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SENSITIVITY OF DISCRETE-TIME KALMAN FILTER TO STATISTICAL MODELING ERRORS

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SUMMARY

The optimum filtering results of Kalman filtering for linear dynamic systems require an exact knowledge of the process noise covariance matrix $\mathbf{Q}_k$, the measurement noise covariance matrix $\mathbf{R}_k$ and the initial error covariance matrix $\mathbf{P}_0$. In a number of practical solutions, $\mathbf{Q}_k$, $\mathbf{R}_k$ and $\mathbf{P}_0$, are either unknown or are known only approximately. In this paper the sensitivity due to a class of errors in statistical modelling employing a Kalman Filter is discussed. In particular, we present a special case where it is shown that Kalman filter gains can be insensitive to scaling of covariance matrices. Some basic results are derived to describe the mutual relations among the three covariance matrices (actual and perturbed covariance matrices), their respective Kalman gain $\mathbf{K}_k$ and the error covariance matrices $\mathbf{P}_k$. It is also shown that system modelling errors, particularly scaling errors of the input matrix, do not perturb the Kalman gain. A numerical example is presented to illustrate the theoretical results, and also to show the Kalman gain insensitivity to less restrictive statistical uncertainties in an approximate sense. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: optimal filtering; Kalman filters; discrete-time linear systems; statistical modelling errors; sensitivity

1. INTRODUCTION

Significant applications of Kalman filtering[1,2] to the problem of state estimation for stochastic systems came into view in the past three decades. Kalman filtering has been widely used in many areas of industrial and government applications such as satellite and submarine navigation, radar, and video tracking systems. With the advancement of high-speed computers, Kalman filter has become more useful even for more complicated applications, for example, application of Kalman filtering to the optimality of information theoretic measures.[3]

The optimum filtering results of Kalman filtering require precise knowledge of the covariance matrices $\mathbf{Q}_k$, $\mathbf{R}_k$ and $\mathbf{P}_0$. In a number of practical solutions, these matrices are either unknown or are known only approximately. Several results which deal with the deviation from the basic assumptions that guarantee optimality, for example, non-Gaussian models of the errors, are presented in the literature, such as in References 4–7, where the robustification of the filter is considered. There are some on-line identification schemes which identify $\mathbf{Q}_k$ and $\mathbf{R}_k$ from the innovation sequence, but their assumptions are rather restrictive and are not applicable for general systems.[8] Other stability considerations were presented,[9–11] where it is shown that

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incorrect values of the noise covariances can cause the filter to diverge; i.e. variance of a linear combination of the estimation error becomes unbounded. Furthermore, it has been shown[10] that if the system is detectable and the filter is designed with only errors in the measurement noise covariance, then the filter divergence will never occur. In addition, it is shown[10] that, if a linear periodic discrete-time system is detectable and the noise covariance matrix is constant, then the one-step predictor error covariance computed from the filter is bounded for any measurement noise covariance. For other related work, the reader may refer to the cited work.[12–17,19] On the other hand, none of the work has explicitly addressed situations where the Kalman gain is insensitive to statistical modelling errors. In this paper, we present special situations of statistical modelling errors where the optimality of the filter is not destroyed.[18] In order to motivate this situation, consider a scalar system $x_{k+1} = \phi_k x_k + w_k$ and the measurement equation $z_k = h_k x_k + v_k$, for $k = 1, 2, \ldots,$ such that $\mathrm{Var}(w_k) = Q_k$, $\mathrm{Var}(v_k) = R_k$, and $\mathrm{Var}(x_0) = P_0$.

Case I: Assume that $R_k = 0$, no measurement errors. One should expect that the measurements should contain the optimal state estimation regardless errors in the values of Q_k and P_0. In fact, for this situation, the Kalman gain $K_k = 1/h_k$ (refer to (3)) is independent of $\mathbf{Q}_k$ and P_0. In general, this does not apply for the n-dimensional system unless the measurement matrix $\mathbf{H}_k$ is square and nonsingular where $\mathbf{K}_k = \mathbf{H}_k^{-1}$.

Case II: Assume that for $k = 1, 2, \ldots, \mathbf{Q}_k = \mathbf{P}_0 = 0$ ($\mathbf{R}_k$ is non-singular), no system noise and no initialization errors. Certainly, measurements are not needed and system update $x_{k+1} = \phi_k x_k$ would generate the optimum state sequence. Consequently, the Kalman algorithm would produce zero Kalman gain, $K_k = 0$ (constant) independent of the measurement variance $\mathbf{R}_k$. These results are based on the system assumption (being scalar) and not on the nature of the statistical uncertainties. This paper accommodates similar results for the n-dimensional linear discrete-time systems where the insensitivity of the Kalman gain depends more on the restriction of the statistical uncertainties rather than system limitations. For instance, it is shown that the Kalman filter gain can be insensitive to scaling of covariance matrices. A typical application where such statistical modelling errors occur can be directly related to deterministic modelling uncertainties. For example, consider a system where the model of the input coupling matrix is a scalar multiple of its nominal value, this can directly result in a scaling error of the system noise covariance matrix. If the measurement noise is negligible, and the system is controllable and observable, then the steady-state Kalman gain matrix is insensitive to such errors (refer to Remark 3, Sections 3, and 4). It is also shown, for small values of the sampling period, that the Kalman gain remains persistent for statistical uncertainties of the form $\mathbf{Q} = \mathbf{MQ}$, where $\mathbf{M}$ is a diagonal positive-definite matrix.

2. MAIN RESULTS

In this section, we introduce our notation, systems considered, and present the theoretical results. The results assume two different linear systems, discrete-time stochastic systems and discrete-time invariant. In the latter we restrict our assumptions on the system and relax the statistical limitations. Specifically, the linear system considered is assumed to be linear time invariant, completely observable and controllable. These additional assumptions allow the steady-state Kalman gain to be independent of the initial covariance matrix $\mathbf{P}_0$. In this case, only the measurement and process covariance matrices $\mathbf{R}$ and $\mathbf{Q}$ are restricted. However, the total independence of initial covariance matrix does not apply for general discrete-time varying linear systems. In each section we present our results in a theorem proceeded by two remarks. The assumptions in the theorems are more restrictive in practice and this is discussed in the next section.

2.1. Linear discrete-time stochastic system

Consider a linear discrete-time stochastic system, e.g. describing an error model, with state-space description and measurement equation governed by

$$\boldsymbol{\delta}\mathbf{x}_{k+1} = \mathbf{\Phi}_k \boldsymbol{\delta}\mathbf{x}_k + \mathbf{\Gamma}_k \mathbf{w}_k \tag{1}$$

$$\mathbf{z}_k = \mathbf{H}_k \boldsymbol{\delta}\mathbf{x}_k + \mathbf{v}_k \tag{2}$$

where $\mathbf{\Phi}_k$, $\mathbf{\Gamma}_k$ and $\mathbf{H}_k$ are $n \times n, n \times p, q \times n$ (known) matrices, $\{\mathbf{w}_k\}$ and $\{\mathbf{v}_k\}$ are zero-mean Gaussian white noise sequences of p- and q-vectors, respectively, such that $\mathrm{Var}(\mathbf{w}_k) = \mathbf{Q}_k$ is positive semi-definite matrix, $\mathrm{Var}(\mathbf{v}_k) = \mathbf{R}_k$ is positive-definite matrix and $E(\mathbf{w}_k \mathbf{v}_l^{\mathrm{T}}) = 0$ for all k and l. The initial (error) state $\boldsymbol{\delta}\mathbf{x}_0$ is also assumed independent of $\mathbf{w}_k$ and $\mathbf{v}_k$ in the sense that $E(\boldsymbol{\delta}\mathbf{x}_0 \mathbf{w}_k^{\mathrm{T}}) = 0$ and $E(\boldsymbol{\delta}\mathbf{x}_0 \mathbf{v}_k^{\mathrm{T}}) = 0$ for all k, and the initial error covariance matrix $\mathbf{P}_0 = \mathrm{Var}(\boldsymbol{\delta}\mathbf{x}_0)$.

The iterative algorithm of the error covariance matrices $\mathbf{P}_k^+ (=\mathbf{P}_k)$, and $\mathbf{P}_k^-$, and the Kalman gain $\mathbf{K}_k$ depend on the initial matrix $\mathbf{P}_0^+ (=\mathbf{P}_0)$, and on the sampled process and measurement noise covariance matrices $\mathbf{Q}_k$ and $\mathbf{R}_k$. Therefore, one should consider the change in $\mathbf{P}_k$, $\mathbf{P}_k^-$, and $\mathbf{K}_k$ due to the change in $\mathbf{P}_0\mathbf{Q}_k$ and $\mathbf{R}_k$. The iterative algorithm of these matrices[22] is as follows:

$$\begin{aligned}
\mathbf{P}_k^- &= \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}^{\mathrm{T}} + \mathbf{\Gamma}_{k-1}\mathbf{Q}_{k-1}\mathbf{\Gamma}_{k-1}^{\mathrm{T}} \\
\mathbf{K}_k &= \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1} \\
\mathbf{P}_k &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\mathbf{P}_k^-
\end{aligned} \tag{3}$$

and the state update, $\boldsymbol{\delta}\mathbf{x}_0^+ = E(\boldsymbol{\delta}\mathbf{x}_0)$

$$\begin{aligned}
\boldsymbol{\delta}\mathbf{x}_k^- &= \mathbf{\Phi}_{k-1}\boldsymbol{\delta}\mathbf{x}_{k-1}^+ && \text{(prediction)} \\
\boldsymbol{\delta}\mathbf{x}_k^+ &= \boldsymbol{\delta}\mathbf{x}_k^- + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}_k\boldsymbol{\delta}\mathbf{x}_k^-) && \text{(correction)}
\end{aligned} \tag{4}$$

where $\mathbf{\Phi}_k$ is the system matrix in the discrete-time domain, $\mathbf{H}_k$ is the measurement matrix, $\mathbf{P}_k = E[(\boldsymbol{\delta}\mathbf{x}_k - \boldsymbol{\delta}\mathbf{x}_k^+)(\boldsymbol{\delta}\mathbf{x}_k - \boldsymbol{\delta}\mathbf{x}_k^+)^{\mathrm{T}}]$, and $\mathbf{P}_k^- = E[(\boldsymbol{\delta}\mathbf{x}_k - \boldsymbol{\delta}\mathbf{x}_k^-)(\boldsymbol{\delta}\mathbf{x}_k - \boldsymbol{\delta}\mathbf{x}_k^-)^{\mathrm{T}}]$.

Theorem 1

Consider the linear system (1), and (2). For $k = 1, 2, \ldots$, let $\mathbf{P}_0, \mathbf{Q}_k, \mathbf{R}_k\,(>0)$ and $\mathbf{K}_k$ be the actual initial, system, measurement covariance, and the associated Kalman gain matrices, respectively, and let $\tilde{\mathbf{P}}_0, \tilde{\mathbf{Q}}_k$ and $\tilde{\mathbf{R}}_k\,(>0)$ be the perturbed matrices, respectively. If $\tilde{\mathbf{P}}_0 = \alpha_1\mathbf{P}_0$, $\tilde{\mathbf{Q}}_{k-1} = \alpha_k\mathbf{Q}_{k-1}$ and $\tilde{\mathbf{R}}_k = \alpha_k\mathbf{R}_k$ with $\alpha_k > 0$ for $k = 1, 2, \ldots$, then the Kalman gain matrices $\tilde{\mathbf{K}}_k$, generated by (3), are independent of α_k; particularly, $\tilde{\mathbf{K}}_k = \mathbf{K}_k$. Moreover, $\tilde{\mathbf{P}}_k^- = \alpha_k\mathbf{P}_k^-$, and $\tilde{\mathbf{P}}_k = \mathbf{P}_k$.

Proof. The proof is done by induction. The first sample gives

$$\tilde{\mathbf{P}}_1^- = \mathbf{\Phi}_0\tilde{\mathbf{P}}_0\mathbf{\Phi}_0^{\mathrm{T}} + \mathbf{\Gamma}_0\tilde{\mathbf{Q}}_0\mathbf{\Gamma}_0^{\mathrm{T}} = \alpha_1(\mathbf{\Phi}\mathbf{P}_0\mathbf{\Phi}^{\mathrm{T}} + \mathbf{\Gamma}_0\mathbf{Q}_0\mathbf{\Gamma}_0^{\mathrm{T}}) = \alpha_1\mathbf{P}_1^- \tag{5}$$

Employing (5), the corresponding Kalman gain is given by

$$\tilde{\mathbf{K}}_1 = \alpha_1\mathbf{P}_1^-\mathbf{H}_1^{\mathrm{T}}(\alpha_1\mathbf{H}_1\mathbf{P}_1^-\mathbf{H}_1^{\mathrm{T}} + \alpha_1\mathbf{R}_1)^{-1} = \mathbf{P}_1^-\mathbf{H}^{\mathrm{T}}(\mathbf{H}_1\mathbf{P}_1^-\mathbf{H}_1^{\mathrm{T}} + \mathbf{R}_1)^{-1} = \mathbf{K}_1 \tag{6}$$

In addition, using (5) and (6), we have

$$\tilde{\mathbf{P}}_1 = (\mathbf{I} - \mathbf{K}_1\mathbf{H}_1)\alpha_1\mathbf{P}_1^- = \alpha_1\mathbf{P}_1 \tag{7}$$

Next, we assume that the results hold for the $(k-1)$th sample, and show that results are also satisfied for the kth sample. Using the induction assumption, we have

$$\tilde{\mathbf{P}}_k^- = \mathbf{\Phi}_{k-1}\tilde{\mathbf{P}}_{k-1}\mathbf{\Phi}_{k-1}^{\mathrm{T}} + \mathbf{\Gamma}_{k-1}\tilde{\mathbf{Q}}_{k-1}\mathbf{\Gamma}_{k-1}^{\mathrm{T}} = \alpha_k(\mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}^{\mathrm{T}} + \mathbf{\Gamma}_{k-1}\mathbf{Q}_{k-1}\mathbf{\Gamma}_{k-1}^{\mathrm{T}}) = \alpha_k\mathbf{P}_k^-$$

$$\tilde{\mathbf{K}}_k = \alpha_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}}(\alpha_k\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \alpha_k\mathbf{R}_k)^{-1} = \mathbf{P}_k^-\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1} = \mathbf{K}_k$$

$$\tilde{\mathbf{P}}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\alpha_k\mathbf{P}_k^- = \alpha_k\mathbf{P}_k \qquad \blacksquare$$

Remark 1

If $\tilde{\mathbf{P}}_0 = \alpha_1\mathbf{P}_0$, $\tilde{\mathbf{Q}}_{k-1} = \alpha_k\mathbf{Q}_{k-1}$ where $\mathbf{H}_k\mathbf{Q}_k\mathbf{H}_k^{\mathrm{T}}$ is non-singular $\forall k$, and $\mathbf{R}_k = 0$ $(=\tilde{\mathbf{R}}_k)$, with $\alpha_k > 0$ for $k = 1, 2, \ldots$, then the Kalman gain matrices $\mathbf{K}_k$, generated by (3), are independent of α_k; in particular, $\tilde{\mathbf{K}}_k = \mathbf{K}_k$. Moreover, $\tilde{\mathbf{P}}_k^- = \alpha_k\mathbf{P}_k^-$, and $\tilde{\mathbf{P}}_k = \mathbf{P}_k$.

The proof is a direct consequence of Theorem 1.

Remark 2

If $\mathbf{Q}_k = 0$ $(=\tilde{\mathbf{Q}}_k)$, $\tilde{\mathbf{P}}_0 = \alpha_1\mathbf{P}_0$, and $\tilde{\mathbf{R}}_k = \alpha_k\mathbf{R}_k$ with $\alpha_k > 0$ for $k = 1, 2, \ldots$, then the Kalman gain matrices $\tilde{\mathbf{K}}_k$, generated by (3), are independent of α_k; specifically, $\tilde{\mathbf{K}}_k = \mathbf{K}_k$. Moreover, $\tilde{\mathbf{P}}_k^- = \alpha_k\mathbf{P}_k^-$, and $\tilde{\mathbf{P}}_k = \mathbf{P}_k$.

This proof is also a direct consequence of Theorem 1.

2.2. Linear discrete-time invariant stochastic system

In this subsection, we consider the special case where all known constant matrices are independent of time. That is, we are going to discuss the insensitivity of Kalman filtering due to incorrect covariance matrices for the time-invariant linear stochastic system with the state-space description:

$$\begin{aligned} \boldsymbol{\delta}\mathbf{x}_{k+1} &= \mathbf{\Phi}\boldsymbol{\delta}\mathbf{x}_k + \mathbf{\Gamma}\mathbf{w}_k \\ \mathbf{z}_k &= \mathbf{H}\boldsymbol{\delta}\mathbf{x}_k + \mathbf{v}_k \end{aligned} \tag{8}$$

Here, $\mathbf{\Phi}$, $\mathbf{\Gamma}$ and $\mathbf{H}$ are known $n\times n$, $n\times p$ and $q\times n$ constant matrices, respectively, with $1 \leqslant p, q \leqslant n$, $\{\mathbf{w}_k\}$ and $\{\mathbf{v}_k\}$ are zero-mean Gaussian white noise sequence with $E(\mathbf{w}_k\mathbf{w}_l^{\mathrm{T}}) = \mathbf{Q}\delta_{kl}$, $E(\mathbf{v}_k\mathbf{v}_l^{\mathrm{T}}) = \mathbf{R}\delta_{kl}$, $E(\mathbf{w}_k\mathbf{v}_l^{\mathrm{T}}) = 0$, $E(\boldsymbol{\delta}\mathbf{x}_0 v_k^{\mathrm{T}}) = 0$, and $E(\boldsymbol{\delta}\mathbf{x}_0\mathbf{w}_k^{\mathrm{T}}) = 0$, where $\mathbf{Q}$ and $\mathbf{R}$ are $p\times p$, and $q\times q$ non-negative and positive-definite symmetric matrices, respectively, independent of k. It is well known[21] that if the linear stochastic system (8) is controllable and observable, then for any initial state $\boldsymbol{\delta}\mathbf{x}_0$ such that $\mathbf{P}_0 \equiv \mathrm{Var}(\boldsymbol{\delta}\mathbf{x}_0)$ is non-negative definite and symmetric, $\mathbf{P}_k^- \to \mathbf{P}^-$ as $k \to \infty$, where $\mathbf{P}^- > 0$ is symmetric and independent of $\mathbf{P}_0$. In addition, $\mathbf{P}^-$ satisfies the matrix Riccati equation

$$\mathbf{P}^- = \mathbf{\Phi}(\mathbf{P}^- - \mathbf{P}^-\mathbf{H}^{\mathrm{T}}(\mathbf{H}\mathbf{P}^-\mathbf{H}^{\mathrm{T}} + \mathbf{R})^{-1}\mathbf{H}\mathbf{P}^-)\mathbf{\Phi}^{\mathrm{T}} + \mathbf{\Gamma}\mathbf{Q}\mathbf{\Gamma}^{\mathrm{T}} \tag{9}$$

Moreover, the Kalman gain matrix converges; i.e. $\mathbf{K}_k \to \mathbf{K}$ as $k \to \infty$, where $\mathbf{K} = \mathbf{P}^-\mathbf{H}^{\mathrm{T}}(\mathbf{H}\mathbf{P}^-\mathbf{H}^{\mathrm{T}} + \mathbf{R})^{-1}$.

Theorem 2

Suppose that the system (8) is observable and controllable. Let $\mathbf{P}_0$, $\mathbf{Q}$ and $\mathbf{R}$ (>0) be the actual initial, system, and measurement covariance matrices, respectively, and let $\tilde{\mathbf{P}}_0$, $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{R}}$ (>0) be the perturbed matrices, respectively. If $\tilde{\mathbf{P}}_0$ is any non-negative definite and symmetric matrix,

$\tilde{\mathbf{Q}} = \alpha\mathbf{Q}$ and $\tilde{\mathbf{R}} = \alpha\mathbf{R}$ with $\alpha > 0$, then the Kalman gain matrices associated with $\mathbf{P}_0$, $\mathbf{Q}$ and $\mathbf{R}$, converge to $\mathbf{K}$, and the ones associated with $\tilde{\mathbf{P}}_0$, $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{R}}$, also converge to $\mathbf{K}$ as $k \to \infty$; i.e. $\mathbf{K}$ is independent of α. Moreover, $\mathbf{P}_k \to \mathbf{P}$, $\mathbf{P}_k^- \to \mathbf{P}^-$, $\tilde{\mathbf{P}}_k \to \alpha\mathbf{P}$, and $\tilde{\mathbf{P}}_k^- \to \alpha\mathbf{P}^-$ as $k \to \infty$.

Proof. The proof for $\mathbf{K}_k \to \mathbf{K}$, $\mathbf{P}_k \to \mathbf{P}$, and $\mathbf{P}_k^- \to \mathbf{P}^-$ can be found in the literature.[21] Say that $\tilde{\mathbf{P}}_k^- \to \tilde{\mathbf{P}}^-$ as $k \to \infty$, then $\tilde{\mathbf{P}}^-$ satisfies (9); i.e.

$$\tilde{\mathbf{P}}^- = \mathbf{\Phi}(\tilde{\mathbf{P}}^- - \tilde{\mathbf{P}}^-\mathbf{H}^{\mathrm{T}}(\mathbf{H}\tilde{\mathbf{P}}^-\mathbf{H}^{\mathrm{T}} + \tilde{\mathbf{R}})^{-1}\mathbf{H}\tilde{\mathbf{P}}^-)\mathbf{\Phi}^{\mathrm{T}} + \mathbf{\Gamma}\tilde{\mathbf{Q}}\mathbf{\Gamma}^{\mathrm{T}} \tag{10}$$

dividing the two sides of (10) by α,

$$\begin{aligned}\frac{1}{\alpha}\tilde{\mathbf{P}}^- &= \mathbf{\Phi}\left(\frac{1}{\alpha}\tilde{\mathbf{P}}^- - \frac{1}{\alpha}\tilde{\mathbf{P}}^-\mathbf{H}^{\mathrm{T}}\left(\mathbf{H}\frac{1}{\alpha}\tilde{\mathbf{P}}^-\mathbf{H}^{\mathrm{T}} + \frac{1}{\alpha}\tilde{\mathbf{R}}\right)^{-1}\mathbf{H}\frac{1}{\alpha}\tilde{\mathbf{P}}^-\right)\mathbf{\Phi}^{\mathrm{T}} + \mathbf{\Gamma}\frac{1}{\alpha}\tilde{\mathbf{Q}}\mathbf{\Gamma}^{\mathrm{T}} \\ &= \mathbf{\Phi}\left(\frac{1}{\alpha}\tilde{\mathbf{P}}^- - \frac{1}{\alpha}\tilde{\mathbf{P}}^-\mathbf{H}^{\mathrm{T}}\left(\mathbf{H}\frac{1}{\alpha}\tilde{\mathbf{P}}^-\mathbf{H}^{\mathrm{T}} + \mathbf{R}\right)^{-1}\mathbf{H}\frac{1}{\alpha}\tilde{\mathbf{P}}^-\right)\mathbf{\Phi}^{\mathrm{T}} + \mathbf{\Gamma}\mathbf{Q}\mathbf{\Gamma}^{\mathrm{T}}\end{aligned} \tag{11}$$

Since the system considered is assumed to be observable and controllable, then the solution of the matrix Riccati equation (9) is unique. Therefore, when comparing (9) and (11), it can be concluded that $\tilde{\mathbf{P}}^- = \alpha\mathbf{P}^-$, or $\tilde{\mathbf{P}}_k^- \to \alpha\mathbf{P}^-$ as $k \to \infty$. Next, we consider the

$$\tilde{\mathbf{K}} = \tilde{\mathbf{P}}^-\mathbf{H}^{\mathrm{T}}(\mathbf{H}\tilde{\mathbf{P}}^-\mathbf{H}^{\mathrm{T}} + \tilde{\mathbf{R}})^{-1} = \alpha\mathbf{P}^-\mathbf{H}^{\mathrm{T}}[\alpha(\mathbf{H}\mathbf{P}^-\mathbf{H}^{\mathrm{T}} + \mathbf{R})]^{-1} = \mathbf{K} \tag{12}$$

Similarly, using (11) and (12), we obtain

$$\tilde{\mathbf{P}} = (\mathbf{I} - \tilde{\mathbf{K}}\mathbf{H})\tilde{\mathbf{P}}^- = (\mathbf{I} - \mathbf{K}\mathbf{H})\,\alpha\mathbf{P}^- = \alpha\mathbf{P} \qquad \blacksquare$$

Remark 3

Suppose that the system (8) is observable and controllable. If $\mathbf{H}\mathbf{Q}\mathbf{H}^{\mathrm{T}}$ is non-singular, $\tilde{\mathbf{P}}_0$ is any non-negative definite and symmetric matrix, $\tilde{\mathbf{Q}} = \alpha\mathbf{Q}$, and $\tilde{\mathbf{R}} = \mathbf{R} = 0$ with $\alpha > 0$, then the Kalman gain matrices associated with $\mathbf{P}_0$, $\mathbf{Q}$ and $\mathbf{R}$, converge to $\mathbf{K}$, and the ones associated with $\tilde{\mathbf{P}}_0$, $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{R}}$, also converge to $\mathbf{K}$ as $k \to \infty$; i.e. $\mathbf{K}$ is independent of α. Moreover, $\mathbf{P}_k \to \mathbf{P}$, $\mathbf{P}_k^- \to \mathbf{P}^-$, $\tilde{\mathbf{P}}_k \to \alpha\mathbf{P}$, and $\tilde{\mathbf{P}}_k^- \to \alpha\mathbf{P}^-$ as $k \to \infty$.

The proof is a direct consequence of Theorem 2.

Remark 4

Suppose that the system (8) is observable and controllable. If $\tilde{\mathbf{P}}_0$ is any non-negative definite and symmetric matrix, $\tilde{\mathbf{Q}} = \mathbf{Q} = 0$, and $\tilde{\mathbf{R}} = \alpha\mathbf{R} = 0$ with $\alpha > 0$, then the Kalman gain matrices associated with $\mathbf{P}_0$, $\mathbf{Q}$ and $\mathbf{R}$, converge to $\mathbf{K}$, and the ones associated with $\tilde{\mathbf{P}}_0$, $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{R}}$, also converge to $\mathbf{K}$ as $k \to \infty$; i.e. $\mathbf{K}$ is independent of α. Moreover, $\mathbf{P}_k \to \mathbf{P}$, $\mathbf{P}_k^- \to \mathbf{P}^-$, $\tilde{\mathbf{P}}_k \to \alpha\mathbf{P}$, and $\tilde{\mathbf{P}}_k^- \to \alpha\mathbf{P}^-$ as $k \to \infty$.

The proof is also a direct consequence of Theorem 2.

3. DISCUSSION

The statistical assumptions of this paper constrain the uncertainties to be a scalar multiple of the actual model which appear quite restrictive in practice and thus deserve further analysis. In this section, we go over the assumptions of Remark 3, and analyse them. Note that in many applications, only the continuous-time statistical and deterministic models are available, furthermore, the system and measurement noise covariance matrices depend on the size of the sampling period (while discretizing). Consequently, a continuous-time linear time-invariant system is considered, discretized, and the main assumptions listed in Section 2.2 are examined. Consider the continuous-time domain where $\mathbf{A}$ represents the error model state matrix, $\mathbf{B}$ the process noise coupling matrix, and $\mathbf{Q}_c$ the covariance matrix. If $\mathbf{Q}$ denotes the covariance matrix of the sampled process, then $\mathbf{Q}$ is given by[20]

$$\mathbf{Q} = \int_0^{T} e^{\mathbf{A}t}\mathbf{B}\mathbf{Q}_c\mathbf{B}^T e^{\mathbf{A}^T t}\,dt \tag{13}$$

If $\mathbf{Q}_c$ is positive definite and diagonal, then $\mathbf{Q}$ is always positive semi-definite and not necessarily diagonal. In many applications, $\mathbf{Q}_c$ is symmetric and positive definite. If $\mathbf{B}$ is non-singular (square) matrix and $\mathbf{H}$ is full row rank matrix (e.g. a single output system), then $\mathbf{Q}$ is positive definite, hence $\mathbf{HQH}^T$ is symmetric positive definite matrix ($\Rightarrow$non-singular). Next, we give an example to illustrate a case where a statistical error of the following form $\tilde{\mathbf{Q}}_c = \alpha\mathbf{Q}_c$ takes place. Note that if $\tilde{\mathbf{Q}}_c = \alpha\mathbf{Q}_c$, then $\tilde{\mathbf{Q}} = \alpha\mathbf{Q}$. This could be easily seen by considering a system modelling error $\tilde{\mathbf{b}} = \alpha'\mathbf{b}$; e.g. $\mathbf{b} = [0\ \ 0\ \ 1{\cdot}5]^T$, and $\tilde{\mathbf{b}} = [0\ \ 0\ \ 2]^T$, thus, $\alpha' = 2/1{\cdot}5$. By examining (13), the corresponding modelling error results in $\tilde{\mathbf{Q}}_c = (a')^2\mathbf{Q}_c$; therefore, $\alpha = (\alpha')^2$. Consequently, such modelling errors do not effect the Kalman gain. By inspecting (9), for the time-invariant case, $\mathbf{\Gamma}_k = \mathbf{\Gamma} = \int_0^T e^{\mathbf{A}\tau}\mathbf{b}\,d\tau$, and $\tilde{\mathbf{\Gamma}}_k = \tilde{\mathbf{\Gamma}} = \int_0^T e^{\mathbf{A}\tau}\tilde{\mathbf{b}}\,d\tau = \alpha'\mathbf{\Gamma}$.

Therefore,

$$\begin{aligned}\tilde{\mathbf{P}}^- &= \mathbf{\Phi}(\tilde{\mathbf{P}}^- - \tilde{\mathbf{P}}^-\mathbf{H}^T(\mathbf{H}\tilde{\mathbf{P}}^-\mathbf{H}^T + \tilde{\mathbf{R}})^{-1}\mathbf{H}\tilde{\mathbf{P}}^-)\,\mathbf{\Phi}^T + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{Q}}\tilde{\mathbf{\Gamma}}^T \\ &= \mathbf{\Phi}(\tilde{\mathbf{P}}^- - \tilde{\mathbf{P}}^-\mathbf{H}^T(\mathbf{H}\tilde{\mathbf{P}}^-\mathbf{H}^T + \tilde{\mathbf{R}})^{-1}\mathbf{H}\tilde{\mathbf{P}}^-)\,\mathbf{\Phi}^T + (\alpha')^4\,\mathbf{\Gamma}\mathbf{Q}\mathbf{\Gamma}^T\end{aligned}$$

The rest follows as in the proof of Theorem 2 (with the setting of Remark 3; i.e. $\mathbf{R} = \tilde{\mathbf{R}} = 0$).

It is also assumed that the measurement errors are known to be zero. One example, of $\mathbf{R} = 0$, is the zero-velocity updates[23,24] while initializing, calibrating and/or orientating used inertial navigation systems (INS). This occurs when the vehicle is at rest, zero-velocity measurements are assumed. The velocity integrated by the INS is compared with zero-velocity 'measurement'. Subsequently, the error is fed to a Kalman filter for state observation with some of state variables being sensor bias terms, orientation errors, etc. In some other applications, for example, event-driven Kalman filter, the measurement and initial covariance matrices $\mathbf{R}$ and $\mathbf{P}_0$, could be neglected when the time between events T is large enough. The investigated system is assumed to be observable and $\mathbf{HQH}^T$ is non-singular. This can be detected by examining (3) and (13). Clearly, since $\mathbf{Q}_c$ is positive definite, then as T increases, the smallest eigenvalue of $\mathbf{Q}$ increases. Therefore,

$$\mathbf{K}_k = \mathbf{P}_k^-\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T + \mathbf{R}_k)^{-1} \approx \mathbf{P}_k^-\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T)^{-1}$$

This situation is well illustrated in the subsequent section, Case 2.

4. AN ILLUSTRATIVE EXAMPLE

In this section we use the real-time tracking[21] as an example to illustrate the preceding results. Let $\mathbf{x}(t)$, $0 \leqslant t < \infty$, denote the trajectory in three-dimensional space of a flying object, where t denotes the time variable. This vector-valued function is discretized with sampling period T to yield $\mathbf{x}_k \equiv \mathbf{x}(kT)$, $k = 0, 1, \ldots$. For practical purposes, $\mathbf{x}(t)$ can be assumed to have continuous first- and second-order derivatives, denoted by $\dot{\mathbf{x}}(t)$, and $\ddot{\mathbf{x}}(t)$, respectively. For small values of T, the position and velocity vectors are given by

$$\mathbf{x}_{k+1} = \mathbf{x}_k + T\dot{\mathbf{x}}_k + \tfrac{1}{2}\, T_k^2 \ddot{\mathbf{x}}_k$$

$$\dot{\mathbf{x}}_{k+1} = \dot{\mathbf{x}}_k + T\mathbf{x}_k$$

where $\dot{\mathbf{x}}_k \equiv \dot{\mathbf{x}}(kT)$, $\ddot{\mathbf{x}}_k \equiv \ddot{\mathbf{x}}(kT)$, for $k = 0, 1, \ldots$. For tracking flying object, position measurement is often available at each time instant. Consequently, the measurement matrix $\mathbf{H} = [\mathbf{I}\ \ \mathbf{0}\ \ \mathbf{0}]$. Note that if the azimuthal angular error and the elevation angular error are also considered, then, using Taylor polynomial approximation, the augmented linear stochastic system can be decoupled into three subsystems with analogous state-space description. In order to facilitate our example even further, we only consider the following tracking model:

$$\begin{bmatrix} \mathbf{x}_{k+1}[1] \\ \mathbf{x}_{k+1}[2] \\ \mathbf{x}_{k+1}[3] \end{bmatrix} = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_k[1] \\ \mathbf{x}_k[2] \\ \mathbf{x}_k[3] \end{bmatrix} + \begin{bmatrix} \mathbf{w}_k[1] \\ \mathbf{w}_k[2] \\ \mathbf{w}_k[3] \end{bmatrix} \tag{14}$$

$$\mathbf{z}_k = [1 \quad 0 \quad 0] \begin{bmatrix} \mathbf{x}_k[1] \\ \mathbf{x}_k[2] \\ \mathbf{x}_k[3] \end{bmatrix} + v_k,$$

where $\mathbf{w}_k \equiv [\mathbf{w}_k[1]\ \ \mathbf{w}_k[2]\ \ \mathbf{w}_k[3]]^{\mathrm{T}}$. It is assumed that the sequences $\{\mathbf{w}_k\}$ and $\{v_k\}$ are zero-mean Gaussian white noise satisfying $E(\mathbf{w}_k\mathbf{w}_l^{\mathrm{T}}) = \mathbf{Q}\delta_{kl}$, $E(v_k v_l) = r\delta_{kl}$, $E(\mathbf{w}_k v_l) = 0$, $E(\mathbf{x}_0 v_k) = 0$, and $E(\mathbf{x}_0\mathbf{w}_k^{\mathrm{T}}) = 0$. We can easily check that the system, described by (14), is completely controllable and observable, so that the results of Theorem 2, Remarks 3 and 4 are applicable. In the following, we use the system presented by (14), where we fix the sampling time $T = 1$ s in Cases 1 and 2, choose different values for the statistical covariance matrices and compare the values of the steady-state Kalman gain and error covariance matrices. Case 3 presents a situation, employing small sampling period, for arbitrary uncertainties except for 'known' zero-measurement errors.

4.1. Case 1

First, we set the initial error covariance matrix and the perturbed one as folows:

$$\mathbf{P}_0 = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \tilde{\mathbf{P}}_0 = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Then, we set a value for the continuous-time process covariance matrix

$$\mathbf{Q}_c = \begin{bmatrix} 1\times 10^{-2} & 0 & 0 \\ 0 & 1\times 10^{-4} & 0 \\ 0 & 0 & 1\times 10^{-6} \end{bmatrix} \tag{15}$$

In order to evaluate the sampled process covariance matrix, we use (13) with the respective state matrix (continuous-time domain)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

to obtain

$$\mathbf{Q}=\begin{bmatrix}1\cdot 00\times 10^{-2} & 5\cdot 06\times 10^{-5} & 1\cdot 69\times 10^{-7}\\ 5\cdot 06\times 10^{-5} & 1\cdot 00\times 10^{-4} & 5\cdot 05\times 10^{-7}\\ 1\cdot 69\times 10^{-7} & 5\cdot 05\times 10^{-7} & 1\cdot 00\times 10^{-6}\end{bmatrix}$$

Next, we choose $\tilde{\mathbf{Q}}_c = 100\mathbf{Q}_c$. This implies that $\tilde{\mathbf{Q}} = 100\mathbf{Q}$. Since $\mathbf{HQH}^{\mathrm{T}} > 0$, we set $R = \tilde{R} = 0$. At steady state, we extract the value for the corresponding Kalman gain vectors

$$\mathbf{K}=\begin{bmatrix}1\\ 1\cdot 63\times 10^{-1}\\ 9\cdot 17\times 10^{-3}\end{bmatrix}=\tilde{\mathbf{K}}$$

In addition, at steady state, the 'prediction' error covariance matrices are found to be as follows:

$$\mathbf{P}^{-}=\begin{bmatrix}1\cdot 18\times 10^{-2} & 1\cdot 94\times 10^{-3} & 1\cdot 08\times 10^{-4}\\ 1\cdot 94\times 10^{-3} & 2\cdot 05\times 10^{-3} & 1\cdot 17\times 10^{-4}\\ 1\cdot 08\times 10^{-4} & 1\cdot 17\times 10^{-4} & 1\cdot 183\times 10^{-5}\end{bmatrix}=\frac{1}{100}\tilde{\mathbf{P}}^{-}$$

and the correction error covariance matrices

$$\mathbf{P}=\begin{bmatrix}1\cdot 31\times 10^{-18} & 1\cdot 88\times 10^{-19} & 8\cdot 45\times 10^{-21}\\ 1\cdot 88\times 10^{-19} & 1\cdot 733\times 10^{-3} & 1\cdot 00\times 10^{-4}\\ 8\cdot 45\times 10^{-21} & 1\cdot 00\times 10^{-4} & 1\cdot 73\times 10^{-5}\end{bmatrix}\approx\frac{1}{100}\tilde{\mathbf{P}}$$

$$\mathbf{P}=\begin{bmatrix}1\cdot 31\times 10^{-16} & 1\cdot 50\times 10^{-17} & 1\cdot 07\times 10^{-18}\\ 1\cdot 50\times 10^{-17} & 1\cdot 733\times 10^{-1} & 1\cdot 00\times 10^{-2}\\ 1\cdot 07\times 10^{-18} & 1\cdot 00\times 10^{-2} & 1\cdot 73\times 10^{-3}\end{bmatrix}$$

As expected, all simulation results strongly agree with the theoretical ones presented in this paper.

4.2. Case 2

Assume that we use the same values for $\mathbf{P}_0$, $\tilde{\mathbf{P}}_0$, $\mathbf{Q}$, and $\tilde{\mathbf{Q}}$ as in Case 1, except for $R = 1\times 10^{-6}$, and $\tilde{R} = 100R = 1\times 10^{-4}$. For these settings, the simulation results perfectly match the theoretical ones presented in Section 2. In order to make the problem more appealing, we set $R = 1\times 10^{-6}$, and $\tilde{R} = 0$. As expected, the steady-state value for the Kalman gain vector $\tilde{\mathbf{K}}$ has exactly the same value as in Case I, and the maximum difference for each component of $\mathbf{K}$ is less than 0·01 per cent. Similar results apply for $(\mathbf{P}^{-}, \tilde{\mathbf{P}}^{-})$ and the diagonal entries of $(\mathbf{P}, \tilde{\mathbf{P}})$. Although the latter results are speculated in Section 3, in the following, a more rigorous approach is conducted. For the system considered in (14), the Kalman gain is found to be

$$\mathbf{K}_k=\frac{1}{\mathbf{P}_k^{-}[1,1]+R}\begin{bmatrix}\mathbf{P}_k^{-}[1,1]\\ \mathbf{P}_k^{-}[1,2]\\ \mathbf{P}_k^{-}[1,3]\end{bmatrix} \tag{16}$$

where the first diagonal entry of $\mathbf{P}_k^-$ is given by

$$\mathbf{P}_k^-[1,1] = \mathbf{P}_{k-1}[1,1] + 2T\mathbf{P}_{k-1}[1,2] + T^2\mathbf{P}_{k-1}[1,3] + T^2\mathbf{P}_{k-1}[2,2] + T^3\mathbf{P}_{k-1}[2,3] + \frac{T^4}{4}\mathbf{P}_{k-1}[3,3] + \mathbf{Q}_{k-1}[1,1] \quad (17)$$

Therefore, for all values of R where $\mathbf{Q}_{k-1}[1,1] \gg R$, the denominator of $\mathbf{K}_k$, $\mathbf{P}_k^-[1,1] + R \cong \mathbf{P}_k^-[1,1]$. Consequently, for all practical purposes and the system described in (14), the Kalman gain can be assumed to be also independent of such small values of the measurement variance R when $\mathbf{Q}_{k-1}[1,1] \gg R$.

4.3. Case 3

Assume that we use the same values for $\mathbf{P}_0, \tilde{\mathbf{P}}_0, \mathbf{R}, \tilde{\mathbf{R}}$ and $\mathbf{Q}$ as in Case 1, except for the sampling period $T = 1 \times 10^{-3}$ second, and

$$\tilde{\mathbf{Q}}_c = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \mathbf{Q}_c$$

The corresponding Kalman gain vectors

$$\mathbf{K} = \begin{bmatrix} 1 \\ 5{\cdot}0559 \\ 12{\cdot}219 \end{bmatrix}, \qquad \text{and} \qquad \tilde{\mathbf{K}} = \begin{bmatrix} 1 \\ 5{\cdot}0569 \\ 12{\cdot}227 \end{bmatrix} \cong \mathbf{K}$$

The difference is less than 0·06 per cent. Furthermore, $\tilde{\mathbf{P}}^- \cong 2\mathbf{P}^-$, and $\tilde{\mathbf{P}} \cong 2\mathbf{P}$. The reason behind these surprising results can be seen by examining (16). First, we use (17), and write expressions for $\mathbf{P}_k^-[1,2]$ and $\mathbf{P}_k^-[1,3]$.

$$\mathbf{P}_k^-[1,2] = \mathbf{P}_k[1,2] + T\mathbf{P}_k[1,3] + T\mathbf{P}_k[2,2] + \frac{3T^2}{2}\mathbf{P}_k[2,3] + \frac{T^3}{2}\mathbf{P}_k[3,3] + \mathbf{Q}[1,2] \quad (18)$$

$$\mathbf{P}_k^-[1,3] = \mathbf{P}_k[1,3] + T\mathbf{P}_k[2,3] + \frac{T^2}{2}\mathbf{P}_k[3,3] + \mathbf{Q}[1,3] \quad (19)$$

Next, we write down the first few terms of the infinite series expansion for the sampled process covariance matrix $\mathbf{Q}$ (13),

$$\mathbf{Q} = \mathbf{B}\mathbf{Q}_c\mathbf{B}^{\mathrm{T}}T + \frac{(\mathbf{A}\mathbf{B}\mathbf{Q}_c\mathbf{B}^{\mathrm{T}} + \mathbf{B}\mathbf{Q}_c\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}})\,T^2}{2!} + \cdots \quad (20)$$

Since $T \ll 1$, then the terms of order T^2 are disregarded. Thus, (18) yields to $\mathbf{Q} \cong \mathbf{Q}_c T$. Thus, the off-diagonal terms of $\mathbf{Q}$ can be neglected; in particular, the two terms $\mathbf{Q}[1,2]$ and $\mathbf{Q}[1,3]$ in equations (18) and (19), respectively. Therefore, by (16)–(19), we see that the Kalman gain $\mathbf{K}_k$ is independent of $\mathbf{Q}$ and the first term $\mathbf{K}_k[1] = 1$. In addition, (refer to (15))

$$\tilde{\mathbf{Q}} \cong \tilde{\mathbf{Q}}_c T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1\times 10^{-2} & 0 & 0 \\ 0 & 1\times 10^{-4} & 0 \\ 0 & 0 & 1\times 10^{-6} \end{bmatrix} 1\times 10^{-3}$$

$$= 2\times 10^{-5}\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{3}{2}\times 10^{-2} & 0\\ 0 & 0 & \frac{4}{2}\times 10^{-4}\end{bmatrix}$$

$$\cong 2\times 10^{-5}\begin{bmatrix}1 & 0 & 0\\ 0 & 1\times 10^{-2} & 0\\ 0 & 0 & 1\times 10^{-4}\end{bmatrix}$$

$$= 2\mathbf{Q}$$

Thus, $\tilde{\mathbf{Q}} \cong 2\mathbf{Q}$. Using the results of Remark 3, we may conclude that $\tilde{\mathbf{P}}^{-} \cong 2\mathbf{P}^{-}$, and $\tilde{\mathbf{P}} \cong 2\mathbf{P}$.

5. CONCLUSION

It was shown that Kalman filter gains can be insensitive to scaling of covariance matrices; i.e. the state estimate remains unchanged or optimal under incorrect noise covariances. Moreover, the error covariance matrices generated by the algorithm were shown to be consistent with the scaled covariance matrices and *not* the state estimates. It was also shown that system modelling errors, particularly scaling errors of the input matrix, do not effect the Kalman gain. By following the ideas given in Sections 3 and 4, one can develop other similar results by considering different systems.

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