Rhodes University

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A STUDY OF THE EXISTENCE OF EQUILIBRIUM IN MATHEMATICAL ECONOMICS

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Abstract

In this thesis we define and study the existence of an equilibrium situation in which producers maximize their profits relative to the production vectors in their production sets, consumers satisfy their preferences in their consumption sets under certain budget constraint, and for every commodity total demand equals total supply. This competitive equilibrium situation is referred to as the Walrasian equilibrium. The existence of this equilibrium is investigated from a various mathematical points of view. These include microeconomic theory, simplicial spaces, global analysis and lattice theory.

KEYWORDS:
Preference relation, Consumption set, Commodity bundle, Utility function, Production, Walrasian equilibrium, Fixed points, Welfare economics, Feasible allocation, Pareto optimality, Simplicial structure, Simplicial spaces, Signatures, Submanifold, Demand and Supply, Posets, Lattices, Complete lattices.

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Contents

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PREFACE

The founders of the mathematical theory of general economic equilibrium are Pareto [42] and Walras [56]. The aims of the mathematical theory in Walras [56] is to explain the price vector and the actions of the various economic agents observed in an economy in terms of an equilibrium resulting from the interaction of those agents through markets for commodities. This theory has been looked at extensively by a number of researchers, namely, Arrow([1], [2]), Debreu([17], [18]), D.Gale and A. Mas-Colell([23], [33]) and Smale [46]. They have developed the theory by making use of diverse fields of Mathematics and in doing so this has lead to a more rigorous mathematical approach to microeconomic theory. The main objective of this thesis is to investigate the existence of equilibrium in microeconomics from different fields in mathematics. We now give detailed description of the contents of the thesis.

In Chapter 1 we investigate the existence of equilibrium from a microeconomy theoretic point of view, which entails the study of preference relations. The existence of equilibrium is brought about by making use of [25] Kakutani's fixed point theorem. In section 1.6, We explore the relationship between welfare economics and equilibrium.

Chapter 2 introduces simplicial spaces which are spaces equipped with a simplicial structure and special nonnegative bivariate functions called signatures. In section 2.2 a result based on signatures and multivalued limit maps is central to this chapter. The subsequent sections pay attention to the applications of simplicial spaces and signatures to Kakutani [25] related fixed point theorems and the existence of equilibrium. A number of the results established with the use of signatures spans through different fields like genetic biology, game theory, mathematical analysis and economics.

Chapter 3 captures the existence of equilibrium from the notion of supply and demand of commodities. The argument is that there is a price vector such that the equality of supply and demand is achieved as demonstrated in [4], [22] and [46]. In contrast to the preceding chapters, this chapter makes use of purely mathematical results on solutions of systems of equations to solve the existence of equilibrium in mathematical economics. In section 3.1 these results are proved by making use of concepts from global analysis. This means that in this chapter there is no appeal made to fixed point theorems.

We introduce the concepts of a partially ordered set and lattices as done in Birkhoff [10], together with the functions that act between these structures in Chapter 4. In [36], [37] and [52] these functions on partially ordered sets and lattices are studied in detail as done in section 4.1 and section 4.2. Section 4.3 discusses the relationship between lattice theory and fixed point theory. The results in [13], [21], [51] and [58] establish the fact that fixed points are elements of complete lattices.

Throughout this thesis, acknowledgements to various authors are given where they are due. As far as we know the following are our own results:

Proposition 2.1.6, **Proposition 2.1.8**, **Proposition 2.3.4** and **Proposition 2.3.5**.

Chapter 1

A Microeconomic Theoretic Approach To The Existence of Walrasian Equilibrium

1.1 Introduction

The main objective in this chapter is to look at a microeconomic theoretic approach to the existence of Walrasian equilibrium. A majority of this work is covered in [3], [6], [33], [41] and [50] with the use of various concepts from microeconomic theory. The principal outcomes of this microeconomic theory entail the qualitative implications on observed demand of changes in the parameters which determine the decision of the economic agents involved. The study of this theory is very important in order to understand a more complicated and to an extent a more realistic microeconomic situation.

We consider an economy which consists of two kinds of economic agents or members, namely, producers and consumers. A finite number and a behavioral rule are assumed for each economic agent. It is usually assumed that each consumer maximizes the level of satisfaction and each producer maximizes profit. An integral part of microeconomics is to focus on each individual economic agent's behaviour and formulate mathematical models which assist in better understanding each economic agent in the economy.

These economic agents are concerned with the consumption and production of commodities which can be divided into goods and services. Each of these commodities is defined by the specification of all its physical characteristics, its availability date and location.

The concept of a consumption set is among the fundamental concepts in a branch of microeconomic theory called consumer theory. We define it here as discussed in [3], [33] and [50] in the following definition.

Definition 1.1.1 *The consumption set X for a consumer is a nonempty subset of the L-dimensional Euclidean space $\mathbb{R}^L$, with the list of different commodities $x_1, \ldots, x_L$ forming a commodity (consumption) bundle, $x = (x_1, x_2, \ldots, x_L) \in X$.*

Let p_ℓ be the unit price for commodity ℓ, where $\ell = 1, 2, 3, \ldots, L$ and $p_\ell \geq 0$.
If the consumer buys a bundle $x = (x_1, \ldots, x_L)$, then the consumer pays

$$\begin{aligned} p \cdot x &= p_1 x_1 + \cdots + p_L x_L \\ &= \sum_{\ell=1}^{L} p_\ell x_\ell \,. \end{aligned}$$

Let w be the consumer's wealth level.

The above setup enables us to define the notions of an affordable commodity bundle, the budget set and that of the demand function.

Definitions 1.1.2 *A commodity bundle $x = (x_1, \ldots, x_L)$ is affordable, given the price vector $p = (p_1, \ldots, p_L)$ and the wealth level w, if*

$$p \cdot x = p_1 x_1 + \cdots + p_L x_L \leq \mathrm{w}.$$

The set of all affordable commodity bundles x satisfying $p \cdot x \leq \mathrm{w}$ is said to be the budget set and is defined by

$$B_{p,\mathrm{w}} = \{x \in X : p \cdot x \leq \mathrm{w}\}.$$

The demand function $x_\ell(p, \mathrm{w})$ is the amount of commodity ℓ demanded, given the prices p and wealth level w.
Hence,

$$x(p, \mathrm{w}) = (x_1(p, \mathrm{w}), x_2(p, \mathrm{w}), \ldots, x_L(p, \mathrm{w}))$$

is the demand function of the consumption bundle $x = (x_1, \ldots, x_L)$.

1.1.1 Preference Relations

The notion of a preference relation is the more basic concept of consumer theory, which can be accepted as a starting point for analyzing consumer behavior. The following definition given in [6], [33], [41] and [50] of a preference relation assists in observing the satisfaction a consumer obtains after consuming a certain number of bundles.

Definition 1.1.3 *A preference relation $\succeq$ which is a binary relation on the consumption set X, is said to be rational, that is, it defines a rational consumer if it possesses the following properties:*

(PR1) Reflexive: $\forall\ x \in X,\ x \succeq x.$

(PR2) Complete: $\forall\ x, y \in X,$ *either* $x \succeq y$ *or* $y \succeq x.$

(PR3) Transitive: $\forall\ x, y, z \in X,$ *if* $x \succeq y$ *and* $y \succeq z$ *then* $x \succeq z.$

The preference relation $x \succeq y$ for $x, y \in X$ means that the bundle x *"is at least as preferable (good) as"* the bundle y. From a mathematical point of view **(PR1)** is an immediate consequence of **(PR2)**.

The relation of a strict preference relation $\succ$ can be derive from the above preference relation $\succeq$ in the following manner.

Definition 1.1.4 *The strict preference relation $\succ$ is defined on the consumption set X such that for any $x, y \in X$, the bundle x is said to be strictly preferred to the bundle y $(x \succ y)$ if $x \succeq y$ and not $y \succeq x$.*

The next result in Nikaido [41] gives a characterization for the strict preference relation $\succ$.

Lemma 1.1.5 *The strict preference relation $\succ$ on the consumption set X satisfies*

(SPR1) *For* $x \notin X,\ x \succ x.$

(SPR2) *For any* $x, y \in X,$ *either* $x \succ y$ *or* $y \succeq x.$

(SPR3) *For any* $x, y, z \in X,$ *if* $x \succeq y,\ \ y \succeq z$ *and either* $x \succ y$ *or* $y \succ z$ *then* $x \succ z.$
This means that for any $x, y, z \in X,$ *if* $x \succ y$ *and* $y \succ z$ *then* $x \succ z.$

Proof [*see* Nikaido [41]].

The indifference relation $\sim$ is another binary relation on the consumption set X, which can be derived from the preference relation $\succeq$ as demonstrated in the following definition.

Definition 1.1.6 *The indifference relation $\sim$ on the consumption set X is defined for any $x, y \in X$ by,*

$$x \sim y \Longleftrightarrow x \succeq y \text{ and } y \succeq x$$

and satisfies the following properties of an equivalence relation on X.

(IR1) Reflexive: $\forall\, x \in X,\ x \sim x.$

(IR2) Symmetric: $\forall\, x, y \in X,\ x \sim y \Longleftrightarrow y \sim x.$

(IR3) Transitive: $\forall\, x, y, z \in X,$ *if* $x \sim y$ *and* $y \sim z$ *then* $x \sim z.$

The indifference relation $\sim$ induces a classification of the elements of X to equivalence classes called the indifference classes. This means the $x \sim y$ for any $x, y \in X$ if and only if they belong to the same indifference class.

Now as done in [33], [50] and [54] we define a continuous preference relation as follows:

Definition 1.1.7 *A preference relation $\succeq$ on a consumption set $X \subseteq \mathbb{R}^L$ is said to be continuous if for every $x \in X$, the upper contour set $\{y \in X : y \succeq x\}$ and the lower contour set $\{y \in X : x \succeq y\}$ are closed sets.*

Remark 1.1.8 The following two sequential continuity conditions give an equivalent way of looking at the above definition of continuity.

(SC1) If $\{x^n\}$ is a sequence in X with $x^n \succeq x$ (respectively, $x \succeq x^n$), then $x^n \to \bar{x}$ implies $\bar{x} \succeq x$ (respectively, $x \succeq \bar{x}$).

(SC2) Let $\{x^n\}$ and $\{\bar{x}^n\}$ be two sequences in X such that $x^n \to x$ and $\bar{x}^n \to \bar{x}$. Suppose $\bar{x}^n \succeq x^n$ (respectively, $x^n \succeq \bar{x}^n$) then $\bar{x} \succeq x$ (respectively, $x \succeq \bar{x}$).

We can also define strict convexity and convexity of a preference relation $\succeq$ on the consumption set X.

Definitions 1.1.9 *A preference relation $\succeq$ on a consumption set X is said to be convex (respectively, strictly convex) if for all $x \in X$, the set $\{x' \in X : x' \succeq x\}$ is convex (respectively, strictly convex).*

The set $C = \{x' \in X : x' \succeq x\}$ is said to be convex, if $\alpha x' + (1-\alpha)y' \succeq x$ whenever $x', y' \in C$ and $\alpha \in [0,1]$.

The set $C = \{x' \in X : x' \succeq x\}$ is said to be strictly convex, if $\alpha x' + (1-\alpha)y' \succ x$ whenever $x', y' \in C$ and $\alpha \in (0,1)$.

The following definitions of a monotone, strongly monotone and locally nonsatiated preference relation $\succeq$ on the consumption set X are presented in [33], [50] and [54]. In this definition the norm refers to the standard Euclidean norm and ">" refers to the componentwise inequality in the Euclidean space.

Definitions 1.1.10 *A preference relation $\succeq$ on X is monotone if $x \in X$ and $y > x$ implies $y \succ x$. It is strongly monotone if $y \geq x$ and $y \neq x$ imply that $y \succ x$.*

The preference relation $\succeq$ on $X \subseteq \mathbb{R}^L$ is locally nonsatiated if for every $x \in X$ and every $\varepsilon > 0$, there is $y \in X$ such that $\|y - x\| \leq \varepsilon$ and $y \succ x$.

Remarks 1.1.11

(i) Strong monotonicity says that if y is larger than x for some commodity and is no less for any other, then y is strictly preferred to x.

(ii) Local nonsatiation says that for any consumption bundle x and any arbitrary small distance away from x, denoted by $\varepsilon > 0$, there is another bundle y within this distance from x that is preferred to x.

1.2 The Utility Function and Production

1.2.1 The Utility Function

The representation of a preference relation by a numerical function has been given a comprehensive attention in [12], [14], [15], [16] and [44]. This concept expresses each consumer's satisfaction and well-being by an index which is a real number. The relationship between the concepts of preference relation and utility function constitutes a major element of the foundations of consumer theory and behavior.

The next definition of a utility function captures the monotonic relationship between the preference relation $\succeq$ on the consumption set X and the corresponding utility function.

Definition 1.2.1 *Let X be a consumption set. A real-valued function $u : X \to \mathbb{R}$ is called a utility function for the preference relation $\succeq$ on X, if*

$$\forall\; x, y \in X,\; x \succeq y \text{ if and only if } u(x) \geq u(y).$$

Remark 1.2.2 It can be shown, with the use of the above definition, that for the indifference relation $\sim$ on X,

$$x \sim y \Longleftrightarrow u(x) = u(y).$$

Clearly from the definition of a utility function it follows that

$$\begin{aligned} x \sim y &\Longleftrightarrow x \succeq y \text{ and } y \succeq x &(1.1)\\ &\Longleftrightarrow u(x) \geq u(y) \text{ and } u(y) \geq u(x) &(1.2)\\ &\Longleftrightarrow u(x) \geq u(y) \geq u(x) &(1.3)\\ &\Longleftrightarrow u(x) = u(y). &(1.4)\end{aligned}$$

The problem about the existence of a utility function that can represent a preference relation $\succeq$ was first solved by Debreu [14] and later generalized by Rader [44]. We need the following concept of a connected space as defined in Engelking [20] and Willard [57] before we state the result by Debreu.

Definition 1.2.3 *Let X be a topological space. Then X is said to be a connected space if it cannot be represented as the union of two disjoint, nonempty, open sets.*

Theorem 1.2.4 *Let X be a connected subset of $\mathbb{R}^L$ and a continuous preference relation $\succeq$ be defined on X. Then there exists a continuous utility function on X for the given preference relation $\succeq$.*

Proof [*see* [14] and [44]].

The following result in [33] follows naturally from the order properties of the reals.

Theorem 1.2.5 *If a preference relation $\succeq$ on X has a utility function $u : X \to \mathbb{R}$ then $\succeq$ is rational.*

Proof.

(i) The reflexivity of $\succeq$ is clear.

(ii) For the completeness of $\succeq$, let $x, y \in X$. Then $u(x), u(y) \in \mathbb{R}$ and $u(x) \geq u(y)$ or $u(y) \geq u(x)$ if and only if $x \succeq y$ or $y \succeq x$. Thus, the relation $\succeq$ is complete.

(iii) For the transitivity of $\succeq$, for any $x, y, z \in X$ with $x \succeq y$ and $y \succeq z$ we have that $u(x) \geq u(y)$ and $u(y) \geq u(z)$. Now from the transitivity of $\geq$ on $\mathbb{R}$, $u(x) \geq u(z)$ if and only if $x \succeq y$. Thus $\succeq$ is transitive, which completes the proof. $\square$

1.2.2 Production

The main idea behind production is to investigate the supply side of an economy, with the intention to study the mechanisms by which commodities consumed by individual consumers are produced. An economy can equivalently be a firm, a collection of firms, the entire national economy or the whole world. In [33], [41] and [50] the production process is composed of a number of production units or firms, which are able to transform some quantities of goods as inputs into some amounts of products as outputs under certain given conditions in the production process.

The study of the production process makes use of set theory and other branches of mathematics. This is evident in [41] and [50] through the following definition of a production technology set or production set.

Definition 1.2.6 *A production set $Y \subseteq \mathbb{R}^L$ is a set consisting of production vectors $y = (-z, q) \in Y$ where the $L - M$ components $z = (z_1, \ldots, z_{L-M})$ form inputs and the M components $q = (q_1, \ldots, q_M)$ form outputs of the production vector $y = (-z_1 \ldots, -z_{L-M}, q_1, \ldots, q_M)$ where the components of z and q are non-negative.*

The production set Y can also satisfy the following properties:

(PS1) Y is nonempty

(PS2) Y is closed *, i.e., if $\{y^n\}$ is a sequence in Y and $y^n \to y$ then $y \in Y$.*

(PS3) Y is convex *, i.e. if $y, y' \in Y$ and $\lambda \in [0, 1]$ then $\lambda y + (1 - \lambda)y' \in Y$.*

(PS4) No free lunch *, i.e., if $y \in Y$ and $y \geq 0$ then $y = 0$.*

(PS5) Free disposal *, i.e., if $y \in Y$ and $y' \leq y$ then $y' \in Y$.*

(PS6) Irreversibility *, i.e., if $y \in Y$ and $y \neq 0$ then $-y \notin Y$.*

(PS7) Proportionality (Constant returns to scale) *, i.e., if $y \in Y$ and $\alpha \geq 0$, $\alpha \in \mathbb{R}$ then $\alpha y \in Y$.*

(PS8) Additivity (Free-entry) *, i.e., if $y \in Y$ and $y' \in Y$ then $y + y' \in Y$.*

We can define the concepts of a profit, profit maximizing production vector and an efficient production vector.

Definitions 1.2.7 *Let $Y \subseteq \mathbb{R}^L$ be a production set and the nonnegative orthant of $\mathbb{R}^L$ be denoted by $\mathbb{R}^L_+ = \{x \in \mathbb{R}^L : x_\ell \geq 0\, for\ \ell = 1, 2, 3, \ldots, L\}$.*
Given a price vector $p = (p_1, \ldots, p_L) \in \mathbb{R}^L_+$ and a production vector $y = (-z, q) \in Y$, the profit generated from implementing $y = (-z_1, \ldots, -z_{L-M}, q_1, \ldots, q_M)$ is $p \cdot y = p \cdot q - p \cdot z$, that is, profit = total revenue - total cost.
A production vector $\bar{y} \in Y$ is said to be a profit maximizing production vector if for any production vector $y \in Y$, $p \cdot \bar{y} \geq p \cdot y$.
A production vector $y \in Y$ is said to be an efficient production vector if there is no other $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.

Remark 1.2.8 From the above definition the non-negative orthant $P_y = y + \mathbb{R}^L_+ = \{y' \in \mathbb{R}^L : y' \geq y\}$ with vertex at y satisfies $P_y \cap Y = \{y\}$.

The next result is important and fundamental in the study of production processes.

Theorem 1.2.9 *If the production vector $y \in Y$ is profit maximizing for some price vector $p \in \mathbb{R}^L_+$, then y is efficient.*

Proof [*see* [33] and [50]]. Suppose not. Then there exist $y' \in Y$ such that $y' \neq y$ and $y' \geq y$. Thus $p \cdot y' > p \cdot y$, contradicting the assumption that y is profit maximizing.

$\square$

In [33] and [50] the converse to the previous result is shown to hold with the added assumption of convexity on the production set. The proof makes use of separating hyperplane theorem for convex sets.

Lemma 1.2.10 [Separating Hyperplane Theorem]. *Suppose that $B \subset \mathbb{R}^N$ is convex and closed, and that $x \notin B$. Then there is $p \in \mathbb{R}^N$ with $p \neq 0$, and a value $c \in \mathbb{R}$ such that $p \cdot x > c$ and $p \cdot y < c$ for every $y \in B$.*
More generally, suppose that the convex sets $A, B \subset \mathbb{R}^N$ are disjoint. Then there is $p \in \mathbb{R}^N$ with $p \neq 0$ and a value $c \in \mathbb{R}$, such that $p \cdot x \geq c$ for every $x \in A$ and $p \cdot y \leq c$ for every $y \in B$. That is, there is a hyperplane that separates A and B on different sides of it.

Proof [*see* [33] and [50]].

Theorem 1.2.11 *Let Y be a convex production set in $\mathbb{R}^L$. Then every efficient production vector $y \in Y$ is a profit maximizing production vector for some nonnegative price vector $p \geq 0$.*

Proof [*see* [33] and [50]]. Suppose that $y \in Y$ is efficient. Let the convex set $P_y^+ = \{y' \in \mathbb{R}^L : y' > y\}$. From **Remark 1.2.8** it follows that $Y \cap P_y^+ = \emptyset$.
By applying **Lemma 1.2.10** on the convex sets Y and P_y^+, there is some $p \neq 0$ such that $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y^+$ and $y'' \in Y$.
From the definition of the set P_y^+, it follows that $p \cdot y' \geq p \cdot y$ for every $y' > y$. Clearly $p \geq 0$ because if $p_\ell < 0$ for some $\ell = 1, 2, 3, \ldots, L$, then $p \cdot y' < p \cdot y$ for some $y' > y$ with $y'_\ell - y_\ell > 0$.
Indeed, we choose y' such that $y'_\ell > y_\ell$ and $y'_k > y_k$, $k \neq \ell$ with y'_k close enough to y_k.

By taking any $y'' \in Y$, we have that $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y^+$. Because y' can be chosen to be arbitrarily close to y, we can conclude that $p \cdot y \geq p \cdot y''$ for any $y'' \in Y$.

Hence, y is profit maximizing for price vector p. $\square$

1.3 The Basic Model and Allocations

The main objective of this section is to formulate a basic model of an economy involving a finite number of economic agents, that is, the consumers and producers or firms. The basic model assumes a situation where we have,

$$\begin{aligned} I &> 0, \text{ consumers} \\ J &> 0, \text{ firms} \\ L &> 0, \text{ commodities (goods)} \end{aligned}$$

where,

- Each consumer $i = 1, 2, 3, \ldots, I$ is characterized by

  1. *A consumption set* $X_i \subset \mathbb{R}^L$.
  2. *A preference relation* $\succeq_i$ *on* X_i *such that,*

     Reflexive $x_i \succeq_i x_i \; \forall \, x_i \in X_i$.

     Complete $x_i \succeq_i \bar{x}_i$ or $\bar{x}_i \succeq_i x_i \; \forall \, x_i, \bar{x}_i \in X_i$.

     Transitive $x_i \succeq_i \bar{x}_i$ and $\bar{x}_i \succeq_i x_i^* \implies x_i \succeq_i x_i^* \; \forall \, x_i, \bar{x}_i, x_i^* \in X_i$.

- Each firm $j = 1, 2, 3, \ldots, J$ is characterised by

  1. *A Production set* $Y_j \subset \mathbb{R}^L$.
  2. *Every* Y_j *is nonempty and closed.*

- The initial resources of commodities, that is, the initial endowments in the economy are given by

  A vector $\bar{e} = (\bar{e_1}, \ldots, \bar{e_L}) \in \mathbb{R}^L$ *with* $\bar{e} = \sum_{i=1}^{I} e_i$ *where,* $e_i \in \mathbb{R}^L$ *is the initial resource bundle of commodities held by the* i^{th} *consumer.*

  Thus, $\bar{e}_\ell$ *is the initial amount of* ℓ^{th} *commodity available in the economy.*

Remarks 1.3.1

(i) In the above mentioned model it is assumed that each of the consumers maximize their satisfaction over the set of commodity bundles they can afford with their income and that each of the producers can maximize their profit with the use of production processes available in their production sets.

(ii) The basic data on preferences, production sets and resources for the economy are given by

$$(\{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \bar{e}).$$

The principal working of the economy is to allocate production and the resulting products among consumers. In [33] and [50] the concepts of an allocation and a feasible allocation are defined as follows:

Definitions 1.3.2 *An allocation $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a specification of a consumption vector $x_i \in X_i$ for each consumer $i = 1, 2, 3, \ldots, I$ and production vector $y_j \in Y_j$ for each firm $j = 1, 2, 3, \ldots, J$.*

An allocation (x, y) is said to be feasible if for every commodity $\ell = 1, 2, 3, \ldots, L$

$$\sum_{i=1}^{I} x_{\ell\, i} = \bar{e}_\ell + \sum_{j=1}^{J} y_{\ell j}$$

i.e.

$$\sum_{i=1}^{I} x_i = \bar{e} + \sum_{j=1}^{J} y_j \ .$$

Having defined the concept of a feasible allocation, the next definition introduces a special feasible allocation, namely, a Pareto [42] optimal feasible allocation in an economy.

Definition 1.3.3 *A feasible allocation (x, y) is Pareto optimal if there is no other feasible allocation (x', y') that Pareto dominates it, that is,*
if there is no feasible allocation (x', y') such that $x'_i \succeq_i x_i$ for all i and $x'_i \succ_i x_i$ for some i.

Given a utility function $u_i : X_i \to \mathbb{R}$ representing the preference relation $\succeq_i$ on X_i, a feasible allocation (x, y) is said to be Pareto optimal if there is no other feasible allocation (x', y') such that $u_i(x'_i) \geq u_i(x_i)$ for all i and $u_i(x'_i) > u_i(x_i)$ for some i.

1.4 Economies and Equilibrium

In the basic model introduced in the preceding section, the behavior and interaction amongst the economic agents, that is, the consumers and firms can lead to different economic situations. This means that the manner in which consumers maximize their satisfaction and producers maximize their profit gives rise to a competitive equilibrium situation in an economy, which is referred to as a Walrasian equilibrium. This Walrasian equilibrium as proposed by Walras [56] is determined by the relationship between the demand and supply of commodities.

The following three subsections define and explore the concept of a competitive equilibrium under various economic situations that result from the interactions amongst the consumers and producers.

1.4.1 Private Ownership Economies

The concept of a private ownership economy is described in the following manner:

- The wealth of consumers is derived from individual endowments of commodities and from ownership shares to profits of the firms. These firms are thought of as being owned by consumers.
- Formally consumer i has
  1. *an initial endowment vector of commodities* $e_i \in \mathbb{R}^L$.
  2. *a claim to a share* $\theta_{ij} \in [0,1]$ *of the profits of firm* j*, where* $\sum_{i=1}^{I} \theta_{ij} = 1$ *for every firm* j.

Having described a private ownership economy, the following is the definition of a Walrasian equilibrium under such an economy.

Definition 1.4.1 *Given a private ownership economy specified by*

$$(\{X_i, \succeq_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{e_i, \theta_{i1}, \ldots, \theta_{iJ}\}_{i=1}^{I}),$$

an allocation (x^*, y^*) *and a price vector* $p = (p_1, \ldots, p_L) \in \mathbb{R}_+^L$ *constitute a Walrasian equilibrium of the private ownership economy* ***(W.E.P.O.E)*** *if,*

(POE1) *For every* j, $y_j^* = (-z_1^*, \ldots, -z_{L-M}^*, q_1^*, \ldots, q_M^*)$ *maximizes profits in* Y_j*; that is,*

$$p \cdot y_j \leq p \cdot y_j^* \text{ for all } y_j \in Y_j\ .$$

(POE2) *For every* i, x_i^* *is maximal for* $\succeq_i$ *in the budget set*

$$\left\{ x_i \in X_i : p \cdot x_i \leq p \cdot e_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j^* \right\}\ .$$

(POE3) $\displaystyle\sum_{i=1}^{I} x_i^* = \bar{e} + \sum_{j=1}^{J} y_j^*$.

1.4.2 General Economies

We introduce the notion of a general economy that

- allows for a more general determination of a consumer's wealth levels than that in a private ownership economy.
- allows for an arbitrary distribution of wealth among consumers.

The definition of a Walrasian equilibrium for the general economy is given in the following manner:

Definition 1.4.2 *Given a general economy specified by*

$$(\{X_i, \succeq_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \bar{e}),$$

an allocation (x^*, y^*) *and a price vector* $p = (p_1, \ldots, p_L) \in \mathbb{R}_+^L$ *constitute a Walrasian equilibrium of a general economy* ***(W.E.G.E)*** *if there is an assignment of wealth levels* $(\mathrm{w}_1, \ldots, \mathrm{w}_I)$ *with* $\displaystyle\sum_{i=1}^{I} \mathrm{w}_i = p \cdot \bar{e} + \sum_{j=1}^{J} p \cdot y_j^*$ *such that,*

(GE1) *For every* j, y_j^* *maximizes profits in* Y_j*; that is,*

$$p \cdot y_j \leq p \cdot y_j^* \text{ for all } y_j \in Y_j\ .$$

(GE2) *For every* i, x_i^* *is maximal for* $\succeq_i$ *in the budget set*

$$\{x_i \in X_i : p \cdot x_i \leq \mathrm{w}_i\}\ .$$

(GE3) $\displaystyle\sum_{i=1}^{I} x_i^* = \bar{e} + \sum_{j=1}^{J} y_j^*$.

1.4.3 Pure Exchange Economies

A pure exchange economy is one in which the only possible production activities are those of free disposal and no production.

- Formally, we

  1. let $J = 1$, *that is, we have one firm.*
  2. let $Y_1 = -\mathbb{R}^L_+$, *that is, there is no production.*
  3. *take* $X_i = \mathbb{R}^L_+$.
  4. *assume that each consumer's preferences are continuous, strictly convex, locally nonsatiated and* $\displaystyle\sum_{i=1}^{I} e_i > 0$.

Lemma 1.4.3 *If the above conditions 1, 2, 3 and 4 are satisfied then the optimal bundle* $x_i^* = x_i(p, p \cdot e_i)$ *is the Walrasian demand function over the Walrasian budget set* $B_{p,\mathrm{w}_i} = \{x_i : p \cdot x_i \leq \mathrm{w}_i\}$.

Proof.
We show that there is a unique optimal bundle $x_i^* \in B_{p,\mathrm{w}_i}$ and it satisfies Walras law, that is, $p \cdot x_i^* = \mathrm{w}_i$. From the continuity of preference relations and the result **Theorem 1.2.4** each $\succeq_i$ is given by a utility function. Since the budget set B_{p,w_i} is compact there is an optimal x_i^* in B_{p,w_i}.
For the uniqueness, suppose we have another optimal bundle $\tilde{x}_i \in B_{p,\mathrm{w}_i}$. By the strict convexity of the relation $\succeq_i$ the bundle $\hat{x}_i = \alpha x_i^* + (1-\alpha)\tilde{x}_i \in B_{p,\mathrm{w}_i}$ for $\alpha \in (0,1)$ and $\hat{x}_i \succ_i x_i^*$, a contradiction.
Now we prove Walras law. If $p \cdot x_i^* < \mathrm{w}_i$ then by the local nonsatiation of $\succeq_i$ there is $\bar{x}_i \in B_{p,\mathrm{w}_i}$ such that $p \cdot \bar{x}_i < \mathrm{w}_i$ and $\bar{x}_i \succ_i x_i^*$, a contradiction. This completes the proof. □

Remark 1.4.4 For pure exchange economies the conditions **(POE1)**, **(POE2)** and **(POE3)** of **Definition 1.4.1** become:

(POE1') The production vector y_1^* maximizes profit in Y_1; that is,

$$p \cdot y_1 \leq p \cdot y_1^* = 0 \text{ for all } y_1 \in Y_1,\ p \geq 0.$$

This condition means that if there is no output then there is no profit.

(POE2') Since $p \cdot y_1^* = 0$ and $\theta_{i1} = 1$. For every i, $x_i^* \in \mathbb{R}_+^L$ is maximal for $\succeq_i$ in the budget set

$$\{x_i \in X_i : p \cdot x_i \leq p \cdot e_i = \mathrm{w}_i\} \ .$$

(POE3') $\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} e_i + y_1^* = \bar{e} + y_1^*.$

The **Definition 1.4.1** and **Lemma 1.4.3** applied to pure exchange economies yields the following definition.

Definition 1.4.5 *Given a pure exchange economy specified by*

$$(\{X_i, \succeq_i\}_{i=1}^{I}, \{Y_1\}, \bar{e}),$$

an allocation $(x^*, y^*) = (x_1^*, \ldots, x_I^*, y_1^*)$ *and a price vector* $p = (p_1, \ldots, p_L) \in \mathbb{R}_+^L$ *constitute a Walrasian equilibrium for the pure exchange economy* ***(W.E.P.E.E)*** *if,*

(PEE1) $y_1^* \leq 0$, $p \cdot y_1^* = 0$ *and* $p \geq 0$.

(PEE2) $x_i^* = x_i(p, p \cdot e_i)$ for all i [*where* $x_i(\cdot)$ *is consumer* i*'s Walrasian demand function, that is,* $p \cdot x_i(p, p \cdot e_i) = \mathrm{w}_i$].

(PEE3) $\sum_{i=1}^{I} x_i^* - \sum_{i=1}^{I} e_i = y_1^*$

Remarks 1.4.6

(i) Clearly an economy that has a **(W.E.P.O.E)** is **(W.E.G.E)**.

This can be done by endowing consumer i with the initial resource of commodity $e_i = x_i^* - (1/I)\sum_{j=1}^{J} y_j^*$, the share $\theta_{ij} = 1/I$ and $\mathrm{w}_i = p \cdot e_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j^*$.

(ii) The terminology "x_i is maximal for $\succeq_i$ in the set B" means that x_i is the i^{th} consumer preference maximizing choice in the set B, that is, $x_i \in B$ and $x_i \succeq_i x_i'$ for all $x_i' \in B$.

(iii) The feasibility condition **(POE3)**, **(GE3)**, **(PEE3)** $\sum_{i=1}^{I} x_i^* = \bar{e} + \sum_{j=1}^{J} y_j^*$ is often replaced by its inequality counterpart $\sum_{i=1}^{I} x_i^* \leq \bar{e} + \sum_{j=1}^{J} y_j^*$ to allow an excess supply of commodities at equilibrium.

The next result follows from the conditions **(PEE1)**, **(PEE2)** and **(PEE3)**.

Theorem 1.4.7 *In a pure exchange economy in which consumer preferences are continuous, strictly convex and locally nonsatiated, $p \in \mathbb{R}^L_+$ is a Walrasian equilibrium price vector if and only if,*

$$\sum_{i=1}^{I}(x_i(p, p \cdot e_i) - e_i) \quad \leq \quad 0\,. \tag{1.5}$$

Proof. The fact that equation (1.5) holds in any Walrasian equilibrium of a pure exchange economy follows from conditions **(PEE1)** to **(PEE3)**.
Suppose $\sum_{i=1}^{I}(x_i(p, p \cdot e_i) - e_i) > 0$.
Multiplying the above by $p \geq 0$ we get,

$$\sum_{i=1}^{I} x_i(p, p \cdot e_i) \cdot p - \sum_{i=1}^{I} e_i \cdot p > 0.$$

By **(PEE3)** and **(PEE1)** the left hand side of this inequality yields $y_1^* \cdot p = 0$, a contradiction.
Conversely, suppose that equation (1.5) holds.
If we let $y_1^* = \sum_{i=1}^{I}(x_i(p, p \cdot e_i) - e_i)$ and $x_i^* = x_i(p, p \cdot e_i)$, then the allocation (x^*, y_1^*) and the price vector p satisfy the conditions **(PEE1)** to **(PEE3)**.
Now, $p \cdot y_1^* = p \cdot \sum_{i=1}^{I}(x_i(p, p \cdot e_i) - e_i) = \sum_{i=1}^{I}(p \cdot x_i(p, p \cdot e_i) - p \cdot e_i) = 0$, this is true because with local nonsatiation we have $p \cdot x_i(p, p \cdot e_i) = p \cdot e_i$ for all i, as in the proof of **Lemma 1.4.3** . $\square$

1.5 The Existence of Walrasian Equilibrium

In this section, we will examine the existence of a Walrasian equilibrium for the economic model where all the economic agents are consumers. This is the pure exchange economy concept discussed in the preceding section, where we had I consumers each described by their preferences and goods that they possess. The consumers trade commodities among themselves in an attempt to maximize their satisfaction, that is, to make themselves better off.

From equation (1.5), we can define the excess demand function for each consumer as well as the aggregate excess demand function of the pure exchange economy.

Definitions 1.5.1 *The excess demand function of consumer i is*

$$z_i(p) = x_i(p, p \cdot e_i) - e_i \tag{1.6}$$

where $x_i(p, p \cdot e_i)$ is consumer i's Walrasian demand function.

The aggregate excess demand function of the economy is

$$z(p) = \sum_{i=1}^{I} z_i(p) \ . \tag{1.7}$$

The domain of this function is the set of nonnegative price vectors that includes all strictly positive vectors.

Remarks 1.5.2 From equations (1.5) ,(1.6) and (1.7) we obtain the following conditions:

(i) The nonnegative price vector $p \in \mathbb{R}^L_+$ is an equilibrium price vector in a pure exchange economy with locally nonsatiated preferences if and only if $z(p) \leq 0$ if and only if $\sum_{i=1}^{I} x_i(p, p \cdot e_i) \leq \sum_{i=1}^{I} e_i$.

(ii) Free goods *:* if $p \in \mathbb{R}^L_+$ is a Walrasian equilibrium price vector and $z_\ell(p) < 0$ then $p_\ell = 0$.

(iii) Desirability *:* if $p_\ell = 0$ then $z_\ell(p) > 0$ for $\ell = 1, 2, 3, \ldots, L$.

(iv) Having strongly monotone preferences, means that any Walrasian equilibrium price vector $p \in \mathbb{R}^L_+ \backslash \{0\}$ must be strictly positive $p = (p_1, \ldots, p_L) > 0$; otherwise consumers would demand an unboundedly large amount of all the free goods.

(v) Equality of demand and supply (clearing of all markets) *:* if all goods are desirable and p^* is a Walrasian equilibrium price vector, then $z(p^*) = 0$.

(vi) For strongly monotone preferences, a price vector $p = (p_1, \ldots, p_L) \in \mathbb{R}^L_+ \setminus \{0\}$ is a Walrasian equilibrium price vector if and only if it clears all markets if and only if it solves the system of L equations in L unknowns,

$$z_\ell(p) = 0 \text{ for all } \ell = 1, 2, 3, \ldots, L . \tag{1.8}$$

OR

$$z(p) = 0 . \tag{1.9}$$

The next result in [33] provides the essential properties of the aggregate excess demand function in pure exchange economies with strongly monotone preferences.

Lemma 1.5.3 *Suppose that for every consumer i, the preference relation $\succeq_i$ on the consumption set $X_i = \mathbb{R}^L_+$ is continuous, strictly convex and strongly monotone. Suppose also that $\sum_{i=1}^{I} e_i > 0$. Then the aggregate excess demand function $z(p)$, defined for all price vectors $p \in \mathbb{R}^L_+ \setminus \{0\}$ satisfies the following properties:*

(AED1) Continuity: *$z(\cdot)$ is continuous.*

(AED2) Homogeneity: *$z(\cdot)$ is homogeneous of degree zero, that is, $z(\alpha p) = z(p)$ for $\alpha > 0$.*

(AED3) Walras' law: *$p \cdot z(p) = 0$ for all $p > 0$.*

(AED4) Lower bound: *There is an $s > 0$ such that $z_\ell(p) > -s$ for every commodity ℓ and all p.*

(AED5) Desirability: *If $p^n \to p$, $p \neq 0$ and $p_\ell = 0$ for some l, then*

$$\max\{z_1(p^n), \ldots, z_L(p^n)\} \to \infty . \tag{1.10}$$

Proof [*see* [33]].

Note 1.5.4 It is important to note that the requirement for equilibrium is that there is no excess demand for any good. The work covered so far indicates that if some good is in excess supply in equilibrium then its price must be zero. Hence, if each good is desirable then equilibrium will be characterized by the equality of demand and supply in the economy.

Now, we are in the appropriate position to provide the result of the existence of Walrasian equilibrium for the pure exchange economy.

Theorem 1.5.5 *Suppose that $z(p)$ is a function defined for all strictly positive price vectors $p \in \mathbb{R}^L_+ \setminus \{0\}$ and satisfying the conditions **(AED1)** to **(AED5)** of **Lemma 1.5.3**, then the system of equations $z(p) = 0$ has a solution. Hence a Walrasian equilibrium exists in any pure exchange economy in which $\sum_{i=1}^{I} e_i > 0$ and every consumer has continuous, strictly convex and strongly monotone preferences.*

Proof [*see* [33]].

We normalize each price vector by dividing each component by the sum of the components. As a result we get

$$\Delta_1 = \left\{ p \in \mathbb{R}^L_+ \; : \; \sum_{\ell=1}^{L} p_\ell = 1 \right\}$$

the unit simplex in $\mathbb{R}^L$. This means that we are restricted to the unit simplex as the domain for the aggregate excess demand function $z(\cdot)$. Because the function $z(\cdot)$ is homogeneous of degree zero according to condition **(AED2)**, this allows us to restrict our search for an equilibrium to price vectors in the unit simplex Δ_1. It is worth noting that the function $z(\cdot)$ is well defined for price vectors in the set

$$\text{Interior}\Delta_1 = \{p \in \Delta_1 : p_\ell > 0 \text{ for all } \ell = 1, 2, 3, \ldots, L\}$$

the interior of the unit simplex Δ_1.

The boundary of the unit simplex is the set denoted by $\partial\Delta_1$ and is given by,

$$\partial\Delta_1 = \{p \in \Delta_1 : p_\ell = 0 \text{ for some } \ell = 1, 2, 3, \ldots, L\}.$$

The proof follows the next four steps:

Step 1: We construct a fixed point correspondence $f(\cdot)$ from Δ_1 to Δ_1.

We define a correspondence

$$f : \Delta_1 \longrightarrow 2^{\Delta_1}.$$

Thus for $p \in \Delta_1$, $f(p) \subseteq \Delta_1$ and for clarity the vectors that are elements of $f(p)$ are denoted by the symbol q.

Case 1: we construct $f(\cdot)$ for $p \in$ InteriorΔ_1.

$$f(p) = \{q \in \Delta_1 : z(p) \cdot q \geq z(p) \cdot q' \text{ for all } q' \in \Delta_1\}\,. \tag{1.11}$$

This condition (1.11) means that given a price vector $p \in \text{Interior}\Delta_1$, the price vector assigned by $f(\cdot)$ is any price vector q that, among the permissible price vectors, maximizes the value of the excess demand vector $z(p)$.
Equivalently the correspondence $f(\cdot)$ for $p \in \text{Interior}\Delta_1$ can be expressed as follows:

$$f(p) = \{q \in \Delta_1 : \; q_\ell = 0 \text{ if } z_\ell(p) < \max\{z_1(p), \ldots, z_L(p)\}\}. \tag{1.12}$$

The maximum of the linear form $L(q) = z(p){\cdot}q$ happens on the face $\Delta(p_{i_1}, p_{i_2}, p_{i_3}, \ldots, p_{i_s})$ of Δ_1, and the value is $z_i(p)$ where $i \in \{i_1, i_2, i_3, \ldots, i_s\}$. So if $q \in \Delta(p_{i_1}, p_{i_2}, p_{i_3}, \ldots, p_{i_s})$ then $q_\ell = 0$ for $\ell \notin \{i_1, i_2, i_3, \ldots, i_s\}$. This shows that optimal q satisfies condition (**1.12**). The converse is obvious.
Clearly if $z(p) \neq 0$ for $p > 0$, then by Walras' law we have that $z_\ell(p) < 0$ for some ℓ and $z_{\ell'}(p) > 0$ for some $\ell' \neq \ell$. Thus, for such a p, any $q \in f(p)$ has $q_\ell = 0$ for some ℓ. Therefore, if $z(p) \neq 0$ then $f(p) \subset \partial\Delta_1 = \Delta_1 \backslash \text{Interior}\Delta_1$. In contrary, if $z(p) = 0$ then $f(p) = \Delta_1$.

Case 2: we construct $f(\cdot)$ for $p \in \partial\Delta_1$.
For $p \in \partial\Delta_1$, we let

$$\begin{aligned} f(p) &= \{q \in \Delta_1 : p \cdot q = 0\} && (1.13)\\ &= \{q \in \Delta_1 : q_\ell = 0 \text{ if } p_\ell > 0\}\,. && (1.14)\end{aligned}$$

Since $p_\ell = 0$ for some ℓ, we have that $f(p) \neq \emptyset$. If $p \in \partial\Delta_1$ then $p \notin f(p)$ because $p \cdot p > 0$ while $p \cdot q = 0$ for all $q \in f(p)$. This means that $f(\cdot)$ has no fixed point for any $p \in \partial\Delta_1$.

Step 2: We argue that a fixed point of $f(\cdot)$ is an equilibrium price vector.
Suppose that $p^* \in f(p^*)$. Then it follows from **Case 2 in Step 1** that $p^* \notin \partial\Delta_1$, which implies that $p^* > 0$. If $z(p^*) \neq 0$, then it follows from **Case 1 in Step 1** that $f(p^*) \subset \partial\Delta_1$, which is incompatible with $p^* \in f(p^*)$ and $p^* > 0$. Hence, if $p^* \in f(p^*)$ we must have $z(p^*)$=0.

Step 3: The fixed point correspondence $f(\cdot)$ is upper semicontinuous and convex valued.
Note that, $f(\cdot)$ is said to be upper semicontinuous in Δ_1, if for any two sequences $p^n \to p$, $q^n \to q$ of points in Δ_1 with $q^n \in f(p^n)$ for all n then $q \in f(p)$.

Case 1: We show upper semicontinuity of $f(\cdot)$ for $p \in$ InteriorΔ_1.
Since $p = (p_1, \ldots, p_L) > 0$ then $p^n > 0$ for n sufficiently large.
Since $q^n \in f(p^n)$,

$$q^n \cdot z(p^n) \geq q' \cdot z(p^n) \text{ for all } q' \in \Delta_1 \,.$$

From the continuity of $z(\cdot)$ and $q^n \to q$ for all $n = 1, 2, \ldots$ we get,

$$q \cdot z(p) \geq q' \cdot z(p) \text{ for all } q' \in \Delta_1 \,.$$

By the definition of $f(p)$ for $p \in$ InteriorΔ_1 we have that,

$$q \in f(p) \,.$$

Case 2: We show upper semicontinuity of $f(\cdot)$ for $p \in \partial\Delta_1$.
Take any ℓ with $p_\ell > 0$. We shall argue that for n sufficiently large we have $q^n_\ell = 0$ and therefore we get $q_\ell = 0$; from this it follows that $q \in f(p)$.

Because $p_\ell > 0$, there is an $\varepsilon > 0$ such that $p^n_\ell > \varepsilon$ for n sufficiently large.
If, in addition, $p^n \in \partial\Delta_1$ then $q^n_\ell = 0$ by the definition of $f(p^n)$.

If, instead $p^n > 0$ using properties **(AED4)** and **(AED5)** of $z(\cdot)$, we can show that

$$z_\ell(p^n) < \max\{z_1(p^n), \ldots, z_L(p^n)\} \tag{1.15}$$

and therefore again $q_\ell^n = 0$ by (1.12). So each time $\lim_{n\to\infty} q_\ell^n = q_\ell = 0$ if $p_\ell = 0$ and hence $q \in f(p)$, by (1.14).

It remains to show the above equation (1.15).
From **(AED5)** $p \neq 0$, $p_\ell = 0$ for some ℓ implies

$$\max\{z_1(p^n), \ldots, z_L(p^n)\} \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

Since $p_\ell^n > \varepsilon$, and for n sufficiently large we have that,

$$\begin{aligned} z_\ell(p^n) &\leq \frac{1}{\varepsilon} p_\ell^n z_\ell(p^n) &(1.16)\\ &= -\frac{1}{\varepsilon} \sum_{\ell' \neq \ell} p_{\ell'}^n z_{\ell'}(p^n) \quad \text{(Walras law)} &(1.17)\\ &< \frac{s}{\varepsilon} \sum_{\ell' \neq \ell} p_{\ell'}^n < \frac{s}{\varepsilon} \quad \text{(AED4)}. &(1.18) \end{aligned}$$

Thus $z_\ell(p^n)$ is bounded. Since

$$\max\{z_1(p^n), \ldots, z_L(p^n)\} \longrightarrow \infty \quad as \quad n \longrightarrow \infty$$

we must have

$$z_\ell(p^n) < \max\{z_1(p^n, \ldots, z_L(p^n)\}$$

for sufficiently large n.
From $p^n > 0$ and the condition (1.12)

$$f(p) = \{q \in \Delta_1 \ : \ q_\ell = 0 \text{ if } z_\ell(p) < \max\{z_1(p), \ldots, z_L(p)\}\}$$

we get that,

$$f(p^n) = \{q^n \in \Delta_1 : q_\ell^n = 0 \text{ if } z_\ell(p^n) < \max\{z_1(p^n), \ldots, z_L(p^n)\}\}$$

such that

$$q^n \in f(p^n) \text{ implies } q_\ell^n = 0$$

for sufficiently large n, and this implies $p \cdot q = 0$ and so $q \in f(p)$.

Case 3: Convexity of $f(\cdot)$ is clear.

Clearly for both cases $p \in \text{Interior}\Delta_1$ and $p \in \partial\Delta_1$, the subset $f(p) \subset \Delta_1$ is a convex set.

Step 4: We show that a fixed point exists.
Kakutani's fixed point theorem says that a convex-valued, upper semicontinuous correspondence from a nonempty, compact, convex set into itself has a fixed point.
Since Δ_1 is a nonempty, compact, and convex set, and since $f(\cdot)$ is a convex-valued upper semicontinuous correspondence from Δ_1 to Δ_1, we conclude that there is a $p^* \in \Delta_1$ with $p^* \in f(p^*)$, by **Step 2** this means we have that $z(p^*) = 0$. Hence this completes the proof.

□

Having proved the preceding result with use of the Kakutani [25] fixed point theorem, we state Brouwer's fixed point theorem which is a fixed point result for functions.

Theorem 1.5.6 [Brouwer's Fixed Point Theorem].
Suppose that $A \subseteq \mathbb{R}^L$ is a nonempty, compact, convex set, and that $f : A \to A$ is a continuous function from A into itself. Then $f(\cdot)$ has a fixed point, that is, there is an $x \in A$ such that $x = f(x)$.

Proof [*see* [19] and [33]].

We can consider a result in which the boundary conditions **(AED4)** and **(AED5)** are eliminated for the aggregate excess demand function $z(p)$ by studying its continuity **(AED1)**, homogeneity **(AED2)** satisfying Walras' law **(AED3)** and defined for nonnegative, nonzero price vectors. This aggregate excess demand function is compatible with locally nonsatiated, continuous and strictly convex preferences. **Remark 1.5.2** says that $p \in \mathbb{R}^L_+$ is an equilibrium price vector in a pure exchange economy with locally nonsatiated preferences if and only if $z(p) \leq 0$, leads to the next result.

Theorem 1.5.7 *Suppose that $z(p)$ is a function defined for all positive price vectors $p \in \mathbb{R}^L_+$ and satisfying conditions **(AED1) → (AED3)** of **Lemma (1.5.3)**. Then there is a price vector p^* such that $z(p^*) \leq 0$.*

Proof [*see* [33] and [54]]. Because of homogeneity of degree zero we can restrict our search for an equilibrium to the unit simplex $\Delta_1 = \left\{ p = (p_1, p_2, \ldots, p_L) \in \mathbb{R}^L_+ \ : \ \sum_{\ell=1}^{L} p_\ell = 1 \right\}$.

Define on Δ_1 the function $z^+(\cdot)$ by $z_\ell^+(p) = \max\{z_\ell(p), 0\}$. Clearly $z^+(\cdot)$ is continuous and that $z^+(p) \cdot z(p) = 0$ implies $z(p) \leq 0$.

We denote $\alpha(p) = \sum_{\ell=1}^{L} [p_\ell + z_\ell^+(p)] = 1 + \sum_{\ell=1}^{L} z_\ell^+(p)$. We have $\alpha(p) \geq 1$ for all p.

Define a continuous function $f(\cdot)$ from the closed, convex set Δ_1 into itself by

$$f(p) = [1/\alpha(p)](p + z^+(p))$$

and component wise,

$$f_\ell(p) = \frac{p_\ell + z_\ell^+(p)}{1 + \sum_{\ell=1}^{L} z_\ell^+(p)} \text{ for all } \ell = 1, 2, 3, \ldots, L\,.$$

By **Theorem 1.5.6** there is a $p^* \in \Delta_1$ such that $p^* = f(p^*)$.

By Walras' law **(AED3)**,

$$0 = p^* \cdot z(p^*) = f(p^*) \cdot z(p^*) = [1/\alpha(p^*)]z^+(p^*) \cdot z(p^*).$$

Therefore, $z^+(p^*) \cdot z(p^*) = 0$, this implies $z(p^*) \leq 0$.

$\square$

The general model was first formulated by Walras [56]. The first proof of existence was due to Wald [55]. A general approach (programming approach) to the existence of equilibrium was provided in [32]. A game theoretic approach to the existence of Walrasian equilibrium was taken in the classic paper of Arrow K. and Debreu G. [1] and later on improved by Gale D. and Mas-Colell A. [23].

1.6 Equilibrium and Welfare Economics

In this section we look at the relationship between Walrasian equilibrium concepts and Pareto optimality. The concept of Pareto optimality as defined in **section 1.3** and in [33], [50] and [54] is the state in which nobody can be better off without making others worse off. The phrases "better off" or "worse off" refer to the welfare of each consumer with respect to the consumer's preferences.

1.6.1 The First Fundamental Theorem of Welfare Economics

The first fundamental theorem of welfare economics addresses conditions when a Walrasian equilibrium realizes a Pareto optimum. The local nonsatiation of preferences is all that is required for the result.

Theorem 1.6.1 (First Fundamental Theorem of Welfare Economics) .
If preferences are locally nonsatiated and if (x^*, y^*, p) constitute a Walrasian equilibrium of a general economy, then the allocation (x^*, y^*) is Pareto optimal. In particular, any Walrasian equilibrium allocation is Pareto optimal.

Proof [*see* [33] and [50]]. Suppose that (x^*, y^*, p) constitute a Walrasian equilibrium of a general economy and that the associated wealth levels are $(\mathrm{w}_1, \ldots, \mathrm{w}_I)$. Recall that

$$\sum_{i=1}^{I} \mathrm{w}_i = p \cdot \bar{e} + \sum_{j=1}^{J} p \cdot y_j^* .$$

The preference maximization part of the definition of a Walrasian equilibrium of a general economy [**(GE2)** of **Definition 1.4.2**] implies that

$$\text{If } x_i \succ_i x_i^* \text{ then } p \cdot x_i > \mathrm{w}_i. \tag{1.19}$$

That is, anything that is strictly preferred by consumer i to x_i^* must be unaffordable to the particular consumer. The significance of the local nonsatiation condition is that with it (1.19) implies that

$$\text{If } x_i \succeq_i x_i^* \text{ then } p \cdot x_i \geq \mathrm{w}_i. \tag{1.20}$$

Clearly (1.20) is true because from local nonsatiation in any neighbourhood of x_i there is $\bar{x}_i$ such that $\bar{x}_i \succ_i x_i \succeq_i x_i^*$ or $\bar{x}_i \succ_i x_i^*$. Now (1.19) implies that $p \cdot \bar{x}_i > \mathrm{w}_i$.

If we let $\bar{x}_i \to x_i$ then $p \cdot x_i \geq \mathrm{w}_i$.
That is, anything that is at least as good as x_i^* is at best just affordable.
Now, we consider an allocation (x, y) that Pareto dominates (x^*, y^*). That is, $x_i \succeq_i x_i^*$ for all i and $x_i \succ_i x_i^*$ for some i. By (1.20), we must have $p \cdot x_i \geq \mathrm{w}_i$ for all i, and by (1.19) $p \cdot x > \mathrm{w}_i$ for some i. Hence,

$$\sum_{i=1}^{I} p \cdot x_i > \sum_{i=1}^{I} \mathrm{w}_i = p \cdot \bar{e} + \sum_{j=1}^{J} p \cdot y_j^* .$$

Moreover, because y_j^* is profit maximizing for firm j at price vector p, we have

$$p \cdot \bar{e} + \sum_{j=1}^{J} p \cdot y_j^* \geq p \cdot \bar{e} + \sum_{j=1}^{J} p \cdot y_j .$$

Thus

$$\sum_{i=1}^{I} p \cdot x_i > p \cdot \bar{e} + \sum_{j=1}^{J} p \cdot y_j . \tag{1.21}$$

But this means that (x, y) cannot be feasible. Indeed, $\sum_{i=1}^{I} x_i = \bar{e} + \sum_{j=1}^{J} y_j$ implies $\sum_{i=1}^{I} p \cdot x_i = p \cdot \bar{e} + \sum_{j=1}^{J} p \cdot y_j$, which contradicts (1.21). We conclude that the equilibrium allocation (x^*, y^*) must be Pareto optimal.

□

The above result illustrates the fact that at any feasible allocation (x, y), the total cost of the consumption bundles $(x_1, \ldots, x_I)$, evaluated at prices p, must be equal to the social wealth at those prices, $p \cdot \bar{e} + \sum_{j=1}^{J} p \cdot y_j$.

1.6.2 The Second Fundamental Theorem of Welfare Economics

The second fundamental theorem of welfare economics gives conditions under which given the fact that an economy is in a Pareto optimal state, we want to know whether or not there exists a price vector such that it can be supported as a Walrasian equilibrium for a general economy with this price vector.

This is done by defining a weakened of version of the Walrasian equilibrium for a general economy which is called the quasi-Walrasian equilibrium for a general economy.

Definition 1.6.2 *Given a general economy specified by*

$$(\{X_i, \succeq_i\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{e}),$$

an allocation (x^*, y^*) *and a price vector* $p = (p_1, \ldots, p_L)$ *constitute a quasi-Walrasian equilibrium of a general economy **(Q.W.E.G.E)** if there is an assignment of wealth levels* $(\mathrm{w}_1, \ldots, \mathrm{w}_I)$ *with* $\sum_{i=1}^{I} \mathrm{w}_i = p \cdot \bar{e} + \sum_{j=1}^{J} p \cdot y_j^*$ *such that*

(QGE1) *For every* j, y_j^* *maximizes profits in* Y_j*; that is,*

$$p \cdot y_j \leq p \cdot y_j^* \text{ for all } y_j \in Y_j .$$

(QGE2) *For every* i*, if* $x_i \succ_i x_i^*$ *then* $p \cdot x_i \geq \mathrm{w}_i$*.*

(QGE3) $\sum_{i=1}^{I} x_i^* = \bar{e} + \sum_{j=1}^{J} y_j^*$.

Remark 1.6.3 In the above definition the preference maximization condition **(GE2)** of **Definition 1.4.2** that anything preferred to x_i^* must cost more than w_i (that is, if $x_i \succ_i x_i^*$ then $p \cdot x > \mathrm{w}_i$) is replaced by the weaker condition **(QGE2)** that anything preferred to x_i^* cannot cost less than w_i (that is, if $x_i \succ_i x_i^*$, then $p \cdot x_i \geq \mathrm{w}_i$).

The next result shows that if all preferences and production sets are convex, any Pareto optimal allocation can be achieved as a quasi-Walrasian equilibrium for a general economy supported with a price vector.

Theorem 1.6.4 (Second Fundamental Theorem of Welfare Economics) . *Consider an economy specified by*

$$(\{X_i, \succeq_i\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{w})$$

and suppose that every Y_j *is convex, every preference relation* $\succeq_i$ *is convex and locally nonsatiated. Then, for every Pareto optimal allocation* (x^*, y^*) *there is a price vector* $p = (p_1, \ldots, p_L) \neq 0$ *such that* (x^*, y^*, p) *constitute a quasi-Walrasian equilibrium for a general economy.*

Proof [*see* [33] and [50]].

We define, for every i, the set V_i of consumption bundles preferred to x_i^*, that is, $V_i = \{x_i \in X_i : x_i \succ_i x_i^*\} \subset \mathbb{R}^L$. Then we define

$$V = \sum_{i=1}^{I} V_i = \left\{ \sum_{i=1}^{I} x_i \in \mathbb{R}^L : x_1 \in V_1, \ldots, x_I \in V_I \right\}$$

and

$$Y = \sum_{j=1}^{J} Y_j = \left\{ \sum_{j=1}^{J} y_j \in \mathbb{R}^L : y_1 \in Y_1, \ldots, y_J \in Y_J \right\}.$$

The set V is the set of aggregate consumption bundles that could be split into I individual consumption bundles, each preferred by its corresponding consumer to x_i^*. The set Y is simply the aggregate production set. The set $Y + \bar{e}$, which geometrically is the aggregate production set with its origin shifted to $\bar{e}$, is the set of aggregate bundles producible with the given production vectors and endowments and usable in principle, for consumption.

The proof follows the next steps,

Step 1: Every set V_i is convex.

Let $x_i, x_i' \in V_i$ and $\alpha \in [0,1]$. Then $x_i \succ_i x_i^*$ and $x_i' \succ_i x_i^*$. We have to show that $\alpha x_i + (1-\alpha)x_i' \succ_i x_i^*$. Because the preferences are complete, we can assume without loss of generality that $x_i \succeq_i x_i'$. Therefore, by the convexity of preferences, we have that $\alpha x_i + (1-\alpha)x_i' \succeq_i x_i'$, which by transitivity yields the desired result, $\alpha x_i + (1-\alpha)x_i' \succ_i x_i^*$.

Step 2: The sets V and $Y + \bar{e}$ are convex.

The sum of any finite number of convex set is convex.

Step 3: $V \cap (Y + \bar{e}) = \emptyset$.

This is a consequence of the Pareto optimality of (x^*, y^*). If there were a vector in both V and $Y + \bar{e}$, then this would mean that, with the given endowments and production vectors, it would be possible to produce an aggregate vector that could be used to give every consumer i a consumption bundle that is preferred to x_i^*. This contradicts Pareto optimality and hence the sets are disjoint.

Step 4: There is $p = (p_1, \ldots, p_L)$ and a number r such that $p \cdot z \geq r$ for every $z \in V$ and $p \cdot z \leq r$ for every for every $z \in Y + \bar{e}$.

This follows directly from **Lemma 1.2.10** .

Step 5: If $x_i \succeq_i x_i^*$ for every i, then $p \cdot \left(\sum_{i=1}^{I} x_i\right) \geq r$.

Suppose that $x_i \succeq_i x_i^*$ for every i. By local nonsatiation, for each consumer i there is a consumption bundle $\hat{x}_i$ arbitrarily close to x_i such that $\hat{x}_i \succ_i x_i$ and therefore by transitivity $\hat{x}_i \in V_i$. Hence, $\sum_{i=1}^{I} \hat{x}_i \in V$, and so $p \cdot \left(\sum_{i=1}^{I} \hat{x}_i\right) \geq r$, which, by taking the limit as $\hat{x}_i \to x_i$, equivalently by local nonsatiation, gives the desired conclusion, $p \cdot \left(\sum_{i=1}^{I} x_i\right) \geq r$.

Step 6: $p \cdot \left(\sum_{i=1}^{I} x_i^*\right) = p \cdot \left(\bar{e} + \sum_{j=1}^{J} y_j^*\right) = r$.

From **Step 5** we have $p \cdot \left(\sum_{i=1}^{I} x_i^*\right) \geq r$. On the other hand,

$\left(\sum_{i=1}^{I} x_i^*\right) = \bar{e} + \sum_{j=1}^{J} y_j^* \in Y + \bar{e}$, and therefore $p \cdot \left(\sum_{i=1}^{I} x_i^*\right) \leq r$. Thus $p \cdot \left(\sum_{i=1}^{I} x_i^*\right) = r$.

Since $\left(\sum_{i=1}^{I} x_i^*\right) = \bar{e} + \sum_{j=1}^{J} y_j^*$, we also have $p \cdot \left(\bar{e} + \sum_{j=1}^{J} y_j^*\right) = r$.

Step 7: For every j,we have $p \cdot y_j \leq p \cdot y_j^*$ for all $y_j \in Y_j$.

For any firm j and $y_j \in Y_j$, we have $y_j + \sum_{h \neq j} y_h^* \in Y$. Therefore,

$$p \cdot (\bar{e} + y_j + \sum_{h \neq j} y_h^*) \leq r = p \cdot (\bar{e} + y_j^* + \sum_{h \neq j} y_h^*).$$

Hence, $p \cdot y_j \leq p \cdot y_j^*$.

Step 8: For every consumer i, if $x_i \succ_i x_i^*$, then $p \cdot x_i \geq p \cdot x_i^*$.

Consider any $x_i \succ_i x_i^*$. Because of **Steps 5 and 6**, we have

$$p \cdot (x_i + \sum_{k \neq i} x_k^*) \geq r = p \cdot (x_i^* + \sum_{k \neq i} x_k^*).$$

Hence, $p \cdot x_i \leq p \cdot x_i^*$.

Step 9: The wealth levels $\mathrm{w}_i = p \cdot x_i^*$ for $i = 1, \dots, I$ support (x^*, y^*, p) is a quasi-Walrasian equilibrium for a general economy.
Conditions **(QGE1)** and **(QGE2)** of **Definition 1.6.3** follow from **Steps 7 and 8**; condition **(QGE3)** follows from the feasibility of the Pareto optimal allocation (x^*, y^*).
Hence **Step 9** completes the proof.

□

Having proved the above result, we pay attention to the next theorem which gives sufficient conditions for the notion of a quasi-equilibrium to be a full equilibrium.

Theorem 1.6.5 *Let the preference relation $\succeq_i$ be continuous on the convex consumption set X_i. Suppose also that the consumption vector $x_i \in X_i$, the price vector p, and the wealth level w_i are such that $x_i \succ_i x_i^*$ implies $p \cdot x_i \geq \mathrm{w}_i$. Then, if there is a consumption vector $x_i' \in X_i$ such that $p \cdot x_i' < \mathrm{w}_i$ [a cheaper consumption for (p, w_i)], then $x_i \succ_i x_i^*$ implies $p \cdot x_i > \mathrm{w}_i$.*

Proof [*see* [33] and [50]]. Suppose $x_i \succ_i x_i^*$ does not imply $p \cdot x_i > \mathrm{w}_i$, meaning that there is an $x_i \succ_i x_i^*$ with $p \cdot x_i = \mathrm{w}_i$. By the cheaper consumption assumption, there exists an $x_i' \in X$ such that $p \cdot x_i' < \mathrm{w}_i$. Then for $\alpha \in [0, 1)$, we have $\alpha x_i + (1 - \alpha) x_i' \in X_i$ and $p \cdot (\alpha x_i + (1 - \alpha) x_i') < \mathrm{w}_i$. But if α is close enough to 1, continuity of $\succeq_i$ implies that $\alpha x_i + (1 - \alpha) x_i' \succ_i x_i^*$, which constitutes a contradiction because we have then found a consumption bundle that is preferred to x_i^* and cost less w_i.

□

Remarks 1.6.6 Having the above results **Theorem 1.6.4** and **Theorem 1.6.5** in hand, we can say that

(i) **Theorem 1.6.5** provides sufficient conditions under which condition **(QGE2)** "$x_i \succ_i x_i^*$ implies $p \cdot x_i \geq \mathrm{w}_i$ " in **Definition 1.6.2** is equivalent to the profit maximization condition **(GE2)** "$x_i \succ_i x_i^*$ implies $p \cdot x_i > \mathrm{w}_i$" in **Definition 1.4.2**.

(ii) Thus, **Theorem 1.6.5** says that a quasi-Walrasian equilibrium is a Walrasian equilibrium.

Chapter 2

Signatures and The Existence of Equilibrium

2.1 Simplicial structure, space and convexity

2.1.1 Introduction

In this chapter, we introduce a class of spaces called simplicial spaces which are topological spaces with an additional structure called a simplicial structure. We also look at the result "theorem on signatures" and the relationship between this result and fixed point theorems, together with the existence of equilibrium in mathematical economics.

Suppose $n \leq m$ where n and m are positive integers. Let $\{p_0, \ldots, p_n\} \subseteq \mathbb{R}^m$ be a collection of linearly independent points of the m - dimensional Euclidean space $\mathbb{R}^m$. This is equivalent to saying that if $t_0, \ldots, t_n \in \mathbb{R}$ and $\sum_{i=0}^{n} t_i p_i = 0$, then $t_0 = \cdots = t_n = 0$.

We define as in [27] the concept of an n-dimensional simplex spanned by the vertices $p_0, \ldots, p_n$ and the k-dimensional face of the n-dimensional simplex.

Definition 2.1.1 *The n-simplex $[p_0, p_1, \ldots, p_n]$ with vertices $p_0, \ldots, p_n$ is the set given by*

$$\left\{ x \in \mathbb{R}^m : x = \sum_{i=0}^{n} t_i p_i,\ \ t_i \geq 0\ \ \forall i = 0, \ldots, n \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}.$$

Definition 2.1.2 *If* $\{p_{i_0}, \ldots, p_{i_k}\} \subseteq \{p_0, \ldots, p_n\}$ *the simplex* $[p_{i_0}, \ldots, p_{i_k}]$ *is said to be a* k*-*$face$ *of the* n*-simplex* $[p_0, \ldots, p_n]$.

We are in a position to define the concepts of an affine continuous map from a simplex to a linear topological space, a singular simplex and a simplicial structure on a topological space.

Definition 2.1.3 *Let* E *be a linear topological space and* $[p_0, p_1, \ldots, p_n]$ *an* n*-simplex . A continuous map*

$$f : [p_0, \ldots, p_n] \longrightarrow E$$

is said to be affine if

$$f\left(\sum_{i=0}^{n} t_i p_i\right) = \sum_{i=0}^{n} t_i f(p_i),$$

where $t_i \geq 0$ *and* $\sum_{i=0}^{n} t_i = 1$.

Definition 2.1.4 *A singular simplex in a topological space* X *is a continuous map*

$$\sigma : [p_0, \ldots, p_n] \longrightarrow X$$

such that

$$\operatorname{vert} \sigma = \{\sigma(p_0), \ldots, \sigma(p_n)\}$$

and

$$\operatorname{im} \sigma = \sigma([p_0, \ldots, p_n]).$$

Definition 2.1.5 *A family* $\mathcal{S}$ *of singular simplices in* X *is a simplicial structure on* X *if*

1. *for any finite collection* $x_0, \ldots, x_n$ *of points in* X *there exists a* $\sigma \in \mathcal{S}$ *such that*

$$\sigma(p_0) = x_0, \ldots, \sigma(p_n) = x_n\ .$$

2. *if $\sigma \in \mathcal{S}$, then any restriction of σ to any face of $[p_0, \ldots, p_n]$ is in $\mathcal{S}$.*

3. *if $l : [q_0, \ldots, q_n] \longrightarrow [p_0, \ldots, p_n]$, with $l(q_i) = p_i$, $i = 0, \ldots, n$ is an affine map then for each*

$$\sigma : [p_0, \ldots, p_n] \longrightarrow X, \text{ such that } \sigma \in \mathcal{S}$$

the composition map

$$\sigma \circ l \in \mathcal{S} .$$

The pair $(X, \mathcal{S})$ is referred to as a simplicial space. The next result seems to follow naturally from the above definition.

Proposition 2.1.6 *Let $(X, \mathcal{S})$ be a simplicial space, Y a topological space and $h : X \rightarrow Y$ be a homeomorphism. Then $h(\mathcal{S})$ is a simplicial structure.*

Proof. We have to show that there is a singular simplex which satisfies conditions of **Definition 2.1.5**.

(1) Let $y_0, \ldots, y_n$ be a finite collection of points in Y and $h^{-1}(y_0), \ldots, h^{-1}(y_n)$ be a finite collection of points in X. Then there exists a singular simplex $\sigma \in \mathcal{S}$ such that

$$h \circ \sigma(p_i) = h(\sigma(p_i)) = h(h^{-1}(y_i)) = y_i, \text{ for each } i = 0 \ldots, n .$$

Thus, $(h \circ \sigma)(p_0) = y_0, \ldots, (h \circ \sigma)(p_n) = y_n$.

(2) If $\sigma \in \mathcal{S}$, we have that $h \circ \sigma \in h(\mathcal{S})$ and since $\sigma|_{[p_{i_0}, \ldots, p_{i_k}]} \in \mathcal{S}$,then $h \circ \sigma|_{[p_{i_0}, \ldots, p_{i_k}]} \in h(\mathcal{S})$.

(3) If $l : [q_0, \ldots, q_n] \longrightarrow [p_0, \ldots, p_n]$, with $l(q_i) = p_i$, $i = 0, \ldots, n$ is an affine map, then for each

$$h \circ \sigma : [p_0, \ldots, p_n] \longrightarrow Y$$

where

$$\sigma \in \mathcal{S} \text{ and the composition map } \sigma \circ l \in \mathcal{S} .$$

This implies that the composition map

$$(h \circ \sigma) \circ l = h \circ (\sigma \circ l)$$

belongs to the set $h(\mathcal{S})$. Thus, $h(\mathcal{S}) = \{h \circ \sigma\}_{\sigma \in \mathcal{S}}$ is a family of singular simplices in Y and also a simplicial structure on Y.

This completes the proof that $(Y, h(\mathcal{S}))$ is a simplicial space. □

Note 2.1.7 .

(i) *If* X is a topological space then $(X, \mathcal{S})$ is a topological simplicial space.

(ii) *If* X is a metric space then $(X, \mathcal{S})$ is a metric simplicial space.

(iii) *If* X is a normed space the $(X, \mathcal{S})$ is a normed simplicial space.

In the next proposition we prove that a product of simplicial spaces is a simplicial space.

Proposition 2.1.8 *If* $\{(X_i, \mathcal{S}_i)\}_{i=0}^{n}$ *is a finite family of simplicial spaces, then the product* $\prod_{i=0}^{n} X_i = X_0 \times \cdots \times X_n$ *is a simplicial space with* $\prod_{i=0}^{n} \mathcal{S}_i = \mathcal{S}_0 \times \cdots \times \mathcal{S}_i$ *as a simplicial structure, induced by the collection of singular simplices*
$\sigma = (\sigma_0, \ldots, \sigma_n)$.

Proof. We have to show that there is a singular simplex which satisfies conditions of **Definition 2.1.5**.

(1) Let $x_0, \ldots, x_r$ be a finite collection of points in $\prod_{i=0}^{n} X_i$ with $x_j = (x_{0j}, \ldots, x_{nj})$ for $j = 0, 1, 2, \ldots, r$. Then there is a singular simplex $\sigma_i \in \mathcal{S}_i$ such that
$\sigma_i(p_0) = x_{0j}, \ldots, \sigma_i(p_n) = x_{nj}$ where $i = 0, 1, 2, \ldots, n$. Thus,

$$\sigma(p_i) = (\sigma_0(p_i), \ldots, \sigma_n(p_i)) = (x_{0j}, \ldots, x_{nj}) = x_j \text{ for each } \; i = 0, 1, 2, \ldots, n.$$

Hence, $\sigma(p_0) = x_0, \ldots, \sigma(p_n) = x_r$.

(2) If $\sigma_i \in \mathcal{S}_i$ for each $i = 0, 1, 2, \ldots, n$, we have that $\sigma = (\sigma_0, \ldots, \sigma_n) \in \prod_{i=0}^{n} \mathcal{S}_i$ and

$$\sigma_i|_{[p_{i_0}, \ldots, p_{i_k}]} \in \mathcal{S}_i$$

implies

$$\sigma|_{[p_{i_0}, \ldots, p_{i_k}]} = (\sigma_0|_{[p_{i_0}, \ldots, p_{i_k}]}, \ldots, \sigma_n|_{[p_{i_0}, \ldots, p_{i_k}]}) \in \prod_{i=0}^{n} \mathcal{S}_i \,.$$

(3) Let, $l = \{l_i\}_{i=0}^{n}$ denote a family of affine maps such that for each $i = 0, 1, 2, \ldots, n$,

$$l_i : [q_0, \ldots, q_n] \to [p_0, \ldots, p_n] \text{ we have that } \; l_i(q_i) = p_i$$

and for each singular simplex

$$\sigma_i \in \mathcal{S}_i$$

the composition map

$$\sigma_i \circ l_i \in \mathcal{S}_i \, .$$

Hence, it follows that the collection $\sigma \circ l = (\sigma_0 \circ l_0, \ldots, \sigma_n \circ l_n) \in \prod_{i=0}^{n} \mathcal{S}_i$.

Thus, $\prod_{i=0}^{n} \mathcal{S}_i$ is a family of singular simplices in $\prod_{i=0}^{n} X_i = X_0 \times \cdots \times X_n$ and also a simplicial structure on $\prod_{i=0}^{n} X_i$.

This proves that $\left(\prod_{i=0}^{n} X_i, \prod_{i=0}^{n} \mathcal{S}_i \right)$ is a simplicial space.

□

We provide the following examples of simplicial structures which generate simplicial spaces.

Examples 2.1.9 .

1. Any convex subset of a linear topological space has a simplicial structure defined by a family of affine maps defined above.

2. In his paper Kulpa [27] proves that any sphere $S^{n-1} \subset \mathbb{R}^n$ has a simplicial structure, even though the sphere S^{n-1} is not convex.

The notion of simplicial convexity is introduced in the papers by Bielawski [9] and Komiya [26] as a generalization of convexity in questions of topology and analysis.

The next definition according to [27] and [29] gives the notion of simplicial convexity.

Definition 2.1.10 *A subset $A \subset X$ of a simplicial space $(X, \mathcal{S})$ is said to be simplicially convex if for each simplex $\sigma \in \mathcal{S}$,* $\mathrm{vert}\sigma \subset A$ *implies* $\mathrm{im}\sigma \subset A$.

Having defined simplicial convexity, the next important result as observed in [27], [28] and [29] plays a crucial role in solving fixed point related results. We also supply the proof of the result.

Theorem 2.1.11 [Theorem on Indexed Families].
Let X be a topological space and $\sigma : [p_0, \ldots, p_n] \longrightarrow X$ be a continuous mapping. For any open covering $\{U_0, \ldots, U_n\}$ of the compact set $\sigma([p_0, \ldots, p_n])$ by non-empty subsets of X, there exists a non-empty subset of indices $\{i_0, \ldots, i_k\} \subseteq \{0, \ldots, n\}$ such that $\sigma([p_{i_0}, \ldots, p_{i_k}]) \cap U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset$.

Proof. For each $i = 0, 1, 2, \ldots, n$ define a function d_i on the n-simplex by

$$d_i(x) = dist(x, [p_0, \ldots, p_n] \setminus \sigma^{-1}(U_i)), \text{ for each } x \in [p_0, \ldots, p_n],$$

where $dist(x, Y) = \inf\{\|x - y\| : y \in Y\}$, $Y \subseteq \mathbb{R}^n$.
By continuity of σ the sets $\sigma^{-1}(U_i)$ are open and hence $[p_0, \ldots, p_n] \setminus \sigma^{-1}(U_i)$ is compact as a closed subset of $[p_0, \ldots, p_n]$.

We know that for a closed subset Y in a metric space, $x \mapsto dist(x, Y)$ is continuous and $dist(x, Y) = 0 \Longleftrightarrow x \in Y$.
Thus the functions d_i are continuous on $[p_0, \ldots, p_n]$ and

$$d_i(x) = 0 \Longleftrightarrow x \notin \sigma^{-1}(U_i)$$

Note that if $d_i(x) = 0$ for $i = 0, \ldots, n$ then $x \notin \sigma^{-1}(U_i)$ or $x \notin \bigcup_{i=1}^{n} \sigma^{-1}(U_i)$, which contradicts the fact that $\{U_0, \cdots, U_n\}$ is a covering of $\sigma([p_0, \cdots, p_n])$.

This clearly indicates that for any $x \in [p_0, \ldots, p_n]$ not all $d_i(x)$ for $i = 0, \cdots, n$ vanish and hence $\sum_{j=0}^{n} d_j(x) > 0$ for $x \in [p_0, \ldots, p_n]$.

The function f given by

$$f(x) = \sum_{i=0}^{n} \left(\frac{d_i(x)}{\sum_{j=0}^{n} d_j(x)} \right) p_i,$$

is a continuous function defined on the n-simplex $[p_0, \ldots, p_n]$ into itself.

According to **Theorem 1.5.6**, there is $a \in [p_0, \ldots, p_n]$ such that $f(a) = a$. Thus

$$a = f(a) = \sum_{i=0}^{n} \left(\frac{d_i(a)}{\sum_{j=0}^{n} d_j(a)} \right) p_i \ . \tag{2.1}$$

Let $\{i_0, \ldots, i_k\}$ be the set of all indices i such that

$$d_i(a) \neq 0\,. \tag{2.2}$$

From equation **(2.1)**, $a \in [p_{i_0}, \ldots, p_{i_k}]$. From equation **(2.2)**,

$$i \in \{i_0, \ldots, i_k\} \iff d_i(a) \neq 0 \iff a \in \sigma^{-1}(U_i)\,.$$

Hence,

$$\sigma(a) \in \sigma([p_{i_0}, \ldots, p_{i_k}]) \cap U_{i_0} \cap \cdots \cap U_{i_k}.$$

This completes the proof.

□

Remark 2.1.12 .

In [28] $\sigma([p_{i_0}, \ldots, p_{i_k}])$ is the convex hull $conv(p_{i_0}, \ldots, p_{i_k})$, that is, the smallest convex set containing the points $p_{i_0}, \ldots, p_{i_k}$.

2.2 Signatures and Multivalued limit maps

The results obtained from the preceding section will be crucial in this section. We shall according to [29] introduce the concept of a signature and that of multivalued limit map.

Definition 2.2.1 *Let X be a simplicial space and Y a topological space. A continuous function $\mu : X \times Y \longrightarrow \mathbb{R}$ is said to be quasi-simplicially convex with respect to the variable $x \in X$ if for each $y \in Y$ and $\varepsilon > 0$ the pseudoball*

$$A(y, \varepsilon) = \{x \in X : \mu(x, y) < \varepsilon\} \text{ is simplicially convex in } X.$$

Moreover, if $\mu(x, y) \geq 0$ for each $(x, y) \in X \times Y$ then it is called a signature.

Definition 2.2.2 *A multivalued map*

$$H : X \longrightarrow 2^Y$$

is called a multivalued limit map if there exists a sequence

$$h_n : X \to Y, n = 1, 2, \ldots$$

of continuous maps such that for each subsequence $\{x_{n_k}\}$ of a sequence $\{x_n\}$ in X, if

$$\lim_{k\to\infty} x_{n_k} = x \quad \text{and} \quad \lim_{k\to\infty} h_{n_k}(x_{n_k}) = y$$

then

$$y \in H(x)\,.$$

Note 2.2.3 . The sequence $\{h_n : n \in \mathbb{N}\}$ is said to be a base for H.

Finally in [11] and [33] the definition of an upper semicontinuous map is given in the following manner.

Definition 2.2.4 *A multivalued map $H : X \longrightarrow 2^Y$ is said to be an upper semicontinuous map if the set*

$$H^{-1}(V) = \{x \in X : H(x) \subset V\}$$

is open in X provided that V is open in Y.

When X and Y are metric spaces the concept of an upper semicontinuous set valued map can be characterized in a sequential manner [[8], [11], [33]].

Definition 2.2.5 *Let X and Y be metric spaces. The set valued map $H : X \to 2^Y$ is said to be upper semicontinuous if for any two convergent sequences $x^n \to x$, and $y^n \to y$ in X and Y respectively with*

$$y^n \in H(x^n) \text{ then } y \in H(x).$$

The following result in [29] shows that under certain conditions there is a relationship between upper semicontinuity and multivalued limit maps and the above **Definition 2.2.5** plays a crucial role in the proof.

Theorem 2.2.6 [29] *Let X be a compact metric space and Y convex compact set in a normed linear space. If $H : X \to 2^Y$ is upper semicontinuous and $H(x) \neq \emptyset$ is a convex compact set for each $x \in X$, then H is a multivalued limit map.*

Proof. We need to define a sequence $\{h_n\}$ satisfying **Definition 2.2.2**.
Let $\{U(x) : x \in X\}$ be an open covering of X given by

$$U(x) = B(x, 1/n) \cap \{y \in X : H(y) \subset B(H(x), 1/n)\},$$

where $B(x, 1/n)$ is the ball centered at x of radius $1/n$, $B(H(x), 1/n)$ is a $1/n$-neighbourhood of the convex compact set $H(x)$ and hence $B(H(x), 1/n)$ is convex. By compactness of X, the open covering $\{U(x) : x \in X\}$ has a finite star-refinement $\{V_0, \dots, V_m\}$, that is, for each $x \in X$ there exists $\bar{x} \in X$ such that
$\bigcup\{V_i : x \in V_i\} \subset U(\bar{x})$. Indeed, a compact space X is paracompact hence the above open covering $\{U(x) : x \in X\}$ has a star-refinement covering and from this star-refinement covering we choose a finite subcovering to get $\{V_0, \dots, V_m\}$. For each $i = 0, 1, 2, \dots, m$, let p_i be an arbitrary point of the set $H(V_i) = \bigcup\{H(x) : x \in V_i\}$.
We define the continuous map $h_n : X \longrightarrow Y$ by

$$h_n(x) = \sum_{i=0}^{m} \left(\frac{d_i(x)}{\displaystyle\sum_{j=0}^{m} d_j(x)} \right) p_i,$$

where $d_i(x) = d(x, X \setminus V_i)$ for each $x \in X$.
For each $x \in X$, let us take $\bar{x} \in X$ such that by the star-refinement condition $\bigcup\{V_i : x \in V_i\} \subset U(\bar{x})$, then $p_i \in B(H(\bar{x}), 1/n)$ whenever $x \in V_i$. Clearly $x \in V_i$ if and only if $d_i(x) \neq 0$, this means that, $p_i \in B(H(\bar{x}), 1/n)$ whenever $d_i(x) \neq 0$. Since $B(H(\bar{x}), 1/n)$ is convex, the convex combination

$$h_n(x) = \sum_{i=0}^{m} \left(\frac{d_i(x)}{\sum_{j=0}^{m} d_j(x)} \right) p_i \in B(H(\bar{x}), 1/n).$$

Thus we have proved that for each $x \in X$ there is $\bar{x} \in X$ such that

$$d(x, \bar{x}) < \frac{1}{n} \text{ and } d(h_n(x), H(\bar{x})) < \frac{1}{n}.$$

We show that the sequence $\{h_n : n = 1, 2, \ldots\}$ is a base for H.
Assume that $\lim_{k\to\infty}(x_{n_k}, h_{n_k}(x_{n_k})) = (x, y)$. From the construction of $\{h_n\}$, it follows that for each x_{n_k} there is $\bar{x}_{n_k}$ such that $d(x_{n_k}, \bar{x}_{n_k}) < \frac{1}{n_k}$ and $d(h_{n_k}(x_{n_k}), H(x_{n_k})) < \frac{1}{n_k}$. The latter means that there exists $\bar{y}_{n_k} \in H(\bar{x}_{n_k})$ such that $d(h_{n_k}(x_{n_k}), \bar{y}_{n_k}) = \|h_{n_k}(x_{n_k}) - \bar{y}_{n_k}\| < \frac{1}{n_k}$. Hence $\lim_{k\to\infty}(\bar{x}_{n_k}, \bar{y}_{n_k}) = (x, y)$. Since H is upper semicontinuous, $y \in H$. Thus the sequence $\{h_n : n = 1, 2, \ldots\}$ is indeed a base for H. □

We are in a position to state and prove the main result of the chapter, that is, the "Theorem on Signatures".

Theorem 2.2.7 [Theorem on Signatures]. *Let M be a family of signatures, that is M is a family of continuous nonnegative functions $\mu : X \times Y \longrightarrow [0, \infty)$, from a product of a compact metric simplicial space $(X, \mathcal{S})$ and a compact metric space Y such that for each finite subset $M_0 \subset M$, $\varepsilon > 0$ and $y \in Y$ the set*

$$\{x \in X : \mu(x, y) < \varepsilon \text{ for each } \mu \in M_0\}$$

is nonempty and simplicially convex.
Then, for each multivalued limit map $H : X \to 2^Y$ there is a point $a \in X$ and $b \in H(a)$ such that

$$\mu(a, b) = 0 \text{ for each } \mu \in M.$$

Proof [*see* [29]]. Firstly we prove the theorem for a continuous map $h : X \to Y$.

(I) Fix $\varepsilon > 0$ and assume that M is finite.
Let for each $y \in Y$, the set

$$A(y) = \{x \in X : \mu(x,y) < \varepsilon \text{ for each } \mu \in M\}$$

is nonempty and simplicially convex.
From the continuity of the μ's, the "dual sets"

$$B(x) = \{y \in Y : \mu(x,y) < \varepsilon \text{ for each } \mu \in M\}$$

are open for each $x \in X$. Moreover

$$x \in A(y) \text{ if and only if } \mu(x,y) < \varepsilon \text{ for each } \mu \in M \text{ if and only if } y \in B(x).$$

Since each set $A(y) \neq \emptyset$, let

$$Y = \bigcup\{B(x) : x \in X\}.$$

Then the family $\{B(x) : x \in X\}$ is an open covering of Y. Then for each $y \in Y$ there is $x \in X$ such that $y \in B(x)$; and $y \in B(x)$ if and only if $\mu(x,y) < \varepsilon$ for all $\mu \in M = M_0$. Hence given y the set $\{x \in X : \mu(x,y) < \varepsilon, \mu \in M\}$ is not empty, so such $x \in X$ does exist.
Compactness of $h(X)$ implies that there is a finite set of points $x_0, \dots, x_n \in X$ such that

$$h(X) \subset B(x_0) \cup \dots \cup B(x_n).$$

We take a singular simplex $\sigma \in \mathcal{S}$, $\sigma : [p_0, \dots, p_n] \to X$, such that $\sigma(p_0) = x_0, \dots, \sigma(p_n) = x_n$. From **Theorem 2.1.11** there is a point $a \in X$ and $0 \leq i_0 \leq \dots \leq i_k \leq n$ such that

$$a \in \sigma([p_{i_0}, \dots, p_{i_k}]) \cap h^{-1}(B(x_{i_0})) \cap \dots \cap h^{-1}(B(x_{i_k})).$$

Then $\eta = \sigma\,|_{[p_{i_0},\dots,p_{i_k}]} \in \mathcal{S}$, $a \in \operatorname{im}\eta$, $h(a) \in B(x_{i_0}) \cap \dots \cap B(x_{i_k})$ and hence vert $\eta = \{x_{i_0}, \dots, x_{i_k}\} \subset A(h(a))$. Since $A(h(a))$ is simplicially convex, we have that

$$a \in A(h(a))$$

and this means that

$$\mu(a, h(a)) < \varepsilon \text{ for each } \mu \in M.$$

(II) From the above result, we have that for each $\varepsilon > 0$ the set

$$K(\varepsilon) = \{x \in X : \mu(x, h(x)) \leq \varepsilon \text{ for each } \mu \in M\}$$

is nonempty and compact. Therefore there is a point

$$a \in \bigcap\{K(\varepsilon) : \varepsilon > 0\}$$

and this implies

$$\mu(a, h(a)) = 0 \text{ for each } \mu \in M.$$

(III) We assume that M is infinite.
For each finite set $M_0 \subset M$, let

$$L(M_0) = \{x \in X : \mu(x, h(x)) = 0 \text{ for each } \mu \in M_0\}.$$

Since X is compact and with $L(M_0)$ being closed we have that $L(M_0)$ is compact. From the finite intersection property, the intersection

$$\bigcap\{L(M_0) : M_0 \text{ is a finite subset of } M\}$$

is nonempty. Clearly, each point $a \in X$ in the intersection satisfies the equation $\mu(a, h(a)) = 0$ for each $\mu \in M$.

(IV) We assume that the sequence $\{h_n : n \in \mathbb{N}\}$ is a base for the multivalued limit map $H : X \to 2^Y$.
For each $n \in \mathbb{N}$ choose a point $a_n \in X$ satisfying

$$\mu(a_n, h_n(a_n)) = 0 \text{ for each } \mu \in M.$$

Now, from the compactness of X and Y there are two points $a \in X$ and $b \in Y$ and a subsequence $\{a_{n_k}\}$ such that

$$\lim_{k\to\infty} a_{n_k} = a \text{ and } \lim_{k\to\infty} h_{n_k}(a_{n_k}) = b.$$

From the definition of the multivalued limit map,

$$b \in H(a).$$

Continuity of the function $\mu \in M$ implies that

$$\mu(a,b) = 0 \text{ for each } \mu \in M.$$

□

Remark 2.2.8 . When $H : X \to Y$ is just a mapping the requirement for X and Y to be metric spaces in **Theorem 2.2.7** is not necessary.

2.3 Applications to Fixed Point Theorems and The Existence of Walrasian Equilibrium

In this section we look at the applications of the theorem on signatures, with special attention on fixed point theorems and the existence of Walrasian equilibrium. To be near classical results we mostly state the theorems in terms of convex sets in normed vector spaces and continuous maps.

Let $M = \{\mu\}$, where $\mu(x,y) = \|x - y\|$ is the metric induced by a norm, we obtain the following fixed point results.

Theorem 2.3.1 [Brouwer-Schauder-Kakutani Theorem].
Let $H : X \to 2^X$ be a multivalued limit map from a convex compact subset X of a normed space. Then H has a fixed point, that is, there exists $a \in X$ such that $a \in H(a)$.

Proof. Let M be given by the singleton

$$M = \{\mu\},$$

where

$$\mu(x, y) = \|x - y\|, \ \ x, y \in X$$

is the metric induced by a norm.

With X being a convex compact subset of a normed space, it does admit a simplicial structure which is induced by affine maps.

Indeed **Theorem 2.2.7**, implies that for each multivalued limit map $H : X \to 2^X$ there are points $a \in X$ and $b \in H(a)$ such that

$$\mu(a, b) = \|a - b\| = 0, \ \text{ for each } \ \mu \in M \ .$$

That is,

$$a = b \ \text{ for each } \ \mu \in M \ .$$

Thus, $a = b \in H(a)$ implies that $a \in H(a)$. We can conclude that there exists a fixed point $a \in X$ such that $a \in H(a)$.

□

The above result can be extended to the next result.

Theorem 2.3.2 [Schauder-Tichonov-Kakutani Theorem]. *Let X be a compact metric simplicial space and M be a singleton set of a continuous function $\mu : X \times X \to [0, \infty)$ quasi-simplicially convex with respect to the first variable such that*

1. *for each $\mu \in M$ and for each $x \in X$, $\mu(x, x) = 0$,*
2. *for each two distinct points $x, y \in X$ there is $\mu \in M$ with $\mu(x, y) > 0$.*

Then any multivalued limit map $H : X \to 2^X$ has a fixed point.

Proof. Let M be the singleton set given by

$$M = \{\mu\},$$

where

$$\mu(x, y) = \|x - y\|, \ \ x, y \in X \ .$$

From the two conditions, it is clear that the continuous function $\mu : X \times X \to [0, \infty)$ is non-negative and also satisfies the conditions of **Theorem 2.2.7**.
This means that $\mu \in M$ is a signature.
Indeed, for each $\varepsilon > 0$ and $y \in X$ where $x \neq y$, the set

$$\{x \in X : \mu(x, y) < \varepsilon \text{ for each } \mu \in M\}$$

is non-empty and simplicially convex.
Moreover, by appealing to **Theorem 2.2.7**, we have that for each multivalued limit map $H : X \to 2^X$ there are points $x \in X$ and $y \in H(x)$ such that

$$\mu(x, y) = 0 \text{ for each } \mu \in M .$$

From the hypothesis $\mu(x, y) = 0$ implies that $x = y$.
This shows that $x \in H(x)$ and any multivalued limit map $H : X \to 2^X$ has a fixed point.

□

Having the two fixed point theorems and **Theorem 2.2.7** in the next result we prove the existence of a Pareto optimal point. Firstly we provide the definition of the utility characterization of a local nonsatiation point just for convenience.

Definition 2.3.3 *Let $X_i \subseteq \mathbb{R}^L$ be a consumption set for $i = 1, 2, 3, \ldots, I$. A continuous utility function $u_i : X_i \to \mathbb{R}$ that represents a preference relation $\succeq_i$ on X_i is said to be locally nonsatiated if for every $x_i \in X_i$ and every $\varepsilon > 0$, there exists $y_i \in X_i$ such that $\|y_i - x_i\| \leq \varepsilon$ and $u_i(y_i) > u_i(x_i)$.*

Now, we state and prove the existence of a Pareto optimal point.

Proposition 2.3.4 [Pareto Optimal Theorem]. *Let X_i be a compact consumption set with a metric induced by the standard Euclidean norm $\| \cdot \|$ and a simplicial structure. If $u_i : X_i \to \mathbb{R}$ is continuous and locally nonsatiated, then for every $x_i' \in X_i$ there exists a point $x_i \in X_i$ such that*

$$u_i(x_i) \geq u_i(x_i') \text{ for all } i$$

and

$$u_i(x_i) > u_i(x_i') \text{ for some } i .$$

Proof. Let M be the singleton set $M = \{\mu\}$, where

$$\mu : X_i \times X_i \to [0, \infty)$$

such that

$$\mu(x_i, y_i) = \|x_i - y_i\| \text{ for } x_i, y_i \in X_i$$

is a continuous metric induced by a norm. Clearly for each $y_i \in X_i$ and $\varepsilon > 0$ the set,

$$\{x_i \in X_i : \mu(x_i, y_i) \leq \varepsilon \text{ for each } \mu \in M\}$$

is simplicially convex because it is convex and it is nonempty because of the compactness of X_i. This means that μ is quasi-simplicially convex. Applying **Theorem 2.2.7** and **Theorem 2.3.2** on the multivalued limit map
$H : X_i \to 2^{X_i}$ there is a fixed point $x_i \in H(x_i) \subset X_i$.
Hence, by the local nonsatiation of the continuous utility function we have that for every $x_i' \in X_i$ and every $\varepsilon > 0$ there exists $x_i \in H(x_i) \subset X_i$ such that

$$\mu(x_i, x_i') = \|x_i - x_i'\| \leq \varepsilon$$

and

$$u_i(x_i) > u_i(x_i') \text{ for some } i.$$

Thus, it follows that

$$u_i(x_i) \geq u_i(x_i') \text{ for all } i.$$

□

Now, we are in a position to prove the existence of equilibrium with the use of **Theorem 2.3.1** or **Theorem 2.3.2**. This will be done in the following manner:

- define the *excess demand map* as done in [33], [54] and also in the first chapter.
- endow the unit simplex with a simplicial structure and
- use the metric induced by a norm in the n-Euclidean space $\mathbb{R}^n$ as signature
- define a *fixed point multivalued map* in terms of the excess demand map.
- and finally use either of the two fixed point results discussed in this section to prove the existence of equilibrium.

The above setup will enable us to prove **Theorem 1.5.5**, the result on the existence of equilibrium for a pure exchange economy, which we now restate as a proposition.

Proposition 2.3.5 *Suppose that $z(p)$ is a function defined for all strictly positive price vectors $p \in \mathbb{R}^L_+ \backslash \{0\}$ and satisfying conditions of **Lemma 1.5.3**. Then the system of equations $z(p) = 0$ has a solution. Hence a Walrasian equilibrium exists in any pure exchange economy in which $\sum_{i=1}^{I} e_i > 0$ and every consumer has continuous, strictly convex and strongly monotone preferences.*

Proof. We begin by normalizing prices in a convenient way. Denote by

$$\Delta_1 = \{p \in \mathbb{R}^L_+ \ : \ \sum_{\ell=1}^{L} p_\ell = 1\}$$

the unit simplex in $\mathbb{R}^L$. This means that we are restricted to the unit simplex as the domain for the aggregate excess demand function $z(\cdot)$. Because the function $z(\cdot)$ is homogeneous of degree zero, according to condition **(AED2)**, this allows us to restrict our search for an equilibrium to price vectors in the unit simplex Δ_1. It is worth noting that the function $z(\cdot)$ is well defined for price vectors in the set

$$\text{Interior}\Delta_1 = \{p \in \Delta_1 : p_\ell > 0 \ \text{ for all } \ \ell = 1, 2, 3, \dots, L\} \ ,$$

the interior of the unit simplex Δ_1.
The boundary of the unit simplex is the set denoted by $\partial\Delta_1$ and is defined as

$$\partial\Delta_1 = \{p \in \Delta_1 : p_\ell = 0 \ \text{ for some } \ \ell = 1, 2, 3, \dots, L\}.$$

The proof follows the next steps:

Step 1: We construct a fixed point multivalued map $H(\cdot)$ from Δ_1 to the power set 2^{Δ_1}.

We define a multivalued map,

$$H : \Delta_1 \longrightarrow 2^{\Delta_1} \ .$$

Thus for $p \in \Delta_1$, $H(p) \subseteq \Delta_1$ and for clarity the vectors that are elements of $H(p)$ are denoted by the symbol q.

Case 1: We construct $H(\cdot)$ for $p \in$ InteriorΔ_1.

$$H(p) = \{q \in \Delta_1 : z(p) \cdot q \geq z(p) \cdot q' \text{ for all } q' \in \Delta_1 .\} \tag{2.3}$$

This condition (2.3) means that given a price vector $p \in \text{Interior}\Delta_1$, the price vector assigned by $H(\cdot)$ is any price vector q that, among the permissible price vectors, maximizes the value of the excess demand vector $z(p)$.
Equivalently the multivalued map $H(\cdot)$ for $p \in \text{Interior}\Delta_1$ can be expressed as follows:

$$H(p) = \{q \in \Delta_1 : \; q_\ell = 0 \text{ if } z_\ell(p) < \max\{z_1(p), \ldots, z_L(p)\}\}. \tag{2.4}$$

Clearly if $z(p) \neq 0$ for $p > 0$, then by Walras' law we have $z_\ell(p) < 0$ for some ℓ and $z_{\ell'}(p) > 0$ for some $\ell' \neq \ell$. Thus, for such a p, any $q \in H(p)$ has $q_\ell = 0$ for some ℓ. Therefore, if $z(p) \neq 0$ then $H(p) \subset \partial\Delta_1 = \Delta_1 \backslash \text{Interior}\Delta_1$. In contrary, if $z(p) = 0$ then $H(p) = \Delta_1$.

Case 2: We construct $H(\cdot)$ for $p \in \partial\Delta_1$.
Then for $p \in \partial\Delta_1$, we let

$$\begin{aligned} H(p) &= \{q \in \Delta_1 : p \cdot q = 0\} && (2.5) \\ &= \{q \in \Delta_1 : q_\ell = 0 \text{ if } p_\ell > 0\}. && (2.6) \end{aligned}$$

Since $p_\ell = 0$ for some ℓ, we have that $H(p) \neq \emptyset$. If $p \in \partial\Delta_1$ then $p \notin H(p)$ because $p \cdot p > 0$ while $p \cdot q = 0$ for all $q \in H(p)$. This means that $H(\cdot)$ has no fixed point for any $p \in \partial\Delta_1$.

Step 2: We argue that a fixed point of $H(\cdot)$ is an equilibrium.
Suppose that $p^* \in H(p^*)$. Then it follows from **Case 2 in Step 1** that $p^* \notin \partial\Delta_1$, which implies that $p^* > 0$. If $z(p^*) \neq 0$, then it follows from **Case 1 in Step 1** that $H(p^*) \subset \partial\Delta_1$, which is incompatible with $p^* \in H(p^*)$ and $p^* > 0$. Hence, if $p^* \in H(p^*)$ we must have $z(p^*)$=0.

Step 3: We show that the fixed point multivalued map $H(\cdot)$ is a multivalued limit map and that the fixed point exists.
It is a known fact that the unit simplex $\Delta_1 \subset \mathbb{R}^L$ is convex and compact, that is, closed and bounded. The convexity condition means that the unit simplex Δ_1 has a

simplicial structure induced by a family of affine maps.
Let M be given by the singleton set $M = \{\mu\}$, where the continuous function

$$\mu : \Delta_1 \times \Delta_1 \to [0, \infty)$$

is the Euclidean metric induced by the Euclidean norm,

$$\mu(p^*, p') = \|p^* - p'\| = \sqrt{\sum_{\ell=1}^{L}(p^*_\ell - p'_\ell)} \text{ for } p^* = (p^*_1, \ldots, p^*_L), p' = (p'_1, \ldots, p'_L) \in \Delta_1.$$

Clearly for each $p' \in \Delta_1$ and $\varepsilon > 0$ the set

$$\{p^* \in \Delta_1 : \mu(p^*, p') < \varepsilon \text{ for each } \mu \in M\}$$

is nonempty and simplicially convex, which means that the function
$\mu : \Delta_1 \times \Delta_1 \to [0, \infty)$ is quasi-simplicially convex.
Applying **Theorem 2.2.7** and **Theorem 2.3.2**, any multivalued map $H : \Delta_1 \to 2^{\Delta_1}$ is a multivalued limit map which has a fixed point $p^* \in H(p^*)$. By **Step 2** this implies that $z(p^*) = 0$. This completes the proof.

□

2.4 Further Applications of The Theorem on Signatures

Now, we are in a position to look at other far reaching consequences of the **Theorem 2.2.7** in solving the other classical known results.

Definition 2.4.1 *Let $(X, \mathcal{S})$ be a simplicial space. A function $f : X \to \mathbb{R}$ is said to be quasi-concave if the set $\{x \in X : f(x) > r\}$ is simplicially convex for each $r \in \mathbb{R}$.*

A function $f : X \to \mathbb{R}$ is said to be quasi-convex if the set $\{x \in X : f(x) < r\}$ is simplicially convex for each $r \in \mathbb{R}$.

The concept of evolutionary stable strategy (ESS) defined below, was introduced by Maynard Smith and Price [35], which is a fundamental notion of modern evolutionarily and genetic biology.

Definition 2.4.2 *A point $a \in X$ is said to be an ESS point for a function $f : X \times X \to \mathbb{R}$ if $f(x,a) \leq f(a,a)$ for each $x \in X$.*

According to Maynard Smith [34] and [35] ESS is a strategy such that, if all members of a population adopt it, then no mutant strategy can invade the population under the influence of natural selection.

The following theorem is an attempt to investigate ESS from a topological point of view, with the use of the **Theorem 2.2.7**.

Theorem 2.4.3 [Maynard Smith Theorem].
Let X be a compact simplicial space and $f : X \times X \to \mathbb{R}$ a continuous function which is quasi-concave with respect to the first variable. Then f has an ESS point.

Proof. Let $\mu(x,y) = -f(x,y) + \sup_{z\in X} f(z,y)$. Fix $y \in X$ and $r > 0$.
Let $s = \sup_{z\in X} f(z,y)$ and $u = s - r$.
Clearly the pseudoball $B(y,r) = \{x \in X : \mu(x,y) < r\}$is simplicially convex because of the quasi-concavity assumption of the function f. The set

$$\begin{aligned} B(y,r) &= \{x \in X : \mu(x,y) < r\} && (2.7) \\ &= \{x \in X : s - r < f(x,y)\} && (2.8) \end{aligned}$$

is simplicially convex. The pseudoball $B(y,r)$ is nonempty due to the continuity of f and the compactness of X. This means that for each point $y \in X$ there is a point $x \in X$ such that $\sup_{x\in X} f(x,y) = f(x,y)$ and consequently $\mu(x,y) = 0$.
Applying **Theorem 2.2.7** to the identity map $h : X \to X, h(x) = x$, we can establish that there is a point $a \in X$ such that $\mu(a,a) = 0$.
Therefore we get $f(a,a) = \sup_{x\in X} f(x,a)$ which implies that $f(x,a) \leq f(a,a)$. Hence, we have that f has an ESS point.

□

Theorem 2.4.4 [Nash Equilibrium Theorem].
Let $X_1, \ldots, X_n$ be compact simplicial spaces and $X = X_1 \times \cdots \times X_n$ be their product. If $f_i : X \to \mathbb{R}$ is a family of continuous functions and each function f_i is quasi-concave

with respect to the variable $x_i \in X_i$ for each $i = 1, 2, 3, \ldots, n$, then there exists a point $a \in X$ such that

$$f_i(a) = \sup_{x \in X_i} f_i(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$$

for each $i = 1, 2, 3, \ldots, n$.

Proof.
Define for each $i = 1, \ldots, n$;

$$\mu_i(x, y) = -f_i(N_i(x, y)) + \sup_{z \in X} f_i(N_i(z, y)),$$

where $N_i : X \times X \to X$ represents the Nash projection,

$$N_i(x, y) = (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)$$

for each $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X$.
Fix $i = 1, 2, 3, \ldots, n$, $y = (y_1, \ldots, y_n) \in X$ and $r > 0$. Let $s = \sup_{x \in X} f_i(N_i(x, y))$ and $u = s - r$.
Since $B_i(y, r) = \{x \in X : \mu_i(x, y) < r\} = \{x \in X : s - r < f_i(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)\}$.
By the quasi-concavity of f_i we have that each pseudoball $B_i(y, r)$ is simplicially convex and therefore the set
$A(y, r) = \bigcap_{i=1}^{n} \{x \in X : \mu_i(x, y) < r$ for each $i = 1, 2, 3, \ldots, n\}$ is simplicially convex.
It remains to show that the set $A(y, r)$ is nonempty. By the compactness for each $i \leq n$ there is a point $a^i \in X$ such that $\mu_i(a^i, y) = 0$. Let $a = (a_1, \ldots, a_n) \in X$ be a unique point such that $a_i = a_i^i$ for each $i \leq n$. Since $N_i(a^i, y) = N_i(a, y)$ it is clear that $\mu_i(a, y) = 0$ for each $i \leq n$, and this implies $a \in A(y, r)$.
Applying **Theorem 2.2.7** to the identity map $h : X \to X$, $h(x) = x$, we infer that there is a point $a \in X$ such that

$$\mu_i(a, a) = 0 \text{ for each } i = 1, 2, 3, \ldots, n.$$

But $N_i(a, a) = (a_1, \ldots, a_{i-1}, a_i, \ldots, a_n) = a$ and therefore we get,

$$f_i(a) = \sup_{x \in X} f_i(N_i(x, a)) = \sup_{x \in X_i} f_i(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$$

for each $i = 1, 2, 3, \ldots, n$.

$\square$

From the above result we can say that a point $a \in X = X_1 \times \cdots \times X_n$, $a = (a_1, \ldots, a_n)$, is said to be *Nash's Equilibrium Point* for the family of functions $f_i : X \to \mathbb{R}$ for each $i = 1, \ldots, n$, if

$$f_i(a_1, \ldots, a_{i-1}, x_i, a_{i-1}, \ldots, a_n) \le f_i(a) \text{ for each } i \le n \text{ and } x_i \in X_i.$$

According to [5], [24], [39] and [53] the equilibrium point $a \in X$ is a feasible joint strategy, X is a set of feasible joint strategies and the functions f_i are payoff functions $\{f_i : i \le n\}$ such that the payoff function $f_i(x)$ is defined on X for each player $i \le n$.

It is important to note that given an equilibrium point, there is no feasible way for any player to strictly improve its utility if the strategies of all the other players remain unchanged.

It can be easily verified that if $a \in X = X_1 \times \cdots \times X_n$ is an ESS point for the function $f : X \times X \to \mathbb{R}$, given by

$$f(x, y) = \sum_{i=0}^{n} f_i(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n),$$

then the point $a \in X$ is Nash's equilibrium point.

Clearly

$$\begin{aligned} f(x, a) &= \sum_{i=0}^{n} f_i(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) && (2.9) \\ &\le \sum_{1=0}^{n} f_i(a) && (2.10) \\ &= f(a, a). && (2.11) \end{aligned}$$

Hence, $a \in X$ is Nash equilibrium point.

The next result as in [28] is called the **Infimum Principle** and it is a general result that helps to derive some classical theorems. We apply the **Theorem 2.2.7** to prove this result.

Theorem 2.4.5 [Infimum Principle].
Let G be a family of continuous quasi-simplicially convex functions $g : X \times Y \to \mathbb{R}$ with respect to the first variable $x \in X$, from a product of a compact metric simplicial space X and a compact metric space Y, such that for each finite subcollection $G_0 \subset G$ and for each point $y \in Y$ there is a point $a \in X$ with

$$g(a, y) = \inf_{x \in X} g(x, y) \text{ for each } g \in G_0,$$

then for each multivalued limit map $H : X \to 2^Y$ there are points $a \in X$ and $b \in H(a)$ such that

$$g(a, b) = \inf_{x \in X} g(x, b) \text{ for each } g \in G.$$

Proof. Define $\mu_g(x, y) = g(x, y) - \inf_{x \in X} g(x, y)$ and let $M = \{\mu_g : g \in G\}$ be a family of signatures.
Fix $y \in Y$ and $r > 0$. The pseudoball

$$\begin{aligned} B(y, r) &= \{x \in X : \mu_g(x, y) < r\} && (2.12)\\ &= \{x \in X : g(x, y) - \inf_{x \in X} g(x, y) < r\} && (2.13)\\ &= \{x \in X : g(x, y) < r + \inf_{x \in X} g(x, y)\} && (2.14) \end{aligned}$$

is simplicially convex due to the quasi-convexity assumption of the functions $g : X \times Y \to \mathbb{R}$. By compactness there is a point $a \in X$ such that $\mu_g(a, y) = 0$ and this implies that $a \in B(y, r)$.
Now, applying **Theorem 2.2.7** to the multivalued limit map $H : X \to 2^Y$ there are points $a \in X$ and $b \in H(a)$ such that

$$\mu_g(a, b) = 0 \quad \text{for each } \mu_g \in M.$$

Hence, we have that $g(a, b) = \inf_{x \in X} g(x, b)$ and this completes the proof.

$\square$

Let $\Delta_n = [e_1, \dots, e_n]$, where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, $e_i(j) = 0$ for $i \neq j$ and $e_i(i) = 1$, denote the $(n-1)$-dimensional standard simplex in the space $\mathbb{R}^n$. The following theorem plays an important role in a proof of the existence of equilibrium points in economic models in the Walras sense as done by Nikaido [41].

Theorem 2.4.6 **[Gale-Nikaido Theorem]**.
Let $H : \Delta_n \to 2^C$ *be an upper semicontinuous map from the standard simplex* Δ_n *such that for each* $x \in \Delta_n$, $H(x)$ *is nonempty compact convex subset of a compact convex set* $C \subset \mathbb{R}^n$. *Suppose further that the Walras law in general holds;*

$$x \cdot y = \sum_{i=1}^{n} x_i y_i \geq 0 \text{ for each } x \in \Delta_n \text{ and } y \in H(x).$$

Then there exist $a \in \Delta_n$ *and* $b \in H(a)$ *such that* $b_i \geq 0$ *for each* $i = 1, \ldots, n$.

Proof.
We apply **Theorem 2.4.5** to $X = \Delta_n$, $Y = C$, the given set valued map H, the function g given by $g(x, y) = x \cdot y$, there is a point $(a, b) \in \Delta_n \times C$ such that $b \in H(a)$ and $a \cdot b = \inf\{x \cdot b : x \in \Delta_n\}$. By Walras law $a \cdot b \geq 0$ and in consequence $0 \leq a \cdot b \leq x \cdot b$ for each $x \in \Delta_n$. Since $e_i \in \Delta_n$, $0 \leq e_i \cdot b = b_i$ for each $i = 1, 2, 3, \ldots, n$.

□

Chapter 3

Global Analysis and The Existence of Equilibrium

3.1 Global Analysis in Economics

3.1.1 Introduction

In this chapter we look at global analysis and calculus foundations such as Sard's Theorem and the Inverse Mapping Theorem to show the existence proof for equilibrium, rather than fixed points techniques as we have done in the preceding chapters. Furthermore there are more classical equilibrium related concepts that can be proved through these calculus foundations together with the global analysis.

The Euclidean n-space is the set $\mathbb{R}^n$ together with the Euclidean distance between the points $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ given by

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \, .$$

The usual basis of $\mathbb{R}^n$ is $e_1, \cdots, e_n$, where

$$e_i = (0, \cdots, 1, \cdots, 0)$$

with 1 in the i^{th} place.
If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, the matrix of T with respect to the usual basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ is the $m \times n$ matrix $A = (a_{ij})$ where $T(e_i) = \sum_{i=1}^{n} a_{ji} e_j$ with the coefficients of $T(e_i)$ appear in the i^{th} column of the matrix.

Now, we are in a position to introduce the concept of a manifold and other concepts which are basic in Global Analysis, that is, Differential Geometry and Differential Topology.

In the book *A Comprehensive Introduction to* **DIFFERENTIAL GEOMETRY**, *Volume I* by Michael Spivak [48] these concepts are defined as follows:

Definition 3.1.1 *A manifold M is a metric space with the following property: If $x \in M$, then there is some neighbourhood $\mathcal{U}$ of x and some integer $n \geq 0$ such that $\mathcal{U}$ is homeomorphic to $\mathbb{R}^n$.*

That is a manifold is supposed to be locally like a certain metric space $\mathbb{R}^n$. We give some simple examples of a manifold.

Examples 3.1.2 .

1. The metric space $\mathbb{R}^n$ is a manifold because for each $x \in \mathbb{R}^n$ we can treat $\mathcal{U}$ as all of $\mathbb{R}^n$.

2. Clearly from the above definition, anything homeomorphic to a manifold is also a manifold.

3. An open ball in $\mathbb{R}^n$ is a manifold.

4. Any open subset V of $\mathbb{R}^n$ is a manifold.

The manifold $\mathbb{R}^n$ is appropriate for the application of global analysis to economics. Since we work in the manifold $\mathbb{R}^n$ we only require submanifolds of $\mathbb{R}^n$.

A definition on the differentiability of the maps $f : \mathbb{R}^n \to \mathbb{R}^m$ is given by Spivak, M [47] in the book *Calculus on Manifolds*, but for our purpose as mentioned above the following definition of a C^r-differentiable map by Lee [30] in the book *Introduction to Smooth Manifolds*, suffices.

Definitions 3.1.3 *Let f be a mapping defined in an open subset $U \subseteq \mathbb{R}^n$ valued in $\mathbb{R}^m$, with $f(x) = (f_1(x), \ldots, f_m(x))$ is said to be C^r-differentiable on U if each f_i has continuous partial derivatives up to the r^{th} order.*

A mapping $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where A is a closed set is said to be C^r-differentiable if there is an open neighbourhood V of A and a C^r function F on V such that $f(x) = F(x)$ for $x \in A$.

Denote by

$$Df(x) = \left[\frac{\partial f_i}{\partial x_j}(x)\right] \; for \;\; i = 1, \cdots, m \;\; and \;\; j = 1, \cdots, n$$

the matrix of partial derivatives at $x \in U$. We say f has rank M at x if the rank of the matrix $Df(x)$ is equal to M.

We say that the map $f : U \subseteq R^n \to \mathbb{R}^m$ is C^∞, that is, each component map f_i possesses continuous partial derivatives of all orders. The terms differentiable or smooth can be used interchangeably to mean C^∞.

It is also important to note that $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the usual bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ is a $m \times n$ matrix called the **Jacobian matrix** of f at x.

Note 3.1.4 . A map f is said to be differentiable on an open (closed) set U in $\mathbb{R}^n$ if f is differentiable at x for each $x \in U$.

Now, we introduce the concept of a singular point and that of a differentiable map as follows:

Definitions 3.1.5 *Let $f : U \to \mathbb{R}^m$ be a differentiable map where U is an open set in $\mathbb{R}^n$. The point $x \in U$ is said to be a singular point if the Jacobian matrix $Df(x)$ has rank less than m.*

The singular values are the images under f of all the singular points.

A point $y \in \mathbb{R}^m$ is said to be a regular value of f if $f^{-1}(y)$ is empty or for each $x \in f^{-1}(y)$, $rank Df(x) = m$, otherwise y is singular.

We obtain the definition of a set with measure zero in the book *Calculus on Manifolds* by Spivak,M [47].

Definition 3.1.6 *A set $A \subset \mathbb{R}^n$ has measure zero if for every $\varepsilon > 0$ there is a sequence $B_1, B_2, B_3, \ldots$ of closed (open) rectangles with*

$$A \subset \bigcup_{\ell=1}^{\infty} B_\ell$$

and

$$\sum_{\ell=1}^{\infty} \upsilon(B_\ell) < \varepsilon,$$

where $\upsilon(B_\ell)$ is the volume of B_ℓ.

In the following definition we introduce the concept of an embedded submanifold of $\mathbb{R}^n$. We achieve this with the aid of the book *Introduction to Smooth Manifolds* by Lee [30].

Definition 3.1.7 *The set B is a submanifold of an open set $U \subset \mathbb{R}^n$ of dimension k if given $x \in B$,one can find a differentiable map $h : N(x) \to \mathcal{O}$ with the following properties:*

(a) h has a differentiable inverse.

(b) $N(x)$ is an open neighbourhood of x in U.

(c) $\mathcal{O}$ is an open set containing 0 in $\mathbb{R}^n$.

(d) $h(N(x) \cap B) = \mathcal{O} \cap C$ where C is a coordinate subspace of $\mathbb{R}^n$ of dimension k, given by $C = \{x \in \mathbb{R}^n : x_{k+1} = 0, \ldots, x_n = 0\}$.

We say B is a C^r submanifold if each map $h : N(x) \to \mathcal{O}$ is a C^r map. It is also important to note that an embedded submanifold B has a topology induced from $\mathbb{R}^n$.

We state the **Inverse Mapping Theorem (IMT)** and **Sard's theorem** respectively as done in [49].

Theorem 3.1.8 [IMT].
If $y \in \mathbb{R}^m$ is a regular value of a C^r map $f : U \to \mathbb{R}^m$, U open in $\mathbb{R}^n$, then either $f^{-1}(y)$ is empty or it is an embedded submanifold B of U of dimension $n - m$.

Proof [*see* Sternberg,S [49]].

Theorem 3.1.9 [Sard's theorem].
If $f : A \to \mathbb{R}^m$ with A open in $\mathbb{R}^n$, f is sufficiently differentiable of class C^r where $r > n - m > 0$, then the set of singular values has measure zero.

Proof [*see* Sternberg,S [49]].

Remark 3.1.10 . The set of regular values has full measure, that is, the complement of the set of regular values has measure zero.

The next results according to [46] are general, purely mathematical theorems about solutions of systems of equations.

Lemma 3.1.11 *Let $\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ denote a closed unit ball in $\mathbb{R}^n$ and its boundary $\partial\mathbb{B}^n = S^{n-1} = \{x \in \mathbb{B}^n : \|x\| = 1\}$ is a unit sphere.*
Suppose that the map $f : \mathbb{B}^n \to \mathbb{R}^n$ is C^2 and satisfies the strong boundary condition:

$$f(x) = -x \text{ for all } x \in \partial\mathbb{B}^n = S^{n-1} . \tag{3.1}$$

Then there is $x^* \in \mathrm{Int}\mathbb{B}^n$ with $f(x^*) = 0$.

Proof [*see* Smale [46]]. According to **Definition 3.1.3** f extends to a C^2 map F on some open neighbourhood $\mathcal{U}$ of $\mathbb{B}^n$ and consider an auxiliary map $g : \mathcal{U} \setminus E \to S^{n-1}$ defined by $g(x) = F(x)/\|F(x)\|$ where $E = \{x \in \mathcal{U} : F(x) = 0\}$ is the solution set, that is, the set of zeros of F. Since g is C^2, **Theorem 3.1.9** implies that the set of regular values of g has full measure in S^{n-1}, hence it is not empty.
Let $y \in S^{n-1}$ be such a regular value. Then, by **Theorem 3.1.8**, the preimage $g^{-1}(y)$ is a 1-dimensional C^2 submanifold or a regular smooth embedded curve. The boundary condition (3.1) implies that $-y$ is the only point on S^{n-1} mapped on y, hence $-y \in g^{-1}(y)$.
Let V be the connected component of $g^{-1}(y)$ containing $-y$. Since g restricted to the sphere S^{n-1} is an antipodal map according to condition (3.1) and since g is constant

along V, the tangent vector to V at $-y$ is transversal to S^{n-1} or not tangent to the unit sphere. The 1 - dimensional submanifold V is a non-singular (simple) arc starting from $-y$ and open at the opposite end.
Let $t \in [\alpha, \beta) \to x(t)$ with $x(\alpha) = -y$, be an onto parametrization of V. Consider the convergent sequence $t_k \to \beta$ as $k \to \infty$. By the compactness of $\mathbb{B}^n$, we suppose that $x(t_k) \to x^0 \in \mathbb{B}^n$ as $k \to \infty$.
Suppose that E is empty.
This is illustrated in the following three cases.

Case I: If $x^0 \in \text{Int}\mathbb{B}^n$ and $g(x^0) = y$, then by the Implicit Function Theorem the curve V extends further beyond β or parametrization $t \to x(t)$ is defined for $\beta' > \beta$, which is a contradiction.

Case II: If $x^0 \in \partial\mathbb{B}^n$ and $x^0 \neq -y$, with g an antipodal map on the boundary $\partial\mathbb{B}^n$ this gives us $g(x^0) \neq y$. This is impossible.

Case III: Let $x^0 = -y$. Then, by the fact that $g^{-1}(y)$ is a submanifold, V is a closed regular curve and $V \cap \partial\mathbb{B}^n = \{-y\}$. Therefore, the tangent vector to V at $-y$ is also tangent to $\partial\mathbb{B}^n$. This leads us to a contradiction due to the fact that this vector is transversal to the sphere $\partial\mathbb{B}^n$.
Thus E is not empty and hence for some $x^* \in \text{Int}\mathbb{B}^n$, $f(x^*) = 0$ or simply put the open end of V must have a limit in $\text{Int}\mathbb{B}^n$ and hence $E \neq \emptyset$.
This gives a geometrically constructive proof of the existence of $x^* \in \text{Int}\mathbb{B}^n$ with $f(x^*) = 0$.

□

We generalize the above result into a case of continuous maps in the following result.

Lemma 3.1.12 *Suppose that the map $f : \mathbb{B}^n \to \mathbb{R}^n$ is continuous and satisfies the condition (3.1):*

$$f(x) = -x \text{ for all } x \in \partial\mathbb{B}^n .$$

Then $f(x^*) = 0$ for some $x^* \in \mathbb{B}^n$.

Proof [*see* Smale [46]]. We define a new continuous map $f_0 : \mathbb{B}_2^n \to \mathbb{R}^n$ by

$$f_0(x) = \begin{cases} f(x) & \text{for } \|x\| \leq 1 \\ -x & \text{for } \|x\| \geq 1, \end{cases}$$

where $\mathbb{B}_2^n = \{x \in \mathbb{R}^n : \|x\| \leq 2\}$. Clearly $f_0(x)$ is continuous on $\mathbb{R}^n$.

We shall construct a sequence of C^∞ functions $\{f_i\}$ on $\mathbb{R}^n$ such that given $\varepsilon_i \to 0$,

(a)$f_i(x) = -x$ for $x \in \partial\mathbb{B}_2^n$.

(b)$\|f_i(x) - f_0(x)\| < \varepsilon_i$ for $x \in \mathbb{B}_2^n$.

Consider a real valued C^∞ function $\varphi_r(x) = \varphi_r(\|x\|) \geq 0$ [***see*** **Narasimhan [38]**] on $\mathbb{R}^n$ such that

$$\varphi_r(x) = 0 \text{ if } \|x\| \geq r$$

and

$$\int_{\mathbb{R}^n} \varphi_r(x)dx = \int \varphi_r(x)dx = 1\ .$$

We choose $r < 1/2$ that depends on ε_i as will be shown later and define

$$f_i(y) = \int f_0(y-x)\varphi_r(x)dx\ .$$

In this formula we integrate each component of f_0. It is known, from the property of convolution that

$$f_i(y) = \int f_0(x)\varphi_r(y-x)dx\ .$$

Since we can differentiate on y under the integral sign, any number of times, the function f_i is C^∞ on $\mathbb{R}^n$.

Let $\|y\| = 2$ or $y \in \partial\mathbb{B}_2^n$. Then for $\|x\| \leq r \leq 1/2$ we have that

$$\|y - x\| \geq |\|y\| - \|x\|| = |2 - \|x\|| \geq 1\ .$$

So,

$$f_0(y-x) = -(y-x) = -y + x$$

and

$$\begin{aligned} f_i(y) &= \int_{\|x\|\leq r} f_0(y-x)\varphi_r(x)dx \\ &= \int_{\|x\|\leq r} (-y+x)\varphi_r(x)dx \\ &= -y\int_{\|x\|\leq r} \varphi_r(x)dx + \int_{\|x\|\leq r} x\varphi_r(x)dx\ . \end{aligned}$$

Now, $\int \varphi_r(x)dx = 1$ and $\int x_k\varphi_r(x)dx = 0$ for all $k = 1, 2, 3, \ldots, n$, due to the fact that $\varphi_r(x)$ is even, that is, $\varphi_r(x) = \varphi_r(-x)$.

This shows that

$$f_i(y) = -y \text{ for } y \in \partial\mathbb{B}_2^n\ .$$

Finally we show that

$$\|f_i(y) - f_0(y)\| \le \varepsilon_i \text{ for all } y \in \mathbb{B}_2^n .$$

Indeed,

$$\begin{aligned}
\|f_i(y) - f_0(y)\| &= \left\| \int f_0(y-x)\varphi_r(x)dx - f_0(x) \int \varphi_r(x)dx \right\| \\
&= \left\| \int [f_0(y-x) - f_0(y)]\varphi_r(x)dx \right\| .
\end{aligned}$$

Given $\varepsilon_i > 0$, we choose a small r such that

$$\|f_0(y-x) - f_0(y)\| < \varepsilon_i$$

whenever

$$\|(y-x) - y\| = \|x\| < r$$

by uniform continuity of f_0 on the compact ball $\mathbb{B}_r^n$.
Thus,

$$\begin{aligned}
\|f_i(y) - f_0(y)\| &\le \int \|f_0(y-x) - f_0(y)\|\varphi_r(x)dx \\
&\le \varepsilon_i \int \varphi_r(x)dx \\
&= \varepsilon_i, \text{ since} \int \varphi_r(x)dx = 1 .
\end{aligned}$$

This means that

$$\|f_i(y) - f_0(y)\| \le \varepsilon_i \text{ if } y \in \mathbb{B}_2^n .$$

Therefore f_i satisfies the assumptions of **Lemma 3.1.11** and has zeros in $\mathbb{B}_2^n$, that is, there exists $x_i \in \mathbb{B}_2^n$ with $f_i(x_i) = 0$.
We need to show that $x_i \in \mathbb{B}^n$.
Indeed,

$$\|f_i(x_i) - f_0(x_i)\| < \varepsilon_i \text{ for } x_i \in \mathbb{B}_2^n .$$

In particular since $f_i(x_i) = 0$, we have that

$$\|f_i(x_i) - f_0(x_i)\| = \|f_0(x_i)\| < \varepsilon_i .$$

Moreover from the strong boundary condition (3.1), we have that

$$\|x\| \ge 1 \text{ implies } f_0(x_i) = -x_i$$

and from the definition of f_0, we have that

$$\|f_i(x_i) - f_0(x_i)\| = \|x_i\| > 1\,.$$

This contradicts the fact that $\|f_i(x_i) - f_0(x_i)\| < \varepsilon_i$. Hence this shows that $\|x_i\| \leq 1$ and from the definition of f_0, we have that $\|f(x_i)\| = \|f_0(x_i)\| < \varepsilon_i$. Thus we have a sequence $\{x_i\}$ in $\mathbb{B}^n$ such that $\|f(x_i)\| < \varepsilon_i$, for every $\varepsilon_i > 0$ as $i \to \infty$.
By compactness of $\mathbb{B}^n$ we choose a convergent subsequence denoted by $\{x_i\}$ again such that $x_i \to x_0$ as $i \to \infty$, $\;x_0 \in \mathbb{B}^n$. Since f is continuous,

$$\lim_{i\to\infty} \|f(x_i)\| = \|f(x_0)\| \leq \lim_{i\to\infty} \varepsilon_i = 0\,.$$

This means that $f(x_0) = 0$.

□

Theorem 3.1.13 *If the continuous mapping $f : \mathbb{B}^n \to \mathbb{R}^n$ satisfies the following boundary condition:*

$$\text{if } x \in \partial\mathbb{B}^n, \text{ then } f(x) \neq \mu x \text{ for any } \mu > 0\,, \tag{3.2}$$

then there is $x^* \in \mathbb{B}^n$ with $f(x^*) = 0$,
where $\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is a closed unit ball in $\mathbb{R}^n$ and its boundary $\partial\mathbb{B}^n = \{x \in \mathbb{B}^n : \|x\| = 1\}$ is a unit sphere.

Proof [*see* Smale [46]].
Suppose that the map $f : \mathbb{B}^n \to \mathbb{R}^n$ is continuous and satisfies (3.2).
We define a continuous map $\hat{f} : \mathbb{B}^n_2 \to \mathbb{R}^n$ such that $\hat{f}(x) = -x$ for $x \in \partial\mathbb{B}^n_2$ as follows:

$$\hat{f}(x) = \begin{cases} f(x) & \text{for } \|x\| \leq 1 \\ (2 - \|x\|)f(x/\|x\|) + (\|x\| - 1)(-x) & \text{for } \|x\| \geq 1\,. \end{cases}$$

It is clear that $\hat{f}(x) = f(x)$ for $\|x\| = 1$ and $\hat{f}(x) = -x$ for $\|x\| = 2$. So f is continuous on $\mathbb{B}^n_2$. Now by **Lemma 3.1.12** there is $x^* \in \mathbb{B}^n_2$ and $x^* \notin \partial\mathbb{B}^n_2$ such that $\hat{f}(x^*) = 0$.
We claim that $\|x^*\| \leq 1$ or $x^* \in \mathbb{B}^n$.
Assume it is not or $1 < \|x^*\| < 2$ then we have that,

$$\begin{aligned} \hat{f}(x^*) &= (2 - \|x^*\|)f(x^*/\|x^*\|) + (\|x^*\| - 1)(-x^*) \\ &= 0 \end{aligned}$$

or

$$\begin{aligned} f(x^*/\|x^*\|) &= \frac{\|x^*\|-1}{2-\|x^*\|}\, x^* \\ &= \frac{\|x^*\|(\|x^*\|-1)}{2-\|x^*\|}\,(x^*/\|x^*\|), \end{aligned}$$

where the scalar factor $\frac{\|x^*\|(\|x^*\|-1)}{2-\|x^*\|}$ is positive and hence the boundary condition (3.2) is violated.

We conclude that,

$$\|x^*\| \leq 1 \text{ and } f(x^*) = \hat{f}(x^*) = 0\,.$$

This completes the proof.

$\square$

In an attempt to move from disks to simplices, the following concepts are important.

Definition 3.1.14 *The vector $p_c = (1/n, \ldots, 1/n)$ is said to be the center of the unit simplex Δ_1, where*

$$\Delta_1 = \left\{ p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} p_i = 1 \right\}$$

and the boundary of Δ_1 is given by,

$$\partial\Delta_1 = \{p \in \Delta_1 : \text{ some } p_i = 0\}\,.$$

Definition 3.1.15 *The hyperplane denoted by Δ_0 is defined as*

$$\Delta_0 = \{p \in \mathbb{R}^n : \sum_{i=1}^{n} p_i = 0\}\,.$$

Definition 3.1.16 .

A ray R through $x \neq 0$ is the set $\{t\, x : t \geq 0$ and $x \in \mathbb{R}^n\}$.

A map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ preserves rays if for any ray R, $x \in R$ then $\phi(x) \in R$.

The next result deals with continuous maps $\phi : \Delta_1 \to \Delta_0$ which satisfy the following boundary condition:

$$\phi(p) \neq \mu(p - p_c),\;\; \mu > 0 \text{ for all } p \in \partial\Delta_1\,. \tag{3.3}$$

Theorem 3.1.17 *If the continuous map $\phi : \Delta_1 \to \Delta_0$ satisfies the boundary condition (3.3), then there is $p^* \in \Delta_1$ with $\phi(p^*) = 0$.*

Proof.

We will construct a ray preserving homeomorphism into the form of the result **Theorem 3.1.13** with the aim of applying the result. Define the map

$$h : \Delta_1 \to \Delta_0, p \mapsto h(p) = p - p_c \,.$$

Clearly $p - p_c \in \Delta_0$ since $\sum_{i=1}^{n}(p_i - p_{c_i}) = 1 - 1 = 0$.

Now let

$$\lambda : \Delta_0 \setminus \{0\} \to \mathbb{R}_+, p \mapsto \lambda(p) = -(1/l)(1/\text{min}_i p_i) \,.$$

From the above definition of λ, it is clear that $p \in \Delta_0 \setminus \{0\}$ if and only if $\text{min}_i p_i < 0$. We denote the intersection $\mathbb{B} = \mathbb{B}^n \cap \Delta_0$. The map

$$\psi : \mathbb{B} \to h(\Delta_1),\ p \mapsto \psi(p) = \begin{cases} \lambda(p/\|p\|)p & \text{for } p \neq 0 \\ 0 & \text{for } p = 0 \end{cases}$$

is a ray preserving homeomorphism.

We show that the map ψ is continuous.

Observe that the continuous function $p \to \text{min}_i p_i = 0$ for $p \in \Delta_0$ if and only if $p = 0$. This shows that ψ is continuous at each $p \in \mathbb{B}$, for $p \neq 0$. Since $p/\|p\|$ lies on the unit sphere in Δ_0 the function $\lambda(p/\|p\|)$ is bounded. This shows that the product $\psi(p) = \lambda(p/\|p\|)p$ has a limit 0 as $p \to 0$.

We show that the map ψ is a ray preserving homeomorphism.

Let $\bar{p} = \bar{\alpha}p$ for $\bar{\alpha} > 0$. Then

$$\begin{aligned} \psi(\bar{p}) &= \lambda(\bar{p}/\|\bar{p}\|)\bar{p} \\ &= \lambda(\bar{\alpha}p/\|\bar{\alpha}p\|)\bar{\alpha}p \\ &= \bar{\alpha}\lambda(p/\|p\|)p \\ &= \beta p\,, \text{ where } \beta = \bar{\alpha}\lambda(p/\|p\|) > 0\,. \end{aligned}$$

Now, we let $\psi(\tilde{p}) = \psi(\bar{p})$ with $\tilde{p} = \tilde{\alpha}p$ and $\bar{p} = \bar{\alpha}p$ on the same ray through p.

So

$$\psi(\tilde{p}) = \tilde{\alpha}\lambda(p/\|p\|)p = \bar{\alpha}\lambda(p/\|p\|)p = \psi(\bar{p})\,.$$

This means that $\tilde{\alpha} = \bar{\alpha}$ and $\tilde{p} = \bar{p}$.

This shows that ψ is ray preserving and one to one.

We have to show that ψ is onto. This is done by establishing the set equality $\psi(\mathbb{B}) = h(\Delta_1)$.
Firstly we show that the set inclusion $\psi(\mathbb{B}) \subseteq h(\Delta_1)$ holds.
If $q \in \psi(\mathbb{B})$ then $q = \psi(p) = \lambda(p/\|p\|)p \in h(\Delta_1)$ for some $p \in \mathbb{B}$. This shows that $\psi(\mathbb{B}) \subseteq h(\Delta_1)$.
Secondly we show the reverse set inclusion $\psi(\mathbb{B}) \supseteq h(\Delta_1)$ holds.
If $q \in h(\Delta_1)$ then $q = h(r) = r - p_c$ for some $r \in \Delta_1$. So for a $p \in \mathbb{B}$, it is clear that $q = r - p_c = \psi(p) = \lambda(p/\|p\|)p \in \psi(\mathbb{B})$. This shows that $\psi(\mathbb{B}) \supseteq h(\Delta_1)$.
We have established the set equality $\psi(\mathbb{B}) = h(\Delta_1)$ and this means that ψ is onto.

Lastly it remains to show that ψ^{-1} is continuous.
Indeed, $\mathbb{B}$ and $h(\Delta_1)$ are compact sets and the map $\psi : \mathbb{B} \to h(\Delta_1)$ is bijective and continuous, and hence the inverse map ψ^{-1} of ψ is continuous.
Thus we can conclude that ψ is a ray preserving homeomorphism.
Consider the composition

$$\alpha : \mathbb{B} \mapsto \Delta_0,\ p \mapsto \alpha(p) = \phi(h^{-1}(\psi(p)))$$

where

$$\mathbb{B} \xrightarrow{\psi} h(\Delta_1) \xrightarrow{h^{-1}} \Delta_1 \xrightarrow{\phi} \Delta_0\ .$$

We show that the composition $\alpha = \phi \circ h^{-1} \circ \psi$ satisfies the boundary condition (3.2) of **Theorem 3.1.13**.
Consider $q \in \partial\mathbb{B}$ and let $p = \psi(q) + p_c = h^{-1}(\psi(q))$. Now by the boundary condition (3.3) and the fact that $p - p_c, q \in \Delta_0$ there is no $\mu > 0$ with $\phi(p) = \mu(p - p_c)$ or with $\mu(p - p_c) = \alpha(q)$. Equivalently this means that there is no $\mu > 0$ with $\alpha(q) = \mu q$, and since ψ is ray preserving that means

$$\alpha(q) \neq \mu q,\ \ \mu > 0\ .$$

This shows that α satisfies the boundary condition (3.2) of **Theorem 3.1.13**.
Hence there is $q^* \in \mathbb{B}$ with $\alpha(q^*) = 0$; or if $p^* = \psi(q^*) + p_c$ then $\phi(p^*) = 0$. This completes the proof. $\square$

3.2 Demand and Supply

The notions of demand and supply have been one of the central concepts of microeconomic theory as can be seen in [22], [33], [54] and in the first chapter. The basic idea of equilibrium theory is to study solutions of the equation $S(p) = D(p)$, that is, "*demand equals supply*".

In his paper, Gale [22] investigates the relationship between demand and supply for a certain good in an economy where the economic agent cannot affect the prices which prevail in the economy. This relationship is such that if for a given price vector the demand for a particular commodity exceeds the available supply then its price undergoes an increase which causes the demand to decrease. On the contrary if supply exceeds demand the price decreases and the demand experiences an increase.

In this way it is assumed that prices regulate themselves to values at which supply and demand balance, and this balance of supply and demand is what the concept of economic equilibrium is based on.

Let us suppose we treat this problem in the context of an economy with L - commodities.
The non-negative orthant $\mathbb{R}^L_+ = \{(x_1, \cdots, x_L) \in \mathbb{R}^L; x_l \geq 0, \forall l = 1, \cdots, L\}$ yields the following two concepts:

Definition 3.2.1 *The space $\mathbb{R}^L_+$ is said to be a commodity space with $x \in \mathbb{R}^L_+$ called a commodity bundle, where $x = (x_1, \cdots, x_L)$ with x_1 measuring the units of the first commodity.*

Definition 3.2.2 *The complement space $\mathbb{R}^L_+ \backslash \{0\}$ is said to be the space of price systems with $p = (p_1, \cdots, p_L) \in \mathbb{R}^L_+ \backslash \{0\}$, the set of prices of the L- commodities.*

Note 3.2.3 The first component p_1 of the L-tuple $p = (p_1, \ldots, p_L)$ is the price of one unit of the first commodity.

We assume that in the economy dealt with, the demand and supply functions are given in the following manner:

Definition 3.2.4 *Let*

$$D : \mathbb{R}^L_+ \backslash \{0\} \to \mathbb{R}^L_+, \;\; p \mapsto D(p)$$

and

$$S : \mathbb{R}^L_+ \backslash \{0\} \to \mathbb{R}^L_+, \;\; p \mapsto S(p)$$

be the demand and supply functions respectively, from the space of price systems to the commodity space.

Remarks 3.2.5

1. The commodity bundle demanded by the consumers and other agents at price $p = (p_1, \ldots, p_L)$ is denoted by $D(p)$.

2. The vector of commodities that would be purchased at price $p = (p_1, \ldots, p_L)$ is represented by $D(p)$.

3. The commodity bundle supplied by the economy at price $p = (p_1, \ldots, p_L)$ is given by $S(p)$.

3.3 The Existence of Equilibrium

The equilibrium problem is to find under suitable conditions on the demand and supply functions

$$D : \mathbb{R}^L_+ \backslash \{0\} \to \mathbb{R}^L_+, \;\; p \mapsto D(p)$$
$$S : \mathbb{R}^L_+ \backslash \{0\} \to \mathbb{R}^L_+, \;\; p \mapsto S(p)$$

a price system $p^* \in \mathbb{R}^L_+ \backslash \{0\}$ such that

$$D(p^*) = S(p^*)$$

The following definition of the excess demand brings us closer to the main objective of the problem of the existence of equilibrium.

Definition 3.3.1 *The excess demand is the mapping given by*

$$Z : \mathbb{R}^L_+ \backslash \{0\} \longrightarrow \mathbb{R}^L, \;\; p \mapsto Z(p) = D(p) - S(p)$$

from the space of price systems to the L - dimensional Euclidean space, satisfying the following conditions:

(EDM1) Continuity: $Z : \mathbb{R}^L_+ \backslash \{0\} \longrightarrow \mathbb{R}^L$ *is continuous.*

(EDM2) Homogeneity: $Z(\lambda p) = Z(p)$ *for all* $\lambda > 0$.

(EDM3) Walras' law: $p \cdot Z(p) = \sum_{\ell=1}^{L} p_\ell Z_\ell(p) = 0$.

(EDM4) Lower bound: $Z(p)$ *is bounded below.*

(EDM5) Desirability: $Z_\ell(p) \geq 0$ *if* $p_\ell = 0$.

Remarks 3.3.2

(i) The condition **(EDM2)** means that $Z(\cdot)$ is homogeneous of degree zero, that is, we are only restricted to the same space of price systems for our price search.

(ii) The **Walras' law** condition **(EDM3)** says that the value of the excess demand is zero, that is, the total value demanded is equal to the total value of the supply.

(iii) The condition **(EDM5)** is a simplified version of the condition **(AED5)** in **Lemma 1.5.3**.

Note 3.3.3 The equilibrium problem is reduced to a search for a price system $p^* \in \mathbb{R}^L_+ \backslash \{0\}$ a solution of the equation $Z(p^*) = 0$.

Theorem 3.3.4 *If an excess demand $Z : \mathbb{R}^L_+ \setminus \{0\} \to \mathbb{R}^L$ satisfies the conditions of **Definition 3.3.1**, then there exist a price system $p^* \in \mathbb{R}^L_+ \setminus \{0\}$ such that*

$$Z(p^*) = 0\,.$$

This price system p^* is given constructively.

Proof.

Let,

$$Z : \mathbb{R}^L_+ \setminus \{0\} \to \mathbb{R}^L, p \mapsto Z(p) = D(p) - S(p)$$

be the excess demand map.

Define a map ϕ from the excess demand as

$$\phi : \Delta_1 \to \Delta_0, p \mapsto \phi(p) = Z(p) - \left(\sum_{\ell=1}^{L} Z_\ell(p)\right) p\,.$$

Component wise, we have that

$$\phi_j(p) \;=\; Z_j(p) - \left(\sum_{\ell=1}^{L} Z_\ell(p)\right) p_j\,. \tag{3.4}$$

We take a sum on both sides of (3.4) and make use of the fact that for any $p \in \Delta_1$, $\sum_{j=1}^{L} p_j = 1$.

$$\begin{aligned}\sum_{j=1}^{L} \phi_j(p) &= \sum_{j=1}^{L} Z_j(p) - \left(\sum_{\ell=1}^{L} Z_\ell(p)\right) \sum_{j=1}^{L} p_j && (3.5)\\ &= \sum_{j=1}^{L} Z_j(p) - \sum_{\ell=1}^{L} Z_\ell(p) && (3.6)\\ &= 0\,. && (3.7)\end{aligned}$$

Hence, the map ϕ is well defined.

The map ϕ is continuous because it is a difference of two continuous maps, $Z(p)$ and $\left(\sum_{\ell=1}^{L} Z_\ell(p)\right) p$.

Also from the definition of $\partial\Delta_1$ if $p \in \partial\Delta_1$, $p_\ell = 0$ for some $\ell = (1, 2, 3, \ldots, L)$ and from the condition **(EDM5)** of the excess demand map, it follows that $\phi_\ell(p) = Z_\ell(p) \geq 0$.

Thus the desirability condition **(EDM5)** of the excess demand map is equivalent to the boundary condition (3.3) for ϕ. Now, by the result **Theorem 3.1.17** there is $p^* \in \Delta_1$, with $\phi(p^*) = 0$, that is, $Z(p^*) = \sum_{\ell=1}^{L} Z_\ell(p^*)p^*$. By taking the dot product on both sides of $Z(p^*) = \sum_{\ell=1}^{L} Z_\ell(p^*)p^*$ by $Z(p^*)$ yields,

$$Z(p^*) = \sum_{\ell=1}^{L} Z_\ell(p^*)p^* \tag{3.8}$$

$$Z(p^*) \cdot Z(p^*) = \sum_{\ell=1}^{L} Z_\ell(p^*)p^* \cdot Z(p^*)\,. \tag{3.9}$$

Walras' Law, that is, $p^* \cdot Z(p^*) = 0$ yields

$$\|Z(p^*)\|^2 = 0 \tag{3.10}$$

$$Z(p^*) = 0\,. \tag{3.11}$$

This completes the proof of our existence theorem.

□

There are cases where equilibrium occurs naturally when for an excess demand map $Z : \mathbb{R}^L_+ \setminus \{0\} \to \mathbb{R}^n$, $D(p) \neq S(p)$ that is "supply does not equal demand". This leads us to the following definition of a free disposal equilibrium.

Definition 3.3.5 *Let $Z : \mathbb{R}^L_+ \setminus \{0\} \to \mathbb{R}^L$ be an excess demand map. Then any price system $p^* \in \mathbb{R}^L_+ \setminus \{0\}$ with $Z(p^*) \leq 0$, $D(p^*) \leq S(p^*)$ is said to be a free disposal equilibrium.*

Note 3.3.6 ([46])

1. This equilibrium is said to be a free disposal equilibrium due to the fact that after eliminating excess supplies, we can establish an equilibrium with $Z(p) = 0$.

2. If $Z : \mathbb{R}^L_+ \setminus \{0\} \to \mathbb{R}^L$ satisfies Walras law, $p \cdot Z(p) = 0$ and $Z(p^*) \leq 0$ then for each ℓ either $Z_\ell(p^*) = 0$ or $p^*_\ell = 0$. Otherwise for some ℓ, $Z_\ell(p^*) > 0$ and $p^*_\ell > 0$ and for all ℓ , $\sum_{\ell=1}^{L} p^*_\ell Z_\ell(p^*) \leq 0$, which contradicts Walras law.

3. The free disposal equilibrium price system satisfies the following weak form of Walras law, $p \cdot Z(p) \leq 0$.

The next result gives existence of a free disposal equilibrium with some relaxed hypothesis of **Theorem 3.3.4**.

Theorem 3.3.7 [Debreu-Gale-Nikaido Theorem].
Let the excess demand map $Z : \mathbb{R}^L_+ \backslash \{0\} \to \mathbb{R}^L$ be continuous and satisfy $p \cdot Z(p) \leq 0$. Then there is $p^* \in \mathbb{R}^L_+ \backslash \{0\}$ with $Z(p^*) \leq 0$.

Proof. Let the function $\beta : \mathbb{R} \to \mathbb{R}$ be defined as

$$\beta(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ t & \text{for } t \geq 0 \,. \end{cases}$$

Define the mappings

$$\bar{Z} : \mathbb{R}^L_+ \backslash \{0\} \to \mathbb{R}^L_+, p \mapsto \bar{Z}_\ell(p) = \beta(Z_\ell(p)) \tag{3.12}$$

and

$$\phi : \Delta_1 \to \Delta_0, p \mapsto \phi(p) = \bar{Z}(p) - \left(\sum_{\ell=1}^{L} Z_\ell(p) \right) \cdot p \,. \tag{3.13}$$

Indeed, by **Theorem 3.1.17** there is $p^* \in \Delta_1$ with $\phi(p^*) = 0$, that is, $\bar{Z}(p^*) = \sum_{\ell=1}^{L} Z_\ell(p^*) \cdot p^*$.

Moreover, by taking inner products on both sides by $Z(p^*) = \sum_{\ell=1}^{L} Z_\ell(p^*) \cdot p^*$ and using the weak form of Walras law, $p^* \cdot Z(p^*) \leq 0$ yields

$$\begin{aligned} \bar{Z}(p^*) \cdot Z(p^*) &= \sum_{\ell=1}^{L} Z_\ell(p^*) \bar{Z}_\ell(p^*) && (3.14)\\ &= \sum_{\ell=1}^{L} Z_\ell(p^*) \beta\left(Z_\ell(p^*)\right) && (3.15)\\ &\leq 0\,. && (3.16) \end{aligned}$$

So, we have that $\sum_{\ell=1}^{L} Z_\ell(p^*) \beta\left(Z_\ell(p^*)\right) \leq 0$ and the definition of the function β yields,

$$t\beta(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ t^2 & \text{for } t \geq 0. \end{cases}$$

Hence, $Z_\ell(p^*) \leq 0$ for all $\ell = 1, 2, 3, \ldots, L$.

$\square$

We are in a position to generalize the two preceding results, for the case where $Z_\ell(p) \to \infty$ as $p_\ell \to 0$. The following result gives the conditions which the excess demand has to satisfy to have a generalized existence of equilibrium.

Theorem 3.3.8 *Let $\mathcal{B} \subseteq \mathbb{R}^L_+ \backslash \{0\}$, $\mathcal{B} = \text{Interior}\mathbb{R}^L_+$ and the excess demand map $Z : \mathcal{B} \to \mathbb{R}^L$ satisfy the following conditions:*

1. *$Z : \mathcal{B} \to \mathbb{R}^L$ is continuous.*
2. *$Z(\lambda p) = Z(p)$, for all $p \in \mathcal{B}$, $\lambda > 0$.*
3. *$p \cdot Z(p) = \sum_{\ell=1}^{L} p_\ell Z_\ell(p) \leq 0$ for all $p \in \mathcal{B}$.*
4. *$\sum_{\ell=1}^{L} Z_\ell(p_k) \to \infty$ if $p_k \to \bar{p} \notin \mathcal{B}$.*

Then there is an equilibrium point $p^* \in \mathcal{B}$ with $Z(p^*) \leq 0$.

Proof.

Let the function $\beta : \mathbb{R} \to \mathbb{R}$ be defined as

$$\beta(t) = \begin{cases} 0 & \text{for } t \le 0 \\ t & \text{for } t \ge 0\,. \end{cases}$$

Let the function $\alpha : \mathbb{R} \to \mathbb{R}$ for a fixed constant $c > 0$ be defined as

$$\alpha(t) = \begin{cases} 0 & \text{for } t \le 0 \\ 1 & \text{for } t \ge c \\ t/c & \text{elsewhere.} \end{cases}$$

Define the map $\bar{Z} : \mathbb{R}^L_+ \to \mathbb{R}^L_+$ by

$$\bar{Z}_\ell(p) = \begin{cases} 1 & \text{for } p \notin \mathcal{B} \\ \left(1 - \alpha\left(\sum_{j=1}^{L} Z_j(p)\right)\right)\beta\left(Z_\ell(p)\right) + \alpha\left(\sum_{j=1}^{L} Z_j(p)\right) & \text{for } p \in \mathcal{B}. \end{cases}$$

We assert that $\bar{Z}$ is a continuous map.

Let $\bar{p} \in \partial\mathcal{B}$ and $p_k \to \bar{p}$. If $p_k \in \mathcal{B}$ then by the hypothesis $\sum_{\ell=1}^{L} Z_\ell(p_k) \to \infty$, there is an integer N such that for $k > N$, $\sum_{\ell=1}^{L} Z_\ell(p_k) > c$. Hence, from the function α we have that $\alpha\left(\sum_{\ell=1}^{L} Z_\ell(p_k)\right) = 1$. This shows that $\lim_{k\to\infty} \bar{Z}_\ell(p_k) = 1$. Clearly if $p_k \notin \mathcal{B}$, $p_k \to \bar{p}$ we have the limit 1.

This proves our assertion that $\bar{Z}_\ell(p)$ is continuous for each $\ell = 1, 2, 3, \cdots, L$.

Again as done in the two preceding results, we define the mapping

$$\phi : \Delta_1 \to \Delta_0, p \mapsto \phi(p) = \bar{Z}(p) - \left(\sum_{\ell=1}^{L} Z_\ell(p)\right) \cdot p\,. \tag{3.17}$$

Indeed, since ϕ satisfies the hypothesis of the result **Theorem 3.1.17** there is $p^* \in \Delta_1$ with $\phi(p^*) = 0$, that is, $\bar{Z}(p^*) = \sum_{\ell=1}^{L} \bar{Z}_\ell(p^*)p^*$.

We suppose that $p^* \in \mathcal{B}$. By taking inner products on both sides by $Z(p^*)$ and using the weak form of Walras law, $p^* \cdot Z(p^*) \le 0$ to obtain, $\bar{Z}(p^*) \cdot Z(p^*) \le 0$, which can be simplified into the form

$$\left(1 - \alpha\left(\sum_{j=1}^{L} Z_j(p^*)\right)\right)\sum_{\ell=1}^{L} Z_\ell(p^*)\beta(Z_\ell(p^*)) + \alpha\left(\sum_{\ell=1}^{L} Z_\ell(p^*)\right)\sum_{\ell=1}^{L} Z_\ell(p^*) \le 0\,.$$

With $t\alpha(t) \geq 0$ for any $t \geq 0$ implies that $\alpha\left(\sum_{\ell=1}^{L} Z_\ell(p^*)\right)\sum_{\ell=1}^{L} Z_\ell(p^*) \geq 0$. We have that

$$\left(1-\alpha\left(\sum_{j=1}^{L} Z_j(p^*)\right)\right)\sum_{\ell=1}^{L} Z_\ell(p^*)\beta(Z_\ell(p^*)) \leq 0, \tag{3.18}$$

and since $\left(1-\alpha\left(\sum_{j=1}^{L} Z_j(p^*)\right)\right) \geq 0$,

$$\sum_{\ell=1}^{L} Z_\ell(p^*)\beta(Z_\ell(p^*)) \leq 0.$$

Therefore $Z_\ell(p^*) \leq 0$ for all $\ell = 1, 2, 3, \cdots, L$.
When $p^* \notin \mathcal{B}$, it follows from the equation $\bar{Z}(p^*) = \sum_{\ell=1}^{L} \bar{Z}_\ell(p^*)p^*$ that $p^* = (1/L, \ldots, 1/L)$ which is in $\mathcal{B}$. This contradiction shows that p^* cannot be outside $\mathcal{B}$. This proves the result. $\square$

Chapter 4

Lattice Theory and Fixed Point Theory

4.1 Basic Concepts in Lattice Theory

4.1.1 Introduction

The main objective in this chapter compared to the preceding chapters is to approach fixed point theory from the point of view of lattice theory as done in Birkhoff [10], Tarski [51], Topkis [53] and Zhou [58]. This kind of a relationship between lattice theory and fixed point theory proves to be useful in establishing some existence problems in game theory. In this section we look at some well known basic concepts in lattice theory.

We introduce some basic concepts in lattice theory. In his book on Lattice Theory Birkhoff [10] introduces the concept of a poset in the following manner.

Definition 4.1.1 *A partially ordered set (poset) P is a set on which a binary relation or ordering denoted by the symbol $\leq$ defined on P satisfies the following conditions,*

(i) Reflexive: *$x \leq x$ for all $x \in P$.*

(ii) Antisymmetric: *$x \leq y$ and $y \leq x$ implies $x = y$ for $x, y \in P$.*

(iii) Transitive: *$x \leq y$ and $y \leq z$ implies $x \leq z$ for $x, y, z \in P$.*

A poset is denoted by $(P, \leq)$.

A binary relation $\leq$ that is reflexive, antisymmetric and transitive is said to be a partial ordering on the set P. The symbol $\leq$ is usually read "included in", "contained in" or "is less than or equal to".

Note 4.1.2 A partially ordered set is a chain if it does not contain an unordered pair of elements.

We give a few examples of partially ordered sets.

Examples 4.1.3 .

1. The usual $\leq$ ordering among the real numbers $\mathbb{R}$, and the extension of $\leq$ to $\mathbb{R}^n$ by $(x_1, x_2, \cdots, x_n) \leq (y_1, y_2, \cdots, y_n)$ if and only if $x_i \leq y_i$ for $i = 1, \cdots, n$, is a partial ordering on both $\mathbb{R}$ and $\mathbb{R}^n$. Thus making them partially ordered sets.

2. The set inclusion in the power set $\mathcal{P}(X)$ of a set X, defined $A \subseteq B$ for $A, B \in \mathcal{P}(X)$ is a partial ordering on $\mathcal{P}(X)$. Hence, the power set with the set inclusion ordering is a partially ordered set.

3. The divisibility $(|)$ among the natural numbers $\mathbb{N}$ is a partial ordering in $\mathbb{N}$. The set of natural numbers $\mathbb{N}$ with divisibility is a partially ordered set.

4. The lexicographic ordering relation $\leq_{lex}$ on $\mathbb{R}^n$ is such that $x \leq_{lex} x'$ in $\mathbb{R}^n$ if either $x = x'$ or if there is some i' with $1 \leq i' \leq n$ such that $x_i = x'_i$ for each $1 \leq i < i'$, and $x_{i'} < x'_{i'}$. The set $\mathbb{R}^n$ with the lexicographic ordering relation $\leq_{lex}$ is a partially ordered set.

We are in a position to define some mappings on a partially ordered set.

Definition 4.1.4 *Let $(P_1, \leq)$ and $(P_2, \leq)$ be any two partially ordered sets.*
A mapping $\theta : P_1 \longrightarrow P_2$ is called an isotone mapping from the poset $(P_1, \leq)$ to $(P_2, \leq)$
if

$$x_1 \leq x_2 \Longrightarrow \theta(x_1) \leq \theta(x_2)$$

for $x_1, x_2 \in P_1$.

In the literature isotone and monotone are used interchangeably.

Definition 4.1.5 *The mapping $\theta : P_1 \longrightarrow P_2$ is called antitone*
if

$$x_1 \leq x_2 \Longrightarrow \theta(x_2) \leq \theta(x_1)$$

for $x_1, x_2 \in P_1$.

A bijective mapping $\theta : P_1 \longrightarrow P_2$ *is called an isomorphism if* θ *and* θ^{-1}*, the inverse of* θ*, are both isotone mappings.*

4.2 Lattices and Functions on Lattices

Suppose that $(L, \leq)$ is a poset and S is a nonempty subset of L. Then an element $u \in L$ (respectively, $\ell \in L$) is said to be an upper bound (respectively, a lower bound) for S in L if $s \leq u$ (respectively, $s \geq \ell$) $\forall\ s \in S$. The set of all upper bounds for S is denoted by U_s. The least element of U is called the least upper bound for S. The least upper bound of S is denoted by $\bigvee S$ and is called the join of S.
Similarly the greatest lower bound of S is denoted by $\bigwedge S$ and is called the meet of S.

Remarks 4.2.1

(i) If S is a finite subset of L then

$$\bigvee S = \vee S = x_1 \vee x_2 \vee x_3 \vee \cdots \vee x_n$$

for some natural number n. In particular if $S = \{x, y\}$ is a two element set we write $\vee S = x \vee y$.

(ii) Similarly

$$\bigwedge S = \wedge S = x_1 \wedge x_2 \wedge \cdots \wedge x_n$$

$$\wedge\{x, y\} = x \wedge y\; for\; x, y \in S.$$

(iii) *1.* $x \leq x \vee y$ and $y \leq x \vee y$

2. If $x \leq u$ and $y \leq u$ for some $u \in L$ then $x \vee y \leq u$

(1) & (2) are the characterizing properties of $x \vee y$

In Birkhoff [10] and Topkis [53] the concept of a lattice is defined in the following way.

Definition 4.2.2 *A partially ordered set* $(L, \leq)$ *is said to be a lattice if every pair of elements* $x, y \in L$ *has a g.l.b and l.u.b with*

$$l.u.b\{x, y\} = x \vee y$$

and

$$g.l.b\{x, y\} = x \wedge y$$

where $\vee$ *is called the join and* $\wedge$ *is called the meet.*

We give some examples of lattices.

Examples 4.2.3

1. Any chain is a lattice.

2. The real line $\mathbb{R}$ is a lattice with $x \vee x' = \max\{x, x'\}$ and $x \wedge x' = \min\{x, x'\}$ for $x, x' \in \mathbb{R}$.

3. For any positive integer n, $\mathbb{R}^n$ is a lattice with $x \vee x' = (x_1 \vee x'_1, \ldots, x_n \vee x'_n)$ and $x \wedge x' = (x_1 \wedge x'_1, \ldots, x_n \wedge x'_n)$ for $x, x' \in \mathbb{R}^n$.

4. For any set X, the power set $\mathcal{P}(X)$ with the set inclusion ordering relation $\subseteq$ is lattice with $A \wedge B = A \cap B$ and $A \vee B = A \cup B$ for $A, B \in \mathcal{P}(X)$. Thus, the meet of the two subsets of X is their intersection and the join of the two subset of X is their union.

In his paper Topkis [52] gives the next definition of sublattice of a lattice.

Definition 4.2.4 *If* N *is a subset of a lattice* $(L, \leq)$ *and* N *contains the join and meet with respect to* $(L, \leq)$ *of each pair of elements of* N*, then* N *is a sublattice of* $(L, \leq)$*.*

Note 4.2.5

1. A sublattice N of a lattice $(L, \leq)$ is a lattice with respect to the partial ordering $\leq$.

2. The symbols L and $(L, \leq)$ will be used interchangeably for denoting a lattice.

3. The lattice $L' = (L, \geq)$ is the dual lattice of $(L, \leq)$, where "$\geq$" is a partial ordering on L.

4. A sublattice N of L is closed if for any subset $M \subset N$, both $\bigwedge M$ and $\bigvee M$ belong to N.

Given the definition of a lattice, one can define a complete lattice as follows:

Definition 4.2.6 *A lattice $\mathcal{L} = (L, \leq)$ is said to be a complete lattice if every non-empty subset $S \subset L$ has a least upper bound $\bigvee S$ and a greatest lower bound $\bigwedge S$ in $\mathcal{L} = (L, \leq)$. A complete lattice has in particular two elements 0 and 1 defined by*

$$0 = \bigwedge S \text{ and } 1 = \bigvee S\,.$$

We are in a position to define and explore as in Quah [43] and Topkis [52] the effects of real valued functions, that is, submodular and supermodular functions on lattices.

Definition 4.2.7 *Let $f : (L, \leq) \longrightarrow \mathbb{R}$ be a real valued function on a lattice $(L, \leq)$ with the property*

$$f(x \wedge y) + f(x \vee y) \leq f(x) + f(y)$$

for all $x, y \in L$. Then f is said to be a submodular function on L.

If L is a chain, then equality holds for all functions f on L. Therefore every function on a chain L is submodular.

Definition 4.2.8 *Let $f : (L, \leq) \longrightarrow \mathbb{R}$ be a real valued function on a lattice $(L, \leq)$ with the property*

$$f(x \wedge y) + f(x \vee y) \geq f(x) + f(y)$$

for all $x, y \in L$. Then f is said to be a supermodular function on L.

Note 4.2.9 (Quah [43],Topkis [52])

1. The function f is submodular if and only if the function $-f$ is supermodular.

2. When $L = \mathbb{R}^n$ and $f \in C^2$, the submodularity of f is equivalent to $\frac{\partial^2 f}{\partial x_i \partial x_j} \leq 0$ for all $i \neq j$ where $1 \leq i, j \leq n$.

3. When $L = \mathbb{R}^n$ and $f \in C^2$, the supermodularity of f is equivalent to $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$ for all $i \neq j$ where $1 \leq i, j \leq n$.

4. The lattice $(\mathbb{R}^n, \leq)$ is defined $x, y \in \mathbb{R}^n$, $x \leq y$ if and only if $x_i \leq y_i$ for $i = 1, 2, \cdots, n$.

4.3 Lattice Theory and Fixed Points

In this section our focus is on the relationship between lattice theory and fixed point theory. In the book *Supermodularity and Complementarity* by Topkis [53] this relationship is dealt with comprehensively. The known concept of a fixed point is given in the following manner. If f is a function from a set X into X and if $x' \in X$ satisfies $f(x') = x'$ then x' is a **fixed point** of f. If f is a set valued map from a set X into the power set $\mathcal{P}(X)$ and if x' in X is in $f(x')$ then x' is a **fixed point** of f.

Having defined a complete lattice, the next fixed point result from Birkhoff [10] on a complete lattice is the lattice theoretic counterpart of **Theorem 1.5.6**.

Theorem 4.3.1 *Let $y = f(x)$ be any isotone function from a complete lattice $\mathcal{L}$ to itself. Then $a = f(a)$ for some $a \in \mathcal{L}$.*

Proof [*see* Birkhoff [10]]. Define a as the least upper bound of the set S of elements $x \in L$ such that $x \leq f(x)$. Since $0 \leq f(0)$, S is nonempty. Since $f(x)$ is isotone, and $a \geq x$ for all $x \in S$, $f(a) \geq f(x) \geq x$ for all $x \in S$; hence, $f(a) \geq \sup S = a$. It follows, since $f(x)$ is isotone, that $f(f(a)) \geq f(a)$, thus $f(a) \in S$. But this implies $f(a) \leq a$, since $a = \sup S$. We conclude that $a = f(a)$. $\square$

Definition 4.3.2 *Let $\mathcal{L} = (L, \leq)$ be a complete lattice. Given any two elements $x, y \in L$ with $x \leq y$, the interval $[x, y]$ is the set $\{z \in L : x \leq z \leq y\}$.*

The next result by Tarski [51] shows that the collection of fixed points of an isotone mapping from a nonempty complete lattice into itself is a nonempty complete lattice.

Theorem 4.3.3 [Tarski's Lattice Theoretic Fixed Point Theorem].
Let

1. *$\mathcal{L} = (L, \leq)$ be a complete lattice,*
2. *f be an isotone mapping from $\mathcal{L}$ to $\mathcal{L}$,*
3. *$\mathcal{R}$ be the set of all fixed points of f.*

Then the set $\mathcal{R}$ is not empty and $(\mathcal{R}, \leq)$ is a complete lattice, in particular we have

$$\bigvee \mathcal{R} = \bigvee \{x \in L : f(x) \geq x\} \in \mathcal{R}$$

and

$$\bigwedge \mathcal{R} = \bigwedge \{x \in L : f(x) \leq x\} \in \mathcal{R}\,.$$

Proof [*see* Tarski [51]].
Let

$$u = \bigvee \{x \in L : f(x) \geq x\}. \tag{4.1}$$

We clearly have $x \leq u$ for every element x with $f(x) \geq x$; hence, the function f being increasing,

$$x \leq f(x) \leq f(u) \text{ and } x \leq f(u)\,.$$

By (4.1) we conclude that

$$u \leq f(u). \tag{4.2}$$

Therefore

$$f(u) \leq f(f(u)),$$

so that $f(u)$ belongs to the set $\{x \in L : f(x) \geq x\}$; consequently, by (4.1),

$$f(u) \geq u. \tag{4.3}$$

Equations (4.2) and (4.3) imply that u is a fixed point of f; hence we conclude by (4.1) that u is the join of all fixed points of f, so that

$$\bigwedge \mathcal{R} = \bigwedge \{x \in L : f(x) \leq x\} \in \mathcal{R}\,. \tag{4.4}$$

Consider the dual complete lattice $\mathcal{L}' = (L, \geq)$ and the isotone mapping f from $\mathcal{L}'$ to $\mathcal{L}'$. The join of any elements in $\mathcal{L}'$ obviously coincides with the meet of these elements in $\mathcal{L}$. Hence, by applying to $\mathcal{L}'$ the result established for $\mathcal{L}$ in (4.4), we conclude that

$$\bigvee \mathcal{R} = \bigvee \{x \in L : f(x) \geq x\} \in \mathcal{R}\,. \tag{4.5}$$

Now let Y be any subset of $\mathcal{R}$ and $[\bigvee Y, 1] = \{x \in L : \bigvee Y \leq x \leq 1\}$ be the interval with endpoints $\bigvee Y$ and 1. The system

$$\mathcal{K} = ([\vee Y, 1], \leq)$$

is a complete lattice. For any $x \in Y$ we have $x \leq \bigvee Y$ and hence

$$x = f(x) \leq f(\vee Y);$$

therefore $\bigvee Y \leq f(\bigvee Y)$. Consequently, $\bigvee Y \leq z$ implies

$$\vee Y \leq f(\vee Y) \leq f(z).$$

Thus, by restricting the domain of f to the interval $[\bigvee Y, 1]$, we obtain an isotone function f' on $[\bigvee Y, 1]$ to $[\bigvee Y, 1]$. By applying formula (4.5) to the complete lattice $\mathcal{K}$ and to f', we conclude that the greatest lower bound v of all fixed points of f' is itself a fixed point of f'. Obviously, v is a fixed point of f, and infact the least fixed point of f which is an upper bound of all elements of Y in the system $(\mathcal{R}, \leq)$. Hence, by passing to the dual complete lattices $\mathcal{L}'$ and $\mathcal{K}'$, we see that there exists also a greatest lower bound of Y in $(\mathcal{R}, \leq)$. Since Y is an arbitrary subset of $\mathcal{R}$, we finally conclude that the system

$$(\mathcal{R}, \leq) \tag{4.6}$$

is a complete lattice. The proof is completed.

□

The above result is generalized in Tarski [51], by the introduction of a commutative set of mappings.

Definition 4.3.4 *A set F of functions is said to be commutative if,*

1. *all the functions of F have a common domain $\mathcal{D}$ and the ranges of all functions of F are subsets of $\mathcal{D}$,*

2. *for any $f, g \in F$ we have fg=gf, that is,*

$$f(g(x)) = g(f(x)) \text{ for every } x \in \mathcal{D}.$$

This commutative definition leads to the following generalized form of **Theorem 4.3.3**.

Theorem 4.3.5 [Tarski's Generalized Lattice Theoretic Fixed Point Theorem]
Let

1. $\mathcal{L} = (L, \leq)$ *be a complete lattice,*

2. F *be any commutative set of isotone mappings from* $\mathcal{L}$ *to* $\mathcal{L}$*,*

3. $\mathcal{R}$ *be the set of all common fixed points of all the functions* $f \in F$*.*

Then the set $\mathcal{R}$ *is not empty and* $(\mathcal{R}, \leq)$ *is a complete lattice, in particular we have*

$$\bigvee \mathcal{R} = \bigvee \{x \in L : f(x) \geq x \text{ for every } f \in F\} \in \mathcal{R}$$

and

$$\bigwedge \mathcal{R} = \bigwedge \{x \in L : f(x) \leq x \text{ for every } f \in F\} \in \mathcal{R}\,.$$

Proof [*see* Tarski [51]].

Clearly **Theorem 4.3.3** is a special case of **Theorem 4.3.5**, given the fact that every singleton commutative set F is obviously commutative.

In her paper Davis [13] gives the characterization of a complete lattice with the use of a fixed point theorem from a complete lattice to itself. We state the result now.

Theorem 4.3.6 *For a lattice* $\mathcal{L} = (L, \leq)$ *to be complete it is necessary and sufficient that every isotone mapping from* $\mathcal{L}$ *to* $\mathcal{L}$ *have a fixed point.*

Proof [*see* Anne C. Davis [13]].

Definition 4.3.7 *Let* $\mathcal{P}(\mathcal{L})$ *be the power set of the complete lattice* $\mathcal{L}$*. A correspondence* f *from* $\mathcal{L}$ *to* $\mathcal{L}$ *is a set valued map from* $\mathcal{L}$ *to* $\mathcal{P}(\mathcal{L})$*. An element* $\ell \in \mathcal{L}$ *is a fixed point of a correspondence* f *if* $\ell \in f(\ell)$*.*

A correspondence f *is isotone if for any* $x \leq y$*, any* $s \in f(x)$*, and any* $t \in f(y)$*, it is true that* $s \wedge t \in f(x)$ *and* $s \vee t \in f(y)$*.*

The next remark follows from the above definition.

Remark 4.3.8 If f is an isotone correspondence, then for any $x \leq y$ and any $s \in f(x)$ there is $t \in f(y)$ with $s \leq t$. Similarly for any $b \in f(y)$ there is $a \in f(x)$ with $a \leq b$.

In his paper Lin Zhou [58] extends **Theorem 4.3.3** to correspondences in the same way that the Kakutani [25] fixed point theorem generalized **Theorem 1.5.6**.

Theorem 4.3.9 [**Zhou's Lattice Theoretic Fixed Point Theorem**].
Let

1. $\mathcal{L} = (L, \leq)$ *be a complete lattice,*

2. $f(\cdot)$ *a correspondence from* $\mathcal{L}$ *to* $\mathcal{L}$*,*

3. $\mathcal{R}$ *be the set of fixed points of* f*.*

If $f(\ell)$ *is a nonempty closed sublattice of* $\mathcal{L}$ *for every* $\ell \in \mathcal{L}$*, and* f *is isotone in* ℓ*, then* $\mathcal{R}$ *is a nonempty complete lattice.*

Proof [***see*** **Lin Zhou** [**58**]].
We follow the argument in **Theorem 4.3.3**.

(i) Let us first show that $\wedge\mathcal{R} = \wedge^{\mathcal{L}}_{r\in\mathcal{R}} r \in \mathcal{R}$. Consider the set

$$C = \{c \in \mathcal{L} : \exists x_c \in f(c) \text{ such that } x_c \leq c\}.$$

C is nonempty since $1 \in C$, where 1 is the greatest element of $\mathcal{L}$.
Let $a = \wedge C = \wedge^{\mathcal{L}}_{c\in C} c$. It is clear that $\mathcal{R} \subset C$. Hence, if $a \in \mathcal{R}$ then $a = \wedge^{\mathcal{L}}_{r\in\mathcal{R}} r \in \mathcal{R}$. We now prove that $a \in \mathcal{R}$ is indeed true.
For any $c \in C$, there is $x_c \in f(c)$ such that $x_c \leq c$. Since the correspondence f is isotone and $a \leq c$, there is $y_c \in f(a)$ such that $y_c \leq x_c \leq c$. Let $y = \wedge^{\mathcal{L}}_{c\in C} y_c$. Because $f(a)$ is a closed sublattice of $\mathcal{L}$, $y \in f(a)$. Clearly $y \leq a$ since $y = \wedge^{\mathcal{L}}_{c\in C} y_c \leq \wedge^{\mathcal{L}}_{c\in C} c = a$.
Then since f is isotone, there is $z \in f(y)$ such that $z \leq y \in f(a)$. Hence, $y \in C$. So we also have $a \leq y$ by the definition of a. Therefore, $a = y \in f(a)$, that is, $a \in \mathcal{R}$.

(ii) Similarly, we can show that $\vee\mathcal{R} = \vee^{\mathcal{L}}_{r\in\mathcal{R}} r \in \mathcal{R}$.

(iii) It is already established in (i) that $\mathcal{R}$ is nonempty.
To show that $\mathcal{R}$ is a complete lattice, we have to show that $\vee U = \vee^{\mathcal{R}}_{r\in U} r$ and $\wedge U = \wedge^{\mathcal{R}}_{r\in U} r$ exists for any $U \subset \mathcal{R}$. It is important to note that $\vee^{\mathcal{R}}_{r\in U} r$ and $\wedge^{\mathcal{R}}_{r\in U} r$ are respectively the greatest and the least elements of U in $\mathcal{R}$ instead of $\mathcal{L}$.
Let us take $b = \vee^{\mathcal{L}}_{r\in U} r$, the greatest element of U in $\mathcal{L}$. For any $r \in U \subset \mathcal{L}$, since $r \in f(r)$ and f is isotone, there is $x_r \in f(b)$ such that $x_r \geq r$. Let $x = \vee^{\mathcal{L}}_{r\in U} x_r$. Clearly $x \geq b$ since $x = \vee^{\mathcal{L}}_{r\in U} x_r \geq \vee^{\mathcal{L}}_{r\in U} r = b$, and $x \in f(b)$ since $f(b)$ is a closed sublattice of $\mathcal{L}$. Because f is isotone, there is $x_r \in f(\ell)$ with $x_r \geq b$ for every $\ell \geq b$. Hence, if we let $\mathcal{L}' = [b, 1]$, and g from $\mathcal{L}'$ to $\mathcal{L}'$ defined by $g(\ell) = f(\ell) \cap [b, 1]$ for all $\ell \in \mathcal{L}'$, then $g(\ell)$ is nonempty for every $\ell \in \mathcal{L}'$. Since both $f(\ell)$ and $[b, 1]$ are closed sublattices of $\mathcal{L}$ for every $\ell \in \mathcal{L}'$, $g(\ell)$ must be a closed sublattice of $\mathcal{L}'$. Also, since both f and h, which assigns each $\ell \in \mathcal{L}' = [b, 1]$, are isotone on $\mathcal{L}$, $g = f \cap h$ is isotone on $\mathcal{L}'$. Hence, $\mathcal{L}'$ and g satisfy the assumptions of the theorem. Therefore, if we let $b' = \vee^{\mathcal{L}'}_{r\in\mathcal{R}'} r$, in which $\mathcal{R}'$ is the set of fixed points of g on $\mathcal{L}'$, then $b' \in \mathcal{L}'$ according to (i). Since $\mathcal{R}' = \mathcal{R} \cap [b, 1]$, b' is indeed the least fixed point that is greater than or equal to b, that is, $b' = \vee^{\mathcal{R}}_{r\in U} r$.
The existence of $\wedge^{\mathcal{R}}_{r\in U} r$ can be proved in a similar manner.

$\square$

The preceding result **Theorem 4.3.9** is useful in establishing the existence of pure equilibrium points in certain noncooperative games.

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