World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:10, No:8, 2016

(λ,μ)-fuzzy Subrings and (λ,μ)-fuzzy Quotient Subrings with Operators

Shaoquan Sun, Chunxiang Liu

Abstract**—**In this paper, we extend the fuzzy subrings with operators to the (λ, μ)-fuzzy subrings with operators. And the concepts of the (λ, μ)-fuzzy subring with operators and (λ, μ)-fuzzy quotient ring with operators are gived, while their elementary properties are discussed.

Keywords**—**Fuzzy subring with operators, (λ,μ)-fuzzy subring with operators, (λ,μ)-fuzzy quotient ring with operators.

I. Introduction

SINCE the concept of the fuzzy set appeared, many scholars have applied it to the ring and obtained many fuzzy theories about the ring. In 1982, Liu [1] first raised the fuzzy subring. After that, [2] and [3] discussed fuzzy quotient ring. Reference [4] proposed the notion of fuzzy subrings and fuzzy quotient ring with operators. Reference [5] defined $(\lambda,\mu)-$ fuzzy subrings. Besides, [6] gave $(\lambda,\mu)-$ intuitionistic fuzzy subgroups with operators. In this paper, we further develop the fuzzy ring theory and give the definition of $(\lambda,\mu)-$ fuzzy subring with operators and $(\lambda,\mu)-$ fuzzy quotient ring with operators, while some elementary properties are discussed.

II. Preliminaries

In this paper, we always assume $0 \le \lambda < \mu \le 1$.

Definition 1. [1] Let A be a fuzzy subset of ring R. Then A is called a fuzzy subring of R if for all $x, y \in R$,

1. $A(x-y) \ge A(x) \wedge A(y)$;
2. $A(xy) \ge A(x) \wedge A(y)$.

Definition 2. [4] Let A be a fuzzy subring of $M-$ring R. Then A is called a $M-$ fuzzy subring of R if for all $x, y \in R$, $m \in M$, $A(mx) \ge A(x)$.

Definition 3. [5] Let A be a fuzzy subset of ring R. Then A is called a $(\lambda,\mu)-$ fuzzy subring of R if for all $x, y \in R$,

1. $A(x+y) \vee \lambda \ge (A(x) \wedge A(y)) \wedge \mu$;
2. $A(-x) \vee \lambda \ge A(x) \wedge \mu$;
3. $A(xy) \vee \lambda \ge (A(x) \wedge A(y)) \wedge \mu$.

Shaoquan Sun is with the College of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao, China (phone: 185-61681686; e-mail: qdsunsaoquan@163.com).

Chunxiang Liu is with the College of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao, China (phone: 178-54204165; e-mail:15864025070@163.com).

Definition 4. [7] A subring R of $M-$ring is said to be an $M-$ subring if for all $\lambda \in M$, $a, b \in R$,

1. $\lambda(a+b) = \lambda a + \lambda b$;
2. $\lambda(ab) = (\lambda a) b$.

Definition 5. [7] Let $f: R \to R'$ be a homomorphism of $M-$ rings. Then f is called a $M-$ homomorphism if for all $x \in R$, $m \in M$, $f(mx) = mf(x)$.

Proposition 1. [5] Let A be a fuzzy subset of R. Then A is a $(\lambda,\mu)-$ fuzzy subring of R iff for all $x, y \in R$,

1. $A(x-y) \vee \lambda \ge (A(x) \wedge A(y)) \wedge \mu$;
2. $A(xy) \vee \lambda \ge (A(x) \wedge A(y)) \wedge \mu$.

Proposition 2. [4] Let S be a nonempty subset of $M-$ring R. If I_s is the characteristic function then S is an $M-$ subring of R iff I_s is an $M-$ fuzzy subring of R.

Proposition 3. [5] Let A be a fuzzy subset of R. Then A is a $(\lambda,\mu)-$ fuzzy subring of R iff for every $\alpha \in (\lambda,\mu]$, A_α is a subring of R when $A_\alpha \ne \varnothing$.

Proposition 4. [8] Let $f: R \to R'$ be a homomorphism of $M-$ rings, A be a fuzzy subring of R, and A' be a fuzzy subring of R'. Then the following statements hold:

1. $f(A)$ is a fuzzy subring of R';
2. $f^{-1}(A')$ is a fuzzy subring of R.

III. $(\lambda,\mu)-$ Fuzzy Subring with Operators

Definition 6. Let A be a fuzzy subring of $M-$ring R. Then A is called a fuzzy subring with thresholds (λ,μ) of operators or a $(\lambda,\mu)-$ fuzzy subring with operators a $(\lambda,\mu)-M-$ fuzzy subring of R if for all $x \in R$, $m \in M$, $A(mx) \vee \lambda \ge A(x) \wedge \mu$, and denoted by a $(\lambda,\mu)-M-$ fuzzy subring of R.

Proposition 5. Let S be a nonempty subset of $M-$ring R. If I_s is the characteristic function then S is an $M-$ subring of R iff I_s is an $(\lambda,\mu)-M-$ fuzzy subring of R.

Proof. According to Proposition 2, I_s is an $M-$ fuzzy subring of R when S is an $M-$ subring of R.

For all $x \in R$, $m \in M$, let $mx \in S$, then

$$I_s(mx)\vee\lambda=1\vee\lambda=1\geq I_s(x)\wedge\mu=1\wedge\mu=\mu.$$

Also $x\notin S$ when $mx\notin S$, and hence

$$I_s(mx)\vee\lambda=0\vee\lambda=\lambda\geq I_s(x)\wedge\mu=0\wedge\mu=0.$$

Thus, I_s is an $(\lambda,\mu)-M-$fuzzy subring of R. Conversely, it is can be obtained from Proposition 2.

Proposition 6. Let A be a $(\lambda,\mu)-M-$fuzzy subring of $M-$ring R. Then the following statements hold:

1. $A\big(m(xy)\big)\vee\lambda\geq A(mx)\wedge A(my)\wedge\mu$;
2. $A\big(m(-x)\big)\vee\lambda\geq A(x)\wedge\mu$.

Proof. (1) For all $x,y\in R$, $m\in M$, we have

$$A\big(m(xy)\big)\vee\lambda=A\big((mx)(my)\big)\vee\lambda\geq A(mx)\wedge A(my)\wedge\mu.$$

Thus, $A\big(m(xy)\big)\vee\lambda\geq A(mx)\wedge A(my)\wedge\mu$.

(2) For all $x\in R$, $m\in M$, we have

$$\begin{aligned}A\big(m(-x)\big)\vee\lambda&=A\big(-(mx)\big)\vee\lambda=\big(A(-(mx))\vee\lambda\big)\vee\lambda\\&\geq\big(A(mx)\wedge\mu\big)\vee\lambda=\big(A(mx)\vee\lambda\big)\wedge\mu\\&\geq A(x)\wedge\mu\wedge\mu=A(x)\wedge\mu.\end{aligned}$$

Thus, $A\big(m(-x)\big)\vee\lambda\geq A(x)\wedge\mu$.

Proposition 7. Let both A and B are $(\lambda,\mu)-M-$fuzzy subring of $M-$ring R. Then $A\cap B$ is a $(\lambda,\mu)-M-$fuzzy subring of R.

Proof. For all $x\in R$, $m\in M$, we have

$$A(mx)\vee\lambda\geq A(x)\wedge\mu;$$

$$B(mx)\vee\lambda\geq B(x)\wedge\mu.$$

Then

$$\begin{aligned}(A\cap B)(mx)\vee\lambda&=\big(A(mx)\wedge B(mx)\big)\vee\lambda=\big(A(mx)\vee\lambda\big)\wedge\big(B(mx)\vee\lambda\big)\\&\geq\big(A(x)\wedge\mu\big)\wedge\big(B(x)\wedge\mu\big)=\big(A(x)\wedge B(x)\big)\wedge\mu\\&=(A\cap B)(x)\wedge\mu.\end{aligned}$$

Thus, $A\cap B$ is a $(\lambda,\mu)-M-$fuzzy subring of R.

Proposition 8. Let A be a $(\lambda,\mu)-M-$fuzzy subring of $M-$ring R. Then A is a $(\lambda,\mu)-M-$fuzzy subring of R iff for every $\alpha\in(\lambda,\mu]$, A_α is a $M-$subring of R when $A_\alpha\neq\varnothing$.

Proof. It is easy to know by Proposition 3 A_α is a subring of R when $A_\alpha\neq\varnothing$ for every $\alpha\in(\lambda,\mu]$ in case of A being an $M-$fuzzy subring of R. Also for all $x\in A_\alpha$, $m\in M$, we have

$$A(x)\geq\alpha.$$

Then

$$A(mx)\geq A(x)\geq\alpha,$$

and hence $mx\in A_\alpha$. Thus, A_α is a $M-$subring of R. Conversely, we get the information from Proposition 3 that A is a $(\lambda,\mu)-$ fuzzy subring of R for every $\alpha\in(\lambda,\mu]$ when $A_\alpha\neq\varnothing$. If there exists $x_0\in R$, $m_0\in M$ such that

$$A(m_0x_0)\vee\lambda<A(x_0)\wedge\mu$$

Let

$$\alpha=A(x_0)\wedge\mu,$$

then for $\alpha\in(\lambda,\mu]$,

$$A(m_0x_0)<\alpha$$

and

$$x_0\in A_\alpha.$$

But $m_0x_0\notin A_\alpha$, so here emerges a contradiction. Hence

$$A(mx)\vee\lambda\geq A(x)\wedge\mu$$

always holds for any $x\in R$, $m\in M$. Therefore, A is a $(\lambda,\mu)-M-$fuzzy subring of R.

Proposition 9. Let $f:R\to R'$ be a $M-$homomorphism of $M-$rings and A be a $(\lambda,\mu)-M-$fuzzy subring of R. Then $f(A)$ is a $(\lambda,\mu)-M-$fuzzy subring of R'.

Proof. It is clear from Proposition 2.4 that $f(A)$ is a fuzzy subring of R'.

For all $y\in R$, $m\in M$, we have

$$
\begin{aligned}
f(A)(my)\vee\lambda &= \sup\left\{A(x)\middle| x\in f^{-1}(my)\right\}\vee\lambda \\
&= \sup\left\{A(x)\middle| f(x)=my\right\}\vee\lambda \\
&\geq \sup\left\{A(x')\middle| f(mx')=my, mx'\in R\right\}\vee\lambda \\
&= \sup\left\{A(x')\vee\lambda\middle| f(mx')=my, mx'\in R\right\} \\
&\geq \sup\left\{A(x')\wedge\mu\middle| f(x')=y, x'\in R\right\} \\
&= \sup\left\{A(x')\middle| f(x')=y, x'\in R\right\}\wedge\mu \\
&= f(A)(y)\wedge\mu.
\end{aligned}
$$

Thus, $f(A)$ is a $(\lambda,\mu)-M-$fuzzy subring of R'.

Proposition 10. Let $f: R\to R'$ be a $M-$homomorphism of $M-$rings and A' be a $(\lambda,\mu)-M-$fuzzy subring of R'. Then $f^{-1}(A')$ is a $(\lambda,\mu)-M-$fuzzy subring of R.

Proof. It is clear from Proposition 4 that $f^{-1}(A')$ is a fuzzy subring of R.

For all $x\in R$, $m\in M$, we have

$$
\begin{aligned}
f^{-1}(A')(mx)\vee\lambda &= A'\left(f(mx)\right)\vee\lambda = A'\left(mf(x)\right)\vee\lambda \\
&\geq A'\left(f(x)\right)\wedge\mu = f^{-1}(A')(x)\wedge\mu.
\end{aligned}
$$

Thus, $f^{-1}(A')$ is a $(\lambda,\mu)-M-$fuzzy subring of R.

IV. $(\lambda,\mu)-$ Fuzzy Quotient Ring with Operators

Let B be a $(\lambda,\mu)-$ fuzzy ideal of ring R. For all $a,b\in R,$ we define a fuzzy set $a+B$ of R as:

$$(a+B)(x) = \left(B(x-a)\vee\lambda\right)\wedge\mu, \forall x\in R.$$

Let $R/B=\{r+B|r\in R\}$. For all $r_1, r_2\in R$, we define them on R/B as:

$$(r_1+B)+(r_2+B) = (r_1+r_2)+B;$$
$$(r_1+B)\cdot(r_2+B) = r_1r_2+B.$$

Reference [2] proved that $(R/B;+,\cdot)$ is a ring.

Proposition 11. Let R be a $M-$ring and B be a $(\lambda,\mu)-$ fuzzy ideal of R. For any $R+B\in R/B$, $m\in M$, we define $m(r+B)=mr+B.$ Then $(R/B;+,\cdot)$ is a $M-$ring.

Proof. First we prove the existence of the definition $m(r+B)=mr+B.$

If $r_1+B=r_2+B,$ then

$$B(r_1-r_2) = B(r_2-r_1) = B(0).$$

$$B(mr_1-mr_2)=B(m(r_1-r_2))\geq B(r_1-r_2)=B(0).$$

Hence, $mr_1+B\supset mr_2+B.$ Similarly, we have

$$B(mr_2-mr_1) = B(m(r_2-r_1))\geq B(r_2-r_1) = B(0).$$

Hence, $mr_2+B\supset mr_1+B.$ Therefore, we have

$$mr_2+B = mr_1+B.$$

Namely,

$$m(r_1+B) = m(r_2+B).$$

Thus, the above definition is reasonable.

On the one hand,

$$
\begin{aligned}
m\left((r_1+B)+(r_2+B)\right) &= m\left((r_1+r_2)+B\right) = m(r_1+r_2)+B \\
&= mr_1+mr_2+B = (mr_1+B)+(mr_2+B) \\
&= m(r_1+B)+m(r_2+B).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
m\left((r_1+B)(r_2+B)\right) &= m(r_1r_2+B) = m(r_1r_2)+B \\
&= (mr_1)r_2+B = (mr_1+B)(r_2+B) \\
&= \left(m(r_1+B)\right)(r_2+B) = r_1(mr_2)+B = (r_1+B)(mr_2+B) \\
&= (r_1+B)\left(m(r_2+B)\right).
\end{aligned}
$$

Thus, R/B is a $M-$ring.

Let R be a $M-$ring, A be a $(\lambda,\mu)-M-$fuzzy subring of R, B be a $(\lambda,\mu)-$ fuzzy ideal of R, and A/B is a fuzzy set of R/B. Now for any $r+B\in R/B$, we define it as:

$$A/B: R/B\to[0,1] \text{ satisfying } A/B(r+B) = \sup_{x+B=r+B} A(x).$$

Reference [4] proved A/B is a $M-$ fuzzy subring of R/B.

Proposition 12. The above fuzzy subset A/B is a $(\lambda,\mu)-M-$fuzzy subring of R/B.

Proof. Let A be a $(\lambda,\mu)-M-$fuzzy subring of R. Then A/B is an $M-$ fuzzy subring of R/B. For any $r+B\in R/B$, $m\in M$, we have

$$A/B\big(m(r+B)\big)\vee\lambda = A/B(mr+B)\vee\lambda = \sup_{x+B=mr+B} A(x)\vee\lambda$$
$$\geq \sup_{my+B=mr+B} A(my)\vee\lambda \geq \sup_{y+B=r+B} A(my)\vee\lambda$$
$$\geq \sup_{y+B=r+B} A(y)\wedge\mu = A/B(r+B)\wedge\mu.$$

Thus, A/B is a $(\lambda,\mu)-M-$fuzzy subring of R/B.

Definition 7. The $(\lambda,\mu)-M-$fuzzy subring A/B is called a $(\lambda,\mu)-$fuzzy quotient ring of A with operators with respect to B, denoted by the $(\lambda,\mu)-M-$fuzzy quotient ring of A with respect to B.

Proposition 13. Let R be a $M-$ring, A be a $(\lambda,\mu)-M-$fuzzy subring of R, B be a $M-$ fuzzy ideal of R, and

$$f: R\to R/B,$$
$$x\to x+B.$$

Then f is a $M-$homomorphism from R to R/B, and $f(A)=A/B.$

Proof. It is clear that f is a homomorphism from R to $R/B.$ For any $x\in R,\ m\in M$, we have

$$f(mx)=mx+B=m(x+B)=m\big(f(x)\big).$$

And for any $a+B\in R/B$, we have

$$f(A)(a+B)=\sup_{f(x)=a+B} A(x)=\sup_{x+B=a+B} A(x)$$
$$=A/B(a+B).$$

Thus, f is a $M-$homomorphism from R to R/B, and $f(A)=A/B.$

REFERENCES

[1] Liu W J. Fuzzy invariant subgroups and Fuzzy ideals (J). Fuzzy Sets and Systems, 1982, 8: 133-139.
[2] Yin Y T. W. Fuzzy ideals and fuzzy quotient rings (J). Fuzzy Math. 1985, 4: 19-26.
[3] Kuraoka T, Nobuaki. On fuzzy quotient-rings included by fuzzy ideals [J]. Fuzzy Sets and Systems, 1992, 47: 381-386.
[4] Shaoquan Sun, Wenxiang Gu. Fuzzy subrings with operators and Fuzzy ideals with operators. Fuzzy Sets and Systems, 2005, 19(2):
[5] Yao B. (λ, μ)-fuzzy subrings normal and (λ, μ)-fuzzy ideals. The Journal of Fuzzy Mathematics. 2007, 15(4): 981-987.
[6] M. Jiang, X.L. Xin, "(λ, μ) Intuitionistic Fuzzy Subrings (Ideals)," Fuzzy Systems and Mathematics, vol. 27, pp. 1-8, 2013.
[7] Quanyan Xiong. Modern algebra (M). Shanghai: Shanghai Scientific and Technical Publishers, 1963.
[8] Yao B. Fuzzy theory of groups and rings (M). Beijing: Science Press, 2008.