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ESTIMATION OF THE CENTRAL MOMENTS OF A RANDOM VECTOR BASED ON THE DEFINITION OF THE POWER OF A VECTOR

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ABSTRACT

The moments of a random vector based on the definition of the power of a vector, proposed by J. Tatar, are scalar and vector characteristics of a multivariate distribution. Analogously to the univariate case, we distinguish the uncorrected and the central moments of a random vector. Other characteristics of a multivariate distribution, i.e. an index of skewness and kurtosis, have been introduced by using the central moments of a random vector. For the application of the mentioned quantities for the analysis of multivariate empirical data, it appears desirable to construct their respective estimators.
This paper presents the consistent estimators of the central moments of a random vector, for which essential characteristics have been found, such as a mean vector and a mean squared error. In these formulas, the relevant orders of approximation have been taken into account.

Key words: central moment of a random vector, estimator, multivariate distribution, power of a vector.

1. Introduction

One of the fundamental characteristics of the univariate random variable is the ordinary (raw, uncorrected) and the central moments (e.g. Shao, 2003, Jakubowski and Sztencel, 2004). Even order moments are measures of dispersion of the distribution of the random variable, the moments of odd order characterize their location. In the analysis of multivariate distributions the product moments (about zero), the central mixed moments or collections thereof, e.g. mean vector, covariance matrix, are considered as classical generalizations of the above quantities (e.g. Johnson, Kotz and Kemp, 1992; Fujikoshi, Ulyanov and Shimizu, 2010). The uncorrected and central moments of a random vector are also considered as the expectations of relevant Kronecker products of a random vector (e.g. Holmquist, 1988). Thus, from this definition, they are size vector quantities

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(collections of product moments (about zero) and central mixed moments of the corresponding orders).

On the basis of the definition of the power of a vector, Tatar (1996, 1999) suggested multivariate generalizations of the uncorrected and the central moments of a random variable, which are different from the above. Let us recall the basic definitions.

Definition 1.1. [Tatar 1996, 1999] Let $(H, R, +, \cdot)$ be a Hilbert space with the inner product $\langle \cdot,\cdot \rangle$. For any vector $v \in H$ and for any number $r \in N_{\circ} = N \cup \{0\}$ the r-th power of the vector v is defined as follows:

$$v^o = 1 \in R \quad \text{and} \quad v^r = \begin{cases} v^{r-1} \cdot v & \text{for } r - \text{odd} \\ \langle v^{r-1}, v \rangle & \text{for } r - \text{even} \end{cases}.$$

Let $L_k^r(\Omega)$ be a space of random vectors whose absolute value raised to the r-th power has finite integral, that is:

$$L_k^r(\Omega) = \left\{ \mathbf{X} : \Omega -> R^k \ \mathbf{:} \ \int_{\Omega} \|\mathbf{X}\|^r \, dP < +\infty \right\}.$$

In the literature, the measure $E\left[\|\mathbf{X}\|^r\right] = \int_{\Omega} \|\mathbf{X}\|^r \, dP$ is sometimes called the moment of the order r of a random vector $\mathbf{X}$ and designated as $E\left[\mathbf{X}^r\right]$ (see Bilodeau and Brenner, 1999). Tatar (2002), however, by analogy with the univariate case, defines this expression as the absolute moment of order r of a random vector $\mathbf{X}$. In this study, we will also regard these values as the absolute moments of a random vector.

Therefore, let us assume that for the vector $\mathbf{X} : \Omega \to R^k$ an absolute moment of order r exists.

Definition 1.2. [Tatar 1996, 1999] The ordinary (raw, uncorrected) moment of order r of the random vector $\mathbf{X} : \Omega \to R^k$ is defined as

$$\alpha_{r,k} = E\left[\mathbf{X}^r\right]. \tag{1.1}$$

Let us note that the uncorrected moment of the first order is the mean vector, that is:

$$\alpha_{1,k} = m = E\mathbf{X}.$$

The definition and basic properties of the central moments of a random vector based on the definition of the power of a vector will be presented in the next section.

2. The central moments of a random vector

Definition 2.1. [Tatar 1996, 1999] The central moment of order r of a random vector $\mathbf{X}:\Omega \to R^k$, for which the absolute moment of order r exists, is given as

$$\mu_{r,k} = E\left[(\mathbf{X} - E\mathbf{X})^r\right] . \qquad (2.1)$$

Remark 2.1. According to the concept of the power of the vector, if r is an even number, then $\alpha_{r,k}, \mu_{r,k} \in R$, which means they are scalar quantities. However, if r is an odd number, then $\alpha_{r,k}, \mu_{r,k} \in R^k$, so we obtain vectors.

Ordinary and central moments of a random vector based on the definition of the power of the vector are defined at an arbitrarily fixed inner product in the R^k space. In the following part of the paper we will consider the Hilbert space R^k where we define the Euclidean inner product $\langle v, w\rangle = \sum_{i=1}^{k} v_i w_i$ where $v = (v_1,...,v_k),\ w = (w_1,...,w_k) \in R^k$.

Remark 2.2. Let us observe that from the basic properties of the power of a vector and multinomial theorem we get the following formulas:

$$\mu_{2l,k} = E\left[\left(\sum_{i=1}^{k}(X_i - EX_i)^2\right)^l\right] = \sum_{l_1+..+l_m=l}\binom{l}{l_1,..,l_m}E\left[\prod_{t=1}^{k}(X_t - EX_t)^{2l_t}\right]$$

and

$$\mu_{2l+1,k} = E\left[\left(\sum_{i=1}^{k}(X_i - EX_i)^2\right)^l \cdot (X_1,...,X_k)\right] =$$

$$\left(\sum_{l_1+..+l_m=l}\binom{l}{l_1,..,l_m}E\left[\prod_{t=1}^{k}(X_t - EX_t)^{2l_t}(X_1 - EX_1)\right],...,\sum_{l_1+..+l_m=l}\binom{l}{l_1,..,l_m}E\left[\prod_{t=1}^{k}(X_t - EX_t)^{2l_t}(X_k - EX_k)\right]\right)$$

where $\binom{l}{l_1,..,l_m} = \frac{l!}{l_1!\cdot...\cdot l_m!}$ is a multinomial coefficient.

By this inner product, the second order central moment is called the total variance of the random vector (see Bilodeau and Brenner, 1999) or the variance of the random vector (see Tatar 1996, 1999). According to these terms, $D^2\mathbf{X}$ will denote the central moment of the second order of a random vector.

Let us recall that the variance of a random vector was used to present multivariate generalization of Chebyshev's inequality (see Osiewalski and Tatar, 1999).

By using the central moments, characteristics of a multivariate distribution such as index of skewness and kurtosis have also been introduced.

Definition 2.2. [Tatar, 2000] The index of skewness of a random vector $\mathbf{X}:\Omega\to R^k$, for which there is an absolute moment of third order, is called

$$\gamma_{1,k}(\mathbf{X})=\frac{\mu_{3,k}}{(\mu_{2,k})^{\frac{3}{2}}}=\frac{E\left[(\mathbf{X}-E\mathbf{X})^3\right]}{(D^2\mathbf{X})^{\frac{3}{2}}}. \qquad (2.2)$$

Definition 2.3. [Budny and Tatar, 2009, Budny, 2009] Kurtosis of a random vector
$\mathbf{X}:\Omega\to R^k$, for which an absolute moment of fourth order exists, is a quantity defined as

$$\beta_{2,k}(\mathbf{X})=\frac{\mu_{4,k}}{\mu_{2,k}^2}=\frac{E\left[(\mathbf{X}-E\mathbf{X})^4\right]}{(D^2\mathbf{X})^2}. \qquad (2.3)$$

Assuming further that the random vector $\mathbf{N}:\Omega\to R^k$ has a multivariate normal distribution, we obtain the form

$$\beta_{2,k}(\mathbf{N})=1+\frac{2\sum_{i=1}^{k}\sigma_i^4+2\sum_{\substack{i,j=1\\ i\neq j}}^{k}\rho_{ij}^2\sigma_i^2\sigma_j^2}{\sum_{i,j=1}^{k}\sigma_i^2\sigma_j^2}. \qquad (2.4)$$

The formula (2.4) was determined (Budny, 2012) using Isserlis theorem (Isserlis, 1919), setting the algorithm for determination of the central mixed moments of the normally distributed random vector.

For the application of the central moments for the analysis of multivariate, empirical data, it appears desirable to construct their respective estimators. The next section will present their form along with a discussion of basic properties.

3. The multivariate sample central moments

3.1. Construction and basic properties

At the beginning let us recall the form of multivariate sample raw moments with their basic properties useful in the next part of this paper (Budny, 2014).

Suppose that $\mathbf{X}^1:\Omega\to R^k,\ldots,\mathbf{X}^n:\Omega\to R^k$ is a random sample from multivariate distribution, i.e. a set of n independent, identically distributed (i.i.d.) random vectors, with a finite r – th absolute moment.

Definition 3.1.1. The multivariate sample raw moment of order r (the estimator of the $r-\text{th}$ raw moment of a random vector) is called:

$$a_{r,k}=\frac{\sum_{i=1}^{n}\left(\mathbf{X}^{i}\right)^{r}}{n}. \qquad (3.1.1)$$

Multivariate sample raw moments are consistent and unbiased estimators, and their central moments satisfy the condition

$$E\left[\left(a_{r,k}-\alpha_{r,k}\right)^{2s}\right]=O\left(n^{-s}\right), \qquad (3.1.2)$$

for all $r,s\in N$.

Let us therefore proceed to formulate the forms of estimators of the central moments of a random vector.

Definition 3.1.2. The multivariate sample central moment of order r (the estimator of the $r-\text{th}$ central moment of a random vector) is defined as

$$m_{r,k}=\frac{\sum_{i=1}^{n}\left(\mathbf{X}^{i}-\overline{\mathbf{X}}\right)^{r}}{n}. \qquad (3.1.3)$$

Remark 3.1.1. According to the definition of the power of a vector: if r is an even number, then $m_{r,k}$ is a univariate random variable, and if r is an odd number, then a random vector is obtained as $m_{r,k}$.

Remark 3.2.1. In the following discussion, while examining the properties of multivariate sample central moments, we will assume, without loss of generality, that the mean vector is a zero vector, i.e. $\alpha_{1,k}=m=0$.

We will begin the analysis of the properties of estimators of the central moments of a random vector by determining the form of their expected values. To do this, we will first introduce some forms of multivariate sample central moments, useful in further considerations.

Theorem 3.1.1. Multivariate sample central moments can be represented as follows:

- estimator of central moments of even order:

$$m_{2s,k}=a_{2s,k}+\sum_{p=1}^{s}\binom{s}{p}a_{2s-2p,k}\overline{\mathbf{X}}^{2p}-2\sum_{p=1}^{s}p\binom{s}{p}\left\langle a_{2s-(2p-1),k},\overline{\mathbf{X}}^{2p-1}\right\rangle+$$

$$+\frac{\sum_{i=1}^{n}\sum_{p=2}^{s}\sum_{l=0}^{p-2}\binom{s}{p}\binom{p}{l}(-1)^{p-l}2^{p-l}\left(\mathbf{X}^{i}\right)^{2(s-p)}\overline{\mathbf{X}}^{2l}\left\langle\mathbf{X}^{i},\overline{\mathbf{X}}\right\rangle^{p-l}}{n}, \qquad (3.1.4)$$

- estimator of central moments of odd order:

$$m_{2s+1,k} =$$
$$= a_{2s+1,k} + \sum_{p=1}^{s} \binom{s}{p} \overline{\mathbf{X}}^{2p} \cdot a_{2s+1-2p,k} - \frac{2\sum_{i=1}^{n}\sum_{p=1}^{s} p\binom{s}{p} \left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle \overline{\mathbf{X}}^{2p-2} \cdot \left(\mathbf{X}^i\right)^{2s+1-2p}}{n} +$$
$$+ \frac{\sum_{i=1}^{n}\sum_{p=2}^{s}\sum_{l=0}^{p-2} \binom{s}{p}\binom{p}{l} (-1)^{p-l} 2^{p-l} \overline{\mathbf{X}}^{2l} \left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle^{p-l} \cdot \left(\mathbf{X}^i\right)^{2s+1-2p}}{n} - m_{2s,k} \cdot \overline{\mathbf{X}} . \quad (3.1.5)$$

Proof: It is easily seen that from the definition of the power of a vector, we get above formulas.

The computation leading to explicit forms of the expected value of multivariate sample central moments is tedious and does not bring any relevant elements for further consideration. So, the next theorem will present their form with appropriate order of approximation (for univariate case - see Cramer 1958, p. 336). Prior to the formulation of this result, let us consider the following lemma.

Lemma 3.1.1. Assume that $r \in N \setminus \{0\}$. Let us consider a multivariate distribution (in population) for which the absolute moment of order $2r$ exists. Then, for every $t \in \{1,...r-1\}$ we get

$$E\left[a_{r-t,k} \circ \overline{\mathbf{X}}^t\right] = O\left(n^{-\frac{t}{2}}\right), \quad (3.1.6)$$

where the operator "$\circ$" is defined (Tatar, 2008) as follows:

$$v^k \circ w^l = \begin{cases} v^k w^l & \text{for } k,l \text{ - even} \\ v^k \cdot w^l & \text{for } k - \text{even, } l - \text{odd} \\ w^l \cdot v^k & \text{gdy } k - \text{odd, } l - \text{even} \\ \left\langle v^k, w^l \right\rangle & \text{gdy } k,l \text{ - odd} \end{cases} .$$

Proof: see Appendix.

Property $(3.1.6)$ will play a key role in the study of property of unbiasedness of multivariate sample central moments. It will be used in the proof of the theorem, which presents forms of their expected values with the relevant order of approximation.

Theorem 3.1.2. Under the assumptions of Lemma 3.1.1.

$$E\left[m_{r,k}\right] = \mu_{r,k} + O\left(n^{-1}\right). \quad (3.1.7)$$

Proof: First, let us consider multivariate sample central moments of even orders. Towards $(3.1.4)$ we get:

$$E\left[m_{2s,k}\right]=E\left[a_{2s,k}\right]+\sum_{p=1}^{s}\binom{s}{p}E\left[a_{2s-2p,k}\overline{\mathbf{X}}^{2p}\right]-2sE\left[\left\langle a_{2s-1,k},\overline{\mathbf{X}}\right\rangle\right]+$$
$$-2\sum_{p=2}^{s}p\binom{s}{p}E\left[\left\langle a_{2s-(2p-1),k},\overline{\mathbf{X}}^{2p-1}\right\rangle\right]+$$
$$+\frac{\sum_{i=1}^{n}\sum_{p=2}^{s}\sum_{l=0}^{p-2}\binom{s}{p}\binom{p}{l}(-1)^{p-l}2^{p-l}E\left[\left(\mathbf{X}^{i}\right)^{2(s-p)}\overline{\mathbf{X}}^{2l}\left\langle\mathbf{X}^{i},\overline{\mathbf{X}}\right\rangle^{p-l}\right]}{n}. \qquad (3.1.8)$$

By the assumption $\alpha_{1,k}=m=0$, an easy computation shows that

$$E\left[\left\langle a_{2s-1,k},\overline{\mathbf{X}}\right\rangle\right]=\frac{\mu_{2s,k}}{n}, \qquad (3.1.9)$$

$$E\left[a_{2s,k}\cdot\overline{\mathbf{X}}\right]=\frac{\mu_{2s+1,k}}{n}. \qquad (3.1.10)$$

Note that for each $i\in\{1,...,n\}$ and $t\in N$ we obtain the following property

$$E\left[\left|\left\langle\mathbf{X}^{i},\overline{\mathbf{X}}\right\rangle\right|^{t}\right]=O\left(n^{-\frac{t}{2}}\right). \qquad (3.1.11)$$

Indeed, due to Minkowski's inequality in the $L_1^t(\Omega)$, Schwarz's inequality in $L_1^2(\Omega)$ and property $(3.1.2)$, applied to the coordinates of univariate sample mean (Cramer, 1958, p. 332), we get:

$$E\left[\left|\left\langle\mathbf{X}^{i},\overline{\mathbf{X}}\right\rangle\right|^{t}\right]=\left(\left(E\left[\left|\sum_{j=1}^{k}X_j^i\overline{X}_j\right|^{t}\right]\right)^{\frac{1}{t}}\right)^{t}\le\left(\sum_{j=1}^{k}\left(E\left[\left|X_j^i\overline{X}_j\right|^{t}\right]\right)^{\frac{1}{t}}\right)^{t}\le$$

$$\le\left(\sum_{j=1}^{k}\left(\left(E\left[\left(X_j^i\right)^{2t}\right]\right)^{\frac{1}{2}}\left(E\left[\left(\overline{X}_j\right)^{2t}\right]\right)^{\frac{1}{2}}\right)^{\frac{1}{t}}\right)^{t}\le\left(\sum_{j=1}^{k}\left(\sqrt{E\left[\left(X_j^i\right)^{2t}\right]}\left(\frac{A}{n^{t}}\right)^{\frac{1}{2}}\right)^{\frac{1}{t}}\right)^{t}=$$

$$=\left(\sum_{j=1}^{k}\left(\sqrt{E\left[\left(X_j^i\right)^{2t}\right]A}\right)^{\frac{1}{t}}\cdot n^{-\frac{1}{2}}\right)^{t}=\frac{C}{n^{\frac{t}{2}}},$$

for each $n\ge n_0$ where $A,C>0$.

Let us also note that by Jensen's inequality and Hölder's inequality, after taking into account properties $(3.1.2)$ and $(3.1.8)$ we have:

$$\left|E\left[\left(\mathbf{X}^i\right)^{2(s-p)}\overline{\mathbf{X}}^{2l}\left\langle\mathbf{X}^i,\overline{\mathbf{X}}\right\rangle^{p-l}\right]\right|\le E\left[\left(\mathbf{X}^i\right)^{2(s-p)}\overline{\mathbf{X}}^{2l}\left|\left\langle\mathbf{X}^i,\overline{\mathbf{X}}\right\rangle\right|^{p-l}\right]\le$$

$$\le\left(E\left[\left(\left(\mathbf{X}^i\right)^2\right)^s\right]\right)^{\frac{s-p}{s}}\left(E\left[\left(\overline{\mathbf{X}}^2\right)^s\right]\right)^{\frac{l}{s}}\left(E\left[\left|\left\langle\mathbf{X}^i,\overline{\mathbf{X}}\right\rangle\right|^s\right]\right)^{\frac{p-l}{s}}\le\frac{C}{n^{\frac{p+l}{2}}}$$

for each $i\in\{1,...,n\}$, $p\in\{2,...,s\}$, $l\in\{0,...,p-2\}$ and $n\ge n_0$ where $C>0$.

Clearly, this leads to the conclusion that

$$\frac{\sum_{i=1}^{n}\sum_{p=2}^{s}\sum_{l=0}^{p-2}\binom{s}{p}\binom{p}{l}(-1)^{p-l}2^{p-l}E\left[\left(\mathbf{X}^i\right)^{2(s-p)}\overline{\mathbf{X}}^{2l}\left\langle\mathbf{X}^i,\overline{\mathbf{X}}\right\rangle^{p-l}\right]}{n}=O\left(n^{-1}\right).\qquad(3.1.12)$$

Finally, the use of the properties in the following order: unbiasedness of multivariate sample raw moments, $(3.1.6)$, $(3.1.9)$ and $(3.1.2)$, to the equation $(3.1.8)$ implies

$$E\left[m_{2s,k}\right]=\mu_{2s,k}+O\left(n^{-1}\right).\qquad(3.1.13)$$

In order to determine the form of the expected value of the multivariate sample central moments of odd orders, let us note that

$$E\left[m_{2s+1,k}\right]=E\left[a_{2s+1,k}\right]+\sum_{p=1}^{s}\binom{s}{p}E\left[\overline{\mathbf{X}}^{2p}\cdot a_{2s+1-2p,k}\right]+$$
$$-\frac{2\sum_{i=1}^{n}\sum_{p=1}^{s}p\binom{s}{p}E\left[\left\langle\mathbf{X}^i,\overline{\mathbf{X}}\right\rangle\overline{\mathbf{X}}^{2p-2}\cdot\left(\mathbf{X}^i\right)^{2s+1-2p}\right]}{n}+$$
$$+\frac{\sum_{i=1}^{n}\sum_{p=2}^{s}\sum_{l=0}^{p-2}\binom{s}{p}\binom{p}{l}(-1)^{p-l}2^{p-l}E\left[\overline{\mathbf{X}}^{2l}\left\langle\mathbf{X}^i,\overline{\mathbf{X}}\right\rangle^{p-l}\cdot\left(\mathbf{X}^i\right)^{2s+1-2p}\right]}{n}-E\left[m_{2s,k}\cdot\overline{\mathbf{X}}\right].$$
$$(3.1.14)$$

At the beginning we shall show that for each $i\in\{1,...,n\}$, $p\in\{2,...,s\}$ and $l\in\{0,...,p-2\}$ there is a property

$$E\left[\overline{\mathbf{X}}^{2l}\left\langle\mathbf{X}^i,\overline{\mathbf{X}}\right\rangle^{p-l}\cdot\left(\mathbf{X}^i\right)^{2s+1-2p}\right]=O\left(n^{-\left(\frac{p+l}{2}\right)}\right).\qquad(3.1.15)$$

Indeed, Jensen's and Hölder's inequalities, the properties $(3.1.2)$ and $(3.1.11)$ imply

$$\left\| E\left[\overline{\mathbf{X}}^{2l} \left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle^{p-l} \cdot \left(\mathbf{X}^i \right)^{2s+1-2p} \right] \right\|^2 \le E\left[\overline{\mathbf{X}}^{4l} \left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle^{2(p-l)} \left(\mathbf{X}^i \right)^{4s+2-4p} \right] \le$$

$$\le \left(E\left[\left(\overline{\mathbf{X}}^2 \right)^{2s-p+l+1} \right] \right)^{\frac{2l}{2s-p+l+1}} \left(E\left[\left(\left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle^2 \right)^{2s-p+l+1} \right] \right)^{\frac{p-l}{2s-p+l+1}} \left(\mu_{4s-2p+2l+2,k} \right)^{\frac{2s+1-2p}{2s-p+l+1}} \le$$

$$\le \left(\mu_{4s-2p+2l+2,k} \right)^{\frac{2s+1-2p}{2s-p+l+1}} \left(\frac{A_1}{n^{2s-p+l+1}} \right)^{\frac{2l}{2s-p+l+1}} \left(\frac{A_2}{n^{2s-p+l+1}} \right)^{\frac{p-l}{2s-p+l+1}} = \frac{C}{n^{p+l}},$$

for each $n \ge n_0$ where $A_1, A_2, C > 0$, which obviously is equivalent to $(3.1.15)$.

This implies, therefore, a property

$$\frac{\sum_{i=1}^{n} \sum_{p=2}^{s} \sum_{l=0}^{p-2} \binom{s}{p} \binom{p}{l} (-1)^{p-l} 2^{p-l} E\left[\overline{\mathbf{X}}^{2l} \left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle^{p-l} \cdot \left(\mathbf{X}^i \right)^{2s+1-2p} \right]}{n} = O\left(n^{-1}\right). \tag{3.1.16}$$

Note that the reasoning analogous to the one carried out above leads to another property expressed as

$$E\left[\left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle \overline{\mathbf{X}}^{2p-2} \cdot \left(\mathbf{X}^i \right)^{2s+1-2p} \right] = O\left(n^{-\left(p-\frac{1}{2}\right)} \right), \tag{3.1.17}$$

for each $i \in \{1,...,n\}$ and $p \in \{1,...,s\}$.

Furthermore, the elementary computation establishes the equality

$$E\left[\left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle \cdot \left(\mathbf{X}^i \right)^{2s-1} \right] = \frac{\mu_{2s+1,n}}{n}, \tag{3.1.18}$$

for every $i \in \{1,...,n\}$.

Owing to the conditions $(3.1.17)$ and $(3.1.18.)$ we get the property

$$\frac{\sum_{i=1}^{n} \sum_{p=1}^{s} p \binom{s}{p} E\left[\left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle \overline{\mathbf{X}}^{2p-2} \cdot \left(\mathbf{X}^i \right)^{2s+1-2p} \right]}{n} = O\left(n^{-1}\right). \tag{3.1.19}$$

The proof of the theorem will be completed by determining the order of approximation of a quantity $E\left[m_{2s,k}\cdot\overline{\mathbf{X}}\right]$, that takes the form

$$E\left[m_{2s,k}\cdot\overline{\mathbf{X}}\right]=E\left[a_{2s,k}\cdot\overline{\mathbf{X}}\right]+\frac{\sum_{i=1}^{n}\sum_{p=1}^{s}\sum_{l=0}^{p}\binom{s}{p}\binom{p}{l}(-1)^{p-l}2^{p-l}E\left[\left(\mathbf{X}^{i}\right)^{2(s-p)}\left\langle\mathbf{X}^{i},\overline{\mathbf{X}}\right\rangle^{p-l}\cdot\overline{\mathbf{X}}^{2l+1}\right]}{n}$$

Reasoning analogous to that shown in the proof of the property $(3.1.15)$ justifies the expression

$$E\left[\left(\mathbf{X}^{i}\right)^{2(s-p)}\left\langle\mathbf{X}^{i},\overline{\mathbf{X}}\right\rangle^{p-l}\cdot\overline{\mathbf{X}}^{2l+1}\right]=O\left(n^{-\left(\frac{p+l+1}{2}\right)}\right) \tag{3.1.20}$$

for each $i\in\{1,...,n\}$, $p\in\{1,...,s\}$ and $l\in\{0,...,p\}$. Thus, we get the condition

$$\frac{\sum_{i=1}^{n}\sum_{p=1}^{s}\sum_{l=0}^{p}\binom{s}{p}\binom{p}{l}(-1)^{p-l}2^{p-l}E\left[\left(\mathbf{X}^{i}\right)^{2(s-p)}\left\langle\mathbf{X}^{i},\overline{\mathbf{X}}\right\rangle^{p-l}\cdot\overline{\mathbf{X}}^{2l+1}\right]}{n}=O\left(n^{-1}\right),$$

which, together with $(3.1.10)$ leads to the property

$$E\left[m_{2s,k}\cdot\overline{\mathbf{X}}\right]=O\left(n^{-1}\right) \tag{3.1.21}$$

Finally, after the consecutive application of unbiasedness of multivariate sample raw moments and the forms $(3.1.6)$, $(3.1.19)$, $(3.1.16)$ and $(3.1.21)$ to the equation (3.14) we get

$$E\left[m_{2s+1,k}\right]=\mu_{2s+1,k}+O\left(n^{-1}\right), \tag{3.1.22}$$

which completes the proof of the theorem.

Corollary 3.1.1 Multivariate sample central moments are asymptotically unbiased estimators of the central moments of a random vector.

3.2. Consistency of the multivariate sample central moments

Our considerations in this section will focus on finding orders of approximations of mean squared errors of multivariate sample central moments, i.e. quantities of the form $E\left[\left(m_{r,k}-\mu_{r,k}\right)^{2}\right]$.

At the beginning we will take into account even-order sample central moments. Let us note that the expression $E\left[\left(m_{2s,k}-\mu_{2s,k}\right)^2\right]$ from the formula $(3.1.4)$ can be presented in the following form:

$$E\left[\left(m_{2s,k}-\mu_{2s,k}\right)^2\right]=E\left[\left(a_{2s,k}-\mu_{2s,k}\right)^2\right]+$$
$$+\frac{\sum_{i=1}^{n}\sum_{p=1}^{s}\sum_{l=0}^{p}\binom{s}{p}\binom{p}{l}(-1)^{p-l}2^{p-l}E\left[\left(a_{2s,k}-\mu_{2s,k}\right)\left(\mathbf{X}^i\right)^{2(s-p)}\overline{\mathbf{X}}^{2l}\left\langle\mathbf{X}^i,\overline{\mathbf{X}}\right\rangle^{p-l}\right]}{n}+$$
$$+\frac{\sum_{i_1,i_2=1}^{n}\sum_{p_1,p_2=1}^{s}\sum_{l_1=0}^{p_1}\sum_{l_2=0}^{p_2}E\left[\prod_{z=1}^{2}\binom{s}{p_z}\binom{p_z}{l_z}(-2)^{p_z-l_z}\left(\mathbf{X}^{i_z}\right)^{2(s-p_z)}\overline{\mathbf{X}}^{2l_z}\left\langle\mathbf{X}^{i_z},\overline{\mathbf{X}}\right\rangle^{p_z-l_z}\right]}{n}. \tag{3.2.1}$$

Based on the property $(3.1.2)$, we have

$$E\left[\left(a_{2s,k}-\mu_{2s,k}\right)^2\right]=O\left(n^{-1}\right). \tag{3.2.2}$$

Let us also note that

$$E\left[\left(a_{2s,k}-\mu_{2s,k}\right)\left(\mathbf{X}^i\right)^{2(s-p)}\overline{\mathbf{X}}^{2l}\left\langle\mathbf{X}^i,\overline{\mathbf{X}}\right\rangle^{p-l}\right]=O\left(n^{-\frac{1+p+l}{2}}\right), \tag{3.2.3}$$

for each $i\in\{1,...,n\}$, $p\in\{1,...,s\}$, $l\in\{0,...,p\}$.

In fact, considering Schwarz's inequality and Hölder's inequality we get

$$E^2\left[\left(a_{2s,k}-\mu_{2s,k}\right)\left(\mathbf{X}^i\right)^{2(s-p)}\overline{\mathbf{X}}^{2l}\left\langle\mathbf{X}^i,\overline{\mathbf{X}}\right\rangle^{p-l}\right]\le$$
$$\le E\left[\left(a_{2s,k}-\mu_{2s,k}\right)^2\right]E\left[\overline{\mathbf{X}}^{4l}\left(\mathbf{X}^i\right)^{4(s-p)}\left\langle\mathbf{X}^{i_z},\overline{\mathbf{X}}\right\rangle^{2(p-l)}\right]\le$$
$$\le E\left[\left(a_{2s,k}-\mu_{2s,k}\right)^2\right]\left(E\left[\overline{\mathbf{X}}^{2d}\right]\right)^{\frac{2l}{d}}\left(E\left[\left(\mathbf{X}^i\right)^{2d}\right]\right)^{\frac{2(s-p)}{d}}\left(E\left[\left\langle\mathbf{X}^i,\overline{\mathbf{X}}\right\rangle^{2d}\right]\right)^{\frac{p-l}{d}},$$

where $d=2s-(p-l)$.

Owing to the properties $(3.1.2)$, $(3.1.11)$ and the condition $m=\alpha_{1,k}=0$, we get

$$E^2\left[\left(a_{2s,k}-\mu_{2s,k}\right)\left(\mathbf{X}^i\right)^{2(s-p)}\overline{\mathbf{X}}^{2l}\left\langle\mathbf{X}^i,\overline{\mathbf{X}}\right\rangle^{p-l}\right]\le$$
$$\le\frac{A_1}{n}\left(\frac{A_2}{n^d}\right)^{\frac{2l}{d}}\left(\mu_{2d,k}\right)^{\frac{2(s-p)}{d}}\left(\frac{A_3}{n^d}\right)^{\frac{p-l}{d}}\le\frac{C}{n^{1+p+l}},$$

where $A_1,A_2,A_3,C>0$, which justifies $(3.2.3)$.

Property $(3.2.3)$ leads to condition

$$\frac{\sum_{i=1}^{n}\sum_{p=1}^{s}\sum_{l=0}^{p}\binom{s}{p}\binom{p}{l}(-1)^{p-l}2^{p-l}E\left[\left(a_{2s,k}-\mu_{2s,k}\right)\left(\mathbf{X}^{i}\right)^{2(s-p)}\overline{\mathbf{X}}^{2l}\left\langle \mathbf{X}^{i},\overline{\mathbf{X}}\right\rangle^{p-l}\right]}{n}=O\left(n^{-1}\right). \tag{3.2.4}$$

Reasoning analogous to that shown above justifies the expression

$$E\left[\prod_{z=1}^{2}\left(\mathbf{X}^{i_z}\right)^{2(s-p_z)}\overline{\mathbf{X}}^{2l_z}\left\langle \mathbf{X}^{i_z},\overline{\mathbf{X}}\right\rangle^{p_z-l_z}\right]=O\left(n^{-\frac{p_1+p_2+l_1+l_2}{2}}\right), \tag{3.2.5}$$

for each $i_1,i_2\in\{1,\dots,n\}$, $p_1,p_2\in\{1,\dots,s\}$, $l_1\in\{0,\dots,p_1\}$, $l_2\in\{0,\dots,p_2\}$.

Therefore,

$$\frac{\sum_{i_1,i_2=1}^{n}\sum_{p_1,p_2=1}^{s}\sum_{l_1=0}^{p_1}\sum_{l_2=0}^{p_2}E\left[\prod_{z=1}^{2}\binom{s}{p_z}\binom{p_z}{l_z}(-2)^{p_z-l_z}\left(\mathbf{X}^{i_z}\right)^{2(s-p_z)}\overline{\mathbf{X}}^{2l_z}\left\langle \mathbf{X}^{i_z},\overline{\mathbf{X}}\right\rangle^{p_z-l_z}\right]}{n}=O\left(n^{-1}\right). \tag{3.2.6}$$

Taking into account the form $(3.2.1)$ and the properties $(3.2.2)$, $(3.2.4)$ and $(3.2.6)$ we get

$$E\left[\left(m_{2s,k}-\mu_{2s,k}\right)^2\right]=O\left(n^{-1}\right). \tag{3.2.7}$$

Now, we will take into consideration the estimators of central moments of odd order. Let us note that on the basis of $(3.1.5)$, the expression $\left(m_{2s+1,k}-\mu_{2s+1,k}\right)^2$ can be presented as the relevant linear combination of the components of the form $\left(a_{2s+1,k}-\mu_{2s+1,k}\right)^2$,

$$\left\langle \left(a_{2s+1,k}-\mu_{2s+1,k}\right),\overline{\mathbf{X}}^{2l}\left\langle \mathbf{X}^{i},\overline{\mathbf{X}}\right\rangle^{p-l}\cdot\left(\mathbf{X}^{i}\right)^{2(s-p)+1}\right\rangle,$$

$$\left\langle \left(a_{2s+1,k}-\mu_{2s+1,k}\right),\left(\mathbf{X}^{i}\right)^{2(s-p)}\left\langle \mathbf{X}^{i},\overline{\mathbf{X}}\right\rangle^{p-l}\cdot\overline{\mathbf{X}}^{2l+1}\right\rangle,$$

$$\left\langle \overline{\mathbf{X}}^{2l_1}\left\langle \mathbf{X}^{i_1},\overline{\mathbf{X}}\right\rangle^{p_1-l_1}\cdot\left(\mathbf{X}^{i_1}\right)^{2(s-p_1)+1},\left(\mathbf{X}^{i_2}\right)^{2(s-p_2)}\left\langle \mathbf{X}^{i_2},\overline{\mathbf{X}}\right\rangle^{p_2-l_2}\cdot\overline{\mathbf{X}}^{2l_2+1}\right\rangle.$$

Moving on to determine the order of approximation of the expected values of these components, we note that

$$E\left[\left(a_{2s+1,k}-\mu_{2s+1,k}\right)^2\right]=O\left(n^{-1}\right). \tag{3.2.8}$$

Indeed, Jensen's inequality implies

$$\left\| E\left[\left(a_{2s+1,k} - \mu_{2s+1,k} \right)^2 \right] \right\| \le E\left[\left(a_{2s+1,k} - \mu_{2s+1,k} \right)^2 \right],$$

so the property $(3.1.2)$ leads to $(3.2.8)$.

Carrying out further analysis we, in turn, apply Jensen's, Schwarz's and Hölder's inequalities and, thanks to them, we obtain

$$\left\| E\left[\left\langle \left(a_{2s+1,k} - \mu_{2s+1,k} \right), \overline{\mathbf{X}}^{2l} \left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle^{p-l} \cdot \left(\mathbf{X}^i \right)^{2(s-p)+1} \right\rangle \right] \right\|^2 \le$$

$$E\left[\left(a_{2s+1,k} - \mu_{2s+1,k} \right)^2 \overline{\mathbf{X}}^{4l} \left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle^{2(p-l)} \left(\mathbf{X}^i \right)^{4(s-p)+2} \right] \le$$

$$\le \left(E\left[\left(a_{2s+1,k} - \mu_{2s+1,k} \right)^{2d} \right] \right)^{\frac{1}{d}} \left(E\left[\left(\overline{\mathbf{X}} \right)^{2d} \right] \right)^{\frac{2l}{d}} \left(E\left[\left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle^{2d} \right] \right)^{\frac{(p-l)}{d}} \left(E\left[\left(\mathbf{X}^i \right)^{2d} \right] \right)^{\frac{2(s-p)+1}{d}},$$

where $d = 2s + 2 - (p - l)$.

This estimation, after taking into account the properties $(3.1.2)$, $(3.1.11.)$ and the fact that $m = \alpha_{1,k} = 0$ leads to

$$\left\| E\left[\left\langle \left(a_{2s+1,k} - \mu_{2s+1,k} \right), \overline{\mathbf{X}}^{2l} \left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle^{p-l} \cdot \left(\mathbf{X}^i \right)^{2(s-p)+1} \right\rangle \right] \right\|^2 \le$$

$$\le \left(\frac{A_1}{n^d} \right)^{\frac{1}{d}} \left(\frac{A_2}{n^d} \right)^{\frac{2l}{d}} \left(\frac{A_3}{n^d} \right)^{\frac{p-l}{d}} \left(\mu_{2d,k} \right)^{\frac{2(s-p)+1}{d}} \le \frac{C}{n^{1+p+l}},$$

which is equivalent to the property

$$E\left[\left\langle \left(a_{2s+1,k} - \mu_{2s+1,k} \right), \overline{\mathbf{X}}^{2l} \left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle^{p-l} \cdot \left(\mathbf{X}^i \right)^{2(s-p)+1} \right\rangle \right] = O\left(n^{-\frac{1+p+l}{2}} \right),$$

for each $i \in \{1,...,n\}$, $p \in \{1,...,s\}$, $l \in \{0,..., p\}$.

A slight change in the proof above shows that

$$E\left[\left\langle \left(a_{2s+1,k} - \mu_{2s+1,k} \right), \left(\mathbf{X}^i \right)^{2(s-p)} \left\langle \mathbf{X}^i, \overline{\mathbf{X}} \right\rangle^{p-l} \cdot \overline{\mathbf{X}}^{2l+1} \right\rangle \right] = O\left(n^{-\frac{2+p+l}{2}} \right),$$

for each $i \in \{1,...,n\}$, $p \in \{1,...,s\}$, $l \in \{0,..., p\}$ and

$$E\left[\left\langle \overline{\mathbf{X}}^{2l_1} \left\langle \mathbf{X}^{i_1}, \overline{\mathbf{X}} \right\rangle^{p_1-l_1} \cdot \left(\mathbf{X}^{i_1} \right)^{2(s-p_1)+1}, \left(\mathbf{X}^{i_2} \right)^{2(s-p_2)} \left\langle \mathbf{X}^{i_2}, \overline{\mathbf{X}} \right\rangle^{p_2-l_2} \cdot \overline{\mathbf{X}}^{2l_2+1} \right\rangle \right] = O\left(n^{-\frac{2+p_1+p_2+l_1+l_2}{2}} \right)$$

for each $i_1, i_2 \in \{1,...,n\}$, $p_1, p_2 \in \{1,...,s\}$, $l_1 \in \{0,..., p_1\}$, $l_2 \in \{0,..., p_2\}$.

The above properties leads, thus, to

$$E\left[\left(m_{2s+1,k}-\mu_{2s+1,k}\right)^2\right]=O\left(n^{-1}\right). \qquad (3.2.9)$$

The next theorem will include the summary of the above consideration.

Theorem 3.2.1. Assume that for any natural numbers $r \geq 2$, a quantity $E\left[\left(m_{r,k}-\mu_{r,k}\right)^2\right]$ exists. Then, the mean squared error of the multivariate sample central moment of order r is of the form

$$\mathrm{MSE}m_{r,k}=E\left[\left(m_{r,k}-\mu_{r,k}\right)^2\right]=O\left(n^{-1}\right).$$

From this theorem and multivariate generalization of Chebyshev's inequality (Osiewalski and Tatar, 1999) we obtain a very important corollary.

Corollary 3.2.1 The multivariate sample central moments are consistent estimators of the central moments of a random vector.

4. Numerical illustration

Multivariate sample central moments, and also their respective functions, can be useful tools for the analysis of multivariate empirical data, for instance for the multivariate financial items, which have been considered in Budny and Tatar (2014).

Budny, Szklarska and Tatar (2014) have presented an analysis of socio-demographic conditions of Poland taking into account the multivariate (four-dimensional) data from the Central Statistical Office of Poland from 2012, and administrative division of the country into counties (*powiats*). The coordinates of the data are: the total marriage rate, the total divorce rate, the total fertility rate and the total mortality rate.

For this data, the second, third and fourth multivariate sample central moments in selected regions of Poland are as follows.

- North region (72 counties of the voivodeships: West Pomeranian, Kuyavian.

Pomernian, Pomeranian, Warmian-Masurian, without city counties):

$$m_{2,k}=1{,}4744, \qquad m_{3,k}=\begin{bmatrix}\text{-0.0022}\\ \text{-0.1227}\\ 0.0171\\ \text{-0.3623}\end{bmatrix}, \qquad m_{4,k}=5.3820.$$

- East region (74 counties of the voivodeships: Podlaskie, Lublin, Subcarpathian.

Lesser Poland, without city counties):

$$m_{2,k} = 2.8700, \qquad m_{3,k} = \begin{bmatrix} -0.3286 \\ 0.2943 \\ -0.0531 \\ 3.3462 \end{bmatrix}, \qquad m_{4,k} = 27.7891.$$

- West region (97 counties of the voivodeships: Greater Poland, Lubusz, Lower.

Silesian, Silesian, Opole, without city counties):

$$m_{2,k} = 1.8058, \qquad m_{3,k} = \begin{bmatrix} -0.2345 \\ 0.1702 \\ -0.0448 \\ 1.1132 \end{bmatrix}, \qquad m_{4,k} = 9.1024.$$

- Central region (71 counties of the voivodeships: Masovian, Świętokrzyskie, Łódź, without city counties):

$$m_{2,k} = 2.3571, \qquad m_{3,k} = \begin{bmatrix} -0.3748 \\ 0.1529 \\ 0.0188 \\ -1.4188 \end{bmatrix}, \qquad m_{4,k} = 14.0671.$$

- City counties (65 counties):

$$m_{2,k} = 2.9811, \qquad m_{3,k} = \begin{bmatrix} -0.0125 \\ -0.0828 \\ 0.00003 \\ 0.69907 \end{bmatrix}, \qquad m_{4,k} = 22.4378.$$

Dispersion of the distribution of the vector of the socio-demographic situation of Poland is measured by the central moments of the even orders. Note that the highest dispersion of test vector was observed in city counties while the smallest in the counties of the northern region (see Budny, Szklarska, Tatar, 2014). Central moments of odd order of multivariate distribution are parameters of location. They can also be considered as a measure of the asymmetry (see Tatar, 2000) and as a vector measures indicate the direction of asymmetry. The above considerations can be supplemented (see Budny, Szklarska, Tatar, 2014) using the functions of the central moments of a random vector, e.g. index of skewness (Tatar 2000) and kurtosis (Budny, Tatar 2009, Budny 2009). Let us mention that the index of skewness shows the direction of asymmetry while its square informs also about the size of the asymmetry.

The problem of estimation of this characteristics of multivariate distribution is left for further study.

5. Conclusions

In this paper we have proposed consistent and asymptotically unbiased estimators of the central moments of a random vector based on the power of a vector. Essential characteristics such as mean vectors and mean squared errors with the relevant orders of approximation have been established for them. The central moments of even order are parameters of dispersion of the distribution of a random vector. The moments of odd order characterize its location. These quantities can be useful tools for the analysis of multivariate data.

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APPENDIX

Proof of lemma 3.11.: The property $(3.1.6)$ will be proved by considering the cases of even and odd powers of random vectors. Reasoning (in each case) will be based on Schwarz's inequality considered in the relevant Hilbert's space $L_k^2(\Omega)$ or $L_1^2(\Omega)$.

At the beginning, let $r = 2s$ and $t = 2p-1$ where $p \in \{1,...s\}$. It is, therefore, necessary to prove the property

$$E\left[\left\langle a_{2s-(2p-1),k}, \overline{\mathbf{X}}^{2p-1}\right\rangle\right] = O\left(n^{-\left(p-\frac{1}{2}\right)}\right), \qquad (1)$$

For the proof we will apply Schwarz's inequality in Hilbert's space $L_k^2(\Omega)$ with the inner product $\langle \mathrm{X}, \mathrm{Y}\rangle_{L_k^2} := E[\langle \mathrm{X}, \mathrm{Y}\rangle]$ that leads to

$$E^2\left[\left\langle a_{2s-(2p-1),k}, \overline{\mathbf{X}}^{2p-1}\right\rangle\right] \leq E\left[a^2_{2s-(2p-1),k}\right] E\left[\overline{\mathbf{X}}^{4p-2}\right].$$

Let us note that $E\left[a^2_{r,k}\right] = D^2 a_{r,k} + \left(E\left[a_{r,k}\right]\right)^2 = O(1)$. Thus, taking into account the property $(3.1.2)$ we get

$$E^2\left[\left\langle a_{2s-(2p-1),k}, \overline{\mathbf{X}}^{2p-1}\right\rangle\right] = O\left(n^{-(2p-1)}\right),$$

which is equivalent to the condition (1).

Now, we consider the even numbers $r = 2s$ and $t = 2p$ where $p \in \{1,...s-1\}$

In view of this, the property

$$E\left[a_{2s-2p,k}\overline{\mathbf{X}}^{2p}\right] = O\left(n^{-p}\right). \qquad (2)$$

requires the proof.

Note that Schwarz's inequality in the space $L_1^2(\Omega)$ of the property $(3.1.2.)$ justifies the estimations

$$E^2\left[a_{2s-2p,k}\overline{\mathbf{X}}^{2p}\right] \leq E\left[a^2_{2s-2p,k}\right] E\left[\overline{\mathbf{X}}^{4p}\right] \leq \frac{C}{n^{2p}},$$

which obviously leads to (2).

In turn, for odd natural numbers $r = 2s - 1$ and $t = 2p - 1$ where $p \in \{1,...s-1\}$ it is necessary to prove the property

$$E\left[a_{2s-2p,k} \cdot \overline{\mathbf{X}}^{2p-1}\right] = O\left(n^{-\left(p-\frac{1}{2}\right)}\right). \qquad (3)$$

For this proof, let us note that Jensen's inequality, Schwarz's inequality and the condition $(3.1.2)$ imply a sequence of inequalities, respectively.