# AN ANALYSIS OF THE INFLUENCE OF QUESTION DESIGN ON PUPILS' APPROACHES TO NUMBER PATTERN GENERALISATION TASKS

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# ABSTRACT

This study is based on a qualitative investigation framed within an interpretive paradigm, and aims to investigate the extent to which question design affects the solution strategies adopted by children when solving linear number pattern generalisation tasks presented in pictorial and numeric contexts.

The research tool comprised a series of 22 pencil and paper exercises based on linear generalisation tasks set in both numeric and 2-dimensional pictorial contexts. The responses to these linear generalisation questions were classified by means of stage descriptors as well as stage modifiers. The method or strategy adopted was carefully analysed and classified into one of seven categories.

A meta-analysis focused on the formula derived for the $n^{th}$ term in conjunction with its justification. The process of justification proved to be a critical factor in being able to accurately interpret the origin of the sub-structure evident in many of these responses. From a theoretical perspective, the central role of justification/proof within the context of this study is seen as communication of mathematical understanding, and the process of justification/proof proved to be highly successful in providing a window of understanding into each pupil's cognitive reasoning.

The results of this study strongly support the notion that question design can play a critical role in influencing pupils' choice of strategy and level of attainment when solving pattern generalisation tasks. Furthermore, this study identified a diverse range of visually motivated strategies and mechanisms of visualisation. An awareness and appreciation for such a diversity of visualisation strategies, as well as an understanding of the importance of appropriate question design, has direct pedagogical application within the context of the mathematics classroom.

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## LIST OF ABBREVIATIONS

| | |
|---|---|
| CCR | Contextual Connectivity Rating |
| CIT | Computer-Integrated Task |
| FET | Further Education and Training |
| GET | General Education and Training |
| LO | Learning Outcome |
| MPRS | Mathematical Processing Response Sheets |
| NCS | National Curriculum Statement |
| QRAS | Question Response Analysis Sheets |
| TSA | Total Stage Attainment |

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# CHAPTER ONE

# INTRODUCTION

## 1.1 MATHEMATICS AND PATTERN

> Patterns are the very essence of mathematics, the language in which it is expressed.
>
> (Sandefur and Camp, 2004:211)

Mathematicians have always been fascinated by the art and science of patterns (Joseph, 2000). In a parallel with the visual arts, to meaningfully engage with a pattern requires a necessary discernment of the principle on which its elements are ordered. Pattern itself does not lie in the individual elements, but rather the rule which governs their mutual relationship (Taylor, 1964:69-70). From relatively humble beginnings, the exploration of patterns, both numeric and pictorial, soon gave rise to an extensive and elaborate mathematical treatment of pattern.

The connection between mathematics and the notion of pattern is prevalent at all levels of mathematical endeavour. Goldin (2002:197) describes mathematics as "the systematic description and study of pattern." Perhaps more generalised and all-encompassing, Steen (1988:616) defines mathematics as "the science of patterns." Pattern, in a broad sense of the word, is by no means restricted to numeric or pictorial patterns, although this is the usual context of the word for most school mathematics syllabi. "The mathematician seeks patterns in number, in space, in science, in computers, and in imagination" (Steen, 1988:616).

## 1.2 NUMBER PATTERNS IN THE CLASSROOM

Working with number patterns or number sequences in the classroom offers valuable opportunities for recognizing, describing, extending and creating patterns (Hargreaves, Threlfall, Frobisher and Shorrocks-Taylor, 1999:67). It has been suggested that these processes have considerable value as a precursor to formal algebra (English and Warren, 1998). Searching for patterns is also an important strategy for mathematical problem solving (Stacey, 1989:147). Furthermore, in their seminal paper on an organising principle for mathematics curricula, Cuoco, Goldenberg and Mark (1996) identify the search for pattern as a critical habit of mind.

The study of pattern has become an integral component across all Grades of the South African school Mathematics curriculum (Department of Education, 2002; Department of Education, 2003b). In the Intermediate Phase (Grades 4-6) the importance of number pattern activities is in "laying the foundation for the study of formal algebra in the Senior Phase while at the same time developing important mathematical thinking skills" (Department of Education, 2003a:37). Number pattern activities in the Senior Phase (Grades 7-9) are essentially an extension of the Intermediate Phase. However, in Grades 8 and 9 there is an expectation that learners "use algebra and algebraic processes in their description of these patterns" (Department of Education, 2003a:39).

South African schools began the phasing in of a revised Mathematics curriculum in the Further Education and Training (FET) band (Grades 10-12) in January 2006. The four general Learning Outcomes (LO) for Mathematics in the FET band, as outlined in the National Curriculum Statement (NCS) (Department of Education, 2003b:12-14), are:

- LO 1 - Number and Number Relationships
- LO 2 - Functions and Algebra
- LO 3 - Space, Shape and Measurement
- LO 4 - Data Handling and Probability

As part of LO 1, learners will "solve problems related to arithmetic, geometric and other sequences and series" as well as "explore real-life and purely mathematical number patterns and problems which develop the ability to generalise, justify and prove" (Department of Education, 2003b:12). There are a variety of different number patterns (more formally understood as sequences or progressions) which fall under the above framework, amongst others: linear or arithmetic sequences, quadratic sequences, power sequences, geometric sequences, and Fibonacci-type sequences (Jacobs, 1970:42-82).

Hargreaves et al. (1999) outline a number of basic processes that may be involved in working with number patterns – these include searching for patterns in a sequence; recognising the existence of a pattern within a sequence; describing sequences orally and in a written form; continuing a sequence; predicting terms in a sequence; testing a rule for a sequence and generalising a rule in words and/or algebraic symbols. The basic processes as outlined by Hargreaves et al. (1999) can all be accomplished numerically – i.e. in terms of patterns presented as a sequence of numerical symbols. However, implicit in the requirement that learners be able to "provide explanations and justifications and attempt to prove conjectures" (Department of Education, 2003b:18) is the necessity that at least some of the pattern questions be set in non-numeric or pictorial contexts. This would appear to be the interpretation of the NCS adopted by textbook developers – see for example de Waal, McAlister, Müller, Wallace and Williams (2005), Pretorius, Potgieter and Ladewig (2005) and Goba and van der Lith (2005).

There are numerous pictorial and practical contexts in which pattern questions can be set, among the most obvious being dot patterns, tiling patterns, matchstick patterns as well as two- and three-dimensional building block patterns. Such pattern tasks usually require some form of generalisation of the pattern, usually in terms of algebraic symbols. It can be argued that setting pattern questions within a pictorial context should allow for greater scope in terms of learner solution strategies, since a pictorial representation can readily be reduced to a purely numeric equivalent provided the pictorial context has been meaningfully understood[1]. However, although pattern problems presented in a pictorial and/or practical context have the potential to widen the scope of solution strategies for some learners, it can be argued that for others this may well create additional complications.

## 1.3 GOALS & OBJECTIVES

The emphasis of the NCS on investigation as a pedagogical approach to number pattern generalisation tasks, as well as its requirement that learners be able to

[1] I use the expression "meaningfully understood" to imply a sufficient understanding of the underlying structure of the pictorial context to allow a learner to both continue the pattern and provide a generalising rule for the pattern, either in words or algebraic symbols.

*investigate* number patterns and hence "make conjectures and generalisations" as well as "provide explanations and justifications and attempt to prove conjectures" (Department of Education, 2003b:18), has important pedagogical implications for classroom practitioners. An understanding of how question design of such pattern generalisation tasks is likely to influence the approach adopted by children would greatly assist teachers in terms of their choice of such activities.

> Under what circumstances and conditions different contexts simplify or complicate this approach to generalization through pattern tasks is an important and still largely unresolved question. Teachers would be greatly assisted by knowing much more about when and how to use particular kinds of pattern tasks and, if possible, with whom.
>
> (Orton, Orton and Roper, 1999:121)

It is within the context of the above statement that the present investigation finds both impetus and import.

Thus, in broad terms, this study aims to qualitatively investigate the solution strategies adopted by children in number pattern generalisation tasks presented in pictorial and numeric contexts. The fundamental research question is: In number pattern generalisation tasks, to what extent does question design, in both pictorial and numeric contexts, affect the solution strategies adopted by learners?

## 1.4 THEORETICAL FRAMEWORK

While embracing the basic tenets of constructivism, central to this study is the fundamental notion that constructivism is a *descriptive* as opposed to *prescriptive* philosophy (Towers and Davis, 2002:314). Built onto this philosophy is the firm belief in the use of both language and notation systems/representations as important mediators in the process of construction – both in terms of their contribution to the organisation of the thinking process itself, as well as the cyclical nature of reflection (Kaput, 1991).

The role of visualisation is also central to the present study, and it is acknowledged that while generalisation problems presented in a pictorial or practical context have the potential to widen the scope of solution strategies for some pupils, for others this may well create additional complications (Orton et al., 1999).

The notions of generalisation, justification and proof are intricately interwoven. Generalisation, by its very nature, can not be separated from justification/proof, and justification is seen as a critical component of the generalisation process. The types of generalisation activities included in this study purposefully include those presented in pictorial contexts, thus allowing for a possible connection to a referential context that has the potential to aid and enhance the generalisation process. The central role of proof within the context of this study is seen as communication of mathematical understanding, and students' justifications of their generalisations are seen to provide "…a window to view their understanding of the general nature of their rules" (Lannin, 2005:251).

## 1.5 METHODOLOGY

This study is based on a qualitative investigation framed within an interpretive paradigm. The essential character underpinning the data analysis is the treatment of all responses, particularly those that are unexpected or idiosyncratic, with a genuine interest in understanding their character and origins – a firm conviction that "the constructions of others … have integrity and sensibility within another's framework" (Confrey, 1990:108).

The present study attempts to interrogate pupils' responses to various linear generalisation tasks from both a technical as well as strategic viewpoint. A case study methodological strategy was adopted and an appropriate group of research participants was identified - the members of a mixed gender, high ability Grade 9 class of 24 learners at an independent school in Grahamstown.

Over a period of 3 months, the 24 research participants each completed a series of 22 pencil and paper exercises based on linear generalisation tasks set in both numeric and 2-dimensional pictorial contexts. For each pattern, participants were required to provide numerical values for the next, $10^{th}$ and $50^{th}$ terms as well as a written articulation of their reasoning at each stage. Participants were also asked to provide an algebraic expression for the $n^{th}$ term as well as to justify their expression. In addition to written responses, individual participants were informally interviewed where the written articulation of their mental reasoning was either ambiguous or required illumination by oral explication.

The responses to the various linear generalisation questions were classified by means of stage descriptors as well as stage modifiers. The method or strategy adopted for determining each of the next, $10^{th}$ and $50^{th}$ terms was carefully analysed and classified into one of seven categories. In addition, a separate framework was used to characterise each pupil's justification of the $n^{th}$ term in terms of the extent to which the justification was linked to the pictorial context. A meta-analysis of the generalisation/justification process was also undertaken. The stage descriptors and modifiers together with the adopted solution strategies and justification characterisation were used to create a rich profile for each research participant as well as for each individual pattern generalisation task.

## 1.6 SIGNIFICANCE

The results of the present study give strong support to the notion that question design can play a key role in influencing which strategies are adopted by pupils when solving pattern generalisation tasks, in both pictorial and purely numeric contexts. This observation is central to the theme of this study, and the notion that different contexts (numeric vs. pictorial) will resonate differently with different pupils. In addition, the process of justification/proof proved to be highly successful in providing a window of understanding into each pupil's general formula. Furthermore, this study identifies a diverse range of visually motivated strategies and mechanisms of visualisation. An awareness and appreciation for such a diversity of visualisation strategies, as well as an understanding of the importance of appropriate question design, has direct pedagogical application within the context of the mathematics classroom.

## 1.7 THESIS OVERVIEW

**Chapter 2 – Literature Review**

A review of past and current research focusing on number pattern generalisation tasks was undertaken in order to inform the present investigation. Key issues arising from the literature review are highlighted and discussed in this chapter.

**Chapter 3 – Theoretical Framework**

This chapter seeks to establish a theoretical framework for the epistemological assumptions that will inform and guide the research process. There are three key elements to this contextual backdrop: constructivism; visualisation; and the interwoven nature of generalisation, justification and proof.

**Chapter 4 - Methodology**

Further theoretical elements pertaining to more practical methodological issues are interrogated in this chapter. The choice of methodology and methodological strategies are justified within the context of the adopted theoretical framework.

**Chapter 5 – Results, Analysis & Discussion**

The results of this study are presented firstly to provide a global overview. A more in-depth analysis then investigates the influence of question design on various parameters. In addition, a meta-analysis of individual responses highlights the diversity of identified mechanisms of visualisation as well as a number of anomalies and idiosyncrasies. Finally, a comparison of two different cognitive styles is undertaken.

**Chapter 6 – Findings & Conclusion**

The final chapter consolidates the findings of this study within the context of the original research question, and with reference to the adopted theoretical framework and methodological choices. In addition, both the limitations and significance of the study are interrogated, and some recommendations for further research are suggested.

# CHAPTER TWO

# LITERATURE REVIEW

## 2.1 INTRODUCTION

A review of past and current research focusing on number pattern generalisation tasks was undertaken in order to inform the present investigation. Pattern as an approach to algebra has fallen under the spotlight for many years (Pegg and Redden, 1990b; English and Warren, 1998; Orton and Orton, 1999; Zazkis and Liljedahl, 2002). Much research has also focused on children's strategies when solving number pattern generalisation tasks (Stacey, 1989; Orton, 1997; Hargreaves, Shorrocks-Taylor and Threlfall, 1998; Orton and Orton, 1999). Issues relating to visualisation (Nixon, 2002), visual reasoning (Hershkowitz, Arcavi and Bruckheimer, 2001), the connection of the visual with the symbolic (Noss, Healy and Hoyles, 1997; Healy and Hoyles, 1999) and the influence of computerised environments (Hershkowitz et al., 2002) have all found voice within the context of pattern generalisation. In addition, a large volume of research emanates from the Pattern in Mathematics Research Group, set up in 1992 at the University of Leeds. Of particular interest to the present investigation are the research findings of Hargreaves et al. (1999) who investigated children's strategies with linear and quadratic sequences; Orton et al. (1999) who were concerned with the perception of pattern within pictorial and practical contexts; and Waring, Orton and Roper (1999) who focused on the notions of proof and justification within the context of pattern.

Recent research focusing on number pattern generalisation tasks would thus seem to encompass a broad range of topics. Key issues arising from the literature review are highlighted and discussed in this chapter.

## 2.2 TYPES OF NUMBER SEQUENCES

### 2.2.1 LINEAR OR ARITHMETIC SEQUENCES

In linear or arithmetic sequences, each term can be obtained by adding (or subtracting) a constant value to the preceding term in the sequence. Linear sequences can in general be written in the form $ax \pm c$ where $a$ and $c$ are constant values, and $x$ is a variable representing the position of the term in the sequence.

Since the difference between any two successive terms is constant, arithmetic sequences can be defined formally as sequences in which the difference between any two successive terms is a constant value (Laridon et al., 1996). Using formal nomenclature, for an arithmetic sequence with first term $a$ and common difference $d$, the general term (or $n^{th}$ term) is given by $T_n = a + (n-1)d$.

By way of example, consider the arithmetic sequence 2 ; 6 ; 10 ; 14 ; … Since the first term is 2 and the common difference is 4, the general term can be expressed as $T_n = 2 + (n-1)4$, which in turn can be simplified to $T_n = 4n - 2$, where $n$ represents the position of the term in the sequence.

### 2.2.2 QUADRATIC SEQUENCES

Quadratic sequences can in general be written in the form $an^2 \pm bn \pm c$ where $a$, $b$ and $c$ are constant values, and $n$ is a variable representing the position of the term. In a quadratic sequence, the difference between successive terms is not constant. However, the difference between the differences (i.e. the second difference) is a constant value.

By way of example, consider the quadratic sequence 3 ; 5 ; 11 ; 21 ; 35 ; …

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Term position: | 1 | | 2 | | 3 | | 4 | | 5 |
| Term value: | 3 | | 5 | | 11 | | 21 | | 35 |
| First difference: | | 2 | | 6 | | 10 | | 14 | |
| Second difference: | | | 4 | | 4 | | 4 | | |

Since the general formula is of the form $T_n = an^2 \pm bn \pm c$, one can readily solve for $a$, $b$ and $c$ by substituting three different points e.g. (1;3) , (2;5) & (3;11) and solving simultaneously. The general formula thus works out to be $2n^2 - 4n + 5$, where $n$ represents the position of the term in the sequence.

### 2.2.3 POWER SEQUENCES

In power sequences, each term is found by raising a counting number to a specific power, e.g. the sequence of squares or cubes. The sequence of squares, 1 ; 4 ; 9 ; 16 ; 25 ; 36 ; 49 ; …, can be written in the form $1^2 ; 2^2 ; 3^2 ; 4^2 ; 5^2 ; 6^2 ; 7^2 ; \ldots$ Likewise, the sequence of cubes, 1 ; 8 ; 27 ; 64 ; 125 ; …, can be written in the form $1^3 ; 2^3 ; 3^3 ; 4^3 ; 5^3 ; \ldots$ Higher power sequences are also possible.

Power sequences are readily recognisable, and their general formulae are of the form $T_n = n^k$ (e.g. $T_n = n^2$ or $T_n = n^3$) where $k$ is a constant and $n$ represents the position of the term in the sequence.

### 2.2.4 GEOMETRIC SEQUENCES

In geometric sequences, each term in the sequence can be obtained by multiplying the preceding term by a constant value, formally known as the common ratio.

Since the ratio of any two successive terms is a constant value, geometric sequences can be defined formally as sequences in which the ratio $r$ of any two successive terms is a constant (Laridon et al., 1996). Using formal nomenclature, for a geometric sequence with first term $a$ and common ratio $r$, where $r = \frac{T_{n+1}}{T_n}$, the general term (or $n^{th}$ term) is given by $T_n = ar^{n-1}$.

By way of example, consider the geometric sequence 3 ; 6 ; 12 ; 24 ; … Since the first term is 3 and the common ratio is 2, the general term can be expressed as $T_n = 3(2)^{n-1}$, where $n$ represents the position of the term in the sequence.

### 2.2.5 FIBONACCI-TYPE SEQUENCES

The Fibonacci sequence (Higgins, 1998:180-199; Livio, 2002:92-123), so dubbed by the French Mathematician Edouard Lucas in the nineteenth century, is one of the most well known recursive sequences. Fibonacci-type sequences, also known as recursive sequences, are characterised by each term being the sum of the two terms immediately preceding it $(T_n = T_{n-1} + T_{n-2})$. Possibly the most famous sequence of this type is the Fibonacci sequence itself: 1 ; 1 ; 2 ; 3 ; 5 ; 8 ; 13 ; 21 ; … In order to calculate the 50$^{th}$ term (for example) of a recursive sequence, one would first have to determine each preceding term by means of an iterative or recursive approach. Such a procedure, while manually rather time-consuming, is well suited to solution by computer.

## 2.3 PICTORIAL AND PRACTICAL CONTEXTS

There are numerous pictorial and practical contexts in which pattern questions can be set (Mason, Graham, Pimm and Gowar, 1985; Orton, Orton et al., 1999), among the most obvious being dot patterns (Kenney, Zawojewski and Silver, 1998), tiling patterns (Lannin, 2004), matchstick patterns (Orton, 1997; English and Warren, 1998; Pegg and Redden, 1990a) as well as two- and three-dimensional building block patterns (Miller, 1991; Nolder, 1991; Abbott, 1992; Pagni, 1992; Lannin, 2003). Polygonal or figurate numbers (e.g. triangular, square, pentagonal and hexagonal numbers) also make use of simple visual patterns to portray numbers (Crookes, 1988; Andrews, 1990; Miller, 1990; Malloy, 1997).

Figure 2.1 shows a typical pictorial representation of a number pattern. This particular example shows the first three terms of the linear sequence 4 ; 7 ; 10 ; … with general term $T_n = 3n + 1$. Typical questions would require pupils to determine, for example, the next, 10$^{th}$, 20$^{th}$ and 100$^{th}$ terms, as well as an expression (either verbal or symbolic) for the n$^{th}$ term and a possible justification for their general formula.

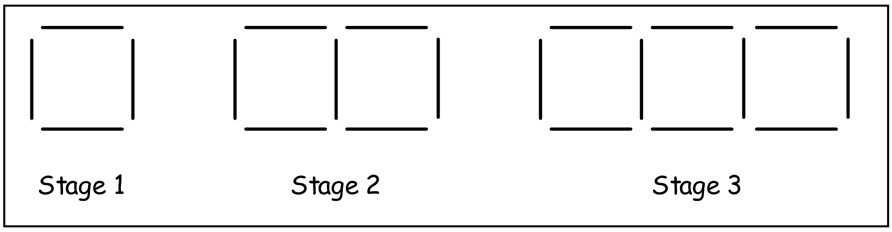



**Figure 2.1** Typical pictorial context for a number pattern

Far from simply being a visual representation of a numeric pattern, number sequences presented in a pictorial/practical context allow for a potentially deeper appreciation of the underlying structure of the pattern, as the pictorial/practical context allows for both a greater depth and scope of interpretation. Furthermore, provided the pictorial context has been meaningfully understood, number patterns presented pictorially are inherently less ambiguous than purely numeric sequences (see discussion of Figure 2.2 and Figure 2.3). Indeed, Clausen (1992) asserts that working directly from a practical or pictorial context is often preferable to an algebraic treatment derived from purely numerical patterns. The use of a pictorial context is also deemed "safer" as it limits the chance that "irrelevant number patterns will mislead one into assuming the truth of an invalid generalization" (Clausen, 1992:18)[2].

By way of explication, it is worth noting that for a finite sequence of numbers there is an infinite number of functions that could generate the sequence. This is equivalent to plotting a finite number of points in the Cartesian Plane where there would obviously be an infinity of curves that could be drawn through the given points (Samson, 2006:8). Thus, no finite sequence of numerical terms uniquely generates the next term in the sequence. It is perhaps also worth noting that mathematicians have studied more than 150 sequences starting 1 ; 2 ; 4 ; … More specifically, they have studied more than 30 sequences starting 1 ; 2 ; 4 ; 8 ; … and at least 9 which start 1 ; 2 ; 4 ; 8 ; 16 ; … (Wells, 1987:3).

Thus, when presented with the numerical sequence 2 ; 6 ; 12 ; … (for example), one needs to be cognizant of the fact that there are an infinite number of ways to continue the sequence, and for each of these numerical patterns would be a

[2] See de Jager (1999) and Clausen (1992) for two different treatments of a classic scenario which calls for both circumspection and caution when searching for patterns – the old-established problem of the number of regions created by joining $n$ points on the circumference of a circle such that there is no point of intersection inside the circle with more than two lines passing through it.

corresponding general formula, complex or otherwise. However, should the same numerical sequence be accompanied by a pictorial representation, the underlying structure of the pattern should become immediately apparent. By way of example, consider the following two pictorial patterns:

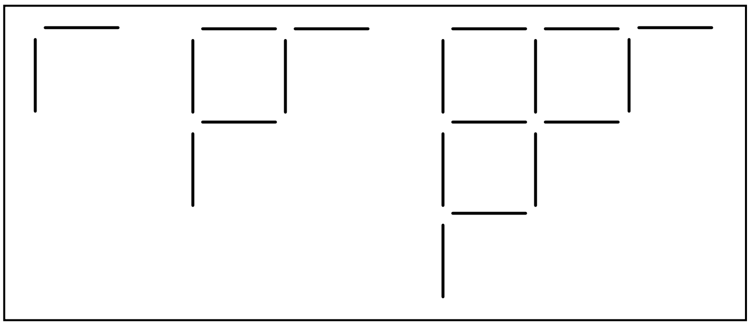


**Figure 2.2** Herring-bone pattern

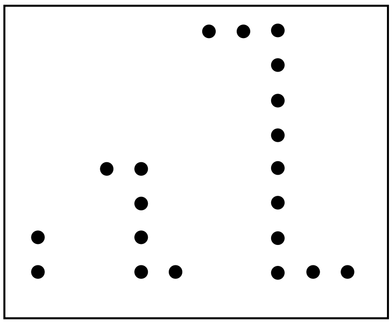


**Figure 2.3** Growing Z-shapes

For both patterns the first three terms are numerically equivalent, *viz*. 2 ; 6 ; 12. However, the underlying structure inherent in the pictorial representation of each of the two numerical sequences points to two very different general terms. Continuing the so-called herring-bone pattern leads to the sequence 2 ; 6 ; 12 ; 20 ; 30 ; 42 ; …, while continuing the growing pattern of Z-shapes leads to the sequence 2 ; 6 ; 12 ; 22 ; 40 ; 74 ; …. Careful analysis of the pictorial context allows one to arrive at the general formulae $T_n = n(n+1)$ and $T_n = 2^n + 2(n-1)$ respectively.

Mathematics education journals abound with number pattern activities and investigations making use of pictorial contexts (Van de Walle and Holbrook, 1986; French, 1990; Onions, 1991; Pagni, 1992; Malloy, 1997; Szetela, 1999; de Mestre, 2001; Lannin, 2004; Farmer and Neumann, 2004; Quinn, 2005). In essence, the use of a pictorial context aims to exploit the visual decoding of the pictorial sequence to give meaning to the symbolic expressions constructed.

Pictorial tasks are often considered to be more elementary than purely symbolic tasks. Orton et al. (1999) comment that this view may be supported by considering Bruner's three stage theory of learning – from the enactive (practical) to the iconic (pictorial) to the symbolic (Baumann, Bloomfield and Roughton, 1997). An alternative view within the context of patterning activities is that a sequence or table of numbers should be sufficiently concrete for pupils old enough to be introduced to algebra, and a pictorial context may simply obfuscate and create additional complications (Orton et al., 1999).

Andrews (1990:13) comments that it is “preferable to offer pupils a situation which can be generalised with reference to the situation itself”. This notion is supported by Hershkowitz et al. (2002) who observed that presenting sequences in a pictorial context tends to encourage generalisation expressed in terms of the independent variable (as opposed to a recursive method), particularly if the pictorial terms are non-consecutive. They comment that as a reflection of the counting method employed to determine specific terms in the sequence, the use of a pictorial context seems to give strong meaning to the general formula. However, in a study by English and Warren (1998), students found it easier to generalise, both verbally and symbolically, when patterns were presented in tabular form as opposed to pictorial form. Of particular interest here is the fact that the tables included both the position of the term (the independent variable) and the term itself (the dependent variable), while it would seem that the pictorial contexts only made mention of the stage number (equivalent to the position of the term). Two central issues thus emerge, the importance of task design/presentation, and the notion that different contexts (numeric, tabular, pictorial) will resonate differently with different pupils.

Based on their studies, Orton et al. (1999) caution that placing a pattern in a pictorial context must not automatically be assumed to be helpful. In addition, some contexts are more difficult than others and the perceived relationship between pattern and context may also be problematic. Furthermore, English and Warren (1998) comment that one of the potential limitations of using pictorial patterning contexts as an approach to algebra is that only positive integer values can be assigned to the variable. They advocate the use of tables of data (where the variable can take on any real value) in conjunction with pictorial contexts to impart the idea of a variable being a generalised number.

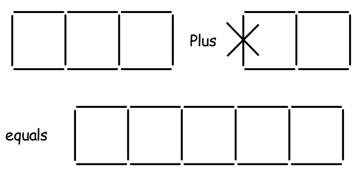
As an aside, Burke and Orton (1999) have also commented that the spatial arrangement of numbers and number sequences are likely to assist the identification and analysis of pattern in certain situations. In this sense, a pictorial context could be interpreted as being nothing more than a spatial arrangement of the individual numerical terms of the pattern.

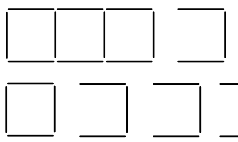
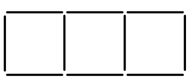
## 2.4 PATTERNING STRATEGIES

Hargreaves et al. (1998, 1999) investigated the strategies used by children aged between 7 and 11 years when asked to provide a verbal generalisation (in natural language) for simple sequences of numerical terms (linear, quadratic and Fibonacci-type sequences). Strategies utilised were: looking for differences (i.e. the numerical difference between consecutive terms); looking at the nature of the differences (usually in terms of odd and even); looking for differences between differences (see Section 2.2.2); looking at the nature of the numbers (usually in terms of odd and even); looking for multiplication tables; combining terms to make other terms.

Healy and Hoyles (1999) devised a classification system of strategies (construction approaches) that purposefully distinguished between iconic (visual) approaches and symbolic strategies. This classification system was used to investigate the connections pupils made between visual and symbolic reasoning while generalising number patterns. The classification system is shown in Table 2.1 which highlights the mathematical equivalence of the symbolic and iconic approaches.

**Table 2.1** Classification of construction approaches

| Symbolic approach | Iconic approach |
|---|---|
| **Counting**<br>Counting individual items in an unstructured way | **Eidetic**<br>Focusing on perceptual rather than mathematical properties of the data |
| **Operating terms**<br>Calculations using known terms to obtain a target term | **Combining diagrams**<br>"Chunking" of known terms to obtain others<br> |

| | |
|---|---|
| **Operating on differences between terms**<br>Calculations based on the numerical difference between consecutive terms | **Inter-term “chunking”**<br>“Chunking” based on a relation between terms<br> |
| **Operating on a variable**<br>Calculations based on a relation between dependent and independent variables | **Intra-term “chunking”**<br>“Chunking” based on a relation within a term<br> |

Adapted from Healy and Hoyles (1999:67)

Stacey (1989) has investigated and analysed the various strategies adopted by children (aged 9 to 13 years) when performing linear generalisation tasks. Some of the problems used were set within a pictorial context. The four methods which were most commonly used were:

- Counting method (successive addition)
- Difference method[3]
- Whole-object method[4]
- Linear method

The counting method (or method of successive addition) represents a recursive approach whereby subsequent terms were determined by successively adding the identified constant difference to previous terms. The difference method (or “difference product”) was based on identifying the common difference and then multiplying it by the term number. Using this approach, the 20$^{th}$ term of the sequence 2 ; 5 ; 8 ; 11 ; … would be incorrectly calculated as being $3 \times 20$. The whole-object method (or “short-cut”) involved the assumption that, for example, the 20$^{th}$ term would be 4 times the 5$^{th}$ term. This approach, although generally incorrect, does produce a correct answer for linear sequences based on direct proportion ($ax \pm c$ where $c = 0$). The linear method acknowledged that both multiplication and addition were involved. Correct use of this method should have always led to the correct answer.

[3] Orton and Orton (1999) have adopted the term “difference product” for this method.

[4] Orton and Orton (1999) have adopted the term “short-cut” for this method, while English and Warren (1998) refer to it as a “ratio” strategy.

Lannin (2003) provides a useful summary of student strategies observed both by himself and other researchers, notably Stacey (1989) and Swafford and Langrall (2000). The strategies are presented in Table 2.2 using Lannin's nomenclature.

**Table 2.2** Student strategies for pattern generalisation

| Strategy | Description |
|---|---|
| Counting | Drawing a picture or constructing a model and physically counting the desired attribute |
| Recursion | Constructing a term by building onto a previous term or terms |
| Whole-object | Constructing larger units by using multiples of smaller units (with or without a correction for over- or undercounting) |
| Contextual | Determining a rule directly from a relationship inherent in the problem context |
| Guess and Check | Guessing a rule without regard as to why the rule may work |
| Rate-adjust | Using a constant rate of change as a multiplying factor followed by an adjustment (adding or subtracting of a constant) to ensure the rule holds for the given terms |

Adapted from Lannin (2003:344)

Lannin (2005) classifies these strategies as being either explicit or non-explicit. Explicit strategies (Whole-object, contextual, guess and check, rate-adjust) allow for the direct calculation of the dependent variable given a specific value of the independent variable. Non-explicit strategies (counting, recursion) do not allow for this, as the calculation of a specific term requires the calculation of all preceding terms.

Choice of strategy is a crucial consideration. Students' difficulties in forming generalisations often result from the inappropriate strategies they used to determine a general rule (English and Warren, 1998). As Orton and Orton (1999) remark, there is little chance of pupils being able to generalise correctly, and in an acceptable algebraic form, if they adopt inappropriate methods. In an investigation of the patterning ability (using a series of matchstick patterns) of mixed-ability pupils aged between 9 and 13 years, Orton (1997) comments that the most common methods which yielded wrong answers were the short-cut and difference product. There were also examples of pupils' own idiosyncratic reasoning. In addition, the method of differencing (recursive strategy) has been found to be particularly popular with children (Stacey, 1989; Hargreaves et al., 1998; Orton and Orton, 1999; Lannin, 2004). It is likely that this is a conditioned/habitual response to typical number pattern activities, or an artefact of the way in which number pattern generalisation tasks are presented. The importance of question design has also been raised and questioned by other researchers (Orton et al., 1999; Swafford and Langrall, 2000). The influence of question design is central to the present study, and is discussed in detail in Section 5.3.

## 2.5 THE PREVALENCE OF THE RECURSIVE STRATEGY

A number sequence can be generated either by using a general formula, where the independent variable represents the *position* of a term, or by relating recursively to the previous term in the sequence. Use of a recursive approach tends to emphasize a local aspect of the relationship, while an explicit formula reflects the relationship in a general way.

A common theme in the research literature relates to the tendency of pupils to generalise recursively rather than using the independent variable in a general formula (Hargreaves et al., 1998; Hershkowitz et al., 2002). Lannin (2004) comments that there would seem to be a natural tendency for pupils to reason recursively when they begin to examine number patterns. English and Warren (1998) found that once students had established a recursive strategy they were reluctant to search for a functional relationship, and Orton et al. (1999) observed that progression to far generalisation tasks usually required a rejection of a such a counting or recursive strategy. MacGregor and Stacey (1993), investigating pattern generalisation tasks presented in table format with 14- and 15-year olds, cite one of the main causes of difficulty in formulating algebraic rules as being pupils' tendency to focus on the recursive patterns of one variable rather than the relationship linking the two variables. Similar observations have been made by other researchers (Orton, 1997). Interestingly, a reliance on differencing (i.e. a recursive strategy) has also been found with adults (Orton and Orton, 1994).

Noss et al. (1997) comment that the tendency of pupils to focus on a recursive strategy shouldn't necessarily be interpreted as pupil failure. They remark that strategies are influenced not only by the nature of the task but also by its presentation. This has been echoed by Frobisher and Threlfall (1999) who comment that presenting a task in sequential stages (e.g. asking for the 10$^{th}$, 20$^{th}$ and 50$^{th}$ terms) often leads pupils to use a step-by-step recursive approach. Hershkowitz et al. (2002) found that the presentation of consecutive terms encouraged recursion, while terms presented non-consecutively tended to encourage generalisation by means of the independent variable. The use of a pictorial context, particularly if non-consecutive terms were presented, also tended to encourage generalisation by means of the independent variable. Hershkowitz et al. (2002) have also made the observation that spreadsheet environments often have the effect of encouraging recursive generalisations.

Lannin (2004) comments that reasoning based on a functional relationship has historically been valued over recursive strategies because of the relative inefficiency and tediousness of iterative procedures. However, with the introduction of technology into the classroom environment, using recursive strategies is far less tedious and time-consuming than it once was. Noss et al. (1997) lend support to this by citing the potential flexibility of the computer environment. Quinn (2005) argues that with the increased speed and power of computers, iterative algorithms are regaining importance. There is thus justification for investigating recursive/iterative strategies. Furthermore, Lannin (2003) argues that in the case of linear sequences the recursive approach provides for a strong connection with the concept of gradient or slope (i.e. the constant increase/decrease in the dependent variable for a unit increase in the independent variable).

## 2.6 CLASSIFYING PATTERNING ABILITY

Patterning ability has often been investigated by providing a sequence of terms, presented either numerically or pictorially, and asking for the calculation of, for example, the 5$^{th}$, 10$^{th}$, 50$^{th}$ and n$^{th}$ terms. While investigating adults' generalisation of quadratic patterns using a similar sequential approach, Orton and Orton (1994) found evidence for a hierarchical classification of patterning ability, at least in the sense that there were no candidates who successfully answered a particular question without having succeeded on all the previous questions.

This led Orton and Orton (1996, 1999) to attempt an investigation into the stages in the development of children's patterning ability using a hierarchical classification system as research instrument. Hierarchical frameworks are well established, both within the general domain of cognition (e.g. Piaget's constructivist theory of cognitive development (Labinowicz, 1980)) and the more specific realm of mathematical thought (e.g. the van Hiele model of levels of geometric thought (van Hiele, 1986)). Notions of hierarchy, both in terms of the learning of mathematics as well as mathematical ability, have nonetheless been subject to criticism (Ernest, 1991). Ernest dismisses the notion of there being an overall mathematical hierarchy by arguing that mathematics is composed of "… a multiplicity of distinct theories, that these cannot be reduced to a single system, and that no one of these is adequate to capture all the truths even in its limited domain of application" (Ernest, 1991:233).

Nonetheless, Orton and Orton (1996, 1999) attempted to develop a hierarchical classification system of patterning ability which they adapted to classify children's (aged 10 to 13) responses to linear generalisation questions. In order for a child to be classified at a particular stage, all the terms within that stage must have been correctly calculated. The various stage descriptors were:

- Stage 0: no progress
- Stage 1: next term provided
- Stage 2: next and $20^{th}$ terms provided
- Stage 3: next, $20^{th}$ and $100^{th}$ terms provided
- Stage 4: next, $20^{th}$, $100^{th}$ and $n^{th}$ terms provided

Stage 4 was further subdivided as follows:

- Stage 4a: a correct verbal statement
- Stage 4b: a creditable attempt at an algebraic expression
- Stage 4c: a correct algebraic representation, but not necessarily the simplest

The specific choice of the $20^{th}$ and $100^{th}$ terms in the above framework is not critical *per se* – these terms have simply been chosen to represent "near generalisation" and "far generalisation" tasks[5]. However, Orton and Orton (1996:87) found that children's responses to linear generalisation questions did not necessarily conform to a strict hierarchy. This led to subdivisions of the stages to cover all possibilities, using the letters *t*, *x* and *y* to indicate errors in the next, 20$^{th}$ or 100$^{th}$ terms respectively. This modification allowed for the classification of a child who made an error in a previous stage but was nonetheless able to progress correctly to a higher stage. A child who gave a correct 100$^{th}$ term but an incorrect 20$^{th}$ term was thus classified at Stage 3*x*, the 3 indicating a correct calculation of the 100$^{th}$ term (Stage 3) and the *x* indicating an error in the calculation of the 20$^{th}$ term. These modified stage descriptors provide a useful picture of children's proficiency in linear generalisation tasks and can possibly be used as an enabling framework for further research.

The purpose of such a sequential approach to pattern generalisation is to scaffold the generalisation process. However, in an investigation of the patterning ability (using a series of matchstick patterns) of mixed-ability pupils aged between 9 and 13 years, Orton (1997) found little evidence to suggest that such a sequential approach assisted in the process of generalisation. Orton et al. (1999) also found that a sequential approach to patterning tasks did not necessarily assist pupils in finding the general term. Even when pupils were successful in determining the 5$^{th}$, 10$^{th}$ and even 50$^{th}$ terms, this did not always lead to an acceptable expression for the general term. An earlier study with adults (Orton and Orton, 1994) also indicated that success with the 5$^{th}$, 10$^{th}$ and 50$^{th}$ terms by no means automatically guarantees success with the n$^{th}$ term.

## 2.7 PATTERNING AS A ROUTE TO ALGEBRA

French echoes the views of numerous mathematics teachers and educationalists when he opines thus:

> The initial encounters that students have with algebra are crucially important in establishing both their attitudes towards the subject

[5] Stacey (1989:150) uses the term "near generalisation" to denote a question which can be solved by step-by-step counting or drawing and "far generalisation" to denote a question which goes beyond reasonable practical limits of such a step-by-step approach.

> and the foundations on which to build their subsequent study of the subject together with its links to the rest of mathematics.
>
> (French, 2002:44)

The use of number patterns as a didactic approach to the introduction of algebra has been advocated by numerous mathematics educators (Mason et al., 1985; Pegg and Redden, 1990b; de Jager, 2004). MacGregor and Quinlan (1992:250) comment that proponents of this approach would seem to expect that algebraic symbolism will emerge as a natural consequence of pupils needing to write their verbal rules in a more succinct manner. There is certainly some evidence to support this notion (Pegg and Redden, 1990a).

From a pedagogic standpoint, French (2002) comments that introducing algebra through what is potentially a wide range of pattern generalisation activities may be effective in assisting pupils to see algebra as both meaningful and purposeful right from the earliest stages. After all, generalisation is one of the core components of mathematical activity. As Mason et al. (1985:8) succinctly put it, “generality is the lifeblood of mathematics and algebra is the language in which generality is expressed”. Thus, algebraic symbolism arising as a natural consequence during pattern generalisation activities is certainly an attractive notion. In addition, from a pedagogic point of view, pattern generalisation activities are a meaningful way of arriving at algebraically equivalent expressions of generality. This lends itself well to exploring the notion of algebraic equivalence in a practical context where pupils would experience the process of negotiation towards meaning (Mason et al., 1985).

Nonetheless, the approach is not without its potential pitfalls. Remarking on their experiences with using the patterning approach to introducing the concept of variable, English and Warren (1998) comment that apart from sound arithmetic skills, flexible and articulate thinking are particularly important to a student’s success with this approach. In addition, although a generalisation can be expressed in numerous ways, both verbally and symbolically, some verbal expressions do not translate as readily as others into an algebraic format, and this may well undermine the process.

## 2.8 VISUALISATION AND VISUAL REASONING

There are many different kinds of visualisation in mathematics[6]. Furthermore, visualisation as both the product and process of creating, interpreting and reflecting upon images, is gaining increased focus in the fields of both mathematics and mathematics education (Zimmermann and Cunningham, 1991; Arcavi, 2003).

Within the realm of pattern generalisation, Hershkowitz et al. (2001) uncovered various "mechanisms" of visualisation in the building of a mathematical generalisation in a pictorial context. They distilled the various visual strategies into the following analytical components: (a) decomposition of a structure into smaller substructures and units, (b) creation of auxiliary constructions, (c) transformation of the whole structure into a different configuration, and (d) recomposition and synthesis. Their results led Hershkowitz et al. (2001) to propose that visualisation can be far more than the intuitive support of higher level reasoning, in that it may well constitute "the essence of rigorous mathematics" (Hershkowitz et al., 2001:255). Although this research was conducted with more mature subjects (in-service teachers), there is evidence to suggest that younger children are also capable of utilising similar visualisation mechanisms (Orton et al., 1999; Waring et al., 1999).

The use of metaphor can also be seen as a type of visualisation. Nolder (1991) reports on the use of metaphors such as "staircases", "wings" and "triangles" by pupils presented with number patterns in 3-dimensional practical context. She comments that although these terms are useful in helping learners communicate their ideas, some metaphors are less helpful than others when it comes to finding an algebraic generalisation of the pattern.

Of particular import to the present investigation is the observation that, although pattern generalisation problems presented in a pictorial or practical context have the potential to widen the scope of solution strategies for some pupils, for others this may well create additional complications. This issue is interrogated in more detail in the following chapter.

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[6] Theoretical aspects pertaining to visualisation are interrogated in Section 3.3.

## 2.9 PROOF AND JUSTIFICATION

In a study of students' generalisations and justifications derived from patterning tasks, Lannin (2005) made use of an algebraic adaptation of the framework of Simon and Blume (1996) for classifying and characterising levels of justification. The first level appeals to an authoritative source (e.g. mathematics teacher or textbook). The second level of justification appeals to inductive or empirical evidence. Levels 3 and 4 display deductive justification based on shared mathematical knowledge, either expressed in terms of specific cases (Level 3) or generality independent of particular instances (Level 4).

In an investigation into the patterning skills of sixth-graders, Lannin (2005) reports that the two types of justification most widely used were empirical justifications and generic examples. The former was generally used by pupils to test their rules. This echoes an earlier study (Lannin, 2003) where a common approach used by pupils when justifying a general formula was to demonstrate that their rule results in the correct values for a few individual cases.

Orton (2004:114) observes that there is evidence to suggest that justifying pattern generalisations is a legitimate approach to proof, and provides pupils with valuable pre-proof experiences en route to more formalised mathematical proofs. This has been echoed by other researchers (Waring et al., 1999).

The interconnectedness of generalisation, justification and proof is dealt with more extensively in the following chapter.

## 2.10 PATTERN SPOTTING

There is a concern amongst mathematics educationalists (Hewitt, 1992; Byatt, 1994; Noss et al., 1997; de Jager, 1999) that pattern generalisation activities are becoming nothing more than rote exercises in the systematic collection and tabulation of data from which a generalised formula may be obtained/conjectured. Indeed, Clausen (1992) mentions that mark schemes structured to assess pupil aptitude for such pattern generalisation tasks often "assume such an approach, and apportion marks accordingly" (Clausen, 1992:17). Noss, Healy and Hoyles capture the situation as follows:

> … attention tends to become focused on the numeric attributes of the output. Worse still, school mathematics becomes constructed – by students and teachers alike – as a stereotypical data-driven 'pattern-spotting' activity in which it is acceptable to search for relationships by constructing tables of numeric data without appreciating any need to understand the structures underpinning them.
>
> (Noss et al., 1997:205)

Of central concern is the notion that much genuine mathematical exploration has become superficial through a preoccupation with tabulating results as a means to describing patterns and arriving at a generalised rule. As such, there is a grave concern that pattern generalisation activities often get reduced to numerical pattern spotting exercises where spotting patterns in the numbers becomes "an activity in its own right and not a means through which insights are gained into the original mathematical situation" (Hewitt, 1992:7). The danger with such an approach is that the focus seems to be "the development of an algebraic relationship, rather than the development of a sense of generality" (Thornton, n.d.:252). As such, the general rule for the pattern becomes divorced from the scenario – pictorial, practical or otherwise – that gave rise to it. Such disconnected algebraic formulation neither illuminates the problem nor provides a means for validating the generated functional relationship (Noss et al., 1997). This becomes particularly problematic in situations where the justification of the general rule assumes significance (Byatt, 1994:25). The ability to justify a general formula is by no means commensurate with a pupil's proficiency in deriving such a generalisation. Indeed, as Hewitt (1992:7) succinctly remarks, the problem with divorcing patterns of numbers from their original context is that any generalised statements become "statements about the results rather than the mathematical situation from which they came".

Byatt (1994) summarises an oft-occurring mode of generalisation and justification in Figure 2.4.

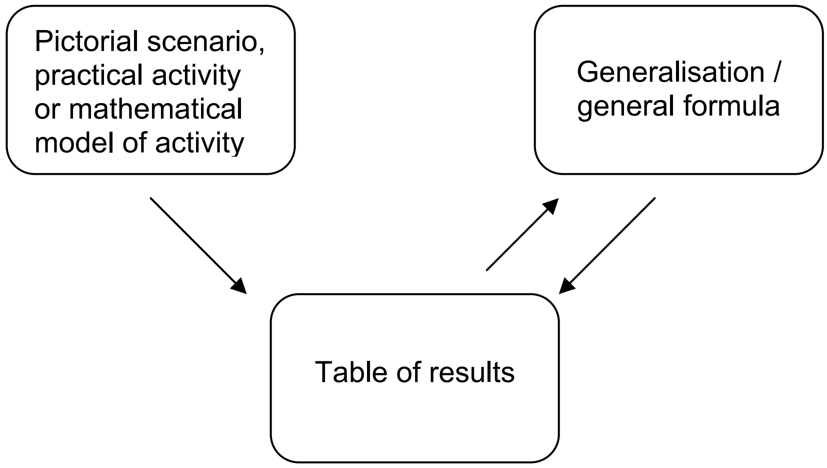


**Figure 2.4** Mode of generalisation and justification, adapted from Byatt (1994:25)

Figure 2.4 represents a mode of justification whereby the original pictorial scenario is simply reduced to a numerical table of results. A pattern is then sought in this table of numbers, with little or no regard for the original context of the question. A crucial missing connection, particularly critical in terms of the justification process, is the link back to the original pictorial/practical context. Such connectivity is critical inasmuch as it pertains to looking for a pattern “in the situation, not in the numbers given in the results” (de Jager, 1999:23). Byatt (1994) asserts that such redirection, at least in part, lies in the skills and incisive questioning of the sensitive practitioner.

There is a diverse mathematical richness that potentially can be extracted from pattern generalisation activities. Hewitt (1992) argues that critical to extracting such mathematical richness is the meaningful engagement with a particular scenario in some depth, rather than a superficial treatment in order to reduce the context to a mere table of numbers. The pedagogical distinction here is the notion of pattern analysis as opposed to pattern recognition. Central to the core of mathematical activity is the concept of generalisation, the formulating of a general formula which describes a scenario, which derives from an in-depth analysis of the given situation. Mathematicians are concerned with pattern analysis, “which involves *analyzing* the situation or the scientific context, *not* just pattern *recognition*” (Dancis, n.d.:1). Furthermore, Roper (1999) points out that searching for patterns without regard for the underlying contextual structure may in fact be counter-productive in terms of encouraging mathematical problem-solving.

## 2.11 VERBAL VS ALGEBRAIC GENERALISATION

Not surprisingly, students often find it easier to express their generalisations verbally than to record them symbolically. English and Warren (1998) comment that this situation generally arises because, although a generalisation can be expressed in numerous ways, both verbally and symbolically, some verbal expressions do not translate as readily as others into an algebraic format. Orton et al. (1999) found that the ability to express a result in words (natural language) was often present when pupils were unable to provide a symbolic (algebraic) expression.

MacGregor and Stacey (1993), investigating pattern generalisation tasks presented in table format with 14- and 15-year olds, cite one of the main causes of difficulty in formulating algebraic rules as being pupils' inability to clearly articulate the structure of perceived patterns and relationship using natural language. The results of their findings suggest that the verbal description is an important and perhaps even necessary part of the process of expressing an algebraic generalisation. In support of this, Franzblau and Warner (2001), in their teaching of sequence notation, found that an important first step in the learning of sequence notation is the writing of descriptions of both explicit and recursive rules in natural language.

## 2.12 TECHNOLOGICAL INFLUENCES

Simple computer programming has been used in the classroom environment for some years as a means of exploring number patterns (Bitter and Edwards, 1989). However, in the past two decades much work has been done by mathematics educators in creating interactive computerised settings with the express purpose of enhancing visualisation as a powerful cognitive support in the learning of mathematical concepts. Lapp (1999) advocates the use of technology (e.g. graphing calculators) as a powerful means of merging representations and forming links between different representations of the same scenario.

The study of polygonal or figurate numbers (e.g. triangular, square, pentagonal and hexagonal numbers) lends itself well to visual strategies. Abramovich, Fujii and Wilson (1994) have demonstrated the usefulness of a multiple-application medium for the study of polygonal numbers by using software tools such as dynamic geometry, a relational grapher, and spreadsheets as a means of enhancing mathematical visualisation by providing a dynamic interplay between geometric, analytical and numerical representations.

Healy and Hoyles (1999) investigated computer-integrated task (CIT) patterning activities making use of different software environments - spreadsheets, and a specially designed Logo microworld called Mathsticks, what Noss et al. (1997) refer to as an *autoexpressive* environment, providing a domain of situated abstraction. Pupils working through the Mathsticks CITs tended to show well-connected iconic and symbolic approaches by the end of the task sequence. Healy and Hoyles (1999) suggest that the fusion of action, visualisation and symbolic representation has the potential to provoke cognitive reorganisation and forge connections between the visual and symbolic representations. The use of a spreadsheet environment was found to be less effective in connecting the visual with the symbolic.

Hoyles, Sutherland and Healy (1991) have investigated how different computer environments promote differing modes of discussion and generalisation within the context of collaborative interaction. The research contrasted a spreadsheet environment with a Logo programming environment. It was found that in the Logo environment the natural language of the pupils and the software tools developed together in a dialectical manner, and served simultaneously as scaffolding towards generalisation. However, in the spreadsheet environment the influences of inter-pupil and pupil-computer interaction were seen to operate at different points in the generalisation process.

## 2.13 SUMMARY OF LITERATURE REVIEW

Recent research focusing on number pattern generalisation tasks would thus seem to encompass a broad range of topics. Key issues arising from the literature review include the various types of number sequences, patterning strategies, pictorial versus practical contexts, the classification of patterning ability, patterning as a route to algebra, visualisation and visual reasoning, notions of proof and justification, the reduction of pattern generalisation activities to rote pattern spotting exercises, the interplay of verbal and algebraic generalisation, and the role of technology.

Key issues highlighted in the literature will be used to inform both the focus and methodology employed in the present investigation, as well as provide a meaningful backdrop to the processes of analysis and interpretation.

# CHAPTER THREE

# THEORETICAL FRAMEWORK

## 3.1 INTRODUCTION

The purpose of this chapter is to establish a theoretical framework for the epistemological assumptions that will inform and guide the research process. There are three key elements to this contextual backdrop: constructivism; visualisation; and the processes of generalisation, justification and proof.

Firstly, rather than provide a general exposé and critique of the constructivist movement, I will attempt to distil only those key epistemological principles and attitudes of mind that have critical bearing on this study. Secondly, the importance of recognising and valuing different cognitive styles is highlighted. Finally, the interwoven nature of generalisation, justification and proof is explored. Further theoretical elements pertaining more to practical methodological issues are interrogated in Chapter 4.

## 3.2 CONSTRUCTIVISM

An important consideration both in terms of the structuring of the research methodology and the associated epistemological assumptions, is the emphasis of the NCS on investigation as a pedagogical approach[7]. Assessment Standard 10.1.3 of the NCS for Grade 10 asserts that learners should be able to *investigate* number patterns (including, but not limited to, linear patterns) and hence "make conjectures and generalisations" as well as "provide explanations and justifications and attempt to prove conjectures" (Department of Education, 2003b:18). Frobisher (1994:169) describes the essence of an investigative approach as "the application of communication, reasoning, operational and recording processes to a study of

[7] "Competence in mathematical process skills such as investigating, generalising and proving is more important than the acquisition of content knowledge for its own sake" (Department of Education, 2003b:9).

the core topics which make up the content of a mathematics curriculum" (Frobisher, 1994:169). In practise, an investigative approach should be characterised by a "spirit of dynamic engagement" on the part of the investigator (Orton and Frobisher, 1996:32). The core epistemological principles that underpin the investigative approach position it comfortably within a constructivist framework.

Broadly speaking, constructivism is "a philosophical perspective on knowledge and learning" (Jaworski, 1994:14). More specifically, constructivism is a theory about the limits of human knowledge, "a belief that all knowledge is necessarily a product of our own cognitive acts" (Confrey, 1990:108). Thus, we construct understanding through our own experiences. Furthermore, in an unavoidable cyclical nature, the character of those experiences is in turn influenced to a large extent by our own cognitive lenses (Confrey, 1990).

The basic tenet of constructivism, *viz.* the notion that knowledge is the result not of the passive reception of information but rather the product of a learner's activity, is embraced by all constructivists (von Glasersfeld, 1991). Nonetheless, the term "constructivism" has been subject to a diversity of interpretations, many of which are seen as being trivialisations of the underlying theory (Towers and Davis, 2002:313). In addition, there is also a continuum of constructivist theories with different emphasis being placed on the role of the individual and the influence of social processes (Lerman, 1994; Ernest, 1994). Radical constructivism prioritizes the individual aspects of learning while social constructivism views mathematics as a social construction. It has also been argued that there is a major division between two types of social constructivism "according to whether Piagetian or Vygotskian theories of mind and learning are adopted as underlying assumptions" (Ernest, 1994:63).

For the purposes of the present study, I will adopt the constructivist stance advocated by Cobb (2000) that there is a reflexive relation between an individual student's mathematical reasoning and the social context of the classroom microculture. However, as the research methodology focuses on the cognitive reasoning of the individual, of more critical importance is the overarching notion that constructivism is a *descriptive* as opposed to *prescriptive* philosophy (Towers and Davis, 2002:314).

Clements and Battista (1990:34) mention one of the basic tenets of constructivism as being the notion that children "create new mathematical knowledge by reflecting on their physical and mental actions". Of particular import to the present investigation is the role of notation systems and representations as mediators in this constructive process and the notion that these mathematically oriented notations contribute to the organisation of the thinking process. Kaput (1991:55) distinguishes between *mental structures* and *notation systems*. While *mental structures* are a "means by which an individual organizes and manages the flow of experience", *notation systems* are "materially realizable cultural or linguistic artefacts shared by a cultural or language community" (Kaput, 1991:55). Such notations can be either consensual or idiosyncratic. When *materially instantiated* (e.g. physical marks on paper), notation systems are used by individuals to "organise the creation and elaboration of their own mental structures" (Kaput, 1991:56). This is accomplished by structuring physical records of prior mental activity and by structuring both physical and mental actions on those records in a cyclical process (Kaput, 1991).

A further tenet of the constructivist view of teaching and learning is the notion that "no one true reality exists, only individual interpretations of the world" (Clements and Battista, 1990:34). A representational perspective can thus possibly be seen to be inconsistent with a constructivist epistemology. However, Kaput (1991) argues that notation systems as a representational framework for mathematical cognition are indeed consistent with constructivism, and highlights the semantic distinction between "representations" and "re-presentations", the latter being a more meaningful interpretation with respect to the constructivist position.

Confrey (1990) asserts that one of the most essential skills for a constructivist educator to embrace is that of approaching "a foreign or unexpected response with a genuine interest in learning its character, its origins, its story and its implications" (Confrey, 1990:108). Furthermore, attempting to see a situation as perceived by another human being should be imbued "with the assumption that the constructions of others … have integrity and sensibility within another's framework" (Confrey, 1990:108). To a constructivist, the notion of knowledge without belief is therefore contradictory. This has particular import within an interpretive research paradigm.

Thus, the present study firmly embraces the notion of the use of both language and notation systems/representations as being important mediators in the process of construction – both in terms of their contribution to the organisation of the thinking process itself, as well as the cyclical nature of reflection. Furthermore:

> From the constructivist point of view, there can be no doubt that reflective ability is a major source of knowledge on all levels of mathematics.… To verbalize what one is doing ensures that one is examining it. And it is precisely during such examination of mental operating that insufficiencies, contradictions, or irrelevancies are likely to be spotted.
>
> (von Glasersfeld, 1991: xviii)

Within the context of the present study, the primary notation system that will be used as mediator and representational framework in the construction process is algebra. According to Mason et al. (1985:1) "algebra is firstly a language – a way of saying and communicating". In essence, algebra is the mathematical language of generalisation. Schoenfeld and Arcavi (1988) advocate the verbalizing of mathematical generalisations prior to the formalising of such generalisations into mathematical language. By summarising observed patterns verbally, the transition from arithmetic to algebra, and in particular to the concept of variable, may well be assisted. Indeed, Radford (2000), investigating students' emergent algebraic thinking during their very first encounter with the algebraic generalisation of pattern, observes that "…the objectification of the general in natural language proved to be fundamental to the rise of the symbolic formula in that the symbolic formula appeared as contracted or abbreviated speech" (Radford, 2000:261). Furthermore, research by Pugalee (2004) has shown that in mathematical problem solving, not only can the process of writing be a useful tool for supporting a metacognitive framework, but it would seem to be more effective than the use of think-aloud processes.

## 3.3 VISUALISATION

> "Objects are concealed from our view, not so much because they are out of the course of our visual ray as because we do not bring our minds and eyes to bear on them…"
>
> Henry David Thoreau, *Autumnal Tints*

Presmeg (1992:38) makes mention of three aspects which could influence the way in which a pupil performs a mathematical task:

- The attributes of the task itself
- The instructions to do the task in a certain way
- The individual differences in cognitive styles of pupils

The first two of these aspects relate to question design and are discussed further in the methodology chapter. However, the third relates to an important theoretical consideration in terms of the present study, based as it is on generalisation tasks set in two different contextual scenarios - numeric and pictorial.

The issue of individual differences in mathematical processing was highlighted in the late 1970s with the publication of the English translation of Krutetskii's book, *The Psychology of Mathematical Abilities in Schoolchildren*. Krutetskii's research focused on the relative role of the verbal-logical and visual-pictorial components of pupils' mental activity. His research suggested that, at least at school level, the two components are not "*necessary* [emphasis mine] components in the structure of mathematical abilities" (Krutetskii, 1976:315). The importance of Krutetskii's research lies in his observation that although the strength or weakness of the two components does not necessarily determine the extent of mathematical ability, it does however play an important role in determining its type.

> A pupil can be mathematically capable with a different correlation between the visual-pictorial and the verbal-logical components, but the given correlation determines what type he belongs to.
>
> Krutetskii (1976:315)

It is worth making note of Krutetskii's remark that "strictly speaking, the verbal-logical component is well-developed in all mathematically able pupils" (Krutetskii, 1976:316). One can thus argue that while the *level* of mathematical ability is determined largely by the verbal-logical component of cognition, the *type* of mathematical giftedness is determined largely by the visual-pictorial component. In the case of the latter component, as Presmeg (1986b:300) highlights, it is not only the ability to use it which determines the type of mathematical giftedness of an individual, but rather the preference for its use.

Krutetskii made use of the relative role of the verbal-logical and visual-pictorial components of pupils' mental activity to categorise mathematics pupils into four groups:

- Analytic – pupils who operate easily with abstract relationships and have no need for visual supports in problem-solving.
- Geometric – pupils who find it necessary to give visual expression to abstract mathematical relationships.
- Abstract-harmonic – pupils who have equally well developed verbal-logical and visual-pictorial components but who are disinclined to use visual supports.
- Pictorial-harmonic – pupils who have equally well developed verbal-logical and visual-pictorial components but who find the use of visual supports helpful.

Visualisation is recognised as being a central component in mathematical activity (Cunningham, 1991; Hershkowitz et al., 2001; Arcavi, 2003). Furthermore:

> Visualization, as both the product and the process of creation, interpretation and reflection upon pictures and images, is gaining increased visibility in mathematics and mathematics education.
>
> (Arcavi, 2003:215)

It has even been suggested that visual thinking may well become "…the primary way of thinking in the future" (Hershkowitz and Markovits, 1992:38).

It has been argued (Thornton, n.d.) that there are at least three reasons to re-evaluate the role of visual thinking in school mathematics. The first is the popular trend to identify mathematics with the study of patterns and the potential danger in the use of technology devaluing algebraic thinking. This concern is echoed by numerous mathematics educationalists (Hewitt, 1992; Byatt, 1994; Noss et al., 1997; de Jager, 1999). The second relates to the importance of establishing connections between different areas of mathematics. The third, and perhaps the most pertinent in terms of the present study, draws on the importance of recognising and valuing different cognitive styles.

Presmeg (1986a, 1992) has identified five different kinds of visual imagery in an attempt to "operationalize" (van Garderen and Montague, 2003:246) such visual imagery. The five types of imagery included in Presmeg's taxonomy (1986a, 1992) can be summarised as follows:

- Concrete, pictorial imagery or mental images
- Pattern imagery showing pure relationships depicted in a visual-spatial scheme
- Memory images of formulae, involving the visual recall of formulae
- Kinaesthetic imagery involving movement and gestures
- Dynamic imagery, involving dynamic transformations of geometric figures

Although Presmeg (1986a) acknowledged that all imagery types have the potential to play a functional role in mathematical problem solving, she considered pattern imagery as being the most essential type, as it identifies the relational aspects of a problem and is thus arguably better suited to abstraction and generalisation. As Thornton points out, the development of such mathematical imagery which focuses on relationships and patterns "is surely one of the principal goals of mathematics education" (Thornton, n.d.:254). Hegarty and Kozhevnikov (1999) distinguish between two different types of visual-spatial representations: *schematic* imagery which focuses on the spatial relationships between objects, and *pictorial* imagery where the focus lies with the visual appearance of the objects themselves. Hegarty and Kozhevnikov (1999) found that the use of schematic representations was positively related to success in mathematical problem solving, whereas use of pictorial representations was negatively correlated with success. This echoes Presmeg's (1986a) ascription of pattern imagery, in which the concrete details are disregarded in favour of pure relationships, as the most essential role in mathematical problem solving.

Kirby and Kosslyn (1992) suggest that inasmuch as image representations are depictive, imagery can be exploited to aid problem-solving. This stems from the notion that, unlike propositional representations, spatial relations in imagery are an emergent property of the depicted perceptual units. Nonetheless, in terms of mathematical problem solving, the fact that the human mind tends to perceive a given visual stimulus as a whole is not without its drawbacks. For a given visual configuration, a specific structure or order is imposed on the mind which may obscure crucial aspects and elements of the problem. In order to overcome this imposed structure "a reorganization of the elements is needed, which then hopefully enables the individual to comprehend how the elements fit together, thus achieving what Gestaltists termed *structural understanding*" (Orton, 2004:78). Far more than merely a theory of form perception, Gestalt theory is pre-eminently a theory of behaviour. However, some of the most interesting phenomena of visual perception were discovered by the Gestalt psychologists, and the founding of the Gestalt movement in the early 20th century is considered one of the most important events in the history of perception (Palmer, 1992:39-40).

The literature concerning Gestalt laws/principles is substantial (see for example Wertheimer, 1938; Katz, 1951; Zusne, 1970:111-135). Helson (1933) was able to distil and articulate 114 such "laws" or propositions. Although only a few of these mention visual form specifically (e.g. the laws of similarity, proximity, symmetry, good continuation, and closed forms), the majority are applicable to visual perception (Zusne, 1970:111). It is important to note that the various Gestalt laws are by no means independent of one another. A number of configurational forces may be in operation at the same time, often in conflict with one another. Thus, although pattern generalisation problems presented in a pictorial or practical context have the potential to widen the scope of solution strategies for some pupils, for others this may well create additional complications. It is only if the underlying structure is perceived in a meaningful or useful way, dependent on the nature of the pictorial scenario, that a visually guided/mediated solution may be more readily accessible. Furthermore, being able to translate both flexibly and competently between the visual and analytic representations of the same scenario is a necessary requirement for visualisation to be cognitively meaningful (Arcavi, 2003).

Fischbein claims that visualisation “not only organizes data at hand in meaningful structures, but … is also an important factor guiding the analytical development of a solution” (Fischbein, as quoted in Hershkowitz et al., 2001:262). Building on from Fischbein is the suggestion by Hershkowitz et al. that visualisation can in some instances take on the role of “the analytical process itself which concludes with a general formal solution” (2001:262). Such analytical components may include:

- Decomposition of a structure into substructures and/or units
- Creation of auxiliary constructions
- Transformation of the original structure into other structures
- Re-composition and synthesis

Cunningham (1991:70) comments that visualisation within the realm of mathematics education not only promotes intuition and understanding, but also allows students to “learn new ways to think about and do their *own* [emphasis mine] mathematics”. There is thus an important connection between visualisation and the constructivist view of teaching and learning.

The role of visualisation is central to the present investigation. From a theoretical perspective, the methodology employed in the data capturing process needs to be sensitive to the relative roles of the verbal-logical and visual-pictorial components of a pupil’s cognitive processes. In addition, both the data capturing and data analysis methodologies should take cognizance of the role of visualisation in the generalisation process. For the purposes of this study, visualisation is understood to incorporate the process of forming images (either mentally or by means of physical instantiation) and using such images to aid mathematical discovery and understanding (Zimmermann and Cunningham, 1991). Furthermore, pattern imagery (Presmeg, 1986a, 1992), which identifies the relational aspects of a problem or scenario, as an emergent property of the depicted perceptual units, is considered the most essential type of imagery for the purposes of abstraction and generalisation.

One of the key questions raised in the intensified study of visualisation is to what extent visual representations can be used in the justification, as opposed to a mere role of evidential support, of a mathematical statement (Hanna, 2000; Brown, 1997). This issue will be explored further in the following section.

## 3.4 GENERALISATION, JUSTIFICATION AND PROOF

The National Curriculum Statement (NCS) for South African schools (Department of Education, 2003b) places fundamental importance on the process skills of generalisation, justification and proof. Furthermore, competence in such skills is regarded as being central to the underlying tenets of the revised South African school Mathematics curriculum.

> Competence in mathematical process skills such as investigating, generalising and proving is more important than the acquisition of content knowledge for its own sake.
>
> (Department of Education, 2003b:9)

As one of the fundamental outcomes of LO1 within the FET band (Grades 10-12) of the NCS, learners will "explore real-life and purely mathematical number patterns and problems which develop the ability to generalise, justify and prove" (Department of Education, 2003b:12).

The concepts of generalisation, justification and proof as mathematical skills are developed over time within the framework of the NCS, with ever increasing complexity and sophistication. In Grade 7 the expectation is for learners to be able to describe, explain and justify observed relationships or rules in their own words. By Grade 8 such explanations and justifications should also be attempted algebraically, while in Grade 9 the algebraic component becomes far more central. By Grade 10, with the commencement of the FET band, learners are expected not only to make generalisations and conjectures, but to be able to explain and justify their generalisations, and attempt to prove their conjectures (Department of Education, 2002, 2003a, 2003b).

It has been observed that mathematical power lies not only in being able to detect, construct, invent, understand and manipulate patterns, but "in being able to communicate these patterns to others" (Goldin, 2002:213). Statements of generality, along with the discovery and investigation of generality, "…are at the very core of mathematical activity" (Lannin, 2005:233). According to Dörfler (as quoted in Zazkis and Liljedahl, 2002:381), generalisation is not only an end in itself, but "…a means of thinking and communicating". Kaput (1999) makes the observation that "generalisation and formalisation are intrinsic to mathematical activity and thinking – they are what make it mathematical" (Kaput, 1999:136). Thus, "generalization as a didactic device cannot avoid the problem of validity" (Radford, as quoted in Lannin, 2005:235). Indeed, generalisation, by its very nature, can not be separated from justification and proof. Hanna (2000:7) comments further that mathematical proofs are at the core primarily conceptual, rather than mere syntactic derivations. Proofs are the "mathematician's way to *display the mathematical machinery* for solving problems and to *justify* that a proposed solution to a problem is indeed a solution" (Rav, 1999:13). Thus, the notions of generalisation, justification and proof are intricately interwoven.

The notion of proof is used with various shades of meaning by mathematicians and mathematics educators. According to Bell (1976), the concept of proof carries three senses within the realm of mathematical meaning. The first is *verification* or *justification*, focusing on the truth of a proposition. The second is *illumination*, in the sense that a proof should convey insight into *why* a proposition is true. The third, the most "characteristically mathematical" (Bell, 1976:24), is that of systematisation – the organisation of results into a deductive system of axioms, major concepts and theorems. Building on from Bell's (1976) distinction between the various functions of proof, de Villiers (1990) outlined the following useful model:

- *Verification* (concerned with the truth of a statement)
- *Explanation* (providing insight into why it is true)
- *Systematisation* (the organisation of various results into a deductive system)
- *Discovery* (the discovery or invention of new results)
- *Communication* (the transmission of mathematical knowledge)

Hanna and Jahnke (1996) further add an additional three functions to the model of de Villiers (1990):

- *Construction* of an empirical theory
- *Exploration* of the meaning of a definition or the consequences of an assumption
- *Incorporation* of a well-known fact into a new framework and thus viewing it from a fresh perspective.

Hanna (2000:8) comments that such a richly diverse view of the concept of mathematical proof could only be the result of a long historical development. As a result of the differing and constantly developing views on the nature and role of proof, translating such a concept into the classroom is far from simple (Hanna and Jahnke, 1996). Nonetheless, "proof is an essential characteristic of mathematics and as such should be a key component in mathematics education" (Hanna and Jahnke, 1996:877).

One could argue that the nature and role of proof and/or justification depends largely on the level of mathematics under consideration (Waring, 2001). With that in mind, Porteous' (1994:5) definition that "a proof of a statement is any adequate expression of the necessity of its truth" is perhaps somewhat more meaningful to the context of the school mathematics classroom. Porteous (1994:5) goes on to specify that in order to count as *adequate*:

- true and accurate statements must be made about the context
- the full domain of applicability of the statement must be addressed – i.e. a general statement requires a general treatment
- the description of why the statement is true must reflect an awareness of the logical necessity of that truth
- the individual, having given the proof, must by that account consider the statement in question to be true

Slomson (1996:11) summarises the situation by succinctly stating that "a mathematical proof is a correct and convincing mathematical argument", while noting that what counts as convincing has changed over the course of time, and will necessarily vary from person to person. Simply put, a proof is nothing more than a "…convincing argument, as judged by competent judges" (Hersh, 1993:389).

It has been argued that "the most significant potential contribution of proof in mathematics education is the communication of mathematical understanding" (Hanna and Jahnke, 1996:878). This resonates with Hersh's (1993) distinction between the different roles of proof in differing contexts. While mathematical proof and justification can both convince and explain, "in mathematical research, its primary role is convincing [whereas] at the high-school or undergraduate level, its primary role is explaining" (Hersh, 1993:398). Expressed differently, within the realm of the classroom, "…the key role of proof is the promotion of mathematical understanding" (Hanna, 2000:5).

Simpson (as cited in Hanna, 2000:9) differentiates between *proof through logic* and *proof through reasoning*, the latter focusing on investigations and heuristic argument. There is thus a view that heuristic techniques are appropriate to developing skills in both reasoning and justification. Formalism and rigour are not necessarily essential components to the process of justification. Indeed, the idiosyncratic and informal reflection of a genuine generalisation should not be seen to diminish the validity of such a justification. As Hersh (1993:391) remarks, "all real-life proofs are to some degree informal". Thus, for the teacher the only significant question when assessing a learner's proof or justification is whether or not the *public performance* which is the proof reflects an adequate *private perception* which generated it (Porteous, 1994:4).

There is a general prevailing acceptance that pictures, although pedagogically important, are nonetheless essentially heuristic devices (Brown, 1997). However, there is some support for the notion that pictures have "…a legitimate role to play as evidence and justification, well beyond a heuristic role" (Brown, 1997:161). As Barwise and Etchemendy (1991:9) claim, "visual forms of representation can be important, not just as heuristic and pedagogic tools, but as legitimate elements of mathematical proofs". The choice of a pictorial context for the generalisation activities in the present study thus provides a referential context for the use of a *generic example*, what Lannin (2005:236) describes as "a particular example that embodies the general characteristics of an argument", serving as a means of justification. The use of the *generic example* will of course only be seen as a legitimate justification for someone who is able to perceive the general nature of the example (Mason and Pimm, 1984).

"Generalization … cannot avoid the problem of validity" (Radford, as quoted in Lannin, 2005:235). Thus, generalisation, by its very nature, can not be separated from justification, and justification should be seen as a critical component of the generalisation process. In terms of the justification of an algebraic model, an argument is deemed acceptable when it connects the algebraic generalisation to a general relation inherent in the original problem context. When the context is provided as a pictorial scheme, the generalisation and associated justification is often based on a visual conceptualisation of the scenario (Lannin, 2005). This type of justification is valued "…because it explains rather than simply convinces" (Lannin, 2005:235).

The types of generalisation activities included in this study purposefully include those presented in pictorial contexts, thus allowing for a possible connection to a referential context that has the potential to aid and enhance the generalisation process. Researchers (Stacey, 1989; Healy and Hoyles, 1999; Goulding, Suggate and Crann, 2000; Hershkowitz et al., 2001; Lannin, 2004) have demonstrated that such activities and contexts encourage the construction of a variety of generalisations. In addition, Friedlander and Hershkowitz (1997:446-447) put forward their belief that problem situations based on the generalisation and justification of patterns help promote an appreciation for "…mathematical reasoning and an understanding of the nature of mathematical proofs".

From the various functions of proof, as outlined by Bell (1976), de Villiers (1990), and Hanna and Jahnke (1996), those seen to be critical in terms of this study are *explanation* and *communication*. Thus, from a theoretical perspective, the central role of proof within the context of this study is seen as communication of mathematical understanding, and students' justifications of their generalisations are seen to provide "…a window to view their understanding of the general nature of their rules" (Lannin, 2005:251). Finally, while mathematical proofs are perhaps "the ultimate in justifications" (Sowder and Harel, 1998:670), proof within the context of the present study will be used in the somewhat broader sense of *justification*.

## 3.5 SUMMARY OF THEORETICAL FRAMEWORK

While embracing the basic tenets of constructivism, central to this investigation is the fundamental notion that constructivism is a *descriptive* as opposed to *prescriptive* philosophy. Built onto this philosophy is the firm belief in the use of both language and notation systems/representations as important mediators in the process of construction – both in terms of their contribution to the organisation of the thinking process itself, as well as the cyclical nature of reflection.

The role of visualisation is central to the present investigation, and it is acknowledged that while generalisation problems presented in a pictorial or practical context have the potential to widen the scope of solution strategies for some pupils, for others this may well create additional complications. Thus, from a theoretical perspective, the methodology employed in the data capturing process needs to be sensitive to the relative roles of the verbal-logical and visual-pictorial components of a pupil's cognitive processes. In addition, both the data capturing and data analysis methodologies should take cognizance of the role of visualisation in the generalisation process.

The notions of generalisation, justification and proof are intricately interwoven. Generalisation, by its very nature, can not be separated from justification/proof, and justification is seen as a critical component of the generalisation process. The types of generalisation activities included in this study purposefully include those presented in pictorial contexts, thus allowing for a possible connection to a referential context that has the potential to aid and enhance the generalisation process. The central role of proof within the context of this study is seen as communication of mathematical understanding, and students' justifications of their generalisations are seen to provide "…a window to view their understanding of the general nature of their rules" (Lannin, 2005:251).

Practical issues of how this theoretical framework is reflected in the methodology of the present research are addressed in the following chapter.

# CHAPTER FOUR

# METHODOLOGY

## 4.1 THEORETICAL CONTEXT

This study is based on a qualitative investigation framed within an interpretive paradigm. According to Cohen and Manion (1994:36), the central endeavour within the context of the interpretive paradigm is "to understand the subjective world of human experience". In an effort to retain the integrity of the phenomenon under investigation, efforts must be made to "get inside" the research subject in order to "understand from within". Furthermore, attempting to see a situation as perceived by another human being should be imbued "with the assumption that the constructions of others … have integrity and sensibility within another's framework" (Confrey, 1990:108). This has particular import within an interpretive research paradigm. Thus, the essential character underpinning the data analysis of the present study is the treatment of all responses, particularly those that are unexpected or idiosyncratic, with a genuine interest in understanding their character and origins.

The epistemological assumptions which provide a theoretical backdrop to this qualitative and interpretive study have been oriented within a constructivist perspective. The choice of the constructivist perspective is based on the emphasis of the NCS on investigation as a pedagogical approach to pattern generalisation tasks. The core epistemological principles that underpin the investigative approach position it comfortably within a constructivist framework. Of particular relevance is the overarching notion that constructivism is a *descriptive* as opposed to *prescriptive* philosophy (Towers and Davis, 2002:314). For the purposes of this study, the constructivist stance as advocated by Cobb (2000) has been adopted, in that there is a reflexive relation between an individual student's mathematical reasoning and the social context of the classroom microculture. However, while acknowledging the reflexive relation between the individual and the social context, within the methodology of this study the focus will fall primarily on the mathematical reasoning of the individual. Of particular import is the role of notation systems and representations as mediators of the constructive process and the notion that these mathematically oriented notations contribute to the organisation of the thinking process (Kaput, 1991). In addition, central to the theoretical orientation that underpins the methodology and task design is the notion that mathematical meanings are developed and strengthened by forging links between alternative expressions of the same mathematical concept, and that successful engagement with the subject depends on the capacity to move flexibly and meaningfully between these different forms of representation.

Of central importance in terms of the methodology of the present study is that the focus here lies on the analysis of the solution strategies themselves, their classification and their cognitive and pedagogical implications. While for some researchers (Noss et al., 1997; Healy and Hoyles, 1999; Radford, 2000; Lannin, 2005) the focus of similar pattern generalisation investigations lies in the actual process of construction of these solutions, this is of much lesser importance in the present study. Thus, the methodology employed here seeks accurately to summarise, analyse and classify the solution strategies themselves without providing detailed protocols of the actual process of construction. A review of recent research into patterning activities served to inform the various classification frameworks.

## 4.2 PILOT STUDY

A small-scale pilot study was conducted prior to commencement of the formal data collection. The purpose of this pilot study was to assess the clarity of the instructions on the mathematical processing response sheets as well as space and time issues relating to written participant responses. Three Grade 9 learners and one Grade 10 learner took part in the pilot study. The four pupils chosen were of mixed mathematical ability. The three questions chosen for the pilot study included purely numeric terms (presented as both a simple sequence of numbers and in tabular form) as well as a pictorial pattern, presented using three consecutive terms. The four learners involved in this pilot study did not take part in the main study. Insights gleaned from the pilot study are addressed elsewhere in this chapter.

## 4.3 THE CASE STUDY AS METHODOLOGICAL STRATEGY

The case study is not a methodological choice *per se*, but rather a choice of the specific object to be studied (Stake, 1994). The small-scale pilot study suggested that one would gain “insight into the [research] question by studying a particular case” (Stake, 1995:3), in this instance a class of high ability learners. Stake (1994, 1995) refers to this type of enquiry as an *instrumental* case study, as opposed to two other broad types of case study which he identifies - *intrinsic* and *collective*. In an *instrumental* case study, the choice of case is made on the basis that it is expected to advance the understanding of the issue under investigation. Although the emphasis of a case study is to optimise understanding of the specific case under scrutiny rather than generalisation beyond that case, a case study can nonetheless be a useful small step towards a larger generalisation, or an increasingly refined generalisation (Cohen and Manion, 1994; Stake, 1994, 1995). The choice of participants for this study was thus guided by the chosen case study methodological strategy.

## 4.4 PARTICIPANTS

The members of a mixed gender, high ability Grade 9 class of 24 learners at an independent school in Grahamstown were chosen as research participants for this study. This purposeful sampling can be justified as follows. Firstly, the purpose of purposeful sampling is “to select information-rich cases whose study will illuminate the questions under study” (Patton, 1990:169). Since the data collection process requires learners to attempt to articulate their own cognitive processes, a high ability group of learners was thought to be more suited to this methodology. The previously mentioned small-scale pilot study seemed to confirm this. In the same pilot study it also became apparent that high ability learners were more likely to progress further in the type of pattern generalisation tasks under investigation and thus more likely to constitute “information-rich cases” (Patton, 1990:169).

Secondly, Grade 9 represents the final year of the General Education and Training (GET) band. Choice of Grade 9 participants thus ensures that responses to tasks are not based on a section of work formally taught at the start of the FET band (Grade 10) but rather the accumulated experiences of prior learning.

## 4.5 ETHICS

The issue of ethics is recognised as playing an important role in any research investigation in the social sciences (Cohen and Manion, 1994:347). Firstly, formal permission was obtained from the headmaster of the school in question for permission to conduct the research. Anonymity of both the school as well as the research participants was assured. Secondly, only those learners who agreed to participate in the study through voluntary informed consent formed part of the research sample. Participants also had the freedom to withdraw from the study at any stage. Participant anonymity has been assured at all times by the use of appropriate pseudonyms when referring to the research participants.

## 4.6 DATA GENERATION

### 4.6.1 MATHEMATICAL PROCESSING RESPONSE SHEETS (MPRS)

Data was generated from a research instrument that used a series of pencil and paper exercises based on linear generalisation tasks set in both numeric and 2-dimensional pictorial contexts. Design considerations for mathematical processing response sheets centred round the key elements of what Frobisher (1994) describes as the essence of an investigative approach - communication, reasoning, operational processes and recording – in an attempt to invoke a sense of dynamic engagement on the part of the investigator (Orton and Frobisher, 1996).

Various pattern generalisation tasks were drawn from the relevant literature, e.g. Stacey (1989), Orton (1997), Healy and Hoyles (1999) and Waring et al. (1999), and adapted to the needs of this study. The extent to which question design affects the solution strategies adopted by pupils was investigated in both numeric and pictorial contexts. For each pattern, participants were required to provide numerical values for the next, $10^{th}$ and $50^{th}$ terms as well as a written articulation of their reasoning at each stage. Participants were also asked to provide an algebraic expression for the $n^{th}$ term as well as to justify their expression. Each task was thus constructed so that participants worked sequentially from relatively small terms to larger ones. It was hoped that this would encourage participants to closely examine the general relations in the problem context, particularly in the case of the "far generalisation" tasks where a recursive strategy would have been impractical.

The use of notation systems within this process was viewed as "contributing to the organization of that person's thinking processes" (Kaput, 1991:54). Sufficient space was thus provided on the mathematical processing response sheets for participants to write both their solution as well as a written articulation of their thought process[8]. This written articulation served a three-fold purpose. Firstly, it was a necessary requirement in order to accurately categorise the adopted solution strategies. Secondly, it was hoped that the material instantiation of participants' cognitive reasoning would scaffold the cyclical process of reflection (Kaput, 1991). Thirdly, a verbal reasoning in natural language, as opposed to mathematical abstraction, was hoped to assist en route to the symbolic formulation of the general term.

Numeric patterns were presented as a simple sequence of numbers (Figure 4.1) as well as tabular form (Figure 4.2). Pictorial patterns were presented using three consecutive terms (Figure 4.3), two non-consecutive terms (Figure 4.4) or one single term[9] (Figure 4.5).

4 ; 7 ; 10 ; 13 ; . . .

**Figure 4.1** Number sequence

| Position | 1 | 2 | 3 | 4 | . . . |
|---|---|---|---|---|---|
| Number | 3 | 5 | 7 | 9 | . . . |

**Figure 4.2** Tabular form

[8] The issue of sufficient space was taken into consideration in the pilot study.

[9] The use of single terms was restricted to cases where a single pictorial term provides an unambiguous explanation of the underlying structure.

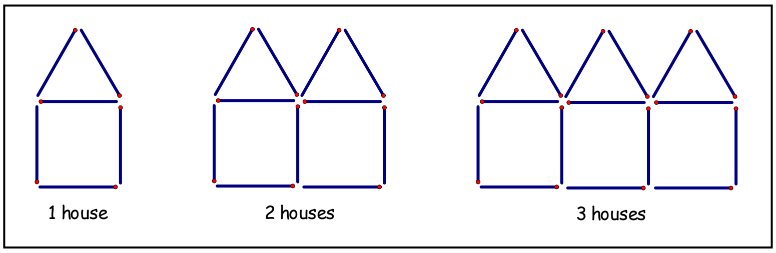


**Figure 4.3** Three consecutive terms

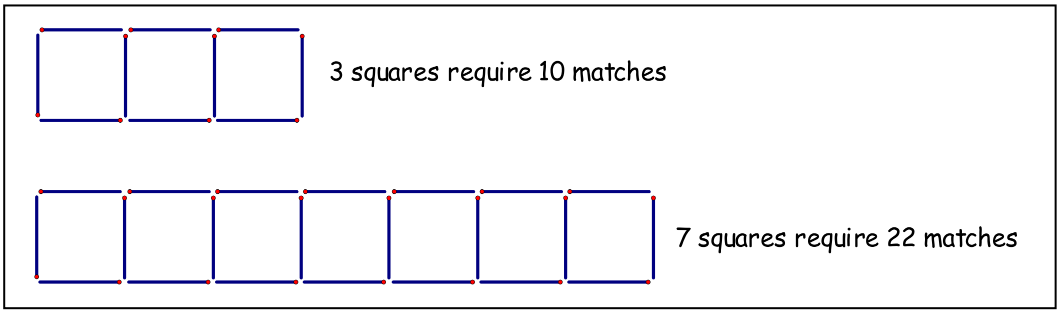


**Figure 4.4** Two non-consecutive terms

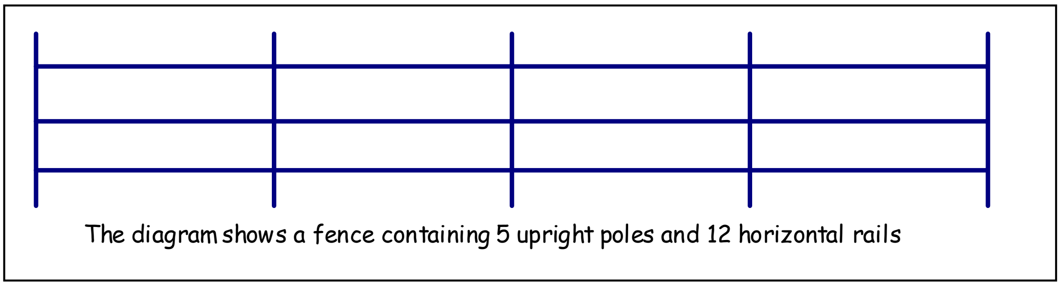


**Figure 4.5** A single pictorial term

In order to compare numeric tasks with pictorial tasks, each numeric pattern has an isomorphic[10] pictorial counterpart. The choice of a pictorial context for the generalisation activities is intended to provide a referential context for the use of a *generic example* in the justification of the general term.

The literature review undertaken to inform this research suggested that linear sequences would be most appropriate in terms of eliciting rich data at all levels of the pattern generalisation process. Accordingly, 22 linear/arithmetic sequences of the type $ax \pm c$ $(c \neq 0)$ were chosen. The choice of sequences with non-zero constant terms was a purposeful attempt at ensuring that choice of an

10 Isomorphic patterns are based on the same general formula. By way of example, the pictorial pattern shown in Figure 4.4 would be isomorphic with the numerical pattern 4 ; 7 ; 10 ; 13 ; . . . since they both have the general formula $T_n = 3n + 1$.

inappropriate strategy (e.g. difference product or whole object) didn't produce a spurious yet numerically correct answer. The 22 sequences were split between pictorial and non-pictorial contexts. The choice of the actual sequences as well as the various pictorial contexts was an attempt to provide sufficient variety for the differing verbal-logical and visual-pictorial reasoning skills likely to be found in the group of 24 Grade 9 pupils.

Responses to the pattern generalisation tasks were recorded on mathematical processing response sheets over a period of three months. On average, one 42 minute school period was set aside approximately every 10 days for the duration of the study. Three pattern generalisation tasks were asked per session, thus allowing for approximately 14 minutes per pattern[11].

In a qualitative study, "research design should be a reflexive process operating through every stage of a project" (Hammersley and Atkinson, 1983 as cited in Maxwell, 1996:2). Thus, through a reflexive process of on-going evaluation, the specific pattern generalisation tasks, as well as the actual question design, remained open to modification and/or development during the course of the study. However, it did not prove necessary to make any alterations to the original decisions.

The mathematical processing response sheets used in the investigation are included as Appendix A. The 22 number patterns (linear sequences) chosen are summarised hereunder, grouped into the following five categories:

- Pictorial scenarios using a single term
- Pictorial sequences with two non-consecutive terms
- Pictorial sequences with three consecutive terms
- Simple numeric sequences
- Numeric sequences in tabular form

For each pattern, participants were required to provide numerical values for the next, $10^{th}$ and $50^{th}$ terms as well as a written articulation of their reasoning at each stage. Participants were also asked to provide an algebraic expression for the $n^{th}$ term as well as to justify their expression.

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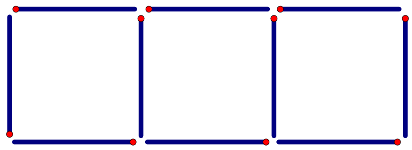
[11] The pilot study suggested that this was sufficient time for most pupils.

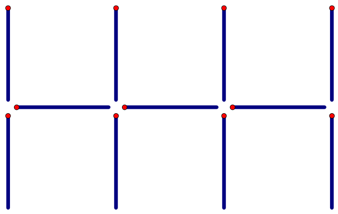
- **PICTORIAL SCENARIOS USING A SINGLE TERM**

**Question 1**



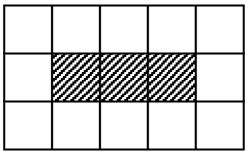
3 squares require 10 matches

**Question 2**



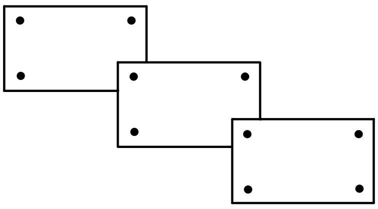
A pattern with 3 **horizontal** matchsticks requires a **total** of 11 matchsticks

**Question 3**



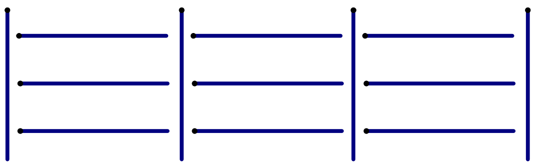
For a row of 3 striped tiles there are 12 white tiles in the border.

**Question 4**



For 3 photos you need 10 drawing pins

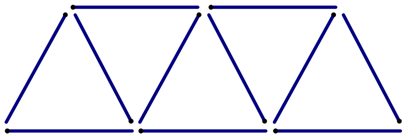
**Question 5**



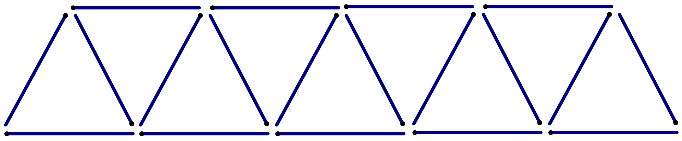
If there are 4 **vertical** matchsticks you need a **total** of 13 matchsticks.

- **PICTORIAL SEQUENCES WITH TWO NON-CONSECUTIVE TERMS**

## Question 6

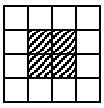


For 3 base matches you need a total of 11 matches

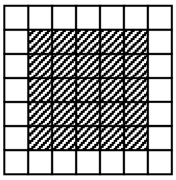


For 5 base matches you need a total of 19 matches.

## Question 7

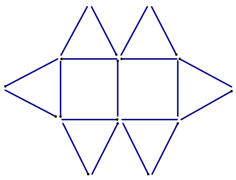


For a 2x2 square of striped tiles, 12 white tiles are needed.

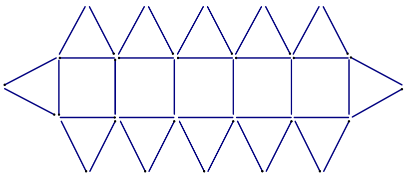


For a 5x5 square of striped tiles, 24 white tiles are needed.

## Question 8

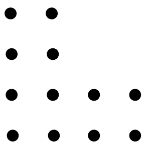


For 2 squares you need a total of 19 matches.

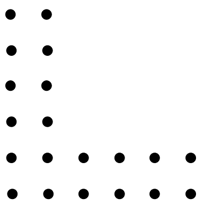


For 5 squares you need a total of 40 matches.

## Question 9

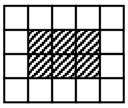


Base is 4 dots long

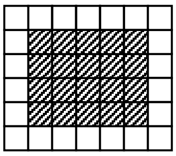


Base is 6 dots long

## Question 10



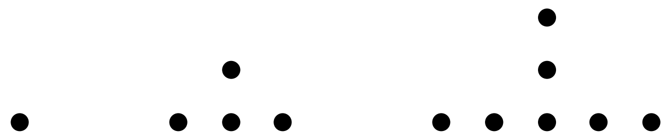
For a 2x3 square of striped tiles, 14 white tiles are needed.



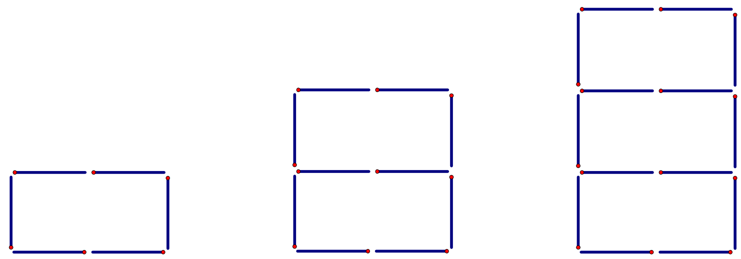
For a 4x5 square of striped tiles, 22 white tiles are needed.

- **PICTORIAL SEQUENCES WITH THREE CONSECUTIVE TERMS**

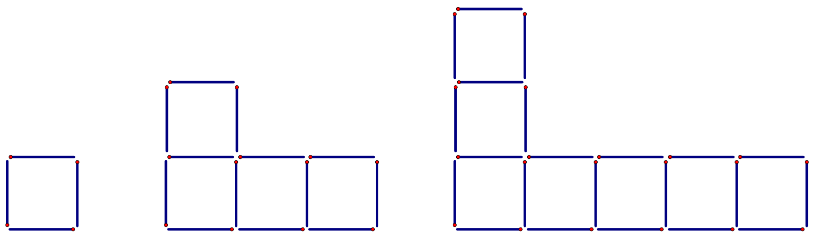
### Question 11



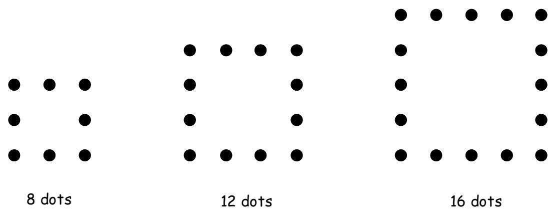
### Question 12



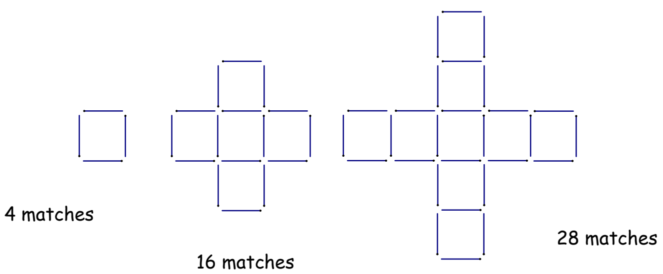
### Question 13



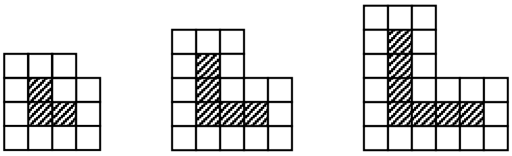
### Question 14




### Question 15



**Question 16**



12 white tiles    16 white tiles    20 white tiles

- **SIMPLE NUMERIC SEQUENCES**

**Question 17**

8 ; 12 ; 16 ; ...

**Question 18**

12 ; 19 ; 26 ; ...

**Question 19**

3 ; 7 ; 11 ; ...

- **NUMERIC SEQUENCES IN TABULAR FORM**

**Question 20**

| 1st | 2nd | 3rd | ... |
|---|---|---|---|
| 4 | 13 | 22 | ... |

**Question 21**

| 1st | 2nd | 3rd | ... |
|---|---|---|---|
| 4 | 7 | 10 | ... |

**Question 22**

| $1^{st}$ | $2^{nd}$ | $3^{rd}$ | … |
|---|---|---|---|
| 4 | 16 | 28 | … |

### 4.6.2 INFORMAL INTERVIEWS

Individual participants were informally interviewed where the written articulation of their mental reasoning was either ambiguous or required illumination by oral explanation. The informal interview (Patton, 2002:342), otherwise known as the unstructured interview (Cohen and Manion, 1994:273), is an open-ended approach to interviewing in which questions flow from the immediate context.

During these interviews participants were given time to read their own written explanations and then asked to provide an oral expansion or explanation of their mental reasoning. These interviews took place as soon as possible after the completed mathematical processing response sheets had been analysed, and in most instances this was one day after completion of the MPRS.

The purpose of these interviews was to provide research participants with the opportunity to further explain or expand on the written articulation of their mental processing. This process of member checking constitutes a form of external validation (Lewis and Ritchie, 2003:276). Since the main outcome of these interviews was to allow for an accurate categorisation of the adopted solution strategy, only field notes were taken. In addition, interviews were restricted to those specific cases where a participant's written articulation of their mental reasoning was either ambiguous or required illumination by oral explanation.

## 4.7 DATA ANALYSIS

The essential character underpinning the data analysis stage was the treatment of all responses, particularly those that were unexpected or idiosyncratic, with a genuine interest in understanding their character and origins. Confrey (1990) asserts that these are some of the most essential skills for a constructivist educator to embrace. Furthermore, the treatment of responses was also imbued with the firm belief that “the constructions of others … have integrity and sensibility within another’s framework” (Confrey, 1990:108).

### 4.7.1 STAGE CLASSIFICATION

For each of the 22 questions, participants were asked to provide numeric values for the next, 10$^{th}$ and 50$^{th}$ terms, as well as an algebraic representation for the n$^{th}$ term. Using the nomenclature of Stacey (1998:150), the 10$^{th}$ and 50$^{th}$ terms represent “near generalization” and “far generalization” tasks respectively. Determining the 10$^{th}$ term thus represents a task which can be accomplished by means of step-by-step counting or drawing, while determining the 50$^{th}$ term represents a task which goes beyond reasonable practical limits of such a step-by-step approach. The n$^{th}$ term denotes an algebraic generalisation of the pattern.

#### 4.7.1.1 Stage descriptors & modifiers

The responses to the various linear generalisation questions were classified by means of stage descriptors as well as stage modifiers. A similar model was used to that employed by Orton and Orton (1996; 1999). The various stage descriptors can be summarised as follows:

- Stage 0: no progress
- Stage 1: next term correctly provided
- Stage 2: next and 10$^{th}$ terms correctly provided
- Stage 3: next, 10$^{th}$ and 50$^{th}$ terms correctly provided
- Stage 4: next, 10$^{th}$, 50$^{th}$ and n$^{th}$ terms correctively provided

The above scheme is not intended as a hierarchical classification system, but rather as a qualitative framework for analysis. Thus, since it is possible for a pupil to correctly determine the 50$^{th}$ term despite having incorrectly determined the 10$^{th}$ term (for example), stage modifiers were used to cover all possibilities. The letters *t*, *x* and *y* were used to indicate errors in the next, 10$^{th}$ or 50$^{th}$ terms respectively.

These modifiers allow for the classification of a pupil who made an error in a previous stage but was nonetheless able to progress correctly to a higher stage. By way of example, a pupil who gave a correct 50$^{th}$ term but an incorrect 10$^{th}$ term would be classified at Stage 3*x;* the 3 indicating a correct calculation of the 50$^{th}$ term (Stage 3) and the *x* indicating an error in the calculation of the 10$^{th}$ term. The absence of a stage modifier (*t*, *x*, *y*) thus implies that all preceding stages were successfully reached. The use of both stage descriptors as well as stage modifiers allowed for both a quantitative as well as qualitative description of the level of attainment of each participant for each pattern generalisation task.

#### 4.7.1.2 Careless errors

Careless numerical slips were ignored provided there was sufficient preceding written evidence to indicate that (i) the slip was in no way intentional and (ii) the absence of the slip would have resulted in the correct answer. By way of example, consider Greg's calculation of the 50$^{th}$ term of Question 6:

$$\begin{aligned} No\ of\ matches &= (base\ matches \times 4) - 1 \\ &= (50 \times 4) - 1 \\ &= 200 - 1 \\ &= 190 \end{aligned}$$

Apart from the careless numerical slip in the final line, both Greg's reasoning and mathematics are faultless. The slip is thus ignored and a final answer of 199 is assumed.

Similarly, careless slips in the algebraic representation of the n$^{th}$ term were also ignored provided there was sufficient written evidence to justify such a decision. By way of example, James's expression for the n$^{th}$ term of Question 12 was "$4n+1$". However, for the calculation of the next, 10$^{th}$ and 50$^{th}$ terms, James made express use of the formula $4n+2$. Furthermore, James justifies his formula for the n$^{th}$ term as follows: "*To get one extra level you are adding four, thus multiplying by 4. In the first shape there were two matches on the bottom, thus adding 2*". In this case there is sufficient evidence to suggest a numerical slip in James's written expression for the n$^{th}$ term, and the correct formula of $4n+2$ is assumed.

Within the question response analysis sheets, the letter "s" has been used to indicate all cases were a slip has been identified and the correct answer assumed.

#### 4.7.1.3 Criteria for Stage 4 classification

In order to be classified at Stage 4, a pupil needed to provide a correct algebraic representation of the general ($n^{th}$) term. There was no necessity for the $n^{th}$ term to be expressed in simplified algebraic form, and all algebraic expressions were accepted provided they were algebraically equivalent to the correct simplified form[12]. By way of example, the general formula for Question 13 is $9n-5$. Each of the following expressions is algebraically equivalent to $9n-5$ and would thus be accepted as a correct response in order to be classified at Stage 4:

- $3(2n+(n-2))+1$
- $4n+5(n-1)$
- $3(3n-2)+1$
- $3[3(n-1)+1]+1$
- $4+9(n-1)$

Subdivision of Stage 4 (as per Orton and Orton, 1996) was not deemed necessary for this high-ability group of pupils. Nonetheless, one minor modifier was needed in the classification of Stage 4 responses. General terms which did not conform to standard algebraic conventions were deemed "creditable attempts" at Stage 4 provided the written expression was sufficiently lucid to be correctly interpreted. By way of example, Sizwe gave the following general term for Question 13: "$n^{th}-1$, $answer \times 9$, $+4$". This is sufficiently lucid to be interpreted as $9(n-1)+4$ in conventional algebraic format, and since this is algebraically equivalent to $9n-5$ it is deemed a creditable attempt at a Stage 4 response. Similarly, Jason's response to the same question, "$[4+(n-1\times 2)\times 3)]+[n-1\times 3]$" is algebraically incorrect in terms of standard algebraic conventions, but closer inspection of his calculation of the $10^{th}$ and $50^{th}$ terms clearly shows his understanding to be $[4+((n-1)\times 2)\times 3]+[(n-1)\times 3]$. Since this is also algebraically equivalent to $9n-5$,

[12] The unsimplified format is actually preferable as this is far more revealing of how the general formula was derived.

it is deemed a creditable attempt at a Stage 4 response. Within the question response analysis sheets, all Stage 4 creditable attempts are indicated by an asterisk (*).

### 4.7.2 STRATEGY/METHOD CLASSIFICATION

The method or strategy adopted for determining each of the next, $10^{th}$ and $50^{th}$ terms was carefully analysed and classified into one of the following seven categories:

- Counting (Co)
- Chunking (Ch)
- Difference Product (DP)
- Explicit (Ex)
- Whole-object uncorrected (Wu)
- Whole-object corrected (Wc)
- Nature of numerical terms (Na)

The literature review revealed very little consistency in the naming of patterning strategies. Although the basic procedural descriptions of various strategies are largely similar, nomenclature seems to be somewhat idiosyncratic. As a result, the seven strategies referred to in this investigation were named after distilling and somewhat modifying the various nomenclatures found in the research literature (Stacey, 1989; English and Warren, 1998; Hargreaves et al., 1998, 1999; Healy and Hoyles, 1999; Orton & Orton, 1999; Swafford and Langrall, 2000; Lannin, 2003, 2005).

Since the method classification is a critical component of this study, each of the seven strategies will now be described in detail. For the purposes of explication, the numeric/pictorial pattern of growing squares of matchsticks shown in Figure 4.6 will be referred to in the description of each method.

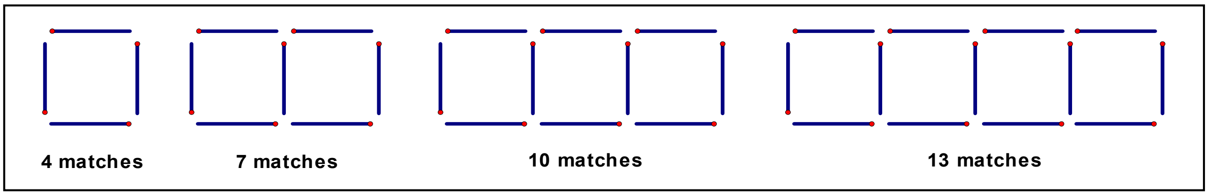



**Figure 4.6** Growing squares of matchsticks

- **COUNTING**

The counting method (or method of successive addition) represents a recursive approach whereby subsequent terms are determined by successively adding the identified constant difference to previous terms.

e.g. The 10th term could be calculated by adding (in a recursive manner) six lots of the constant difference (3) to the 4th term (13).

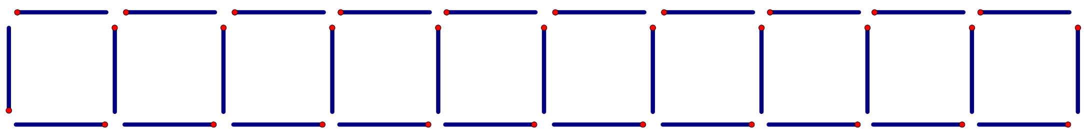
$$T_{10} = 13+3+3+3+3+3+3$$
$$= 31$$

e.g. The use of a table, constructed in a recursive manner (i.e. where each term is determined successively from its preceding term by addition of the constant difference), would also be classified as a counting method.

| Term | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Matches | 4 | 7 | 10 | 13 | 16 | 19 | 22 | 25 | 28 | 31 |

In a pictorial context, the counting method can be realised by simply drawing a diagram of the required pictorial term and counting individual elements.

e.g.



31 matches

- **CHUNKING**

The chunking method is similar to the counting method. However, instead of the successive addition of the constant difference in a *recursive* manner, "chunks" of the constant difference are added to a given term.

e.g. The 10th term could be determined by adding a "chunk" of six lots of the constant difference (3) to the 4th term (13).

$$\begin{aligned} T_{10} &= 13 + 6 \times 3 \\ &= 13 + 18 \\ &= 31 \end{aligned}$$

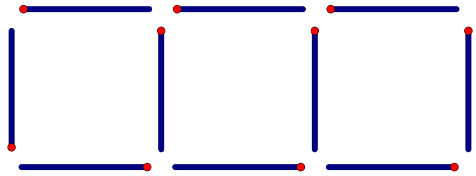
- **DIFFERENCE PRODUCT**

The difference product method is based on identifying the common difference and then multiplying it by the position number of the desired term. Using this approach, the 10th term of the sequence 4 ; 7 ; 10 ; 13 ; … would be incorrectly calculated as being $3 \times 10 = 30$, where 3 is the common difference and 10 is the position number of the desired term. This approach, although generally incorrect, does produce a correct answer for linear sequences based on direct proportion ($T_n = an$ where $a$ is a constant) e.g. 3 ; 6 ; 9 ; 12 ; ….

- **EXPLICIT**

The explicit method refers to a strategy where a general formula is first derived for the nth term and the desired term is then calculated directly from the general formula by using the independent variable (i.e. the position of the term). Provided the general term has been correctly formulated, the explicit method will yield any number of algebraically equivalent expressions for the nth term. In the examples that follow, the general terms of $T_n = 2n + (n+1)$, $T_n = 1 + n \times 3$, $T_n = 4n - (n-1)$ and $T_n = 3n + 1$ are all algebraically equivalent.

e.g.

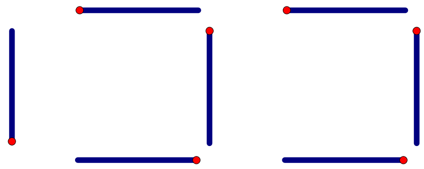


**Term 3 (i.e. n = 3)**
"n+1" vertical matches
"2n" horizontal matches

By subdividing the structure into vertical and horizontal matches, one can arrive at the following general formula: $T_n = 2n + (n+1)$. For the 10th term:

$$\begin{aligned} T_{10} &= 20 + (10+1) \\ &= 31 \end{aligned}$$

e.g.



**Term 2 (i.e. n = 2)**
1 starting match
"n" groups of 3 matches

By subdividing the structure into one initial starting match followed by the addition of $n$ groups of three matches, one can arrive at the following general formula: $T_n = 1 + n \times 3$. For the 10th term:

$$\begin{aligned} T_{10} &= 1 + 10 \times 3 \\ &= 31 \end{aligned}$$

e.g. By studying the given pictorial terms one could argue that the 5th term (for example) contains five squares, giving a total of $5 \times 4 = 20$ matches. However, four of these matches would have been counted twice due to overlapping, and would need to be subtracted. For five squares there are thus $20 - 4 = 16$ matches. By generalizing this reasoning one could argue that the nth term (containing $n$ squares) would contain $4n$ matches of which $n-1$ would need to be subtracted due to overlapping, thus arriving at the general formula: $T_n = 4n - (n-1)$. For the 10th term:

$$\begin{aligned} T_{10} &= 4 \times 10 - (10-1) \\ &= 40 - 9 \\ &= 31 \end{aligned}$$

e.g. One could also derive a general formula (explicit strategy) given purely numeric terms. By considering the numeric sequence 4, 7, 10, 13, … one could argue that since the common difference is 3, the general term must be of the form "$3n \pm c$" where $c$ is a constant. Since $3n \pm c$ must equal 4 for the first term (where $n = 1$), one readily arrives at the general formula $T_n = 3n + 1$. For the $10^{th}$ term:

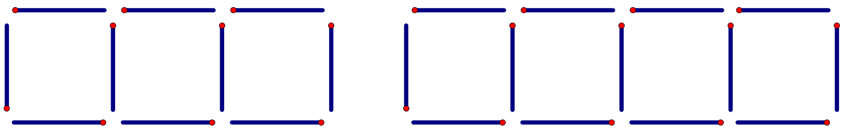
$$T_{10} = 3(10) + 1$$
$$= 31$$

Lannin (2003) differentiates between a "contextual" strategy and a "rate-adjust" strategy. Both of these strategies result in an explicit formula for determining the numerical value of any term from the independent variable. In the case of the contextual strategy, the general formula is derived from the context of the problem situation (first 3 examples), while the "rate-adjust" strategy stems from an essentially numeric or abstract argument (last example). However, the distinction between the two strategies does not take into account those pupils who made use of a blend of both abstract and contextual elements. Thus, within the methodology of the present investigation, no distinction is made between the two strategies. However, the extent to which the justification of the general term was specifically linked to the pictorial (as opposed to numerical) context is rated separately. This characterisation is discussed in the next section.

- **WHOLE-OBJECT UNCORRECTED**

The whole-object method involves the assumption that, for example, the $10^{th}$ term would be 2 times the $5^{th}$ term. A slight variation of the whole-object method involves the assumption that, for example, the $10^{th}$ term can be arrived at by simply adding the $4^{th}$ and $6^{th}$ terms. This approach, although generally incorrect, does produce a correct answer for linear sequences based on direct proportion ($T_n = an$ where $a$ is a constant) e.g. 3 ; 6 ; 9 ; 12 ; …. For linear sequences not based on direct proportion ($T_n = an \pm c$ where $a$ and $c$ are both constants), the uncorrected whole-object method will always result in an incorrect answer (more specifically, an over-count) as it does not compensate for overlapping units.

e.g.




Using an uncorrected whole-object method, the 20th term could be incorrectly calculated as being 5 times the 4th term:

$$\begin{aligned} T_{20} &= 5 \times T_4 \\ &= 5 \times 13 \\ &= 65 \end{aligned}$$

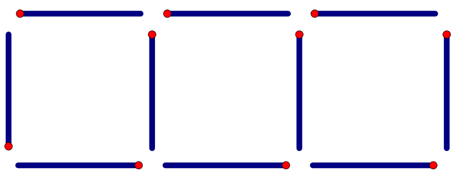
e.g. Using a slight variation of the whole-object method, the 7th term of the given sequence could be incorrectly calculated by simply adding the 3rd and 4th terms:

$$\begin{aligned} T_7 &= T_3 + T_4 \\ &= 10 + 13 \\ &= 23 \end{aligned}$$

- **WHOLE-OBJECT CORRECTED**

The corrected whole-object method is similar to the uncorrected whole-object method. However, in the corrected method the final answer is adjusted to compensate for overlapping units. If done accurately, the corrected whole-object method should always result in the correct answer.

e.g.

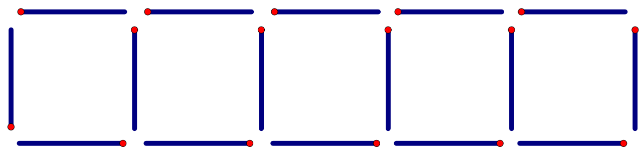


**Term 3**
10 matches

Since the 3rd term contains 10 matches, the 6th term could be calculated by adding together two units of ten matches and then subtracting one match to compensate for overlapping:

$$\begin{aligned} T_6 &= 2 \times T_3 - 1 \\ &= 2 \times 10 - 1 \\ &= 19 \end{aligned}$$

e.g.



**Term 5**
16 matches

Since the 5th term contains 16 matches, the 20th term could be calculated by adding together four units of 16 matches and then subtracting 3 matches to compensate for overlapping:

$$\begin{aligned} T_{20} &= 4 \times T_5 - 3 \\ &= 4 \times 16 - 3 \\ &= 61 \end{aligned}$$

Within the classification methodology of this study, the whole-object method is regarded as a local (as opposed to global) strategy, and this distinction is very important. The above examples show a local treatment of the corrected whole-object method which results in a numerical value of a specific term. If the whole-object method is treated in a more abstract global sense to arrive at an explicit generalised formula for $T_n$, then the strategy is classified as being explicit - e.g. by arguing that the nth term, containing $n$ squares, would contain $4n$ matches of which $n-1$ would need to be subtracted due to overlapping, thus yielding general formula: $T_n = 4n - (n-1)$.

- **NATURE OF NUMERICAL TERMS**

This method involves identifying a property applicable to some or all of the numerical terms in a given sequence.

e.g. Given the sequence 4 ; 7 ; 10 ; 13 ; 16 ; 19 ; … one might observe that the pattern is even, odd, even, odd… and (for example) that between any two adjacent even numbers there are two "missing" even numbers (e.g. 12 and 14 are "missing" between 10 and 16). Alternatively, from the given numerical terms, one could make the spurious observation that every alternate number is a prime number.

### 4.7.3 JUSTIFICATION CLASSIFICATION

In each question, pupils were asked to justify their general formula – i.e. to explain why their formula for the $n^{th}$ term works. Simon and Blume (1996) highlight four levels of justification. The first level appeals to an authoritative source (e.g. mathematics teacher or textbook). The second level of justification appeals to inductive or empirical evidence. Levels 3 and 4 display deductive justification based on shared mathematical knowledge, either expressed in terms of specific cases (Level 3) or generality independent of particular instances (Level 4). In a study of students' generalisations and justifications derived from patterning tasks, Lannin (2005) made use of an algebraic adaptation of the framework of Simon and Blume (1996).

Lannin's (2005) adaptation was considered as a possible framework for the present study. However, the framework soon proved to be unsuited to the data collection methodology of this study and had to be abandoned. However, a far more pertinent aspect of the justification process was an analysis of the extent to which pupils used the pictorial scenario as a referential context for the use of a *generic example* in their justification of the general term. To this end, responses were rated in terms of whether or not the justification was specifically linked to the pictorial (as opposed to numerical) context – a contextual connectivity rating (CCR). Only those questions that had a pictorial element (Questions 1 - 16) were rated. Scores of 1, ½ or 0 were awarded depending on the extent to which the pictorial context featured in the justification.

**Table 4.1** Contextual connectivity rating (CCR) for justification of $T_n$

| CCR | Description |
|---|---|
| **1** | Justification makes express reference (either diagrammatically or verbally) to the pictorial context |
| ½ | Justification either makes only partial reference to the pictorial context or makes use of both pictorial and numerical elements |
| **0** | Justification is purely numerically based and contains no reference to the pictorial context |

Furthermore, since the processes of generalisation and justification are so intricately interwoven, it soon became apparent that a meta-analysis of the generalisation process, as articulated by each pupil in the justification stage, would yield a rich tapestry of fascinating data centred on the pictorial scenario as referential context. The meta-analysis (Section 5.4.1) entailed carefully relating the pictorial scenario to the structure of the general term.

## 4.8 VALIDITY

The main threat to validity in the study is ambiguity or lack of clarity with respect to participants' written articulation of their own reasoning processes. Individual participants were informally interviewed where written responses were either ambiguous or required illumination by oral explanation. Member checking was thus used as a form of external validation (Lewis and Ritchie, 2003:276). In order to ensure valid comparisons of numeric and pictorial tasks, it was ensured that each numeric pattern had an isomorphic pictorial counterpart, as previously discussed. Furthermore, the order in which the various tasks were presented to the research participants was of critical importance in terms of validity considerations. The questions needed to be presented in such a way that a task under consideration was influenced as little as possible by the question design of previously experienced tasks *within this study*. In order to ensure this, the following order was strictly adhered to: pictorial scenarios using a single term; pictorial sequences with two non-consecutive terms; pictorial sequences with three consecutive terms; simple sequence of numbers; numeric sequences in tabular form. The validity of the individual tasks themselves was ensured by virtue of each question having been drawn from relevant and applicable literature.

## 4.9 SUMMARY OF METHODOLOGY

The present study attempts to interrogate pupils' responses to various linear generalisation tasks from both a technical as well as strategic viewpoint. Over a period of 3 months, 24 Grade 9 pupils from a mixed gender, high ability class each completed a series of 22 pencil and paper exercises based on linear generalisation tasks set in both numeric and 2-dimensional pictorial contexts. More specifically, numeric patterns were presented as a simple sequence of numbers as well as in tabular form, while pictorial patterns were presented using three consecutive terms, two non-consecutive terms, or one single term. For each pattern, participants were required to provide numerical values for the next, $10^{th}$ and $50^{th}$ terms as well as a written articulation of their reasoning at each stage. Participants were also asked to provide an algebraic expression for the $n^{th}$ term as well as to justify their expression. In addition to written responses, individual participants were informally interviewed where the written articulation of their mental reasoning was either ambiguous or required illumination by oral explication.

The responses to the various linear generalisation questions were classified by means of stage descriptors as well as stage modifiers. The method or strategy adopted for determining each of the next, $10^{th}$ and $50^{th}$ terms was carefully analysed and classified into one of seven categories. In addition, a separate framework was used to characterise each pupil's justification of the $n^{th}$ term in terms of the extent to which the justification was linked to the pictorial context. A meta-analysis of the generalisation/justification process was also undertaken. The stage descriptors and modifiers together with the adopted solution strategies and justification characterisation were used to create a rich profile for each research participant as well as for each individual pattern generalisation task.

# CHAPTER FIVE

# RESULTS, ANALYSIS & DISCUSSION

## 5.1 INTRODUCTION

Responses to all questions were carefully analysed and categorised in terms of strategy/method employed, stages successfully attained, and the contextual connectivity behind the justification of the general term. The results of this analysis were summarised on Question Response Analysis Sheets (QRAS) for each of the 22 questions, and these appear in Appendix B. These summary sheets were used to give a global view of the results.

Secondly, the influence of question design on strategy choice, stage progress, contextual connectivity, and the diversity of expressions for the general term was analysed.

Thirdly, a meta-analysis of the Stage 4 responses was prompted by the diversity of algebraic representations of the general term. The meta-analysis focused on the sub-structure evident in the formula derived for the general term in conjunction with its justification. In addition, the diverse nature of visually mediated solutions and visual strategies uncovered by the meta-analysis are highlighted and discussed.

Finally, a vignette of anomalous and idiosyncratic approaches is presented and discussed, and a comparison of two different cognitive styles is undertaken in order to highlight the divergent influence of question design on different pupils.

## 5.2 GLOBAL PICTURE

Responses to all questions were carefully analysed and categorised in terms of (a) strategy choice, (b) stage classification, and (c) contextual connectivity rating (CCR). The results of this analysis were summarised on Question Response

Analysis Sheets (QRAS) for each of the 22 questions, and these appear in Appendix B. These summary sheets were used to give a global view of the results.

### 5.2.1 STRATEGY CHOICE

For each of the 22 number patterns, the 24 research participants were asked individually to determine numerical values for the next, 10th and 50th terms along with a written articulation of their reasoning behind each, yielding a total of 1584 responses ($22 \times 24 \times 3$). Each of these responses was carefully analysed and categorised into one of the seven strategies previously described (Section 4.7.2): counting, chunking, difference product, explicit, whole-object uncorrected, whole-object corrected, nature of numerical terms. Table 5.1 shows the global picture of the extent to which each of these strategies was utilised[13].

The total number of strategies identified (1658) is in excess of the 1584 individual responses on account of a number of pupils using more than one strategy within a particular stage. 84 individual instances of more than one strategy being used for a single stage were identified. There were also 10 instances of a question not having been attempted at a particular stage, for which no analysis or characterisation was possible.

**Table 5.1** Global picture of strategy utilisation

| | | STAGE[14] | | | |
|---|---|---|---|---|---|
| | | **1 (next)** | **2 (10th)** | **3 (50th)** | **TOTAL** |
| **STRATEGY** | **Counting (Co)** | 255 | 36 | 0 | 291 |
| | **Chunking (Ch)** | 0 | 6 | 2 | 8 |
| | **Difference Product (DP)** | 3 | 7 | 6 | 16 |
| | **Explicit (Ex)** | 329 | 488 | 504 | 1321 |
| | **Whole-object uncorrected (Wu)** | 0 | 8 | 10 | 18 |
| | **Whole-object corrected (Wc)** | 0 | 1 | 1 | 2 |
| | **Nature of numerical terms (Na)** | 2 | 0 | 0 | 2 |
| | **TOTAL** | 589 | 546 | 523 | 1658 |

13 Stage 0 is indicative of no progress having been made, and is consequently not included in Table 5.1 as no characterisation was possible. Similarly, Stage 4 is not included in Table 5.1 as it represents an algebraic generalisation rather than a numerical result.

14 See Section 4.7.1.1

In terms of calculating the next term in a given sequence, two strategies clearly dominated: counting (43%) and explicit (56%). The counting strategy represents a recursive or iterative approach where the common difference is simply added to an existing term, either numerically or pictorially, in order to calculate the following term in the sequence. It is thus not surprising that this strategy proved popular at Stage 1, what Lannin (2004:217) remarks as being an almost natural tendency. The explicit strategy makes use of a general formula for $T_n$ in terms of the independent variable $n$, the position of the term in the sequence. The general formula could derive from either the pictorial context, or could be purely numerically driven. The explicit strategy thus requires the construction of a general formula which embodies the underlying structure of the pictorial/numeric pattern. In this regard it is perhaps a little surprising that so many pupils used this strategy at the Stage 1, where the counting strategy would have been arguably a faster and more direct approach. Together, the explicit and counting strategies account for 99% of all strategies attempted at Stage 1.

At Stage 2, the explicit strategy alone represents approximately 89% of all responses, while the counting strategy accounts for just less than 7%. Stage 2 represents what Stacey (1989:150) refers to as a “near generalisation” task, where a step-by-step counting procedure would still be within the bounds of feasibility. It is thus perhaps a little surprising that, already at this point, so many pupils rejected a recursive approach in favour of an explicit strategy, since the recursive approach would have required at most an additional 6 iterations from Stage 1.

Stage 3 is what Stacey (1989:150) refers to as a “far generalisation” task – a question that goes beyond reasonable practical limits of a step-by-step counting/drawing approach. At Stage 3, the explicit strategy accounts for almost 96% of all responses, while the counting strategy was not employed at all.

In total, the explicit strategy represents almost 80% of all strategies used, while a step-by-step counting procedure accounts for 17.5%. The remaining five strategies account for less than 3% of all responses. The predominance of the explicit strategy is surprising in view of the common theme in the research literature (e.g. Hargreaves et al., 1998; MacGregor and Stacey, 1993; Hershkowitz et al., 2002) which relates to the tendency of pupils to generalise recursively rather than using the independent variable, i.e. the explicit strategy. Furthermore, English and Warren (1998) found that once students had established a recursive strategy they were reluctant to search for a functional relationship. This is certainly not the case in the present study, where the tendency to generalise recursively drops markedly after Stage 1 – from 43% at Stage 1 to just less than 7% at the Stage 2. Associated with this is a marked increase in the number of pupils using an explicit strategy after Stage 1 – from 56% at Stage 1 to just over 89% at Stage 2. These shifts are a clear indication of pupils changing from a counting strategy to an explicit strategy.

A particularly interesting observation is the number of instances where two different strategies were employed in the same stage, both resulting in the same correct numerical answer. Equally interesting is the fact that the only two strategies used in combination were the counting and explicit strategies, although considering the preponderance of those two methods this is not too surprising. In total, 84 separate instances were noted where these two strategies were used in combination, 63 at Stage 1, and 21 at Stage 2. In some instances (19 of the 84) this merely amounted to extending the numerical sequence by means of the constant difference and using this to confirm the answer obtained from the explicit strategy. However, in the vast majority of cases (65 of the 84), the combined strategy entailed checking the answer obtained from the explicit strategy by means of drawing the required pictorial term and counting the required elements (matches, dots etc.). In addition to merely acting as a check, the physical act of drawing a pictorial representation of the *desired* term could also possibly have served as a meaningful *specific* reference for investigating the general structure underlying the pictorial context. This may well have assisted some pupils in the generalisation process in terms of seeing the general in the particular and hence moving towards an algebraic expression for the general term.

### 5.2.2 STAGE CLASSIFICATION

The responses to the various linear generalisation questions were classified by means of the stage descriptors (Stages 0 – 4) and stage modifiers (*t*, *x* and *y*) previously described (Section 4.7.1.1). It is important to keep in mind that Stages 0 through 4 do not represent a hierarchical structure. Being classified at a particular stage is not dependent on providing correct responses for all the previous stages, hence the need for the stage modifiers. While not being a hierarchical structure, the stages do however represent increased levels of difficulty, requiring numerical values for the next (Stage 1), $10^{th}$ (Stage 2) and $50^{th}$ (Stage 3) terms, and finally an algebraic expression for the $n^{th}$ term (Stage 4). Stage 0 is indicative of no progress having been made.

Table 5.2 shows the global picture for the stage classification of all 24 research participants for each of the 22 questions. All stage modifiers have been included, and asterisks indicate general terms which were given in non-standard algebraic format but which were nonetheless sufficiently lucid to be deemed "creditable attempts" at Stage 4. In order to highlight all incorrect responses, Stages 0 through 3 have been shaded along with Stage 4 responses which include a stage modifier (i.e. an incorrect response at one of the previous stages). In addition, an average "total stage attainment" (TSA) value is indicated for each research participant as well as for each individual question. This is an attempt to ascribe a numerical value to the level of attainment of the research participants as a whole to each of the 22 questions, as well as the level of success attained on average by each of the 24 research participants. The TSA value was calculated for each individual question by awarding 1 point for a correct Stage 1 response, 2 points for a correct Stage 2 response, 3 points for a correct Stage 3 response, and 4 points for a correct Stage 4 response. The highest obtainable score for a single question is thus 10 (1+2+3+4) for a pupil who correctly answered all four stages[15]. The intention of this global picture is twofold. Firstly it is intended to give an overview of how well the research participants, as a whole, fared in each of the questions. Secondly, it is intended to show the spread of levels of attainment for each of the 24 research participants. Table 5.2 represents this global picture.

[15] By way of further example, a pupil classified at Stage 3 would be awarded a mark of 6 (1+2+3+0), while a pupil classified at Stage 4y (having made an error at Stage 3 but nonetheless managing a correct Stage 4 response) would be awarded a mark of 7 (1+2+0+4).

**Table 5.2** Global picture of stage classification

| | Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Q7 | Q8 | Q9 | Q10 | Q11 | Q12 | Q13 | Q14 | Q15 | Q16 | Q17 | Q18 | Q19 | Q20 | Q21 | Q22 | Average TSA (10) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Alex** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4* | 4 | 0 | 0 | 4 | 4 | 4 | 4 | 1 | 4 | 4 | 1 | 4 | 8.27 |
| **Bianca** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 1 | 9.59 |
| **Carol** | 3 | 3 | 3 | 4 | 0 | 4 | 4 | 4 | 4 | 4 | 4 | 3 | 1 | 1 | 1 | 1 | 4 | 1 | 2 | 4 | 4 | 4 | 6.45 |
| **Dana** | 4 | 4 | 4 | 4 | 4 | 4 | 4x | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4t | 4 | 1 | 4 | 4 | 4 | 4 | 4 | 9.45 |
| **Greg** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 10.00 |
| **Hannah** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 0 | 4 | 1 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 9.14 |
| **Helen** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 10.00 |
| **James** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 10.00 |
| **Jason** | 4y | 4xy | 4 | 4y | 1 | 4xy | 4 | 4 | 4 | 4 | 4 | 4 | 4*t | 4 | 0 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 8.36 |
| **Julian** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4* | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 10.00 |
| **Kyle** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4* | 4 | 4 | 4 | 4t | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 9.95 |
| **Lisa** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 0 | 4 | 4 | 4 | 4 | 9.36 |
| **Lucas** | 4 | 4 | 4 | 4 | 1 | 4 | 4 | 4 | 4 | 3 | 4 | 0 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 8.95 |
| **Mark** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4* | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 10.00 |
| **Mary** | 4 | 4 | 4 | 4 | 0 | 4 | 4 | 4 | 1 | 0 | 1 | 4 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 4 | 0 | 4.41 |
| **Nell** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 1 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 9.59 |
| **Owen** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 0 | 4 | 4 | 4 | 0 | 4 | 4 | 4 | 4 | 4 | 4 | 9.09 |
| **Phil** | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 1 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 9.41 |
| **Richard** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 1 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4y | 4 | 4 | 9.45 |
| **Ryan** | 1 | 4xy | 1 | 4xy | 1 | 4 | 4 | 0 | 0 | 3 | 4 | 4 | 3 | 1 | 4 | 4 | 4 | 4 | 4 | 4* | 4 | 1 | 6.23 |
| **Sizwe** | 4 | 4 | 3 | 4 | 0 | 4 | 0 | 0 | 4* | 1 | 4* | 4* | 4* | 4* | 4* | 4* | 4 | 4 | 4 | 4 | 4 | 4* | 8.05 |
| **Sonya** | 4*y | 4 | 4 | 0 | 4 | 0 | 4y | 4 | 4 | 4 | 0 | 4* | 1 | 4 | 4 | 4 | 1 | 1 | 4 | 4 | 4 | 4 | 7.14 |
| **Sue** | 4x | 4 | 4 | 4 | 4 | 4txy | 4 | 4 | 4 | 3 | 4 | 4 | 1 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 9.05 |
| **Ted** | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4y | 4 | 4 | 3 | 1 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 9.27 |
| **Average TSA (10)** | 8.96 | 9.42 | 9.29 | 9.25 | 7.63 | 9.13 | 9.38 | 8.79 | 9.08 | 8.54 | 9.21 | 8.42 | 6.29 | 8.83 | 8.38 | 8.83 | 8.88 | 8.08 | 9.29 | 9.50 | 9.63 | 8.83 | |

On the whole, the research participants fared well with the majority of the questions. In fact, in 17 of the 22 questions more than 80% of the research participants were able to provide a correct Stage 4 response. Question 13 (average TSA = 6.29) was the most poorly answered question by some considerable margin. Question 21 (average TSA = 9.63) proved to be the best answered question. The lowest average TSA for an individual research participant was 4.41, which was markedly lower than the next lowest value of 6.23. At the other end of the scale, there were 5 pupils who managed to give correct responses at all four stages in all 22 questions.

The high general level of success, in terms of reaching a correct Stage 4 response, can be ascribed to at least two reasons. Firstly, all of the questions chosen for this study were linear/arithmetic sequences - i.e. of the form $ax \pm c$ where $a$ and $c$ are constants. This was a purposeful decision, as the literature review suggested that the alternative (quadratic sequences) would prove too problematic for pupils in the age group under investigation[16]. It thus seemed reasonable to postulate that linear sequences would allow research participants to progress further in the type of pattern generalisation tasks envisaged, and would thus more likely constitute "information-rich cases" (Patton, 1990:169). Secondly, the high success rate can in part be ascribed to the fact that the research participants represent a group of high ability learners. Once again this was a purposeful decision based on the argument that a high ability group of learners would be better suited to a methodology in which the data collection process required learners to attempt to articulate their own cognitive reasoning[17].

### 5.2.3 CONTEXTUAL CONNECTIVITY RATING

For each question, the pupils' written justifications of their general terms were rated in terms of whether or not the justification was specifically linked to the pictorial (as opposed to numerical) context. This rating has been dubbed the "Contextual Connectivity Rating" (CCR). Only those questions that had a pictorial element

[16] Other types of number sequences – power, geometric, and Fibonacci-type sequences – were not considered, as they lacked sufficient variety for the purposes of this investigation.
[17] A small-scale pilot study seemed to confirm this, and the high Stage 4 success rate indicated in Table 5.2 certainly seems to validate both of the decisions taken.

(Questions 1 - 16) were rated. Scores of 1, ½ or 0 were awarded depending on the extent to which the pictorial context featured in the justification. For a score of 1 to have been awarded, the justification must have made explicit reference (either diagrammatically or verbally) to the pictorial context. A score of 0 indicates that the justification is purely numerically based and contains no reference to the pictorial context. A score of ½ is indicative of those justifications that either made only partial reference to the pictorial context or where both pictorial and numerical elements play a role in the justification.

Table 5.3 shows the average CCR for each research participant as well as each of the 16 questions that were set in a pictorial/practical context. All general term justifications were awarded a CCR, irrespective of whether or not the general term represented a correct Stage 4 response. The CCR is thus not a measure of success at Stage 4 (algebraic generalisation), but rather a measure of the extent to which the Stage 4 response made reference, either correctly or incorrectly, to the pictorial context.

**Table 5.3** Global picture of average contextual connectivity ratings

| Pupil | AVE CCR |
|---|---|
| Alex | 1.00 |
| Bianca | 0.88 |
| Carol | 0.91 |
| Dana | 0.94 |
| Greg | 0.75 |
| Hannah | 0.81 |
| Helen | 0.88 |
| James | 0.72 |
| Jason | 0.78 |
| Julian | 0.50 |
| Kyle | 0.78 |
| Lisa | 0.81 |
| Lucas | 0.91 |
| Mark | 0.81 |
| Mary | 0.38 |
| Nell | 0.81 |
| Owen | 0.94 |
| Phil | 0.59 |
| Richard | 0.47 |
| Ryan | 0.22 |
| Sizwe | 0.25 |
| Sonya | 0.78 |
| Sue | 0.56 |
| Ted | 0.69 |

| Question | AVE CCR |
|---|---|
| 1 | 0.88 |
| 2 | 0.77 |
| 3 | 0.90 |
| 4 | 0.92 |
| 5 | 0.85 |
| 6 | 0.85 |
| 7 | 0.81 |
| 8 | 0.77 |
| 9 | 0.81 |
| 10 | 0.75 |
| 11 | 0.50 |
| 12 | 0.90 |
| 13 | 0.52 |
| 14 | 0.56 |
| 15 | 0.42 |
| 16 | 0.23 |

The average CCR values for individual research participants show a good spread – from 1.00 (Alex) to 0.22 (Ryan). Alex was the only pupil to make explicit reference to the pictorial context in all 16 questions. The average CCR values for individual questions were also well spread – from 0.92 (Question 4) to 0.23 (Question 16). There is a definite downward trend of the CCR as the questions progress from 1 through 16. This is an important observation, and its significance is discussed later in this chapter.

Figure 5.1 shows a scatter plot of average TSA against average CCR for each of the 24 research participants. The intention of the graph is to investigate the correlation between these two parameters. At best one could say that there is a very weak positive correlation between average TSA and average CCR, but even this statement should be treated with due circumspection. While a higher CCR seems to correspond to a higher average TSA for *some* pupils, the exact opposite is true for others. This observation is central to the theme of the present study, and the notion that different contexts (numeric vs. pictorial) will resonate differently with different pupils. While a pictorial context may be helpful to some pupils, for others it may simply create additional complications. Furthermore, some pupils may simply opt to convert a pictorial pattern into a numerical equivalent, and give no further thought to the pictorial context. This theme is expanded on as the chapter unfolds.

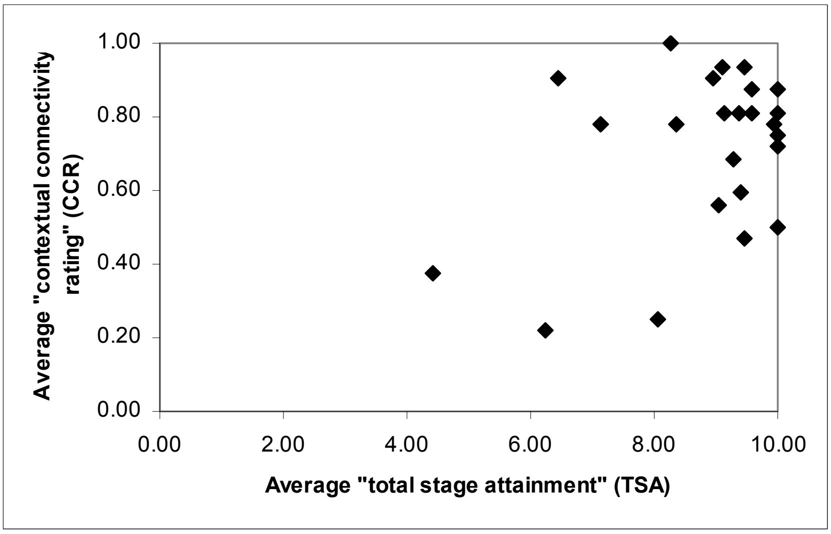



**Figure 5.1** Scatter plot of average TSA versus CCR

## 5.3 INFLUENCE OF QUESTION DESIGN

Each of the 22 pattern generalisation tasks used in this study fell into one of 6 different question design formats. The structure of these 6 formats was guided by insights gleaned from the literature review. As previously discussed, the order in which the various tasks were presented to the research participants was of critical importance in terms of validity considerations. The questions needed to be presented in such a way that a task under consideration was influenced as little as possible by the question design of previously experienced tasks *within this study*. In order to ensure this, the following order was strictly adhered to: pictorial scenarios using a single term; pictorial sequences with two non-consecutive terms; pictorial sequences with three consecutive terms; simple sequence of numbers; numeric sequences in tabular form.

The six different question design formats are summarised below, where the dependent variable refers to the numerical value of the term itself, while the independent variable refers to the position of the term in the sequence.

**Questions 1 - 5:** A single pictorial term in which the underlying structure is unambiguous. Both dependent and independent variable mentioned in the context of the picture.

**Questions 6 – 10:** Two non-consecutive pictorial terms. Both dependent and independent variable mentioned in the context of the picture.

**Questions 11 – 13:** Three consecutive purely pictorial terms.

**Questions 14 – 16:** Three consecutive pictorial terms with numerical value of dependent variable indicated.

**Questions 17 – 19:** Three consecutive purely numeric terms (dependent variable indicated).

**Questions 20 – 22:** Three consecutive purely numeric terms in table format (dependent and independent variables indicated).

The influence of question design on (a) strategy choice, (b) stage progress, (c) contextual connectivity, and (d) the diversity of expressions for the general term was analysed.

### 5.3.1 INFLUENCE OF QUESTION DESIGN ON STRATEGY

MacGregor and Stacey (1993), cite one of the main causes of difficulty in formulating algebraic rules as being pupils' tendency to focus on the recursive patterns of one variable rather than the relationship linking the two variables. Similar observations have been made by other researchers (e.g. Orton, 1997). This part of the analysis focuses on the extent to which question design either attracts or discourages a recursive approach.

The counting strategy (recursive approach) was used in one of two different modes in this investigation, either (a) on its own as sole strategy, or (b) in combination with an explicit strategy. Table 5.4 shows the percentage of total responses using a counting strategy (as *sole* strategy) for Stages 1, 2 and 3. The value under the "Total" column indicates the number of responses using the counting strategy as a percentage of the total responses (using any strategy) for Stages 1, 2 and 3 combined. The rationale behind considering only those responses that used counting as the *sole* strategy was the fact that when counting and explicit strategies were used in combination, the counting strategy was used simply to check the answer derived from the explicit strategy, and was thus not critical to a correct response at that stage.

**Table 5.4** Percentage of total responses using counting as sole strategy

| | **STAGE** | | | |
|---|---|---|---|---|
| **QUESTIONS** | **Next (Stage 1)** | **$10^{th}$ (Stage 2)** | **$50^{th}$ (Stage 3)** | **Total** |
| **1 – 5** | 36.7 % | 2.5 % | 0.0 % | 13.1 % |
| **6 – 10** | 17.5 % | 0.0 % | 0.0 % | 5.8 % |
| **11 – 13** | 38. 9 % | 2.8 % | 0.0 % | 13.9 % |
| **14 – 16** | 47.2 % | 2.8 % | 0.0 % | 16.6 % |
| **17 – 19** | 51.4 % | 8.3 % | 0.0 % | 19.9 % |
| **20 – 22** | 40.3 % | 1.4 % | 0.0 % | 13.9 % |

Table 5.4 reveals some interesting trends. There is a dramatic drop in the number of pupils using the counting strategy when two non-consecutive pictorial terms are used instead of one single pictorial term. There could be two possible reasons for this. Firstly, a single pictorial term may not be a sufficient scaffold to enable some

pupils to derive a general expression. A second diagram, physically drawn by the pupil, may have been necessary in order to see the general structure underlying the pictorial context. Thus, using a counting strategy at Stage 1 may have been a necessary prerequisite to moving to an explicit strategy at Stage 2. Secondly, questions incorporating two non-consecutive pictorial terms tended to have slightly bigger physical structures compared to the single term scenario, and drawing the next diagram in such a case may have been considered impractical by some pupils.

There is a dramatic increase in the number of pupils using the counting strategy when three consecutive purely pictorial terms are used instead of two non-consecutive pictorial terms. This increase is even more pronounced when the three consecutive terms are accompanied by an indication of the dependent variable. The initial increase could be a result of two possibilities. Firstly, the fact that the three consecutive pictorial terms are the first three terms in the sequence, the physical structures of the pictorial representations are a little less complex than in the case of the two non-consecutive terms. This may have encouraged pupils to simply draw the next term rather than looking for an explicit strategy. Secondly, because the three consecutive terms give a physical representation of growth, pupils may have been drawn to the recursive nature of the pattern and simply added the common difference to the third term in order to obtain a numerical value for the next term. This seemed to be slightly more often the case than simply drawing the next term and counting the number of elements. The even greater increase when the three consecutive terms are accompanied by an indication of the dependent variable can be explained in terms of the common difference having been made somewhat more explicit by the inclusion of the dependent variable and pupils thus being drawn even more towards a recursive strategy.

The simple presentation of three consecutive purely numeric terms resulted in the highest proportion of pupils opting for the recursive strategy. Just over 51% of all responses at Stage 1 made use of the counting strategy in the three questions presented in this format. Furthermore, just over 8% of the responses at Stage 2 also made use of the counting strategy, far more than in any other question design. Once again, the common difference becomes immediately clear from the given terms, and pupils seem to have been drawn towards this, and used a recursive approach as a result. Interestingly, when the three consecutive numeric terms are put into table format, which necessarily includes the independent variable, there is a slight drop in the tendency to pattern recursively. One can only surmise that the explicit presence of both dependent and independent variables assisted some pupils in seeing a general relation between the two and hence being more inclined to use an explicit strategy over a recursive approach.

Table 5.5 shows the percentage of total responses using an explicit strategy at Stages 1, 2 and 3. The value under the "Total" column indicates the number of responses using the explicit strategy as a percentage of the total responses (using any strategy) for Stages 1, 2 and 3 combined. As a result of the preponderance of the counting and explicit strategies being the main two strategies employed, the picture is essentially the negative image of Table 5.4.

**Table 5.5** Percentage of total responses using an explicit strategy

| | **STAGE** | | | |
|---|---|---|---|---|
| **QUESTIONS** | **Next** (Stage 1) | **10th** (Stage 2) | **50th** (Stage 3) | **Total** |
| **1 – 5** | 63.3 % | 88.3 % | 92.5 % | 81.4 % |
| **6 – 10** | 81.7 % | 95.0 % | 96.7 % | 91.1 % |
| **11 – 13** | 61.1 % | 95.8 % | 98.6 % | 85.2 % |
| **14 – 16** | 51.4 % | 93.1 % | 95.8 % | 80.1 % |
| **17 – 19** | 44.4 % | 86.1 % | 91.7 % | 74.1 % |
| **20 – 22** | 58.3 % | 97.2 % | 98.6 % | 84.7 % |

The question design that seems to best encourage an explicit strategy is that using two non-consecutive pictorial terms – just over 90% of all responses used an explicit strategy for this type of question. One can conjecture that the presence of two non-consecutive terms makes the common difference less obviously noticeable, encouraging pupils to make use of an explicit rather than recursive strategy. Interestingly, almost 82% of pupils made use of an explicit strategy already at Stage 1 with this type of question.

The question design that least encouraged an explicit strategy was the simple presentation of three consecutive numeric terms with no indication of the independent variable. The use of *consecutive* numeric terms seems to draw attention to the common difference and hence toward a recursive approach. However, when a tabular format is used, including an indication of the independent variable, recursion seems to lose favour to an explicit strategy. Once again one can only surmise that the explicit presence of both dependent and independent variables assisted some pupils in seeing a general relation between the two and hence being more inclined to use an explicit strategy over a recursive approach.

The above observations lend support to Hershkowitz et al. (2002) who found that the presentation of consecutive terms encouraged recursion, while terms presented non-consecutively tended to encourage generalisation by means of the independent variable. Hershkowitz et al. (2002) also found that the use of a pictorial context, particularly if non-consecutive terms were presented, tended to encourage explicit generalisations. Table 5.5 clearly shows the influence of using two non-consecutive pictorial terms with respect to encouraging an explicit approach to patterning. Furthermore, the results give strong support to the notion that question design can play a key role in influencing which strategies are adopted by pupils when solving pattern generalisation tasks.

### 5.3.2 INFLUENCE OF QUESTION DESIGN ON STAGE PROGRESS

Table 5.6 shows the average "Total Stage Attainment" (TSA) values for each of the six different question designs. The average TSA values are indicative of the level of attainment/progress made by the research participants as a whole. The calculation of these values has been described in Section 5.1.2.

**Table 5.6** Average TSA per question type

| QUESTIONS | Average TSA |
|---|---|
| 1 – 5 | 8.91 |
| 6 – 10 | 8.98 |
| 11 – 13 | 7.97 |
| 14 – 16 | 8.68 |
| 17 – 19 | 8.75 |
| 20 – 22 | 9.32 |

Although the majority of the average TSA values lie fairly close to one another, of interest are the highest and lowest values, which are well distanced from the rest of the cluster. The highest level of attainment (average TSA = 9.32) was achieved on those questions presented purely numerically, in tabular format. The explicit presence of both the dependent and independent variable, along with the fact that the terms were consecutive and hence made the common difference easier to recognise, all seem to have allowed for greater overall attainment. This finds resonance with a study by English and Warren (1998) where students found it easier to generalise, both verbally and symbolically, when patterns were presented in tabular form as opposed to pictorial form.

The lowest level of attainment (average TSA = 7.97) was achieved on those questions presented as three consecutive purely pictorial terms. In these questions, no mention was made of either the dependent or independent variable. This is an interesting observation when taken in conjunction with Table 5.5. Question designs making use of (a) three consecutive purely pictorial terms, and (b) three consecutive purely numeric terms in tabular format show almost identical values for the percentage of *total* responses using an explicit strategy (85.2% vs. 84.7%). However, there is a marked difference in level of attainment in these two question types (7.97 for the former, 9.32 for the latter). This adds weight to the notion that a pictorial representation is only of benefit if the underlying structure can be clearly seen. Despite the fact that pupils made almost equal use of an explicit strategy in the two question types, the lower level of success in the purely pictorial context would seem to suggest the use of explicit strategies based on misinterpretation of the general structure inherent in the pictorial context. Thus, while a purely pictorial context may be useful to some pupils, to others it may well create complications. A contextualised indication of both the dependent and independent variable (e.g. for 2 squares you will need 7 matchsticks) in conjunction with the pictorial representation (Questions 1-5 and 6-10) seemed to be most successful in alleviating this problem.

### 5.3.3 INFLUENCE OF QUESTION DESIGN ON CONTEXTUAL CONNECTIVITY

Table 5.7 shows the average "Contextual Connectivity Rating" (CCR) for each of the four different question designs that were based on a pictorial context (i.e. Questions 1-16). The CCR (which has been described more fully in section 4.7.3) is indicative of the extent to which the justification of the general term makes reference to the pictorial context.

**Table 5.7** Average CCR per question type

| QUESTIONS | Average CCR |
|---|---|
| 1 – 5 | 0.86 |
| 6 – 10 | 0.80 |
| 11 – 13 | 0.64 |
| 14 – 16 | 0.40 |
| 17 – 19 | - |
| 20 – 22 | - |

The results shown in Table 5.7 reveal a fascinating trend. The effect of presenting consecutive terms (Questions 11-13 and 14-16) seems to have a big influence on moving pupils' $T_n$ justifications away from the referential context (the pictorial representation) toward a more numerically based argument. This effect is even more pronounced in those questions (14-16) where the pictorial context is presented in conjunction with values for the dependent variable. The most likely explanation for this observation is that consecutive terms attract attention to the common difference, hence away from the underlying general structure inherent in the pictorial context, and thus to a more numeric approach to extracting and justifying the general formula for $T_n$.

There is also a slight decrease in the average CCR value when moving from questions involving a single pictorial term (Questions 1-5) to those making use of two non-consecutive pictorial terms (Questions 6-10). It is worth keeping in mind that both these question types make contextualised reference to both the dependent and independent variables. Thus, the slight decrease can probably be ascribed to the presence of more numeric points of reference.

### 5.3.4 INFLUENCE OF QUESTION DESIGN ON DIVERSITY OF EXPRESSIONS FOR $T_n$

Table 5.8 shows the average number of $T_n$ variations per question type. This gives an indication of the diversity of responses in formulating a general algebraic expression for the n^th^ term. Only correct Stage 4 responses have been considered.

**Table 5.8** Average number of $T_n$ variations per question type

| QUESTIONS | Average number of correct $T_n$ variations |
|---|---|
| 1 – 5 | 3.6 |
| 6 – 10 | 6.2 |
| 11 – 13 | 4.7 |
| 14 – 16 | 6.0 |
| 17 – 19 | 2.7 |
| 20 – 22 | 2.3 |

The dramatic drop in the number of correct $T_n$ variations for those questions incorporating purely numeric terms is both expected and understandable, since the lack of a referential (pictorial) context severely limits the scope of readily identifiable variations in $T_n$. Without a pictorial frame of reference, expressions for $T_n$ can only be derived from purely numeric considerations, the resulting expressions usually taking the form $a+(n-1)d$ or $dn+(a-d)$, or those deriving fortuitously from a guess-and-check approach.

The increase in the number of correct $T_n$ variations when moving from questions involving a single pictorial term (Questions 1-5) to those making use of two non-consecutive pictorial terms (Questions 6-10) can probably be ascribed to pupils' enhanced appreciation of the underlying general structure inherent in the pictorial context as a result of the additional term. The same argument could be applied when moving from two pictorial terms (Questions 6-10) to three pictorial terms (Questions 11-13 and 14-16). The value of 4.7 (Questions 11-13) is thus somewhat anomalous, and is probably a result of the specific questions chosen for that particular design type. Responses to Stage 4 in question 13 gave rise to seven different $T_n$ variations, while Question 11 and Question 12 had only 4 and 3 respectively. It is worth bearing in mind that some pictorial designs yield fewer accessible (easily identifiable) expressions for $T_n$, and this is likely to have been the situation in this case.

Comparing the number of correct $T_n$ variations per question type with average CCR values should be treated with extreme caution. There is no reason to assume that a high CCR value implies a high diversity of $T_n$ variations. The CCR value relates to the contextualisation of the justification for the $n^{th}$ term, but the justification itself is not necessarily an indication of the approach used to derive the algebraic expression for $T_n$. It is thus hardly surprising that there is little correlation between the average CCR values per question type and the average number of $T_n$ variations per question type.

Nonetheless, the diversity of algebraic representations of $T_n$ derived for each question was most informative, and prompted a meta-analysis of the generalised formulae in conjunction with their justification. The results of the meta-analysis are discussed later in the chapter, but it is worthwhile at this point listing the various expressions for $T_n$ in order to gain an idea of their remarkable richness and diversity (Tables 5.9 – 5.14). Only correct Stage 4 responses have been included, and where necessary, pupils' written responses for the $n^{th}$ term have been converted into equivalent expressions in standard algebraic format. The number of occurrences of each expression is indicated, along with the total number of correct Stage 4 responses for each question.

**Table 5.9** Pupils' algebraic expressions for $T_n$ (Questions 1-5)

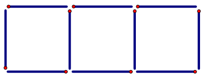
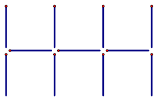
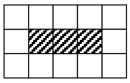
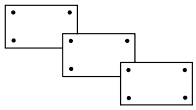
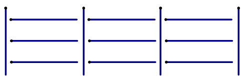
| | Question | Expression for $T_n$ | Tally | Total |
|---|---|---|---|---|
| Q1 |  3 squares require 10 matches | $3n+1$ | 18 | |
| | | $2n+(n+1)$ | 1 | |
| | | $4+3(n-1)$ | 1 | 20 |
| Q2 |  A pattern with 3 **horizontal** matchsticks requires a **total** of 11 matchsticks | $3n+2$ | 21 | |
| | | $4n-(n-2)$ | 1 | |
| | | $5+3(n-1)$ | 1 | 23 |
| Q3 |  For a row of 3 striped tiles there are 12 white tiles in the border. | $2n+6$ | 16 | |
| | | $2(n+2)+2$ | 2 | |
| | | $3(n+2)-n$ | 1 | |
| | | $8+2(n-1)$ | 1 | |
| | | $[n+2(n+2)+2]-n$ | 1 | 21 |
| Q4 |  For 3 photos you need 10 drawing pins | $3n+1$ | 21 | |
| | | $4n-(n-1)$ | 1 | |
| | | $3(n-1)+4$ | 1 | 23 |
| Q5 |  If there are 4 **vertical** matchsticks you need a **total** of 13 matchsticks. | $4n-3$ | 7 | |
| | | $3(n-1)+n$ | 8 | |
| | | $4(n-1)+1$ | 1 | |
| | | $3n+n-3$ | 2 | 18 |

**Table 5.10** Pupils' algebraic expressions for $T_n$ (Questions 6-10)

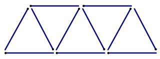
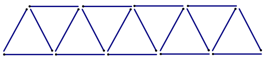
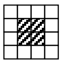
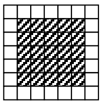
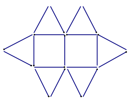
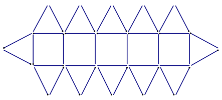
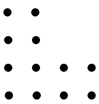
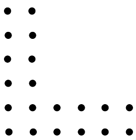
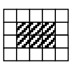
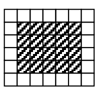
| Question | Expression for $T_n$ | Tally | Total |
|---|---|---|---|
| **Q6**  For 3 base matches you need a total of 11 matches  For 5 base matches you need a total of 19 matches. | $4n-1$ | 11 | |
| | $3n+(n-1)$ | 9 | |
| | $2n+n+(n-1)$ | 2 | |
| | $5n-n-1$ | 1 | 23 |
| **Q7**  For a 2x2 square of striped tiles, 12 white tiles are needed.  For a 5x5 square of striped tiles, 24 white tiles are needed. | $4n+4$ | 15 | |
| | $2(n+2)+2n$ | 2 | |
| | $2n+2n+4$ | 1 | |
| | $4(n+1)$ | 2 | |
| | $(n+2)^2-n^2$ | 2 | |
| | $2(2n)+4$ | 1 | 23 |
| **Q8**  For 2 squares you need a total of 19 matches.  For 5 squares you need a total of 40 matches. | $7n+5$ | 16 | |
| | $6n+(n+1)+4$ | 1 | |
| | $[3(n-1)+4]+[2(2n+2)]$ | 1 | |
| | $3n+3n+6+n-1$ | 1 | |
| | $4n+3n+5$ | 1 | |
| | $(3n+1)+4n+4$ | 1 | 21 |
| **Q9**  Base is 4 dots long  Base is 6 dots long | $4n-4$ | 7 | |
| | $2n+2(n-2)$ | 5 | |
| | $4(n-2)+4$ | 1 | |
| | $4(n-1)$ | 3 | |
| | $2n+(2n-4)$ | 1 | |
| | $(2n-1)+(2n-3)$ | 3 | |
| | $3n+(n-4)$ | 1 | |
| | $n+(n-1)+(2n-3)$ | 1 | 22 |
| **Q10**  For a 2x3 square of striped tiles, 14 white tiles are needed.  For a 4x5 square of striped tiles, 22 white tiles are needed. | $2n+2(n+1)+4$ | 9 | |
| | $n+(n+1)+n+(n+1)+4$ | 1 | |
| | $2(n+1)+2(n+1+1)$ | 1 | |
| | $4(n+1)+2$ | 3 | |
| | $2n+2n+2+4$ | 1 | |
| | $(n+2)(n+3)-n(n+1)$ | 1 | |
| | $2[n+(n+1)]+4$ | 2 | 18 |

**Table 5.11** Pupils' algebraic expressions for $T_n$ (Questions 11-13)

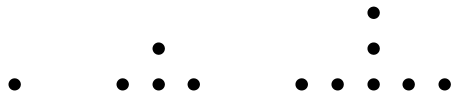
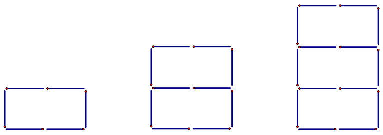
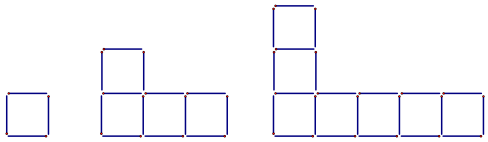
| Question | Expression for $T_n$ | Tally | Total |
|---|---|---|---|
| **Q11**  | $3n-2$ | 12 | |
| | $n+2(n-1)$ | 2 | |
| | $3(n-1)+1$ | 7 | |
| | $(n-1)+2(n-1)+1$ | 1 | 22 |
| **Q12**  | $4n+2$ | 16 | |
| | $4(n-1)+6$ | 2 | |
| | $2n+2(n+1)$ | 1 | 19 |
| **Q13**  | $9n-5$ | 6 | |
| | $3(2n+(n-2))+1$ | 1 | |
| | $4n+5(n-1)$ | 1 | |
| | $3(3n-2)+1$ | 1 | |
| | $3[3(n-1)+1]+1$ | 2 | |
| | $4+9(n-1)$ | 2 | |
| | $4+3(2(n-1))+3(n-1)$ | 1 | 14 |

**Table 5.12** Pupils' algebraic expressions for $T_n$ (Questions 14-16)

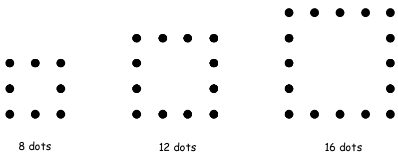
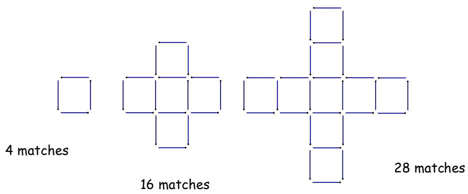
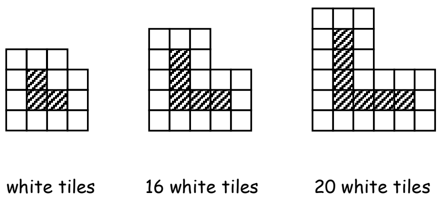
| Question | Expression for $T_n$ | Tally | Total |
|---|---|---|---|
| **Q14**   | $4n+4$ | 12 | |
| | $2(n+2)+2n$ | 4 | |
| | $4(n+1)$ | 2 | |
| | $4(n+2)-4$ | 1 | |
| | $4(n-1)+8$ | 2 | 21 |
| **Q15**   | $12n-8$ | 6 | |
| | $[4(3(n-1))]+4$ | 3 | |
| | $12(n-1)+4$ | 8 | |
| | $3n+3(n-1)+1+3n+3(n-2)$ | 1 | |
| | $3(4n-4)+4$ | 1 | |
| | $[3(4(n-1))]+4$ | 1 | 20 |
| **Q16**   | $4n+8$ | 8 | |
| | $2(2n+1)+6$ | 3 | |
| | $2(n+n+1)+6$ | 1 | |
| | $4(n+2)$ | 5 | |
| | $4(n+1)+4$ | 1 | |
| | $4(n-1)+12$ | 2 | |
| | $4(n+3)-4$ | 1 | 21 |

**Table 5.13** Pupils' algebraic expressions for $T_n$ (Questions 17-19)

| | Question | Expression for $T_n$ | Tally | Total |
|---|---|---|---|---|
| Q17 | 8 ; 12 ; 16 ; … | $4n+4$ | 15 | |
| | | $4(n+1)$ | 5 | |
| | | $4(n-1)+8$ | 1 | 21 |
| Q18 | 12 ; 19 ; 26 ; … | $7n+5$ | 17 | |
| | | $7(n+1)-2$ | 1 | |
| | | $7(n-1)+12$ | 1 | 19 |
| Q19 | 3 ; 7 ; 11 ; … | $4n-1$ | 21 | |
| | | $3n+(n-1)$ | 1 | 22 |

**Table 5.14** Pupils' algebraic expressions for $T_n$ (Questions 20-22)

| | Question | Expression for $T_n$ | Tally | Total |
|---|---|---|---|---|
| Q20 | 1st, 2nd, 3rd, … : 4, 13, 22, … | $9n-5$ | 20 | |
| | | $9(n-1)+4$ | 3 | 23 |
| Q21 | 1st, 2nd, 3rd, … : 4, 7, 10, … | $3n+1$ | 20 | |
| | | $4n-(n-1)$ | 1 | |
| | | $4+3(n-1)$ | 2 | 23 |
| Q22 | 1st, 2nd, 3rd, … : 4, 16, 28, … | $12n-8$ | 18 | |
| | | $4+12(n-1)$ | 3 | 21 |

### 5.3.5 QUESTION DESIGN – FURTHER CONSIDERATIONS

The 6 purely numeric patterns (Questions 17-19 and 20-22) each have an isomorphic pictorial counterpart for comparison purposes. Patterns based on the same general formula can thus be compared in different contexts. Although the influence of question design on strategy, progress, contextual connectivity and diversity of $T_n$ expressions has already been discussed in some detail, there are a number of smaller observations which can be made by comparing isomorphic pairs.

One of the most striking comparisons is for the pair of questions based on the general expression $9n-5$: Question 13 (three consecutive pictorial terms) and Question 20 (three consecutive numeric terms in table format). For Question 13, although only 14 of the 24 pupils managed to provide a correct algebraic expression for the $n^{th}$ term, 7 different expressions were realised. For Question 20, only 2 different forms of the general term were realised. However, 23 of the 24 pupils managed to arrive at a correct algebraic expression. Thus, although the pictorial context for this particular question would seem to have encouraged diversity of responses at Stage 4, far fewer pupils managed to reach this level.

This was not always the situation, however. Consider the comparison of Question 8 (two non-consecutive pictorial terms) and Question 18 (three consecutive numeric terms), both of which are based on the general formula $7n+5$. In Question 8 there were 6 different forms provided for the general term, while 21 of the 24 pupils managed to provide correct Stage 4 responses. However, in Question 18, only 3 different expressions were realised for the general terms, while at the same time there was a slight *drop* in the number of pupils who successfully reached Stage 4 – from 21 to 19. For this particular number pattern the specific pictorial context chosen seems to have been successful in both eliciting a greater variety of expressions for the general term, as well as allowing more pupils to reach Stage 4.

Another interesting comparison is that of Question 1 and Question 4, both based on the general formula $3n+1$. Both questions are of the same type – a pictorial representation of a single term. However, Question 4 is more of a *practical* than merely pictorial context – photos being pinned to a board. Both questions resulted in only 3 different expressions for the $n^{th}$ term, although the *practical* context allowed 2 more pupils to successfully reach Stage 4 (23 pupils, compared with 21 pupils in Question 1). However, one striking difference is the number of pupils using a counting strategy as sole strategy at Stage 1 – 50% for Question 1 compared with 21% for Question 4. This is a marked difference, and could well be ascribed to the *practical* context affording a more *dynamic* visualisation of the scenario, and hence encouraging an explicit strategy earlier on.

The above three observations are intended to give some idea of the complex interplay between the number pattern itself, the nature of the question design and the *specific* pictorial context chosen. This interwoven complexity, and its interpretation and treatment with the diverse cognitive skills of each individual pupil (verbal-logical and visual-pictorial components), will ultimately be manifested in choice of strategy, progress through the different stages, contextual connectivity, and the diversity of $n^{th}$ term expressions. There is thus a high degree of interconnectedness, and correlations between different aspects should be treated with due caution.

## 5.4 MECHANISMS OF VISUALISATION

As discussed in Section 3.3, for the purposes of this study, visualisation is understood to incorporate the process of forming images (either mentally or by means of physical instantiation) and using such images to aid mathematical discovery and understanding (Zimmermann and Cunningham, 1991). Furthermore, pattern imagery (Presmeg, 1986a, 1992), which identifies the relational aspects of a problem or scenario, is considered the most essential type of imagery for the purposes of abstraction and generalisation.

### 5.4.1 META-ANALYSIS OF $T_n$ EXPRESSIONS

A meta-analysis of the Stage 4 responses was prompted by the diversity of algebraic representations of $T_n$ (Section 5.3.4). The meta-analysis focused on the formula derived for the $n^{th}$ term in conjunction with its justification. The process of justification proved to be a critical factor in being able to accurately interpret the origin of the sub-structure evident in many of the Stage 4 responses. The use of a pictorial context allowed pupils to make use of a *generic example* within this reference frame as a means of scaffolding the justification process. From a theoretical perspective, the central role of proof within the context of this study is seen as communication of mathematical understanding. The process of justification/proof was highly successful in providing a window of understanding into each pupil's general formula. The results of the meta-analysis reveal a number of fascinating visually driven generalisations and gives strong support for the use of a pictorial context to enhance both visual approaches to generalisation and justification, as well as intensifying the diversity of resulting general solutions.

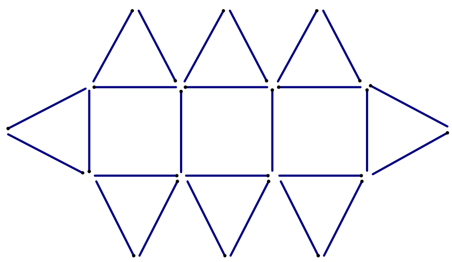
Two different questions have been chosen to represent the meta-analysis – Question 8 (involving a dot pattern) and Question 9 (a matchstick pattern). The meta-analysis of the general solutions to these two specific questions (chosen for their rich diversity of general solutions) gives a good vignette of the rich spectrum of visually driven explicit strategies evinced by the pictorial context. It is clear that visualisation played an important role for many pupils in the structuring of the algebraic (symbolic) representation of the general formula. Given that these two questions were presented in a pictorial context, the use of visual strategies is perhaps not surprising. However, what is surprising is the immense *diversity* of those visual strategies. Equally interesting is the fact that visualisation played very little role for some pupils, who favoured a numerically based derivation of the general formula (e.g. by using a table and searching for a likely formula to link the dependent and independent variables).

Visually mediated solutions to Questions 8 and 9 that came to light in the meta-analysis are described in detail hereunder:

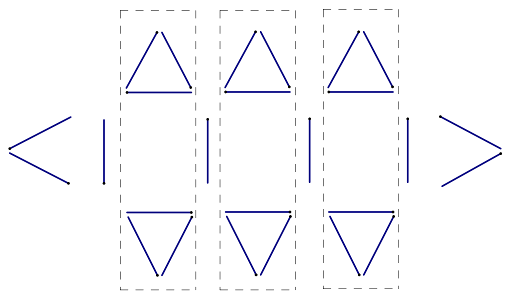
#### 5.4.1.1 Question 8 ($T_n = 7n + 5$)

Table 5.10 in Section 5.3.4 contains a summary of $T_n$ expressions generated for Question 8. For the purposes of explication, the following diagram shows the *generic example* (incidentally the 3rd term of the sequence) which will be used in all descriptions. The diagram is characterised by the nth term containing $n$ squares.

$n = 3$

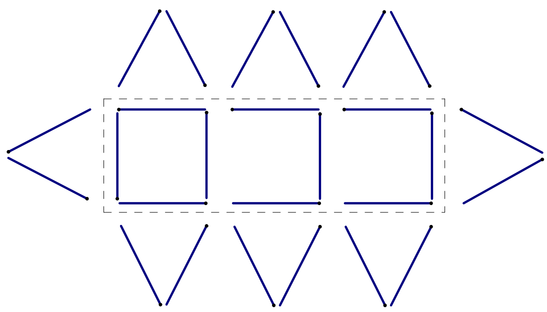


- **General formula** $6n+(n+1)+4$



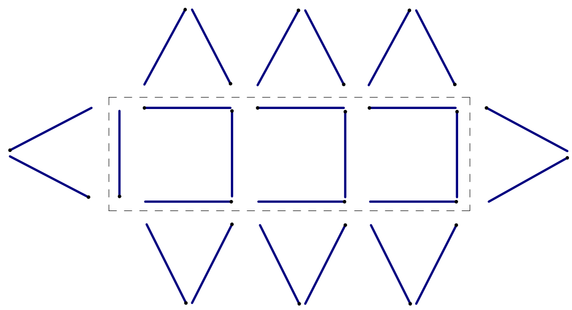
With this visual strategy, Alex sub-divided the whole into smaller units – triangles, V-shapes, and single vertical matches. The triangles were not visualised separately, but rather in pairs situated vertically above and below one another. For the n[th] pattern there will be $n$ pairs of triangles, each pair requiring 6 matches, and thus $6n$ matches in total. The n[th] term there will also contain $n+1$ vertical matches. Finally the V-shapes at either end, which are constants and thus independent of which term is under consideration, will always require 4 matches. The total count thus comes to $6n+(n+1)+4$.

- **General formula** $[3(n-1)+4]+[2(2n+2)]$



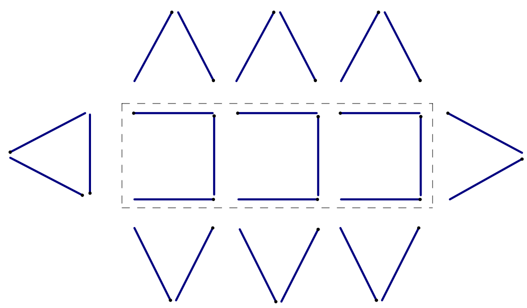
In this visual scheme, Carol subdivided the structure into squares and triangles. The central portion of the pattern is identical to Question 1, and was further subdivided into an initial square and a series of sideways U-shapes, each containing 3 matches. By starting with one square of 4 matches, $n$ squares in total would require an additional $3(n-1)$ matches, yielding a total of $3(n-1)+4$ matches. To complete the overall picture, Carol reasoned as follows. For $n$ squares there are a total of $2n+2$ perimeter matches - $n$ on top, $n$ below, and 1 on either side. Each of these perimeter matches requires an additional 2 matches to construct the triangles, i.e. $2(2n+2)$ matches. The total count for the complete diagram is thus $[3(n-1)+4]+[2(2n+2)]$ matches.

- **General formula** $(3n+1)+4n+4$



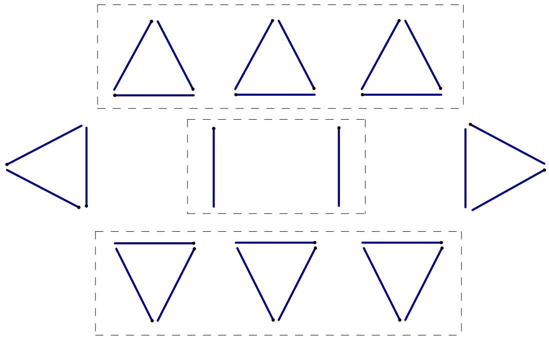
Nell visualised the problem in a similar way to Carol, but her somewhat different approach led to a slightly different expression for the n^th term. Nell also subdivided the structure into squares and triangles, but further subdivided the inner portion into a single vertical match followed by $3n$ sideways U-shapes, requiring a total of $3n+1$ matches. For the remainder of the structure, each square has two associated triangles – one above and one below – requiring 4 matches per pair and thus $4n$ matches in total. In addition, 4 extra matches are required to complete the triangles at either end. The triangles thus require $4n+4$ matches, and the count for the structure as a whole comes to $(3n+1)+4n+4$.

- **General formula** $4n+3n+5$



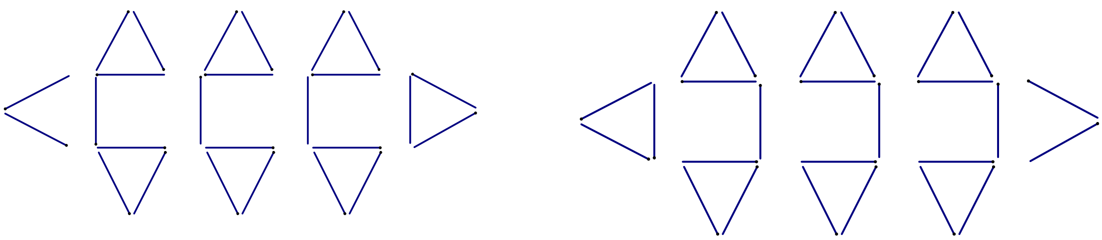
Sonya's approach was almost identical to Nell's, but a slightly different visualisation led to a slight variation in general formula. Instead of subdividing the inner structure into a single match followed by $n$ sideways U-shapes, Sonya let the single match form part of a triangle on the left. The inner portion thus contains $n$ sideways U-shapes requiring a total of $3n$ matches. For the remainder of the structure, each square has two associated triangles – one above and one below – requiring 4 matches per pair and thus $4n$ matches in total. 5 additional matches are required for the triangle at the left and the V-shape on the right, giving a final formula of $4n+3n+5$.

- **General formula** $3n+3n+6+n-1$



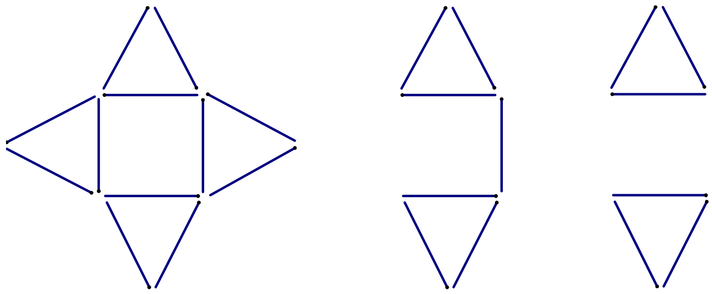
Helen subdivided the overall structure into two different component parts – triangles and vertical lines. What is particularly interesting in this instance is that after visually deconstructing the diagram into triangles, the squares become "negative space" as the matches that originally formed them have been apportioned to different component parts. Nonetheless, Helen made use of these squares to scaffold her reasoning. For each square there are 2 triangles, 1 above and 1 below. For $n$ squares there are thus $n$ triangles on top, each requiring $3n$ matches, and another $n$ triangles below, also requiring $3n$ matches. In addition, 6 matches are required to form the triangles at either end, and $n-1$ matches are needed for the vertical lines. The total count is thus $3n+3n+6+n-1$.

- **General formula** $7n+5$



A popular visual strategy (of which there were two varieties) was to subdivide the structure into a V-shape at one end, a triangle at the other, and the remainder into the 7-match additive portion – i.e. the basic unit which is effectively inserted into the structure to progress from one term to the next. For the n^th shape there are $n$ of these basic units comprising 7 matches each, thus $7n$ matches, and an additional 5 for the V-shape and triangle at the two ends. The final count is thus $7n+5$.

- **General formula** $12+7(n-1)$



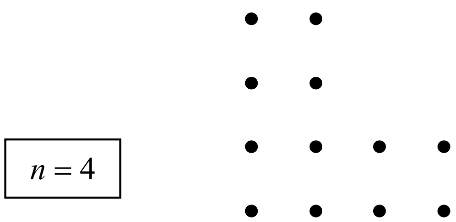
Ryan was unable to give the correct algebraic expression stemming from his visual reasoning, which in itself was partly faulty, but his visual strategy is worth noting nonetheless. From the given two terms, Ryan deduced that the first term in the sequence resembled a star shape comprising 12 matches. Since the second term was given (19 matches) he deduced the shape of the 7-match segment needing to be added. From this point on his visualisation became faulty, as he reasoned that only multiples of 6 matches (in the form of an upper and lower triangle) needed to be added for subsequent terms. This deconstruction of his visual reasoning at least explains his general formula $7+12+(6\times n-2)$, which in itself doesn't conform to standard algebraic convention. Nonetheless, his initial visual reasoning has the potential to create another variation for the general term, $12+7(n-1)$.

- **Mechanisms of visualisation (Question 8)**

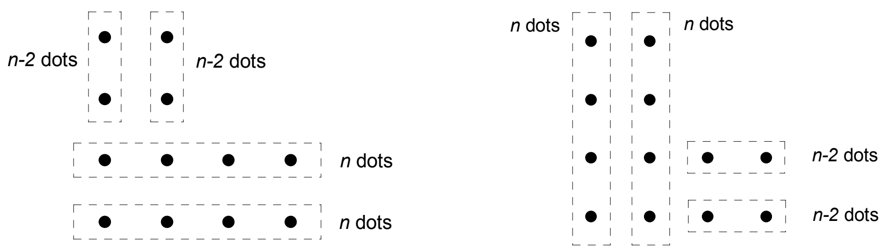
A number of mechanisms of visualisation become apparent from this meta-analysis, and are quite revealing in terms of the subtlety and complexity of the visual reasoning evident in the generalisation strategies. Most visual strategies began by deconstructing a *generic example* into a number of component parts. In some instances these component parts were further subdivided into even smaller parts. This decomposition of the *generic example* is essentially a retro-synthesis of the whole into perceived component parts. The complexity of these subdivisions ranged from single matches, V-shapes (2 matches), U-shapes (3 matches), squares (4 matches) and finally an odd shaped 7-match additive unit. Once separated into component parts, the visualisation process became one of reconstruction by means of multiplying the various parts by the frequency of their appearance, and finally summing the various multiples and constants together to arrive at a final general term. In general, the greater the number of different component parts, the greater will be the complexity of the derived general expression.

#### 5.4.1.2 Question 9 ($T_n = 4n - 4$)

Table 5.10 in Section 5.3.4 contains a summary of $T_n$ expressions generated for Question 9. For the purposes of explication, the following diagram shows the *generic example* (incidentally the 4th term of the sequence) which will be used in all descriptions. The diagram is characterised by the n<sup>th</sup> term containing $n$ dots along the base.

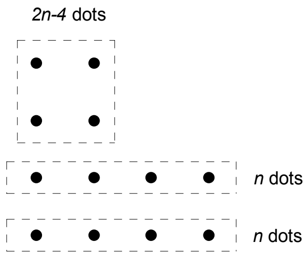



- **General formula** $2n + 2(n-2)$



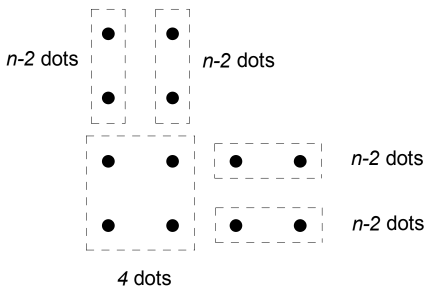

Despite the diagram being described as a “double L”, a popular visual strategy was to deconstruct it into vertical columns of dots and horizontal rows of dots. The first variation on this theme was to subdivide the whole into 2 rows of $n$ dots, requiring a total of $2n$ dots, as well as 2 vertical columns of $n-2$ dots (each column being 2 dots shorter than the horizontal rows), requiring a total of $2(n-2)$ dots. This yields a final total of $2n+2(n-2)$. Alternatively, but equivalently, the whole could be divided into 2 columns of $n$ dots, and 2 rows of $n-2$ dots.

- **General formula** $2n+(2n-4)$



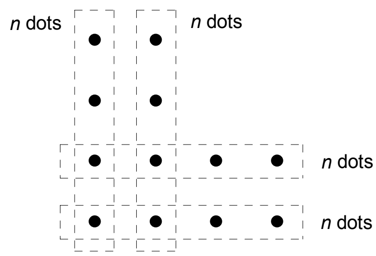

In this similar strategy, Julian subdivided the structure into 2 horizontal rows of $n$ dots, requiring a total of $2n$ dots. He then reasoned that the number of remaining vertical dots would always be 4 less than the total number of horizontal dots, i.e. $2n-4$, yielding an overall total of $2n+(2n-4)$.

- **General formula** $4(n-2)+4$



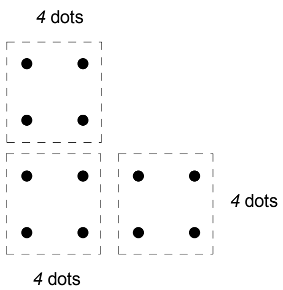

In a somewhat more symmetrical approach, Bianca isolated the 4 bottom corner dots and treated them as a constant. She then subdivided the remaining dots into 2 rows and 2 columns, each containing $n-2$ dots. In total there are thus 4 groups of $n-2$ dots, along with the 4 corner dots, yielding the general formula $4(n-2)+4$.

- **General formula** $4n-4$



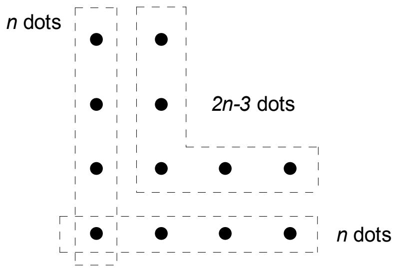

In an equally symmetrical approach, a number of pupils visualised the structure as being 4 overlapping rows of dots – 2 lying vertically and 2 horizontally – each comprising $n$ dots. This yields a total of $4n$ dots. However, as a result of the overlapping, there are 4 dots that have effectively been counted twice, and these need to be subtracted from the $4n$, giving a final count of $4n-4$ dots.

- **General formula** $4(n-1)$




James visualised the scenario somewhat differently, and his visualisation may well have been an artefact of the two given pictorial terms having an even number of dots along their base. As a result, James noticed that the entire structure could be subdivided into blocks of 4 dots, and that the quantity of these blocks would always be $n-1$. This led him to structure his general formula as $4(n-1)$. It is worth recalling at this point that the function of a *generic example* is to show the general in the particular. Thus, strictly speaking, James's justification doesn't make use of a *generic example*, as the next pictorial term in the sequence couldn't be split into a similar arrangement of blocks of 4 dots.

- **General formula** $(2n-1)+(2n-3)$ **and** $n+(n-1)+(2n-3)$




A number of pupils visualised the whole as being composed of an inner and outer L-shape. The outer L-shape was then visually deconstructed into 2 overlapping rows of dots – one lying vertically, the other horizontally – each containing $n$ dots. Correcting for overlapping thus gave the outer L-shape a total of $2n-1$ dots. The number of dots in the inner L-shape was then given as $2n-3$, justified by the visual observation that the inner L-shape would always be 2 dots shorter than the outer L-shape. This brought the final tally of dots to $(2n-1)+(2n-3)$. Using a similar approach, Sonya arrived at the general formula $n+(n-1)+(2n-3)$.

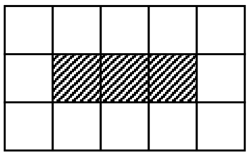
- **Mechanisms of visualisation (Question 9)**

The mechanisms of visualisation identified in Question 9 can also be described in terms of deconstruction and reconstruction, as discussed in Question 8. In most instances the structure was visualised in terms of rows and columns of dots of varying length. The component part was thus the single dot, its frequency of appearance being the length of the columns and rows. Another type of substructure comprising single dot components was the L-shape. Visualisation strategies in this particular question also had to frequently take into account a correcting mechanism for overlapping dots. There is also evidence to suggest, given James's deconstruction, that the presence or absence of *specific* terms may well attract or discourage a particular visually motivated strategy.

### 5.4.2 FURTHER EVIDENCE OF THE ROLE OF VISUALISATION

Throughout this investigation there was much evidence of visualisation playing a leading role both in terms of the justification of pupils' general rules and the actual structuring of the general rules themselves. The visual strategies used in Questions 8 and 9 have been described in detail. What follows is a small selection of some of the more interesting and unexpected visual strategies found in the other 14 questions based in pictorial contexts, with an aim at highlighting the diverse nature of visually motivated strategies.

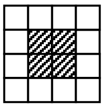
- **Question 3**



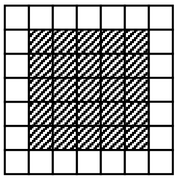
For a row of 3 striped tiles there are 12 white tiles in the border.

In this question, most pupils' visual strategy made use of the *global* observation that each striped tile had 2 white tiles associated with it, one above and another below. Thus, for $n$ striped tiles there would be $2n$ white tiles. Adding on the extra 3 at either end gave the formula $2n+6$. There were one or two minor variations on this theme, but the idea was much the same. Interestingly, Lisa first worked out the total number of tiles, both striped and white combined. Since $n$ represents the number of striped tiles, each row contains $n+2$ tiles, giving a total of $3(n+2)$. From this she then subtracted the central portion of $n$ striped tiles, giving a final tally of $3(n+2)-n$. Ryan saw the situation quite differently. He worked backwards from the given term and realised that the first term of the sequence would be a single striped tile surrounded by 8 white tiles. Rather than a *global* strategy, his visual strategy then focused on the *recursive* aspect of the growing sequence. His visual imagery at this point became *dynamic* as opposed to *static*, as he saw that moving from one term to the next would require the insertion of an additional striped tile along with 2 additional white tiles. Although he was unable to express this algebraically, his visual reasoning leads to the expression $8+2(n-1)$.

- **Question 7**



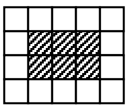
For a 2x2 square of striped tiles, 12 white tiles are needed.



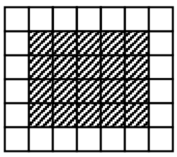
For a 5x5 square of striped tiles, 24 white tiles are needed.

Sonya's visual reasoning for this particular question involved a dynamic appreciation for the growth process of the given pictorial structure. Sonya focused on the recursive aspect of moving from one term to the next. She reasoned that for each additional row or column that gets inserted, an additional 2 white tiles would be needed. Since the visualisation focused on the expanding nature of the pictorial representations of the terms, the imagery is once again *dynamic*. This visualisation helped her realise that for each row of striped tiles there are 2 white tiles, one at either end. The same reasoning applies for the columns of striped tiles. Visually, the striped tiles thus form part of two different overlapping structures, rows and columns. Since there are $n$ rows and $n$ columns, this would require $2n \times 2$ white tiles. Adding on the 4 corner tiles yields a final tally of $2n \times 2 + 4$. Sonya's strategy in this question is quite similar to Ryan's treatment of Question 3, the added complexity of Question 7 being a result of the terms expanding in two directions instead of just one.

- **Question 10**



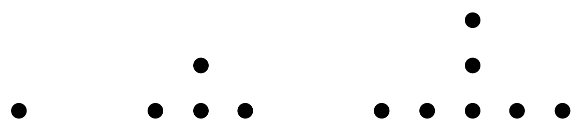
For a 2x3 square of striped tiles, 14 white tiles are needed.



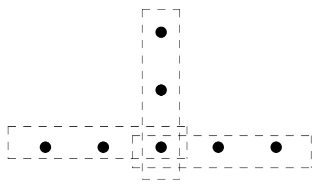
For a 4x5 square of striped tiles, 22 white tiles are needed.

In this question, Phil visualised the scenario as a rectangle of striped tiles overlapping, or nested inside, a rectangle of white tiles. His visual strategy was thus to first work out the number of tiles in the larger rectangle and then to subtract the number of tiles in the smaller rectangle. For an $n$ by $n+1$ rectangle of striped tiles, the size of the bigger rectangle would be $n+2$ by $n+3$. Phil's final formula was thus $[(n+2)(n+3)]-[n\times(n+1)]$. Lisa also followed a similar visual strategy for this question.

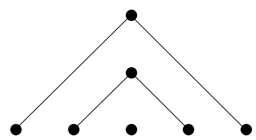
- **Question 11**



A number of pupils approached this problem by viewing the structure as three rows of dots radiating out from a single central dot. Since each of these radial arms contains $n$ dots, there would be a total of $3n$ dots. Correcting for overlapping leads to the formula $3n-2$.



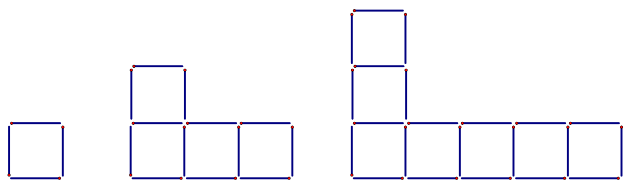
3 overlapping radial arms



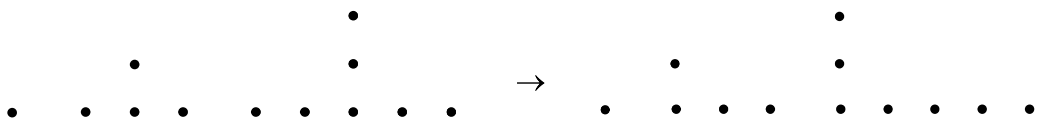
Greg's strategy

A number of other pupils arrived at an identical formula, but based on purely numeric observations. Greg had a slightly different approach, nonetheless yielding the same general formula. He visualised the entire structure as being composed of inverted V-shapes nested inside one another. For the n$^{th}$ term in the sequence there would be $n$ nested V-shapes, each containing 3 dots, with the exception of the smallest (first) V-shape which only contains a single dot. This visualisation led directly to the formula $3n-2$. The visual strategy employed by Greg makes use of what Hershkowitz et al. (2001:263) refer to as "auxiliary constructions". Greg's physical addition of straight lines between the 3 dots added in the creation of each subsequent term allowed him to view the overall structure quite differently. The use of "auxiliary constructions" has thus transformed the whole in such a way that different and quite unexpected visual patterns have emerged. This re-organisation of the whole is what led Greg to his generalisation of the pattern.

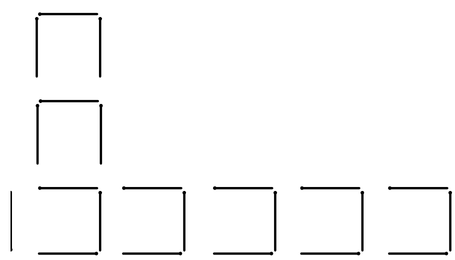
- **Question 13**



Question 11 and Question 13 were completed during the same session. This was fortuitous, as it provoked Lisa into visually reorganising the structures given in Question 11 so that the intrinsic similarity to the pattern in Question 13 became apparent:



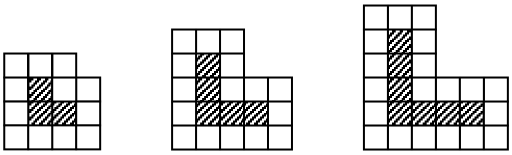
This visual similarity prompted Lisa to make use of the previously derived formula $3n-2$. This formula then prompted a visual deconstruction of the structures in Question 13 into U-shaped components of 3 matches, and a single match on the far left to close off the structure:



Since each dot (in Question 11) has effectively been replaced by 3 matches (in Question 13), Lisa simply multiplied the previously derived formula ($3n-2$) by 3, and added on the additional match needed on the far left to complete the structure. This gave the final formula $(3n-2)\times 3+1$. In this scenario it was the juxtaposition of two different questions that prompted a visual strategy in one that may not otherwise have been realised. In addition, the deconstruction process was prompted by both visual imagery as well as a symbolic representation of an associated pattern.

Using a similar visual strategy, but not relating it to Question 11, Owen first focused on the number of "boxes" formed by the matches, i.e. a *macro-structure*, for which he derived the formula $3(n-1)+1$. He then deconstructed the whole in an identical manner to Lisa, allowing him to produce the formula $3(3(n-1)+1)+1$.

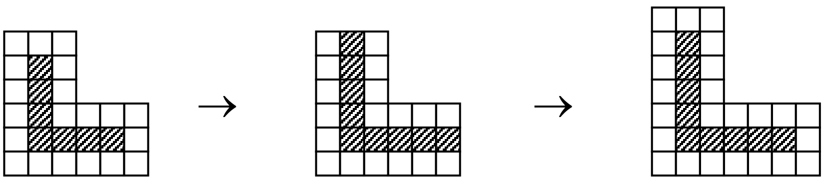
- **Question 16**



12 white tiles    16 white tiles    20 white tiles

Kyle began this question using a visual strategy, but this proved to be algebraically unhelpful, and he very quickly rejected it in favour of a numerical trial-and-improvement approach to arrive at a general formula.

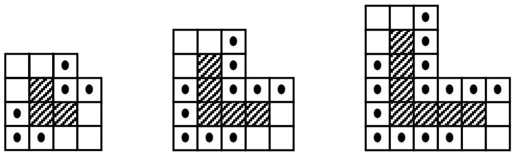
Kyle's initial visual strategy focused on the recursive relation between terms. He visualised moving from one term to another by subtracting two white tiles, replacing them with striped tiles, and adding on 6 further white tiles to close the two ends of the structure. Thus:



$20$ white tiles $\quad -2 \quad +6 \quad = \quad 24$

Kyle then rejected this visual approach and used the numerical values of the dependent and independent variables to arrive at the formula $4(n+2)$. In this particular instance it was the rejection of an inappropriate, or at least unhelpful, visual strategy that allowed Kyle to progress to Stage 4 of the generalisation process.

Lisa's visual strategy for this same question was prompted by the L-shape of the striped tiles:



For each term in the sequence there are 2 L-shapes made from white tiles, each equivalent in size to the striped L-shape. Once the L-shapes have been accounted for, there are always 6 remaining white tiles. For the n^th^ shape in the sequence, each L-shape contains $2n+1$ tiles. Thus, taking the 2 white-tiled L-shapes along with the remaining 6 white tiles gives a total of $(2n+1)\times 2+6$. Lisa's visual strategy in this question was prompted by a substructure embedded in the whole, even though that substructure didn't contain elements that needed to be counted in the final tally.

## 5.5 ANOMALIES AND IDIOSYNCRASIES

Orton and Orton (1999:120) mention that one of the four main obstacles to successful generalisation is the fact that "idiosyncratic methods are adopted by individual pupils in unpredictable ways". The purpose of this section is to highlight some of the anomalous and idiosyncratic approaches that were observed in this study.

### 5.5.1 TRIAL AND ERROR NUMERIC APPROACHES

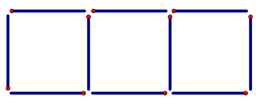
The numeric strategy of trial and error, sometimes referred to as "guess and check", was a common approach to problems when pupils were trying to find a functional relationship between dependent and independent variable using a numeric strategy. Sometimes elements of the pictorial context acted as prompts in these numeric approaches. On other occasions, despite the presence of a pictorial frame of reference, pupils simply ignored the pictorial context and relied purely on the numbers themselves. This led to some surprising algebraic expressions that had no relation to the number pattern under investigation, but which somewhat incidentally yielded correct terms in one or two instances of the independent variable. In such cases, pupils tended to justify their expression for the general formula by citing the one or two instances where the expression had worked. This form of justification, based on empirical evidence, which appeals through the correctness of particular examples, would be classified at Level 2 in Lannin's (2005) algebraic adaptation of the framework of Simon and Blume (1996). Four particular examples of unexpected expressions arising from such trial and error approaches are shown below:

**Table 5.15** Trial and error numeric approaches

| Question | Pupil | Correct $T_n$ | Pupil's $T_n$ | Terms for which pupil's $T_n$ is correct |
|---|---|---|---|---|
| 1 | Ryan | $3n+1$ | $3+n\times n$ | $T_1=4$ and $T_2=7$ |
| 8 | Sizwe | $7n+5$ | $n^2+15$ | $T_2=19$ and $T_5=40$ |
| 9 | Mary | $4n-4$ | $(n\times 3)-1+4$ | $T_7=24$ |
| 10 | Sizwe[18] | $4n+6$ | $n(n+1)\div 3+16$ | $T_5=26$ and $T_6=30$ |

### 5.5.2 ALGEBRAICALLY UNHELPFUL GENERALISATIONS

Although generalisations can be expressed in numerous ways, both verbally and symbolically, some verbal expressions do not translate as readily as others into an algebraic format. The case in point is an example of just such a problem.



3 squares require 10 matches

In Question 1, Sonya's generalisation strategy entailed building onto the pre-existing structure given in the original question, using a chunking strategy. For 10 squares, Sonya worked out the number of matches required as follows:

$10-3=7$ (i.e. an additional 7 squares need to be added)

$7\times 3=21$ (since 3 additional matches are needed to make an extra square)

$21+10=31$ (21 additional matches added to the original 10)

[18] This is an algebraic representation of Sizwe's verbal description of his general term. Interestingly, in those cases where $n(n+1)$ is not divisible by 3, Sizwe simply ignored the remainder in the final answer. Based on this approach, his formula also gives the correct answer for the 4th term, $T_4=22$.

To complicate matters further, Sonya then attempted to symbolically generalise her own specific generalisation. Letting $x$ be the number of existing squares, she proceeded as follows:

$$n - x = y$$

$$y \times 3 = b$$

$$b + x = your\ \ answer$$

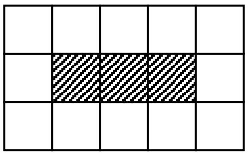
Interestingly, Phil approached this same problem in an identical manner, chunking multiples of 3 matches onto the pre-existing structure of 10. For his Stage 4 response, Phil gives the general formula $[(n-1)\times 3]-6$. On closer inspection, although this isn't a correct expression for $T_n$, it turns out to be a correct expression for the number of *additional* matches needed to be added onto the original 10.

### 5.5.3 REDEFINING THE INDEPENDENT VARIABLE

In a number of instances, pupils altered the meaning of the independent variable. Invariably this was done in cases where $n$ was redefined to mean either $n+1$ or $n-1$ in the original context. There are three possible reasons for this anomalous behaviour. Firstly, it may stem from a lack of desire to work with the more complex expressions $n+1$ or $n-1$. Secondly, it may simply stem from an inability to use these more complex notations. Thirdly, and what would seem to be a likely scenario, pupils may have internally redefined the independent variable, so that incorrect expressions for $T_n$ arising from this redefinition are nonetheless correct in the pupil's frame of reference.

Occasionally this redefinition was made explicit, e.g. Sizwe's redefinition of the independent variable in Question 3 to become the "next consecutive no. of striped tiles"[19]. However, in the overwhelming majority of cases, the redefinition was entirely implicit.

[19] This is a strange alteration to have made given the context of the question, but entirely plausible in the light of Sizwe's generalisation strategy being a numerically driven trial and error approach.



For a row of 3 striped tiles there are 12 white tiles in the border.

In the same question, Carol arrived at the general formula $8+2n$ for the number of white tiles needed to put a border around a row of $n$ striped tiles. Carol's formula makes sense in terms of her justification. She started by realising that the first shape in the sequence would be a single striped tile surrounded by 8 white tiles. She reasoned that for every additional striped tile in the shape, 2 additional white tiles would be needed. Her general formula, $8+2n$, thus reflects the original 8 tiles plus 2 for every *additional* striped tile. Within her own frame of reference, the formula is entirely correct, but she has effectively redefined the independent variable. In Carol's frame of reference, $n$ no longer represents the number of striped tiles in the whole structure, but rather the $n-1$ *additional* striped tiles added onto the first shape in the sequence. Replacing $n$ with $n-1$ in Carol's formula yields $8+2(n-1)$ which simplifies to $2n+6$, the correct general term in the original context.

### 5.5.4 ADDING OR REMOVING THE FIRST TERM

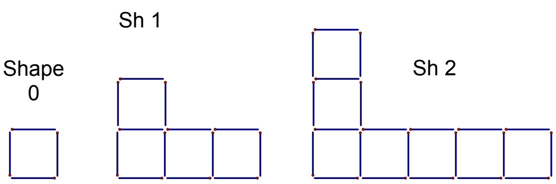
The effect of adding an additional first term, or removing the existing one, is identical to redefining the independent variable. However, I have classified the process differently for two reasons. Firstly, the alteration lies not in the *meaning* of the algebraic symbols, but in the *physical structure* of the pattern itself. Secondly, the alteration is both explicitly and purposefully carried out.

Consider Dana's alteration of Question 17:

| | | |
|---|---|---|
| 8 ; 12 ; 16 ; … | → | 4 ; 8 ; 12 ; 16 ; … |
| | | 1 2 3 4 |
| Original question | | Dana's alteration |

The original sequence was 8 ; 12 ; 16 ; … Not only has Dana added in an additional first term, but she has quite expressly labelled the new term as being the first term of the sequence. All of Dana's calculations are based on her new sequence, and are thus constantly 4 short of the correct numerical answers. Her expression for the general term is $4n$, which is also 4 short of the correct general expression $4n+4$ since the n[th] term of the original sequence is in fact term $n+1$ of Dana's sequence. There seems to be no rational explanation for Dana's decision to alter the sequence, as both her numeric and algebraic skills are exceptional and she has successfully generalised far more complex sequences than this particular one.

As an example of the removal of the first term, consider Sonya's alteration of Question 13:

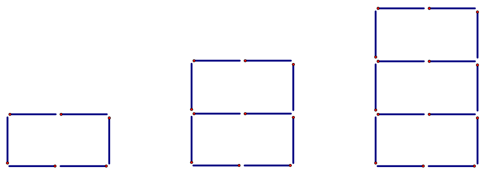



Instead of physically scratching out the first term, Sonya has simply called the first term "Shape 0", thus effectively starting the sequence at the second term of the original pattern. Having done this, Sonya found it much easier to arrive at a general expression, $(9 \times n) + 4$. This may well have been a deliberate attempt to avoid having to work with the numeric expression $n-1$, as the general term of the original sequence, based on Sonya's reasoning, would have been $9(n-1)+4$. In addition to labelling the sketch, Sonya makes express mention of her alteration of the pattern in her justification of her general term: "9 matches are needed to get to the next shape and 4 are needed for the original block. Do not count the original block as shape one!"

### 5.5.5 GENERAL EXPRESSIONS BASED ON SPURIOUS OBSERVATIONS

Numerically based trial and error strategies were occasionally based on spurious observations. In Question 19, Mary noticed that the sequence 3, 7, 11, … was composed entirely of prime numbers, which prompted her to give the next term in the sequence, Stage 1, as being 13. This led her to the general formula $(n \times 3) + 1$, which yielded prime numbers for $n = 10$ and $n = 50$, Stages 2 and 3 respectively. Her justification of the general expression simply amounted to "all prime numbers". By focusing on a spurious and incidental observation, Mary didn't even consider using a simple counting strategy, based on the difference between consecutive terms, to calculate the next term in the sequence. In addition, the fact that her general formula only held true for one of the original 3 terms didn't seem to deter her from using it to calculate the $10^{th}$ and $50^{th}$ terms of the sequence.

In Question 12, Mary also made use of a somewhat coincidental yet entirely spurious observation to arrive at a general formula, in this case the *correct* general formula.



Mary noticed that the first shape in the sequence contained 4 horizontal matches and 2 vertical matches. This prompted her to search for a working formula using a trial and error approach based on the numbers 4 and 2. This eventually led her to the formula $(n \times 4) + 2$, which seemed to work for the given terms. She thus adopted the formula and used it to calculate the $10^{th}$ and $50^{th}$ terms in the sequence. In this particular case, Mary managed to progress right through to Stage 4 of the generalisation process despite her initial visual prompt being spurious and incidental.

### 5.5.6 LOOKING FOR PATTERNS WITHIN PATTERNS

Ryan took a somewhat unusual approach to generalising Question 22:

| 1 | 4 | 7 | 10 | 13 |
|---|---|---|---|---|
| $1^{st}$ | $2^{nd}$ | $3^{rd}$ | … | |
| 4 | 16 | 28 | … | |

The numbers above the table were added in by Ryan after he realised that 4, 16 and 28 were the $1^{st}$, $4^{th}$ and $7^{th}$ multiples of 4 respectively. He then saw the recursive nature of his new pattern, which he extended to the $5^{th}$ term. He correctly calculated the $4^{th}$ term of the original sequence by multiplying the $4^{th}$ term of his sequence by 4. However, in order to calculate both the $10^{th}$ and $50^{th}$ terms he made use of an erroneous uncorrected whole-object strategy. After extending his new sequence to 13 (the $5^{th}$ term), Ryan reasoned that the $10^{th}$ term would simply be double the $5^{th}$ term. He thus doubled 13 to get 26 and then multiplied 26 by 4 (giving an answer of 104) to give the $10^{th}$ term of the original sequence. He then reasoned that the $50^{th}$ term of the original sequence would simply be 5 times the $10^{th}$ term, giving an answer of 520.

### 5.5.7 FIXATION WITH THE FORMULA

As discussed in the literature review, much attention has focused on pupils' reliance on the method of differencing, i.e. the use of a recursive strategy to move from a given term to the next, and so on in an iterative manner. Orton and Orton (1999:120) mention that one of the four main obstacles to successful generalisation is a "fixation with a recursive approach [which] can seriously obstruct progress towards the universal rule".

In Question 10, Mary handed back a largely blank response sheet. Apart from one or two minor calculations, nothing had been written at all, and no answers were supplied for any of the 4 stages. When interviewed shortly afterwards, she explained that she couldn't do this particular question because she "couldn't find the formula". Here we have a situation where a pupil was so fixated on an expression for the general term that she was unable to employ a simple counting strategy to work out the next term in the sequence.

## 5.6 A COMPARISON OF COGNITIVE STYLES

Throughout the analysis it became apparent that question design did not have a uniform influence on all research participants. While some pupils favoured a pictorial context, others found greater resonance with a purely numeric context. In addition, for some pupils the context of the generalisation problems had little to no effect. By way of example, two research participants who were influenced differently by the question design were Mary and Ryan. An analysis of their progress over the course of the 22 questions reveals some interesting cognitive differences.

**Table 5.16** Pupil profile - Ryan

| **Question** | **Method** | | | **Overall stage descriptor** | **CCR** | **TSA** |
|---|---|---|---|---|---|---|
| | **Next** | **10th** | **50th** | | | |
| **1** | Co | Wu | Wu | 1 | 0 | 1 |
| **2** | Co | Wu | Wu | 4xy | 0 | 5 |
| **3** | Co | Wu | Wu | 1 | 1 | 1 |
| **4** | Co | Wu | Wu | 4xy | 0 | 5 |
| **5** | Co | Wu | Wu | 1 | 0.5 | 1 |
| **6** | Ex | Ex | Ex | 4 | 0 | 10 |
| **7** | Ex | Ex | Ex | 4 | 0 | 10 |
| **8** | Co | Ch | Ex | 0 | 1 | 0 |
| **9** | Ex | Ex | Ex | 0 | 1 | 0 |
| **10** | Ex | Ex | Ex | 3 | 0 | 6 |
| **11** | Ex | Ex | Ex | 4 | 0 | 10 |
| **12** | Co | Co/Ex | Ex | 4 | 0 | 10 |
| **13** | Co/Ex | Ex | Ex | 3 | 0 | 6 |
| **14** | Co | DP | DP | 1 | 0 | 1 |
| **15** | Co | Ex | Ex | 4 | 0 | 10 |
| **16** | Co | Ex | Ex | 4 | 0 | 10 |
| **17** | Ex | Ex | Ex | 4 | - | 10 |
| **18** | Ex | Ex | Ex | 4 | - | 10 |
| **19** | Ex | Ex | Ex | 4 | - | 10 |
| **20** | Ex | Ex | Ex | 4* | - | 10 |
| **21** | Ex | Ex | Ex | 4 | - | 10 |
| **22** | Na | Wu | Wu | 1 | - | 1 |
| **AVERAGE** | - | - | - | 2.86 | 0.22 | 6.23 |

Ryan made use of 6 different strategies during the course of the study – counting, explicit, chunking, difference product, whole-object uncorrected, and nature of numerical terms. The explicit strategy was used most frequently (40 instances), with the counting and uncorrected whole-object strategies each being used 12 times. Only twice did Ryan use a counting strategy in conjunction with an explicit strategy, the double method being used to confirm the numerical result. In addition, there was one occasion in which he used three different methods in the same question, one for each stage. Interestingly, there were only five occurrences of this happening in the entire study.

A brief glance at Table 5.16 shows a very clear trend, a trend which becomes even more pronounced when average TSA values are compared for the different question designs:

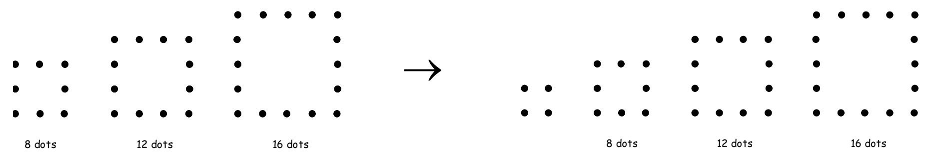
**Table 5.17** Average TSA values per question type - Ryan

| Question design | Questions | Average TSA |
|---|---|---|
| Single pictorial term | 1 – 5 | 2.60 |
| Two non-consecutive pictorial terms | 6 – 10 | 5.20 |
| Three consecutive pictorial terms | 11 – 16 | 7.83 |
| Three consecutive numeric terms | 17 – 22 | 8.50 |

Ryan fared most poorly with the generalisation problems presented as a single pictorial term. For all five he used a counting strategy to determine the next term, and then focused on the numerical terms using an erroneous (uncorrected) whole-object strategy to determine the $10^{th}$ and $50^{th}$ terms. In two of the five questions he managed to arrive at a correct general term, but did not use his expression to calculate numerical values for any of the preceding three stages. In addition, he was unable to justify these algebraic expressions.

He fared somewhat better with the patterns presented as two non-consecutive pictorial terms, focusing almost exclusively on an explicit strategy. Interestingly, the three questions for which his Stage 4 justification was purely numerically based (CCR = 0) were much better handled than the two questions in which he attempted to base his justification on the underlying structure of the pictorial context. In the case of Questions 8 and 9, it would seem that the pictorial frame of reference acted as a distraction, effectively complicating the generalisation process.

When the pictorial terms were presented as three consecutive terms, Ryan was even more successful with the generalisation process. The only question with which he had no success was Question 14, in which he extended the pictorial pattern backwards in order to generate a new first term:




Original question Ryan's modification

From the modified sequence, Ryan arrived at the general term $4 \times n$. Although this is a correct general expression for the modified sequence, it does not hold true for the original pattern. In this particular case it would seem that Ryan's fixation on the numbers enticed him into creating a new first term in order to convert the pattern into the sequence of multiples of 4.

For the last six questions, presented as purely numeric terms, Ryan managed excellently. The exception was the very last question, where he attempted to use a short-cut approach based on multiples of 4, and resorted to an erroneous (uncorrected) whole-object method, as described in Section 5.5.6.

**Table 5.18** Pupil profile - Mary

| Question | Method | | | Overall stage descriptor | CCR | TSA |
|---|---|---|---|---|---|---|
| | Next | 10th | 50th | | | |
| **1** | Co | Ex | Ex | 4 | 1 | 10 |
| **2** | Co/Ex | Ex | Ex | 4 | 0.5 | 10 |
| **3** | Co/Ex | Ex | Ex | 4 | 1 | 10 |
| **4** | Co/Ex | Ex | Ex | 4 | 0.5 | 10 |
| **5** | Ex | Ex | Ex | 0 | 0.5 | 0 |
| **6** | Co | Ex | Ex | 4 | 1 | 10 |
| **7** | Ex | Ex | Ex | 4 | 0 | 10 |
| **8** | Ex | Ex | Ex | 4 | 0 | 10 |
| **9** | Co/Ex | Ex | Ex | 1 | 0 | 1 |
| **10** | - | - | - | 0 | - | 0 |
| **11** | Co | Ex | Ex | 1 | 0 | 1 |
| **12** | Co/Ex | Ex | Ex | 4 | 1 | 10 |
| **13** | Ex | Ex | Ex | 0 | 0 | 0 |
| **14** | Ex | Ex | Ex | 1 | 0.5 | 1 |
| **15** | Ex | Ex | Ex | 0 | 0 | 0 |
| **16** | Ex | Ex | Ex | 1 | 0 | 1 |
| **17** | Co | DP | DP | 1 | - | 1 |
| **18** | Co | DP | DP | 1 | - | 1 |
| **19** | Na | Ex | Ex | 0 | - | 0 |
| **20** | Co | Ex | Ex | 1 | - | 1 |
| **21** | Ex | Ex | Ex | 4 | - | 10 |
| **22** | Co | Ex | Ex | 0 | - | 0 |
| **AVERAGE** | - | - | - | 1.95 | 0.38 | 4.41 |

Mary made use of 4 different strategies during the course of the study – counting, explicit, difference product, and nature of numerical terms. The explicit strategy was used most frequently (51 instances), followed by the counting strategy (12 instances). Mary used a counting strategy in conjunction with an explicit strategy five times. Each time that the counting strategy was used in a pictorial context, Mary either drew the next term in the sequence or adjusted a pre-existing term by adding additional elements (e.g. matches or dots) to it.

A brief glance at Table 5.18 shows a very clear trend, a trend which becomes even more pronounced when average TSA values are compared for the different question designs:

**Table 5.19** Average TSA values per question type - Mary

| Question design | Questions | Average TSA |
|---|---|---|
| Single pictorial term | 1 – 5 | 8.00 |
| Two non-consecutive pictorial terms | 6 – 10 | 6.20 |
| Three consecutive pictorial terms | 11 – 16 | 2.17 |
| Three consecutive numeric terms | 17 - 22 | 2.17 |

Mary fared best with the generalisation problems presented as a single pictorial term. In all five questions her justification was linked, or at least partially linked, to the pictorial context. Interestingly, the only question of the five where her visual reasoning was faulty was also the only question of the five where she didn't physically draw the next term in the sequence. One can only speculate that had she done so she may well have experienced a better understanding of the pictorial context and thus been able to have correctly interpreted the scenario.

She fared slightly less well with the patterns presented as two non-consecutive pictorial terms. Mary's general term for Question 6 was justified in terms of the pictorial context. Those for Questions 7 and 8 were not justified in terms of the pictorial context, but there was evidence of the pictorial patterns having assisted in the generalisation process. In Question 9, Mary made no reference to the given pictorial terms, and focused primarily on the given numbers. She managed to determine the next term in the sequence by means of a recursive approach, and then searched for a general formula which would afford a numerical answer. She adopted this erroneous formula in order to calculate numerical answers for Stage 2 and 3, and justified the algebraic expression on the basis of it having worked in one particular instance. In Question 10, Mary's focus became so fixated on determining an algebraic expression to account for the two given terms, that she was unable even to employ a simple counting strategy to determine the next term in the sequence. She was unable to progress with this question simply because she "couldn't find the formula".

When the pictorial terms were presented as three consecutive terms, Mary seems to have largely ignored the pictorial context and rather focused on the numerical aspects of the situation, either making use of the common difference in her algebraic formula, or applying a trial and error approach by comparing the dependent and independent variables. Rather than simply focusing on one diagram and trying to find a global strategy, Mary seems to have been distracted by the consecutive nature of the diagrams in these particular questions.

Mary fared equally poorly for the last six questions, presented as three consecutive purely numeric terms. Once again, her focus seemed to be drawn to either the difference between the terms, or to numerical properties of the numbers themselves. This prompted Mary to take unjustified and inappropriate short-cuts in determining her responses to the various stages of the generalisation process. As an additional complication, Mary seemed to struggle conceptually with the algebraic notions of $n-1$ and $n+1$. Without a pictorial frame of reference, which may well have provided some scaffolding for this conceptual difficulty, the purely numerical/algebraic context is likely to have exacerbated this problem.

In summary, Ryan and Mary responded differently to the various pattern generalisation problems, a difference most likely arising from a tension between their preferred modes of cognitive processing, and the context and design of the patterning tasks. While Ryan became more successful as the problems moved away from a pictorial context into a more numerical one, for Mary this change in question design had exactly the opposite effect. For other pupils, the context of the generalisation problems had little to no effect on their success rate. A graphical summary of the comparison between Ryan and Mary is given in Figure 5.2.

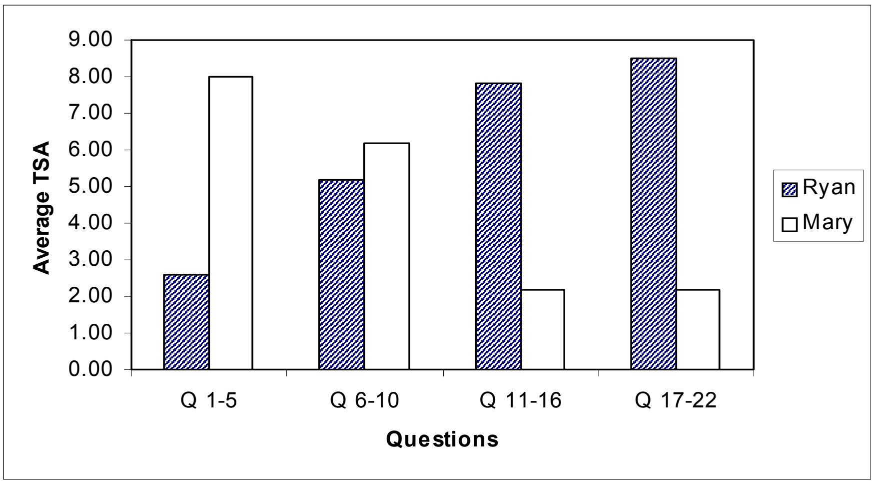



**Figure 5.2** Comparison of average TSA vs. question design for Ryan and Mary

# CHAPTER SIX

# FINDINGS & CONCLUSION

## 6.1 INTRODUCTION

The purpose of this final chapter is to consolidate the findings of this study within the context of the original research question, and with reference to the adopted theoretical framework and methodological choices. In addition, both the limitations and significance of the study are interrogated, and some recommendations for further research are suggested.

## 6.2 REVIEW OF THEORETICAL FRAMEWORK

While embracing the basic tenets of constructivism, central to this study is the fundamental notion that constructivism is a *descriptive* as opposed to *prescriptive* philosophy (Towers and Davis, 2002:314). Built onto this philosophy is the firm belief in the use of both language and notation systems/representations as important mediators in the process of construction – both in terms of their contribution to the organisation of the thinking process itself, as well as the cyclical nature of reflection (Kaput, 1991).

The role of visualisation is also central to the present study, and it is acknowledged that while generalisation problems presented in a pictorial or practical context have the potential to widen the scope of solution strategies for some pupils, for others this may well create additional complications (Orton et al., 1999). Thus, from a theoretical perspective, the methodology employed in the data capturing process is sensitive to the relative roles of the verbal-logical and visual-pictorial components of a pupil's cognitive processes. In addition, both the data capturing and data analysis methodologies take cognizance of the role of visualization in the generalisation process.

The notions of generalisation, justification and proof are intricately interwoven. Generalisation, by its very nature, can not be separated from justification/proof, and justification is seen as a critical component of the generalisation process. The types of generalisation activities included in this study purposefully include those presented in pictorial contexts, thus allowing for a possible connection to a referential context that has the potential to aid and enhance the generalisation process. The central role of proof within the context of this study is seen as communication of mathematical understanding, and students' justifications of their generalisations are seen to provide "…a window to view their understanding of the general nature of their rules" (Lannin, 2005:251).

## 6.3 REVIEW OF METHODOLOGY

This study is based on a qualitative investigation framed within an interpretive paradigm. According to Cohen and Manion (1994:36), the central endeavour within the context of the interpretive paradigm is "to understand the subjective world of human experience". Furthermore, attempting to see a situation as perceived by another human being should be imbued "with the assumption that the constructions of others … have integrity and sensibility within another's framework" (Confrey, 1990:108). Thus, the essential character underpinning the data analysis of the present study is the treatment of all responses, particularly those that are unexpected or idiosyncratic, with a genuine interest in understanding their character and origins.

The present study attempts to interrogate pupils' responses to various linear generalisation tasks from both a technical as well as strategic viewpoint. A small-scale pilot study conducted prior to commencement of the main study suggested that one would gain "insight into the [research] question by studying a particular case" (Stake, 1995:3), in this instance a class of high ability learners. A case study methodological strategy was accordingly adopted and an appropriate group of participants was identified.

Over a period of 3 months, 24 Grade 9 pupils from a mixed gender, high ability class each completed a series of 22 pencil and paper exercises based on linear generalisation tasks set in both numeric and 2-dimensional pictorial contexts. More specifically, numeric patterns were presented as a simple sequence of numbers as well as in tabular form, while pictorial patterns were presented using three consecutive terms, two non-consecutive terms, or one single term. For each pattern, participants were required to provide numerical values for the next, $10^{th}$ and $50^{th}$ terms as well as a written articulation of their reasoning at each stage. Participants were also asked to provide an algebraic expression for the $n^{th}$ term as well as to justify their expression. In addition to written responses, individual participants were informally interviewed where the written articulation of their mental reasoning was either ambiguous or required illumination by oral explication.

The responses to the various linear generalisation questions were classified by means of stage descriptors as well as stage modifiers. The method or strategy adopted for determining each of the next, $10^{th}$ and $50^{th}$ terms was carefully analysed and classified into one of seven categories. In addition, a separate framework was used to characterise each pupil's justification of the $n^{th}$ term in terms of the extent to which the justification was linked to the pictorial context. A meta-analysis of the generalisation/justification process was also undertaken. The stage descriptors and modifiers, together with the adopted solution strategies and justification characterisation, were used to create a rich profile for each research participant as well as for each individual pattern generalisation task.

## 6.4 FINDINGS OF THIS STUDY

### 6.4.1 STRATEGY CHOICE – A GLOBAL PICTURE

In terms of calculating the next term in a given sequence, two strategies clearly dominated: counting (43%) and explicit (56%). The counting strategy represents a recursive or iterative approach, what Lannin (2004:217) remarks as being an almost natural tendency. It is thus not surprising that this strategy proved popular at Stage 1. The explicit strategy, however, requires the construction of a general formula in terms of the independent variable. In this regard it is perhaps a little surprising that so many pupils used this strategy at Stage 1, where a simple counting strategy would have been a more direct approach.

At Stage 2, the explicit strategy became far more dominant (89%). Stage 2 represents what Stacey (1989:150) refers to as a “near generalisation” task, where a step-by-step counting procedure would still be within the bounds of practicality. It is interesting that, already at this point, so many pupils rejected a recursive approach in favour of an explicit strategy, since the recursive approach would have required at most an additional 6 iterations from Stage 1.

Stage 3 is what Stacey (1989:150) refers to as a “far generalisation” task – a question that goes beyond reasonable practical limits of a step-by-step counting/drawing approach. At Stage 3, the explicit strategy accounted for almost 96% of all responses, while the counting strategy was not employed at all.

In total, the explicit strategy represented almost 80% of all strategies used, while a step-by-step counting procedure accounted for 17.5%. The predominance of the explicit strategy is surprising in view of the common theme in the research literature (MacGregor and Stacey, 1993; Hargreaves et al., 1998; Hershkowitz et al., 2002) which relates to the tendency of pupils to generalise recursively rather than using the independent variable, i.e. the explicit strategy. Furthermore, English and Warren (1998) found that once students had established a recursive strategy they were reluctant to search for a functional relationship. This is certainly not the case in the present study, where there is a clear indication of pupils changing from a counting strategy to an explicit strategy when moving from Stage 1 to Stage 2.

A particularly interesting observation is the number of instances where two different strategies were employed in the same stage, both resulting in the same correct numerical answer. The only two strategies used in combination were the counting and explicit strategies. In total, 84 separate instances were noted where these two strategies were used in combination. In some instances this merely amounted to extending the numerical sequence by means of the constant difference and using this to confirm the answer obtained from the explicit strategy. However, in the vast majority of cases the combined strategy entailed checking the answer obtained from the explicit strategy by means of drawing the required pictorial term and counting the required elements (matches, dots etc.). In addition to merely acting as a check, the physical act of drawing a pictorial representation of the *desired* term could also possibly have served as a meaningful *specific* reference for investigating the general structure underlying the pictorial context. This may well have assisted some pupils in the generalisation process in terms of seeing the general in the particular and hence moving towards an algebraic expression for the general term.

### 6.4.2 STAGE CLASSIFICATION – A GLOBAL PICTURE

On the whole, the research participants fared well with the majority of the questions. In 17 of the 22 questions more than 80% of the research participants were able to provide a correct Stage 4 response, and 5 pupils managed to give correct responses at all four stages in all 22 questions. It is important to bear in mind that Stages 0 through 4 do not represent a hierarchical structure. Although the stages do represent an increasing level of difficulty, being classified at a particular stage is not dependent on providing correct responses for all the previous stages.

The high general level of success, in terms of reaching a correct Stage 4 response, can be ascribed to at least two reasons. Firstly, all of the questions chosen for this investigation were linear/arithmetic sequences - i.e. of the form $ax \pm c$ where $a$ and $c$ are constants. This was a purposeful decision, as the literature review suggested that linear sequences would allow the research participants to progress further in the type of pattern generalisation tasks envisaged, and would thus more likely constitute "information-rich cases" (Patton, 1990:169). Secondly, the high success rate can in part be ascribed to the fact that the research participants represent a group of high ability learners. Once again this was a purposeful decision based on the argument that a high ability group of learners would be better suited to a methodology in which the data collection process required learners to attempt to articulate their own cognitive reasoning.

### 6.4.3 CONTEXTUAL CONNECTIVITY RATING – A GLOBAL PICTURE

The CCR proved to be a useful mechanism for ascribing a numerical value to the extent to which pupils' written justifications of their general terms were specifically linked to the pictorial (as opposed to numerical) context. The average CCR values for individual research participants showed a good spread – from 1.00 to 0.22. Only one pupil made express reference to the pictorial context in all 16 questions. The average CCR values for individual questions were also well spread – from 0.92 to 0.23. There was a definite downward trend of CCR values as the questions progressed from 1 through 16. This is an important observation, and its significance is discussed later in this chapter.

A comparison of CCR with TSA showed that while a higher CCR seems to correspond to a higher average TSA for *some* pupils, the exact opposite was true for others. This observation is central to the theme of this study, and the notion that different contexts (numeric vs. pictorial) resonate differently with different pupils. While a pictorial context may be helpful to some pupils, for others it may simply create additional complications. Furthermore, some pupils may simply opt to convert a pictorial pattern into a numerical equivalent, and give no further thought to the pictorial context.

### 6.4.4 INFLUENCE OF QUESTION DESIGN ON STRATEGY

MacGregor and Stacey (1993), cite one of the main causes of difficulty in formulating algebraic rules as being pupils' tendency to focus on the recursive patterns of one variable rather than the relationship linking the two variables. Similar observations have been made by other researchers (e.g. Orton, 1997). Question design seems to have an important influence on pupils' choice of strategy, and the findings of this study would suggest that the tendency to pattern recursively could be discouraged by careful question design.

There was a dramatic drop in the number of pupils using the counting strategy when two non-consecutive pictorial terms were used instead of a single pictorial term. There are two possible reasons for this. Firstly, a single pictorial term may not be a sufficient scaffold to enable some pupils to derive a general expression. A second diagram, physically drawn by the pupil, may have been necessary in order to see the general structure underlying the pictorial context. Thus using a counting strategy at Stage 1 may be a necessary prerequisite to moving to an explicit strategy at Stage 2 for some pupils. Secondly, one can not ignore the fact that questions incorporating two non-consecutive pictorial terms tended to have slightly bigger physical structures compared to the single term scenario, and drawing the next diagram in such a case may have been considered impractical by some pupils.

There was a dramatic increase in the number of pupils using the counting strategy when three consecutive purely pictorial terms were used instead of two non-consecutive pictorial terms. This increase was even more pronounced when the three consecutive terms were accompanied by an indication of the dependent variable. The initial increase could be a result of two possibilities. Firstly, the fact that the three consecutive pictorial terms are the first three terms in the sequence, the physical structures of the pictorial representations are a little less complex than in the case of the two non-consecutive terms. This may have encouraged pupils to simply draw the next term rather than looking for an explicit strategy. Secondly, because the three consecutive terms give a physical representation of growth, pupils may have been drawn to the recursive nature of the pattern and simply added the common difference to the third term in order to obtain a numerical value for the next term. This seemed to be slightly more often the case than simply drawing the next term and counting the number of elements. The even greater increase when the three consecutive terms are accompanied by an indication of the dependent variable can be explained in terms of the common difference having been made somewhat more explicit by the inclusion of the dependent variable and pupils thus being drawn even more towards a recursive strategy.

The simple presentation of three consecutive purely numeric terms resulted in the highest proportion of pupils opting for the recursive strategy – both in Stage 1 and Stage 2. Once again, the common difference becomes immediately clear from the given terms, and pupils seem to have been drawn towards this, and used a recursive approach as a result. Interestingly, when the three consecutive numeric terms were put into table format with the inclusion of the independent variable, there was a slight drop in the tendency to pattern recursively. One can only surmise that the explicit presence of both dependent and independent variables assisted some pupils in seeing a general relation between the two and hence being more inclined to use an explicit strategy over a recursive approach.

The above observations lend support to Hershkowitz et al. (2002) who found that the presentation of consecutive terms encouraged recursion, while terms presented non-consecutively tended to encourage generalisation by means of the independent variable. Hershkowitz et al. (2002) also found that the use of a pictorial context, particularly if non-consecutive terms were presented, tended to encourage explicit generalisations. The results of the present study give strong support to the notion that question design can play a key role in influencing which strategies are adopted by pupils when solving pattern generalisation tasks.

### 6.4.5 INFLUENCE OF QUESTION DESIGN ON STAGE PROGRESS

The highest level of progress was achieved on those questions presented purely numerically, in tabular format. The explicit presence of both the dependent and independent variable, along with the fact that the terms were consecutive and hence made the common difference easier to recognise, all seem to have allowed for greater overall attainment. This finds resonance with a study by English and Warren (1998) where students found it easier to generalise, both verbally and symbolically, when patterns were presented in tabular form as opposed to pictorial form.

The lowest level of progress was achieved on those questions presented as three consecutive purely pictorial terms where no mention was made of either the dependent or independent variable. This adds weight to the notion that a pictorial representation is only of benefit if the underlying structure can be clearly seen and correctly interpreted. Thus, while a purely pictorial context may be useful to some pupils, to others it may well create complications. A contextualised indication of both the dependent and independent variable (e.g. for 2 squares you will need 7 matchsticks) in conjunction with the pictorial representation seemed to be most successful in alleviating this problem.

### 6.4.6 INFLUENCE OF QUESTION DESIGN ON DIVERSITY OF $T_n$ EXPRESSIONS

The dramatic drop in the number of correct $T_n$ variations for those questions incorporating purely numeric terms is both expected and understandable, since the lack of a referential (pictorial) context severely limits the scope of readily identifiable variations in $T_n$. Without a pictorial frame of reference, expressions for $T_n$ can only be derived from purely numeric considerations, the resulting expressions usually taking the form $a+(n-1)d$ or $dn+(a-d)$, or those deriving fortuitously from a guess-and-check approach.

The increase in the number of correct $T_n$ variations when moving from questions involving a single pictorial term to those making use of two non-consecutive pictorial terms can probably be ascribed to pupils' enhanced appreciation of the underlying general structure inherent in the pictorial context as a result of the additional term. The same argument could be applied when moving from two pictorial terms to three pictorial terms, although it is worth bearing in mind that some pictorial designs yield fewer accessible (easily identifiable) expressions for $T_n$.

The richness and diversity of algebraic representations of $T_n$ derived for each question was most informative, and prompted a meta-analysis of the generalised formulae in conjunction with their justification. The results of the meta-analysis are discussed later in the chapter.

### 6.4.7 QUESTION DESIGN – FURTHER CONSIDERATIONS

Pupils' responses gave evidence of the complex interplay between the number pattern itself, the nature of the question design and the *specific* pictorial context chosen. This interwoven complexity, and its interpretation and treatment with the diverse cognitive skills of each individual pupil, will ultimately be manifested in choice of strategy, progress through the different stages, contextual connectivity, and the diversity of $T_n$ expressions. There is thus a high degree of interconnectedness, and correlations between different aspects should be treated with due circumspection.

### 6.4.8 MECHANISMS OF VISUALISATION

A meta-analysis of the Stage 4 responses was prompted by the diversity of algebraic representations of $T_n$. The meta-analysis focused on the formula derived for the $n^{th}$ term in conjunction with its justification. The process of justification proved to be a critical factor in being able to accurately interpret the origin of the sub-structure evident in many of the Stage 4 responses. The use of a pictorial context allowed pupils to make use of a *generic example* within this reference frame as a means of scaffolding the justification process. From a theoretical perspective, the central role of proof within the context of this study is seen as communication of mathematical understanding. The process of justification/proof proved to be highly successful in providing a window of understanding into each pupil's general formula. The results of the meta-analysis revealed a diverse number of fascinating visually driven generalisations and gives strong support for the use of a pictorial context to enhance both visual approaches to generalisation and justification, as well as intensifying the diversity of resulting general solutions. While visualisation played an important role for many pupils in the structuring of the general formula, it is worth bearing in mind that for others visualisation played very little role, a numerically based derivation of the general formula being favoured.

A number of mechanisms of visualisation became apparent from the meta-analysis, and are quite revealing in terms of the subtlety and complexity of the visual reasoning evident in the generalisation strategies. Most visual strategies began by deconstructing a *generic example* into a number of component parts. In some instances these component parts were further subdivided into even smaller parts. This decomposition of the *generic example* is essentially a retro-synthesis of the whole into perceived component parts. For questions involving matchsticks, the complexity of the subdivisions ranged from single matches (vertical or horizontal), V-shapes (2 matches), U-shapes (3 matches), triangles (3 matches), squares (4 matches) and oddly shaped question-specific additive units. For questions involving dots or tiles, the structure was most often visualised in terms of rows and columns of dots/tiles of varying length. The component part was thus the single dot/tile, its frequency of appearance being the length of the columns and rows. Once separated into component parts, the visualisation process became one of reconstruction by means of multiplying the various parts by the frequency of their appearance, and finally summing the various multiples and constants together to arrive at a final general term. Visualisation strategies also had to frequently take into account a correcting mechanism for overlapping units.

Throughout this study there was much evidence of visualisation playing a leading role both in terms of the justification of pupils' general rules and the actual structuring of the general rules themselves. The essential character underpinning the data analysis of this study was the treatment of all responses, particularly those that were unexpected or idiosyncratic, with a genuine interest in understanding their character and origins. The data analysis process was thus imbued with the firm conviction that "the constructions of others … have integrity and sensibility within another's framework" (Confrey, 1990:108). This has critical bearing within an interpretive research paradigm, and allowed for the identification of a diverse range of other visually motivated strategies:

- The use of "negative space" to scaffold visual reasoning
- *Global* versus *recursive* observations
- *Dynamic* versus *static* visual imagery
- The use of "auxiliary constructions" (Hershkowitz et al., 2001:263)
- The visualisation of overlapping or *nested* structures
- Visual transformation or re-organisation of the original pictorial structure
- Identification of an imbedded *macro-structure*
- Visual strategies prompted by an otherwise irrelevant substructure element

In addition, there was also evidence to suggest that the presence or absence of *specific* terms may well attract or discourage a particular visually motivated strategy.

### 6.4.9 ANOMALIES AND IDIOSYNCRASIES

Orton and Orton (1999:120) mention that one of the four main obstacles to successful generalisation is the fact that "idiosyncratic methods are adopted by individual pupils in unpredictable ways". A number of anomalous and idiosyncratic approaches were observed in this study:

- Numeric strategies based on a trial and error approach
- "Guess and check" strategies based on spurious observations
- Approaches based on algebraically unhelpful generalisations
- General terms based on an implicitly redefined independent variable
- Physical alteration of the given pattern by addition or removal of term 1
- Looking for patterns within patterns
- Lack of progress due to a fixation with determining the general formula

## 6.5 LIMITATIONS

A case study approach was adopted as methodological strategy for this study. Accordingly, the members of a mixed gender, high ability Grade 9 class of 24 learners were chosen as research participants - "information-rich cases whose study will illuminate the questions under study" (Patton, 1990:169). This purposeful sampling was justified in terms of a small-scale pilot study undertaken prior to the commencement of the main study.

Although the emphasis of a case study is to optimise understanding of the specific case under scrutiny rather than generalisation beyond that case, a case study can nonetheless be a useful small step towards a larger generalisation, or an increasingly refined generalisation (Cohen and Manion, 1994; Stake, 1994, 1995). Thus, although any general trends or patterns observed in the course of this study are only relevant to the group of 24 research participants who took part in the study, such "generalisations" could be broadened or increasingly refined by future research involving further samples from the larger population.

Furthermore, pupils' responses gave evidence of the complex interplay between the number pattern itself, the nature of the question design and the *specific* numeric/pictorial context chosen. Choice of strategy, level of stage progression, contextual connectivity, and the diversity of $T_n$ expressions are a manifestation of this interwoven complexity in conjunction with the diverse cognitive skills of each individual pupil. There is thus a high degree of interconnectedness, and correlations between different aspects should be treated with due circumspection.

## 6.6 SIGNIFICANCE

The emphasis of the NCS on investigation as a pedagogical approach to number pattern generalisation tasks, as well as its requirement that learners be able to *investigate* number patterns and hence "make conjectures and generalisations" as well as "provide explanations and justifications and attempt to prove conjectures" (Department of Education, 2003b:18), has important pedagogical implications for classroom practitioners. An understanding of how question design of such pattern generalisation tasks is likely to influence the approach adopted by children would greatly assist teachers in terms of their choice of such activities. It is within this pedagogical context that this study finds practical significance and import.

The results of the present study give strong support to the notion that question design can play a key role in influencing which strategies are adopted by pupils when solving pattern generalisation tasks, in both pictorial and purely numeric contexts. This observation is central to the theme of this study, and the notion that different contexts (numeric vs. pictorial) will resonate differently with different pupils. While a pictorial context may be helpful to some pupils, for others it may simply create additional complications. Nonetheless, the use of a pictorial context allowed pupils to make use of a *generic example* within this reference frame as a means of scaffolding the justification process. From a theoretical perspective, the central role of justification/proof within the context of this study is seen as communication of mathematical understanding, and the process of justification/proof proved to be highly successful in providing a window of understanding into each pupil's general formula. Furthermore, this study identified a diverse range of visually motivated strategies. An awareness and appreciation for such a diversity of visualisation strategies has direct pedagogical application within the context of the classroom discourse.

## 6.7 RECOMMENDATIONS FOR FURTHER RESEARCH

The present study focused on a high ability group of learners. It would be interesting to repeat this study using a lower ability group of research participants. This would in all likelihood require a modification of the data collection protocol, however, as the small-scale pilot study suggested that lower ability pupils would probably struggle to sufficiently articulate a written explanation of their cognitive reasoning.

It would be equally interesting to repeat the present study with other high ability groups of learners, possibly with an augmented selection of patterning questions. This would serve to broaden and/or increasingly refine any localised "generalisations" identified in the present study. In addition, this would add further insight into the complex interplay between the number pattern, the nature of the question design and the *specific* numeric/pictorial context chosen.

## 6.8 CONCLUDING COMMENTS

The connection between mathematics and the notion of pattern is prevalent at all levels of mathematical endeavour. Goldin (2002:197) describes mathematics as "the systematic description and study of pattern." Furthermore, searching for patterns is an important strategy for mathematical problem solving (Stacey, 1989:147).

The study of pattern has become an integral component across all Grades of the South African school Mathematics curriculum (Department of Education, 2002, 2003b). In addition, the process of searching for patterns, making and testing conjectures, and formulating and justifying generalisations are essential in both mathematical thinking and the generation of mathematical knowledge (Thompson, 1985).

The results of this study strongly support the notion that question design can play a critical role in influencing pupils' choice of strategy and level of attainment when solving pattern generalisation tasks. Furthermore, this study identified a diverse range of visually motivated strategies and mechanisms of visualisation. An awareness and appreciation for such a diversity of visualisation strategies, as well as an understanding of the importance of appropriate question design, has direct pedagogical application within the context of the mathematics classroom.

Finally, "at the most basic level," as Adler (2005:2) succinctly puts it, "we have yet to understand how to make mathematics learnable by all children." It is hoped that this study will add, in some small way, to that growing discourse.

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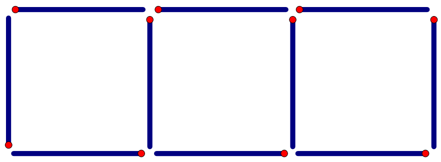
**Appendix A – MPRS Question 1**

NAME: ____________________________

Look at the diagram below. 10 matchsticks have been used to make a row of 3 squares.



3 squares require 10 matches

- How many matchsticks will you need to make a row of 4 squares? Explain and/or show HOW you got to your answer.

- How many matchsticks will you need for a row of 10 squares? Explain and/or show HOW you got to your answer.

- How many matchsticks will you need for a row of 50 squares? Explain and/or show HOW you got to your answer.

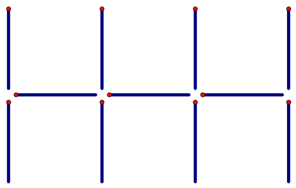
- Give an algebraic "rule" or "formula" to work out the number of matchsticks needed for a row of "n" squares .

- Justify your formula – i.e. explain WHY your formula works.

**Appendix A – MPRS Question 2**

NAME: ______________________________

Look at the diagram below. 11 matchsticks have been used to make the pattern.



A pattern with 3 **horizontal** matchsticks requires a **total** of 11 matchsticks

- How many matchsticks will you need in **total** if there were 4 horizontal matchsticks? Explain and/or show HOW you got to your answer.

- How many matchsticks will you need in **total** if there were 10 horizontal matchsticks? Explain and/or show HOW you got to your answer.

- How many matchsticks will you need in **total** if there were 50 horizontal matchsticks? Explain and/or show HOW you got to your answer.

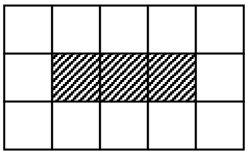
- Give an algebraic "rule" or "formula" to work out the number of matchsticks needed in **total** for a pattern containing "n" horizontal matchsticks .

- Justify your formula – i.e. explain WHY your formula works.

**Appendix A – MPRS Question 3**

NAME: ___________________________

Look at the diagram below. 12 white tiles have been used to build a border all the way round a row of 3 striped tiles.



For a row of 3 striped tiles there are 12 white tiles in the border.

- How many **white** tiles will you need to put a border around a row of 4 striped tiles? Explain and/or show HOW you got to your answer.

- How many **white** tiles will you need to put a border around a row of 10 striped tiles? Explain and/or show HOW you got to your answer.

- How many **white** tiles will you need to put a border around a row of 50 striped tiles? Explain and/or show HOW you got to your answer.

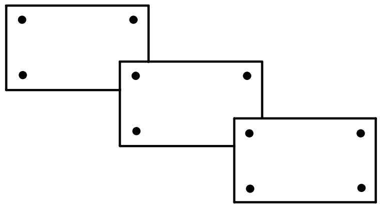
- Give an algebraic "rule" or "formula" to work out the number of **white** tiles you will need to put a border around a row of "n" striped tiles .

- Justify your formula – i.e. explain WHY your formula works.

**Appendix A – MPRS Question 4**

NAME: ___________________________

Look at the following diagram. Photos have been pinned to a board as shown below.



For 3 photos you need 10 drawing pins

- How many drawing pins will you need to display 4 photos in the same way? Explain and/or show HOW you got to your answer.

- How many drawing pins will you need to display 10 photos in the same way? Explain and/or show HOW you got to your answer.

- How many drawing pins will you need to display 50 photos in the same way? Explain and/or show HOW you got to your answer.

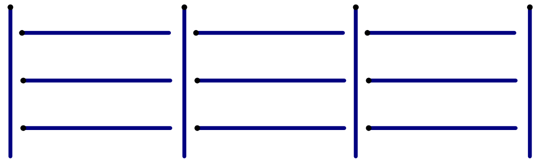
- Give an algebraic "rule" or "formula" to work out the number of drawing pins you will need for "n" photos.

- Justify your formula – i.e. explain WHY your formula works.

**Appendix A – MPRS Question 5**

NAME: ______________________________

Look at the following diagram. A "fence" has been built using matchsticks.



If there are 4 **vertical** matchsticks you need a **total** of 13 matchsticks.

- How many matchsticks will you need in **total** if there are 5 vertical matchsticks? Explain and/or show HOW you got to your answer.

- How many matchsticks will you need in **total** if there are 10 vertical matchsticks? Explain and/or show HOW you got to your answer.

- How many matchsticks will you need in **total** if there are 50 vertical matchsticks? Explain and/or show HOW you got to your answer.

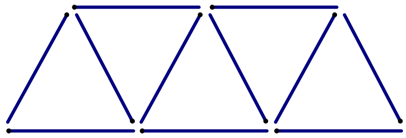
- Give an algebraic "rule" or "formula" to work out the **total** number of matchsticks you will need if there are "n" vertical matchsticks.

- Justify your formula – i.e. explain WHY your formula works.

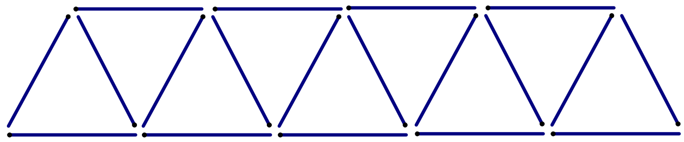
**Appendix A – MPRS Question 6**

NAME: ____________________________

Look at the following diagrams. “Bridges” have been built using matchsticks.



For 3 base matches you need a total of 11 matches



For 5 base matches you need a total of 19 matches.

- How many matchsticks will you need in **total** if there are 6 base matchsticks? Explain and/or show HOW you got to your answer.

- How many matchsticks will you need in **total** if there are 10 base matchsticks? Explain and/or show HOW you got to your answer.

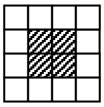
- How many matchsticks will you need in **total** if there are 50 base matchsticks? Explain and/or show HOW you got to your answer.

- Give an algebraic “rule” or “formula” to work out the **total** number of matchsticks you will need if there are “n” base matchsticks.

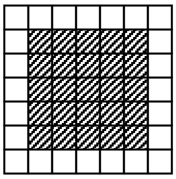
- Justify your formula – i.e. explain WHY your formula works.

NAME: ______________________________

Look at the following diagrams. A border of white tiles has been built around squares of striped tiles.



For a 2x2 square of striped tiles, 12 white tiles are needed.



For a 5x5 square of striped tiles, 24 white tiles are needed.

- How many white tiles will you need to put a border around a 6x6 square of striped tiles? Explain and/or show HOW you got to your answer.

- How many white tiles will you need to put a border around a 10x10 square of striped tiles? Explain and/or show HOW you got to your answer.

- How many white tiles will you need to put a border around a 50x50 square of striped tiles? Explain and/or show HOW you got to your answer.

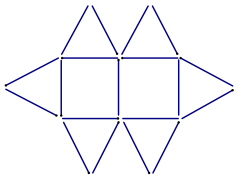
- Give an algebraic "rule" or "formula" to work out many white tiles you will need to put a border around an "nxn" square of striped tiles.

- Justify your formula – i.e. explain WHY your formula works.

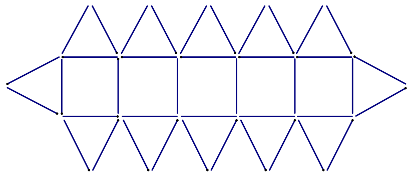
**Appendix A – MPRS Question 8**

NAME: ____________________________

Look at the following diagrams containing squares and triangles built from matchsticks.



For 2 squares you need a total of 19 matches.



For 5 squares you need a total of 40 matches.

- How many matchsticks will you need in **total** if there are 6 squares?
  Explain and/or show HOW you got to your answer.

- How many matchsticks will you need in **total** if there are 10 squares?
  Explain and/or show HOW you got to your answer.

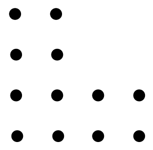
- How many matchsticks will you need in **total** if there are 50 squares?
  Explain and/or show HOW you got to your answer.

- Give an algebraic "rule" or "formula" to work out the **total** number of matchsticks you will need if there are "n" squares.

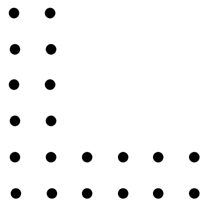
- Justify your formula – i.e. explain WHY your formula works.

NAME: ______________________________

Look at the following "double L" diagrams made from dots.



Base is 4 dots long



Base is 6 dots long

- How many dots will you need in **total** if the base is 7 dots long?
  Explain and/or show HOW you got to your answer.

- How many dots will you need in **total** if the base is 10 dots long?
  Explain and/or show HOW you got to your answer.

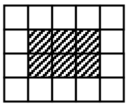
- How many dots will you need in **total** if the base is 50 dots long?
  Explain and/or show HOW you got to your answer.

- Give an algebraic "rule" or "formula" to work out the **total** number of dots you will need if the base is "n" dots long.

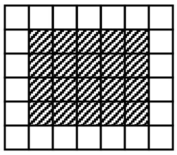
- Justify your formula – i.e. explain WHY your formula works.

NAME: ___________________________

Look at the following diagrams. A border of white tiles has been built around rectangles of striped tiles. The rectangles are always one tile longer than they are wide.



For a 2x3 square of striped tiles, 14 white tiles are needed.



For a 4x5 square of striped tiles, 22 white tiles are needed.

- How many white tiles will you need to put a border around a 5x6 square of striped tiles? Explain and/or show HOW you got to your answer.

- How many white tiles will you need to put a border around a 10x11 square of striped tiles? Explain and/or show HOW you got to your answer.

- How many white tiles will you need to put a border around a 50x51 square of striped tiles? Explain and/or show HOW you got to your answer.

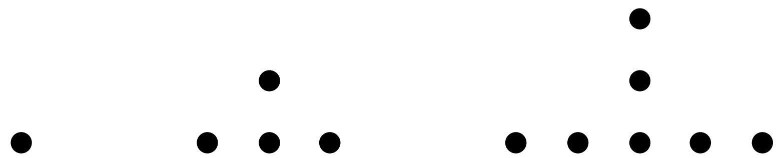
- Give an algebraic "rule" or "formula" to work out many white tiles you will need to put a border around an "nx(n+1)" square of striped tiles.

- Justify your formula – i.e. explain WHY your formula works.

**Appendix A – MPRS Question 11**

NAME: ____________________________

Look at the following growing pattern of dots.



- How many dots will you need for the next shape in the sequence?
  Explain and/or show HOW you got to your answer.

- How many dots will you need for the $10^{th}$ shape in the sequence?
  Explain and/or show HOW you got to your answer.

- How many dots will you need for the $50^{th}$ shape in the sequence?
  Explain and/or show HOW you got to your answer.

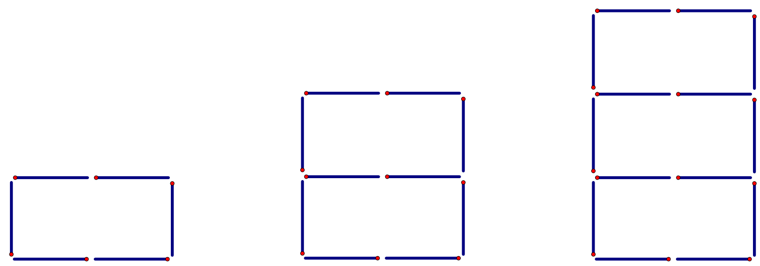
- Give an algebraic "rule" or "formula" to work out how many dots you will need for the $n^{th}$ shape in the sequence.

- Justify your formula – i.e. explain WHY your formula works.

## Appendix A – MPRS Question 12

NAME: ______________________________

Look at the following growing pattern of "towers" made from matchsticks.



- How many matchsticks will you need for the next shape in the sequence?
  Explain and/or show HOW you got to your answer.

- How many matchsticks will you need for the $10^{th}$ shape in the sequence?
  Explain and/or show HOW you got to your answer.

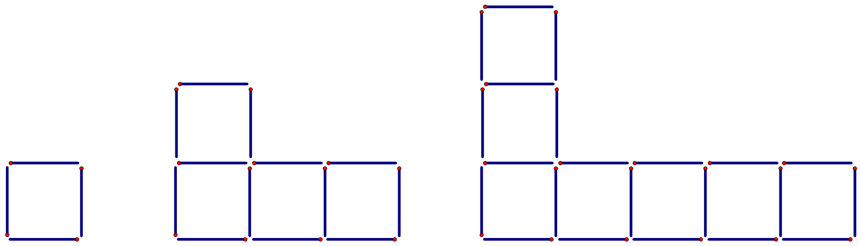
- How many matchsticks will you need for the $50^{th}$ shape in the sequence?
  Explain and/or show HOW you got to your answer.

- Give an algebraic "rule" or "formula" to work out how many matchsticks you will need for the $n^{th}$ shape in the sequence.

- Justify your formula – i.e. explain WHY your formula works.

NAME: ______________________________

Look at the following growing pattern of "skew L shapes" made from matchsticks.



- How many matchsticks will you need for the next shape in the sequence?
  Explain and/or show HOW you got to your answer.

- How many matchsticks will you need for the $10^{th}$ shape in the sequence?
  Explain and/or show HOW you got to your answer.

- How many matchsticks will you need for the $50^{th}$ shape in the sequence?
  Explain and/or show HOW you got to your answer.

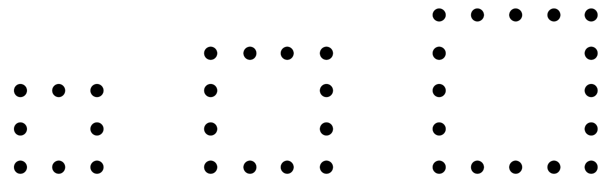
- Give an algebraic "rule" or "formula" to work out how many matchsticks you will need for the $n^{th}$ shape in the sequence.

- Justify your formula – i.e. explain WHY your formula works.

**Appendix A – MPRS Question 14**

NAME: ____________________________

Look at the following growing pattern of "hollow squares" made from dots.



8 dots 12 dots 16 dots

- How many dots will you need for the next shape in the sequence?
  Explain and/or show HOW you got to your answer.

- How many dots will you need for the $10^{th}$ shape in the sequence?
  Explain and/or show HOW you got to your answer.

- How many dots will you need for the $50^{th}$ shape in the sequence?
  Explain and/or show HOW you got to your answer.

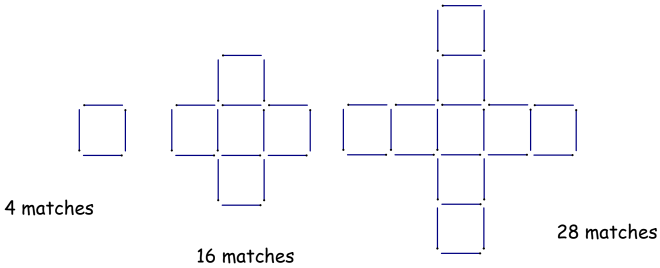
- Give an algebraic "rule" or "formula" to work out how many dots you will need for the $n^{th}$ shape in the sequence.

- Justify your formula – i.e. explain WHY your formula works.

**Appendix A – MPRS Question 15**

NAME: ____________________________

Look at the following growing pattern of "crosses" made from matchsticks.




- How many matchsticks will you need for the next shape in the sequence? Explain and/or show HOW you got to your answer.

- How many matchsticks will you need for the $10^{th}$ shape in the sequence? Explain and/or show HOW you got to your answer.

- How many matchsticks will you need for the $50^{th}$ shape in the sequence? Explain and/or show HOW you got to your answer.

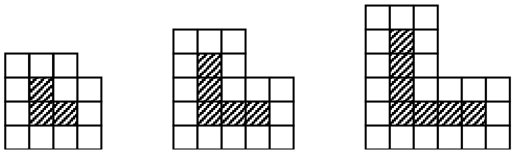
- Give an algebraic "rule" or "formula" to work out how many matchsticks you will need for the $n^{th}$ shape in the sequence.

- Justify your formula – i.e. explain WHY your formula works.

**Appendix A – MPRS Question 16**

NAME: ______________________________

Look at the following diagrams. A border of white tiles has been built around growing "L shapes" of striped tiles.



12 white tiles 16 white tiles 20 white tiles

- How many white tiles will you need in the next diagram? Explain and/or show HOW you got to your answer.

- How many white tiles will you need in the $10^{th}$ diagram? Explain and/or show HOW you got to your answer.

- How many white tiles will you need in the $50^{th}$ diagram? Explain and/or show HOW you got to your answer.

- Give an algebraic "rule" or "formula" to work out many white tiles you will need in the $n^{th}$ diagram.

- Justify your formula – i.e. explain WHY your formula works.

NAME: ____________________________

Look at the following sequence of numbers:

8 ; 12 ; 16 ; …

- What is the next number in the sequence? Explain and/or show HOW you got to your answer.

- What is the $10^{th}$ number in the sequence? Explain and/or show HOW you got to your answer.

- What is the $50^{th}$ number in the sequence? Explain and/or show HOW you got to your answer.

- Give an algebraic "rule" or "formula" to work out the $n^{th}$ number in the sequence.

- Try to explain HOW you arrived at your formula.

NAME: ____________________________

Look at the following sequence of numbers:

12 ; 19 ; 26 ; …

- What is the next number in the sequence? Explain and/or show HOW you got to your answer.

- What is the $10^{th}$ number in the sequence? Explain and/or show HOW you got to your answer.

- What is the $50^{th}$ number in the sequence? Explain and/or show HOW you got to your answer.

- Give an algebraic "rule" or "formula" to work out the $n^{th}$ number in the sequence.

- Try to explain HOW you arrived at your formula.

**Appendix A – MPRS Question 19**

NAME: ____________________________

Look at the following sequence of numbers:

3 ; 7 ; 11 ; ...

- What is the next number in the sequence? Explain and/or show HOW you got to your answer.

- What is the $10^{th}$ number in the sequence? Explain and/or show HOW you got to your answer.

- What is the $50^{th}$ number in the sequence? Explain and/or show HOW you got to your answer.

- Give an algebraic "rule" or "formula" to work out the $n^{th}$ number in the sequence.

- Try to explain HOW you arrived at your formula.

**Appendix A – MPRS Question 20**

NAME: ______________________________

Look at the following sequence of numbers:

| 1st | 2nd | 3rd | … |
|---|---|---|---|
| 4 | 13 | 22 | … |

- What is the next number in the sequence? Explain and/or show HOW you got to your answer.

- What is the 10th number in the sequence? Explain and/or show HOW you got to your answer.

- What is the 50th number in the sequence? Explain and/or show HOW you got to your answer.

- Give an algebraic "rule" or "formula" to work out the $n^{th}$ number in the sequence.

- Try to explain HOW you arrived at your formula.

**Appendix A – MPRS Question 21**

NAME: ____________________________

Look at the following sequence of numbers:

| $1^{st}$ | $2^{nd}$ | $3^{rd}$ | … |
|---|---|---|---|
| 4 | 7 | 10 | … |

- What is the next number in the sequence? Explain and/or show HOW you got to your answer.

- What is the $10^{th}$ number in the sequence? Explain and/or show HOW you got to your answer.

- What is the $50^{th}$ number in the sequence? Explain and/or show HOW you got to your answer.

- Give an algebraic "rule" or "formula" to work out the $n^{th}$ number in the sequence.

- Try to explain HOW you arrived at your formula.

NAME: ______________________________

Look at the following sequence of numbers:

| 1st | 2nd | 3rd | ... |
|---|---|---|---|
| 4 | 16 | 28 | ... |

- What is the next number in the sequence? Explain and/or show HOW you got to your answer.

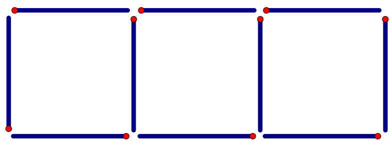
- What is the $10^{th}$ number in the sequence? Explain and/or show HOW you got to your answer.

- What is the $50^{th}$ number in the sequence? Explain and/or show HOW you got to your answer.

- Give an algebraic "rule" or "formula" to work out the $n^{th}$ number in the sequence.

- Try to explain HOW you arrived at your formula.

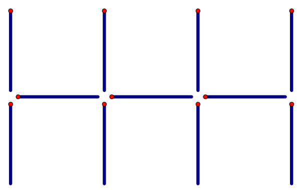
# QUESTION 1



3 squares require 10 matches

| | Stage classification | | | | | Method classification | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term ($t$) | Stage 2: $10^{th}$ term ($x$) | Stage 3: $50^{th}$ term ($y$) | Stage 4: $n^{th}$ term | Overall stage descriptor | Method for next term | Method for $10^{th}$ term | Method for $50^{th}$ term | Formula for $n^{th}$ term | Contextual Connectivity Rating |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $2x+(x+1)$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ | 1 |
| Carol | ✓ | ✓ | ✓ | × | 3 | Co | Ex | Ex | $4+3n$ | 1 |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3\times n+1$ | 1 |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $(n\times 3)+1$ | 1 |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 3+1$ | 1 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $x\times 3+1$ | 1 |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ | 1 |
| Jason | ✓ | ✓ | × | ✓ | 4y | Co | Co | Wu | $n3+1$ | 1 |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n\times 3+1$ | 1 |
| Kyle | ✓s | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ | 1 |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n\times 3+1$ | 1 |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ | 1 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Co Ex | Ex | $(n\times 3)+1$ | 1 |
| Mary | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(N\times 3)+1$ | 1 |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ | ½ |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ | 1 |
| Phil | ✓ | ✓ | ✓ | × | 3 | Co | Ch | Ch | $[(n-1)\times 3]-6$ | 0 |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $3x+1$ | 1 |
| Ryan | ✓ | × | × | × | 1 | Co | Wu | Wu | $3+n\times n$ | 0 |
| Sizwe | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $x\times 3+1$ | 1 |
| Sonya | ✓ | ✓ | × | ✓* | 4*y | Co | Ch | Ch | $n-x=y; y\times 3=b; b+x=your\ answer$ | ½ |
| Sue | ✓ | × | ✓ | ✓ | 4x | Co | DP | Ex | $(3\times n)+1$ | 1 |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4+3(n-1)$ | 1 |

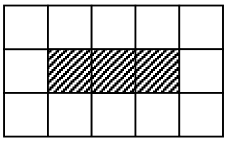
## QUESTION 2



A pattern with 3 **horizontal** matchsticks requires a **total** of 11 matchsticks

| | **Stage classification** | | | | | **Method classification** | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term ($t$) | Stage 2: $10^{th}$ term ($x$) | Stage 3: $50^{th}$ term ($y$) | Stage 4: $n^{th}$ term | Overall stage descriptor | Method for next term | Method for $10^{th}$ term | Method for $50^{th}$ term | Formula for $n^{th}$ term | Contextual Connectivity Rating |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $(N\times 3)+2$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+2$ | 1 |
| Carol | ✓ | ✓ | ✓ | × | 3 | Co | Ex | Ex | $5+3n$ | 1 |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+2$ | 1 |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $(n\times 3)+2$ | 1 |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 3+2$ | 1 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 3+2$ | 1 |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+2$ | 1 |
| Jason | ✓ | × | × | ✓ | 4xy | Co | Wu | Wu | $3n+2$ | 1 |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n\times 4)-(n-2)$ | 0 |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $3x+2$ | 1 |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n\times 3)+2=3n+2$ | 1 |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+2$ | 1 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Co Ex | Ex | $(3\times n)+2$ | 1 |
| Mary | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $(n\times 3)+2$ | ½ |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+2$ | ½ |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $2+3(n)$ *or* $5+(n-1)3$ | 1 |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $3n+2$ | 0 |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $2+(x\times 3)$ | 1 |
| Ryan | ✓ | × | × | ✓ | 4xy | Co | Wu | Wu | $(n\times 3)+2$ | 0 |
| Sizwe | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $x\times 3+2$ | 0 |
| Sonya | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n\times 3)+2$ | 1 |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n\times 3)+2$ | ½ |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(3\times n)+2$ | 1 |

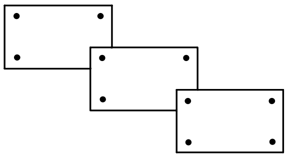
## QUESTION 3



For a row of 3 striped tiles there are 12 white tiles in the border.

| | **Stage classification** | | | | | **Method classification** | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term ($t$) | Stage 2: $10^{th}$ term ($x$) | Stage 3: $50^{th}$ term ($y$) | Stage 4: $n^{th}$ term | Overall stage descriptor | Method for next term | Method for $10^{th}$ term | Method for $50^{th}$ term | Formula for $n^{th}$ term | Contextual Connectivity Rating |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(N\times 2)+6$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $2n+6$ | 1 |
| Carol | ✓ | ✓ | ✓ | × | 3 | Ex | Ex | Ex | $8+2n$ | 1 |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $2(n+2)+2$ | 1 |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Ex | Ex | $(n\times 2)+6$ | 1 |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 2+6$ | 1 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Ex | Ex | $2(x+2)+2$ | 1 |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $2n+6$ | 1 |
| Jason | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ch | Ex | $2n+6$ | 1 |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n\times 2+6$ | 1 |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Ex | Ex | $2n+6$ | 1 |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $3(n+2)-n$ | 1 |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $2n+6$ | 1 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Co<br>Ex | Ex | $(n\times 2)+6$ | 1 |
| Mary | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Ex | Ex | $(n\times 2)+6$ | 1 |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Co<br>Ex | Ex | $2n+6$ | 1 |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $2n+6$ | 1 |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $2n+6$ | 1 |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $8+2(x-1)$ | 0 |
| Ryan | ✓ | × | × | × | 1 | Co | Wu | Wu | $(n+8)+2$ | 1 |
| Sizwe | ✓ | ✓ | ✓ | × | 3 | Co | Ex | Ex | $x\times 2+4$ | 0 |
| Sonya | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times 2)+6$ | 1 |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co | Ex | $[n+(n+2)+(n+2)+2]-n$ | ½ |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Ex | Ex | $2n+6$ | 1 |

## QUESTION 4

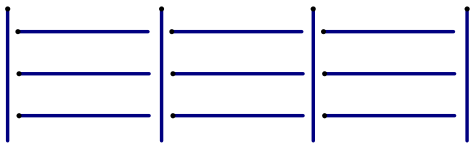


For 3 photos you need 10 drawing pins

| | Stage classification | | | | | Method classification | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term (*t*) | Stage 2: $10^{th}$ term (*x*) | Stage 3: $50^{th}$ term (*y*) | Stage 4: $n^{th}$ term | Overall stage descriptor | Method for next term | Method for $10^{th}$ term | Method for $50^{th}$ term | Formula for $n^{th}$ term | Contextual Connectivity Rating |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(N \times 3)+1$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ | 1 |
| Carol | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ | 1 |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ | 1 |
| Greg | ✓ | ✓ | ✓s | ✓ | 4 | Co<br>Ex | Ex | Ex | $(n \times 3)+1$ | 1 |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 3+1$ | 1 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 3+1$ | 1 |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ | 1 |
| Jason | ✓ | ✓ | × | ✓ | 4y | Co | Ex | Ex | $3n+1$ | 1 |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n \times 4-(n-1)$ | 1 |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Ex | Ex | $3n+1$ | 1 |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 3)+1$ | 1 |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ | 1 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Ex | Ex | $(n \times 3)+1$ | 1 |
| Mary | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Ex | Ex | $(n \times 3)+1$ | ½ |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Co<br>Ex | Ex | $3n+1$ | 1 |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ | 1 |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ | 1 |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ | 1 |
| Ryan | ✓ | × | × | ✓ | 4xy | Co | Wu | Wu | $(n \times 3)+1$ | 0 |
| Sizwe | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $x \times 3+1$ | 1 |
| Sonya | × | × | × | × | 0 | Ex | Ex | Ex | $(n \times 3)+2$ | 1 |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n \times 3)+1$ *or* $3n+1$ | ½ |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Ex | Ex | $3(n-1)+4$ | 1 |

**Appendix B – QRAS 5**

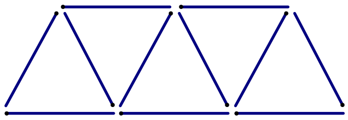
## QUESTION 5



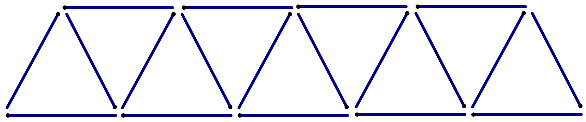
If there are 4 **vertical** matchsticks you need a **total** of 13 matchsticks.

| | Stage classification | | | | | Method classification | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term ($t$) | Stage 2: $10^{th}$ term ($x$) | Stage 3: $50^{th}$ term ($y$) | Stage 4: $n^{th}$ term | Overall stage descriptor | Method for next term | Method for $10^{th}$ term | Method for $50^{th}$ term | Formula for $n^{th}$ term | Contextual Connectivity Rating |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(3\times(N-1))+N$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3(n-1)+n$ | 1 |
| Carol | × | × | × | × | 0 | Ex | Ex | Ex | $5+4n$ | 1 |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n+3\times(n-1)$ | 1 |
| Greg | ✓ | ✓ | ✓ | ✓s | 4 | Ex | Ex | Ex | $(n\times4)-3$ | 1 |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n\times4-3$ | 1 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n+([n-1]\times3)$ | 1 |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n-1)\times3+n$ | 1 |
| Jason | ✓ | × | × | × | 1 | Co | Ch | Ex | $4n+1$ | 1 |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n\times4-3$ | 1 |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Ex | Ex | $4n-3$ | 1 |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $3(n-1)+n$ | 1 |
| Lucas | ✓ | × | × | × | 1 | Co | Ex | Ex | $4n+1$ | 1 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Co<br>Ex | Ex | $(n\times4)-3$ | 1 |
| Mary | × | × | × | × | 0 | Ex | Ex | Ex | $(n\times3)+1$ | ½ |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Co<br>Ex | Ex | $4n-3$ | 1 |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4(n-1)+1$ | 1 |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $3(n-1)+n \quad = \quad 4n-3$ | 1 |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n-3$ | 0 |
| Ryan | ✓ | × | × | × | 1 | Co | Wu | Wu | $n\times4+1$ | ½ |
| Sizwe | × | × | × | × | 0 | Co | Ex | Ex | $x\times3+1$ | 1 |
| Sonya | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times3)+n-3$ | 1 |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co | Ex | $3n+(n-3)$ | 0 |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3(n-1)+n \quad or \quad 4n-3$ | ½ |

## QUESTION 6



For 3 base matches you need a total of 11 matches

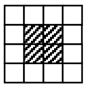


For 5 base matches you need a total of 19 matches.

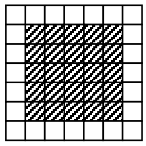
| Pupil | Stage classification | | | | | Method classification | | | | Contextual Connectivity Rating |
|---|---|---|---|---|---|---|---|---|---|---|
| | Stage 1: next term ($t$) | Stage 2: 10th term ($x$) | Stage 3: 50th term ($y$) | Stage 4: nth term | Overall stage descriptor | Method for next term | Method for 10th term | Method for 50th term | Formula for nth term | |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(N \times 3) + N - 1$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n(3) + (n-1)$ *or* $2n + n + (n-1)$ | 1 |
| Carol | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n + (n-1)$ | 1 |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n + (n-1)$ | 1 |
| Greg | ✓ | ✓ | ✓s | ✓ | 4 | Ex | Ex | Ex | $(n \times 4) - 1$ | 1 |
| Hannah | ✓ | ✓ | ✓s | ✓ | 4 | Ex | Ex | Ex | $n \times 4 - 1$ | 1 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 3 + (n-1)$ | 1 |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4x - 1$ | 1 |
| Jason | ✓ | × | × | ✓ | 4xy | Co | Wu | Wu | $n3 + n - 1$ | 1 |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 3 + (n-1)$ | 1 |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $4n - 1$ | 1 |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n - 1$ | 0 |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n - 1$ | 1 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $(n \times 4) - 1$ | 1 |
| Mary | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n \times 4) - 1$ | 1 |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $3n + (n-1)$ | 1 |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n - 1$ | 1 |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n - 1$ | 1 |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n + (n-1) + (n \times 2)$ | 1 |
| Ryan | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(N \times 4) - 1$ | 0 |
| Sizwe | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $x \times 5 - x - 1$ | 0 |
| Sonya | × | × | × | × | 0 | Ex | Ex | Ex | $3n + 2$ | 1 |
| Sue | × | × | × | ✓ | 4txy | Co | Co Ex | Ex | $(3 \times x) + x - 1$ *or* $3x + (x-1)$ | ½ |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 3) + (n-1)$ | 1 |

## Appendix B – QRAS 7

# QUESTION 7



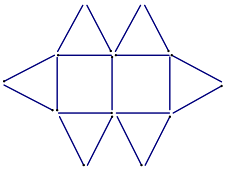
For a 2x2 square of striped tiles, 12 white tiles are needed.



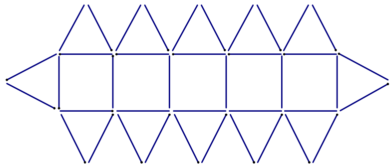
For a 5x5 square of striped tiles, 24 white tiles are needed.

| | Stage classification | | | | | Method classification | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term (*t*) | Stage 2: 10$^{th}$ term (*x*) | Stage 3: 50$^{th}$ term (*y*) | Stage 4: n$^{th}$ term | Overall stage descriptor | Method for next term | Method for 10$^{th}$ term | Method for 50$^{th}$ term | Formula for n$^{th}$ term | Contextual Connectivity Rating |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(N \times 4) + 4$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n(4) + 4$ | 1 |
| Carol | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n + 4$ | 1 |
| Dana | ✓ | × | ✓ | ✓ | 4x | Ex | Ex | Ex | $2(n+2) + 2n \quad or \quad 4n + 4$ | 1 |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 2) + (n \times 2) + 4$ | 1 |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 4 + 4$ | 1 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n+1) \times 4$ | 1 |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n + 4$ | 1 |
| Jason | ✓ | ✓ | ✓ | ✓ | 4 | Co | Wc | Wc | $4n + 4$ | 1 |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 4 + 4$ | 1 |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $4n + 4$ | 1 |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n+2)^2 - n^2$ | 1 |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4(n+1)$ | 1 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $(n \times 4) + 4$ | 1 |
| Mary | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n + 4$ | 0 |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n + 4 \quad or \quad 4(n+1)$ | 1 |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n + 4$ | 1 |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(2+n)(2+n) - (n \times n)$ | ½ |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n + 4$ | 1 |
| Ryan | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 4) + 4$ | 0 |
| Sizwe | × | × | × | × | 0 | Co | - | - | - | - |
| Sonya | ✓ | ✓ | × | ✓ | 4y | Ex | Ex | Ex | $(2n \times 2) + 4$ | 1 |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $[(n+2)2] + 2n$ | 1 |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n + 4$ | 0 |

# QUESTION 8



For 2 squares you need a total of 19 matches.



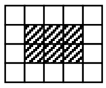
For 5 squares you need a total of 40 matches.

| | Stage classification | | | | | Method classification | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term ($t$) | Stage 2: 10th term ($x$) | Stage 3: 50th term ($y$) | Stage 4: nth term | Overall stage descriptor | Method for next term | Method for 10th term | Method for 50th term | Formula for nth term | Contextual Connectivity Rating |
| Alex | ✓ | ✓ | ✓s | ✓ | 4 | Ex | Ex | Ex | $(N \times 6)+(N+1)+4$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $7n+5$ | ½ |
| Carol | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $[3(n-1)+4]+[2(2n+2)]$ | 1 |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $7x+5$ | 1 |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Ex | Ex | $(n \times 7)+5$ | 0 |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 7+5$ | 1 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 3)+(n \times 3)+(2 \times 3)+n-1$ | 1 |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $7n+5$ | 1 |
| Jason | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n \times 7+5$ | 1 |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 7+5$ | 1 |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Ex | Ex | $7n+5$ | 1 |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $7n+5$ | 1 |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $7n+5$ | 1 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Ex | Ex | $(n \times 7)+5$ | 1 |
| Mary | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $7x+5$ | 0 |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Co<br>Ex | Ex | Ex | $7n+5$ | 1 |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $7(n)+5$ | 1 |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $7n+5$ | 1 |
| Richard | ✓ | × | × | × | 1 | Co | Ex | Ex | $n+(n \times 7)$ | 0 |
| Ryan | × | × | × | × | 0 | Co | Ch | Ex | $7+12+(6 \times n-2)$ | 1 |
| Sizwe | × | × | × | × | 0 | Ex | Ex | Ex | $x^2+15$ | 0 |
| Sonya | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n+3n+5$ | 1 |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $7x+5$ | 0 |
| Ted | ✓s | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(3n+1)+4n+4=7n+5$ | 1 |

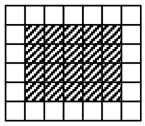
## QUESTION 9


| | Stage classification | | | | | Method classification | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term ($t$) | Stage 2: $10^{th}$ term ($x$) | Stage 3: $50^{th}$ term ($y$) | Stage 4: $n^{th}$ term | Overall stage descriptor | Method for next term | Method for $10^{th}$ term | Method for $50^{th}$ term | Formula for $n^{th}$ term | Contextual Connectivity Rating |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(N\times2)+((N-2)\times2)$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $4(n-2)+4$ | 1 |
| Carol | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $2n+2(n-2)$ | 1 |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $4n-4$ | 1 |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times4)-4$ | 1 |
| Hannah | ✓ | ✓ | ✓s | ✓ | 4 | Co | Ex | Ex | $4n-4$ | 1 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times2+(n-2)\times2$ | 1 |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4(n-1)$ | 1 |
| Jason | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ch | Ex | $(n2)+(n-2)2$ | 1 |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $2n+(2n-4)$ | 1 |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n-4$ | 1 |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(2n-1)+(2n-3)$ | 1 |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n-4$ | 1 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Co Ex | Ex | $(n\times2)+(n-2)\times2$ | 1 |
| Mary | ✓ | × | × | × | 1 | Co Ex | Ex | Ex | $(n\times3)-1+4$ | 0 |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n-4$ | 1 |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4(n-1)$ | 0 |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n-4$ | 1 |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $3n+(n-4)$ | 0 |
| Ryan | × | × | × | × | 0 | Ex | Ex | Ex | $(n+n)+(n-1+n)$ | 1 |
| Sizwe | ✓ | ✓ | ✓ | ✓* | 4* | Co | Ex | Ex | $x-1,\times4$ | 0 |
| Sonya | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n+(n-1)+(2n-3)$ | 1 |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co Ex | Ex | $[(n\times2)-1]+[(n\times2)-3]$ | ½ |
| Ted | ✓ | ✓ | × | ✓ | 4y | Ex | Ex | Ex | $(2n-1)+(2n-3)=4n-4$ | 1 |

# QUESTION 10



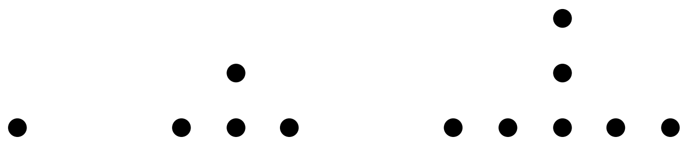
For a 2x3 square of striped tiles, 14 white tiles are needed.



For a 4x5 square of striped tiles, 22 white tiles are needed.

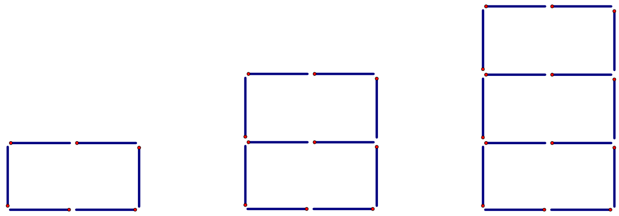
| Pupil | Stage classification | | | | | Method classification | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Stage 1: next term ($t$) | Stage 2: $10^{th}$ term ($x$) | Stage 3: $50^{th}$ term ($y$) | Stage 4: $n^{th}$ term | Overall stage descriptor | Method for next term | Method for $10^{th}$ term | Method for $50^{th}$ term | Formula for $n^{th}$ term | Contextual Connectivity Rating |
| Alex | ✓ | ✓ | ✓s | ✓* | 4* | Ex | Ex | Ex | $(N\times 2)+(N+1\times 2)+4$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $2n+2(n+1)+4$ | 1 |
| Carol | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n+(n+1)+n+(n+1)+4$ | 1 |
| Dana | ✓ | ✓ | ✓ | ✓s | 4 | Ex | Ex | Ex | $2(n+1)+2(n)+4$ | 1 |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times 2)+((n+1)\times 2)+4$ | 1 |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $2(n+n+1)+4$ | 1 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $2(n+1)+2(n+1+1)$ | 1 |
| James | ✓ | ✓s | ✓ | ✓ | 4 | Ex | Ex | Ex | $4(n+1)+2$ | 1 |
| Jason | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(2n)+(n+1)2+4$ | 1 |
| Julian | ✓ | ✓ | ✓ | ✓* | 4* | Ex | Ex | Ex | $base\ number\times 4+2$ | 0 |
| Kyle | ✓ | ✓ | ✓ | ✓* | 4* | Co | Ex | Ex | $(n\times 2+y\times 2)+4$ | 1 |
| Lisa | ✓ | ✓ | ✓ | × | 3 | Ex | Ex | Ex | $(n\times(n+1))-((n+2)\times(n+3))$ | 1 |
| Lucas | ✓ | ✓ | ✓ | × | 3 | Ex | Ex | Ex | $2(n+1+n+1)$ | 1 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $(n\times 2)+(n+1)\times 2+4$ | 1 |
| Mary | × | × | × | × | 0 | - | - | - | - | - |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $2n+2n+2+4=4n+6$ | 1 |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $2(n)+2(n+1)+4=4n+6$ | 1 |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $[(n+2)(n+3)]-[n\times(n+1)]$ | ½ |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $2(n+(n+1))+4$ | 1 |
| Ryan | ✓ | ✓ | ✓ | × | 3 | Ex | Ex | Ex | $n\times 2\ \ (n\times 2)+4$ | 0 |
| Sizwe | ✓ | × | × | × | 1 | Ex | Ex | Ex | Verbal description | 0 |
| Sonya | ✓ | ✓s | ✓s | ✓ | 4 | Ex | Ex | Ex | $2n+2(n+1)+4$ | 1 |
| Sue | ✓ | ✓ | ✓ | × | 3 | Co Ex | Ex | Ex | $[(n+2)2]+2n$ | ½ |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4(n+1)+2=4n+6$ | 0 |

# QUESTION 11



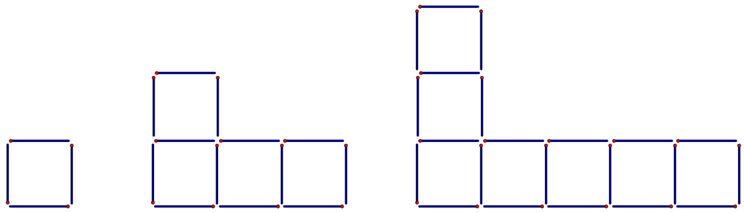
| | Stage classification | | | | | Method classification | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term (*t*) | Stage 2: $10^{th}$ term (*x*) | Stage 3: $50^{th}$ term (*y*) | Stage 4: $n^{th}$ term | Overall stage descriptor | Method for next term | Method for $10^{th}$ term | Method for $50^{th}$ term | Formula for $n^{th}$ term | Contextual Connectivity Rating |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $N+(2\times(N-1))$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $3(n-1)+1$ | 1 |
| Carol | ✓ | ✓s | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n-2$ | 1 |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n^{th}+2(n^{th}-1)$ | 1 |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times3)-2$ | 1 |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n\times3-2$ | 0 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n-1)\times3+1$ | 1 |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n-2$ | 0 |
| Jason | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n-1)\times3+1$ | 0 |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times3-2$ | 0 |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $3n-2$ | ½ |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co | Ex | $n\times3-2$ | 0 |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $3(n-1)+1$ | 1 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co | Ex | $(n\times3)-2$ | 0 |
| Mary | ✓ | × | × | × | 1 | Co | Ex | Ex | $(n\times3)+1$ | 0 |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $3n-2$ | ½ |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n-2$ | 1 |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $3n-2$ | 1 |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $3\times(n-1)+1$ | 0 |
| Ryan | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times3)-2$ | 0 |
| Sizwe | ✓ | ✓ | ✓ | ✓* | 4* | Co | Ex | Ex | $n^{th}-1, answer\times3, +1$ | 0 |
| Sonya | × | × | × | × | 0 | Co Ex | Ex | Ex | $(3\times n)+1$ | 0 |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Co Ex | Ex | $3(n-1)+1$ | 1 |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n-1)+2(n-1)+1=3n-2$ | 1 |

## QUESTION 12



| | Stage classification | | | | | Method classification | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term ($t$) | Stage 2: 10th term ($x$) | Stage 3: 50th term ($y$) | Stage 4: nth term | Overall stage descriptor | Method for next term | Method for 10th term | Method for 50th term | Formula for nth term | Contextual Connectivity Rating |
| Alex | × | × | × | × | 0 | Ex | Ex | Ex | $((N-1)\times 6)+4$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(4\times n)+2$ | 1 |
| Carol | ✓ | ✓ | ✓ | × | 3 | Ex | Ex | Ex | $4n+6$ | 1 |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n+2$ | 1 |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times 4)+2$ | 1 |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n\times 4+2$ | 1 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 4+2$ | 1 |
| James | ✓ | ✓s | ✓ | ✓s | 4 | Ex | Ex | Ex | $4n+2$ | 1 |
| Jason | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 4+2$ | 1 |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 4+2$ | 0 |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n+2$ | 1 |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n\times 4+2$ | 1 |
| Lucas | × | × | × | × | 0 | Ex | Ex | Ex | $3n+1$ | 1 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $(4\times n)+2$ | 1 |
| Mary | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $(n\times 4)+2$ | 1 |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n+2$ | 1 |
| Owen | × | × | × | × | 0 | Ex | Ex | Ex | $3n+1$ | 1 |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n+2$ | 1 |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n+2$ | 1 |
| Ryan | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co Ex | Ex | $(n\times 4)+2$ | 0 |
| Sizwe | ✓ | ✓ | ✓ | ✓* | 4* | Co | Ex | Ex | $n^{th}-1, answer\times 4, +6$ | ½ |
| Sonya | ✓ | ✓ | ✓ | ✓* | 4* | Co | Ex | Ex | $(n-1\times 4)+6$ | 1 |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(2n)+[2(n+1)]$ | 1 |
| Ted | ✓ | ✓ | ✓ | × | 3 | Ex | Ex | Ex | $6+(n\times 4)=4n+6$ | 1 |

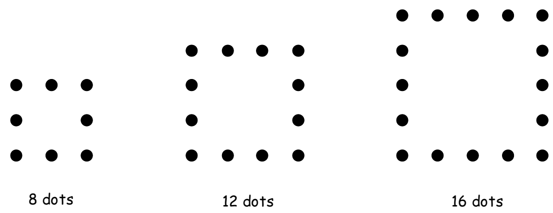
## QUESTION 13



| | Stage classification | | | | | Method classification | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term (*t*) | Stage 2: $10^{th}$ term (*x*) | Stage 3: $50^{th}$ term (*y*) | Stage 4: $n^{th}$ term | Overall stage descriptor | Method for next term | Method for $10^{th}$ term | Method for $50^{th}$ term | Formula for $n^{th}$ term | Contextual Connectivity Rating |
| Alex | × | × | × | × | 0 | Ex | Ex | Ex | $(4\times N)+((N-1)\times 2)+6+(3\times(N-1))$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(9\times n)-5$ | ½ |
| Carol | ✓ | × | × | × | 1 | Ex | Ex | Ex | $2n\times 3+4=6n+4$ | 1 |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $3(2n^{th}+(n^{th}-2))+1$ | 1 |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times 9)-5$ | 0 |
| Hannah | × | × | × | × | 0 | Co | Co Ex | Ex | $(n\times 3+n\times 3+3(n-1))-1$ | 1 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $9n-5$ | 0 |
| James | ✓ | ✓ | ✓s | ✓ | 4 | Ex | Ex | Ex | $9n-5$ | 0 |
| Jason | × | ✓ | ✓ | ✓* | 4*t | Ex | Ex | Ex | $[4+(n-1\times 2)\times 3)]+[n-1\times 3]$ | 1 |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 9-5$ | 0 |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n+(n-1)\times 5$ | ½ |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(3n-2)\times 3+1=9n-5$ | 1 |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3(3(n-1)+1)+1$ | 1 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $(9\times n^{th})-5$ | 0 |
| Mary | × | × | × | × | 0 | Ex | Ex | Ex | $(n\times 4)+2$ | 0 |
| Nell | ✓ | × | × | × | 1 | Co | - | - | - | - |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3(3(n-1)+1)+1=9n-5$ | 1 |
| Phil | ✓ | × | × | × | 1 | Co | Ex | Ex | $2(3n)+7$ | ½ |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4+(n-1)9$ | ½ |
| Ryan | ✓ | ✓s | ✓ | × | 3 | Co Ex | Ex | Ex | $(n\times 9)+4$ | 0 |
| Sizwe | ✓ | ✓ | ✓ | ✓*s | 4* | Co | Ex | Ex | $n^{th}-1, answer\times 9, +4$ | 0 |
| Sonya | ✓ | × | × | × | 1 | Ex | Ex | Ex | $(9\times n)+4$ | 1 |
| Sue | ✓ | × | × | × | 1 | Co Ex | Co Ex | Ex | $(n\times 2)+(n+1)+3(n+1)$ | ½ |
| Ted | ✓ | × | × | × | 1 | Ex | Ex | Ex | $3(3n-2)+4$ | 1 |

**Appendix B – QRAS 14**

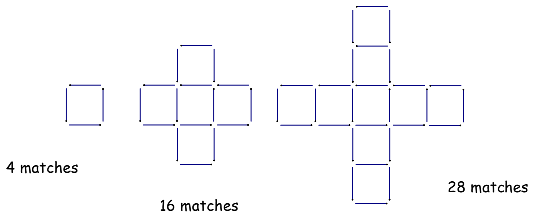
# QUESTION 14




| | Stage classification | | | | | Method classification | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term ($t$) | Stage 2: 10th term ($x$) | Stage 3: 50th term ($y$) | Stage 4: nth term | Overall stage descriptor | Method for next term | Method for 10th term | Method for 50th term | Formula for nth term | Contextual Connectivity Rating |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(N \times 4)+4$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(4 \times n)+4$ | 1 |
| Carol | ✓ | × | × | × | 1 | Co | Ex | Ex | $8+4n$ | ½ |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $2(n+2)+2n=4(n^{th})+4$ | 1 |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 4)+4$ | 1 |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n+1) \times 4$ | 1 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 4+4$ | 1 |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n+4$ | 0 |
| Jason | ✓ | ✓ | ✓ | ✓s | 4 | Co | Ex | Ex | $4+(n \times 4)$ | 0 |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 4+4$ | 0 |
| Kyle | × | ✓ | ✓ | ✓ | 4t | Co | Ex | Ex | $4(n+2)-4$ | ½ |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co | Ex | $n \times 4+4$ | 0 |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $2(n+2)+2n$ | 1 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 4)+4$ | 1 |
| Mary | ✓ | × | × | × | 1 | Ex | Ex | Ex | $(n \times 4)-4$ | ½ |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $2(n+2)+n+n=4n+4$ | 1 |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n+4$ | 1 |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n+4$ | 0 |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4+n \times 4$ | 0 |
| Ryan | ✓ | × | × | × | 1 | Co | DP | DP | $4 \times n$ | 0 |
| Sizwe | ✓ | ✓ | ✓ | ✓* | 4* | Co | Ex | Ex | $n^{th}+1, multiply\ answer\ by\ 4$ | ½ |
| Sonya | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n-1) \times 4+8$ | ½ |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co<br>Ex | Ex | $2(n+2)+2n$ | 1 |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n-1 \times 4)+8=4n-4+8=4n+4$ | 0 |

**Appendix B – QRAS 15**

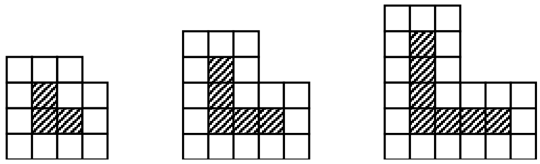
## QUESTION 15




| Pupil | Stage classification | | | | | Method classification | | | | Contextual Connectivity Rating |
|---|---|---|---|---|---|---|---|---|---|---|
| | Stage 1: next term ($t$) | Stage 2: $10^{th}$ term ($x$) | Stage 3: $50^{th}$ term ($y$) | Stage 4: $n^{th}$ term | Overall stage descriptor | Method for next term | Method for $10^{th}$ term | Method for $50^{th}$ term | Formula for $n^{th}$ term | |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(((N-1)\times 3)\times 4)+4$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $[(n-1)\times 12]+4$ | 1 |
| Carol | ✓ | × | × | × | 1 | Co | Ex | Ex | $4n\times 3+4=12n+4$ | ½ |
| Dana | × | ✓ | ✓ | ✓ | 4t | Ex | Ex | Ex | $3n^{th}+3(n^{th}-1)+1+3n^{th}+3(n^{th}-2)$ | 1 |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times 12)-8$ | 0 |
| Hannah | ✓ | × | × | × | 1 | Co | - | - | - | - |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n-1)\times 3\times 4+4$ | 1 |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $12n-8$ | 0 |
| Jason | × | × | × | × | 0 | DP | DP | DP | $(3\times 4)\times n$ | ½ |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 12-8$ | 0 |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $12n-8$ | 0 |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(4n-4)\times 3+4$ | 1 |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $12n-8$ | ½ |
| Mark | ✓ | ✓ | ✓ | ✓* | 4* | Co | Ex | Ex | $(n-1)\times 4=(answer\times 3)+4$ | 1 |
| Mary | × | × | × | × | 0 | Ex | Ex | Ex | $(n\times 4)-4$ | 0 |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $12(n-1)+4=12n-8$ | ½ |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4(3(n-1))+4$ | 1 |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $12n-8$ | 0 |
| Richard | ✓ | ✓ | ✓ | ✓s | 4 | Co | Ex | Ex | $4+(12(n-1))$ | 0 |
| Ryan | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(12\times(n-1))+4$ | 0 |
| Sizwe | ✓ | ✓ | ✓ | ✓* | 4* | Co | Ex | Ex | $n^{th}-1, answer\times 12, answer+4$ | 0 |
| Sonya | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $12(n-1)+4$ | ½ |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $[12(n-1)]+4$ | 0 |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $12(n-1)+4=12n-8$ | ½ |

**Appendix B – QRAS 16**

## QUESTION 16



12 white tiles    16 white tiles    20 white tiles

| Pupil | Stage classification | | | | | Method classification | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Stage 1: next term ($t$) | Stage 2: 10th term ($x$) | Stage 3: 50th term ($y$) | Stage 4: nth term | Overall stage descriptor | Method for next term | Method for 10th term | Method for 50th term | Formula for nth term | Contextual Connectivity Rating |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(((N\times2)+1)\times2)+6$ | 1 |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $[(n+n+1)2]+6$ | 0 |
| Carol | ✓ | × | × | × | 1 | Co | Ex | Ex | $4n+12$ | ½ |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4x+8$ | 0 |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times4)+8$ | 0 |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $4n+8$ | 0 |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $n\times4+8$ | 0 |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n+2)\times4$ | ½ |
| Jason | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n+2)\times4$ | 0 |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n+2)\times4$ | 0 |
| Kyle | ✓ | ✓ | ✓s | ✓ | 4 | Co | Ex | Ex | $4(n+2)$ | 0 |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(2n+1)\times2+6$ | 1 |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n+8$ | 0 |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times4)+8$ | 0 |
| Mary | ✓ | × | × | × | 1 | Ex | Ex | Ex | $(n\times4)-4$ | 0 |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $2(2n+1)+6=4n+8$ | 1 |
| Owen | × | × | × | × | 0 | Ex | Ex | Ex | $4(2(n)+1)=8n+4$ | 1 |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co | Ex | $4(2+n)$ *or* $4n+8$ | 0 |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $8+4n$ | 0 |
| Ryan | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4\times(n+1)+4$ | 0 |
| Sizwe | ✓ | ✓ | ✓ | ✓* | 4* | Co | Ex | Ex | $n-1, answer\times4, answer+12$ | 0 |
| Sonya | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times4)+8$ | 0 |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4(n+3)-4$ | ½ |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4(n-1)+12=4n+8$ | 0 |

## QUESTION 17

# 8 ; 12 ; 16 ; …

| | Stage classification | | | | | Method classification | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term ($t$) | Stage 2: 10th term ($x$) | Stage 3: 50th term ($y$) | Stage 4: nth term | Overall stage descriptor | Method for next term | Method for 10th term | Method for 50th term | Formula for nth term |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(N+1)\times 4$ |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4(n+1)$ |
| Carol | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co Ex | Ex | $4n+4$ |
| Dana | ✓ | × | × | × | 1 | DP | DP | DP | $4(n^{th})$ |
| Greg | ✓ | ✓ | ✓s | ✓ | 4 | Ex | Ex | Ex | $(n\times 4)+4$ |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4\times n+4$ |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 4+4$ |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n+4$ |
| Jason | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 4+4$ |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 4+4$ |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n\times 4)+4$ |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co | Ex | $n\times 4+4=4n+4$ |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n+4$ |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n+1)\times 4$ |
| Mary | ✓ | × | × | × | 1 | Co | DP | DP | $(n\times 4)$ |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n+4$ |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4(n+1)$ |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co Ex | Ex | $4n+4$ |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n+4$ |
| Ryan | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(4\times n)+4$ |
| Sizwe | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n\times 4+4$ |
| Sonya | ✓ | × | × | × | 1 | DP | DP | DP | $n\times 4$ |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co | Ex | $(n+1)4$ |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4(n-1)+8 \quad or \quad 4n+4$ |

## QUESTION 18

$$12 ; 19 ; 26 ; \ldots$$

| | Stage classification | | | | | Method classification | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term ($t$) | Stage 2: $10^{th}$ term ($x$) | Stage 3: $50^{th}$ term ($y$) | Stage 4: $n^{th}$ term | Overall stage descriptor | Method for next term | Method for $10^{th}$ term | Method for $50^{th}$ term | Formula for $n^{th}$ term |
| Alex | ✓ | × | × | × | 1 | Co | Ex | Ex | $(N\times 7)-2$ |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n\times 7)+5$ |
| Carol | ✓ | × | × | × | 1 | Co | Co Ex | Ex | $7n+7$ |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $7(n^{th})+5$ |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times 7)+5$ |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $7n+5$ |
| Helen | ✓ | ✓ | ✓s | ✓ | 4 | Ex | Ex | Ex | $n\times 7+5$ |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $7n+5$ |
| Jason | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 7+5$ |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 7+5$ |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co Ex | Ex | $(7n)+5$ |
| Lisa | × | × | × | × | 0 | Co | Co | - | - |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $7n+5$ |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(7\times n^{th})+5$ |
| Mary | ✓ | × | × | × | 1 | Co | DP | DP | $(n\times 7)$ |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $7n+5$ |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $7n+5$ |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co | Ex | $7n+5$ |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $7n+7-2$ |
| Ryan | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(7\times(n+1))-2$ |
| Sizwe | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n\times 7+5$ |
| Sonya | ✓ | × | × | × | 1 | Co | Ex | Ex | $(7\times n)-2$ |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co Ex | Ex | $(n\times 7)+5$ |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $7(n-1)+12=7n+5$ |

## QUESTION 19

3 ; 7 ; 11 ; …

| Pupil | Stage classification | | | | | Method classification | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Stage 1: next term ($t$) | Stage 2: 10th term ($x$) | Stage 3: 50th term ($y$) | Stage 4: nth term | Overall stage descriptor | Method for next term | Method for 10th term | Method for 50th term | Formula for nth term |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(N \times 4) - 1$ |
| Bianca | ✓ | ✓ | ✓s | ✓ | 4 | Co | Ex | Ex | $(4 \times n) - 1$ |
| Carol | ✓ | ✓ | × | × | 2 | Co | Co | Wu | - |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4x - 1$ |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 4) - 1$ |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 4 - 1$ |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 4 - 1$ |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n - 1$ |
| Jason | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n - 1$ |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $n \times 3 + (n - 1)$ |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n - 1$ |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co | Ex | $4n - 1$ |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n - 1$ |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n^{th} \times 4) - 1$ |
| Mary | × | × | × | × | 0 | Na | Ex | Ex | $(n \times 3) + 1$ |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n - 1$ |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n - 1$ |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4n - 1$ |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4(n) - 4 + 3$ |
| Ryan | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(4 \times n) - 1$ |
| Sizwe | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n \times 4 - 1$ |
| Sonya | ✓s | ✓ | ✓s | ✓ | 4 | Ex | Ex | Ex | $(4 \times n) - 1$ |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $(n \times 4) - 1$ |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4n - 1$ |

## QUESTION 20

| 1st | 2nd | 3rd | … |
|---|---|---|---|
| 4 | 13 | 22 | … |

| | Stage classification | | | | | Method classification | | | |
|---|---|---|---|---|---|---|---|---|---|
| Pupil | Stage 1: next term ($t$) | Stage 2: 10th term ($x$) | Stage 3: 50th term ($y$) | Stage 4: nth term | Overall stage descriptor | Method for next term | Method for 10th term | Method for 50th term | Formula for nth term |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(N \times 9) - 5$ |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 9) - 5$ |
| Carol | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $9n - 5$ |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $9x - 5$ |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 9) - 5$ |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $9n - 5$ |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 9 - 5$ |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $9n - 5$ |
| Jason | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n \times 9 - 5$ |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 9 - 5$ |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $9n - 5$ |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $9n - 5$ |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $9n - 5$ |
| Mark | ✓ | ✓s | ✓ | ✓ | 4 | Co | Ex | Ex | $(n^{th} \times 9) - 5$ |
| Mary | ✓ | × | × | × | 1 | Co | Ex | Ex | $(n \times 3) + 3 + 9$ |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $9n - 5$ |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $9n - 5$ |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $9n - 5$ |
| Richard | ✓ | ✓ | × | ✓ | 4y | Co | Ex | Ex | $4 + (n-1) \times 9$ |
| Ryan | ✓ | ✓ | ✓ | ✓* | 4* | Ex | Ex | Ex | $(9 \times n - 1) + 4$ |
| Sizwe | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n^{th} \times 9, -5$ |
| Sonya | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 9) - 5 = 9n - 5$ |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n^{th} \times 9) - 5$ |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4 + 9n - 9 = 9n - 5$ |

## QUESTION 21

| 1st | 2nd | 3rd | … |
|---|---|---|---|
| 4 | 7 | 10 | … |

| | Stage classification | | | | | Method classification | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Pupil** | Stage 1: next term ($t$) | Stage 2: 10th term ($x$) | Stage 3: 50th term ($y$) | Stage 4: nth term | Overall stage descriptor | Method for next term | Method for 10th term | Method for 50th term | Formula for nth term |
| Alex | ✓ | × | × | × | 1 | Co | Ex | Ex | $(N \times 3) - 1$ |
| Bianca | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 4) - (n-1)$ *or* $3n+1$ |
| Carol | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $3n+1$ |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3x+1$ |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 3)+1$ |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 3+1$ |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $3n+1$ |
| Jason | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n \times 3+1$ |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n \times 3+1$ |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co Ex | Ex | $3n+1$ |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n^{th} \times 3)+1$ |
| Mary | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 3)+1$ |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3n+1$ |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $3n+1$ |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4+(n-1) \times 3$ |
| Ryan | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n \times 3)+1$ |
| Sizwe | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $n^{th} \times 3+1$ |
| Sonya | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(3 \times n)+1 = 3n+1$ |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(n^{th} \times 3)+1$ |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $3(n-1)+4 = 3n+1$ |

## QUESTION 22

| 1st | 2nd | 3rd | ... |
|---|---|---|---|
| 4 | 16 | 28 | ... |

| Pupil | Stage classification | | | | | Method classification | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Stage 1: next term ($t$) | Stage 2: 10th term ($x$) | Stage 3: 50th term ($y$) | Stage 4: nth term | Overall stage descriptor | Method for next term | Method for 10th term | Method for 50th term | Formula for nth term |
| Alex | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(N\times 12)-8$ |
| Bianca | ✓ | × | × | × | 1 | Co | Ex | Ex | $(n\times 12)+4$ |
| Carol | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $12n-8$ |
| Dana | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $12x-8$ |
| Greg | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times 12)-8$ |
| Hannah | ✓ | ✓ | ✓ | ✓ | 4 | Co Ex | Ex | Ex | $12n-8$ |
| Helen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 12-8$ |
| James | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $12n-8$ |
| Jason | ✓ | ✓ | ✓s | ✓ | 4 | Co | Ex | Ex | $n\times 12-8$ |
| Julian | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $n\times 12-8$ |
| Kyle | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $12n-8$ |
| Lisa | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co | Ex | $12n-8$ |
| Lucas | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $12n-8$ |
| Mark | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $(12\times n^{th})-8$ |
| Mary | × | × | × | × | 0 | Co | Ex | Ex | $(n\times 4)+1$ |
| Nell | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $12n-8$ |
| Owen | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $12n-8$ |
| Phil | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $12n-8$ |
| Richard | ✓ | ✓ | ✓ | ✓ | 4 | Co | Ex | Ex | $4+(n-1)\times 12$ |
| Ryan | ✓ | × | × | × | 1 | Na | Wu | Wu | $4\times N+12$ |
| Sizwe | ✓ | ✓ | ✓ | ✓* | 4* | Co | Ex | Ex | $n^{th}\times 12,-8$ |
| Sonya | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $(n\times 12)-8$ |
| Sue | ✓ | ✓ | ✓ | ✓ | 4 | Co | Co Ex | Ex | $12(n^{th}-1)+4$ |
| Ted | ✓ | ✓ | ✓ | ✓ | 4 | Ex | Ex | Ex | $4+12(n-1)=12n-8$ |