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# APPLICATION OF THE DIFFERENTIAL TRANSFORM METHOD TO THE FREE VIBRATION ANALYSIS OF FUNCTIONALLY GRADED TIMOSHENKO BEAMS

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In this study, free vibration characteristics of a functionally graded Timoshenko beam that undergoes flapwise bending vibration is analysed. The energy expressions are derived by introducing several explanotary figures and tables. Applying Hamilton's principle to the energy expressions, governing differential equations of motion and boundary conditions are obtained. In the solution part, the equations of motion, including the parameters for rotary inertia, shear deformation, power law index parameter and slenderness ratio are solved using an efficient mathematical technique, called the differential transform method (DTM). Natural frequencies are calculated and effects of several parameters are investigated.

*Keywords:* differential transform method, functionally graded beam, Timoshenko beam

## 1. Introduction

The concept of Functionally Graded Materials (FGMs) was originated from a group of material scientists in Japan as means of preparing thermal barrier materials (Loy *et al.*, 1999). FGMs are special composites that have continuous variation of material properties in one or more directions to provide designers with the ability to distribute strength and stiffness in a desired manner to get suitable structures for specific purposes in engineering and scientific fields such as design of aircraft and space vehicle structures, electronic and biomedical installations, automobile sector, defence industries, nuclear reactors, electronics, transportation sector, etc. As a consequence, it is important to understand static and dynamic behavior of FGMs, so it has been an area of intense research in the recent years. Especially, functionally graded beam (FGB) structures have become a fertile area of research since beam structures have been widely used in aeronautical, astronautical, civil, mechanical and other kinds of installations. Several research papers provide a good introduction and further references on the subject (Alshorbagy *et al.*, 2011; Chakraborty *et al.*, 2003; Giunta *et al.*, 2011; Huang and Li, 2010; Kapuria *et al.*, 2008; Lai *et al.*, 2012; Li, 2008; Loja *et al.*, 2012; Lu and Chen, 2005; Thai and Vo, 2012; Wattanasakulpong *et al.*, 2012; Zhong and Yu, 2007).

Due to the increasing application trend of FGMs, several beam theories have been developed to examine the response of FGBs. The Classical Beam Theory (CBT), i.e. Euler Bernoulli Beam Theory, is the simplest theory that can be applied to slender FGBs. The first order shear deformation theory (FSDT), i.e. Timoshenko Beam Theory, is used for the case of either short beams or high frequency applications to overcome the limitations of the CBT by accounting for the tranverse shear deformation effect. Bhimaraddi and Chandrashekhara (1991) derived laminated composite beam equations of motion using the first-order shear deformation plate theory (FSDPT). Dadfarnia (1997) developed a new beam theory for laminated composite beams using the assumption that the lateral stresses and all derivatives with respect to the lateral coordinate in the plate equations of motion are ignored.

In this study, which is an extension of the author's previous works (Kaya and Ozdemir Ozgumus, 2007; Kaya and Ozdemir Ozgumus, 2010; Ozdemir Ozgumus and Kaya, 2013), free vibration analysis of a functionally graded Timoshenko beam that undergoes flapwise bending vibrations is performed. At the beginning of the study, expressions for both kinetic and potential energies are derived in a detailed way by using explanatory tables and figures. In the next step, governing differential equations of motion are obtained applying Hamilton's principle. In the solution part, the equations of motion, including the parameters for rotary inertia, shear deformation, power law index parameter and slenderness ratio are solved using an efficient mathematical technique, called the differential transform method (DTM). Natural frequencies are calculated and effects of the parameters, mentioned above, are investigated. The calculated results are compared with the ones in open literature. Consequently, it is observed that there is a good agreement between the results which proves the correctness and accuracy of the DTM.

## 2. Beam model

The governing differential equations of motion are derived for the free vibration analysis of a functionally graded Timoshenko beam model with a right-handed Cartesian coordinate system which is represented by Fig. 1.

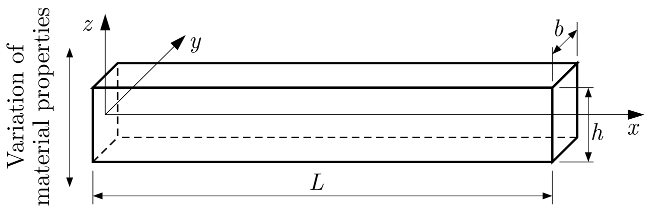


Fig. 1. Functionally graded beam model and the coordinate system

Here a uniform, functionally graded Timoshenko beam of length $L$, height $h$ and width $b$ which has the cantilever boundary condition at point $O$ is shown. The $xyz$-axes constitute a global orthogonal coordinate system with the origin at the root of the beam. The $x$-axis coincides with the neutral axis of the beam in the undeflected position, the $y$-axis lies in the width direction and the $z$-axis lies in the depth direction.

## 3. Formulation

### 3.1. Functionally graded beam formulation

Material properties of the beam, i.e. modulus of elasticity $E$, shear modulus $G$, Poisson's ratio $\nu$ and material density $\rho$ are assumed to vary continuously in the thickness direction $z$ as a function of the volume fraction, and the properties of the constituent materials according to a simple power law.

According to the rule of mixture, the effective material property $P(z)$ can be expressed as follows

$$P(z) = P_t V_t + P_b V_b \tag{3.1}$$

where $P_t$ and $P_b$ are the material properties at the top and bottom surfaces of the beam while $V_t$ and $V_b$ are the corresponding volume fractions. The relation between the volume fractions is given by

$$V_t + V_b = 1 \tag{3.2}$$

The volume fraction of the top constituent of the beam $V_t$ is assumed to be given by

$$V_t = \Big(\frac{z}{h} + \frac{1}{2}\Big)^k \qquad\qquad k \geqslant 0 \tag{3.3}$$

where $k$ is a non-negative power law index parameter that dictates the material variation profile through the beam thickness.

Considering Eqs. (3.1)-(3.3), the effective material property can be rewritten as follows

$$P(z) = (P_t - P_b)\Big(\frac{z}{h} + \frac{1}{2}\Big)^k + P_b \tag{3.4}$$

It is evident from Eq.(4) that when $z = h/2$, $E = E_t$, $\nu = \nu_t$, $G = G_t$, $\rho = \rho_t$ and when $z = -h/2$, $E = E_b$, $\nu = \nu_b$, $G = G_b$ and $\rho = \rho_b$.

### 3.2. Displacement field and strain field

The cross-sectional and the longitudinal views of a Timoshenko beam that undergoes extension and flapwise bending deflections are given in Figs. 2a and 2b, respectively. Here, the reference point is chosen, and is represented by $P_0$ before deformation and by $P$ after deformation.

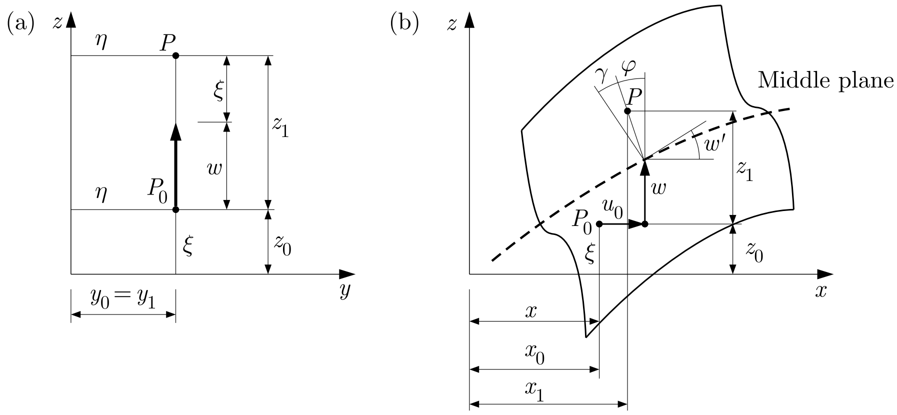


Fig. 2. (a) Cross-sectional view, (b) longitudinal view of the Timoshenko beam

Here, $\eta$ is the offset of the reference point from the $z$-axis, $\xi$ is the offset of the reference point from the middle plane, $x$ is the offset of the reference point from the $z$-axis, $u_0$ is the elongation, $w$ is the flapwise bending displacement, $\varphi$ is the rotation due to bending and $\gamma$ is the shear angle.

Considering Figs. 2a and 2b, the coordinates of the reference point are obtained as follows:
— before deflection (coordinates of $P_0$)

$$x_0 = x \qquad\qquad y_0 = \eta \qquad\qquad z_0 = \xi \tag{3.5}$$

— after deflection (coordinates of $P$)

$$x_1 = x + u_0 + \xi\varphi \qquad\qquad y_1 = \eta \qquad\qquad z_1 = w + \xi \tag{3.6}$$

The position vectors of the reference point are represented by $\mathbf{r}_0$ and $\mathbf{r}_1$ before and after deflection, respectively. Therefore, $d\mathbf{r}_0$ and $d\mathbf{r}_1$ can be written as follows

$$\begin{aligned} d\mathbf{r}_0 &= dx\mathbf{i} + d\eta\mathbf{j} + d\xi\mathbf{k} \\ d\mathbf{r}_1 &= [(1 + u_0' + \xi\varphi')]dx\mathbf{i} + d\eta\mathbf{j} + (w'dx + d\xi)\mathbf{k} \end{aligned} \tag{3.7}$$

where $(\cdot)'$ denotes differentiation with respect to the spanwise coordinate $x$.

The classical strain tensor $\varepsilon_{ij}$ may be obtained by using the following equilibrium equation given by Eringen (1980)

$$d\mathbf{r}_1 \cdot d\mathbf{r}_1 - d\mathbf{r}_0 \cdot d\mathbf{r}_0 = 2[dx \quad d\eta \quad d\xi][\varepsilon_{ij}] \begin{bmatrix} dx \\ d\eta \\ d\xi \end{bmatrix} \tag{3.8}$$

where

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{x\eta} & \varepsilon_{x\xi} \\ \varepsilon_{\eta x} & \varepsilon_{\eta\eta} & \varepsilon_{\eta\xi} \\ \varepsilon_{\xi x} & \varepsilon_{\xi\eta} & \varepsilon_{\xi\xi} \end{bmatrix} \tag{3.9}$$

Substituting Eqs. (3.7) into Eq. (3.8), the components of the strain tensor $\varepsilon_{ij}$ are obtained as follows

$$\begin{aligned} &\varepsilon_{xx} = u_0' + \frac{(u_0')^2}{2} + \frac{(w')^2}{2} + u_0'\varphi'\xi + \varphi'\xi + \frac{(\varphi')^2}{2}\xi^2 \\ &\gamma_{x\eta} = 0 \qquad\qquad \gamma_{x\xi} = (w' + \varphi) + \varphi\varphi'\xi - u_0'\varphi \end{aligned} \tag{3.10}$$

where $\varepsilon_{xx}$, $\gamma_{x\eta}$ and $\gamma_{x\xi}$ are the axial strain and the shear strains, respectively.

In this work, only $\varepsilon_{xx}$, $\gamma_{x\eta}$ and $\gamma_{x\xi}$ are used in the calculations because, as noted by Hodges and Dowell (1974) for long slender beams, the axial strain $\varepsilon_{xx}$ is dominant over the transverse normal strains $\varepsilon_{\eta\eta}$ and $\varepsilon_{\xi\xi}$. Moreover, the shear strain $\gamma_{\eta\xi}$ is by two orders smaller than the other shear strains $\gamma_{x\xi}$ and $\gamma_{x\eta}$. Therefore, $\varepsilon_{\eta\eta}$, $\varepsilon_{\xi\xi}$ and $\gamma_{\eta\xi}$ are neglected.

In order to obtain simpler expressions for the strain components given by Eqs. (3.10), higher order terms can be neglected, so an order of magnitude analysis is performed by using the ordering scheme taken from Hodges and Dowell (1974) and introduced in Table 1.

**Table 1.** Ordering scheme for the Timoshenko beam model

| Term | Order |
|---|---|
| $w'$ | $O(\varepsilon)$ |
| $\varphi$ | $O(\varepsilon)$ |
| $w' + \varphi$ | $O(\varepsilon^2)$ |
| $u_0'$ | $O(\varepsilon^2)$ |
| $\varphi'$ | $(\varepsilon^2)$ |

Hodges and Dowell (1974) used the formulation for an Euler-Bernoulli beam, so in this study their formulation is modified for the Timoshenko beam, and a new expression $w' + \varphi = O(\varepsilon^2)$ is added to their ordering scheme as a contribution to literature.

Considering Table 1, Eqs. (3.10) are simplified as follows

$$\varepsilon_{xx} = u_0' + \frac{(u_0')^2}{2} + \frac{(w')^2}{2} + \varphi'\xi \qquad \gamma_{x\eta} = 0 \qquad \gamma_{x\xi} = w' + \varphi \tag{3.11}$$

### 3.3. Potential energy

The expression for potential energy is given by

$$U = \frac{1}{2}\int_0^l \int_A (\sigma_{xx}\varepsilon_{xx} + \tau_{x\xi}\gamma_{x\xi})\, dA\, dx = \frac{b}{2}\int_0^l \int_{-h/2}^{h/2} (\sigma_{xx}\varepsilon_{xx} + \tau_{x\xi}\gamma_{x\xi})\, d\xi\, dx \tag{3.12}$$

The axial force $N$, the bending moment $M$ and the shear force $Q$ that act on a laminate at the midplane are expressed as follows (Kollar and Springer, 2003)

$$N = b \int_{-h/2}^{h/2} \sigma \, dz \qquad M = b \int_{-h/2}^{h/2} z\sigma \, dz \qquad Q = b \int_{-h/2}^{h/2} \tau \, dz \tag{3.13}$$

Substituting Eqs. (3.11) into Eq. (3.12) and considering Eqs. (3.13), the following expression is obtained

$$U = \frac{1}{2} \int_0^l \Big\{ N_x \Big[ u_0' + \frac{(w')^2}{2} \Big] + M_x \varphi' + Qz(w' + \varphi) \Big\} \, dx \tag{3.14}$$

where

$$N_x = \overline{A}_{11} u_0' + \overline{B}_{11} \varphi' \qquad M_x = \overline{B}_{11} u_0' + \overline{D}_{11} \varphi' \qquad Q = \overline{A}_{55} \gamma_{x\xi} \tag{3.15}$$

Here, the stiffness coefficients are obtained as follows

$$[\overline{A}_{11} \;\; \overline{B}_{11} \;\; \overline{D}_{11}] = \int_A E(z)[1 \;\; z \;\; z^2] \, dA \qquad \overline{A}_{55} = K \int_A G(z) \, dA \tag{3.16}$$

where $K$ is defined as the shear correction factor that takes the value of $K = 5/6$ for rectangular cross sections.

Substituting Eqs. (3.15) into Eq. (3.14) gives

$$U = \frac{1}{2} \int_0^l [\overline{A}_{11}(u_0')^2 + 2\overline{B}_{11} u_0' \varphi' + \overline{D}_{11}(\varphi')^2 + \overline{A}_{55}(w' + \varphi)] \, dx \tag{3.17}$$

Referring Eq. (3.17), variation of the potential energy is obtained as follows

$$\delta U = \int_0^l [(\overline{A}_{11} u_0' + \overline{B}_{11} \varphi') \delta u_0' + (\overline{B}_{11} u_0' + \overline{D}_{11} \varphi') \delta \varphi' + \overline{A}_{55}(w' + \varphi)(\delta w' + \delta \varphi)] \, dx \tag{3.18}$$

### 3.4. Kinetic energy

The position vector of the point $P$ shown in Fig. 2 is given by

$$\mathbf{r} = (x + u_0 + \xi \varphi)\mathbf{i} + w\mathbf{k} \tag{3.19}$$

Considering Eq. (3.19), the velocity vector of this point is obtained as follows

$$\mathbf{V} = \frac{\partial \mathbf{r}}{\partial t} = (\dot{u}_0 + \xi \dot{\varphi})\mathbf{i} + \dot{w}\mathbf{k} \tag{3.20}$$

Hence, the velocity components are

$$V_x = \dot{u}_0 + \xi \dot{\varphi} \qquad V_y = 0 \qquad V_z = \dot{w} \tag{3.21}$$

The kinetic energy expression is given by

$$T = \frac{1}{2} \int_0^l \int_A \rho(z)(V_x^2 + V_y^2 + V_z^2) \, dA \, dx = \frac{b}{2} \int_0^l \int_{-h/2}^{h/2} \rho(z)(V_x^2 + V_y^2 + V_z^2) \, d\xi \, dx \tag{3.22}$$

where $\rho(z)$ is the effective material density.

Substituting the velocity components into Eq. (3.22) and taking the variaiton of kinetic energy gives

$$\delta T = \int\limits_0^l [I_1(\dot{u}_0\delta\dot{u}_0 + \dot{w}\delta\dot{w}) + I_2(\dot{u}_0\delta\dot{\varphi} + \dot{\varphi}\delta\dot{u}_0) + I_3\dot{\varphi}\delta\dot{\varphi}]\, dx \tag{3.23}$$

where $I_1$, $I_2$ and $I_3$ are the inertial characteristics of the beam given by

$$[I_1 \ \ I_2 \ \ I_3] = \int\limits_A \rho(z)[1 \ \ z \ \ z^2]\, dA \tag{3.24}$$

### 3.5. Equations of motion and the boundary conditions

Hamilton's principle is expressed as follows

$$\int\limits_{t1}^{t2} \delta(U - T)\, dt = 0 \tag{3.25}$$

Substituting Eqs. (3.18) (3.23) into Eq. (3.25) gives the equations of motion and the boundary conditions as follows:
— equations of motion

$$\begin{aligned} &\overline{A}_{11}u_0'' + \overline{B}_{11}\varphi'' = I_1\ddot{u}_0 + I_2\ddot{\varphi} \qquad \overline{A}_{55}(w'' + \dot{\varphi}') = I_1\ddot{w} \\ &\overline{D}_{11}\varphi'' + \overline{B}_{11}u_0'' - \overline{A}_{55}(w' + \varphi) = I_2\ddot{u}_0 + I_3\ddot{\varphi} \end{aligned} \tag{3.26}$$

— boundary conditions

$$\begin{aligned} x = 0 \qquad & u_0(0,t) = w(0,t) = \varphi(0,t) = 0 \\ x = L \qquad & \overline{A}_{11}u_0'(L,t) + \overline{B}_{11}\varphi'(L,t) = 0 \qquad \overline{A}_{55}[w'(L,t) + \varphi(L,t)] = 0 \\ & \overline{D}_{11}\varphi(L,t) + \overline{B}_{11}u_0'(L,t) - \overline{A}_{55}(w' + \varphi) = 0 \end{aligned} \tag{3.27}$$

In order to investigate free vibration of the beam model considered in this study, a sinusoidal variation of $u_0$, $w$ and $\varphi$ with a circular natural frequency $\omega$ is assumed, and the functions are approximated as

$$u_0(x,t) = \overline{u}(x)\mathrm{e}^{\mathrm{i}\omega t} \qquad w(x,t) = \overline{w}(x)\mathrm{e}^{\mathrm{i}\omega t} \qquad \varphi(x,t) = \overline{\varphi}(x)\mathrm{e}^{\mathrm{i}\omega t} \tag{3.28}$$

Substituting Eqs. (3.28) into the equations of motion, i.e. Eqs. (3.26), and into the boundary conditions, i.e. Eqs.(3.27), the following dimensionless equations are obtained as follows:
— equations of motion

$$\begin{aligned} &\gamma^2\widetilde{u}^{**} + \alpha^2\widetilde{\varphi}^{**} + \lambda^2(\widetilde{u} + \mu^2\widetilde{\varphi}) = 0 \qquad \frac{\widetilde{w}^{**} + \widetilde{\varphi}}{\tau^2} + \lambda^2\widetilde{w} = 0 \\ &\tau^2(\alpha^2\widetilde{u}^{**} + \widetilde{\varphi}^{**} + \mu^2\lambda^2\widetilde{u}) + (r^2\tau^2\lambda^2 - 1)\widetilde{\varphi} - \widetilde{w}^{*} = 0 \end{aligned} \tag{3.29}$$

— boundary conditions

$$\begin{aligned} x = 0 \qquad & \widetilde{u}(0,t) = \widetilde{w}(0,t) = \widetilde{\varphi}(0,t) = 0 \\ x = L \qquad & \gamma^2\widetilde{u}^{*} + \alpha^2\widetilde{\varphi}^{*}(L,t) = 0 \qquad \frac{1}{\tau^2}[\widetilde{w}^{*}(L,t) + \widetilde{\varphi}(L,t)] = 0 \\ & \alpha^2\widetilde{u}^{*}(L,t) + \widetilde{\varphi}^{*}(L,t) = 0 \end{aligned} \tag{3.30}$$

Here, the dimensionless parameters are defined as

$$\widetilde{w} = \frac{\overline{w}}{L} \qquad \widetilde{u} = \frac{\overline{u}}{L} \qquad \widetilde{\varphi} = \varphi \qquad \gamma^2 = \frac{\overline{A}_{11}L^2}{\overline{D}_{11}} \qquad \tau^2 = \frac{\overline{D}_{11}}{\overline{A}_{55}L^2}$$
$$\lambda^2 = \frac{I_1 L^4 \omega^2}{\overline{D}_{11}} \qquad \mu^2 = \frac{I_2}{I_1 L} \qquad r^2 = \frac{I_3}{I_1 L^2} \qquad \alpha^2 = \frac{\overline{B}_{11}L}{\overline{D}_{11}} \tag{3.31}$$

## 4. Differential Transform Method

The Differential Transform Method (DTM) is a transformation technique based on the Taylor series expansion and is a useful tool to obtain analytical solutions of differential equations. In this method, certain transformation rules are applied, and the governing differential equations and the boundary conditions of the system are transformed into a set of algebraic equations in terms of the differential transforms of the original functions, and the solution of these algebraic equations gives the desired solution of the problem.

Consider a function $f(x)$ which is analytical in a domain $\mathcal{D}$ and let $x = x_0$ represent any point in $\mathcal{D}$. The function $f(x)$ is then represented by a power series whose center is located at $x_0$. The differential transform of the function $f(x)$ is given by

$$F[k] = \frac{1}{k!}\Big(\frac{d^k f(x)}{dx^k}\Big)_{x=x_0} \tag{4.1}$$

where $f(x)$ is the original function and $F[k]$ is the transformed function. The inverse transformation is defined as

$$f(x) = \sum_{k=0}^{\infty}(x - x_0)^k F[k] \tag{4.2}$$

Combining Eq. (4.1) and Eq. (4.2), we get

$$f(x) = \sum_{k=0}^{\infty}\frac{(x - x_0)^k}{k!}\Big(\frac{d^k f(x)}{dx^k}\Big)_{x=x_0} \tag{4.3}$$

Considering Eq. (4.3), it is noticed that the concept of differential transform is derived from the Taylor series expansion. However, the method does not evaluate the derivatives symbolically.

In actual applications, the function $f(x)$ is expressed by a finite series and Eq. (4.3) can be written as follows

$$f(x) = \sum_{k=0}^{m}\frac{(x - x_0)^k}{k!}\Big(\frac{d^k f(x)}{dx^k}\Big)_{x=x_0} \tag{4.4}$$

which means that the rest of the series

$$f(x) = \sum_{k=m+1}^{\infty}\frac{(x - x_0)^k}{k!}\Big(\frac{d^k f(x)}{dx^k}\Big)_{x=x_0} \tag{4.5}$$

is negligibly small. Here, the value of m depends on the convergence of natural frequencies.

Theorems that are frequently used in the transformation procedure are introduced in Table 2, and theorems that are used for boundary conditions are introduced in Table 3.

After applying DTM to Eqs. (3.29) and (3.30), the transformed equations of motion and boundary conditions are obtained as follows:

**Table 2.** DTM theorems used for equations of motion

| Original function | Transformed function |
|---|---|
| $f(x) = g(x) \pm h(x)$ | $F[k] = G[k] \pm H[k]$ |
| $f(x) = \lambda g(x)$ | $F[k] = \lambda G[k]$ |
| $f(x) = g(x)h(x)$ | $F[k] = \sum_{l=0}^{k} G[k-l]H[l]$ |
| $f(x) = \frac{d^n g(x)}{dx^n}$ | $F[k] = \frac{(k+n)!}{k!} G[k+n]$ |
| $f(x) = x^n$ | $F[k] = \delta(k-n) = \begin{cases} 0 & \text{if} \quad k \neq n \\ 1 & \text{if} \quad k = n \end{cases}$ |

**Table 3.** DTM theorems used for boundary conditions

| $x = 0$ | | $x = 1$ | |
|---|---|---|---|
| Original B.C. | Transformed B.C. | Original B.C. | Transformed B.C. |
| $\frac{df(0)}{dx} = 0$ | $F(0) = 0$ | $f(1) = 0$ | $\sum_{k=0}^{\infty} F(k) = 0$ |
| $\frac{df}{dx}(0) = 0$ | $F(1) = 0$ | $\frac{df}{dx}(1) = 0$ | $\sum_{k=0}^{\infty} kF(k) = 0$ |
| $\frac{d^2 f}{dx^2}(0) = 0$ | $F(2) = 0$ | $\frac{d^2 f}{dx^2}(1) = 0$ | $\sum_{k=0}^{\infty} k(k-1)F(k) = 0$ |
| $\frac{d^3 f}{dx^3}(0) = 0$ | $F(3) = 0$ | $\frac{d^3 f}{dx^3}(1) = 0$ | $\sum_{k=0}^{\infty}(k-1)(k-2)kF(k) = 0$ |

— equations of motion

$$
\begin{aligned}
&\gamma^2(k+1)(k+2)U[k+2] + \alpha^2(k+1)(k+2)\varphi[k+2] + \lambda^2(U[k] + \mu^2\varphi[k]) = 0 \\
&\frac{1}{\tau^2}(k+1)(k+2)W[k+2] + \lambda^2 W[k] + \frac{1}{\tau^2}(k+1)\varphi[k+1] = 0 \\
&\alpha^2(k+1)(k+2)U[k+2] + (k+1)(k+2)\varphi[k+2] + \lambda^2\mu^2 U[k] + \Big(r^2\lambda^2 - \frac{1}{\tau^2}\Big)\varphi[k] \\
&\qquad - \frac{1}{\tau^2}(k+1)W[k+1] = 0
\end{aligned}
\tag{4.6}
$$

— boundary conditions

$$
\begin{aligned}
&x = 0 \qquad U[k] = W[k] = \varphi[k] = 0 \\
&x = L \qquad \gamma^2 \sum_{k=0}^{\infty} kU[k] + \alpha^2 \sum_{k=0}^{\infty} k\varphi[k] = 0 \qquad \frac{1}{\tau^2}\Big(\sum_{k=0}^{\infty}(kW[k] + \varphi[k])\Big) = 0 \\
&\qquad\qquad \alpha^2 \sum_{k=0}^{\infty} kU[k] + \sum_{k=0}^{\infty} k\varphi[k] = 0
\end{aligned}
\tag{4.7}
$$

## 5. Results and discussions

In the numerical analysis, two cases are studied. In the first case, natural frequencies of a pure aluminum Timoshenko beam with simply-simply supported (SS) end conditions and, in the second case, a fuctionally graded Timoshenko beam with clamped free (CF) boundary conditions are calculated. Effects of the slenderness ratio $L/h$ and the power law index parameter $k$ on the natural frequencies are investigated. The results are presented in related tables. In order to validate the calculated results, comparisons with the studies in open literature are made and a very good agreement between the results is observed, which proves the correctness and accuracy of the Differential Transform Method. It is believed that the tabulated results can be used as references by other researchers to validate their results.

**Case 1.** Pure aluminum simply supported beam

**Table 4.** Material properties of the aluminum Timoshenko beam

| Property | Aluminum (Al) |
|---|---|
| Elasticity modulus $E$ | 70 GPa |
| Material density $\rho$ | 2700 kg/m$^3$ |
| Poisson's ratio $\nu$ | 0.23 |

Variation of the first five natural frequencies of the S-S pure aluminum Timoshenko beam with respect to the slenderness ratio $L/h$ is given in Table 5. When the calculated results are compared with the ones given by Sina *et al.* (2009), a very good agreement between the results is observed.

**Table 5.** Dimensionless natural frequencies of the pure aluminum Timoshenko beam

| Frequency | Slenderness ratio $L/h$ | | | | | |
|---|---|---|---|---|---|---|
| $\lambda = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$ | 10 | 20 | 30 | 40 | 50 | 100 |
| Fundamental | 2.87896 | 2.91515 | 2.92204 | 2.92447 | 2.92559 | 2.92709 |
| Sina *et al.* (2009) | 2.879 | – | 2.922 | – | – | 2.927 |
| 2nd NF | 10.9963 | 11.5159 | 11.6224 | 11.6606 | 11.6784 | 11.7024 |
| 3rd NF | 23.1528 | 25.3995 | 25.9107 | 26.0988 | 26.1876 | 26.3078 |
| 4th NF | 32.2814 | 43.9854 | 45.4901 | 46.0634 | 46.3380 | 46.7137 |
| 5th NF | 38.0919 | 64.5627 | 69.9845 | 71.3222 | 71.9741 | 72.8788 |

**Case 2.** FG Timoshenko beam

The FG beam is made of aluminum (Al) at the top and alumina ($Al_2O_3$) at the bottom. The effective beam properties change through the beam thickness according to the power law. The material properties of the FG beam are displayed in Table 6.

**Table 6.** Material properties of the FG beam

| Property | Aluminum (Al) | Alumina ($Al_2O_3$) |
|---|---|---|
| Elasticity modulus $E$ | 70 GPa | 380 GPa |
| Material density $\rho$ | 2702 kg/m$^3$ | 3960 kg/m$^3$ |
| Poisson's ratio $\nu$ | 0.3 | 0.3 |

Variation of the fundamental natural frequency of the C-F functionally graded Timoshenko beam according to the power law exponent for $L/h = 20$ is given in Table 7. When the calculated results are compared with the ones given by Şimsek (2010), a very good agreement between the results is observed.

**Table 7.** Dimensionless fundamental frequencies of the C-F FG Timoshenko beam

| Frequency | Power law exponent $k$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\lambda = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$ | 0 | 0.2 | 0.5 | 1 | 2 | 5 | 10 | Full metal |
| Fundamental | 1.94955 | 1.81407 | 1.66026 | 1.50103 | 1.36966 | 1.30373 | 1.26493 | 1.01297 |
| Şimsek (2010) | 1.94957 | 1.81456 | 1.66044 | 1.50104 | 1.36968 | 1.30375 | 1.26495 | 1.01297 |

In Table 8, variation of the dimensionless natural frequencies of the C-F functionally graded Timoshenko beam with respect to the power law exponent $k$ and the slenderness ratio $L/h$ is presented.

**Table 8.** Variation of the dimensionless natural frequencies of the C-F functionally graded Timoshenko beam with respect to the power law exponent $k$ and the slenderness ratio $L/h$

| | $k$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $L/h$ | 0 | 0.2 | 0.5 | 1 | 2 | 5 | 10 | Full metal |
| 3 | 1.80329 | 1.6829 | 1.54468 | 1.39885 | 1.27348 | 1.19956 | 1.15684 | 0.936972 |
| | 8.21514 | 7.72196 | 7.12189 | 6.44425 | 5.77913 | 5.22620 | 4.96530 | 4.26852 |
| | 9.06941 | 8.675 | 8.23136 | 7.7081 | 7.05971 | 6.19459 | 5.64279 | 4.71239 |
| | 17.9802 | 16.9703 | 15.7499 | 14.3438 | 12.9094 | 11.6647 | 11.0314 | 9.34237 |
| | 26.4033 | 25.1248 | 23.5417 | 21.6204 | 19.4397 | 17.1022 | 15.8590 | – |
| 4 | 1.86385 | 1.73735 | 1.59278 | 1.44141 | 1.31344 | 1.24242 | 1.20111 | 0.96844 |
| | 9.42868 | 8.8443 | 8.15412 | 7.39468 | 6.68493 | 6.15895 | 5.8858 | 4.89906 |
| | 12.0925 | 11.5548 | 10.931 | 10.1889 | 9.27327 | 8.07187 | 7.35535 | 6.28319 |
| | 21.5877 | 20.3297 | 18.8307 | 17.1413 | 15.4804 | 14.1093 | 13.3796 | 11.2168 |
| | 34.5928 | 32.7263 | 30.451 | 27.8043 | 25.0665 | 22.6209 | 21.4285 | 18.0754 |
| 5 | 1.89441 | 1.76476 | 1.61692 | 1.46276 | 1.33353 | 1.26419 | 1.2237 | 0.98432 |
| | 10.2025 | 9.55154 | 8.79239 | 7.97167 | 7.23018 | 6.72424 | 6.44766 | 5.30114 |
| | 15.1157 | 14.4404 | 13.6505 | 12.7061 | 11.5406 | 10.0258 | 9.14605 | 7.85398 |
| | 24.2839 | 22.8225 | 21.0981 | 19.1875 | 17.3683 | 15.9509 | 15.1753 | 12.6177 |
| | 40.3144 | 38.0031 | 35.2454 | 32.1226 | 29.0171 | 26.3775 | 24.9471 | 20.9484 |
| 10 | 1.93806 | 1.80382 | 1.65126 | 1.49308 | 1.36215 | 1.29547 | 1.25629 | 1.007 |
| | 11.6155 | 10.8294 | 9.92996 | 8.98688 | 8.18692 | 7.73783 | 7.4778 | 6.03531 |
| | 30.2314 | 28.5306 | 26.2115 | 23.7423 | 21.5708 | 19.8231 | 18.204 | 15.708 |
| | 30.5505 | 28.8901 | 27.3023 | 25.3998 | 23.0609 | 20.4233 | 19.5414 | 15.8738 |
| | 55.4176 | 51.8978 | 47.8058 | 43.39 | 39.4041 | 36.6649 | 35.1095 | 28.7945 |
| 15 | 1.94655 | 1.81139 | 1.65791 | 1.49895 | 1.3677 | 1.30157 | 1.26267 | 1.01141 |
| | 11.9506 | 11.1299 | 10.1949 | 9.22147 | 8.40836 | 7.97825 | 7.72685 | 6.20943 |
| | 32.4399 | 30.247 | 27.7372 | 25.1042 | 22.8681 | 21.6068 | 20.8769 | 16.8555 |
| | 45.347 | 43.3142 | 40.921 | 38.0455 | 34.4964 | 29.9319 | 27.3397 | 23.5619 |
| | 60.9894 | 56.9502 | 52.3063 | 47.394 | 43.1404 | 40.553 | 39.057 | 31.6896 |
| 20 | 1.94955 | 1.81407 | 1.66026 | 1.50103 | 1.36966 | 1.30373 | 1.26493 | 1.01297 |
| | 12.0753 | 11.2415 | 10.293 | 9.30821 | 8.49032 | 8.06791 | 7.8202 | 6.27423 |
| | 33.2016 | 30.9307 | 28.3406 | 25.6387 | 23.3723 | 22.1529 | 21.4418 | 17.2513 |
| | 60.4627 | 57.7447 | 54.1446 | 49.0496 | 44.62 | 39.8473 | 36.4316 | 31.4159 |
| | 63.4443 | 59.1681 | 54.6767 | 50.7988 | 46.1299 | 42.3338 | 40.8514 | 32.9651 |

In Fig. 3, convergence of the first five natural frequencies with respect to the number of terms $N$ used in DTM application is shown, where $L/h = 5$ and $k = 0.5$. To evaluate up to the fifth natural frequency to five-digit precision, it has been necessary to take 45 terms.

Additionally, it is seen that higher modes appear when more terms are taken into account in DTM application. Thus, depending on the order of the required modes, one must try a few values for the term number at the beginning of the calculations in order to find the adequate number of terms. For instance, only $N = 50$ is enough for the results given in Tables 5, 7 and 8.

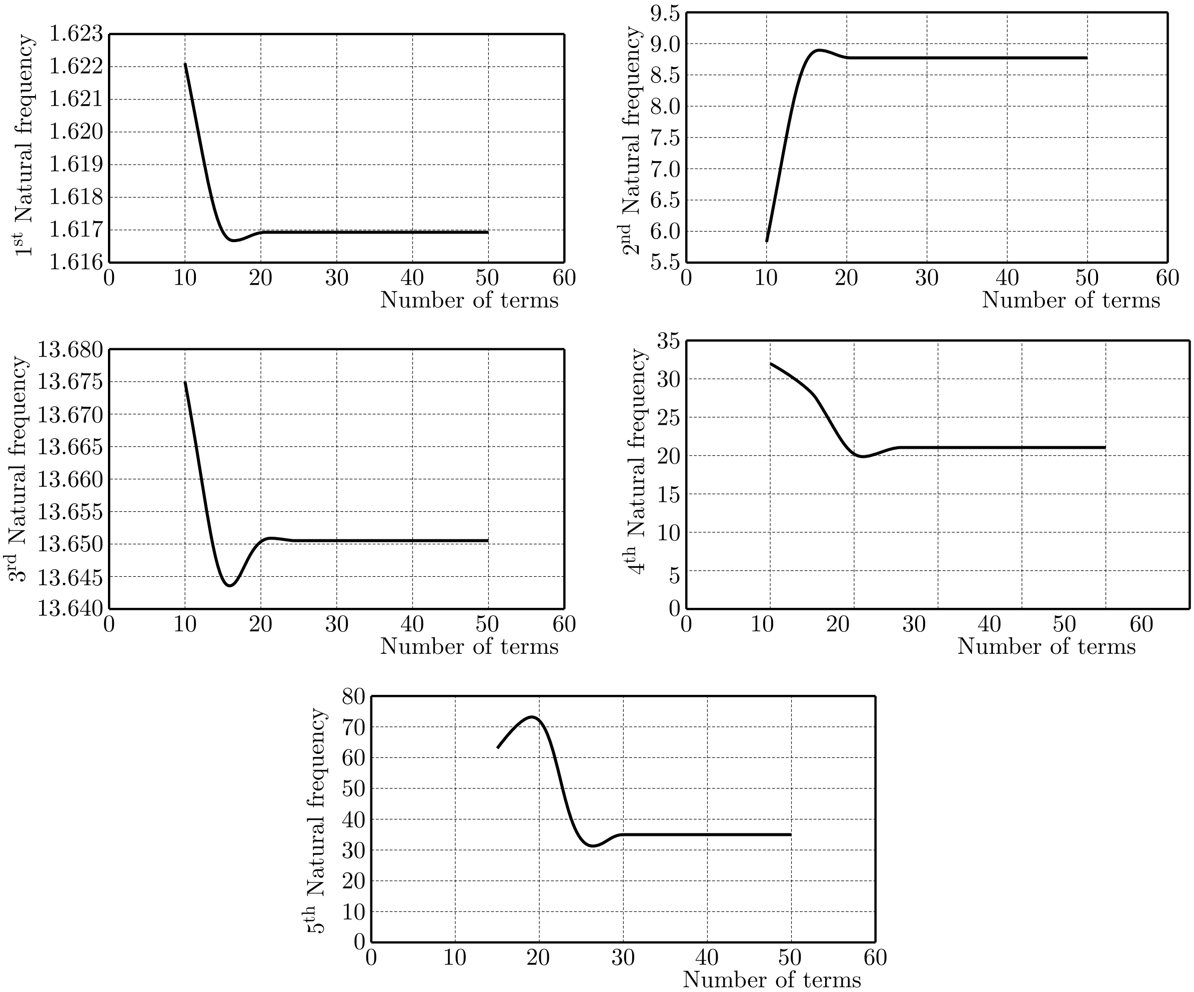


Fig. 3. Convergence of the first five natural frequencies with respect to the number of terms $N$

## 6. Conclusion

In this study, formulation of a functionally graded Timoshenko beam that undergoes flapwise bending vibration is derived by introducing several explanotary figures and tables. Applying Hamilton's principle to the obtained energy expressions, governing differential equations of motion and the boundary conditions are derived. In the solution part, the equations of motion, including the parameters for rotary inertia, shear deformation, power law index parameter and slenderness ratio are solved using an efficient mathematical technique, called the differential transform method (DTM). Natural frequencies are calculated and effects of the above mentioned parameters are investigated.

Considering the calculated results,the following conclusions are reached:

- As the slenderness ratio $L/h$ increases, the natural frequencies increase;
- The effect of the slenderness ratio on the frequencies is negligible forlong FG beams (i.e., $L/h \geqslant 20$);
- The natural frequencies decrease as the value of the power-law exponent $k$ increases.

## 7. Future work

According to the author's knowledge, the Differential Transform Method has not been applied to functionally graded Timoshenko beams in literature before. Therefore, this gap is aimed to be fulfilled in this paper. However, in this study, a functionally graded Timoshenko beam with a power-law gradient is considered and the efficiency of DTM has not been examined for other gradients such as exponent gradient (Tang *et al.*, 2014; Hao and Wei, 2016; Li *et al.*, 2013; Wang *et al.*, 2016). The examination of the DTM efficiency for other gradient types can be considered as a challenging future work.

## References


1. Alshorbagy A.E., Eltaher M.A., Mahmoud F.F., 2011, Free vibration characteristics of a functionally graded beam by finite element method, *Applied Mathematical Modelling*, **35**, 412-425
2. Bhimaraddi A., Chandrashekhara K., 1991, Some observation on the modeling of laminated composite beams with general lay-ups, *Composite Structures*, **19**, 371-380
3. Chakraborty A., Gopalakrishnan S., Reddy J.N., 2003, A new beam finite element for the analysis of functionally graded materials, *International Journal of Mechanical Sciences*, **45**, 519-539
4. Dadfarnia M., 1997, Nonlinear forced vibration of laminated beam with arbitrary lamination, M.Sc. Thesis, Sharif University of Technology
5. Deng H.D., Wei C., 2016, Dynamic characteristics analysis of bi-directional functionally graded Timoshenko beams, *Composite Structures*, in press
6. Eringen A.C., 1980, *Mechanics of Continua*, Robert E. Krieger Publishing Company, Huntington, New York
7. Giunta G., Crisafulli D., Belouettar S., Carrera E., 2011, Hierarchical theories for the free vibration analysis of functionally graded beams, *Composite Structures*, **94**, 68-74
8. Hodges D.H., Dowell E.H., 1974, Nonlinear equations of motion for the elastic bending and torsion of twisted nonuniform rotor blades, *NASA Technical Report*, NASA TN D-7818
9. Huang Y., Li X.F., 2010, A new approach for free vibration of axially functionally graded beams with non-uniform cross-section, *Journal of Sound and Vibration*, **329**, 2291-2303
10. Kapuria S., Bhattacharyya M., Kumar A.N., 2008, Bending and free vibration response of layered functionally graded beams: a theoretical model and its experimental validation, *Composite Structures*, **82**, 390-402
11. Kaya M.O., Ozdemir Ozgumus O., 2007, Flexural-torsional-coupled vibration analysis of axially loaded closed-section composite Timoshenko beam by using DTM, *Journal of Sound and Vibration*, **306**, 495-506
12. Kaya M.O., Ozdemir Ozgumus O., 2010, Energy expressions and free vibration analysis of a rotating uniform timoshenko beam featuring bending-torsion coupling, *Journal of Vibration and Control*, **16**, 6, 915-934
13. Kollar L.R., Springer G.S., 2003, *Mechanics of Composite Structures*, Cambridge University Press, United Kingdom
14. Lai S.K., Harrington J., Xiang Y., Chow K.W., 2012, Accurate analytical perturbation approach for large amplitude vibration of functionally graded beams, *International Journal of Non-Linear Mechanics*, **47**, 473-480
15. Li X.F., 2008, A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler-Bernoulli beams, *Journal of Sound and Vibration*, **318**, 1210-1229

16. Li X.F., Kang Y.A., Wu J.X., 2013, Exact frequency equation of free vibration of exponentially funtionally graded beams, *Applied Acoustics*, **74**, 3, 413-420

17. Loja M.A.R., Barbosa J.I., Mota Soares C.M., 2012, A study on the modelling of sandwich functionally graded particulate composite, *Composite Structures*, **94**, 2209-2217

18. Loy C.T., Lam K.Y., Reddy J.N., 1999, Vibration of functionally graded cylinderical shells, *International Journal of Mechanical Science*, **41**, 309-324

19. Lu C.F., Chen W.Q., 2005, Free vibration of orthotropic functionally graded beams with various end conditions, *Structural Engineering and Mechanics*, **20**, 465-476

20. Ozdemir Ozgumus O., Kaya M.O., 2013, Energy expressions and free vibration analysis of a rotating Timoshenko beam featuring bending-bending-torsion coupling, *Archive of Applied Mechanics*, **83**, 97-108

21. Sina S.A., Navazi H.M., Haddadpour H., 2009, An analytical method for free vibration analysis of functionally graded beams, *Materials and Design*, **30**, 741-747

22. Şimsek M., 2010, Fundamental frequency analysis of functionally graded beams by using different higher-order beam theories, *Nuclear Engineering and Design*, **240**, 697-705

23. Tang A.Y., Wu J.X., Li X.F., Lee K.Y., 2014, Exact frequency equations of free vibration of exponentially non-uniform functionally graded Timoshenko beams, *International Journal of Mechanical Sciences*, **89**, 1-11

24. Thai H.T., Vo T.P., 2012, Bending and free vibration of functionally graded beams using various higher-order shear deformation beam theories, *International Journal of Mechanical Sciences*, **62**, 57-66

25. Wang Z., Wang X., Xu G., Cheng S., Zeng T., 2016, Free vibration of two directional functionally graded beams, *Composite Structures*, **135**, 191-198

26. Wattanasakulpong N., Prusty B.G., Kelly D.W., Hoffman M., 2012, Free vibration analysis of layered functionally graded beams with experimental validation, *Materials and Design*, **36**, 182-190

27. Zhong Z., Yu T., 2007, Analytical solution of a cantilever functionally graded beam, *Composites Science and Technology*, **67**, 481-488

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