Dynamic pricing with online learning of customer arrival rate and acceptable price

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Abstract

In many industries, the price of commodities are adjusted by taking into account the current level of inventory and the distribution of future demand. This has motivated studies in the area of *dynamic pricing*, which among others have been successfully applied in airline industry. A common assumption in these studies is that the distribution of future demand is known in advance. While sometimes this distribution can be learned from historical data in advance, there are cases where the selling scenario has unique characteristics and the learning can only be achieved as the selling process is going on. In this note we will use Bayesian learning to update our belief about two distributions: The distribution of customer arrivals and the distribution of acceptable price for customers.

Dynamic pricing is the practice of adjusting the price of a product in a timely fashion. It can be best described through an example in airline reservation, which is a major application domain for such techniques. The price of airline seats (in the same class, e.g. economy) depends on the time to departure and the number of empty seats. For example, if the departure time is approaching and there are a lot of vacant seats, it is reasonable to decrease the price.

Products like airline seats are called *perishable products*. Such products have three characteristics which makes them good targets for dynamic pricing:

1. *fixed inventory*: The quantity of product is fixed.
2. *fixed sale period*: There is a deadline for sale.
3. *high profit margin*: The marginal cost for each item is small, and most of revenue turns into profit.

There are two sources of randomness in demand: **1)** customer arrival rate, **2)** customer acceptable price distribution.

It is often assumed that the distributions of these random effects are known. A notable exception is the work of Lin [2006], where he assumes a prior on customer arrival rate and updates the distribution according to the observed arrivals as the sales goes on. In this note, we build on this work and in addition to the arrival rate, we also learn the customer acceptable price.

The rest of this note is organized as follows: Section 1 summarizes the work of Lin [2006]. In section 3 we introduce our extension which allows learning the acceptable price.

1 The dynamic pricing problem

Before dealing with the learning problems, we will formulate and solve the dynamic pricing problem assuming that we know the true distributions for customer arrival and customer acceptable price.

It is worth emphasizing that the motivation for one of the model assumptions will become clear in later sections. In the model it is assumed that the total number of remaining customers follows a negative binomial distribution. As we will see later, this results from assuming a gamma prior on the parameter of a Poisson distribution.

1.1 The model

These are the model assumptions: The seller has to sell a stock of items in the time horizon $[0, T]$. Customers arrive according to a Poisson distribution with the rate Λ. At each point in time, the number of remaining customers follows a Gamma distribution with given time-dependent parameters. After their arrival, a customer will buy an item if the product price is below their acceptable price, and will leave empty-handed otherwise. The acceptable price of customers are i.i.d random variables with a continuous CDF denoted by F. More formally, when the seller sets the price p, the probability that an arriving customer buys the product is $\bar{F} \equiv 1 - F(p)$.

1.2 Solving the pricing problem by recursion

In this section, we summarize the method proposed by Lin [2006] for solving the dynamic pricing problem with the assumptions mentioned earlier. More specifically, the total number of customers follows a negative binomial distribution with parameters (c, α). For a price $p \geq 0$, the probability that a customer buys the item is $\bar{F}(p) \equiv 1 - F(p)$.

We denote the inverse of $\bar{F}$ by $\bar{F}^{-1}(v), 0 \leq v \leq 1$. We also denote the total expected revenue from an arriving customer by $g(v) \equiv v\bar{F}^{-1}(v)$.

Let $J(s, c, \alpha)$ denote the optimal expected revenue when there are s items to sell, and the total customers follows a negative binomial distribution with parameters (c, α). Then the following recursion holds (see appendix 4.1 for details):

$$
\begin{aligned}
J(s, c, \alpha) &= \alpha J(s, c-1, \alpha) + (1-\alpha) \max_v \Big(v\big(\bar{F}^{-1}(v) + J(s-1, c, \alpha)\big) + (1-v)J(s, c, \alpha)\Big) \\
&= \alpha J(s, c-1, \alpha) + (1-\alpha)J(s, c, \alpha) \\
&\qquad\qquad + (1-\alpha) \max_v \Big(g(v) - v\big(J(s, c, \alpha) - J(s-1, c, \alpha)\big)\Big)
\end{aligned} \tag{1}
$$

The optimization problem consists of finding a v that minimizes $g(v)p - v\big(J(s, c, \alpha) - J(s-1, c, \alpha)\big)$. Let us denote the optimal policy by $v^*(s, c, \alpha)$. Taking the derivative with respect to v we obtain:

$$
g'\big(v^*(s, c, \alpha)\big) = J(s, c, \alpha) - J(s-1, c, \alpha) \tag{2}
$$

Substituting $J(s, c, \alpha) = g'\big(v^*(s, c, \alpha)\big) + J(s-1, c, \alpha)$ back in equation 1 shows that $v^*(s, c, \alpha)$ satisfies the following equation:

$$
\alpha\big(g'(v) + J(s-1, c, \alpha)\big) = \alpha J(s, c-1, \alpha) + (1-\alpha)(g(v) - vg'(v)) \tag{3}
$$

Equation 26 gives a way to recursively compute the value of $J(s, c, \alpha)$ in terms of $J(s-1, c, \alpha)$ and $J(s, c-1, \alpha)$. The base cases for this recursion are:

$$
J(s, 0, \alpha) = J(0, c, \alpha) = 0, \quad c \geq 0, s \geq 0 \tag{4}
$$

Now that we have a means for choosing the price given the negative binomial distribution for the number of customers, we can plug-in the learned distribution of equation 10 into this formula. The optimal value for variable v at time t is equal to:

$$
v^*\Big(s, k+i+1, \frac{a+t}{a+T}\Big) \tag{5}
$$

To implement this policy, the seller needs to set a price at each time step. At time t, if there exist s items to sell and i customers have shown up so far, the optimal price is given by:

$$\bar{F}^{-1}\Big(v^*\big(s, k+i+1, \frac{a+t}{a+T}\big)\Big) \tag{6}$$

2 Learning the arrival rate

Since the conjugate prior for Poisson distribution is Gamma distribution, Lin [2006] assumes a Gamma prior on the arrival rate of customers (i.e. the parameter Λ). With a Gamma prior with parameters (k, a), the distribution of arrival rate will be given by equation 7.

$$f_\Lambda(\lambda) = \frac{ae^{-a\lambda}(a\lambda)^{k-1}}{\Gamma(k)} \tag{7}$$

It is known that a Gamma mixture of Poisson distributions gives rise to a negative binomial distribution. In our setting this is the case for the total number of customers, N.

$$\begin{aligned} P(N=n) &= \int_0^\infty P(N=n|\Lambda=\lambda) f_\Lambda(\lambda) d\lambda \\ &= \int_0^\infty e^{-\lambda T}\frac{(\lambda T)^n}{n!}\frac{ae^{-a\lambda}(a\lambda)^{k-1}}{\Gamma(k)} d\lambda \\ &= \frac{\Gamma(n+k)}{n!\Gamma(k)}\Big(\frac{a}{a+T}\Big)^k\Big(\frac{T}{a+T}\Big)^n \end{aligned} \tag{8}$$

If k is an integer, equation 8 is equivalent to a negative binomial distribution with parameters $(k, {}^{a}/_{a+T})$:

$$P(N=n) = \binom{n+k-1}{n}\Big(\frac{a}{a+T}\Big)^k\Big(\frac{T}{a+T}\Big)^n \tag{9}$$

In general, when a Gamma prior with parameters (α, β) is assumed for the rate of a Poisson distribution, observing n values x_i from the distribution gives a Gamma posterior with parameters $(\alpha + \sum x_i, \beta + n)$. Using this property we can update the arrival rate each time that a new customer arrives. More formally, if i customers have shown up before time t and the $i+1$-th customer arrives at time t, then the posterior of Λ will be a Gamma distribution with parameters $(k+i+1, a+t)$. This in turn enables us to re-calculate the distribution of the number of future customers that will show up in the interval $(t, T]$:

$$\begin{aligned} & P\big(N(T) - N(t) = n \mid N(t^-) = i, \text{arrival at time } t\big) \\ & \quad = \int_0^\infty \frac{e^{-(T-t)\lambda}\big((T-t)\lambda\big)^n}{n!} \frac{(a+t)e^{-(a+t)\lambda}\big((a+t)\lambda\big)^{k+i}}{(k+i)!} d\lambda \\ & \quad = \binom{n+k+i}{n} \Big(\frac{a+t}{a+T}\Big)^{k+i+1} \Big(\frac{T-t}{a+T}\Big)^n \end{aligned} \tag{10}$$

3 Learning the acceptable price

The work of Lin [2006] updates only the arrival rate of customers according to the observations and assumes that the distribution of acceptable price (denoted by $F(p)$ earlier) is known. However, it is admitted in that study that this assumption is not always valid. An example mentioned there is introduction of a seasonal garment. In this case is difficult to predict how much money the customers are willing to pay. The author then suggests this question as a possible future research direction.

This problem is investigated by Babaki [2008]. The challenge is raised by the fact that there were no direct observations of the acceptable price. Assume that the seller sets a price p and the acceptable price for a customer is $\hat{p}$. If the customer buys the item, we will only know that $p \leq \hat{p}$. Similarly, if the customer does not buy the item, we will know that $p > \hat{p}$.

The method used by Babaki [2008] is a small modification to a method that Berk et al. [2007] proposed for Bayesian learning with censored data in a different setting. In section 3.1 we will summarize the part from the work of Berk et al. [2007] which deals with Gamma distributed censored data.

3.1 Bayesian learning with censored data

Assume that the random variable X has a Gamma distribution with parameters (α, β). However, instead of directly observing X, we observe $M = \min(s, X)$. The density of the observed variable M can be written as:

$$f_M(y \mid \alpha, \beta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta y} y^{\alpha-1} & \text{if } y < s \\ \int_s^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha-1}\, dx & \text{if } y = s \end{cases} \tag{11}$$

We will present a theorem from Berk et al. [2007] (the proof is presented in appendix 4.2):

Theorem 3.1. *Suppose random variable M has a gamma distribution with known shape parameter α and random scale parameter β, where the prior distribution for β is also gamma, with shape and scale parameters γ and τ respectively. Then*

a) *The posterior distribution of β is given as:*

$$f_M(y \mid \alpha, \gamma, \tau) = \begin{cases} \frac{\beta^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} e^{-(\tau+y)\beta}(\tau+y)^{\alpha+y} & \text{if } y < s \\ \frac{\beta^{\alpha+\gamma-1}}{\Gamma(\gamma)\Gamma(\alpha)} \frac{e^{-\tau\beta}\tau^{\gamma}}{\bar{F}_{B(\alpha,\gamma)}(s/(s+\tau))} \int_s^\infty e^{-\beta u} u^{\alpha-1}\, du & \text{if } y = s \end{cases} \tag{12}$$

b) *The first two moments of the posterior distribution of β under censoring are given by:*

$$m_1 \equiv E(\beta \mid M = s, \alpha, \gamma, \tau) = \frac{\gamma}{\tau} \frac{\bar{F}_{B(\alpha,\gamma+1)}(s/(s+\tau))}{\bar{F}_{B(\alpha,\gamma)}(s/(s+\tau))} \tag{13}$$

$$m_2 \equiv E(\beta^2 \mid M = s, \alpha, \gamma, \tau) = \frac{\gamma(\gamma+1)}{\tau^2} \frac{\bar{F}_{B(\alpha,\gamma+2)}(s/(s+\tau))}{\bar{F}_{B(\alpha,\gamma)}(s/(s+\tau))} \tag{14}$$

where B is a beta random variable with parameters (α, γ).

When there is no censoring (that is, $y < s$), the posterior of β is a gamma distribution with shape parameter $\gamma^* = \alpha+\gamma$ and scale parameter $\tau^* = \tau+y$. In the case of censoring, the posterior is not a gamma distribution anymore. In this case, we approximate the posterior by a gamma distribution such that the first and second moments of this gamma distribution are equal to the moments of the true posterior.

Recall that for a gamma distribution with parameters (γ, τ) the first moment is given by $m_1 = \gamma/\tau$ and the second moment is given by $m_2 = \gamma(\gamma+1)/\tau^2$. Hence in the case of censored observations, we approximate the posterior with a gamma distribution with the following parameters:

$$\tau^* = \frac{m_1}{m_2 - m_1^2} \tag{15}$$

$$\gamma^* = \frac{m_1^2}{m_2 - m_1^2} \tag{16}$$

and the posterior distribution will have the following form:

$$f(x \mid \alpha, \gamma^*, \tau^*) = \frac{\Gamma(\alpha+\gamma^*)}{\Gamma(\alpha)\Gamma(\gamma^*)} \frac{1}{x} \Big(\frac{x}{\tau^*+x}\Big)^{\alpha} \Big(\frac{\tau^*}{\tau^*+x}\Big)^{\gamma^*} \tag{17}$$

3.2 Bayesian Learning of Acceptable Price

In this section we modify the method proposed by Berk et al. [2007] to adapt it to the problem of learning the acceptable price. This section summarizes the main contribution of Babaki [2008].

We assume that the acceptable price of customers follows a gamma distribution with parameters (α, β). Let us denote the event that a customer buys and item at price p is by B_p^+ and its complement by B_p^-. The probabilities of these events are given by:

$$P(B_p^+ \mid \alpha, \beta) = \int_p^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha-1}\, dx \tag{18}$$

$$P(B_p^- \mid \alpha, \beta) = \int_{-\infty}^p \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha-1}\, dx \tag{19}$$

The following theorem from Babaki [2008] shows that we can approximate the posterior distribution for acceptable price in a manner similar to the method proposed by Berk et al. [2007] (the proof is very similar to the proof of theorem 3.1 and is dropped for brevity):

Theorem 3.2. *Suppose the acceptable price has a gamma distribution with known shape parameter α and random scale parameter β, where the prior distribution for β is also gamma, with shape and scale parameters γ and τ respectively. Then*

a) *If a customers buys an item at price p, the posterior distribution of β is given by:*

$$f(\beta \mid p, \alpha, \gamma, \tau) = \frac{\beta^{\alpha+\gamma-1}}{\Gamma(\gamma)\Gamma(\alpha)} \frac{e^{-\tau\beta}\tau^\gamma}{\bar{F}_{B(\alpha,\gamma)}(p/(p+\tau))} \int_p^\infty e^{-\beta u} u^{\alpha-1}\, du \tag{20}$$

b) *If a customer buys an item at price p, the first and second moments of posterior distribution of β are given by:*

$$m_1 \equiv E(\beta \mid p, \alpha, \gamma, \tau) = \frac{\gamma}{\tau} \frac{\bar{F}_{B(\alpha,\gamma+1)}(p/(p+\tau))}{\bar{F}_{B(\alpha,\gamma)}(p/(p+\tau))} \tag{21}$$

$$m_2 \equiv E(\beta^2 \mid p, \alpha, \gamma, \tau) = \frac{\gamma(\gamma+1)}{\tau^2} \frac{\bar{F}_{B(\alpha,\gamma+2)}(p/(s+\tau))}{\bar{F}_{B(\alpha,\gamma)}(p/(p+\tau))} \tag{22}$$

c) *If a customers refuses to buy an item at price p, the posterior distribution of β is given by:*

$$f(\beta \mid p, \alpha, \gamma, \tau) = \frac{\beta^{\alpha+\gamma-1}}{\Gamma(\gamma)\Gamma(\alpha)} \frac{e^{-\tau\beta}\tau^\gamma}{\bar{F}_{B(\alpha,\gamma)}(p/(p+\tau))} \int_{-\infty}^p e^{-\beta u} u^{\alpha-1}\, du \tag{23}$$

d) *If a customer refuses to buy an item at price p, the first and second moments of posterior distribution of β are given by:*

$$m_1 \equiv E(\beta \mid p, \alpha, \gamma, \tau) = \frac{\gamma}{\tau} \frac{F_{B(\alpha,\gamma+1)}(p/(p+\tau))}{F_{B(\alpha,\gamma)}(p/(p+\tau))} \tag{24}$$

$$m_2 \equiv E(\beta^2 \mid p, \alpha, \gamma, \tau) = \frac{\gamma(\gamma+1)}{\tau^2} \frac{F_{B(\alpha,\gamma+2)}(p/(p+\tau))}{F_{B(\alpha,\gamma)}(p/(p+\tau))} \tag{25}$$

where B is a beta random variable with parameters (α, γ).

After obtaining the moments using this theorem, we can approximate the posterior distribution of acceptable price using equations 16 and 17.

4 Appendices

4.1 The recurrence relation 1

In this section we will see why the following recurrence relation holds:

$$\begin{aligned} J(s,c,\alpha) &= \alpha J(s,c-1,\alpha) \\ &\quad + (1-\alpha)\max_v \Big(v\big(\bar{F}^{-1}(v) + J(s-1,c,\alpha)\big) + (1-v)J(s,c,\alpha) \Big) \end{aligned} \tag{26}$$

Recall that the total number of remaining customers follows a negative binomial distribution with parameters (c,α). The distribution of this variable is

$$P(X=i) = \binom{i+c-1}{i} \alpha^c (1-\alpha)^i \quad i = 0,1,2,\dots \tag{27}$$

The random variable X can be interpreted as the number of failures before the cth success. Note that each *'success'* will reduce the first parameter of the negative binomial distribution by one.

At each time, there is an α chance for *'success'*, that is, a chance that the distribution of remaining customers changes into a negative binomial with parameters $(c-1,\alpha)$. This is reflected by $\alpha J(s,c-1,\alpha)$ in equation 26. Otherwise, we should set a price for the arriving customer. We do this indirectly by setting the probability that the customer will accept the price. If the customer accepts the price, we will immediately obtain a profit of $\bar{F}^{-1}(v)$ which we sum with expected future profit $J(s-1,c,\alpha)$. Otherwise we will only have the expected future profit $J(s,c,\alpha)$.x

4.2 Proof of theorem 3.1

To simplify the proof, we first present this lemma from Berk et al. [2007]:

Lemma 4.1. *Let*

$$I(\alpha,\gamma,\tau,s) = \int_s^\infty \frac{x^{\alpha-1}\tau^\gamma}{(\tau+x)^{\alpha+\gamma}}\,dx \tag{28}$$

Then

$$I(\alpha,\gamma,\tau,s) = \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} P(B > {}^{s}/_{(s+\tau)}) = \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \bar{F}_{B(\alpha,\gamma)}({}^{s}/_{(s+\tau)}) \tag{29}$$

where B is a beta random variable with parameters α, γ with the tail probability given by $\bar{F}_{B(\alpha,\gamma)}({}^{s}/_{(s+\tau)})$.

Proof.

$$\int_s^\infty \frac{x^{\alpha-1}\tau^\gamma}{(\tau+x)^{\alpha+\gamma}}\,dx = \int_s^\infty \Big(\frac{x}{\tau+x}\Big)^{\alpha-1}\Big(1-\frac{x}{\tau+x}\Big)^{\gamma-1}\frac{\tau}{(\tau+x)^2}\,dx \tag{30}$$

By substituting $y = \frac{x}{\tau+x}$, we obtain:

$$\begin{aligned} I(\alpha,\gamma,\tau,s) &= \int_{s/s+\tau}^\infty y^{\alpha-1}(1-y)^{\gamma-1}\,dy \\ &= \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\int_{s/s+\tau}^\infty \frac{y^{\alpha-1}(1-y)^{\gamma-1}}{B(\alpha,\gamma)}\,dy \\ &= \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\bar{F}_{B(\alpha,\gamma)}(s/(s+\tau)) \end{aligned} \tag{31}$$

□

Now we can prove the theorem 3.1:

Proof. The joint distribution of M and β is given by

$$f_M(y,\beta \mid \alpha,\gamma,\tau) = \begin{cases} \frac{\beta^{\alpha+\gamma-1}}{}e^{-(\tau+y)\beta}(\tau+y)^{\alpha+\gamma} & \text{if } y < s \\ \int_s^\infty \frac{\beta^\alpha\tau^\gamma}{\Gamma(\alpha)\Gamma(\gamma)}e^{-\beta(\tau+x)}x^{\alpha-1}\beta^{\gamma-1}\,dx & \text{if } y = s \end{cases} \tag{32}$$

and the marginal distribution of M is given by

$$f_M(y \mid \alpha,\gamma,\tau) = \begin{cases} \frac{\Gamma(\alpha+y)}{\Gamma(\alpha)\Gamma(y)}\frac{y^{\alpha-1}\tau^\gamma}{(\tau+y)^{\alpha+\gamma}} & \text{if } y < s \\ \frac{\Gamma(\alpha+y)}{\Gamma(\alpha)\Gamma(y)}\int_s^\infty \frac{x^{\alpha-1}\tau^\gamma}{(\tau+x)^{\alpha+\gamma}}\,dx & \text{if } y = s \end{cases} \tag{33}$$

The proof of part **a** of theorem 3.1 is obtained by taking the ratio of these two functions. To prove part **b** note that

$$\begin{aligned} m_1 &= \int_0^\infty \beta\frac{\beta^{\alpha+\gamma-1}}{\Gamma(\gamma)\Gamma(\alpha)}\frac{e^{-\tau\beta}\tau^\gamma}{\bar{F}_{B(\alpha,\gamma)}(s/s+\tau)}\int_s^\infty e^{-\beta u}u^{\alpha-1}\,du\,d\beta \\ &= \frac{\tau^\gamma}{\Gamma(\gamma)\Gamma(\alpha)\bar{F}_{B(\alpha,\gamma)}(s/s+\tau)}\int_s^\infty u^{\alpha-1}\int_0^\infty \beta^{\alpha+y}e^{-(\tau+u)\beta}\,d\beta\,du \\ &= \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma)\Gamma(\alpha)\bar{F}_{B(\alpha,\gamma)}(s/s+\tau)}\int_s^\infty \frac{\tau^\gamma u^{\alpha-1}}{(\tau+u)^{\alpha+\gamma+1}}\,du \\ &= \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma)\Gamma(\alpha)\bar{F}_{B(\alpha,\gamma)}(s/s+\tau)}\frac{1}{\tau}I(\alpha,\gamma+1,\tau,s) \end{aligned} \tag{34}$$

The result follows using lemma 4.1. The second moment m_2 is obtained similarly. □

References

Behrouz Babaki. Dynamic pricing of perishable asset using demand learning. Master's thesis, Sharif University of Technology, 2008. Available Only in Persian Language.

Emre Berk, Ülkü Gürler, and Richard A. Levine. Bayesian demand updating in the lost sales newsvendor problem: A two-moment approximation. *European Journal of Operational Research*, 182(1):256–281, 2007.

Kyle Y. Lin. Dynamic pricing with real-time demand learning. *European Journal of Operational Research*, 174(1):522–538, 2006.