving that M is a 2-multiplication module.

Now suppose that M is a 2-multiplication module. Let $A \triangleleft_R M$. Since R is zerosymmetric, A is an R -submodule of M . Since M is a 2-multiplication module, we have $\tilde{A}M = A$ and the proof is complete. ■

At the outset, we did state that all near-rings in this thesis will be zerosymmetric. This condition, although not specifically repeated in Proposition 151, is a necessity for the truth of this proposition. We justify our claim with the following counter example:

Example 152 *Let R be the near-ring constructed on $K_4 = \{0, 1, 2, 3\}$ with multiplication on R defined by the following table:*

×	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	0	1	2	3
3	1	0	3	2

Then, clearly, R is not zerosymmetric. Let $M = {}_R R$. Then the R -submodules of M are $\{0\}$, $\{0, 1\}$ and M with $\widetilde{\{0\}}M = \{0\}$, $\widetilde{\{0, 1\}}M = \{0, 1\}$ and $\widetilde{M}M = M$. So M is a 2-multiplication module.

However, $\{0, 2\}$ is an R -ideal of M with $\widetilde{\{0, 2\}}M = \{0\}M = \{0\}$. Hence M is not a 0-multiplication module.

Lemma 153 *If C is an ideal of R and R has a multiplicative identity 1, then $\tilde{C} = C$.*

Proof. Let $x \in \tilde{C} = (C : R)$. Then $xR \subseteq C$ and since $1 \in R$, we have $x = x.1 \in C$. So $\tilde{C} \subseteq C$.

On the other hand, if $x \in C$, then since C is an ideal of R , we have $xR \subseteq C$ which implies that $x \in \tilde{C}$. ■

Lemma 154 *Let M be an R -module. Then:*

- (a) *an R -ideal C of M is a 0-multiplication ideal if and only if there exists an ideal I of R such that $IM = C$.*
- (b) *an R -submodule C of M is a 2-multiplication submodule if and only if there exists an R -subgroup I of R such that $IM = C$.*
- (c) *an element m of M is a c -multiplication element if and only if there exists an element $i \in R$ such that $iM = m$.*

Proof. (a) Let C be a 0-multiplication R -ideal of M . Since we know that $(C : M)$ is an ideal of R , we can set $I = (C : M)$. Then, by definition of a multiplication set, $IM = \tilde{C}M = C$.

Conversely, if $I \triangleleft R$ such that $IM = C$, we have $I \subseteq \tilde{C}$. This implies that $C = IM \subseteq \tilde{C}M \subseteq C$ and hence $\tilde{C}M = C$. Therefore C is a 0-multiplication R -ideal.

(b) Here we note that if C is an R -submodule of M , then $(C : M)$ is an R -subgroup of R . Hence, we once again set $I = (C : M)$ and the rest of the proof follows as in (a).

(c) If $m \in M$ is a c -multiplication element then $\tilde{m}M = m$. Hence there exists $i \in \tilde{m} \subseteq R$ such that $iM = m$.

Conversely, let $m \in M$ and suppose that there exists an $i \in R$ such that $iM = m$. Then $iM = m$ implies $i \in \tilde{m}$ and hence we have:

$$iM = m \implies \tilde{m}M = m.$$

So m is a c -multiplication element. ■

Lemma 155 *Let M be an R -module. Then the following are equivalent:*

(a) M is a c -multiplication R -module.

(b) For every subset C of M we have $IM = C$ for some subset I of R .

Proof. (a) \Rightarrow (b): Let $C \subseteq M$ and let $I = \bigcup_{x \in C} \tilde{x}$. Then if $a \in I$, it follows that $a \in \tilde{x}$ for some $x \in C$. Hence $aM = x \in C \implies aM \subseteq C \implies a \in \tilde{C}$. So $I \subseteq \tilde{C}$ and hence, by definition of a c -multiplication module, $IM = C$.

(b) \Rightarrow (a): Follows directly from part (c) of the previous lemma by considering the elements of M as singleton subsets of M . ■

In view of the above definitions and lemmas, we are now in a position to achieve the objectives of this section.

Proposition 156 *Let P be an R -ideal of a 0-multiplication R -module M such that \tilde{P} is a 0-prime ideal of R . Then P is a 0-prime R -ideal of M .*

Proof. Let $A \triangleleft R$ and $B \triangleleft_R M$ such that $AB \subseteq P$. Since M is a 0-multiplication module, we know that $\tilde{B}M = B$. Hence we have:

$$\tilde{A}\tilde{B}M = AB \subseteq P \text{ which implies that } \tilde{A}\tilde{B} \subseteq \tilde{P}.$$

Since \tilde{P} is a 0-prime ideal of R , we have that $A \subseteq \tilde{P}$ or $\tilde{B} \subseteq \tilde{P}$. If $A \subseteq \tilde{P}$, then $AM \subseteq P$ and we are done. If $\tilde{B} \subseteq \tilde{P}$, then $B = \tilde{B}M \subseteq \tilde{P}M = P$. So $B \subseteq P$ and once again we are done. ■

The following proposition can be proved in exactly the same way as the previous proposition by choosing A to be a left R -subgroup of R and B to be an R -submodule of M .

Proposition 157 *Let P be an R -ideal of a 2-multiplication R -module M such that \tilde{P} is a 2-prime ideal of R . Then P is a 2-prime R -ideal of M .*

Proposition 158 *Let P be an R -ideal of a c -multiplication R -module M such that \tilde{P} is a 3-prime (resp. c -prime) ideal of R . Then P is a 3-prime (resp. c -prime) R -ideal of M .*

Proof. Suppose that \tilde{P} is a 3-prime ideal of R . Let $a \in R$ and $m \in M$ such that $aRm \subseteq P$. Since M is a c -multiplication module, $aR\tilde{m}M = aRm \subseteq P$ which implies that $aR\tilde{m} \subseteq \tilde{P}$. Since \tilde{P} is a 3-prime ideal of R , $a \in \tilde{P}$ or $\tilde{m} \subseteq \tilde{P}$. If $a \in \tilde{P}$ then $aM \subseteq P$ and the proof is complete. If $\tilde{m} \subseteq \tilde{P}$, then $\tilde{m} \subseteq (P : M)$ which implies that $\tilde{m}M \subseteq P$. So $m = \tilde{m}M \in P$ and once again we are done.

Now suppose that \tilde{P} is a c -prime ideal of R . Let $a \in R$ and $m \in M$ such that $am \in P$. Then $a\tilde{m}M = am \in P$ and the rest of the proof follows as for the 3-prime case. ■

The following proposition follows easily from Definition 109 and Propositions 156, 157 and 158:

Proposition 159 *Let P be an R -ideal of M . Then:*

- (a) *If M is a 0-multiplication R -module and $\tilde{P} \triangleleft R$ is 0- s -prime, then $P \triangleleft_R M$ is 0- s -prime.*
- (b) *If M is a 2-multiplication R -module and $\tilde{P} \triangleleft R$ is 2- s -prime, then $P \triangleleft_R M$ is 2- s -prime.*
- (c) *If M is a c -multiplication R -module and $\tilde{P} \triangleleft R$ is 3- s -prime, then $P \triangleleft_R M$ is 3- s -prime.*

Proposition 160 *Let P be an R -ideal of a c -multiplication R -module M such that \tilde{P} is a strongly prime ideal of R . Then P is a strongly prime R -ideal of M .*

Proof. Let $m \in M \setminus P$. Then if $t \in \tilde{m}$, we get $tM = \{m\} \not\subseteq P$ and hence $t \notin (P : M) = \tilde{P}$. Since $\tilde{P} \triangleleft R$ is strongly prime, there exists a finite subset F of R such that $a \in R$ and $aFt \subseteq \tilde{P}$ implies that $a \in \tilde{P}$.

Therefore, if $r \in R$ such that $rFm \subseteq P$, we get:

$$rFtM \subseteq P \implies rFt \subseteq \tilde{P} \implies r \in \tilde{P}$$

Hence $rM \subseteq P$ implies that P is strongly prime R -ideal of M . ■

Corollary 161 *Suppose that M is a ν -multiplication faithful R -module where $\nu = 0, 2, c$. Then M is ν -prime if and only if R is ν -prime.*

Furthermore, if M is a c -multiplication faithful R -module, then:

- (a) M is 3-prime if and only if R is 3-prime.
- (b) M is strongly prime if and only if R is strongly prime.

Finally, we conclude this section with the following definition and a further characterization of prime modules.

Definition 162 *Let M be an R -module. Then:*

- (a) M is called a **0-fully faithful R -module** if all nonzero proper R -ideals of M are faithful R -modules.
- (b) M is called a **2-fully faithful R -module** if all nonzero proper R -submodules of M are faithful R -modules.

Theorem 163 *M is a ν -prime R -module if and only if M is a ν -fully faithful $\frac{R}{(0 : M)_R}$ -module, where $\nu = 0$ or 2.*

Proof. We illustrate the proof for $\nu = 0$. The other case can be proved in a similar way.

Suppose that M is a 0-prime R -module. Let $0 \neq P \triangleleft_R M$. Since M is 0-prime, by Corollary 62, $(0 : M) = (0 : P)$. Furthermore, we note that P is a faithful $\frac{R}{(0 : P)_R}$ -module. But $\frac{R}{(0 : P)_R} = \frac{R}{(0 : M)_R}$. Hence P is a faithful $\frac{R}{(0 : M)_R}$ -module and it follows by definition that M is a 0-fully faithful $\frac{R}{(0 : M)_R}$ -module.

Conversely, let $0 \neq P$ be an R -ideal of M . Clearly, $(0 : M) \subseteq (0 : P)$. If $(0 : P) \subsetneq (0 : M)$, then there exists $x \in (0 : P)$ with $x \notin (0 : M)$. Since P is a faithful $\frac{R}{(0 : M)_R}$ -module, it must also follow that $xP \neq 0$ which is a contradiction. Hence $(0 : P) = (0 : M)$ and, once again by Corollary 62, M is a 0-prime R -module. ■

Chapter 3

SPECIAL RADICALS OF NEAR-RING MODULES

INTRODUCTION

In this chapter, we define special radical classes of near-ring modules and immediately thereafter, we establish that a special class of near-ring modules leads to the construction of a special class of near-rings and, in turn, a special class of near-rings leads to the construction of a special class of near-ring modules. We, then, show that the classes of 2-prime, 3-prime, c -prime, strongly prime and s -prime near-ring modules form \mathcal{A} -special classes (Andrunakievich special) with respect to our definition of a special class.

However, we first begin with the following observations with regard to general classes of near-ring modules.

3.1 General classes of near-ring modules

For each near-ring R , let \mathcal{M}_R be a class (possibly empty) of R -modules M with $RM \neq 0$. Then we define:

$$\rho(R) = \cap\{(0 : M)_R : M \in \mathcal{M}_R\}.$$

Now let $\mathcal{M} = \cup\{\mathcal{M}_R : R \text{ is a near-ring}\}$. Then we have the following definition:

Definition 164 *The class \mathcal{M} is called a general class of near-ring modules if it satisfies the following conditions:*

- (G1) *If $I \triangleleft R$ and $M \in \mathcal{M}_{\frac{R}{I}}$, then $M \in \mathcal{M}_R$.*
- (G2) *If $M \in \mathcal{M}_R$ and $I \triangleleft R$ such that $I \subseteq (0 : M)_R$, then $M \in \mathcal{M}_{\frac{R}{I}}$.*
- (G3) *If $\rho(R) = 0$, then $\mathcal{M}_I \neq \emptyset$ for all nonzero ideals I of R .*
- (G4) *If $\mathcal{M}_I \neq \emptyset$ whenever $0 \neq I \triangleleft R$, then $\rho(R) = 0$.*

In view of the above definition, we record the following observations made by Veldsman in [23]:

- (a) Let $\mathcal{R} = \{R : \text{there exists } M \in \mathcal{M}_R \text{ such that } (0 : M)_R = 0\} \cup \{0\}$. Then \mathcal{R} is a Kurosh-Amitsur radical class.
- (b) If the class \mathcal{M} satisfies (G1) and (G2), then \mathcal{R} is a Hoehnke radical class.

Now let \mathcal{T} be a class of near-rings that is closed under homomorphic images. For the near-ring R , let \mathcal{M}_R be a class of near-ring modules and let:

$$\mathcal{M} = \cup\{\mathcal{M}_R : R \text{ is a near-ring}\}.$$

Then we have the following definition:

Definition 165 *The class \mathcal{M} is called a \mathcal{T} -general class if it satisfies:*

- (T1) *If $I \triangleleft R$ and $M \in \mathcal{M}_{\frac{R}{I}}$, then $M \in \mathcal{M}_R$.*
- (T2) *If $M \in \mathcal{M}_R$ and $I \triangleleft R$ with $I \subseteq (0 : M)_R$, then $M \in \mathcal{M}_{\frac{R}{I}}$.*
- (T3) *If $R \in \mathcal{T}$ and $\rho(R) = 0$, then $\mathcal{M}_I \neq \emptyset$ for every $0 \neq I \triangleleft R$.*
- (T4) *If $R \in \mathcal{T}$ and $\mathcal{M}_I \neq \emptyset$, then whenever $0 \neq I \triangleleft R$, we have $\rho(R) = 0$.*

Definition 166 *If \mathcal{M} is a \mathcal{T} -general class, then the class:*

$$\mathcal{R} = \{R : \text{there exists } M \in \mathcal{M}_R \text{ with } (0 : M)_R = 0\} \cup \{0\}$$

is called a \mathcal{T} -radical class.

3.2 \mathcal{T} -special classes of near-ring modules

In order to construct the definition of a special class of near-ring modules, we recall the following result which we stated as Proposition 33 in the Chapter 1:

Let R be a near-ring and $I \triangleleft R$. Let $r \in R$ and $m \in M$. Then:

If M is an $\frac{R}{I}$ -module, then with respect to $rm = (r + I)m$, M becomes an R -module and $I \subseteq (0 : M)_R$.

If M is an R -module and $I \subseteq (0 : M)_R$, then M is an $\frac{R}{I}$ -module with respect to $(r + I)m = rm$.

In both cases, we have that $(0 : M)_{\frac{R}{I}} = \frac{(0 : M)_R}{I}$.

Now let \mathcal{T} be a nonempty class of all zerosymmetric right near-rings which is closed under homomorphic images. For each near-ring R , let \mathcal{M}_R be a class of R -modules (possibly empty). Let $\mathcal{M} = \cup\{\mathcal{M}_R : R \text{ is a near-ring}\}$. We introduce the notion of a \mathcal{T} -special class of near-ring modules.

Definition 167 *A class $\mathcal{M} = \cup\mathcal{M}_R$ of near-ring modules is called a \mathcal{T} -special class if it satisfies the following conditions:*

(M1) *If $M \in \mathcal{M}_R$ and $I \triangleleft R$ with $IM = 0$, then $M \in \mathcal{M}_{\frac{R}{I}}$.*

(M2) *If $I \triangleleft R$ and $M \in \mathcal{M}_{\frac{R}{I}}$, then $M \in \mathcal{M}_R$.*

(M3) *If $M \in \mathcal{M}_R$ and $I \triangleleft R \in \mathcal{T}$ with $IM \neq 0$, then $M \in \mathcal{M}_I$.*

(M4) *If $M \in \mathcal{M}_R$, then $RM \neq 0$ and $\frac{R}{(0 : M)_R}$ is a 2-prime near-ring.*

(M5) *If $I \triangleleft R \in \mathcal{T}$ and $M \in \mathcal{M}_I$, then there exists an R -module $N \in \mathcal{M}_R$ such that $(0 : N)_I \subseteq (0 : M)_I$.*

(M6) *If $K \triangleleft I \triangleleft R \in \mathcal{T}$ and there exists a faithful $\frac{I}{K}$ -module $M \in \mathcal{M}_{\frac{I}{K}}$, then $K \triangleleft R$.*

Definition 168 *A class \mathcal{F} of near-rings is called a \mathcal{T} -special class if the following conditions are satisfied:*

(R1) *If $R \in \mathcal{F}$, then R is 2-prime.*

(R2) If $R \in \mathcal{F} \cap \mathcal{T}$ and $I \triangleleft R$, then $I \in \mathcal{F}$.

(R3) If $K \triangleleft I \triangleleft R \in \mathcal{T}$ and $\frac{I}{K} \in \mathcal{F}$, then $K \triangleleft R$.

(R4) If $I \triangleleft \cdot R$ and $I \in \mathcal{F}$, then $R \in \mathcal{F}$ (ie. \mathcal{F} is closed under essential extensions).

In the previous chapter, we have seen that there were numerous relationships between a near-ring and its modules. In particular, prime R -ideals of the R -module M led to prime ideals of R and, under certain conditions, the converses also existed. It is, therefore, natural to assume that there is a relationship between special radicals of near-rings and special radicals of their modules. In the two theorems that follow, we show the construction of a special class of near-rings from a special class of near-ring modules and the reversal of the process.

Theorem 169 Let $\mathcal{M} = \cup_R \mathcal{M}_R$ be a \mathcal{T} -special class of near-ring modules. Then $\mathcal{F} = \{R : \text{there exists } M \in \mathcal{M}_R \text{ with } (0 : M)_R = 0\} \cup \{0\}$ is a \mathcal{T} -special class of near-rings.

Proof. **(R1):** Let $R \in \mathcal{F}$. Then there exists an $M \in \mathcal{M}_R$ with $(0 : M)_R = 0$. From **(M4)** we have that $R = \frac{R}{(0 : M)_R}$ is a 2-prime near-ring.

(R2): Let $R \in \mathcal{F} \cap \mathcal{T}$ and $I \triangleleft R$. Then there exists $M \in \mathcal{M}_R$ such that $(0 : M)_R = 0$. If $I = 0$, then $I \in \mathcal{F}$ and we are done. If $I \neq 0$, then $IM \neq 0$ (for if $IM = 0$, we have that $I \subseteq (0 : M)_R = 0 \implies I = 0$). Hence, from **(M3)**, it now follows that $M \in \mathcal{M}_I$. Furthermore, $(0 : M)_I \subseteq (0 : M)_R = 0$. Therefore $I \in \mathcal{F}$.

(R3): Let $K \triangleleft I \triangleleft R \in \mathcal{T}$ with $\frac{I}{K} \in \mathcal{F}$. Since $\frac{I}{K} \in \mathcal{F}$, there exists an $\frac{I}{K}$ -module M (ie. $M \in \mathcal{M}_{\frac{I}{K}}$) such that $(0 : M)_{\frac{I}{K}} = 0$. From **(M6)**, it follows that $K \triangleleft R$.

(R4): Let $I \triangleleft \cdot R$ and suppose that $I \in \mathcal{F}$. So there exists $M \in \mathcal{M}_I$ with $(0 : M)_I = 0$. From **(M5)**, there exists $N \in \mathcal{M}_R$ such that:

$$(0 : N)_I \subseteq (0 : M)_I = 0.$$

But $0 = (0 : N)_I = (0 : N)_R \cap I$. Since $I \triangleleft R$ and $(0 : N)_R \triangleleft R$, we have that $(0 : N)_R = 0$. Hence we have that $R \in \mathcal{F}$. ■

Theorem 170 *Let \mathcal{F} be a \mathcal{T} -special class of near-rings and for the near-ring R , let $\mathcal{M}_R = \{M : M \text{ is an } R\text{-module, } RM \neq 0 \text{ and } \frac{R}{(0 : M)_R} \in \mathcal{F}\}$. If $\mathcal{M} = \cup \mathcal{M}_R$, then \mathcal{M} is a \mathcal{T} -special class of near-ring modules.*

Proof. **(M1):** Let $M \in \mathcal{M}_R$ with $I \triangleleft R$ such that $IM = 0$. Since $M \in \mathcal{M}_R$ we have $RM \neq 0$ and $\frac{R}{(0 : M)_R} \in \mathcal{F}$. Since $I \subseteq (0 : M)_R$, it follows that M is also an $\frac{R}{I}$ -module with $(0 : M)_{\frac{R}{I}} = \frac{(0 : M)_R}{I}$. Furthermore, since $RM \neq 0$, we also have that $\left(\frac{R}{I}\right)M \neq 0$. Now, $\frac{\frac{R}{I}}{(0 : M)_{\frac{R}{I}}} \cong \frac{\frac{R}{I}}{\frac{(0 : M)_R}{I}} \cong \frac{R}{(0 : M)_R} \in \mathcal{F}$. Hence $M \in \mathcal{M}_{\frac{R}{I}}$ and **(M1)** is satisfied.

(M2): Let $I \triangleleft R$ and $M \in \mathcal{M}_{\frac{R}{I}}$. Then we know that M is an $\frac{R}{I}$ -module and $I \subseteq (0 : M)_R$. Since $\left(\frac{R}{I}\right)M \neq 0$, we have that $RM \neq 0$. Moreover, $\frac{R}{(0 : M)_R} \cong \frac{\frac{R}{I}}{\frac{(0 : M)_R}{I}} = \frac{\frac{R}{I}}{(0 : M)_{\frac{R}{I}}} \in \mathcal{F}$. Hence $M \in \mathcal{M}_R$.

(M3): Let $M \in \mathcal{M}_R$ and $I \triangleleft R \in \mathcal{T}$ with $IM \neq 0$. Then M is an I -module with $IM \neq 0$. Furthermore,

$$\frac{I}{(0 : M)_I} = \frac{I}{(0 : M)_R \cap I} \cong \frac{(I + (0 : M)_R)}{(0 : M)_R} \triangleleft \frac{R}{(0 : M)_R} \in \mathcal{F}.$$

We also know that $R \in \mathcal{T}$ and \mathcal{T} is homomorphically closed. Hence we have that $\frac{R}{(0 : M)_R} \in \mathcal{F} \cap \mathcal{T}$. So, from **(R2)**, it follows that $\frac{I}{(0 : M)_I} \in \mathcal{F}$, and hence $M \in \mathcal{M}_I$.

(M4): Let $M \in \mathcal{M}_R$. Then $RM \neq 0$ and $\frac{R}{(0 : M)_R} \in \mathcal{F}$. Since \mathcal{F} is \mathcal{T} -special, from **(R1)** it follows that $\frac{R}{(0 : M)_R}$ is a 2-prime near-ring.

(M5): Let $I \triangleleft R \in \mathcal{T}$ such that $M \in \mathcal{M}_I$. Since $(0 : M)_I \triangleleft I \triangleleft R$ and $\frac{I}{(0 : M)_I} \in \mathcal{F}$, it follows from **(R3)** that $(0 : M)_I \triangleleft R$. Now let $\frac{K}{(0 : M)_I}$ be the

ideal of $\frac{R}{(0 : M)_I}$ which is maximal with respect to $\frac{I}{(0 : M)_I} \cap \frac{K}{(0 : M)_I} = 0$. Then it is well known that $\frac{I}{(0 : M)_I} \cong \frac{I+K}{K} \triangleleft \frac{R}{K}$. Since $\frac{I}{(0 : M)_I} \in \mathcal{F}$ and \mathcal{F} is essentially closed, we have that $\frac{R}{K} \in \mathcal{F}$. Now $\frac{R}{K}$ is an R -module; thus we show that $H = \frac{R}{K}$ is the required R -module.

Clearly, $R\left(\frac{R}{K}\right) \neq 0$. We show that $\left(0 : \frac{R}{K}\right)_R = K$. So let $x \in K$. Then $x(r+K) = xr + K = K$ for all $r \in R$. Therefore $x \in \left(0 : \frac{R}{K}\right)_R$. Conversely, let $x \in \left(0 : \frac{R}{K}\right)_R$. Then $xR \subseteq K$. Since $\frac{R}{K} \in \mathcal{F}$, $\frac{R}{K}$ is a 2-prime near-ring and hence K is a 2-prime ideal of R . But $xR \subseteq K$ and K is 2-prime implies that $x \in K$. Hence we have that $\left(0 : \frac{R}{K}\right)_R = K$.

Now $\frac{R}{\left(0 : \frac{R}{K}\right)_R} = \frac{R}{K} \in \mathcal{F}$ and $R\left(\frac{R}{K}\right) \neq 0$. Hence $H = \frac{R}{K} \in \mathcal{M}_R$.

Finally, we show that $\left(0 : \frac{R}{K}\right)_I \subseteq (0 : M)_I$. Let $x \in \left(0 : \frac{R}{K}\right)_I$. Since $I \triangleleft R$, we have that $xR \subseteq I$. Furthermore, $x\left(\frac{R}{K}\right) = 0 \implies xR \subseteq K$. Hence $xR \subseteq I \cap K$, and from the definition of $\frac{K}{(0 : M)_I}$, we get $xR \subseteq I \cap K \subseteq (0 : M)_I$. Hence $xRM = 0$. Now $xIM \subseteq xRM = 0$ implies $xI \subseteq (0 : M)_I$. Since $(0 : M)_I$ is a 2-prime ideal of I (since $\frac{I}{(0 : M)_I} \in \mathcal{F}$), we get $x \in (0 : M)_I$. So $\left(0 : \frac{R}{K}\right)_I \subseteq (0 : M)_I$ and **(M5)** is satisfied.

(M6): Let $K \triangleleft I \triangleleft R \in \mathcal{T}$ and $M \in \mathcal{M}_{\frac{I}{K}}$ be a faithful $\frac{I}{K}$ -module. Since $M \in \mathcal{M}_{\frac{I}{K}}$ and M is faithful, we have that $\frac{I}{K} = \frac{I}{(0 : M)_{\frac{I}{K}}} \in \mathcal{F}$. So, from **(R3)**, it follows that $K \triangleleft R$. ■

Proposition 171 *Let \mathcal{M} be a \mathcal{T} -special class of near-ring modules and suppose $I \triangleleft R \in \mathcal{R}_0$. Let \mathcal{F} be the corresponding \mathcal{T} -special class of near-rings. Then $\frac{R}{I} \in \mathcal{F}$ if and only if $I = (0 : M)_R$ for some $M \in \mathcal{M}_R$.*

Proof. Suppose $I \triangleleft R \in \mathcal{R}_0$ and $\frac{R}{I} \in \mathcal{F}$. Then there exists $M \in \mathcal{M}_{\frac{R}{I}}$

such that $(0 : M)_{\frac{R}{I}} = 0$. So it follows from **(M2)** that $M \in \mathcal{M}_R$. Since $(0 : M)_{\frac{R}{I}} = \frac{(0 : M)_R}{I}$, it also follows that $\frac{(0 : M)_R}{I} = 0$. Hence $I = (0 : M)_R$ as required.

Conversely, suppose that $I = (0 : M)_R$ for some $M \in \mathcal{M}_R$. Then by **(M1)**, we have $M \in \mathcal{M}_{\frac{R}{I}}$. Furthermore, $(0 : M)_{\frac{R}{I}} = \frac{(0 : M)_R}{I} = \frac{I}{I} = 0$. Hence $\frac{R}{I} \in \mathcal{F}$. ■

Booth and Groenewald [5] have already shown that the class, $\mathcal{M}_e = \cup \mathcal{M}_R$ where $\mathcal{M}_R = \{M : M \text{ is an equiprime } R\text{-module}\}$, is a special class of near-ring modules if R belongs to the class of zerosymmetric near-rings. In the results that follow, we prove that similar constructions of classes with respect to 2-prime, 3-prime, c -prime, strongly prime and s -prime near-ring modules result in special classes of the respective near-ring modules. However, we restrict R to the class of \mathcal{A} -near-rings (Andrunakievich near-rings). In each case, we show that the six conditions of Definition 167 are satisfied. Although proofs of these conditions for the various special classes may seem to be repetitive, we only omit those parts which are exactly the same.

Proposition 172 *Let R be an \mathcal{A} -near-ring. Let $\mathcal{M}_R = \{M : M \text{ is a 2-prime } R\text{-module}\}$ and let $\mathcal{M}_2 = \cup \mathcal{M}_R$. Then \mathcal{M}_2 is an \mathcal{A} -special class of near-ring modules.*

Proof. **(M1)** : Let $M \in \mathcal{M}_R$ and $I \triangleleft R$ such that $IM = 0$. Then M is an $\frac{R}{I}$ -module with respect to $(r + I)m = rm$ for $r \in R$ and $m \in M$. Now let $A \leq \frac{R}{I}$ and $B \leq_R M$ such that $AB = 0$. Then $A = \frac{L}{I}$ for some left R -subgroup, L of R , and hence $\left(\frac{L}{I}\right)B = 0$. So for all $l \in L$, we have $lB = (l + I)B = 0$ implies that $LB = 0$. Since M is a 2-prime R -module, it follows that $LM = 0$ or $B = 0$. But, again by definition of the scalar operation in $\frac{R}{I}$, we have that $LM = (L + I)M = \left(\frac{L}{I}\right)M = AM$. So $AM = 0$ or $B = 0$ whence M is a 2-prime $\frac{R}{I}$ -module and therefore $M \in \mathcal{M}_{\frac{R}{I}}$.

(M2) : Let $I \triangleleft R$ and let $M \in \mathcal{M}_{\frac{R}{I}}$. Then M is an R -module with respect to $rm = (r+I)m$ for $r \in R$ and $m \in M$. Now let $A \leq R$ and $B \leq_R M$ such that $AB = 0$. Then $\frac{A}{I} \leq \frac{R}{I}$ and for all $a \in A$, we have: $(a+I)B = aB = 0$. Hence $\left(\frac{A}{I}\right)B = 0$ and since M is a 2-prime $\frac{R}{I}$ -module, it follows that $\left(\frac{A}{I}\right)M = 0$ or $B = 0$. But for all $a \in A$, $(a+I)M = aM$. Hence $AM = 0$ or $B = 0$ and therefore $M \in \mathcal{M}_R$.

(M3) : Let $M \in \mathcal{M}_R$ and $I \triangleleft R \in \mathcal{A}$ with $IM \neq 0$. Then, since R is an \mathcal{A} -near-ring and M is a 2-prime R -module, by Proposition 94, we have that M is a 2-prime I -module. Hence $M \in \mathcal{M}_I$.

(M4) : Let $M \in \mathcal{M}_R$, then by definition of a 2-prime R -module, we have that $RM \neq 0$. Now, since M is a 2-prime R -module, $(0 : M)_R$ is a 2-prime ideal of R and so $\frac{R}{(0 : M)_R}$ is a 2-prime near-ring.

(M5) : Let $I \triangleleft R \in \mathcal{A}$ and let $M \in \mathcal{M}_I$. Since M is a 2-prime I -module, by Corollary 80, $(0 : M)_I$ is a 2-prime ideal of I . So $(0 : M)_I \triangleleft I \triangleleft R$. Since R is an \mathcal{A} -near-ring and $\frac{I}{(0 : M)_I} \in \mathcal{M}_R$, from [3, Lemma 1] we have that $(0 : M)_I \triangleleft R$. Now choose $\frac{K}{(0 : M)_I} \triangleleft \frac{R}{(0 : M)_I}$ maximal with respect to $\frac{I}{(0 : M)_I} \cap \frac{K}{(0 : M)_I} = 0$. Then $\frac{I}{(0 : M)_I} \cong \frac{I+K}{K} \triangleleft \frac{R}{K}$. Since $\frac{I}{(0 : M)_I} \triangleleft \frac{R}{K}$ and $\frac{I}{(0 : M)_I}$ is a 2-prime near-ring, $\frac{R}{K}$ is also a 2-prime near-ring. By Corollary 91, $\frac{R}{K}$ is also a 2-prime R -module and so $\frac{R}{K} \in \mathcal{M}_R$.

Let $N = \frac{R}{K}$. To show that $(0 : N)_I \subseteq (0 : M)_I$, the proof follows as in Theorem 170.

(M6) : Let $K \triangleleft I \triangleleft R \in \mathcal{A}$ and suppose that there exists a faithful $\frac{I}{K}$ -module $M \in \mathcal{M}_{\frac{I}{K}}$. Since M is a faithful $\frac{I}{K}$ -module, $(0 : M)_{\frac{I}{K}} = 0$. But M is a 2-prime $\frac{I}{K}$ -module; therefore $0 = (0 : M)_{\frac{I}{K}}$ is a 2-prime ideal of $\frac{I}{K}$ which implies that $\frac{I}{K} \cong \frac{I}{(0 : M)_{\frac{I}{K}}}$ is a 2-prime near-ring. Hence K is a 2-prime ideal of I . Since I is an \mathcal{A} -ideal of R , it follows from [3, Lemma 1] that $K \triangleleft R$. ■

Corollary 173 *If \mathcal{M}_2 is an \mathcal{A} -special class of near-ring modules, then the \mathcal{A} -special radical induced by \mathcal{M}_2 on a near-ring R is given by:*

$$\begin{aligned}\mathcal{P}_2(R) &= \cap\{(0 : M)_R : M \text{ is a 2-prime } R\text{-module}\}. \\ &= \cap\{I \triangleleft R : I \text{ is a 2-prime ideal of } R\}.\end{aligned}$$

Proposition 174 *Let R be an \mathcal{A} -near-ring. Let $\mathcal{M}_R = \{M : M \text{ is a 3-prime } R\text{-module}\}$ and let $\mathcal{M}_3 = \cup\mathcal{M}_R$. Then \mathcal{M}_3 is an \mathcal{A} -special class of near-ring modules.*

Proof. (M1) : Let $M \in \mathcal{M}_R$ and let $I \triangleleft R$ such that $IM = 0$. Then M is an $\frac{R}{I}$ -module with respect to $(r + I)m = rm$ where $r \in R$ and $m \in M$. Now let $a + I \in \frac{R}{I}$ (where $a \in R$) and $m \in M$ such that $(a + I)\frac{R}{I}(m) = 0$. Then for all $r \in R$, we have that $arm = (ar + I)m = (a + I)(r + I)m = 0$. Hence $aRm = 0$. Since M is a 3-prime R -module, it follows that $aM = 0$ or $m = 0$ whereby $(a + I)M = 0$ or $m = 0$. Thus M is a 3-prime $\frac{R}{I}$ -module and so $M \in \mathcal{M}_{\frac{R}{I}}$.

(M2) : Let $I \triangleleft R$ such that $M \in \mathcal{M}_{\frac{R}{I}}$. Then M is an R -module with respect to $rm = (r + I)m$ where $r \in R$ and $m \in M$. Let $a \in R$ and $m \in M$ such that $aRm = 0$. Then for all $r \in R$, we have that:

$$(a + I)(r + I)m = (ar + I)m = arm = 0.$$

So $(a + I)\frac{R}{I}m = 0$ and since M is a 3-prime $\frac{R}{I}$ -module, it follows that $aM = (a + I)M = 0$ or $m = 0$. Therefore M is a 3-prime R -module and so $M \in \mathcal{M}_R$.

(M3) : Let $M \in \mathcal{M}_R$ and $I \triangleleft R \in \mathcal{A}$ such that $IM \neq 0$. By Proposition 95, M is a 3-prime I -module. Hence $M \in \mathcal{M}_I$.

(M4) : If $M \in \mathcal{M}_R$, then by definition of a 3-prime R -module, we have that $RM \neq 0$. Furthermore, by Corollary 80, $(0 : M)_R$ is a 3-prime ideal of R whence $\frac{R}{(0 : M)_R}$ is a 3-prime near-ring. But any 3-prime near-ring is also 2-prime. Hence $\frac{R}{(0 : M)_R}$ is a 2-prime near-ring.

(M5) : Let $I \triangleleft R \in \mathcal{A}$ and $M \in \mathcal{M}_I$. Since M is a 3-prime I -module,

$(0 : M)_I$ is a 3-prime ideal of I . The rest of the proof follows as in Proposition 172 by replacing 2-prime with 3-prime.

(M6) : Let $K \triangleleft I \triangleleft R \in \mathcal{A}$ and let $M \in \mathcal{M}_{\frac{I}{K}}$ be a faithful $\frac{I}{K}$ -module. Since M is a faithful $\frac{I}{K}$ -module, $(0 : M)_{\frac{I}{K}} = 0$. But M is a 3-prime $\frac{I}{K}$ -module. Hence $0 = (0 : M)_{\frac{I}{K}}$ is a 3-prime ideal of $\frac{I}{K} \implies \frac{I}{K}$ is a 3-prime near-ring $\implies K$ is a 3-prime ideal of I . Again by [3, Lemma 1], it follows that, since I is an \mathcal{A} -ideal of R , $K \triangleleft R$. ■

Corollary 175 *If \mathcal{M}_3 is an \mathcal{A} -special class of near-ring modules, then the \mathcal{A} -special radical induced by \mathcal{M}_3 on a near-ring R is given by:*

$$\mathcal{P}_3(R) = \cap \{(0 : M)_R : M \text{ is a 3-prime } R\text{-module}\}.$$

Proposition 176 *Let R be an \mathcal{A} -near-ring. Let $\mathcal{M}_R = \{M : M \text{ is a } c\text{-prime } R\text{-module}\}$ and let $\mathcal{M}_c = \cup \mathcal{M}_R$. Then \mathcal{M}_c is an \mathcal{A} -special class of near-ring modules.*

Proof. (M1) : Let $M \in \mathcal{M}_R$ and let $I \triangleleft R$ such that $IM = 0$. Then M is an $\frac{R}{I}$ -module with respect to $(r + I)m = rm$ where $r \in R$ and $m \in M$. Now let $a + I \in \frac{R}{I}$ (where $a \in R$) and $m \in M$ such that $0 = (a + I)m = am$. Since M is a c -prime R -module, it follows that $0 = aM = (a + I)M$ or $m = 0$. Thus M is a c -prime $\frac{R}{I}$ -module and so $M \in \mathcal{M}_{\frac{R}{I}}$.

(M2) : Let $I \triangleleft R$ such that $M \in \mathcal{M}_{\frac{R}{I}}$. Then M is an R -module with respect to $rm = (r + I)m$ where $r \in R$ and $m \in M$. Let $a \in R$ and $m \in M$ such that $0 = am = (a + I)m$. Since M is a c -prime $\frac{R}{I}$ -module, it follows that $0 = (a + I)M = aM$ or $m = 0$. Therefore M is a c -prime R -module and so $M \in \mathcal{M}_R$.

(M3) : Let $M \in \mathcal{M}_R$ and $I \triangleleft R \in \mathcal{A}$ such that $IM \neq 0$. By Proposition 97, M is a c -prime I -module. Hence $M \in \mathcal{M}_I$.

(M4) : If $M \in \mathcal{M}_R$, then by definition of a c -prime R -module, we have that $RM \neq 0$. Furthermore, by Corollary 80, $(0 : M)_R$ is a c -prime ideal of

R whence $\frac{R}{(0 : M)_R}$ is a c -prime near-ring. But any c -prime near-ring is also 2-prime. Hence $\frac{R}{(0 : M)_R}$ is a 2-prime near-ring.

(M5) : Let $I \triangleleft R \in \mathcal{A}$ and $M \in \mathcal{M}_I$. Since M is a c -prime I -module, $(0 : M)_I$ is a c -prime ideal of I . The rest of the proof follows as in Proposition 172 by replacing 2-prime with c -prime.

(M6) : Let $K \triangleleft I \triangleleft R \in \mathcal{A}$ and let $M \in \mathcal{M}_{\frac{I}{K}}$ be a faithful $\frac{I}{K}$ -module. Since M is a faithful $\frac{I}{K}$ -module, $(0 : M)_{\frac{I}{K}} = 0$. But M is a c -prime $\frac{I}{K}$ -module. Hence $0 = (0 : M)_{\frac{I}{K}}$ is a c -prime ideal of $\frac{I}{K} \implies \frac{I}{K}$ is a c -prime near-ring. So K is a c -prime ideal of I and hence, by [3, Lemma 1], it follows that, since I is an \mathcal{A} -ideal of R , $K \triangleleft R$. ■

Corollary 177 *If \mathcal{M}_c is an \mathcal{A} -special class of near-ring modules, then the \mathcal{A} -special radical induced by \mathcal{M}_c on a near-ring R is given by:*

$$\mathcal{P}_c(R) = \cap \{(0 : M)_R : M \text{ is a } c\text{-prime } R\text{-module}\}.$$

Proposition 178 *Let R be an \mathcal{A} -near-ring. Let $\mathcal{M}_R = \{M : M \text{ is a strongly prime } R\text{-module}\}$ and let $\mathcal{M}_{sp} = \cup \mathcal{M}_R$. Then \mathcal{M}_{sp} is an \mathcal{A} -special class of near-ring modules.*

Proof. (M1) : Let $M \in \mathcal{M}_R$ and $I \triangleleft R$ such that $IM = 0$. Then M is an $\frac{R}{I}$ -module with respect to $(r + I)m = rm$ where $r \in R$ and $m \in M$. Let $0 \neq m \in M$. Since M is a strongly prime R -module, there exists a finite set $F = \{r_1, r_2, \dots, r_n\} \subseteq R$ such that $x \in R$ and $xFm = 0$ implies that $xM = 0$. Let $F_1 = \{r_1 + I, r_2 + I, \dots, r_n + I\}$. Then F_1 is finite and $F_1 \subseteq \frac{R}{I}$. If $a + I \in \frac{R}{I}$ (where $a \in R$) such that $(a + I)F_1m = 0$, then we have:

$$aFm = (a + I)F_1m = 0$$

Since $aFm = 0$, we have $0 = aM = (a + I)M$. So M is a strongly prime $\frac{R}{I}$ -module. Hence $M \in \mathcal{M}_{\frac{R}{I}}$.

(M2) : Let $I \triangleleft R$ and let $M \in \mathcal{M}_{\frac{R}{I}}$. Then M is an R -module with respect to $rm = (r + I)m$ where $r \in R$ and $m \in M$. Now let $0 \neq m \in M$. Since M is a

strongly prime $\frac{R}{I}$ -module, there exists a finite set $F = \{r_1 + I, r_2 + I, \dots, r_n + I\}$ contained in $\frac{R}{I}$ such that $x + I \in \frac{R}{I}$ (where $x \in R$) and $(x + I)Fm = 0$ implies $(x + I)M = 0$. Let $F_1 = \{r_1, r_2, \dots, r_n\} \subseteq R$ and let $a \in R$ such that $aF_1m = 0$. Then we have:

$$(a + I)Fm = aF_1m = 0$$

Since $(a + I)Fm = 0$ and $a \in R$, it follows that $0 = (a + I)M = aM$. Hence M is strongly prime R -module implies $M \in \mathcal{M}_R$.

(M3) : Let $M \in \mathcal{M}_R$ and $I \triangleleft R \in \mathcal{A}$ such that $IM \neq 0$. Then, by Proposition 144, M is a strongly prime I -module. Hence $M \in \mathcal{M}_I$.

(M4) : If $M \in \mathcal{M}_R$, then, by the definition of a strongly prime R -module, $RM \neq 0$. Since M is a strongly prime R -module, by Corollary 140, $(0 : M)_R$ is a strongly prime, and hence a 2-prime, ideal of R . Therefore it follows that $\frac{R}{(0 : M)_R}$ is a 2-prime near-ring.

(M5) : Let $I \triangleleft R \in \mathcal{A}$ and $M \in \mathcal{M}_I$. Since M is a strongly prime I -module, $(0 : M)_I$ is a strongly prime ideal of I . The rest of the proof follows as in Proposition 172 by replacing 2-prime with strongly prime.

(M6) : Let $K \triangleleft I \triangleleft R \in \mathcal{A}$ and let $M \in \mathcal{M}_{\frac{I}{K}}$ be a faithful $\frac{I}{K}$ -module. Since M is a faithful $\frac{I}{K}$ -module, $(0 : M)_{\frac{I}{K}} = 0$. But M is a strongly prime $\frac{I}{K}$ -module. Hence $0 = (0 : M)_{\frac{I}{K}}$ is a strongly prime ideal of $\frac{I}{K} \implies \frac{I}{K}$ is a strongly prime near-ring $\implies K$ is a strongly prime ideal of I . So by [3, Lemma 1], it follows that, since I is an \mathcal{A} -ideal of R , $K \triangleleft R$. ■

Corollary 179 *If \mathcal{M}_{sp} is an \mathcal{A} -special class of near-ring modules, then the \mathcal{A} -special radical induced by \mathcal{M}_{sp} on a near-ring R is given by:*

$$\mathcal{P}_{sp}(R) = \cap \{(0 : M)_R : M \text{ is a strongly prime } R\text{-module}\}.$$

Recall that if M is an R -module with $RM \neq 0$, then M is said ν -s-prime (here we only consider $\nu = 2, 3$) if M satisfies:

(a) M is a ν -prime R -module.

(b) $\frac{R}{(0 : M)_R}$ contains no nonzero nil ideals.

Now let R be an \mathcal{A} -near-ring. For $\nu = 2, 3$, let $\mathcal{M}_R = \{M : M \text{ is a } \nu\text{-}s\text{-prime } R\text{-module}\}$ and let $\mathcal{M}_{\nu s} = \cup \mathcal{M}_R$.

We want to show that \mathcal{M}_{2s} and \mathcal{M}_{3s} are \mathcal{A} -special classes of near-ring modules. However, these two classes differ with respect to the condition that \mathcal{M}_{2s} contains 2-prime R -modules while \mathcal{M}_{3s} contains 3-prime R -modules but they share a common condition that, in both classes, $\frac{R}{(0 : M)_R}$ must have no nonzero nil ideals (that is, $\aleph\left(\frac{R}{(0 : M)_R}\right) = 0$). Furthermore, we have already shown in Proposition 172 and Proposition 174 that the class \mathcal{M}_2 (consisting of 2-prime R -modules) and the class \mathcal{M}_3 (consisting of 3-prime R -modules) are \mathcal{A} -special classes of near-ring modules.

In view of these observations, we consider the two classes simultaneously in the next proposition. Furthermore, note that to conclude that $\mathcal{M}_{\nu s}$ is an \mathcal{A} -special class of near-ring modules, we need only show that conditions (M1) to (M6) of Definition 167 are satisfied with respect to condition (b) above. Hence the proof of Proposition 181 should be read in conjunction with Propositions 172 and 174.

Lemma 180 *Let R be an \mathcal{A} -near-ring and let $I \triangleleft R$. Then:*

(a) $\aleph(I) \subseteq \aleph(R)$

(b) *If $I \triangleleft \cdot R$ such that $\aleph(I) = 0$, then $\aleph(R) = 0$.*

Proof. (a): From Lemma 117, $\aleph(I) \subseteq I \cap \aleph(R)$. Hence $\aleph(I) \subseteq \aleph(R)$.

(b): Suppose that A is a nonzero nil ideal of R . Since $I \triangleleft \cdot R$ we have that $A \cap I \neq 0$. Furthermore, $A \cap I \triangleleft I$. Since $\aleph(I) = 0$, $A \cap I$ cannot be a nil ideal of I which implies that there exists $x \in A \cap I \subset A$ such that $x^m \neq 0$ for all $m \in \mathbb{N}$. This contradicts the fact that A is a nil ideal of R . Hence $\aleph(R) = 0$. ■

Proposition 181 *Let R be an \mathcal{A} -near-ring. For $\nu = 2, 3$, let $\mathcal{M}_R = \{M : M \text{ is an } \nu\text{-}s\text{-prime } R\text{-module}\}$ and let $\mathcal{M}_{\nu s} = \cup \mathcal{M}_R$. Then $\mathcal{M}_{\nu s}$ is an \mathcal{A} -special class of near-ring modules.*

Proof. (M1) : Let $M \in \mathcal{M}_R$ and $I \triangleleft R$ with $IM = 0$. Then $M \in \mathcal{M}_R$ implies that $\frac{R}{(0 : M)_R}$ contains no nonzero nil ideals. But

$$\frac{\frac{R}{I}}{(0 : M)_{\frac{R}{I}}} = \frac{\frac{R}{I}}{\frac{(0 : M)_R}{I}} \cong \frac{R}{(0 : M)_R}.$$

Hence $\frac{R}{(0 : M)_{\frac{R}{I}}}$ contains no nonzero nil ideals and, thus, $M \in \mathcal{M}_{\frac{R}{I}}$.

(M2) : If $I \triangleleft R$ and $M \in \mathcal{M}_{\frac{R}{I}}$, $\frac{R}{(0 : M)_{\frac{R}{I}}}$ contains no nonzero nil ideals.

So $\frac{R}{(0 : M)_R} \cong \frac{R}{(0 : M)_{\frac{R}{I}}}$ has no nonzero nil ideals implies that $M \in \mathcal{M}_R$.

(M3) : Let $M \in \mathcal{M}_R$ and $I \triangleleft R \in \mathcal{A}$ with $IM \neq 0$. Then $\frac{R}{(0 : M)_R}$ contains no nonzero nil ideals. Now:

$$\frac{I}{(0 : M)_I} = \frac{I}{(0 : M)_R \cap I} \cong \frac{I + (0 : M)_R}{(0 : M)_R} \triangleleft \frac{R}{(0 : M)_R}.$$

Since R is an \mathcal{A} -near-ring, from Lemma 180(a), $\frac{I}{(0 : M)_I}$ will also not contain any nonzero nil ideals. Hence $M \in \mathcal{M}_I$.

(M4) : If $M \in \mathcal{M}_R$, then, by the definition of a ν - s -prime R -module, we have $RM \neq 0$ and M is a ν -prime R -module. In either case, whether $\nu = 2$ or 3, M is a 2-prime R -module. Hence $(0 : M)_R$ is a 2-prime ideal of R which implies that $\frac{R}{(0 : M)_R}$ is a 2-prime near-ring.

(M5) : Let $I \triangleleft R \in \mathcal{A}$ and $M \in \mathcal{M}_I$. Choose $N = \frac{R}{K}$ as per construction in Theorem 170 and, by using similar methods, it can be again proved that $\frac{I}{(0 : M)_I} \triangleleft \frac{R}{K}$. Since $M \in \mathcal{M}_I$, $\frac{I}{(0 : M)_I}$ contains no nonzero nil ideals. Hence, from Lemma 180(b), $\frac{R}{K}$ contains no nonzero nil ideals. But we already know that $K = \left(0 : \frac{R}{K}\right)_R$. Hence $\frac{R}{(0 : \frac{R}{K})_R}$ contains no nonzero nil ideals implying that $\frac{R}{K} \in \mathcal{M}_R$.

(M6) : Let $K \triangleleft I \triangleleft R \in \mathcal{A}$ and let $M \in \mathcal{M}_{\frac{I}{K}}$ be a faithful $\frac{I}{K}$ -module. Since M is a faithful $\frac{I}{K}$ -module, $(0 : M)_{\frac{I}{K}} = 0$. But M is a ν - s -prime $\frac{I}{K}$ -

module. Hence $0 = (0 : M)_{\frac{I}{K}}$ is a ν - s -prime ideal of $\frac{I}{K} \implies \frac{I}{K}$ is a ν - s -prime near-ring $\implies K$ is a ν - s -prime ideal of I . So by [3, Lemma 1], it follows that, since I is an \mathcal{A} -ideal of R , $K \triangleleft R$. ■

Corollary 182 *Let $\nu = 2, 3$. If $\mathcal{M}_{\nu s}$ is an \mathcal{A} -special class of near-ring modules, then the \mathcal{A} -special radical induced by $\mathcal{M}_{\nu s}$ on a near-ring R is given by:*

$$\mathcal{P}_{\nu s}(R) = \cap \{(0 : M)_R : M \text{ is a } \nu\text{-}s\text{-prime } R\text{-module}\}.$$

Chapter 4

LINKS TO THE GENERALISED GROUP NEAR-RING

INTRODUCTION

In 1989, Le Riche, Meldrum and Van der Walt [17] introduced the general notion of a group near-ring, $R[G]$. Recently, Groenewald and Lee [14] extended this idea to what they referred to as the generalised semigroup near-ring, denoted by $R[S : M]$. Here R is a zerosymmetric right near-ring with identity 1, S is a semigroup and M is any faithful left R -module. By considering M^S as the cartesian direct sum of $|S|$ copies of $(M, +)$, they defined the generalised semigroup near-ring as follows:

For $r \in R$, $s \in S$ and $\mu \in M^S$, let $[r, s]$ be the function defined by $([r, s]\mu)(h) = r\mu(hs)$ where $h \in S$. $R[S : M]$ was then defined as the subnear-ring of all mappings from M^S to M^S generated by the set $\{[r, s] : r \in R, s \in S\}$.

In this chapter, we define the generalised group near-ring, $R[G : M]$, following the definition provided by Groenewald and Lee but by choosing G to be a

group rather than a semigroup. Although, we provide some general results on $R[G : M]$ (which may be construed as a degeneralisation of some of the results obtained in [14]), our main intention here is to investigate the relationships between the prime ideals of $R[G : M]$ and that of the base near-ring R and/ or the underlying module M . To this end, we begin with an R -ideal P of M . We define the ideals P^* and P^+ of $R[G : M]$ and proceed to show that if P^* is a ν -prime ideal ($\nu = 0, 2, 3$), then P is a ν -prime R -ideal of M and consequently $\tilde{P} = (P : M)_R$ is a ν -prime ideal of R . However, it turns out that a prime condition in R and/ or M does not, in general, imply that prime condition in $R[G : M]$. We demonstrate this by providing various situations in which $R[G : M]$ fails to preserve the prime condition of R and/ or M .

However, we end this chapter on a positive note by showing that if R is a near-field and G is an ordered group, then $R[G : M]$ is 2-prime (where M is the specific R -module ${}_R R$) and consequently $R[G : M]$ is also ν -prime for $\nu = 0, 2, 3$.

Throughout this chapter, R will denote a zerosymmetric right near-ring with multiplicative identity 1, G will be a (multiplicatively written) group with neutral element e , M will be a faithful left R -module, and M^G will denote the cartesian direct sum of $|G|$ copies of M .

4.1 Some General Results

In this section, we provide some general results on the generalised group near-ring. The construction of some of these results is based on the results given in [17]. Therefore, we will omit some of the trivial proofs.

It is well known that the set of all mappings from the group M^G into itself is a right near-ring with addition defined pointwise and multiplication defined by composition of functions. Now, for $r \in R$, $g \in G$ and $\mu \in M^G$, let the function $[r, g] : M^G \rightarrow M^G$ be defined by $([r, g]\mu)(h) = r\mu(hg)$ where $h \in G$. (Note that, since M is an R -module, $r\mu(hg) \in M$).

Definition 183 *The generalised group near-ring, denoted by $R[G : M]$, is de-*

fined to be the subnear-ring of all mappings from M^G to M^G generated by the set $\{[r, g] : r \in R, g \in G\}$.

Remark 184 *We note that:*

- (a) *If we let M be the special case, $M = {}_R R$, then $R[G : M]$ is simply the group near-ring, $R[G]$, as defined by Le Riche, Meldrum and Van der Walt in [17].*
- (b) *If $A : M^G \longrightarrow M^G$, $B : M^G \longrightarrow M^G$ are elements of $R[G : M]$ and $\mu \in M^G$, then under the natural operations:*

$$(A + B)\mu = A\mu + B\mu$$

$$(AB)\mu = A(B\mu)$$

M^G is an $R[G : M]$ -module

Definition 185 *For $\mu \in M^G$, we define:*

- (a) *the support of μ by: $\text{supp}(\mu) = \{g \in G : \mu(g) \neq 0\}$.*
- (b) *$M^{(G)}$ as the subgroup of M^G consisting of elements with finite support.*

It is quite clear from the above definition that if G is a finite group, then $M^{(G)} \cong M^G$. In view of this and the following definition and lemma, which is a direct generalisation of [17, Lemma 2.6], $R[G : M]$ could just as well have been defined as a subnear-ring of all mappings from $M^{(G)}$ to $M^{(G)}$.

Definition 186 *Let $\mu \in M^G$, $X \subseteq G$. Then we define $\mu_X : M^G \longrightarrow M^G$ by $\mu_X(h) = \mu(h)$ if $h \in X$ and $\mu_X(h) = 0$ if $h \in G \setminus X$.*

Lemma 187 *Let $h \in G, \mu \in M^G$ and $A \in R[G : M]$. Then there exists a finite set X (independent of μ) such that for all $X' \supseteq X$, we have:*

$$(A\mu)(h) = (A\mu_{X'})(h)$$

Theorem 188 *$M^{(G)}$ is a faithful $R[G : M]$ -submodule of M^G .*

Proof. Let $A \in R[G : M]$ and $\mu \in M^{(G)}$ be arbitrary. We use induction on the complexity, $c(A)$ of A , to show that $A\mu \in M^{(G)}$.

If $c(A) = 1$, then $A = [r, g]$ for some $r \in R, g \in G$. So for all $h_i \in G$, we get:

$$(A\mu)(h_i) = ([r, g]\mu)(h_i) = r\mu(h_i g)$$

Since $\mu \in M^{(G)}$, $\mu(h_i g) \neq 0$ for finitely many i and hence, $r\mu(h_i g) \neq 0$ for finitely many i . So $A\mu \in M^{(G)}$.

Now suppose that $c(A) = n > 1$ and assume that for any $W \in R[G : M]$, where $c(W) < n$, it holds that $W\mu \in M^{(G)}$.

Since $c(A) > 1$, $A = B + C$ or $A = BC$ for some $B, C \in R[G : M]$ with $c(B), c(C) < n$.

If $A = B + C$, then:

$$A\mu = (B + C)\mu = B\mu + C\mu \in M^{(G)}, \text{ since } B\mu, C\mu \in M^{(G)}$$

If $A = BC$, then:

$$A\mu = (BC)\mu = B(C\mu) \in M^{(G)} \text{ since } B(\alpha) \in M^{(G)} \text{ and } \alpha = C\mu \in M^{(G)}$$

Hence $A\mu \in M^{(G)}$ in all cases. Since $A \in R[G : M]$ and $\mu \in M^{(G)}$ were arbitrary, it follows that $R[G : M]M^{(G)} \subseteq M^{(G)}$. So $M^{(G)}$ is an $R[G : M]$ -submodule of M^G .

To show that $M^{(G)}$ is faithful, let $0 \neq A \in R[G : M]$. Then there exists a $\mu \in M^G$ and an $h \in G$ such that $(A\mu)(h) \neq 0$. But by Lemma 187, there exists a finite set X such that $0 \neq (A\mu)(h) = (A\mu_X)(h)$. Since X is finite, we have that $\mu_X \in M^{(G)}$ such that $A\mu_X \neq 0$. Hence $M^{(G)}$ is a faithful $R[G : M]$ -submodule of M^G . ■

Lemma 189 *Let $r, r_1, r_2 \in R$ and $g, g_1, g_2 \in G$. Then the following apply in $R[G : M]$:*

- (a) *If G has identity element e then $R[G : M]$ has identity element $[1, e]$.*
- (b) $[r_1, g] + [r_2, g] = [r_1 + r_2, g]$.

(c) $[r_1, g_1] \cdot [r_2, g_2] = [r_1 r_2, g_1 g_2]$.

(d) $[1, g]$ is a unit in $R[G : M]$.

(e) the map $r \rightarrow [r, e]$ is an embedding of R into $R[G : M]$.

Proof. The proofs follow similar methods as in [17]. ■

Proposition 190 $R[G : M]$ is a zerosymmetric near-ring.

Proof. Let $\bar{0} : M^G \rightarrow M^G$ be the zero mapping in $R[G : M]$. We use induction on $c(A)$ where $A \in R[G : M]$ is arbitrary, to show that $A\bar{0} = \bar{0}$.

If $c(A) = 1$, then $A = [r, g]$ for some $r \in R, g \in G$. Let $\mu \in M^G$. Then for all $h \in G$, we get:

$$(A\bar{0}\mu)(h) = ([r, g]\bar{0}\mu)(h) = r\bar{0}\mu(hg) = r0.$$

Since R is zerosymmetric, $r0 = 0$. Hence $A\bar{0} = \bar{0}$.

If $c(A) > 1$, then $A = B + C$ or $A = BC$ where $c(B) < c(A)$, $c(C) < c(A)$.

We assume that $B\bar{0} = \bar{0}$ and $C\bar{0} = \bar{0}$.

If $A = B + C$, then:

$$A\bar{0} = (B + C)\bar{0} = B\bar{0} + C\bar{0} = \bar{0} + \bar{0} = \bar{0}.$$

If $A = BC$, then:

$$A\bar{0} = (BC)\bar{0} = B(C\bar{0}) = B\bar{0} = \bar{0}.$$

Hence $R[G : M]$ is a zerosymmetric near-ring. ■

Proposition 191 Let P be an R -ideal of the R -module M . Let $\mu \in P^G$ and $\alpha \in M^G$. Then for any $A \in R[G : M]$, there exists a $\beta \in P^G$ such that $A(\alpha + \mu) = A\alpha + \beta$.

Proof. Again we use induction on the $c(A)$.

If $c(A) = 1$, then $A = [r, g]$ for some $r \in R$ and $g \in G$. So for $h \in G$, we get:

$$\begin{aligned} & [A(\alpha + \mu)](h) \\ &= ([r, g](\alpha + \mu))(h) \end{aligned}$$

$$\begin{aligned}
&= r(\alpha + \mu)(hg) \\
&= r(\alpha(hg) + \mu(hg)) \\
&= [r(\alpha(hg) + \mu(hg)) - r\alpha(hg)] + r\alpha(hg) \\
&= p + r\alpha(hg) \text{ for some } p \in P. \\
&= ([r, g]\alpha)(h) + p' \text{ for some } p' \in P.
\end{aligned}$$

Now define $\beta \in P^G$ by $\beta(h) = p'$ for $h \in G$ and the result follows.

Now suppose that $c(A) > 1$, and suppose that the result holds for any $D \in R[G : M]$ with $c(D) < c(A)$. Since $c(A) > 1$, we have $A = B + C$ or $A = BC$ with $c(C) < c(A)$, $c(B) < c(A)$ and $B, C \in R[G : M]$.

If $A = B + C$, then:

$$\begin{aligned}
&A(\alpha + \mu) \\
&= [B + C](\alpha + \mu) \\
&= B(\alpha + \mu) + C(\alpha + \mu) \\
&= B\alpha + \beta_1 + C\alpha + \beta_2 \text{ for some } \beta_1, \beta_2 \in P^G \\
&= B\alpha + C\alpha + \beta_3 \text{ for some } \beta_3 \in P^G \\
&= (B + C)\alpha + \beta_3 \\
&= A\alpha + \beta_3.
\end{aligned}$$

If $A = BC$, then:

$$\begin{aligned}
&A(\alpha + \mu) \\
&= [BC](\alpha + \mu) \\
&= B[C(\alpha + \mu)] \\
&= B[C\alpha + \beta_1] \text{ for some } \beta_1 \in P^G \\
&= B(C\alpha) + \beta_2 \text{ for some } \beta_2 \in P^G \\
&= (BC)\alpha + \beta_2 \\
&= A\alpha + \beta_2.
\end{aligned}$$

This completes the proof by induction. ■

Proposition 192 *Let P be an R -ideal of the R -module, M . Then P^G is an $R[G : M]$ -ideal of the $R[G : M]$ -module, M^G .*

Proof. Clearly, P^G is a subgroup of M^G . Furthermore, if $\mu \in P^G$ and $\nu \in M^G$, then:

$$(\nu + \mu - \nu)(g) = \nu(g) + \mu(g) - \nu(g) \in P$$

for all $g \in G$ since $\nu(g) \in M$, $\mu(g) \in P$ and P is a normal subgroup of M .

Hence P^G is a normal subgroup of M^G .

We need to show that if $A \in R[G : M]$, $\mu \in M^G$ and $\beta \in P^G$, then $A(\mu + \beta) - A\mu \in P^G$. But, from Proposition 191, there exists $\theta \in P^G$ with the property that $A(\mu + \beta) = A\mu + \theta$. Hence $A(\mu + \beta) - A\mu = \theta \in P^G$, thus proving that P^G is an $R[G : M]$ -ideal of M^G . ■

Definition 193 Let P be a subset of the R -module M . Then we define the subset, P^* of $R[G : M]$ by:

$$P^* = \{A \in R[G : M] : A\mu \in P^G \text{ for all } \mu \in M^G\}.$$

Proposition 194 If P is an R -ideal of M , then P^* is an ideal of $R[G : M]$.

Proof. Since P is an R -ideal of M , by Proposition 192, we know that P^G is an $R[G : M]$ -ideal of M^G . Hence $(P^G : M^G)_{R[G : M]}$ is an ideal of the near-ring $R[G : M]$. But by definition, $P^* = (P^G : M^G)_{R[G : M]}$. So the result follows. ■

Another way of constructing an ideal in $R[G : M]$ lies in the following definition:

Definition 195 Let P be an R -ideal of the R -module M . Then we define P^+ to be the ideal in $R[G : M]$ generated by the set: $\{[a, e] : a \in (P : M)_R\}$.

Immediately, we have the following:

Lemma 196 If P is an R -ideal of M , then $P^+ \subseteq P^*$.

Proof. Let $\mu \in M^G$ and $g \in G$. Then for every $[a, e] \in P^+$, we have $([a, e]\mu)(g) = a\mu(g) \in P$ since $a \in (P : M)_R$ and $\mu(g) \in M$. Hence $[a, e] \in P^*$. Since P^* is an ideal of $R[G : M]$, all elements generated by the set

$$\{[a, e] : a \in (P : M)_R\}$$

must also belong to P^* . Therefore $P^+ \subseteq P^*$. ■

Lemma 197 *If A, B and C are R -ideals of the R -module M with the property that $(A : M)_R B \subseteq C$, then $A^+ B^* \subseteq C^*$.*

Proof. Let $U \in A^+, V \in B^*$ and $\rho \in M^G$. Then we need to show that $(UV)(\rho) \in C^G$. But we know that $V(\rho) \in B^G$. Therefore, all we need to show is that $U(\beta) \in C^G$ for all $\beta \in B^G$. We use induction on the complexity of U .

If $c(U) = 1$, then $U = [a, g]$ for some $a \in (A : M)_R$ and $g \in G$. So for all $h \in G$ and $\beta \in B^G$, we get:

$$(U(\beta))(h) = ([a, g]\beta)(h) = a\beta(hg) \in (A : M)_R B \subseteq C.$$

Hence $U(\beta) \in C^G$.

Now suppose that $c(U) = m$, where $m \in \mathbb{N}$, $m \geq 2$ and suppose that $W(\beta) \in C^G$ for all $\beta \in B^G$ if $W \in A^+$ and $c(W) < m$. Since $U \in A^+$ (which is a generated ideal), we have four possibilities:

(i) : $U = U_1 + U_2$ where $U_1, U_2 \in A^+$ and $c(U_1), c(U_2) < m$. Then for all $\beta \in B^G$, we get: $U(\beta) = (U_1 + U_2)(\beta) = U_1(\beta) + U_2(\beta) \in C^G$, by induction.

(ii) : $U = U_1 X$ where $U_1 \in A^+$ and $X \in R[G : M]$ with $c(X) < m$. Then $U(\beta) = (U_1 X)(\beta)$. Since, by Proposition 192, B^G is an $R[G : M]$ -ideal of M^G , for all $\beta \in B^G$ we have $X(\beta) \in B^G$. Hence, by induction, it follows that $U(\beta) = (U_1 X)(\beta) \in C^G$.

(iii) : $U = X(Y + U_1) - XY$ where $X, Y \in R[G : M]$ and $U_1 \in A^+$ with $c(U_1) < m$. Then for all $\beta \in B^G$, we have:

$$U(\beta) = [X(Y + U_1) - XY](\beta) = X(Y(\beta) + U_1(\beta)) - X(Y(\beta)).$$

Since $U_1(\beta) \in C^G$, $Y(\beta) \in M^G$ and C^G is an $R[G : M]$ -ideal of M^G , it follows that $U(\beta) \in C^G$.

(iv) : $U = X + U_1 - X$ where $U_1 \in A^+$ with $c(U_1) < m$ and $X \in R[G : M]$.

Then: $U(\beta) = (X + U_1 - X)(\beta) = X(\beta) + U_1(\beta) - X(\beta)$.

Since $U_1(\beta) \in C^G$ and $(C^G, +)$ is a normal subgroup of $(M^G, +)$, we must have that $U(\beta) \in C^G$. ■

At this point we would like to point out that although the definitions of A^+ and A^* in $R[G : M]$ are equivalent to the definitions of A^+ and A^* in $R[G]$, that is when we take $M = {}_R R$, we need to be weary of how we apply these definitions.

Corollary 198 *If A is an ideal of the near-ring R , and B and C are R -ideals of the R -module M such that $AB \subseteq C$, then $A^+B^* \subseteq C^*$.*

Proof. Since R has identity, it follows from Lemma 153 that $(A : R)_R = A$. So $AB \subseteq C$ implies that $(A : R)_R B \subseteq C$, and hence the rest of the proof follows in exactly the same way as the previous proof. ■

The ideals, A^* and A^+ of $R[G : M]$, both have important applications when we investigate the prime relations between M (or R) and $R[G : M]$ in the next section. Of importance, also, will be the construction of (left) $R[G : M]$ -subgroups of $R[G : M]$. We begin with the following simple lemma:

Lemma 199 *If $A \in R[G : M]$, then $R[G : M]A$ is a left $R[G : M]$ -subgroup of $R[G : M]$.*

Proof. Let $BA, CA \in R[G : M]A$ where $B, C \in R[G : M]$. Then since $B - C \in R[G : M]$, we have:

$$BA - CA = (B - C)A \in R[G : M]A$$

Furthermore, $R[G : M](R[G : M]A) = (R[G : M]R[G : M])A \subseteq R[G : M]A$. Hence the proof is complete. ■

We know that if M is an R -module, then for any $m \in M$, Rm is an R -submodule of M . In the following lemma, we show that $(Rm)^*$ is a left $R[G : M]$ -subgroup of $R[G : M]$:

Lemma 200 *Let M be an R -module and let $m \in M$. Then $(Rm)^*$ is a left $R[G : M]$ -subgroup of $R[G : M]$.*

Proof. Let $A, B \in (Rm)^*$, and let $\mu \in M^G$, $h \in G$. Then:

$$((A - B)\mu)(h) = (A\mu)(h) - (B\mu)(h).$$

Since $(A\mu)(h) \in Rm$ and $(B\mu)(h) \in Rm$, and since Rm is an R -submodule of M , we have that $(A\mu)(h) - (B\mu)(h) \in Rm$. Hence $A - B \in (Rm)^*$.

Now let $A \in R[G : M]$. We, once again, show by induction on the complexity of A that $A(Rm)^* \subseteq (Rm)^*$.

If $c(A) = 1$, then $A = [r, g]$ for some $r \in R$ and $g \in G$. Then, for all $\mu \in M^G$, $h \in G$ and $B \in (Rm)^*$, we have:

$$((AB)\mu)(h) = (([r, g]B)\mu)(h) = r((B\mu)(hg)) \in rRm \subseteq Rm.$$

So $AB \in (Rm)^*$. Since $B \in (Rm)^*$ was arbitrary, $A(Rm)^* \subseteq (Rm)^*$.

Now let $c(A) > 1$. Then $A = D + E$ or $A = DE$ where $D, E \in R[G : M]$, $c(D) < c(A)$, $c(E) < c(A)$, $D(Rm)^* \subseteq (Rm)^*$ and $E(Rm)^* \subseteq (Rm)^*$.

If $A = D + E$, then:

$$A(Rm)^* = (D+E)(Rm)^* = D(Rm)^* + E(Rm)^* \subseteq (Rm)^* + (Rm)^* \subseteq (Rm)^*.$$

If $A = DE$, then:

$$A(Rm)^* = (DE)(Rm)^* = D((E(Rm)^*)) \subseteq D(Rm)^* \subseteq (Rm)^*.$$

Since $A \in R[G : M]$ was arbitrary, we have that $R[G : M](Rm)^* \subseteq (Rm)^*$, thus proving that $(Rm)^*$ is a left $R[G : M]$ -subgroup of $R[G : M]$. ■

We now introduce another $R[G : M]$ -submodule of the $R[G : M]$ -module M^G which we use in the next section, and which could have wide applications in group near-ring theory.

Definition 201 Let M be an R -module of any near-ring R and let G be any group. Then we define the diagonal of M^G by:

$$d(M^G) = \{\mu \in M^G : \mu(g_1) = \mu(g_2) \text{ for all } g_1, g_2 \in G\}$$

Lemma 202 $d(M^G)$ is a nonzero, proper left $R[G : M]$ -submodule of M^G .

Proof. It is clear from the definition that $d(M^G)$ is a nonzero, proper subset of M^G . Now let $g_1, g_2 \in G$.

(i) : If $\mu, \nu \in d(M^G)$, then:

$$(\mu - \nu)(g_1) = \mu(g_1) - \nu(g_1) = \mu(g_2) - \nu(g_2) = (\mu - \nu)(g_2).$$

So $\mu - \nu \in d(M^G)$ and hence $d(M^G)$ is a subgroup of M^G .

(ii) : We need to show that $R[G : M](d(M^G)) \subseteq d(M^G)$. Let $A \in R[G : M]$ and $\mu \in d(M^G)$. We use induction on $c(A)$ to show that $A\mu \in d(M^G)$.

If $c(A) = 1$, then $A = [r, g]$ for some $r \in R$ and $g \in G$. Then:

$$(A\mu)(g_1) = ([r, g]\mu)(g_1) = r\mu(g_1g) - \mu(g_1) = r\mu(g_2g) - \mu(g_2) = ([r, g]\mu)(g_2) = (A\mu)(g_2)$$

Let $c(A) > 1$. Then $A = B + C$ or $A = BC$ where $c(B), c(C) < c(A)$ with $(B\mu)(g_1) = (B\mu)(g_2)$ and $(C\mu)(g_1) = (C\mu)(g_2)$.

If $A = B + C$, then:

$$\begin{aligned} (A\mu)(g_1) &= ((B + C)\mu)(g_1) \\ &= (B\mu)(g_1) + (C\mu)(g_1) \\ &= (B\mu)(g_2) + (C\mu)(g_2) \\ &= ((B + C)\mu)(g_2) \\ &= (A\mu)(g_2). \end{aligned}$$

If $A = BC$, then:

$$\begin{aligned} (A\mu)(g_1) &= ((BC)\mu)(g_1) \\ &= B((C\mu)(g_1)) \\ &= B((C\mu)(g_2)) \\ &= ((BC)\mu)(g_2) \\ &= (A\mu)(g_2). \end{aligned}$$

Therefore in all three cases, we have that $A\mu \in d(M^G)$ and consequently it follows that $d(M^G)$ is an $R[G : M]$ -submodule of M^G . ■

Corollary 203 *For any near-ring R and any group G , $d(R^G)$ is a nonzero, proper left $R[G]$ -subgroup of R^G .*

4.2 Prime Relations between $R[G : M]$, M and R

If $P \triangleleft R$, it has already been proved [6, Theorem 4.9] that whenever P^* is a 0-prime ideal of the group near-ring $R[G]$, P is a 0-prime ideal of the base near-ring, R . However, the reverse implication is still an open problem. Now let P be an R -ideal of the R -module M . In this section, we show that if $\nu = 0, 2, 3$ and P^* is a ν -prime ideal of $R[G : M]$, then P is a ν -prime R -ideal of M . This, then, leads to a link between P^* and a ν -prime ideal of R .

Furthermore, we provide situations under which $R[G : M]$ is not ν -prime ($\nu = 2, 3, c$), independent of the prime condition imposed on M and/ or R . Nevertheless, we end this section with some faith in the fact that if R is a near-field and G is an ordered group, then $R[G]$ (the specific case of $R[G : M]$ where $M = {}_R R$) is 2-prime. This result was taken from [13], a paper by Groenewald, Meyer and the current author.

We begin this section with the following lemma:

Lemma 204 *Let P be an R -ideal of a 2-multiplication module M such that P is not 3-prime. Then there exists an $m \in M$ such that:*

(a) $(Rm)^+ \not\subseteq P^*$

(b) $(Rm)^+ \not\subseteq P^+$

Proof. (a) : Since P is not 3-prime, there exists $a \in R$ and $m \in M$ such that $aRm \subseteq P$ but $a \notin (P : M)_R = \tilde{P}$ and $m \notin P$. Consider $[a, e] \in R[G : M]$. Then:

(i) If $[a, e] \in P^*$, then $([a, e]\mu)(g) \in P$ for all $\mu \in M^G$ and $g \in G$. So $a\mu(g) \in P$ for all $\mu \in M^G$ and $g \in G \implies aM \subseteq P$ which contradicts that $a \notin (P : M)_R$. Hence $[a, e] \notin P^*$.

(ii) If $Rm \subseteq P$, then since $1 \in R$ it follows that $1.m = m \in P$ which contradicts that $m \notin P$. So $Rm \not\subseteq P$.

(iii) If $\widetilde{Rm} \subseteq \widetilde{P}$, then $\widetilde{Rm}M \subseteq \widetilde{P}M$. Since M is a 2-multiplication module, we have that $Rm = \widetilde{Rm}M \subseteq \widetilde{P}M = P$ which contradicts that $Rm \not\subseteq P$. So $\widetilde{Rm} \not\subseteq \widetilde{P}$.

Now suppose that $(Rm)^+ \subseteq P^*$. Then for all $[b, e] \in (Rm)^+$, $[b, e] \in P^*$. This means that for all $b \in \widetilde{Rm}$, we have that for all $\mu \in M^G$ and $g \in G$:

$$([b, e]\mu)(g) = b\mu(g) \in P$$

which implies that $bM \subseteq P$, that is $b \in \widetilde{P}$. So $\widetilde{Rm} \subseteq \widetilde{P}$ which contradicts (iii) above. Hence $(Rm)^+ \not\subseteq P^*$.

(b): Follows as a direct consequence of (a) since $P^+ \subseteq P^*$. ■

Proposition 205 *If P is an R -ideal of a 2-multiplication module M such that P^* is a 3-prime ideal of $R[G : M]$, then P is a 3-prime R -ideal of M .*

Proof. If P is not 3-prime, then there exists an $a \in R$ and $m \in M$ such that $aRm \subseteq P$ but $aM \not\subseteq P$ and $m \notin P$. Hence, by Lemma 204, $(Rm)^+ \not\subseteq P^*$.

Now let $W \in (Rm)^+ \setminus P^*$. We prove by induction on the complexity of $A \in R[G : M]$ that $(AW\mu)(h) \in Rm$ for all $\mu \in M^G$, $h \in G$.

If $c(A) = 1$, then $A = [t, g]$ for some $t \in R$ and $g \in G$. So:

$$(AW\mu)(h) = ([t, g]W\mu)(h) = t((W\mu)(hg)).$$

Since $W \in (Rm)^+ \subseteq (Rm)^*$, we have $t(W\mu)(hg) \in tRm \subseteq Rm$ and consequently $(AW\mu)(h) \in Rm$.

Now let $c(A) > 1$. Then $A = B + D$ or $A = BD$ where $c(B), c(D) < c(A)$, $(BW\mu)(h) \in Rm$ and $(DW\mu)(h) \in Rm$.

If $A = B + D$, then:

$$(AW) = (B + D)W = (BW) + (DW) \in (Rm)^* + (Rm)^* = (Rm)^*.$$

So $((AW)\mu)(h) \in Rm$.

If $A = BD$, then:

$(AW) = (BD)W = B(DW) \in B(Rm)^* \subseteq (Rm)^*$ since $(Rm)^*$ is a left $R[G : M]$ -subgroup of $R[G : M]$.

So, once again, we have $((AW)\mu)(h) \in Rm$.

Since A was arbitrary, it follows that $(R[G : M]W\mu)(h) \subseteq Rm$. Furthermore, $[a, e](R[G : M]W\mu)(h) = a(R[G : M]W\mu)(h) \subseteq aRm \subseteq P$.

Hence $[a, e]R[G : M]W \subseteq P^*$. Since P^* is a 3-prime ideal of $R[G : M]$ and $W \notin P^*$, it follows that $[a, e] \in P^*$.

So for all $\mu \in M^G$ and $g \in G$, $([a, e]\mu)(g) = a\mu(g) \in P$ which implies that $aM \subseteq P$. Therefore we have a contradiction, and consequently it follows that P is 3-prime. ■

Corollary 206 *If P is an R -ideal of a 2-multiplication module M such that P^* is a 3-prime ideal of $R[G : M]$, then \tilde{P} is a 3-prime ideal of the near-ring R .*

Proof. Follows from the fact that if P is a 3-prime R -ideal of M , then \tilde{P} is a 3-prime ideal of R . ■

It is well known that if R is a near-ring with multiplicative identity 1, then R is 2-prime if and only if R is 3-prime. From Proposition 75, it follows that the same is also true, in general, for any R -module M . Now $R[G : M]$ is a near-ring constructed from the near-ring R and $R[G : M]$ has identity $[1, e]$. Hence, we have the following result:

Proposition 207 *Let \mathbb{P} be an ideal of $R[G : M]$. Then \mathbb{P} is 2-prime if and only if \mathbb{P} is 3-prime.*

Proof. Suppose that \mathbb{P} is 3-prime. Let \mathbb{A} and \mathbb{B} be left $R[G : M]$ -subgroups of $R[G : M]$ such that $\mathbb{A}\mathbb{B} \subseteq \mathbb{P}$. Then, if $A \in \mathbb{A}$ and $B \in \mathbb{B}$, we have:

$$AR[G : M]B \subseteq \mathbb{A}R[G : M]\mathbb{B} \subseteq \mathbb{A}\mathbb{B} \subseteq \mathbb{P}$$

Since \mathbb{P} is 3-prime, it follows that $A \in \mathbb{P}$ or $B \in \mathbb{P}$. But A and B were arbitrary elements of \mathbb{A} and \mathbb{B} respectively. Hence $\mathbb{A} \subseteq \mathbb{P}$ or $\mathbb{B} \subseteq \mathbb{P}$ which implies that \mathbb{P} is 2-prime.

Conversely, suppose that \mathbb{P} is 2-prime and let $A, B \in R[G : M]$ such that $AR[G : M]B \subseteq \mathbb{P}$. Then:

$$(R[G : M]A)(R[G : M]B) = R[G : M](AR[G : M]B) \subseteq R[G : M]\mathbb{P} \subseteq \mathbb{P}.$$

By Lemma 199, $R[G : M]A$ and $R[G : M]B$ are left $R[G : M]$ -subgroups of $R[G : M]$. Hence, since \mathbb{P} is 2-prime, it follows that $R[G : M]A \subseteq \mathbb{P}$ or $R[G : M]B \subseteq \mathbb{P}$.

In particular, since $R[G : M]$ has identity $[1, e]$, we get:

$$[1, e]A \in \mathbb{P} \text{ or } [1, e]B \in \mathbb{P} \implies A \in \mathbb{P} \text{ or } B \in \mathbb{P}.$$

Therefore \mathbb{P} is 3-prime. ■

As immediate consequences of Proposition 205 and Proposition 207, we have the following two results:

Corollary 208 *If P is an R -ideal of a 2-multiplication module M such that P^* is a 2-prime ideal of $R[G : M]$, then P is a 2-prime R -ideal of M .*

Corollary 209 *If P is an R -ideal of a 2-multiplication module M such that P^* is a 2-prime ideal of $R[G : M]$, then \tilde{P} is a 2-prime ideal of the near-ring R .*

Proposition 210 *Let P be an R -ideal of M and assume that P^* is a 0-prime ideal of $R[G; M]$. Then P is a 0-prime R -ideal of M .*

Proof. Let $A \triangleleft R$ and $B \triangleleft_R M$ such that $AB \subseteq P$. Then, by Corollary 198, $A^+B^* \subseteq P^*$. Since P^* is a 0-prime ideal of $R[G : M]$ and A^+ and B^* are ideals of $R[G : M]$, we have that $A^+ \subseteq P^*$ or $B^* \subseteq P^*$.

If $A^+ \subseteq P^*$, then $[a, e] \in P^*$ for all $a \in A$. So for all $\mu \in M^G$, $h \in G$ and $a \in A$, $a\mu(h) = ([a, e]\mu)(h) \in P$ which implies that $AM \subseteq P$.

If $B^* \subseteq P^*$, then $B^+ \subseteq P^*$. Hence we also have $B \subseteq P$.

This proves that P is a 0-prime R -ideal. ■

Corollary 211 *Let P be an R -ideal of a monogenic (or tame) R -module M and assume that P^* is a 0-prime ideal of $R[G; M]$. Then \tilde{P} is a 0-prime ideal of R .*

Proof. If M is monogenic (or tame) and P is a 0-prime R -ideal of M , then we know that \tilde{P} is a 0-prime ideal of R . Therefore the result follows from the previous proposition. ■

Thus far we have shown that if P is an R -ideal of M (a 2-multiplication module in some cases) such that P^* is a ν -prime ideal in $R[G : M]$, then P is a ν -prime R -ideal of M , where $\nu = 0, 2, 3$. In the next part of this section, we investigate the reverse situation of some of these implications by considering the primeness of $R[G : M]$ under certain conditions imposed on R and/ or M .

Theorem 212 *Let R be any near-ring, M a nonzero R -module and G be a cyclic group of order n . Then $R[G : M]$ is not completely prime.*

Proof. Let $G = \{e = g^0, g^1, g^2, \dots, g^{n-1}\}$ and consider the following two elements of $R[G : M]$:

$$U = [1, e] + [1, g] + \dots + [1, g^{n-1}], \text{ and}$$

$$V = [1, e] + [-1, g], \text{ where } g \neq e.$$

Then, if we define $\mu \in M^G$ by:

$$\mu(e) = m \neq 0, \text{ and}$$

$$\mu(g^i) = 0 \text{ for all } i = 1, 2, \dots, n-1,$$

we have that:

$$(U\mu)(e) = (([1, e] + [1, g] + \dots + [1, g^{n-1}])\mu)(e) = \mu(e) + \mu(g) + \dots + \mu(g^{n-1}) = m$$

$$(V\mu)(e) = (([1, e] + [-1, g])\mu)(e) = \mu(e) - \mu(g) = m$$

Since $m \neq 0$, it follows that both U and V are nonzero.

However, we show that $UV = 0$. Let $\mu \in M^G$ be arbitrary. Then:

$$(V\mu)(g^i) = (([1, e] + [-1, g])\mu)(g^i) = \mu(g^i) - \mu(g^{i+1}).$$

Hence:

$$(UV\mu)(g^i)$$

$$= (([1, e] + [1, g] + \dots + [1, g^{n-1}])V\mu)(g^i)$$

$$= (V\mu)(g^i) + (V\mu)(g^{i+1}) + \dots + (V\mu)(g^{i+n-1})$$

$$\begin{aligned}
&= \mu(g^i) - \mu(g^{i+1}) + \mu(g^{i+1}) - \mu(g^{i+2}) + \dots + \mu(g^{i+n-1}) - \mu(g^{i+n}) \\
&= \mu(g^i) - \mu(g^i), \text{ since } g^{i+n} = g^i \\
&= 0.
\end{aligned}$$

So $UV = 0$, but neither U nor V are 0, thus proving that $R[G : M]$ is not completely prime. ■

Theorem 213 *Let M be an R -module with $|M| \geq 2$. Suppose that R contains a nonzero element c with the property that $cm_1 + cm_2 = cm_2 + cm_1$ for all $m_1, m_2 \in M$. If G is finite with $|G| \geq 2$, then $R[G : M]$ is not 3-prime.*

Proof. Let $G = \{e = g_1, g_2, \dots, g_n\}$ where $n \geq 2$ and consider the elements:

$$\begin{aligned}
U &= [c, e] + [c, g_2] + \dots + [c, g_n], \text{ and} \\
V &= [1, e] + [-1, g_2].
\end{aligned}$$

Then as in the proof of Theorem 212, we can show that both $U \neq 0$ and $V \neq 0$.

Now let $\mu \in M^G$ be arbitrary. Then for $1 \leq k \leq n$, we have:

$$\begin{aligned}
&(U\mu)(g_k) \\
&= (([c, e] + [c, g_2] + \dots + [c, g_n])\mu)(g_k) \\
&= c\mu(g_k g_1) + c\mu(g_k g_2) + \dots c\mu(g_k g_n), \text{ where } e = g_1.
\end{aligned}$$

Since $\mu(g_k g_i) \in M$ for all $i = 1, 2, \dots, n$, it follows that:

$$c\mu(g_k g_1) + c\mu(g_k g_2) = c\mu(g_k g_2) + c\mu(g_k g_1) \text{ for all } g_1, g_2 \in G.$$

Hence, by repeated re-arrangement of the $c\mu(g_k g_i)$, we conclude that for all $i, k = 1, 2, \dots, n$:

$$(U\mu)(g_k) = (U\mu)(g_i).$$

By the definition of $d(M^G)$, it follows that $U\mu \in d(M^G)$. Since $d(M^G)$ is an $R[G : M]$ -submodule of M^G , we have that $R[G : M]U\mu \subseteq d(M^G)$. So, for all $\nu \in d(M^G)$ and for all $k = 1, 2, \dots, n$, we get:

$$\begin{aligned}
&(V\nu)(g_k) \\
&= (([1, e] + [-1, g_2])\nu)(g_k)
\end{aligned}$$

$$\begin{aligned}
&= \nu(g_k) - \nu(g_k g_2) \\
&= 0.
\end{aligned}$$

Therefore $V(R[G : M])U = 0$, but $U \neq 0$ and $V \neq 0$ which implies that $R[G : M]$ is not 3-prime. ■

Corollary 214 *If M is an abelian R -module and G is a finite group, then $R[G : M]$ is not 3-prime.*

Corollary 215 *If M is an abelian R -module and G is a finite group, then $R[G : M]$ is not completely prime.*

Proof. If $R[G : M]$ is completely prime, then we know that $R[G : M]$ is 3-prime. Hence the result follows from the contrapositive statement. ■

Corollary 216 *If M is an abelian R -module and G is a finite group, then $R[G : M]$ is not 2-prime.*

Proof. The result follows from the fact that $R[G : M]$ is 2-prime if and only if it is 3-prime. ■

Now let $(M, +)$ be a group. Then it is well known that $R = M_0(M)$ is a near-ring of all zerosymmetric mappings from M to M . Furthermore, M is an R -module with respect to $(r_1 + r_2)m = r_1(m) + r_2(m)$ and $(r_1 r_2)m = r_1(r_2(m))$ where $r_1, r_2 \in R$ and $m \in M$. In view of this we have the following result:

Theorem 217 *Let $R = M_0(M)$ where $(M, +)$ is a group with $|M| \geq 2$. If G is a finite group, then $R[G : M]$ is not 3-prime.*

Proof. Fix $0 \neq m' \in M$ and let $c \in R$ be such that:

$$c(m) = \begin{cases} m' & \text{if } m \neq 0 \\ 0 & \text{if } m = 0 \end{cases}$$

Now let $r_1, r_2 \in R$ and $m \in M$. Then:

$$c(r_1(m)) + c(r_2(m))$$

$$\begin{aligned}
&= \left\{ \begin{array}{ll} c(m') + c(m') & \text{if } r_1(m) \neq 0 \text{ and } r_2(m) \neq 0 \\ c(m') & \text{if } r_1(m) \neq 0 \text{ and } r_2(m) = 0 \\ c(m') & \text{if } r_1(m) = 0 \text{ and } r_2(m) \neq 0 \\ 0 & \text{if } r_1(m) = 0 \text{ and } r_2(m) = 0 \end{array} \right\} \\
&= \left\{ \begin{array}{ll} m' + m' & \text{if } r_1(m) \neq 0 \text{ and } r_2(m) \neq 0 \\ m' & \text{if } r_1(m) \neq 0 \text{ and } r_2(m) = 0 \\ m' & \text{if } r_1(m) = 0 \text{ and } r_2(m) \neq 0 \\ 0 & \text{if } r_1(m) = 0 \text{ and } r_2(m) = 0 \end{array} \right\} \\
&= c(r_2(m)) + c(r_1(m))
\end{aligned}$$

Hence, for all $m_1, m_2 \in M$, we have that $cm_1 + cm_2 = cm_2 + cm_1$ and the result follows from Theorem 213. ■

Corollary 218 *Let $R = M_0(M)$ where $(M, +)$ is a group with $|M| \geq 2$. If G is a finite group, then $R[G : M]$ is neither c -prime nor 2-prime.*

In the preceding theorems, we demonstrated many negative results. In the final part of this section, we present a positive result which could have a great impact on future research on group near-rings. To this end, we let R be a near-field and we consider the specific R -module, $M = {}_R R$. We then show that $R[G : M]$ (or more specifically $R[G]$, the notation that we will use for the rest of this chapter) is 2-prime.

However, before we proceed we recall that $R^{(G)} = \{\mu \in R^G : \mu(g) \neq 0 \text{ for at most finitely many } g \in G\}$ is a subgroup of R^G . Furthermore, we also noted that $R[G]$ could just as well be defined as a subnear-ring of all mappings from $R^{(G)}$ to $R^{(G)}$.

Lemma 219 *Let R be a near-field and G be an ordered group. Let $\mu \in R^{(G)}$, $\mu \neq 0$. Then for any finite set $X = \{x_1, x_2, \dots, x_n\} \subset G$ with $x_1 < x_2 < \dots < x_n$ and any $q_1, q_2, \dots, q_n \in R$, there exists a $V \in R[G]$ such that $(V\mu)(x_i) = q_i$ for $i = 1, 2, \dots, n$.*

Proof. Since $\mu \in R^{(G)}$ and $\mu \neq 0$, there exists $k_1 < \dots < k_m$ in G and nonzero $r_1, \dots, r_m \in R$ such that:

$$\mu(g) = \begin{cases} r_i & \text{if } g = k_i, 1 \leq i \leq m \\ 0 & \text{otherwise} \end{cases}$$

Now, define $\eta = [r_1^{-1}, k_1]\mu$ and let $\eta_1 = [q_1, x_1^{-1}]\eta$. Define η_i inductively by:

$$\eta_i = [q_i - \eta_{i-1}(x_i), x_i^{-1}]\eta + \eta_{i-1}, 2 \leq i \leq n.$$

We show by induction that $\eta_n(x_i) = q_i$ for $1 \leq i \leq n$.

If $n = 1$, then:

$$\begin{aligned} \eta_1(x_1) &= ([q_1, x_1^{-1}]\eta)(x_1) \\ &= ([q_1, x_1^{-1}][r_1^{-1}, k_1]\mu)(x_1) \\ &= q_1([r_1^{-1}, k_1]\mu)(e) \\ &= q_1 r_1^{-1} \mu(k_1) \\ &= q_1 r_1^{-1} r_1 \\ &= q_1 \end{aligned}$$

Now assume that $\eta_{i-1}(x_i) = q_i$.

Then, if $n = i$, we have:

$$\begin{aligned} \eta_i(x_i) &= ([q_i - \eta_{i-1}(x_i), x_i^{-1}]\eta + \eta_{i-1})(x_i) \\ &= ([q_i - \eta_{i-1}(x_i), x_i^{-1}][r_1^{-1}, k_1]\mu)(x_i) + (\eta_{i-1})(x_i) \\ &= (q_i - q_i)([r_1^{-1}, k_1]\mu)(e) + q_i \\ &= q_i \end{aligned}$$

Hence, $\eta_n(x_i) = q_i$ for $1 \leq i \leq n$. We note that the proof also shows the existence of $V \in R[G]$ such that $(V\mu)(x_i) = q_i$ for $i = 1, 2, \dots, n$. ■

Theorem 220 *Let R be a near-field and G be an ordered group. Then $R[G]$ is 2-prime.*

Proof. Suppose that $R[G]$ is not 2-prime. Then there exists left $R[G]$ -subgroups \mathcal{I} and \mathcal{J} of $R[G]$, both nonzero, such that $\mathcal{I}\mathcal{J} = 0$.

Let $0 \neq U \in \mathcal{I}$ and $0 \neq V \in \mathcal{J}$. Then there exists $\mu, \nu \in R^{(G)}$ such that $U\mu \neq 0$ and $V\nu \neq 0$.

Since $U\mu \neq 0$, there exists nonzero $s_1, \dots, s_n \in R$ and group elements l_1, \dots, l_n in G such that:

$$U\mu(g) = \begin{cases} s_i & \text{if } g = l_i, 1 \leq i \leq n \\ 0 & \text{otherwise} \end{cases}$$

Similarly, since $V\nu \neq 0$, there exists nonzero $r_1, \dots, r_m \in R$ and group elements k_1, \dots, k_m in G such that:

$$V\nu(g) = \begin{cases} r_i & \text{if } g = k_i, 1 \leq i \leq m \\ 0 & \text{otherwise} \end{cases}$$

By Lemma 187, there is a finite set $X = \{x_1, \dots, x_t\} \subseteq G$ such that:

$$0 \neq s_1 = U\mu(l_1) = U\mu_X(l_1)$$

where $\mu_X(g) = \mu(g)$ if $g \in X$ and $\mu_X(g) = 0$ otherwise.

Let $q_i = \mu(s_i)$, $1 \leq i \leq t$. Then, by Lemma 219, there exists $V_1 \in R[G]$ such that $V_1V\nu(x_i) = q_i$ for $i = 1, 2, \dots, t$. Let $\xi = V_1V\nu$. Then $\xi_X = \mu_X$, and again by Lemma 187, we have $0 \neq s_1 = U\mu_X(l_1) = U\xi_X(l_1) = UV_1V\nu(l_1)$.

Hence, $0 \neq UV_1V \in \mathcal{I}\mathcal{J}$ is a contradiction, thus implying that $R[G]$ is 2-prime ■

Corollary 221 *If R is a near-field and G is an ordered group, then $R[G]$ is ν -prime for $\nu = 0, 2, 3$.*

Proof. From the previous theorem, we know that $R[G]$ is 2-prime. Since any 2-prime near-ring is also 0-prime, $R[G]$ is 0-prime. Furthermore, since $R[G]$ has identity $[1, e]$, $R[G]$ is 2-prime if and only if it is 3-prime. Hence $R[G]$ is also 3-prime. ■

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