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A PERSPECTIVE ON VIBRATION-INDUCED SIZE SEGREGATION OF GRANULAR MATERIALS

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Abstract

Segregation of particulate mixtures is a problem of great consequence in industries involved with the handling and processing of granular materials in which homogeneity is generally required. While there are several factors that may be responsible for segregation in bulk solids, it is well accepted that nonuniformity in particle size is a fundamental contributor. When the granular material is exposed to vibrations, the question of whether or not convection is an essential ingredient for size segregation is addressed by distinguishing between the situation where vibrations are not sufficiently energetic to promote a mean flow of the bulk solid, and those cases where a convective flow does occur. Based on experimental and simulation results in the literature, as well as dynamical systems analysis of a recent model of a binary granular mixture, it is proposed that "void-filling" beneath large particles is a universal mechanism promoting segregation, while convection essentially provides a means of mixing enhancement.

Keywords: Granular Materials, Microstructure, Nonlinear Dynamics, Particulate, Segregation, Discrete Element Simulation

1. Introduction

Segregation is a process in which a homogeneous bulk solid composed of various constituents or species becomes spatially non-uniform as a result of relative movement within the material (Rosato and Blackmore 2000). Its occurrence is of universal concern to the industrial sector involved with the processing and handling of bulk solids, as well as to geologists in exploring the nature of geophysical flows. (See for example, (Davies 1988; Savage and Hutter 1991; Savage 1992; McElwaine and Nishimura 2000). Within the confines of industry, the critical issue when segregation occurs is that it poses a formidable impediment to the general requisite of creating and maintaining homogeneous mixtures. An inability to achieve and maintain a well-mixed condition of a bulk solid throughout its processing history can lead to serious flaws in the properties of an end product, leading to unfavorable economic consequences (Carson, Royal et al. 1986; Bates 1997). The magnitude of the problem becomes evident when one simply considers the breadth of industries where segregation is an issue, i.e., chemicals, food, agriculture, pharmaceuticals, mining, materials, energy, munitions, and electronics.

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Studies of segregation under a variety of flow situations have appeared throughout the literature on granular systems. The fact that the phenomenon is rather ubiquitous in bulk solids where particle size differences are present indicates that this property is a primary trigger. Below we highlight several relevant papers that demonstrate some of the salient features of vibration-induced segregation. For a comprehensive classification of the literature up to 1976, the reader is also referred to the review of Cook et al. (Cooke, Stephens et al. 1976).

Perhaps J. C. Williams (Williams 1963) carried out one of the first qualitative studies on the effect of vibration frequency on the motion of a single large sphere in a bed of sand that was oscillated vertically. He attributes the sphere's observed upward trajectory to the "locking" effect of the overburden pressure that it exerts on the column of materials directly beneath it, thereby preventing it from moving down. If the large particle does experience an upward movement during the vibration, smaller particles could easily move beneath it and become locked. In a subsequent paper (Williams 1976), he highlights particle properties that can promote segregation - namely particle size, density, shape and elasticity - and he also describes three mechanisms of segregation - trajectory segregation, percolation of fines and the rise of coarse particles due to vibration. A series of articles (Olsen and Rippie 1964; Rippie, Olsen et al. 1964; Faiman and Rippie 1965; Rippie, Faiman et al. 1967) on the behavior of systems of steel spheres subjected to vertical sinusoidal vibrations explored the influence of particle size and distribution, particle density, agitation intensity, and particle shape on energy requirements. From the experiments, the authors are able to model the segregation process via first-order kinetics, with the system evolving to an equilibrium state characterized by a balance between mixing and separation. In addition, they find evidence indicating that the presence of an intermediate particle size reduces the segregation rate in the system, an effect that was also seen in Monte Carlo simulations (Rosato, Lan et al. 1991). An investigation (Ahmad and Smalley 1973) quantifying the effects of vibration frequency and acceleration on the time required for a single large particle in a sand bed to rise to the surface illustrated several interesting trends that are important to note. With an increase of acceleration at constant frequency, this rise time was reduced; however, within certain acceleration ranges, an increase of frequency at constant acceleration resulted in a longer rise time. Although size effects were dominant within their experimental conditions, it was observed that the higher the material density of the large particle, the lower was its tendency to rise. A subsequent experiment by Harwood (Harwood 1977) using radioactive tracer powders demonstrated the importance of the material properties as well as vibration amplitude and frequency.

A crucial aspect of the phenomenon is that segregation is often governed by a combination of mechanisms, which themselves depend on the nature of the flow, particle properties and environmental conditions. Consider, for example, those particle features which can influence segregation behavior: particles sizes and distributions, shape, morphology, contact friction, elasticity, brittleness, density, chemical affinities, ability to absorb moisture, and magnetic properties. An additional complexity is brought about by the interplay between different time and length scales. These scales depend upon the means by which energy is imparted to the bulk solid causing it to flow, such as through vibrations (Evesque and Rajchenbach 1989; Fauve, Douady et al. 1989; Laroche, Douady et al. 1989; Clement and Rajchenbach 1991; Clement, Duran et al. 1992; Gallas, Herrmann et al. 1992; Taguchi 1992; Brennen, Ghosh et al. 1993; Hunt, Hsiau et al. 1994; Lan and Rosato 1995; Aoki, Akiyama et al. 1996; Clement, Vanel et al. 1996; Knight 1997; Lan and Rosato 1997), shearing (Lun, Savage et al. 1984;

Campbell and Brennen 1985; Savage 1991; Savage and Dai 1991; Rosato and Kim 1994; Karion and Hunt 2000), and gravity (Fowler 1960; Fowler 1961; Campbell 1990; Walton 1991; Furll 1992). Consequently, individual researchers have had to focus their efforts on narrow ranges of conditions in an attempt to identify, isolate or enhance dominant mechanisms. As an example, we refer to the pioneering work of Bridgwater and his colleagues (Bridgwater, Sharpe et al. 1969; Bridgwater and Ingram 1971; Scott and Bridgwater 1975; Bridgwater 1976; Cooke, Stephens et al. 1976; Scott and Bridgwater 1976; Cooke, Bridgwater et al. 1978; Stephens and Bridgwater 1978; Stephens and Bridgwater 1978; Stephens, Cooke et al. 1978; Cooke and Bridgwater 1979; Foo and Bridgwater 1983; Bridgwater, Foo et al. 1985; Bridgwater 1999).

The theme of this paper is on segregation that takes place as a consequence of vertical vibrations applied to bulk solids whose constituent particles are of non-uniform size. Within this context, we review the literature describing the dynamics of a single large particle or intruder in a vertically vibrated bed of mono-sized particulates. While this behavior is not sufficient to predict segregation patterns in systems where size distributions exist, it is not unreasonable to expect that this may provide some measure of the tendency of a specific bulk solid to become spatially inhomogeneous. We remark that industrial systems often employ a combination of vertical and horizontal vibrations, such as in screeners. However, there have been relatively few studies on only horizontal shaking (Herrmann 1994; Liffman, Metcalfe et al. 1997). Equally important is the effect of particle density on segregation in vibrated systems, where again there does not appear to be a large body of literature. (See for example, (Rippie, Olsen et al. 1964; Ohtsuki, Kinoshita et al. 1995; Shinbrot and Muzzio 1998; Eggert and Wu 2000)). An understanding of the relationship between size and density in segregating systems is unclear. Shinbrot and Muzzio propose that the sinking of large, low density particles in strongly vibrated deep beds may be promoted by the important role that inertia plays in causing heavy objects to lift off the bed, in contrast to lighter objects that are more inclined to exhibit more chaotic motions. In situations where the flow is promoted by gravity, such as in chute flows inclined at slopes much greater than the bed-friction angle, recent experiments (Vallance and Savage 2000) indicate that density segregation is far weaker than that promoted by particle size. Another consideration is segregation that is promoted by particle shape. Although we shall focus on size effects in this paper, recent studies have shown that particles shapes – such as angular, needle and plate-like – have some interesting effects on segregation and mixing. More information on this topic can be found in (Matsen 1983; Wanibe and Itoh 1999; Brown, Clarke et al. 2000).

In this paper, we consider two extreme or limiting flow cases: (1) the vibrations are not energetic enough to promote a mean flow within the bulk solid; and (2) the vibrations induce a mean convective flow. This paper is organized as follows. In Section 2, an introduction to discrete element simulations is given since this technique is now widely used in investigations of the behavior of particle assemblies. In the next section a discussion of the effects of vibrations on granular materials is presented, which primarily emphasizes convection and densification. Section 4 examines the relationship between segregation, convection and densification. In the conclusion, it is proposed that void-filling is a universal sorting mechanism which is present with or without convection.

2. Discrete Element Simulations

The development of the “discrete element” simulation method and the considerable growth in its application over approximately the last ten years due to advances in computing technology has been instrumental in enhancing our ability to understand the physics governing the complex behavior of granular or particulate flows. Besides furnishing a means of interpreting developing theories, these simulations provide invaluable information on the evolution of internal microstructure and its relationship to observable or measurable phenomena. Although the method is actually an outgrowth of molecular dynamics modeling first reported by Alder and Wainwright (Alder and Wainwright 1956) and later extended by Ashurst and Hoover (Ashurst and Hoover 1973) for nonequilibrium problems of dense fluid transport, it was independently developed by Cundall (Cundall 1974) in the 1970’s. Flows of practical interest all involve inelastic particles that are far from their equilibrium state. Therefore, nonequilibrium statistical theories are required to model flowing granular solids, and not the more usual thermodynamic equilibrium theories of gas dynamics.

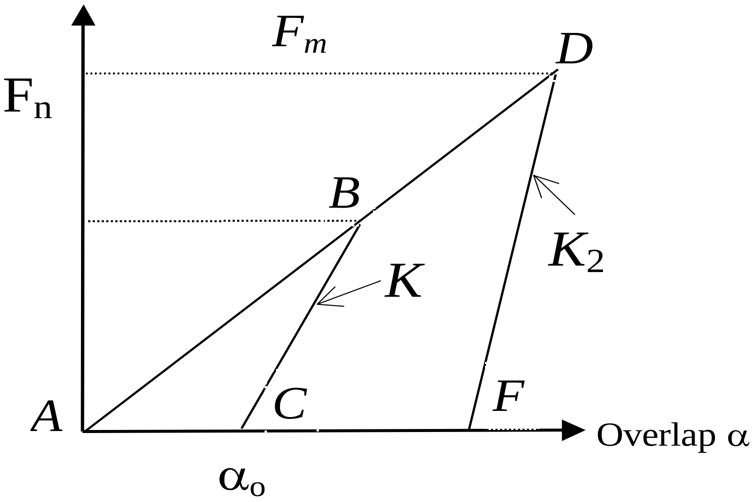

Fig. 1: Normal force model of Walton and Braun (1986).

In contrast to energy-conserving gas molecules which behave essentially as smooth, perfectly elastic particles, macroscopic granular particles suffer collisional energy losses and can spin as a result of the transfer of tangential momentum. Therefore, it is necessary to employ force models that approximate the behavior of real colliding granules (Walton 1992; Schafer, Dippel et al. 1996). With the prescription of appropriate contact laws based on idealized dissipative collisions, the equations of motion of the system of particles can be numerically integrated to yield the complete specification of the history of all particle positions and velocities. This data may be used to compute transport properties, which can be correlated with the evolution of the microstructure.

In general, force models for granular flows in the absence of an interstitial fluid can be categorized as using either “hard” or “soft” particles. Rigid “hard” particle models. (for example, (Campbell and Brennen 1985; Campbell 1990; Savage 1991; Walton 1991)) are based on instantaneous, binary collisions, and are restricted to systems dominated by collisions, i.e., those in which continuous contacts are not characteristic. Numerical studies employing these particles have also been termed “event driven” simulations (Lubachevsky 1991). The main feature is that post-collisional kinematics are computed from a collision operator which is a function of interaction parameters (i.e., coefficient of friction, and normal and tangential restitution coefficients) and the pre-collisional velocities and spins of the particles (Walton 1992). Soft models on the other hand admit forces that are proportional to a small, allowed overlap α between particles. Within this class, there are "spring-dashpot" (SD) and hysteretic-like models. SD interactions (Luding 1994) employ an elastic restoring force in the normal direction $F_n^e = -K\alpha^{\beta}$, which for Hertzian contact between elastic spheres of diameter d is given by $F_n^e = \left(E\sqrt{d}/(1-\nu^2)\right)\ \alpha^{3/2}$, where E is Young's modulus and ν is Poisson's ratio. Energy loss associated with particle interaction in the normal direction is obtained through a damping force $F_n^d = -D\alpha^{\lambda}\dot{\alpha}$, and selection of the constants β and λ will determine the variation of the restitution coefficient e with impact velocity. Nonphysical behavior may result with certain choices of the constants β and λ. For example, by using β = 3/2 and λ = 0, an increase of e with relative impact velocity occurs in contradistinction to the behavior of real materials (Goldsmith 1960; Johnson 1987). In the discussions in Section 3 and 4 of this paper, several discrete element investigations are described. These employ the soft sphere models derived by Walton and Braun (Walton and Braun 1986; Walton and Braun 1986), which have been shown to reproduce (Walton 1992) the almost linear loading behavior for spherical surfaces that experience plastic deformation of the order of one percent of a particle diameter. In the direction along the line of centers of the contacting spheres, loading and unloading springs of stiffness K_1 and K_2, respectively, determine the normal force. This can be expressed as,

$F_n = \begin{cases} K_1\alpha^{\beta}, & \text{for loading} \\ K_2(\alpha-\alpha_o)^{\beta}, & \text{for unloading} \end{cases}$, where α_o is the overlap remaining when $F_n = 0$. For the linear case (β = 1), Figure 1 illustrates the paths for the normal force versus overlap α. Loading is along path *AB*, and if unloading occurs at *B*, it follows the line *BC* to α_o. At this point, if there is no reloading (which would follow *CBD* with unloading along *DECA)*, particles continue to separate at constant velocity to a zero overlap. The linear model produces a constant restitution coefficient given by $e = \sqrt{K_1/K_2}$ if the unloading paths (lines *BC* and *DF* in Fig. 1) have the same slope K_2. A variable restitution coefficient as a function of the relative approach velocity v_o may be obtained by allowing K_2 to be linearly dependent on the maximum force realized before unloading $K_2 = K_1 + cF_m$. This is given by $e = \sqrt{\Omega_o/(cv_o + \Omega_o)}$, where the parameter c is empirically determined and $\Omega_o \equiv \sqrt{2K_1/m}$. We remark that models that yield the theoretically predicted behavior $e \propto v_o{}^{-1/4}$ (Johnson 1987), namely plastic deformation in the contact region, have been developed (Ning and Thornton 1993), capturing the rebound of particles from surfaces after elastic-plastic impact has occurred.

In the tangential direction, Walton and Braun (Walton and Braun 1986) approximated Mindlin's and Deresiewicz's (Mindlin and Deresiewicz 1953) theory for elastic spheres subjected to tangential loading, resulting in a one-dimensional "incrementally slipping" friction force model for disks which was subsequently extended to three dimensions for spheres (Walton and Braun 1986). Although this does not include the effects of contacts that experience rotation coupled with tangential sliding, particles can rotate due to the transmission of tangential impulse. In accordance with the theory, full sliding of the contact occurs when the limiting force μF_n is attained as a result of increasing elastic strain. This takes place when the contact tangential stiffness, given by

$$K_{\tan} = \begin{cases} K_0\left[1-(T-T^*)/\left(\mu F_n - T^*\right)\right]^{1/3}, & T \text{ increasing} \\ K_0\left[1-(T^*-T)/\left(\mu F_n + T^*\right)\right]^{1/3}, & T \text{ decreasing} \end{cases}$$

decreases to zero. In the latter expression, F_n and T are the magnitudes of the current normal and tangential forces, respectively, $K_0 \approx 0.8K_1$ is an initial tangential stiffness, μ is the friction coefficient and T^* is the turning point value of T. We remark that the recent paper of Vu-Quoc and Zhang (Vu-Quoc and Zhang 1999) reports on an improved tangential force model for elastic frictional contacts based on Walton and Braun's original model.

3. Some Important Features of Vibrated Granular Materials

Under rapid deformation conditions, granular materials appear to respond much like fluids. Numerous examples of this behavior are visible in the industrial handling and processing of particulates (i.e., screening, sieving and powder mixing), as well as in naturally occurring events, such as landslides and avalanches. The main feature of these highly energetic flows is that momentum and vibrational kinetic energy are transmitted from boundaries or a free surface to the interior through impacts or collisions between the granules. Consequently, the solids fraction (i.e., the volume fraction occupied by solids) will be substantially different from its value when the material is at rest, while the granular temperature (Ogawa 1978) (i.e., the kinetic energy associated with the fluctuating or deviatoric particle velocities) can display significant spatial variations. These latter characteristics reveal the inherent nonequilibrium nature of these flows.

When a vessel containing granular materials is "strongly" vibrated, fluid-like behavior can occur, with the possible appearance of arching, surface waves and convection, such as in (Savage 1988; Douady, Fauve et al. 1989; Fauve, Douady et al. 1989; Clement and Rajchenbach 1991; Clement, Duran et al. 1992; Taguchi 1992; Lan and Rosato 1995; Wassgren 1996; Knight 1997; Lan and Rosato 1997). On the other hand, the application of small amplitude/high frequency oscillations (Vanel, Rosato et al. 1997; Rosato and Yacoub 2000) or taps (Takahashi and Suzuki 1986; Knight, Frandrich et al. 1995; Nowak 1997) can promote subtle changes in the microstructure resulting in an increase in bulk density. Such an effect has important applications for instance in industries where there is major interest in reducing costs associated with transportation of bulk solids over long distances.

The fluid like nature of vibrated particulates is revealed in the experiments of Hunt et al. (Hunt, Hsiau et al. 1994) who reported a striking volume increase of the bed at relative accelerations $\Gamma \equiv a\omega^2/g \approx 2$, where a is the displacement amplitude, ω is the vibration frequency and g is gravitational acceleration. To explain this phenomenon, Brennen et al. (Brennen, Ghosh et al. 1993) offer a plausible theory that is based on an analogy with the bifurcation of the solution for a single sphere bouncing on a vibrating plate, for which the critical value of relative acceleration, as a function of the normal restitution coefficient (e), is given by $\Gamma_{cr} \equiv \left(a\omega_{cr}^2\right)/g = \pi(1-e)/(1+e)$. By replacing e with an assumed bulk restitution coefficient of the bed $\varepsilon = 0.25$, the computed value of $\Gamma_{cr} = 1.88$ was found to be in good agreement with the experimental results. Discrete element simulations (Lan and Rosato 1995) demonstrated the same expansion in a bed of smooth, inelastic spheres with a computed bulk restitution $\varepsilon = 0.32$.

Kinetic theory models have provided some insight into behavior produced by vibrations by utilizing the correspondence between the motion of molecules of dense gases and the fluctuating movement of granular particles that experience energy dissipation through inelastic collisions (Ogawa 1978). One such model is that of Richman and Martin (Richman and Martin 1992) who report the steady-state depth profiles of solids fraction and granular temperature in a bed of uniform, smooth, inelastic spheres that are subjected to vibrations induced by a bumpy floor having a Maxwellian velocity distribution about a zero mean speed. Their theory predicts that the granular temperature profile monotonically decreases upward from the floor, while the solids fraction achieves a maximum at some intermediate location along the depth. Analogous discrete element simulations (Lan and Rosato 1995) at large floor accelerations ($\Gamma \cong 56.5$) agreed quite well with the theoretical predictions, but departed significantly from the theory at lower accelerations (e.g., $\Gamma \cong 2.84, 5.7$). It was inferred from the computations that the deviation at low accelerations was a consequence of a violation of one or more of the assumptions implicit in the kinetic theory (such as the Maxwellian distribution of "peculiar" velocities, and an equal collision frequency between the boundary and flow particles and the flow particles themselves).

Relevant to the topic of this paper is the mean convective flow that develops when particles are vigorously shaken, and, at the opposite extreme, the densification that occurs due to energetic, low amplitude vibrations. Savage (Savage 1988) experimentally studied convection using polystyrene particles in a rectangular vessel that was excited via a vibrating flexible floor membrane with non-uniform amplitude across its width (maximum in the center and a zero at the side walls). The two slowly recirculating vortices identified in the experiments were attributed to "acoustic" waves transmitted through the bed from the floor. Here, particles traveled upwards along the centerline of the box and downward along the walls. A maximum in the magnitudes of the streaming velocities versus base frequency was observed over a range of floor membrane amplitudes. The decrease of these velocities beyond critical values of floor acceleration was explained as a loss of bed contact with the base thereby resulting in a less effective energy transfer. Streaming flow velocities were predicted through an analysis of a kinetic theory based model previously developed by Jenkins and Savage (Jenkins and Savage 1983).

Taguchi (Taguchi 1992) developed a discrete element simulation of vibration-induced convection using inelastic, frictional disks. In contrast to the flexible oscillating floor employed in the Savage experiments, the entire rectangular box of particles was sinusoidally vibrated. It was found that convection "strength" was minimal when $\Gamma = a\omega^2 / g \leq 1.2$. He conjectured that two stages in a single shaking cycle contributed to the development of this flow. When the upward acceleration of the floor was greater than gravity, particles are pushed together causing a pressure to be developed in the bed. During the interval when downward acceleration of floor becomes greater than *g*, particles are in free flight, so that the vertical component of this pressure is relaxed. However, the constraint of the walls would tend to prevent a release of the horizontal pressure component, thus promoting relative horizontal sliding motion between particles. It was argued that this sliding induced a horizontal flow, which would meet at the center of the cell and then turn upwards to create the observed flow pattern. Subsequent simulations of Gallas et al. (Gallas, Herrmann et al. 1992) employed rigid and flexible vibrating floors where, in the latter case, the computational cell had periodic sidewall conditions and a floor whose amplitude varied along its width with a zero value at the ends. The resulting flow field, which was similar to that in the experiments of Savage (Savage 1988), was attributed to the spatial form of the floor excitations. Upon replacing the floor by a rigid, sinusoidally oscillating plane with frictionless sidewalls, particles flowed downward in the center of the cell and upwards along the walls. The authors argued that this pattern was a consequence of horizontal flows (generated by a transfer of part of the relative vertical momentum into the horizontal direction), being forced upwards at the sidewalls that prohibited the horizontal flow. A reversal of the flow pattern occurred when sidewalls were made frictional - particles moved upwards in the center and down along the sidewalls. They suggested that the additional mechanism of a strong downward shear drag force, resulting from pressure exerted on the walls by the compressed packing, was responsible for the downward motion. It was argued that a combined effect of internal horizontal flows and wall shear drag was responsible for a convection pattern when sidewalls were frictional.

Knight et al. (Knight 1997) examined granular convection in non-rectangular vessels using both photographic and MRI techniques to obtain images of the flow. When the walls were tilted outward from the vertical beyond a transition angle, the flow was quite different from that which occurred with vertical walls (i.e., particles move upward in the center and downward in a stream near the walls). In this case, particles moved down in center, accompanied by a substantial upward flow within the triangular region formed by the angled sides. This is explained in terms of a “ratchet mechanism” that opposes the downward wall shear that was hypothesized earlier by Taguchi as the origin of convection. Reversal of the flow direction from up to down in the center can also occur in rectangular vessels with smooth walls. This phenomenon, which was identified earlier in discrete element simulations (Gallas, Herrmann et al. 1992; Lan 1994), was observed in the experiments of Aoki et al. (Aoki, Akiyama et al. 1996).

At this juncture it is worthwhile noting that segregation behavior in vibrated containers where convection is present is strongly influenced by the depth of the filling material (Poschel and Herrmann 1995). Thus segregation patterns that are promoted principally by convection will be necessarily dependent on the amount of filling material. A great deal more research is needed in order to devise scaling relations that could be used in industrial process operations.

It is possible to increase the bulk density of a container of granular solids either by applying vibrations with relatively small displacement amplitudes ($a/d < 0.1$) and high frequencies (Vanel, Rosato et al. 1997; Rosato and Yacoub 2000), or through tapping (Takahashi and Suzuki 1986; Knight, Frandrich et al. 1995; Nowak 1997). We demonstrate this in the following simple experiment. A 6.35 cm diameter acrylic cylinder was rigidly mounted onto an electromagnetic shaker so that a piston could actuate the cylinder’s floor, which was free to move up and down. The cylinder was filled with uniform 3.175 mm diameter (d) acrylic spheres to a poured depth $H = 9.515$ cm. A series of studies over a range of amplitudes ($0.0375 \le a/d \le 0.24$) and frequencies (25 Hz $\le f \le$ 100 Hz) was carried out to determine the change in bulk solids fraction from the poured state after 10 minutes of vibration. These vibration parameters corresponded to relative accelerations $\Gamma \equiv a(2\pi f)^2/g$

between 0.94 and 11.54. A principal finding is presented in Fig. 2, which shows the variation of solids fraction with relative acceleration. For each fixed value of *a*/*d*, data points were obtained by varying the frequency. In general, for amplitudes *a*/*d* less than approximately 0.1, an increase in relative acceleration produced densification within the bed, with each curve exhibiting a different maximum value depending on Γ.

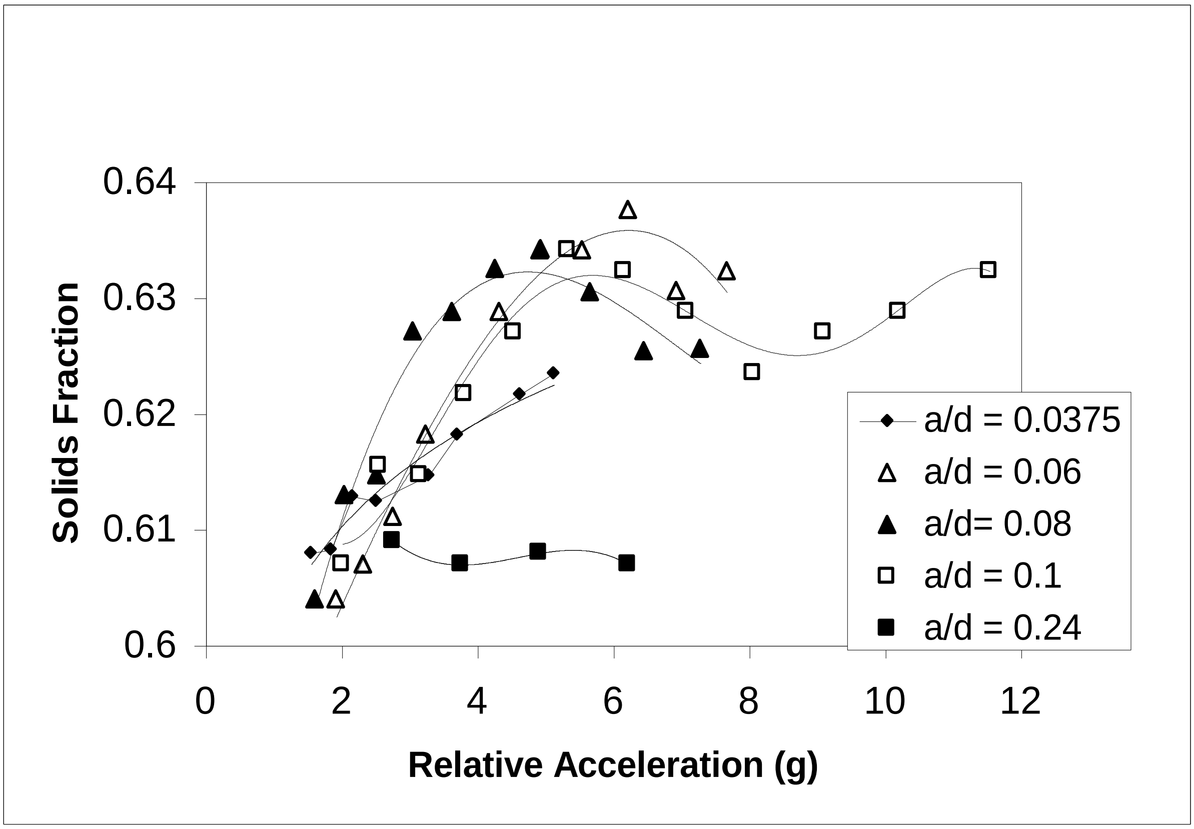

Fig. 2: Densification behavior depicted by solids fraction versus relative acceleration. Lines are drawn to indicate trends.

We have carried out several preliminary discrete element studies (Rosato and Yacoub 2000) of the densification process in a narrow computational cell whose dimensions were chosen so as to promote the development of an orthorhombic structure, which has a maximum solids fraction of 0.6046. The initial packing was obtained by allowing particles to settle under gravity until a stable assembly formed having a height of approximately 7.5*d* and solids fraction $\nu_0 \approx 0.53$. Densification was achieved through the application of two minutes of vertical sinusoidal vibrations to the floor with *a*/*d* = 0.1 so that no mean motion occurred in the bed. Two case studies (Γ = 1.02 and 6.4) were selected corresponding to *f* = 20 and 50 Hz, respectively. Figure 3 presents the evolution of the solids fraction, which is fundamentally in agreement with the phenomenology reported in (Nowak 1997).

4. Convection and Segregation

Depending on the nature of the particle flow, certain segregation mechanisms may become dominant while others remain latent in some sense (Bates 1997; Desilva, Dyroy et al. 2000). In the case of vibrated beds of particles, there has been some controversy concerning the relationship between segregation and convection, particularly on the mechanism responsible for the rise of a large particle to the surface (Jullien, Meakin et al. 1992; Knight, Jaeger et al. 1993; Dippel and Luding 1995; Gallas, Herrmann et al. 1996; Brone and Muzzio 1997; Lan and Rosato 1997; Vanel, Rosato et al. 1997). Perhaps part of the difficulty lies in the fact that often the terms segregation and convection are associated with each other in a manner that implies one cannot exist without the other (See for example (Poschel and Herrmann 1995). However, it is essential to make a clear distinction between those mechanisms that actually promote size sorting, and those that enhance mixing. Here we address the question of whether or not convection is an essential ingredient for vibration induced size segregation.

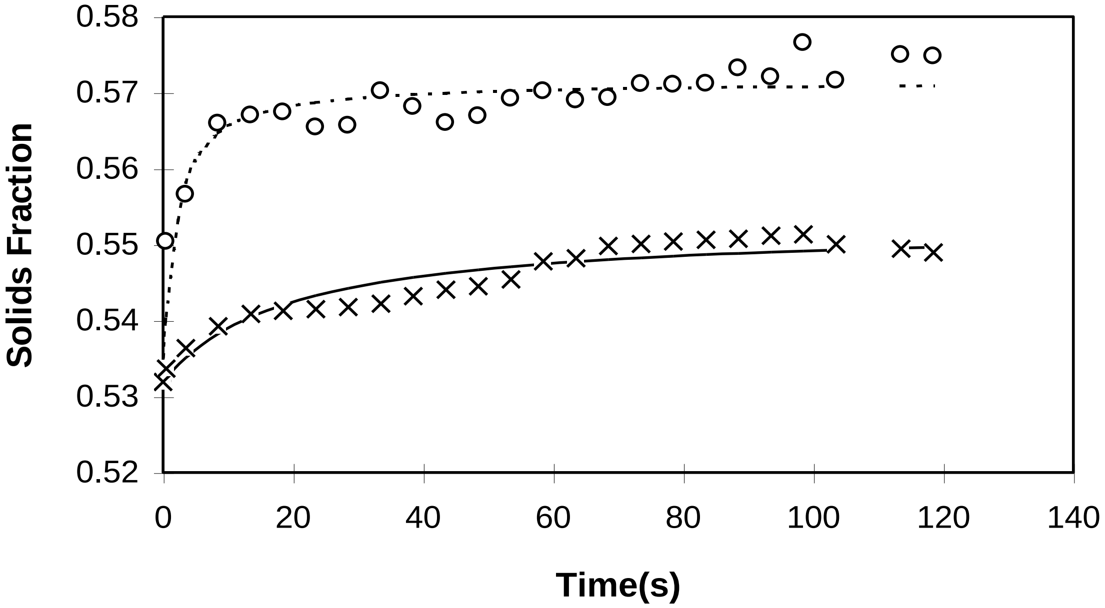

Fig. 3: Evolution of the solids fraction in the bed's center for Γ= 1.02 ("**x**" *f* = 20 Hz) and Γ = 6.4 ("O" *f* = 50 Hz). Solids lines are least square fits to the phenomenological model of Knight et al. (Knight, Frandrich, Lau, Jaeger & Nagel 1995) (Reprinted from Rosato & Yacoub 2000 with permission from Elsevier Science).

In the 1980's, it was (Rosato, Prinz et al. 1986; Rosato, Strandburg et al. 1997) deduced from Monte Carlo computer simulations that the rising of a large particle in a vibrated bed was due to a void created beneath the large particle that could easily be filled by smaller particles. Thus the larger particle could only move down if many smaller particles simultaneously were displaced from beneath it, an event of low probability[1]. In 1993, experimental findings (Knight, Jaeger et al. 1993) were reported that pointed to another mechanism responsible for the upward motion of the large particle. A 35 mm cylindrical vessel filled with 2 mm glass spheres was subjected to oscillations, each consisting of a vertical 30 Hz sinusoidal "tap", followed by a one second relaxation period. From the results, it was deduced that the observed rise of a large sphere from the bottom of the bed, which depended on the sphere's depth below the surface, was controlled by convection cycles. They also found that small particles rose at the same mean rate as the large sphere. Upon reaching the surface, the large sphere migrated to the cylinder walls where it was either trapped or re-entrained in the flow depending on its size relative to the downward flowing stream near the wall. Consequently, the authors conclude that

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[1] This is commonly known as the 'Brazil Nuts' theory.

the motion of the large particle was principally due to convection and that for the conditions investigated in their experiment, the filling of voids and/or internal particle rearrangements did not play a major role in the particle's upward ascent. However, they also observe that separation of a sufficiently large particle from the bulk occurred at the bed's surface due to its ability to move down the narrow stream adjacent to the cylinder wall. We conjecture that this was due to the same void-filling mechanism previously reported. That is, spheres move down the side of the container, making way for others to follow them. However, it is relatively improbable for the spaces opened up (i.e., space made available to allow other spheres to move down) to be of sufficient size to accommodate the large sphere. Consequently, the small spheres move beneath the large sphere, which then remains at the surface.

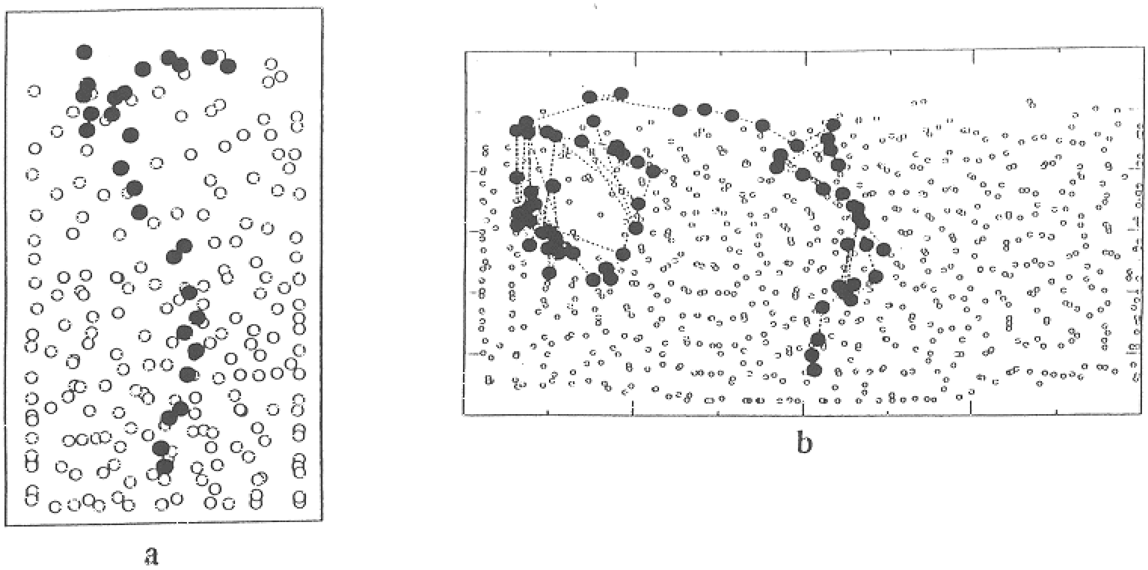

Fig. 4: Projected trajectory of the center of the large particle from discrete element simulations (Lan &Rosato 1997) for (a) width = 6*d*, and (b) width = 20*d*.

The viewpoint that convection is in effect a mechanism that promotes mixing rather than separation is supported by experiments (Brone and Muzzio 1997) in which systems of glass spheres vibrated in cylindrical vessel experienced either radial segregation or mixing depending on the frequency f of vibration. At frequencies less than approximately 19 Hz, radial segregation was commensurate with surface heaping, while a mixed state was attained in the absence of surface heaping at $f \approx\geq 22$ Hz. Additionally, the system could be reversibly driven between radially segregated and mixed states by simply adjusting the frequency in an appropriate manner. Their results demonstrate that the ability of the system to re-entrain large particles from the surface depends on the width of the downward flow stream near the walls.

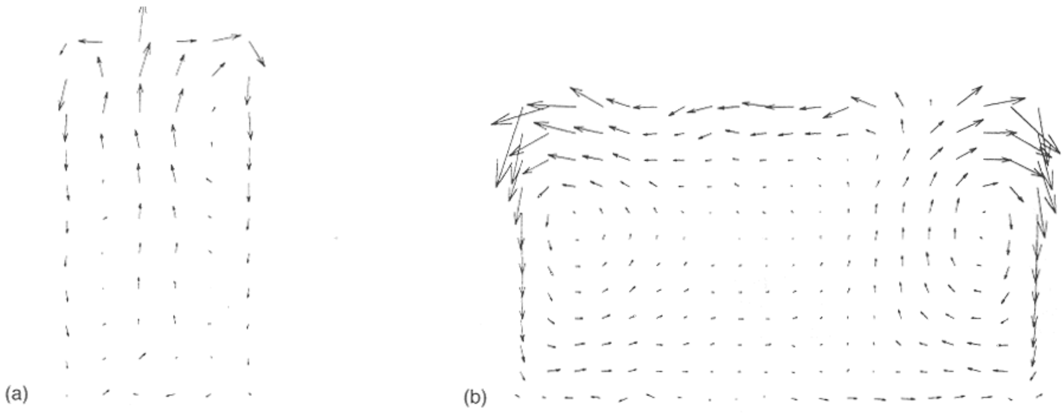

Fig. 5: Long time velocity field spatially averaged through the thickness of the rectangular computational cell for (a) width = 6*d* and (b) width = 20*d*. (Lan & Rosato 1997).

Discrete element studies (Lan and Rosato 1997) have provided further confirmation of the latter experimental observations with regards to the filling of voids beneath a larger particle trapped at the surface as well as the importance of the width of the downward streams near the walls. In Fig. 4a,b the projected trajectory of the center of a large sphere (D/d = 3) is shown, corresponding to the long time averaged velocity fields of Fig. 5a,b. In the presence of convection, surface trapping is evident in the narrow cell, while re-entrainment of the large particle occurs in the wider cell. During the large particle's ascent to the surface, results revealed a strong correlation between its coordination number (i.e., number of near neighbor contacts) and velocity. Consequently, an upward displacement of this particle occurred when it was closely surrounded by small particles. A careful inspection of the process showed the large particle undergoing a sequence of up and down displacements, having a net upward tendency over a time period during which it reached the surface. There appeared to be somewhat of a ratchet-like motion, reminiscent of the arching effect model proposed by Duran et al. (Duran, Rajchenbach et al. 1993). Further evidence that this might occur was found from an analysis of the motion of particles using modern dynamical systems theory: (Blackmore, Samulyak et al. 1999) showed that the dynamical system of a two species granular mixture in a vibrating bed exhibits chaotic mixing regimes for certain amplitude and frequency ranges. They conclude that this leads to a tendency for the smaller particles to occupy gaps that evolve beneath the large particle, thereby causing the larger particles to migrate upward in the bed. The above discussion suggests that void filling may be an important contributor to the ascent of the large particle, although a more detailed simulation study is needed to link the kinematics and dynamics of the process.

In the absence of convection, a large particle will rise to the free surface, albeit much more slowly than when convection is present. This was reported by Duran et al. (Duran, Rajchenbach et al. 1993) who carried out experiments with disks, from which they developed an "arching effect" model to explain their observations. Subsequently, Vanel et al. (Vanel, Rosato et al. 1997) performed full three dimensional experiments using 3 mm acrylic spheres in a cylindrical container that was agitated through sinusoidal oscillations of a floor piston at displacement amplitude a and vibration frequency f. The dimensionless time Tf required for a large intruder particle of diameter $D = 8d$ to rise to the surface from the floor was measured. Three behavioral regimes characterized by the piston vibration parameters were identified. For frequencies below $f_{cr} \cong 15$ Hz, strong convection and surface heaping was evident for which $Tf \sim e^{f}$. In the second regime ($15\,\text{Hz} \prec f \prec 40\,\text{Hz}$), surface heaping was not observed and $Tf \sim (\Gamma\text{-}\Gamma_c)^{-\beta}[e^{(f/f_c)^2} - 1]$, where $\beta \cong 2$, and $\Gamma_c \cong 1$. The latter scaling predicts a variation of Tf with Γ at fixed frequencies consistent with earlier observations (Ahmad and Smalley 1973). Most interesting is the last regime, where at small relative amplitudes ($a/d \leq 0.25$) and frequencies between approximately 40 and 75 Hz (corresponding to $\Gamma \geq 4$), there is a minimal dependence of Tf on Γ. In this range, the bed experiences an increase in bulk density, and no convection occurs. The large intruder slowly rises to the surface at a rate that depends on $\Phi \equiv D/d$ as shown in Fig. 6 (Vanel, Rosato et al. 1997), in striking contrast to the rate independent behavior that occurs in the presence of convection. Such a dependence on Φ is a consequence of the extent of the disturbance of the particle structure in the vicinity of the large particle. These results tend to support the early hypothesis of Brown (Brown 1939) who suggested that segregation under vibration occurred due to locally abnormal packing conditions in the neighborhood of a large particle. The latter results provide evidence that segregation can occur even when convection is absent.

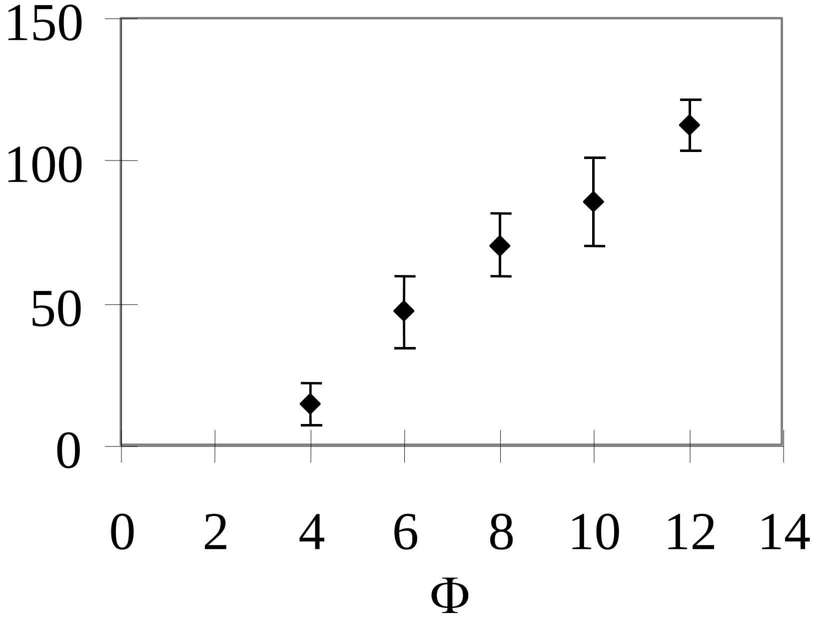

Fig. 6: Mean rise velocity (x10^{-6} m/s) of the large sphere as a function of Φ (f = 50 Hz, a/d = 0.1) (Vanel, Rosato & Dave 1997).

5. Summary and Conclusions

Segregation of homogeneous mixtures of granular materials due to vibrations is an important and not well-understood phenomenon. A large particle will rise to the surface of a vibrated vessel of particles with or without convection, although at drastically different rates. In both cases, the filling of voids beneath the intruder appears to be an essential mechanism during the process. In the situation where vibrations are such as to cause an increase in bulk density and no mean bulk flow (i.e., at small amplitudes $a/d < 0.1$ and large frequencies), a large particle ascends very slowly to the surface at a rate that depends strongly on the size ratio $\Phi = D/d$. Here the movement of smaller particles into cavities formed beneath the intruder plays a dominant role in the overall process. In contrast, when vibrations are sufficiently energetic to induce an upward current in the center and a descending stream near the wall, experiments and discrete element simulations reveal that the rise velocity is independent of Φ. We suggest that since convection causes the large and small particles to rise at the same rate, it can't be a mechanism that is responsible for a large particle to move away from smaller granules. At the surface, a large particle may be re-entrained in the flow, or it may be trapped at the wall depending on its size relative to the width of the downward stream. In this case, smaller spheres easily move down under the large particle, thereby causing it to remain at the surface. During the ascent process itself, discrete element simulations and dynamical systems models have provided evidence that the filling of voids beneath the large particle may also be operative even when a strong convective flow exists. Clearly, a great deal more research is needed in order to resolve some of the issues addressed in this paper.

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