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TRANSMUTED KUMARASWAMY DISTRIBUTION

Muhammad Shuaib Khan[1], **Robert King**[2], **Irene Lena Hudson**[3]

ABSTRACT

The Kumaraswamy distribution is the most widely applied statistical distribution in hydrological problems and many natural phenomena. We propose a generalization of the Kumaraswamy distribution referred to as the transmuted Kumaraswamy (TKw) distribution. The new transmuted distribution is developed using the quadratic rank transmutation map studied by Shaw et al. (2009). A comprehensive account of the mathematical properties of the new distribution is provided. Explicit expressions are derived for the moments, moment generating function, entropy, mean deviation, Bonferroni and Lorenz curves, and formulated moments for order statistics. The TKw distribution parameters are estimated by using the method of maximum likelihood. Monte Carlo simulation is performed in order to investigate the performance of MLEs. The flood data and HIV/ AIDS data applications illustrate the usefulness of the proposed model.

Key words: Kumaraswamy distribution, moments, order statistics, parameter estimation, maximum likelihood estimation.

1. Introduction

The Kumaraswamy probability distribution was originally proposed by Poondi Kumaraswamy (1980) for double bounded random processes for hydrological applications. The Kumaraswamy double bounded distribution denoted by $Kw(\alpha,\theta)$ distribution is a family of continuous probability distributions defined on the interval [0,1] with cumulative distribution function given by

$$G_{Kw}(x;\alpha,\theta) = 1-(1-x^{\alpha})^{\theta}\,, \tag{1}$$

and probability density function (pdf) corresponding to (1) given by

$$g_{Kw}(x;\alpha,\theta) = \alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\,, \tag{2}$$

[1] School of Mathematical and Physical Sciences, The University of Newcastle, Callaghan, NSW 2308, Australia. E-mail: Muhammad.S.Khan@newcastle.edu.au.
[2] School of Mathematical and Physical Sciences, The University of Newcastle, Callaghan, NSW 2308, Australia. E-mail: robert.king@newcastle.edu.au.
[3] School of Mathematical and Physical Sciences, The University of Newcastle, Callaghan, NSW 2308, Australia. E-mails: Irene.Hudson@newcastle.edu.au, Irenelena.hudson@gmail.com.

where $\alpha > 0$ and $\theta > 0$ are the shape parameters. The *Kw* probability density function has the same basic properties as the beta distribution. According to Jones (2009) and Cordeiro et al. (2010, 2012) it depends in the same way as the beta distribution on the values of its parameters: it is unimodal for $\alpha > 1$ and $\theta > 1$; uniantimodal for $\alpha < 1$ and $\theta < 1$; increasing for $\alpha > 1$ and $\theta \leq 1$; decreasing for $\alpha \leq 1$ and $\theta > 1$ and constant for $\alpha = \theta = 1$. Jones (2009) investigated not only properties of the Kumaraswamy distribution but also some similarities and differences between the beta and *Kw* distributions. According to Jones (2009) the Kumaraswamy distribution has several advantages over the beta distribution, such as a simple normalizing constant, simple explicit formulae for the distribution and quantile functions which do not involve any special functions, a simple formula for random variable generation, explicit formulae for L-moments and simpler formulae for moments of order statistics. On the other hand, the beta distribution has simpler formulae for moments and the moment generating function, a one-parameter sub-family of symmetric distributions, simpler moment estimation and more ways of generating the distribution through physical processes.

The *Kw* distribution is applicable to a number of hydrological problems and many natural phenomena whose process values are bounded on both sides. Cordeiro and de Castro (2009) studied a new class of the Kumaraswamy generalized distributions (denoted by the *Kw*–G distribution) based on the Kumaraswamy distribution (denoted by *Kw* distribution). They derived almost all formulas for the probability characteristics of the *Kw*–G distribution. In hydrology and related areas, the *Kw* distribution has received considerable interest, see Cordeiro et al. (2010, 2012), Fletcher and Ponnambalam (1996), Ganji et al. (2006), Ponnambalam et al. (2001), Sundar and Subbiah (1989) and Seifi et al. (2000). According to Nadarajah (2008), many papers in the hydrological literature have used this distribution because it is deemed to be a "better alternative" to the beta distribution; see, for example, Koutsoyiannis and Xanthopoulos (1989).

This article introduces a new three-parameter distribution which is a generalized two-parameter Kumaraswamy distribution, called the transmuted Kumaraswamy distribution and denoted by TKw. Recently the generalization of parametric models by transforming an appropriate model, into a more general model, by adding a shape parameter has been intensively studied for many different families of lifetime distributions. Aryal and Tsokos et al. (2009, 2011) considered the following transmuted extreme value distributions: the transmuted Gumbel distribution to model climate data, the transmuted Weibull distribution and their applications to analyse real data sets. Recently Khan et al. (2013 a, b, c) developed the transmuted modified Weibull distribution, the transmuted generalized inverse Weibull distribution and the transmuted generalized exponential distribution. More recently Khan et al. (2014) studied the characteristics of the transmuted Inverse Weibull distribution. Ashour et al. (2013), Elbatal et al. (2013) and Aryal (2013) studied the transmuted Lomax distribution, the transmuted quasi Lindley distribution and the transmuted log-logistic distribution with a discussion on some properties of this family. The most recent families of the transmuted Rayleigh distribution, the transmuted generalized Rayleigh distribution and the transmuted Lindley distribution are derived in studies of Merovici (2013 a, b) & (2014). More recently Ahmad et al. (2015) also studied the transmuted Kumaraswamy distribution and discussed some mathematical results.

Using the quadratic rank transmutation map proposed by Shaw et al. (2009), we develop the three-parameter TKw distribution. According to this approach a random variable X is said to have a transmuted distribution if its cumulative distribution function (cdf) satisfies the relationship

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2, \qquad |\lambda| \leq 1 \tag{3}$$

and

$$f(x) = g(x)[(1+\lambda) - 2\lambda G(x)], \tag{4}$$

where $G(x)$ is the cdf of the base distribution, $g(x)$ and $f(x)$ are the corresponding probability density functions (pdf) associated with $G(x)$ and $F(x)$, respectively. It is important to note that at $\lambda = 0$ we have the distribution of the base random variable.

The paper is organized as follows. In Section 2, we present the analytical shapes of the probability density and hazard functions of the model under study. A range of mathematical properties are considered in Section 3, specifically we demonstrate the quantile functions, moment estimation and moment generating function. Maximum likelihood estimates (MLEs) of the unknown parameters are discussed in Section 4. Entropy and mean deviations are derived in Section 5 and 6. The probability density function (pdf) of order statistics and their moments are derived in Section 7. In Section 8 we evaluate the performance of MLEs using Simulation. Two applications of the TKw distribution to the flood data and HIV/AIDS data are illustrated in Section 9. In Section 10, concluding remarks are addressed.

2. Transmuted Kumaraswamy distribution

A random variable X is said to have transmuted Kw probability distribution denoted by $TKw(x;\alpha,\theta,\lambda)$ with parameters $\alpha,\theta > 0$ and $-1 \leq \lambda \leq 1$, $x \in (0,1)$, if its pdf and cdf are given by

$$f_{TKw}(x;\alpha,\theta,\lambda) = \alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\{1-\lambda+2\lambda(1-x^{\alpha})^{\theta}\}, \tag{5}$$

and

$$F_{TKw}(x;\alpha,\theta,\lambda) = \left[1-(1-x^{\alpha})^{\theta}\right]\left[1+\lambda(1-x^{\alpha})^{\theta}\right], \tag{6}$$

where α and θ are the shape parameters and λ the transmuting parameter, representing the different patterns of the subject distribution. The TKw distribution approaches the Kw distribution when the transmuted parameter $\lambda = 0$.

If X has $TKw(\mathrm{x}; \alpha, \theta, \lambda)$distribution, then the reliability function (RF), hazard function and cumulative hazard function corresponding to (5) are given by

$$R_{TKw}(x;\alpha,\theta,\lambda) = 1 - \left[1-(1-x^{\alpha})^{\theta}\right]\left[1+\lambda(1-x^{\alpha})^{\theta}\right], \tag{7}$$

$$h_{TKw}(x;\alpha,\theta,\lambda) = \frac{\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\left\{1-\lambda+2\lambda(1-x^{\alpha})^{\theta}\right\}}{1-[1-(1-x^{\alpha})^{\theta}][1+\lambda(1-x^{\alpha})^{\theta}]}, \tag{8}$$

and

$$H_{TKw}(x;\alpha,\theta,\lambda) = \int_0^x \frac{\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\left\{1-\lambda+2\lambda(1-x^{\alpha})^{\theta}\right\}}{1-[1-(1-x^{\alpha})^{\theta}][1+\lambda(1-x^{\alpha})^{\theta}]}\,dx,$$

$$H_{TKw}(x;\alpha,\theta,\lambda) = -\ln\left|1-\left[1-(1-x^{\alpha})^{\theta}\right]\left[1+\lambda(1-x^{\alpha})^{\theta}\right]\right|. \tag{9}$$

Figure 1 shows some possible shapes of probability density function of the TKw distribution for selected values of the parameters α, θ and λ, and the hazard function of the TKw distribution for the same value of the parameters, respectively.

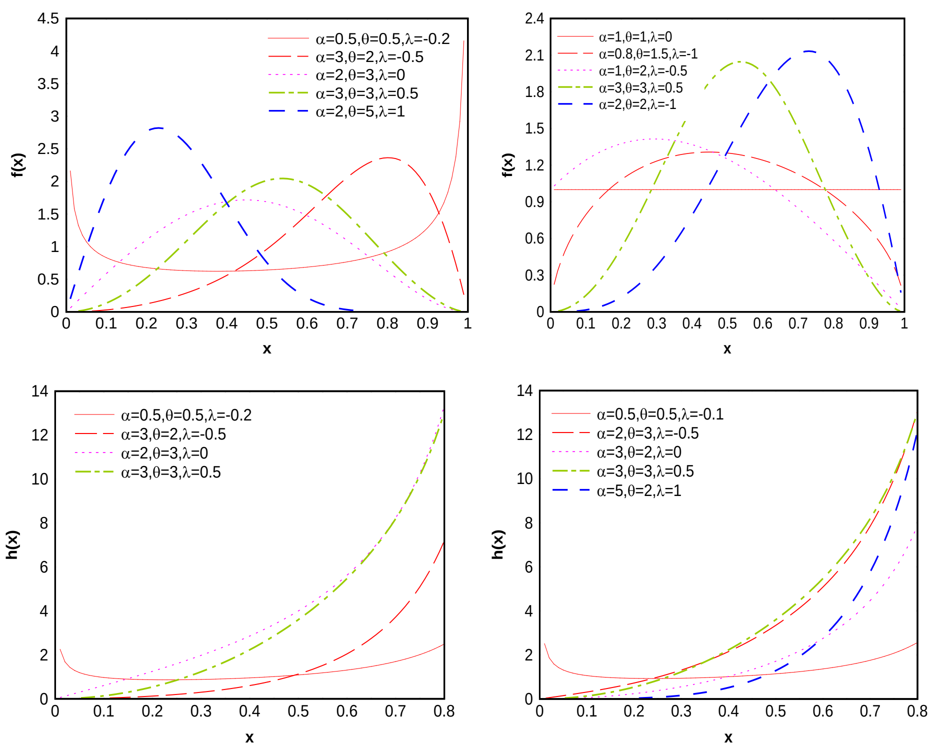


Figure 1. Plots of the TKw PDF & HF for some parameter values

Figure 1 also illustrates the TKw instantaneous failure rates. These failure rates are defined with different choices of parameters. For all choices of parameters the distribution has a monotonically increasing and decreasing behaviour of hazard rates. The transmuted beta and TKw distributions share their main special cases. T-Beta $(a, 1, \lambda)$ and $TKw(\alpha, 1, \lambda)$ distributions are both power function distributions. The TKw distribution approaches the transmuted uniform distribution $a = \theta = 1, \lambda \leq 1$, uniform distribution for $a = \theta = 1, \lambda = 0$.

3. Moments and quantiles

This section presents expressions for the moments, moment generating function and quantiles of the TKw distribution.

Theorem 1: If X has the $Tkw(x; \alpha, \theta, \lambda)$ distribution with $|\lambda| \leq 1$, then the k^{th} moment of X is given as follows

$$E\left(X^{\mathrm{k}}\right) = (1-\lambda)\theta\beta\left(\frac{\mathrm{k}}{\alpha}+1, \theta\right) + 2\lambda\theta\beta\left(\frac{\mathrm{k}}{\alpha}+1, 2\theta\right).$$

Proof: Let X have a Tkw distribution, then the k^{th} moment of X is given as

$$E\left(X^{\mathrm{k}}\right) = \int_0^1 \alpha\theta x^{k+\alpha-1}(1-x^{\alpha})^{\theta-1}\left\{1-\lambda+2\lambda(1-x^{\alpha})^{\theta}\right\}dx,$$

$$E\left(X^{\mathrm{k}}\right) = (1-\lambda)\int_0^1 \alpha\theta x^{k+\alpha-1}(1-x^{\alpha})^{\theta-1}dx$$

$$+ 2\lambda\int_0^1 \alpha\theta x^{k+\alpha-1}(1-x^{\alpha})^{2\theta-1}\,dx.$$

Finally, we obtain the k^{th} moment of the Tkw distribution as

$$E\left(X^{\mathrm{k}}\right) = (1-\lambda)\theta\ \psi_{1,\mathrm{k}} + 2\lambda\theta\ \psi_{2,\mathrm{k}}, \qquad (10)$$

where $\psi_{j,\mathrm{k}}$ is introduced for simplicity as

$$\psi_{j,\mathrm{k}} = \beta\left(\frac{\mathrm{k}}{\alpha}+1, j\theta\right), \quad j = 1, 2.$$

The expressions for the expected value and variance are

$$\mathrm{E}(X) = (1-\lambda)\theta\ \psi_{1,1} + 2\lambda\theta\ \psi_{2,1}, \qquad (11)$$

and

$$Var(X) = (1-\lambda)\theta\ \psi_{1,2} + 2\lambda\theta\ \psi_{2,2} - \left\{(1-\lambda)\theta\ \psi_{1,1} + 2\lambda\theta\ \psi_{2,1}\right\}^2. \qquad (12)$$

The coefficient of variation, skewness and kurtosis measures can now be calculated using the following relationships

$$CV(X) = \frac{\sqrt{Var(X)}}{\mathrm{E}(X)},$$

$$\mathrm{Skewness}(X) = \frac{\mathrm{E}\left(\mathrm{X} - \mathrm{E}(X)\right)^3}{[Var(X)]^{\frac{3}{2}}},$$

and

$$\mathrm{Kurtosis}(X) = \frac{\mathrm{E}\left(\mathrm{X} - \mathrm{E}(X)\right)^4}{[Var(X)]^2}.$$

The mean, variance and coefficient of variation can be obtained using equations (11) and (12). The relationship between α and the mean is shown in Figure 2.

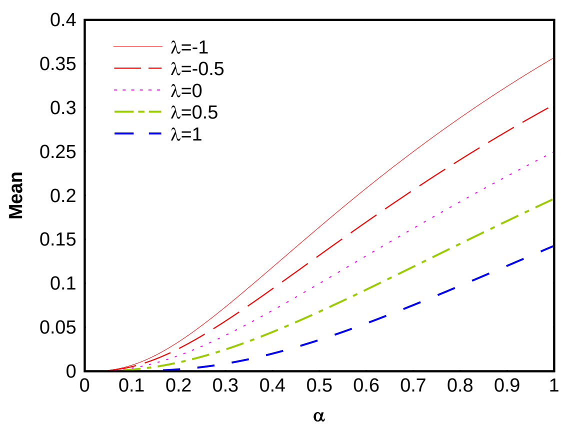
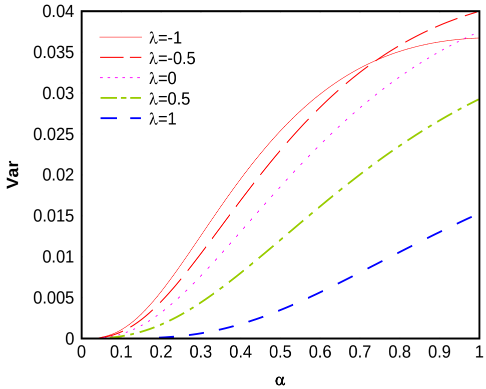
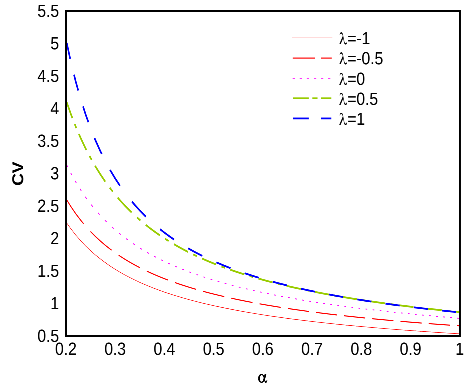


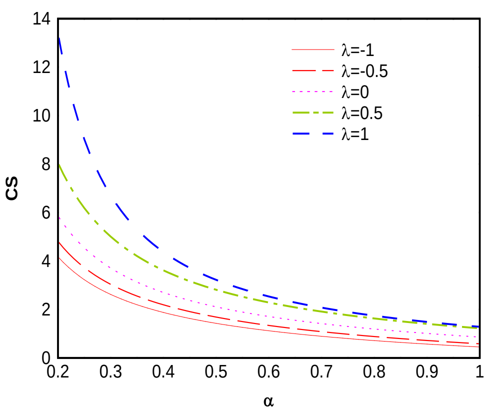
Figure 2. α vs mean



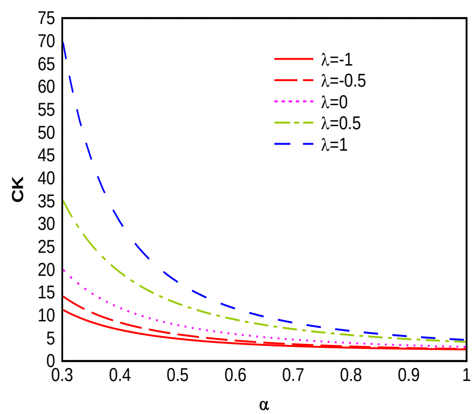
(a)α vs Var_{TKw}



(b) α vs CV_{TKw}



(c) α vs CS_{TKw}



(d) α vs CK_{TKw}

Figure 3. Plots of the TKw distribution α vs coefficients

The relationship between α and the variance is represented in Figure 3a. It is clear that, as α increases, the variance of the subject distribution also increases. The relationship between α and the coefficient of variation (CV_{TKw}) is shown in Figure 3b, which shows that as the parameter α increases, the coefficient of variation of the subject distribution is decreasing. Graphical representation of skewness (CS_{TKw}) and kurtosis (CK_{TKw}) when $\theta = 3$ and $\lambda = -1, -0.5, 0, 0.5, 1$ as a function of α are illustrated in Figures 3c and 3d, respectively. This shows the coefficients of skewness and kurtosis have a negative relationship with α.

Theorem 2: If X has the $Tkw(x;\alpha,\theta,\lambda)$ distribution with $|\lambda|\leq 1$, then the moment generating function of X say $M_x(t)$ is given as follows

$$M_x(t)=(1-\lambda)\sum_{m=0}^{\infty}\frac{t^m}{m!}\theta\beta\left(\frac{m}{\alpha}+1,\theta\right)+2\lambda\sum_{m=0}^{\infty}\frac{t^m}{m!}\theta\beta\left(\frac{m}{\alpha}+1,2\theta\right).$$

Proof: Let X has a Tkw distribution, then the moment generating function of X is given as

$$M_x(t)=\int_0^1\alpha\theta\exp(tx)\,x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\left\{1-\lambda+2\lambda(1-x^{\alpha})^{\theta}\right\}dx.$$

Using the Taylor series of function e^{tx} reduces the above to

$$M_x(t)=(1-\lambda)\sum_{m=0}^{\infty}\frac{t^m}{m!}\int_0^1\alpha\theta x^{m+\alpha-1}(1-x^{\alpha})^{\theta-1}dx+$$
$$2\lambda\sum_{m=0}^{\infty}\frac{t^m}{m!}\int_0^1\alpha\theta x^{m+\alpha-1}(1-x^{\alpha})^{2\theta-1}\,dx.$$

By solving the above integral we obtain

$$M_x(t)=(1-\lambda)\sum_{m=0}^{\infty}\frac{t^m}{m!}\theta\beta\left(\frac{m}{\alpha}+1,\theta\right)+2\lambda\sum_{m=0}^{\infty}\frac{t^m}{m!}\theta\beta\left(\frac{m}{\alpha}+1,2\theta\right),\qquad(13)$$

which completes the proof.

Theorem 3: The qth quantile x_q of the Tkw random variable is given by

$$x_q=\left[1-\left\{1-\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda q}}{2\lambda}\right\}^{\frac{1}{\theta}}\right]^{\frac{1}{\alpha}},\qquad 0<q<1.\qquad(14)$$

Proof: The qth quantile x_q of the Tkw distribution is defined as

$$q=P(X\leq x_q)=F(x_q),\qquad x_q\geq 0.$$

Using the distribution function of the Tkw distribution we have

$$q=F(x_q)=(1+\lambda)\left[1-\left(1-x_q^{\alpha}\right)^{\theta}\right]-\lambda\left[1-\left(1-x_q^{\alpha}\right)^{\theta}\right]^2,$$

that is

$$\lambda\left[1-\left(1-x_q^{\alpha}\right)^{\theta}\right]^2-(1+\lambda)\left[1-\left(1-x_q^{\alpha}\right)^{\theta}\right]+q=0.$$

Consider this as a quadratic in $1-\left(1-x_q^{\alpha}\right)^{\theta}$ as

$$\Delta=1+(2-4q)\lambda+\lambda^2.$$

It has roots $\frac{(1+\lambda)-\sqrt{\Delta}}{2\lambda}$. These exist if Δ is positive. Consider the following cases. If $\lambda=-1$ then Δ reduces to

$$\Delta = 4q > 0, \quad if\ q > 0.$$

If $\lambda = 1$ then Δ takes the form

$$\Delta = 4(1-q) > 0, \quad if\ q > 0.$$

Otherwise for $-1 < \lambda < 1$, consider the roots of Δ, as a quadratic form in λ, are

$$\lambda = -1 + 2q \pm 2\sqrt{q^2 - q}\,,$$

Therefore, $q^2 - q < 0$ for $0 < q < 1$. So the only real roots could occure for $q = 0\ or\ 1$.

If $q = 0$ then roots = -1, contradiction between $(-1 < \lambda < 1)$, and if $q = 1$ then roots = 1, contradiction between $(-1 < \lambda < 1)$. Thus, there are no real roots of Δ as a quadratic in λ. Therefore, Δ has the same sign in the range $-1 \le \lambda \le 1$, hence $\Delta > 0$.

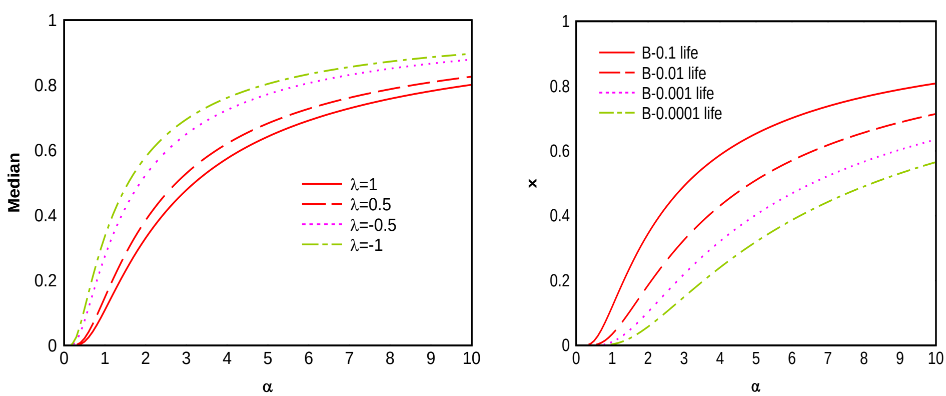
Since $\Delta \ge 0$, then

$$1-\left(1-x_q^{\alpha}\right)^{\theta} = \frac{(1+\lambda)-\sqrt{\Delta}}{2\lambda}.$$

Finally, we obtain the qth quantile x_{q} of the Tkw distribution as

$$x_{\mathrm{q}} = \left[1-\left\{1-\frac{(1+\lambda)-\sqrt{\Delta}}{2\lambda}\right\}^{\frac{1}{\theta}}\right]^{\frac{1}{\alpha}},$$

which completes the proof.



(a) Median (b) B-lives

Figure 4. Plots of the Quantiles for the TKw distribution for some parameter values

Using the method of inversion we can generate random variables from the Tkw distribution. One can use equation (14) to generate random numbers when the parameters α, θ, λ are known.

Hence, the median of Tkw distribution is given by

$$x_{0.5} = \left[1 - \left\{1 - \frac{(1+\lambda) - \sqrt{1+\lambda^2}}{2\lambda}\right\}^{\frac{1}{\theta}}\right]^{\frac{1}{\alpha}}. \tag{15}$$

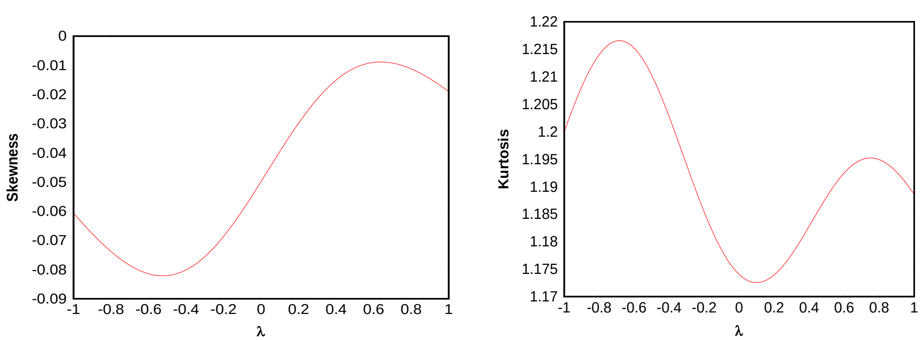
To demonstrate the effect of the shape parameter on the median and the percentile life (or B-life) as a function of α, they are calculated using quantiles and shown in Figure 4a and 4b, respectively. It can be concluded that as the shape parameter α increases the behaviour of median and percentile life (or B-life) also increases. To illustrate the effect of the transmuting parameter, on skewness and kurtosis, we also consider the measure based on quantiles. Skewness and kurtosis are calculated by using the relationship of Bowley ($\mathcal{B}$) and Moors ($\mathcal{M}$). The Bowley skewness is one of the earliest skewness measures defined by the average of the quantiles minus median, divided by the half of the interquantile range given by (see Kenney and Keeping (1962))

$$\mathcal{B} = \frac{\mathcal{Q}(3/4) + \mathcal{Q}(1/4) - 2\mathcal{Q}(2/4)}{\mathcal{Q}(3/4) - \mathcal{Q}(1/4)}. \tag{16}$$

The Moors kurtosois is based on octiles and given by Moors (1998)

$$\mathcal{M} = \frac{\mathcal{Q}(3/8) - \mathcal{Q}(1/8) + \mathcal{Q}(7/8) - \mathcal{Q}(5/8)}{\mathcal{Q}(6/8) - \mathcal{Q}(2/8)}. \tag{17}$$

Figures 5a and 5b respectively illustrate the graphical representation of the Bowley ($\mathcal{B}$) skewness and Moors ($\mathcal{M}$) kurtosis as a function of the λ for $\alpha = \theta = 3$.



(a) Skewness (b) kurtosis

Figure 5. Plots of the Bowley skewness and Moors kurtosis for the TKw distribution

4. Parameter estimation

If the parameters of the TKw distribution are not known, then the maximum likelihood estimates, MLEs, of the parameters are given as follows.

Let $x_1, x_2, \ldots, x_n$ be the random samples of size n from the TKw distribution. Then the log-likelihood function of (5) is given by

$$\mathcal{L} = \mathrm{n}\ln\alpha + \mathrm{n}\ln\theta + (\alpha - 1)\sum_{\mathrm{i}=1}^{\mathrm{n}} \ln x_{\mathrm{i}} + (\theta - 1)\sum_{\mathrm{i}=1}^{\mathrm{n}} \ln(1 - x_{\mathrm{i}}{}^{\alpha}) + \sum_{\mathrm{i}=1}^{\mathrm{n}} \ln\{1 - \lambda + 2\lambda(1 - x_{\mathrm{i}}{}^{\alpha})^{\theta}\}. \quad (18)$$

By differentiating (18) with respect to α, θ and λ, then equating it to zero, we obtain the estimating equations

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \frac{\mathrm{n}}{\alpha} + \sum_{\mathrm{i}=1}^{\mathrm{n}} \ln x_{\mathrm{i}} - (\theta - 1)\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{x_{\mathrm{i}}{}^{\alpha}\ln x_{\mathrm{i}}}{1 - x_{\mathrm{i}}{}^{\alpha}} - \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{2\lambda\theta(1 - x_{\mathrm{i}}{}^{\alpha})^{\theta-1} x_{\mathrm{i}}{}^{\alpha}\ln x_{\mathrm{i}}}{\{1 - \lambda + 2\lambda(1 - x_{\mathrm{i}}{}^{\alpha})^{\theta}\}},$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{\mathrm{n}}{\theta} + \sum_{\mathrm{i}=1}^{\mathrm{n}} \ln(1 - x_{\mathrm{i}}{}^{\alpha}) - \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{2\lambda(1 - x_{\mathrm{i}}{}^{\alpha})^{\theta}\ln(1 - x_{\mathrm{i}}{}^{\alpha})}{\{1 - \lambda + 2\lambda(1 - x_{\mathrm{i}}{}^{\alpha})^{\theta}\}},$$

$$\frac{\partial\mathcal{L}}{\partial\lambda} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{-1 + 2(1 - x_{\mathrm{i}}{}^{\alpha})^{\theta}}{\{1 - \lambda + 2\lambda(1 - x_{\mathrm{i}}{}^{\alpha})^{\theta}\}}.$$

The maximum likelihood estimator $\widehat{\omega} = \left(\widehat{\alpha}, \widehat{\theta}, \widehat{\lambda}\right)^{\mathrm{T}}$ of $\omega = (\alpha, \theta, \lambda)^{\mathrm{T}}$ is obtained by solving this nonlinear system of equations. These solutions will yield the ML estimators $\widehat{\alpha}, \widehat{\theta}$ and $\widehat{\lambda}$. Here we used a nonlinear optimization algorithm such as the quasi-Newton algorithm to numerically maximize the log-likelihood function given in (18). The required numerical evaluations were implemented using the R language. Under the conditions that are fulfilled for parameters in the interior of the parameter space, but not on the boundary, the asymptotic distribution of the element of the 3×3 observed information matrix for the TKw distribution is

$$\sqrt{\mathrm{n}}(\widehat{\omega} - \omega) \sim N_3(0, V^{-1}).$$

where V is the expected information matrix. Thus, the expected information matrix is

$$V^{-1} = -\mathrm{E}\begin{bmatrix} \frac{\partial^2 \mathcal{L}}{\partial \alpha^2} & \frac{\partial^2 \mathcal{L}}{\partial \alpha\, \partial \theta} & \frac{\partial^2 \mathcal{L}}{\partial \alpha\, \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \alpha\, \partial \theta} & \frac{\partial^2 \mathcal{L}}{\partial \theta^2} & \frac{\partial^2 \mathcal{L}}{\partial \theta\, \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \alpha\, \partial \lambda} & \frac{\partial^2 \mathcal{L}}{\partial \theta\, \partial \lambda} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{bmatrix}. \tag{19}$$

By solving the expected information matrix, these solutions will yield the asymptotic variance and covariances of these ML estimators for $\widehat{\alpha}$, $\widehat{\theta}$ and $\hat{\lambda}$. By using (19), approximate $100(1-\gamma)\%$ confidence intervals for α, θ and λ can be determined as

$$\widehat{\alpha} \pm Z_{\frac{\gamma}{2}}\sqrt{\widehat{V}_{11}}, \qquad \hat{\theta} \pm Z_{\frac{\gamma}{2}}\sqrt{\widehat{V}_{22}}, \qquad \hat{\lambda} \pm Z_{\frac{\gamma}{2}}\sqrt{\widehat{V}_{33}}\,,$$

where $Z_{\frac{\gamma}{2}}$ is the upper γth percentile of the standard normal distribution.

5. Entropy

The original definition of entropy was defined by Rényi (1961) for a variable X with the probability density function $f(x)$ continuous on [0,1]. Then, the integrated probability is

$$P_{n,k} = \int_{k/n}^{(k+1)/n} f(x)dx\,, \qquad k = 0,1,\dots,n-1.$$

By defining this as a discrete mass function $P_n = P_{n,k}$, it is possible to show that (see Principe (2009))

$$\lim_{n\to\infty}(I_n(P_n) - \log n) = \frac{1}{1-\rho}\log\left\{\int f(x)^{\rho}dx\right\}.$$

The entropy of a random variable X with density $f(x)$ is a measure of variation of the uncertainty. A large entropy value indicates greater uncertainty in the data. By definition the Rényi entropy is defined as

$$I_R(\rho) = \frac{1}{1-\rho}\log\{\int f(x)^{\rho}dx\}, \tag{20}$$

where $\rho > 0$ and $\rho \neq 1$. The Rényi entropy for the TKw variable X with density $f(x)$ is given by the following theorem.

Theorem 4: If a random variable X has the Tkw distribution, then the Rényi entropy of X, $I_R(\rho)$ is given by

$$I_R(\rho) = \frac{\rho}{1-\rho}\log(\alpha) + \frac{\rho}{1-\rho}\log(\theta) + \frac{\rho}{1-\rho}\log(1+\lambda) + \frac{1}{1-\rho}\log$$

$$\left[\sum_{k,\ell=0}^{\infty}(\alpha\theta)^{\rho}\left(\frac{2\lambda}{1+\lambda}\right)^{k}(1+\lambda)^{\rho}\mathcal{W}_{k,\ell,\rho}\,\beta\left\{\frac{\rho}{\alpha}(\alpha-1)+1, \rho(\theta-1)+\theta\ell+1\right\}\right].$$

Proof: Rényi entropy is defined in equation (20). So, to complete the proof, we first evaluate the integral as

$$\int_0^1 f(x)^{\rho}dx = \int_0^1 \alpha^{\rho}\theta^{\rho}x^{\rho(\alpha-1)}(1-x^{\alpha})^{\rho(\theta-1)}\left\{1+\lambda-2\lambda\left[1-(1-x^{\alpha})^{\theta}\right]\right\}^{\rho}dx$$

$$= \sum_{k,\ell=0}^{\infty}(\alpha\theta)^{\rho}\left(\frac{2\lambda}{1+\lambda}\right)^{k}(1+\lambda)^{\rho}\mathcal{W}_{k,\ell,\rho}\int_0^1 x^{\rho(\alpha-1)}(1-x^{\alpha})^{\rho(\theta-1)+\theta\ell}dx,$$

where

$$\mathcal{W}_{k,\ell,\rho} = \frac{(-1)^{k+\ell}\Gamma(\rho+1)\Gamma(k+1)}{k!\ell!\,\Gamma(\rho+1-k)\Gamma(k+1-\ell)}.$$

The above integral reduces to

$$\int_0^1 f(x)^{\rho}dx = \sum_{k,\ell=0}^{\infty}(\alpha\theta)^{\rho}\left(\frac{2\lambda}{1+\lambda}\right)^{k}(1+\lambda)^{\rho}\mathcal{W}_{k,\ell,\rho}\;\beta\left\{\frac{\rho}{\alpha}(\alpha-1)+1, \rho(\theta-1)+\theta\ell+1\right\}. \quad (21)$$

By using equation (21), equation (20) can be simplified to

$$I_R(\rho) = \frac{\rho}{1-\rho}\log(\alpha) + \frac{\rho}{1-\rho}\log(\theta) + \frac{\rho}{1-\rho}\log(1+\lambda) + \frac{1}{1-\rho}\log$$

$$\left[\sum_{k,\ell=0}^{\infty}(\alpha\theta)^{\rho}\left(\frac{2\lambda}{1+\lambda}\right)^{k}(1+\lambda)^{\rho}\mathcal{W}_{k,\ell,\rho}\,\beta\left\{\frac{\rho}{\alpha}(\alpha-1)+1, \rho(\theta-1)+\theta\ell+1\right\}\right]$$

which completes the proof.

6. Mean deviation

The mean deviation, about the mean and the median, can be used as measures of the degree of scatter in a population. Let $\mu = E(X)$ and M be the mean and the median of the Tkw distribution given by (11) and (15) equations, respectively.

The mean deviation about the mean, and about the median, can be calculated as

$$\delta_1(X) = \mathrm{E}|X - \mu| = \int_0^1 |X - \mu|\, f(x)dx,$$

and

$$\delta_2(X) = \mathrm{E}|X - \mathrm{M}| = \int_0^1 |X - \mathrm{M}|\, f(x)dx,$$

respectively. Hence, we obtain the following equations (Cordeiro et al. (2013))

$$\delta_1 = 2\mu \mathrm{F}(\mu) - 2\psi(\mu) \quad \text{and} \quad \delta_2 = \mu - 2\psi(\mathrm{M}), \tag{22}$$

where $\psi(\mathrm{q})$ can be obtained from (5) by

$$\psi(\mathrm{q}) = (1-\lambda)\int_0^{q^\alpha} \theta z^{\frac{1}{\alpha}}(1-z)^{\theta-1}dz + 2\lambda\int_0^{q^\alpha} \theta z^{\frac{1}{\alpha}}(1-z)^{2\theta-1}\, dz. \tag{23}$$

One can easily compute these integrals numerically in software such as MAPLE, MATLAB, Mathcad, R and others and hence obtain the mean deviation about the mean and about the median, as desired. The mean deviation can be also used to determine the Bonferroni and Lorenz curves which have application in econometrics, finance, insurance and others.

By using equation (23), the Bonferroni curve can be calculated from the following equation

$$\mathrm{B(P)} = \frac{1}{\mathrm{P}\mu}\left\{(1-\lambda)\int_0^{q^\alpha} \theta z^{\frac{1}{\alpha}}(1-z)^{\theta-1}dz + 2\lambda\int_0^{q^\alpha} \theta z^{\frac{1}{\alpha}}(1-z)^{2\theta-1}\, dz\right\},$$

and the Lorenz curve can be calculated from the following equation

$$\mathrm{L(P)} = \frac{1}{\mu}\left\{(1-\lambda)\int_0^{q^\alpha} \theta z^{\frac{1}{\alpha}}(1-z)^{\theta-1}dz + 2\lambda\int_0^{q^\alpha} \theta z^{\frac{1}{\alpha}}(1-z)^{2\theta-1}\, dz\right\}.$$

7. Order statistics

If $X_{1:n}, \dots, X_{n:n}$ denote the order statistics of a random sample $X = (X_1, \dots, X_n)$ from the TKw distribution with cumulative distribution function $\mathrm{F}(x)$, and probability density function $f(x)$, then the 1[st] order and n[th] order probability density functions are given by

$$f_{1:n}(x) = \mathrm{n}\left\{1 - \left[1 - (1 - x^\alpha)^\theta\right]\left[1 + \lambda(1 - x^\alpha)^\theta\right]\right\}^{\mathrm{n}-1}$$

$$\times\, \alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\{1-\lambda+2\lambda(1-x^{\alpha})^{\theta}\}, \tag{24}$$

and

$$f_{n:n}(x) = \mathrm{n}\{[1-(1-x^{\alpha})^{\theta}][1+\lambda(1-x^{\alpha})^{\theta}]\}^{\mathrm{n}-1}$$
$$\times\, \alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\{1-\lambda+2\lambda(1-x^{\alpha})^{\theta}\}. \tag{25}$$

Theorem 5: The probability density function and the k^{th} moment of *r*th order statistic $X_{r:n}$ of the random sample X from the TKw distribution are given by

$$f_{r:n}(x) = \mathrm{n}\binom{\mathrm{n}-1}{\mathrm{r}-1}\sum_{\mathrm{k}=0}^{\mathrm{n-r}}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} w_{k,\ell,m,\lambda}\ \left\{(1-\lambda)\tau^{(1)}_{\alpha,\theta,m,\lambda}+2\lambda\tau^{(2)}_{\alpha,\theta,m,\lambda}\right\}.$$

$$\mu_{\mathrm{k}}^{(\mathrm{r:n})} = \mathrm{n}\binom{\mathrm{n}-1}{\mathrm{r}-1}\sum_{\mathrm{k}=0}^{\mathrm{n-r}}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} w_{k,\ell,m,\lambda}\ \left[(1-\lambda)\theta\beta\left\{\frac{\mathrm{k}}{\alpha}+1,\theta(m+1)\right\}\right.$$
$$\left.+2\lambda\theta\beta\left\{\frac{\mathrm{k}}{\alpha}+1,\theta(m+2)\right\}\right].$$

Proof: From Balakrishnan and Nagaraja (1992), the pdf of *r*th order statistics is given by

$$f_{r:n}(x) = \frac{[\mathrm{F}(x)]^{\mathrm{r}-1}[1-\mathrm{F}(x)]^{\mathrm{n-r}}\mathrm{f}(x)}{\mathrm{B}(\mathrm{r},\mathrm{n}-\mathrm{r}+1)}\,, \tag{26}$$

where $\mathrm{B}(.,.)$ is the beta function. Using (26) the pdf of $x_{(r)}$ is given by

$$f_{r:n}(x) = \frac{1}{\mathrm{B}(\mathrm{r},\mathrm{n}-\mathrm{r}+1)}\sum_{\mathrm{k}=0}^{\mathrm{n-r}}\binom{\mathrm{n}-\mathrm{r}}{\mathrm{k}}(-1)^{\mathrm{k}}\left(\mathrm{F}(x)\right)^{\mathrm{r+k}-1}\mathrm{f}(x). \tag{27}$$

Substituting (5) and (6) in (27) we obtain

$$f_{r:n}(x) = \mathrm{n}\binom{\mathrm{n}-1}{\mathrm{r}-1}\sum_{\mathrm{k}=0}^{\mathrm{n-r}}\binom{\mathrm{n}-\mathrm{r}}{\mathrm{k}}(-1)^{\mathrm{k}}\{[1-(1-x^{\alpha})^{\theta}][1+\lambda(1-x^{\alpha})^{\theta}]\}^{\mathrm{r+k}-1}$$
$$\times\, \alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\{1-\lambda+2\lambda(1-x^{\alpha})^{\theta}\}.$$

The above expression reduces to the pdf of the *r*th order statistic as

$$f_{r:n}(x) = \mathrm{n}\binom{\mathrm{n}-1}{\mathrm{r}-1}\sum_{\mathrm{k}=0}^{\mathrm{n-r}}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} w_{k,\ell,m,\lambda}\ \left\{(1-\lambda)\tau^{(1)}_{\alpha,\theta,m,\lambda}+2\lambda\tau^{(2)}_{\alpha,\theta,m,\lambda}\right\}, \tag{28}$$

where

$$w_{k,\ell,m,\lambda} = \binom{\mathrm{n}-\mathrm{r}}{\mathrm{k}}\binom{\mathrm{r}+\mathrm{k}-1}{\ell}\binom{\mathrm{r}+\mathrm{k}+\ell-1}{m}(-1)^{\mathrm{k}+\ell+m}\left(\frac{\lambda}{1+\lambda}\right)^{\ell}(1+\lambda)^{\mathrm{r+k}-1}$$
$$\tau^{(\mathrm{g})}_{\alpha,\theta,m,\lambda} = \alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta(m+\mathrm{g})-1},\quad g=1,2.$$

Using (28) the k^{th} moment of rth order statistic of $x_{(r)}$ is given by

$$\mu_{k}^{(r:n)} = n\binom{n-1}{r-1}\sum_{k=0}^{n-r}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} w_{k,\ell,m,\lambda}\left[(1-\lambda)\int_{0}^{1}\alpha\theta x^{k+\alpha-1}(1-x^{\alpha})^{\theta(m+1)-1}\,dx\right.$$
$$\left.+2\lambda\int_{0}^{1}\alpha\theta x^{k+\alpha-1}(1-x^{\alpha})^{\theta(m+2)-1}\,dx\right].$$

Therefore, by solving the above integral the k^{th} moment of the rth order statistic of the TKw distribution can be obtained as

$$\mu_{k}^{(r:n)} = n\binom{n-1}{r-1}\sum_{k=0}^{n-r}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} w_{k,\ell,m,\lambda}\left[(1-\lambda)\theta\beta\left\{\frac{k}{\alpha}+1,\theta(m+1)\right\}\right.$$
$$\left.+2\lambda\theta\beta\left\{\frac{k}{\alpha}+1,\theta(m+2)\right\}\right], \tag{29}$$

which completes the proof.

Theorem 6: The probability density function and the k^{th} moment of the median order statistic of a random sample X from the TKw distribution are given by

$$g(\tilde{x}) = \frac{(2m+1)!}{m!\,m!}\sum_{k=0}^{m}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty} w_{k,\ell,m,n,\lambda}\left\{(1-\lambda)\mathcal{E}_{\alpha,\theta,n,\lambda}+2\lambda\mathcal{J}_{\alpha,\theta,n,\lambda}\right\}$$

and

$$\tilde{\mu}_{k}^{(r:n)} = \frac{(2m+1)!}{m!\,m!}\sum_{k=0}^{m}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\sum_{n=0}^{\infty} w_{k,\ell,m,n,\lambda}\left[(1-\lambda)\theta\beta\left\{\frac{k}{\alpha}+1,\theta(n+1)\right\}\right.$$
$$\left.+2\lambda\theta\beta\left\{\frac{k}{\alpha}+1,\theta(n+2)\right\}\right].$$

Proof: Let $X,\ldots,X_n$ be independently and identically distributed ordered random variables from the TKw distribution having median order X_{m+1} probability density function given by

$$g(\tilde{x}) = \frac{(2m+1)!}{m!\,m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^{k}\{F(\tilde{x})\}^{m+k}f(\tilde{x}). \tag{30}$$

Substituting (5) and (6) in (30) we obtain

$$g(\tilde{x}) = \frac{(2m+1)!}{m!\,m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^{k}\left\{\left[1-(1-\tilde{x}^{\alpha})^{\theta}\right]\left[1+\lambda(1-\tilde{x}^{\alpha})^{\theta}\right]\right\}^{m+k}$$
$$\times\,\alpha\theta\tilde{x}^{\alpha-1}\left(1-\tilde{x}^{\alpha}\right)^{\theta-1}\left\{1-\lambda+2\lambda\left(1-\tilde{x}^{\alpha}\right)^{\theta}\right\}.$$

The above expression reduces to the pdf of the median as

$$g(\tilde{x}) = \frac{(2m+1)!}{m!\,m!} \sum_{k=0}^{m} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} w_{k,\ell,m,n,\lambda} \left\{(1-\lambda)\mathcal{J}_{\alpha,\theta,n,\lambda}^{(1)} + 2\lambda \mathcal{J}_{\alpha,\theta,n,\lambda}^{(2)}\right\}, \quad (31)$$

where

$$w_{k,\ell,m,n,\lambda} = \binom{m}{k}\binom{m+k}{\ell}\binom{m+k+\ell}{n}(-1)^{k+\ell+n}\left(\frac{\lambda}{1+\lambda}\right)^{\ell}(1+\lambda)^{m+k}$$

$$\mathcal{J}_{\alpha,\theta,n,\lambda}^{(h)} = \alpha\theta\tilde{x}^{\alpha-1}(1-\tilde{x}^{\alpha})^{\theta(n+h)-1}, \qquad h = 1, 2.$$

Using (31) the k^{th} moment of the rth median $\tilde{x}_{(r)}$ given by

$$\tilde{\mu}_k^{(r:n)} = \frac{(2m+1)!}{m!\,m!} \sum_{k=0}^{m} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} w_{k,\ell,m,n,\lambda} \left[(1-\lambda)\int_0^1 \alpha\theta\tilde{x}^{k+\alpha-1}(1-\tilde{x}^{\alpha})^{\theta(n+1)-1}\,d\tilde{x} \right.$$

$$\left. +2\lambda\int_0^1 \alpha\theta\tilde{x}^{k+\alpha-1}(1-\tilde{x}^{\alpha})^{\theta(n+2)-1}\,d\tilde{x}\right].$$

Therefore, the above integral reduces to the k^{th} moment of the median as follows

$$\tilde{\mu}_k^{(r:n)} = \frac{(2m+1)!}{m!\,m!} \sum_{k=0}^{m} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} w_{k,\ell,m,n,\lambda} \left[(1-\lambda)\theta\beta\left\{\frac{k}{\alpha}+1, \theta(n+1)\right\} \right.$$

$$\left. +2\lambda\theta\beta\left\{\frac{k}{\alpha}+1, \theta(n+2)\right\}\right], \quad (32)$$

which completes the proof.

8. Simulation

This section evaluates the performance of the MLEs for the three parameters α, θ and λ of the TKw distribution by using Monte Carlo simulation. The simulation of the TKw distribution can be performed by using (14). The samples of the TKw distribution were generated for different sizes $n = 25, 50, 75, 100, 200, 300, 400, 500$ for fixed choice of parameters $\alpha = 3, \theta = 3$ and $\lambda = 0.5$. The estimates of the unknown parameters has been obtained by using BFGS method to minimize the total log-likelihood function. The estimated values of the parameters α, θ, λ with their corresponding standard error, bias and mean square error (MSE) are displayed in Table 1. The plot in Figure 6 evaluates the overall performance of the TKw distribution for simulated data sets that show the exact densities and histogram for some selected values of parameters.

Table 1. Mean, standard Error, Bias and MSE of the TKw distribution

n	Parameter	Mean	S.E	Bias	MSE
25	α	3.1119	0.6526	0.1119	0.4384
	θ	3.2504	1.3394	0.2504	1.8566
	λ	0.2605	0.6013	-0.2395	0.4189
50	α	2.9603	1.0269	-0.0397	1.0561
	θ	3.6296	0.8641	0.6296	1.1430
	λ	-0.2737	0.8336	-0.7737	1.2935
75	α	3.2675	0.3535	0.2675	0.1965
	θ	3.1488	0.9566	0.1488	0.9372
	λ	0.6362	0.3023	0.1362	0.1099
100	α	2.6158	0.4246	-0.3842	0.3278
	θ	3.1376	0.5942	0.1376	0.3720
	λ	0.0164	0.5084	-0.4836	0.4923
200	α	2.9212	0.2045	-0.0788	0.0480
	θ	3.0396	0.8866	0.0396	0.7876
	λ	0.4882	0.4256	-0.0118	0.1812
300	α	3.0556	0.1983	0.0556	0.0424
	θ	3.6425	0.6270	0.6425	0.8059
	λ	0.3285	0.3121	-0.1715	0.1268
400	α	2.8835	0.1679	-0.1165	0.0417
	θ	3.0600	0.4560	0.0600	0.2115
	λ	0.3212	0.2736	-0.1788	0.1068
500	α	2.9709	0.1579	-0.0291	0.0257
	θ	3.2659	0.4845	0.2659	0.3054
	λ	0.3331	0.2789	-0.1669	0.1056

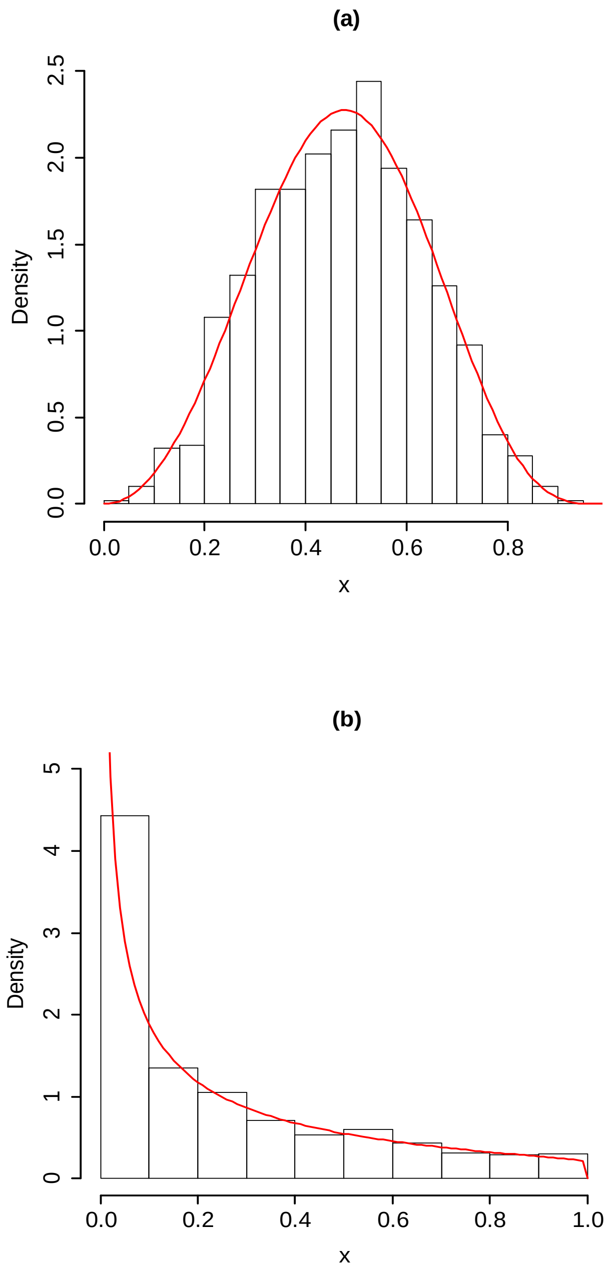


Figure 6. Plots of the TKw densities for simulated data sets:
(a) $\alpha = 3, \theta = 3, \lambda = 1$ and (b) $\alpha = 0.5, \theta = 1, \lambda = 0.5$.

9. Applications

In this section we provide two data analyses in order to assess the goodness-of-fit of the TKw distribution. The first data set is from Dumonceaux and Antle, [32]; with respect to the flood data with 20 observations 0.265, 0.269, 0.297, 0.315, 0.3235, 0.338, 0.379, 0.379, 0.392, 0.402, 0.412, 0.416, 0.418, 0.423, 0.449, 0.484, 0.494, 0.613, 0.654, 0.74. The summary statistics for the TKw distribution are given in Table 2. The MLEs and the values of the maximized log-likelihoods for the TKw and Kw distributions are given in Table 3.

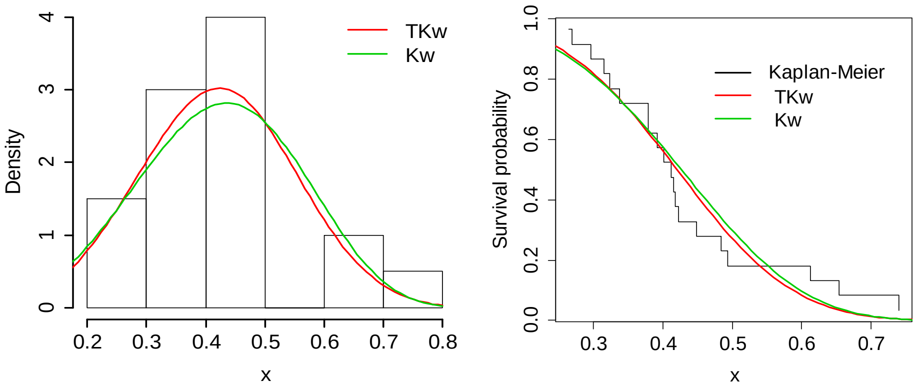
Table 2. Summary Statistics for the TKw and Kw distributions for flood data

Distribution	Median	Coefficient of Quartile Deviation	Bowley's skewness	Moors($\mathcal{M}$) kurtosis
TKw	0.4207	4.7328	-0.0048	1.2212
Kw	0.4268	4.4817	-0.0176	1.2019

In order to compare the distributions, we consider the Kolmogorov-Smirnov (K-S) test, Cramér-von Mises and Anderson-Darling goodness-of-fit statistics for the flood data. Table 3 gives the MLEs of the unknown parameters and the corresponding standard errors of the TKw and Kw distributions. These results indicate that the TKw distribution provides an adequate fit for the flood data.

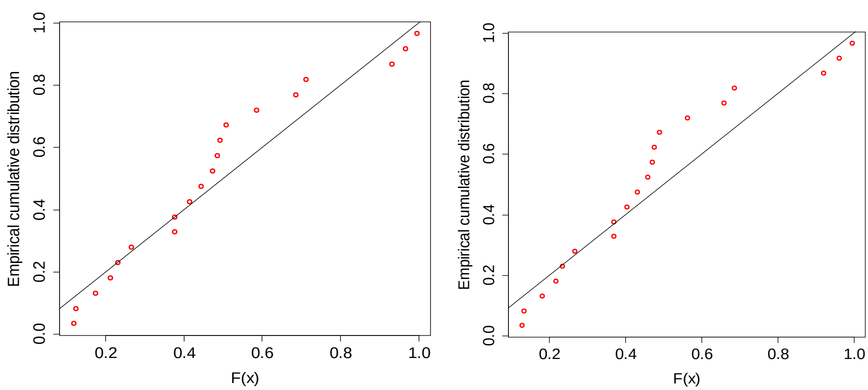
Table 3. MLEs of the unknown Parameters for the flood data with the corresponding standard errors in parenthesis and the goodness-of-fit measures, The K-S test, Cramér-von Mises and Anderson-Darling goodness-of-fit.

Distribution	Parameter Estimates			K-S test	$\mathcal{W}$	$\mathcal{A}$
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$			
TKw	3.7252 (0.6489)	10.9575 (6.0334)	0.6143 (0.3752)	0.1930	0.1409	0.8408
Kw	3.3631 (0.6033)	11.7882 (5.3589)	-	0.2109	0.1658	0.9722



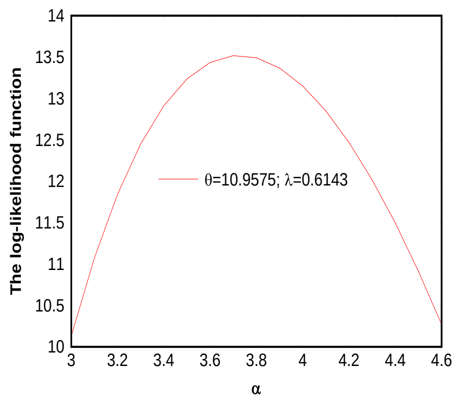
(a) Estimated pdf (b) Estimated Survival function

Figure 7. Estimated densities and survival functions of *TKw* and *Kw* models fitted to flood data

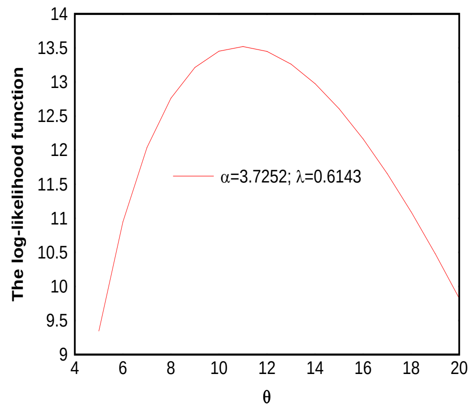


(a) TKw (b) Kw

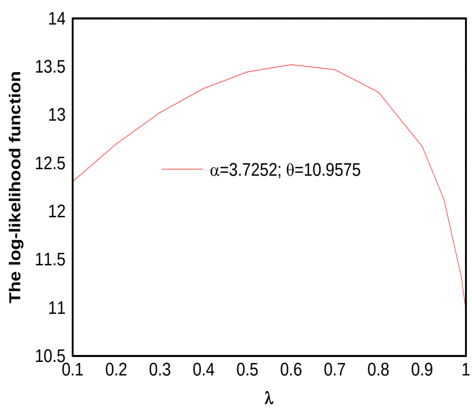
Figure. 8 P-P Plots of the (a) TKw and (b) Kw models fitted to flood data.



(a) $\mathcal{L}$ for α



(b) $\mathcal{L}$ for θ



(c) $\mathcal{L}$ for λ

Figure 9. The profile of the log-likelihood function for α, θ and λ for flood data

Based on the plots of the estimated TKw and Kw densities the relative histogram of the flood data suggests that the fit of the proposed model performs better than the baseline distribution shown in Figure 7(a). Furthermore, the empirical survival function and the fitted survival functions are plotted in Figure 7(b). By comparing the fitted models of these two distribution we have supporting evidence that the TKw distribution provides a good fit for flood data. We have supporting evidence that the Kolmogorov-Smirnov (K-S) distance between the empirical and fitted TKw distribution is smaller than the Kw distribution. Table 3 also indicates that the Cramér-von Mises test statistics and Anderson-Darling goodness statistics have the smallest values for the TKw distribution for the flood data with regard to the Kw distribution. Based on these three goodness-of-fit measures we conclude that the TKw distribution provides a better fit than the Kw lifetime distribution. Figure 8 displays the P-P Plots of the TKw distribution and

Kw distribution. The P-P Plot distance between the ab-line and the fitted model of the TKw distribution and Kw distribution are very similar and close to the baseline model. Figure 9 illustrates the profile of the log-likelihood function for the TKw distribution with parameters α, θ and λ fitted to the flood data and exhibits the unique maximum for these parameters. Based on these results we conclude that the TKw tends to provide a relatively better fit than the Kw distribution for the flood data.

The second data set has been collected from Joint United Nations programme on HIV/ AIDS (UNAIDS), for Infants born to HIV+ women receiving a virological test for HIV within 2 months of birth. The data set consists of 906 observations for the years 2009-13 and is freely available online at http://data.un.org/Data.aspx?d=UNAIDS&f=inID%3a41. The summary statistics for the TKw and Kw distributions for HIV data are given in Table 4.

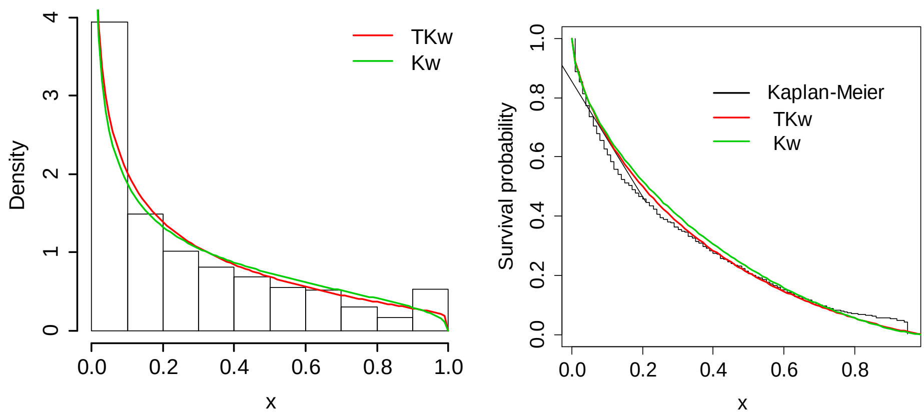
Table 4. Summary Statistics for the TKw and Kw distributions for HIV data

Distribution	Median	Coefficient of Quartile Deviation	Bowley's skewness	Moors($\mathcal{M}$) kurtosis
TKw	0.2002	1.3272	0.2754	1.1491
Kw	0.2146	1.3137	0.2529	1.0884

The MLEs and the values of the maximized log-likelihoods for the TKw and Kw distributions for HIV data are given in Table 5, with the MLEs of the unknown parameters and the corresponding standard errors of the TKw and Kw distributions. The standard error estimates obtained using the observed information matrix appear to be smaller than the parameter estimates. In order to compare the distributions, we consider the AIC (Akaike Information Criterion) for the HIV/ AIDS data. These results indicate that the TKw distribution provides better fit for the HIV/ AIDS data. Furthermore, we applied the LR statistics in order to verify which model fits better for the HIV data. The hypotheses to be tested are $H_0\colon \omega = (\alpha, \theta, \lambda)^{\mathrm{T}}$ versus H_A: H_0 *is not true*, and the LR statistics reduces to $\Lambda = 2\{l(\widehat{\omega}) - l(\widetilde{\omega})\}$=52.8088, where $\widehat{\omega}$ is the MLE of ω under H_0. The null hypothesis is rejected as the p-value=1.7048E-12 < $\alpha = 0.05$.

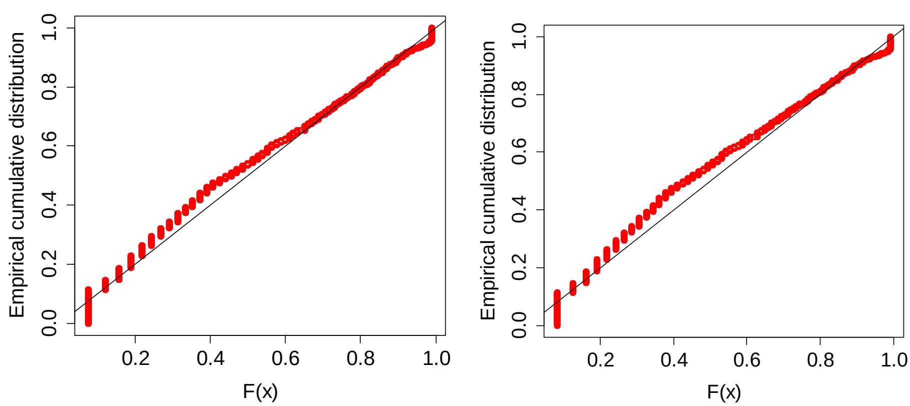
Table 5. MLEs of the unknown Parameters for the HIV/ AIDS data with the corresponding standard errors in parenthesis and the goodness-of-fit measure AIC

Distribution	Parameter Estimates			AIC
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	
TKw	0.6724 (0.0251)	1.1222 (0.0837)	0.5491 (0.0882)	-633.605
Kw	0.6096 (0.0237)	1.3961 (0.0629)	-	-607.201



(a) Estimated pdf (b) Estimated Survival function

Figure 10. Estimated densities and survival functions of TKw and Kw models fitted to HIV data



(a) TKw (b) Kw

Figure 11. P-P Plots of the (a) TKw and (b) Kw models fitted to HIV data

Finally, in order to assess whether the proposed model is appropriate for HIV data we display the visualization of the estimated TKw and Kw densities and the relative histogram for the HIV/ AIDS data suggests that the fit of the proposed model performs better than the baseline distribution shown in Figure 10(a). Furthermore, the plots of the empirical survival function and the fitted survival functions are shown in Figure 10(b). Both figures suggest that the TKw

distribution provides a good fit for HIV/ AIDS data. Moreover, the graphical comparison of the PP-Plots corresponding to these fits confirms our claim as demonstrated in Figure 11.

10. Concluding remarks

In this paper we have presented a new generalization of the Kw distribution, called the TKw distribution. This generalization is obtained by transforming the two parameter Kw model through the quadratic rank transmuted map technique. The properties of the proposed distribution are discussed. We obtain the analytical shapes of the density and hazard functions of the TKw distribution. We also consider mean deviations, Bonferroni and Lorenz curves and Rényi entropy. Maximum likelihood estimation is discussed within the framework of asymptotic log-likelihood inferences including confidence intervals. The three parameter TKw distribution produced monotonically increasing and decreasing hazard rates. In terms of the statistical significance of the model adequacy, the TKw distribution leads to a better fit than the Kw distribution.

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REFERENCES

ASHOUR, S. K., ELTEHIWY, M. A., (2013). Transmuted Lomax Distribution, American Journal of Applied Mathematics and Statistics, 1(6), pp. 121−127.

AHMAD, A., AHMAD, S. P., AHMED, A., (2015). Characterization and estimation of transmuted Kumaraswamy distribution, Mathematical Theory and Modeling, Vol. 5, No. 9, 168−174.

ARYAL, G. R., TSOKOS, C. P., (2011). Transmuted Weibull distribution: A Generalization of the Weibull Probability Distribution. European Journal of Pure and Applied Mathematics, Vol. 4, No. 2, 89−102.

ARYAL, G. R., TSOKOS. C. P., (2009). On the transmuted extreme value distribution with applications. Nonlinear Analysis: Theory, Methods and applications, Vol. 71, 1401−1407.

ARYAL, G. R., (2013). Transmuted Log-Logistic Distribution, J. Stat. Appl. Pro. 2, No. 1, 11−20.

BALAKRISHNAN, A. N., NAGARAJA, H. N., (1992). A first course in order statistics. New York: Wiley-Interscience.

CORDEIRO, G. M., ORTEGA, E. M. M., NADARAJAH, S., (2010). The Kumaraswamy Weibull distribution with application to failure data. J Frankl Inst 347, 1399–1429.

CORDEIRO, G. M., NADARAJAH, S., ORTEGA, E. M. M., (2012). The Kumaraswamy Gumbel distribution, Stat Methods Appl 21: 139–168.

CORDEIRO G. M., GOMES, A. E., QUEIROZ DA-SILVA, C., ORTEGA, E. M. M., (2013). The beta exponentiated Weibull distribution, Journal of Statistical Computation and Simulation. 83, 1, 114–138.

CORDEIRO, G. M., DE CASTRO M., (2009). A new family of generalized distributions, Journal of Statistical Computation & Simulation, Vol. 00, No. 00, August, 1–17.

DUMONCEAUX, R., ANTLE, C. E., (1973). Discrimination between the log-normal and the Weibull distributions. Technometrics 15 (4), 923–926.

ELBATHAL, I., ELGARHY, M., (2013). Transmuted Quasi Lindley Distributions: A generalization of the Quasi Lindley Distribution. International Journal of Pure and Applied Sciences and Technology , 18(2), pp. 59–69.

FLETCHER, S. C., PONNAMBALAM, K., (1996). Estimation of reservoir yield and storage distribution using moments analysis. J Hydrol 182, 259–275.

GANJI, A., PONNAMBALAM, K., KHALILI, D., (2006). Grain yield reliability analysis with crop water demand uncertainty. Stoch Environ Res Risk Assess 20, 259–277.

JONES, M. C., (2009). Kumaraswamy's distribution: a beta-type distribution with some tractability advantages. Stat Methodol 6, 70–91.

KENNEY, J. F., KEEPING, E. S., (1962). Mathematics of Statistics. Princeton, NJ.

KUMARASWAMY, P., (1980). Generalized probability density-function for double-bounded random-processes. J Hydrol 46, 79–88.

KHAN, M. S., KING, R., (2013a). Transmuted Modified Weibull Distribution: A Generalization of the Modified Weibull Probability Distribution, European Journal of Pure And Applied Mathematics, Vol. 6, No. 1, 66–88.

KHAN, M. S., KING, R., (2013b). Transmuted generalized Inverse Weibull distribution, Journal of Applied Statistical Sciences, Vol. 20, No. 3, 15–32.

KHAN, M. S., KING, R., HUDSON, I., (2013c). Transmuted generalized Exponential distribution, 57th Annual Meeting of the Australian Mathematical Society, Australia.

KHAN, M., SHUAIB, KING, R., HUDSON, I., (2014). Characterizations of the transmuted Inverse Weibull distribution, ANZIAM J., Vol. 55 (EMAC2013 at the Queensland University of Technology from 1st – 4th December 2013), C197–C217.

KOUTSOYIANNIS, D, XANTHOPOULOS, T., (1989). On the parametric approach to unit hydrograph identification. Water Resour Manag 3, 107–128.

MOORS, J. A., (1998). A quantile alternative for kurtosis. Journal of the Royal Statistical Society, D, 37, 25–32.

MEROVCI, F., (2013a). Transmuted Rayleigh distribution. Austrian Journal of Statistics, Vol. 42, No. 1, 21–31.

MEROVCI, F., (2013b). Trasmuted Lindley distribution. Int. J. Open Problems Compt. Math, 6, 63–72.

MEROVCI, F., (2014). Transmuted generalized Rayleigh distribution, J. Stat. Appl. Pro. 3, No. 1, 9–20.

NADARAJAH, S., (2008). On the distribution of Kumaraswamy. J Hydrol 348, 568–569.

PRINCIPE, J. C., (2009). http://www.cnel.ufl.edu/courses/EEL6814/renyis_entropy.pdf.

PONNAMBALAM, K, SEIFI, A, VLACH, J., (2001). Probabilistic design of systems with general distributions of parameters. Int J Circuit Theory Appl 29, 527–536.

RÉNYI, A., (1961). On measures of entropy and information. University of California Press, Berkeley, California, 547–561.

SUNDAR, V., SUBBIAH, K., (1989). Application of double bounded probability density-function for analysis of ocean waves. Ocean Eng 16, 193–200.

SEIFI, A., PONNAMBALAM, K., VLACH, J., (2000). Maximization of manufacturing yield of systems with arbitrary distributions of component values. Ann Oper Res 99, 373–383.

SHAW, W. T., BUCKLEY, I. R. C., (2009). The alchemy of probability distributions: beyond Gram–Charlier expansions, and a skew-kurtotic normal distribution from a rank transmutation map. Technical report, http://arxiv.org/abs/0901.0434.