International Journal of Engineering Research and Modern Education (IJERME)
Impact Factor: 6.525, ISSN (Online): 2455 - 4200
(www.rdmodernresearch.com) Volume 2, Issue 2, 2017

MAJORIZATION OF MATRICES

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Cite This Article: S. Manimaran, V. Chinnadurai & A. Swaminathan, "Majorization of Matrices", International Journal of Engineering Research and Modern Education, Volume 2, Issue 2, Page Number 65-68, 2017.

Abstract:

It is shown that under certain conditions the column majorization of matrices is reversed for the column majorization of their corresponding Moore-Penrose inverses and preserved for the column majorization of their powers. The condition for column majorization of block matrices is determined.

Index Terms: Majorization & Moore-Penrose inverse

1. Introduction:

Let $C^{m\times n}$ denote the space of $m\times n$ complex matrices. If $A\in C^{m\times n}$ then the Moore-Penrose inverse A^{+} of A is the unique solution to the equations:

$$AXA = A,\ \ XAX = X,\ \ (AX)^{*} = AX \ \text{and}\ \ (XA)^{*} = XA \ [2, p.7].$$

A square matrix A is called EP if $R(A) = R(A^{*})$ or equivalently $AA^{+} = A^{+}A,$ where $R(A)$ denotes the range space of A.A matrix A is called EP_r if A is EP and is of rank r.

Let $R^{m\times n}$ denotes the space of $m\times n$ real matrices. For any column vector $x\in R^{n}$, let $x_{[1]}, x_{[2]},\ldots x_{[n]}$ denote the coordinates of x arranged in decreasing order of magnitude: $x_{[1]}\geq x_{[2]}\geq \ldots \geq x_{[n]}$. We shall write $x\downarrow = (x_{[1]}, x_{[2]},..x_{[n]})^{t}$ where t denotes the transpose. If $x, y\in R^{n}$, we say that y is majorized by x, denote$y \prec$ x if

$$\sum_{i=1}^{k} y_i = \sum_{i=1}^{k} x_i,\quad for\ 1\leq k\leq n-1,\ \ \text{and}\ \ \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i$$

Also y is said to be majorized by x if and only if there exists a doubly stochastic matrix M such that $y = Mx$ [4, p.7-12]. Throughout this paper we consider only real matrices.

2. Majorization of Matrices:

The majorization of vectors is extended to matrices as follows:

Definition 1:

Let A and B be $m\times n$ real matrices. Then A is said to be column majorized by B, denoted by $A\prec^{c} B$ if and only if $A = MB$ where M is a doubly stochastic matrix of order m.

We note that the column majorization of matrices is equivalent to the majorization of the transpose of the corresponding matrices [4, p.430].

$$\text{i.e. } A\prec^{c} B \Leftrightarrow A = MB, \quad \text{where M is doubly stochastic}$$

$$\Leftrightarrow A = MB \quad \text{for all i, where } A_i \text{ is the } i^{th} \text{ column of A}$$

$$\Leftrightarrow A_i^{t} = B_i^{t} \quad for\, all\ i$$

$$\Leftrightarrow A^{t} \prec B^{t}.$$

Lemma 1:

Let A, B be EP_r matrices and $A\prec^{c} B$. Then $R(A) = R(B)$ and $(AB)^{+} = B^{+}A^{+}$.

Proof:

$$A\prec^{c} B \Rightarrow A = MB \Rightarrow N(B) \subseteq N(A) \quad where\ N(A), \text{ denotes the null space of } A.$$

Since A and B are EP_r, $R(A) = R(B)$, then by theorem 3 of [1], AB is EP_r and by theorem 4 of [1], we get $(AB)^+ = B^+A^+$.

Lemma 2:

Let A be an EP matrix. Then

$(i)\ A \prec^c A^+A \Leftrightarrow A^+A \prec^c A^+$

$(ii)\ A^+A \prec^c A \Leftrightarrow A^+ \prec^c A^+A$

Proof:

$$. (i)\ A \prec^c A^+A \Leftrightarrow A = MA^+A$$

$$\Leftrightarrow AA^+ = MA^+AA^+ \ (on\ post\ multiplication\ by\ A^+\ or\ A)$$

$$\Leftrightarrow AA^+ = MA^+ \ (since\ A\ is\ EP\ and\ \ A^+AA^+ = A^+)$$

$$\Leftrightarrow A^+A \prec^c A^+$$

Hence the result (i). Similarly (ii) can be proved.

Remark 1:

In particular, if A is nonsingular, then $A \prec^c I \Leftrightarrow I \prec^c A^{-1}$. Hence $A \prec^c I \Rightarrow A \prec^c A^{-1}$. If the column sums of A is one, then $I \prec^c A^{-1}$, however A need not be a doubly stochastic matrix.

Remark 2:

We note that the condition on A cannot be relaxed in the above lemma 2. For example,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$AA^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^+A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Here A is not EP, however $A \prec^c A^+A\ and\ A^+A \prec^c A^+$.

Remark 3:

For any EP matrix A, if $A \prec^c I$, then $A = AA^+A \prec^c A^+A$. Hence by lemma 2 $A \prec^c A^+A \prec^c A^+$
In particular if A is EP and doubly stochastic, then $A \prec^c I$ holds automatically and hence $A \prec^c A^+A \prec^c A^+$

Theorem 1:

If A is EP_r and B is symmetric idempotent with rank r, then

$$(i)\ A \prec^c B \Leftrightarrow B^+ \prec^c A^+$$

$$(ii)\ B \prec^c A \Leftrightarrow A^+ \prec^c B^+$$

$$(iii)\ A \prec^c B \Rightarrow A \prec^c A^+$$

$$(iv)\ B \prec^c A \Rightarrow A^+ \prec^c A$$

Proof:

Since A is EP_r and B is symmetric idempotent with rank r, both A and B are EP_r. By lemma 1, $R(A) = R(B) \Rightarrow A^+A = B^+B = B$ (since A^+A is the projection onto $R(A)$ along $N(A)$. Then (i) and (ii) follow from lemma 2. (iii) and (iv) follow from (i) and (ii) respectively and $B = B^+$.

Corollary 1:

Let A be EP_r ,B be symmetric idempotent with rank r and $A \prec^c B$. Then $A^n \prec^c B \prec^c (A^n)^+$ for any positive integer n.

Proof:

Since $A \prec^c B$ and by theorem 1 $(i), B = B^+ \prec^c A^+$. Hence, $A \prec^c A^+$ follows from theorem 1 (iii). SinceB is symmetric idempotent and by theoremB2 of [4p.433],

$A^2 \prec^c A^\dagger A = B^\dagger B = B$. Thus, $A^n \prec^c B$ is true for n=2. Now, $A^3 \prec^c BA = BB^\dagger A = AA^\dagger A = A \prec^c B$. Hence $A \prec^c B$. Thus , it is true for n=3. By continuing in this manner we can show that $A \prec^c B \Rightarrow A^n \prec^c B$ for any positive integer n. $B \prec^c (A^n)^\dagger$ follows from $A^n \prec^c B$ and theorem 1(i). Hence the corollary.

Theorem 2:

Let A and B be EP_r matrices such that $AB = BA$. Then $A \prec^c B \Leftrightarrow B^+ \prec^c A^+$.

Proof:

Since A, B are EP_r and $A \prec^c B$, by lemma 1, $R(A) = R(B) = R(B^+)$. Since A and B^+ are EP_r and $R(A) = R(B^+)$, by theorem 3 and 4 of [1], AB^+ is EP_r and $(AB^+)^+ = BA^+$

Thus, $AB^+ = B^+A$ (by theorem 2 of [5]) (1)

$$A \prec^c B \Rightarrow AB^+ \prec^c BB^+ \quad \text{(by theorem } B2 \text{ of [4,p.433])}$$
$$\Rightarrow B^+A \prec^c BB^+ \quad \text{(by (1))}$$
$$\Rightarrow BB^+ \prec^c (B^+A)^+ \quad \text{(by theorem 1(i) applied to } BB^+)$$
$$\Rightarrow B^+B \prec^c A^+B$$
$$\Rightarrow B^+BB^+ \prec^c A^+BB^+$$
$$\Rightarrow B^+ \prec^c A^+$$

Conversely, since A^+ and B^+ are EP_r. And by theorem 2 of [5], $BA^+ = A^+B$ is of rank r. In the above part, replacing A by B^+ and B by A^+ and using $(A^+)^+ = A, (B^+)^+ = B$,

We get $B^+ \prec^c A^+ \Rightarrow A \prec^c B$. Hence the theorem.

REMARK 4:

The condition on A and B that they have the same rank, but $AB \neq BA$ cannot hold in theorem 2. For example, consider,

$$A = 1/3\begin{pmatrix} 12 & 3 & 3 \\ 11 & 12 & 10 \\ 7 & 12 & 11 \end{pmatrix} : \quad B = \begin{pmatrix} 5 & 0 & 0 \\ 3 & 6 & 5 \\ 2 & 3 & 3 \end{pmatrix}$$

$$A^+ = 1/45\begin{pmatrix} 12 & 3 & -6 \\ -51 & 111 & -87 \\ 48 & -123 & 111 \end{pmatrix}; \quad B^+ = 1/15\begin{pmatrix} 3 & 0 & 0 \\ 1 & 15 & -25 \\ -3 & -15 & 30 \end{pmatrix}$$

Here $AB \neq BA$, however $A \prec^c B$, A and B are EP_r with rank 3 and $B^+ \prec^c A^+$.

Corollary 2:

Let A and B be EP_r matrices and $A \prec^c B$ such that $AB = BA$. Then $A^n \prec^c B^n$ for any positive integer n.

Proof:

By theorem $B2$ of [4, p.433] and $A \prec^c B$ we have $AB^+ \prec^c BB^+$. Since BB^+ is symmetric idempotent by theorem 1 (iii), we get

$$AB^+ \prec^c (AB^+)^+ = BA^+. \text{ By lemma 1, } AB^+A \prec^c BA^+A = B.$$

Using (1), we see that $AAB^+ \prec^c B$. By lemma 1, $A^+A = B^+B$.

Hence $A^2B^+B \prec^c B^2 \Rightarrow A^2 \prec^c B^2$. Thus the corollary is true for n=2.

Now, $A^2 \prec^c B^2 \Rightarrow A^3 \prec^c B^2A = AB^2 \prec^c BB^2 = B^3$

(Since $A \prec^c B \Rightarrow AB^2 \prec^c BB^2$). Thus it is true for $n = 3$. By continuing in this manner, we can show that $A \prec^c B \Rightarrow A^n \prec^c B^n$ for any positive integer n. Hence the corollary.

Theorem 3:

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $rank\ M = rank\ A$ and $L = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$ with

$rank\ L = rank\ E$ be $n \times n$ real matrices such that $A^{+}B = E^{+}F$ (2)

Then $A \prec^{c} E$ and $C \prec^{c} G$ .i.e., $A = R_1 E$ and $C = R_2 G \Leftrightarrow M \prec^{c} L$ with doubly stochastic matrix R of the form $\begin{pmatrix} O & R_1 \\ O & R_2 \end{pmatrix}$

Proof:

Since $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $rank\ M = rank\ A,$ where A is $k \times k$. and D is $(n-k)\times(n-k)$ matrices, by corollary in [3] it follows that $N(A) \subseteq N(C),\ \ N(A^{*}) \subseteq N(B^{*})$ and $D = CA^{+}B$ or equivalently, $C = CA^{+}A,\ B = AA^{+}B$ and $D = CA^{+}B$.

For $L,$ we have $G = GE^{+}E,\ \ F = EE^{+}F$ and $H = GE^{+}F.$ (3)

Now, $A \prec^{c} E$ and $C \prec^{c} G \Leftrightarrow \begin{pmatrix} A & O \\ C & O \end{pmatrix} \prec^{c} \begin{pmatrix} E & O \\ G & O \end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} A & O \\ C & O \end{pmatrix} \begin{pmatrix} I & A^{+}B \\ O & I \end{pmatrix} \prec^{c} \begin{pmatrix} E & O \\ G & O \end{pmatrix} \begin{pmatrix} I & A^{+}B \\ O & I \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \prec^{c} \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

$$\Leftrightarrow M \prec^{c} L$$

Hence the theorem.

References:

1. Baskett .T.S. and Katz .I.J., Theorems on products of EP_r matrices, Linear Algebra Appl. 2 (1969), 87-103.
2. Ben Israel .A. and Greville .T.N.E., Generalized Inverses: Theory and Applications, Wiely (1974).
3. Carlson .D., Haynsworth .E. and Markham .Th., A generalization of the Schur complement by means of the Moore-Penrose Inverse, SIAM J. APPL. Math. 26 (1974), 169-175.
4. Marshall .A.W. and Olkin .I., Inequalities :Theory of majorization and its Applications, Academic Press (1979).
5. Pearl .M.H., On generalized Inverses of matrices, Proc. Camb. Phil. Soc. 62 (1966), 673-677.