International J.Math. Combin. Vol.2(2017), 70-83

Rainbow Connection Number in the Brick Product Graphs $C(2n, m, r)$

K.Srinivasa Rao

(Department of Mathematics, Shri Pillappa college of Engineering, Bangalore, India)

R.Murali

(Department of Mathematics, Dr.Ambedkar Institute of Technology, Bangalore, India)

E-mail: srinivas.dbpur@gmail.com, muralir2968@gmail.com

Abstract: Let G be a nontrivial connected graph on which is defined a coloring $c : E(G) \to \{1, 2, \cdots, k\},\ k \in N$ of the edges of G, where adjacent edges may be colored the same. A path in G is called a rainbow path if no two edges of it are colored the same. G is rainbow connected if G contains a rainbow $u - v$ path for every two vertices u and v in it. The minimum k for which there exists such a k-edge coloring is called the rainbow connection number of G, denoted by $rc(G)$. In this paper we determine $rc(G)$ for the brick product $C(2n, m, r)$ associated with the even cycle C_{2n} for $m = 1$ and odd $r \geq 5$ such that $n = r + 1, r + 2$ and $n \geq r + 3$. We also discuss the critical property of the graphs with $n = r + 1$ and $n = r + 2$ with respect to rainbow coloring.

Key Words: Diameter, edge-coloring, rainbow path, Smarandacely H-rainbow connected, rainbow connected, rainbow connection number, rainbow critical graph, brick product.

AMS(2010): 05C15.

§1. Introduction

All graphs considered in this paper are simple, finite and undirected. Let G be a nontrivial connected graph with an edge coloring $c : E(G) \to \{1, 2, \cdots, k\},\ k \in N$, where adjacent edges may be colored the same. A path in G is called a *rainbow path* if no two edges of it are colored the same. An edge colored graph G is said to be *Smarandacely H-rainbow connected to a graph H* if for any two vertices in a subgraph $G' \preceq G$ isomorphic to H, there is always a rainbow path in G connecting them. Particularly, an edge colored graph G is said to be *rainbow connected* if for any two vertices in G, there is a rainbow path in G connecting them, i.e., Smarandacely G-rainbow connected. Clearly, if a graph is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge coloring that makes it rainbow connected, i.e., a coloring such that each edge has a distinct color. The minimum k for which there exist a rainbow k-coloring of G is called the *rainbow connection number* of G, denoted by $rc(G)$.

[1]Received July 14, 2016, Accepted May 22, 2017.

Let c be a rainbow coloring of G. For any two vertices u and v of G a *rainbow* $u-v$ *geodesic* in G is a rainbow $u-v$ path of length $d(u,v)$ where $d(u,v)$ is the distance between u and v. G is termed *strongly rainbow connected* if G contains a rainbow $u-v$ geodesic for every two vertices u and v in G and in this case the coloring c is called a *strong rainbow coloring* of G. The minimum k for which there exists a coloring $c: E(G) \to \{1,2,\cdots,k\}$, $k \in N$ of the edges of G such that G is strongly rainbow connected is called the *strong rainbow connection number* of G, denoted by $src(G)$. Thus $rc(G) \leq src(G)$ for every connected graph G.

The rainbow connection number and the strong rainbow connection number are defined for every connected graph G, since every coloring that assigns distinct colors to the edges of G is both a rainbow coloring and a strong rainbow coloring and G is rainbow connected and strongly rainbow connected with respect to some coloring of the edges of G.

The concept of rainbow connectivity and strong rainbow connectivity were first introduced by Chartrand et al. [2] in 2008 as a means of strengthening the connectivity. In [2], the authors computed the rainbow connection numbers of several graph classes (complete graphs, trees, cycles, wheels and complete bipartite graphs). In [5] and [6] K.Srinivasa Rao and R.Murali, determined $rc(G)$ and $src(G)$ of the stacked book graph, the grid graph, the prism graph etc. They also discussed the critical property of these graphs with respect to rainbow coloring. In [7], the authors determined $rc(G)$ of brick product graphs associated with even cycles and also discussed the critical property of these graphs with respect to rainbow coloring. An overview about rainbow connection number can be found in a book of Li and Sun in [4] and a survey by Li et.al. in [3].

§2. Definitions

Definition 2.1 *A graph G is said to be rainbow critical if the removal of any edge from G increases the rainbow connection number of G, i.e. if $rc(G) = k$ for some positive integer k, then $rc(G-e) > k$ for any edge e in G.*

The brick product of even cycles was introduced in a paper by B.Alspach et.al. [1] in which the Hamiltonian laceability properties of brick products was explored.

Definition 2.2 (Brick product of even cycle) *Let m, n and r be positive integers. Let $C_{2n} = v_0, v_1, v_2, \cdots, v_{(2n-1)}, v_0$ denote a cycle of order $2n$. The (m,r)-brick product of C_{2n} denoted by $C(2n,m,r)$ is defined as follows:*

For $m = 1$, we require that r be odd and greater than 1. Then, $C(2n,m,r)$ is obtained from C_{2n} by adding chords $v_{2k}(v_{2k+r})$, $k = 1,2,\cdots,n$, where the computation is performed under modulo $2n$.

For $m > 1$, we require that $m+r$ be even. Then, $C(2n,m,r)$ is obtained by first taking disjoint union of m copies of C_{2n} namely, $C_{2n}(1), C_{2n}(2), C_{2n}(3), \cdots, C_{2n}(m)$ where for each $i = 1,2,\cdots,m$, $C_{2n}(i) = v_{i1}, v_{i2}, v_{i3},, \cdots, v_{i(2n)}$. Next, for each odd $i = 1,2,\cdots,m-1$ and each even $k = 0,1,\cdots,2n-2$, an edge(called a brick edge) drawn to join v_{ik} to $v_{(i+1)k}$, whereas, for each even $i = 1,2,\cdots,m-1$ and each odd $k = 1,2,\cdots,2n-1$, an edge (also called a brick *edge) is drawn to join* v_{ik} *to* $v_{(i+1)k}$.

Finally, for each odd $k = 1, 2, \cdots, 2n-1$, *an edge (called a hooking edge)is drawn to join* v_{1k} *to* $v_{m(k+r)}$. *An edge in* $C(2n, m, r)$ *which is neither a brick edge nor a hooking edge is called a flat edge.*

The brick products $C(10, 1, 5)$ and $C(14, 1, 5)$ are shown in the Figure 1.

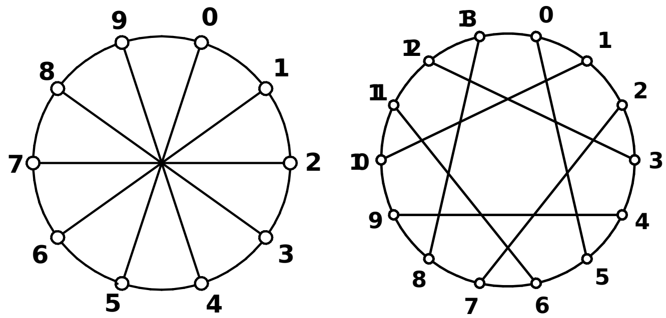

Figure 1 The brick product $C(10, 1, 5)$ and $C(14, 1, 5)$.

In the next section, we determine the values of $rc(G)$ for the brick product graph $C(2n, m, r)$ for $m = 1$ and $n = r + 1$, $n = r + 2$ and $n \geq r + 3$ for odd $r \geq 5$. In our results, we denote the vertices of the cycle C_{2n} as $v_0, v_1, \cdots, v_{2n-1}, v_{2n} = v_0$.

§3. Main Results

Theorem 3.1 *Let* $G = C(2n, m, r)$. *Then for* $m = 1$ *and odd* $r \geq 5$,

$$rc(G) = \begin{cases} \frac{n}{2} + 1 & \text{for } n = r + 1 \\ \lceil \frac{n}{2} \rceil & \text{for } n = r + 2 \end{cases}$$

Proof We consider the vertex set of G as $V(G) = \{v_0, v_1, \cdots, v_{2n-1}, v_{2n} = v_0\}$ and the edge set of G as $E(G) = \{e_i : 1 \leq i \leq 2n\} \cup \{e_i^{'} : 1 \leq i \leq n\}$, where e_i is the *cycle edge* (v_{i-1}, v_i) and $e_i^{'}$ is the *brick edge* (v_{2k}, v_{2k+r}), $k = 0, 1, \cdots, n$. Here $2k + r$ is computed modulo $2n$. We prove this result in different cases as follows.

Case 1. $n = r + 1$.

Since $diam(G) = \frac{n}{2}$, it follows that $rc(G) \geq \frac{n}{2}$. We define a $\frac{n}{2}$ coloring $c : E(G) \to \{1, 2, \cdots, \frac{n}{2}\}$ on the cycle edges of G as

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & \text{if } \frac{n}{2} + 1 \leq i \leq n \\ i - n & \text{if } n + 1 \leq i \leq \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if } \frac{3n}{2} + 1 \leq i \leq 2n \end{cases}$$

Also, we consider the assignment of colors to the brick edges in two different ways.

1. $$c(e_i^{'}) = \begin{cases} i & \text{if} \quad 1 \le i \le \frac{n}{2} \\ i - \frac{n}{2} & \text{if} \quad \frac{n}{2} + 1 \le i \le n \end{cases}$$

This coloring will not give a rainbow $v_0 - v_{n+1}$, $v_1 - v_n$ path $\forall n$ and this is true for other pair of vertices.

2. $$c(e_i^{'}) = \frac{n}{2} \quad \text{if} \quad 1 \le i \le n.$$

This coloring also will not give a rainbow $v_0 - v_n$ path $\forall n$ and this is true for other pair of vertices. (This is illustrated in Figure 2).

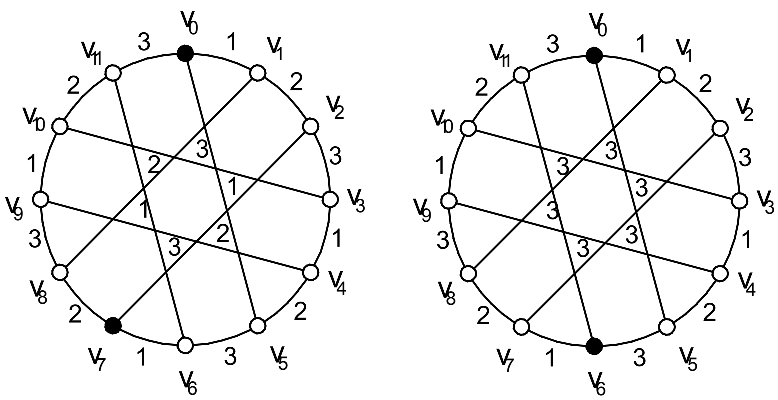

Figure 2 Assignment of colors in $C(12, 1, 5)$ in two different ways.

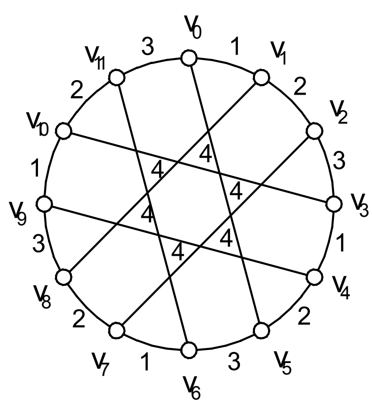

Figure 3 Assignment of colors in $C(12, 1, 5)$.

Accordingly, we define a coloring $c : E(G) \to \{1, 2, \cdots, \frac{n}{2} + 1\}$ and assign the colors to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if} \quad 1 \le i \le \frac{n}{2} \\ i - \frac{n}{2} & \text{if} \quad \frac{n}{2} + 1 \le i \le n \\ i - n & \text{if} \quad n + 1 \le i \le \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if} \quad \frac{3n}{2} + 1 \le i \le 2n \end{cases}$$

and

$$c(e_i^{'}) = \frac{n}{2} + 1 \quad \text{if} \quad 1 \le i \le n$$

From the above assignment, it is clear that for every two distinct vertices $x, y \in V(G)$, there exists an $x - y$ rainbow path. Hence $rc(G) \le \frac{n}{2} + 1$. This proves $rc(G) = \frac{n}{2} + 1$. An illustration for the assignment of colors in $C(12, 1, 5)$ is provided in Figure 3.

Case 2. $n = r + 2$.

Since $diam(G) = \lceil \frac{n}{2} \rceil$, it follows that $rc(G) \ge \lceil \frac{n}{2} \rceil$. In order to show that $rc(G) \le \lceil \frac{n}{2} \rceil$, we construct an edge coloring $c : E(G) \to \{1, 2, \cdots, \lceil \frac{n}{2} \rceil\}$ and assign the colors to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if} \quad 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ i - \lfloor \frac{n}{2} \rfloor & \text{if} \quad \lceil \frac{n}{2} \rceil \le i \le n - 1 \\ i - n & \text{if} \quad n + 1 \le i \le \lfloor \frac{3n}{2} \rfloor \\ i - \lfloor \frac{3n}{2} \rfloor & \text{if} \quad \lceil \frac{3n}{2} \rceil \le i \le 2n - 1 \\ \lceil \frac{n}{2} \rceil & \text{elsewhere} \end{cases}$$

and

$$c(e_i^{'}) = \lceil \frac{n}{2} \rceil \quad \text{if} \quad 1 \le i \le n$$

From the above assignment, for any two vertices $x, y \in V(G)$, we obtain a rainbow $x - y$ path in G. Hence $rc(G) \le \lceil \frac{n}{2} \rceil$.

This proves $rc(G) = \lceil \frac{n}{2} \rceil$. An illustration for the assignment of colors in $C(14, 1, 5)$ is provided in Figure 4. □

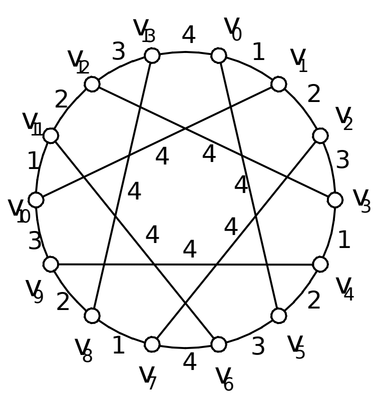

Figure 4 Assignment of colors in $C(14, 1, 5)$.

The critical nature of the brick product graph in Theorem 3.1 has been observed for particular values of n. This is illustrated in our next result.

Lemma 3.2 *Let $G = C(2n, m, r)$, where $m = 1$ and odd $r \geq 5$. Then G is rainbow critical for $n = r + 1$ and $r + 2$.*

$$i.e., \quad rc(G - e) = n \quad for \quad n = r + 1 \ \ and \ \ n = r + 2.$$

Proof We prove this result in two cases.

Case 1. $n = r + 1$.

From the Theorem 3.1, $rc(G) = \frac{n}{2} + 1$ for $n = r+1$. Consider the graph G and let $e \in E(G)$ be the any edge in G. Deletion of the edge e in G does not yield a rainbow path between the end vertices of e in $G - e$. For illustration see Figure 5.

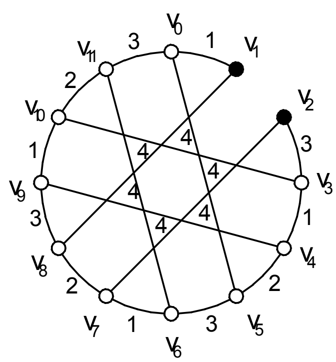

Figure 5 Assignment of colors in $C(12, 1, 5)$.

This shows $rc(G - e) > \frac{n}{2} + 1$. We define a rainbow coloring $c : E(G) \to \{1, 2, \cdots, \frac{n}{2} + 2\}$ to the edges of G as defined in Theorem 3.1. i.e.,

$$c(e_i) = \begin{cases} i & \text{if } \ 1 \leq i \leq \frac{n}{2} + 1 \\ i - (\frac{n}{2} + 1) & \text{if } \ \frac{n}{2} + 2 \leq i \leq n + 2 \\ i - (n + 2) & \text{if } \ n + 3 \leq i \leq \frac{3n}{2} + 3 \\ i - (\frac{3n}{2} + 3) & \text{if } \ \frac{3n}{2} + 4 \leq i \leq 2n \end{cases}$$

and

$$c(e_i^{'}) = \frac{n}{2} + 2 \ \text{ if } \ \ 1 \leq i \leq n$$

From the above assignment of colors, we fail to get a rainbow path between any two arbitrary vertices in $G - e$. Without loss of generality let $e = v_1 v_2$. An illustration is provided in Figure 6.

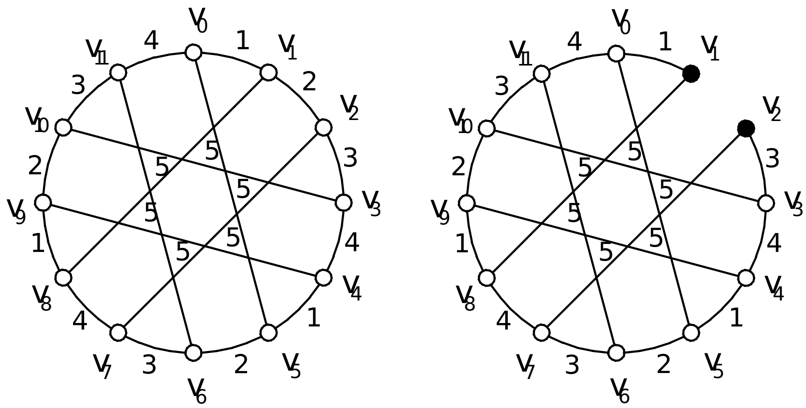

Figure 6 Assignment of colors in $C(12,1,5)$.

Accordingly, we assign a $(\frac{n}{2}+\frac{r+1}{2})=(\frac{n}{2}+\frac{n}{2})=n$ rainbow coloring $c:E(G)\rightarrow\{1,2,\cdots,n\}$ to the edges of G and since the cycle C_{2n} is a sub graph of G, assign the colors to the edges of cycle as in Theorem 3.1. That is,

$$c(e_i)=\begin{cases} i & \text{if} \quad 1\leq i\leq n \\ i-n & \text{if} \quad n+1\leq i\leq 2n \end{cases}$$

Next, we will consider the assignment of colors to the brick edge as

$$c(e_i^{'})=\begin{cases} (2i+4)\ mod\ n & \text{if} \quad 1\leq i\leq \frac{n}{2} \\ [(2i+4)\ mod\ n]-\frac{n}{2} & \text{if} \quad \frac{n}{2}+1\leq i\leq n \end{cases}$$

From the above assignment, it is clear that for any two vertices $x,y\in V(G)$, there exists a rainbow $x-y$ path in $G-e$ and hence $rc(G-e)\leq n$. This proves $rc(G-e)=n$. An illustration for the assignment of colors in $C(12,1,5)$ is provided in Figure 7.

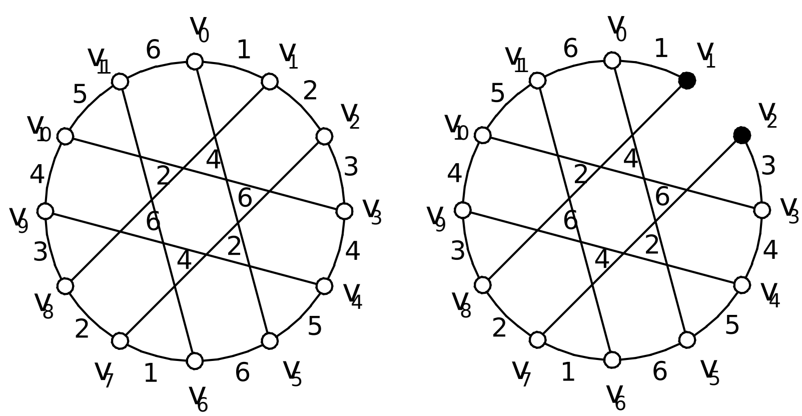

Figure 7 Assignment of colors in $C(12,1,5)$.

Case 2. $n = r + 2$.

The proof for $n = r + 2$ is the same as Case 1.

As in Case 1, in this case also we fail to get the $x-y$ rainbow path for different combinations of assignment of $n-1$ colors to the edges of $G - e$.

Accordingly, we construct a rainbow coloring $c : E(G) \to \{1, 2, \cdots, n\}$ to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if} \quad 1 \le i \le n \\ i - n & \text{if} \quad n + 1 \le i \le 2n \end{cases}$$

and

$$c(e_i^{'}) = \begin{cases} n + 1 - i & \text{if} \quad 1 \le i \le 2 \\ i & \text{if} \quad i = 3 \\ i + 1 & \text{if} \quad 4 \le i \le n - 3 \\ i - (n - 3) & \text{if} \quad n - 2 \le i \le n - 1 \\ 4 & \text{if} \quad i = n \end{cases}$$

From the above assignment for any two vertices $x, y \in V(G)$, we obtain a rainbow $x - y$ path in $G - e$. This proves $rc(G - e) = n$. Hence the proof. An illustration for the assignment of colors in $C(14, 1, 5)$ is provided in Figure 8.

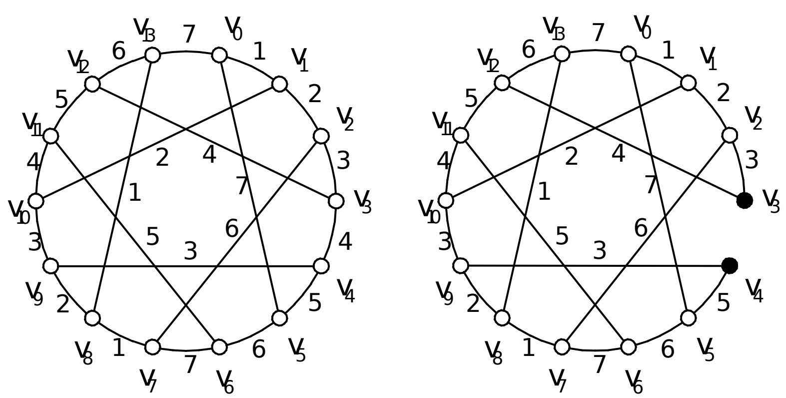

Figure 8 Assignment of colors in $C(14, 1, 5)$.

For $r \ge 5$ and $n \ge r + 3$, we have the following result.

Theorem 3.3 *Let $G = C(2n, m, r)$. Then for $m = 1$, odd $r \ge 5$ and $n \ge r + 3$,*

$$rc(G) = \begin{cases} \frac{n}{2} + 1 & \text{for} \quad n = r + 3 \\ \lceil \frac{n}{2} \rceil + 1 & \text{for} \quad r + 4 \le n \le 2r \\ (n - r + 1) - x\lfloor \frac{r}{2} \rfloor & \text{for} \quad (2r + 1) + xr \le n \le 3r + xr \\ (2r + 2) + x(\lfloor \frac{r}{2} \rfloor + 1) & \text{for} \quad (3r + 1) + xr \le n \le 4r + xr \\ & \text{where} \quad x = 0, 2, 4 \cdots \end{cases}$$

Proof We consider the vertex set $V(G)$ and the edge set $E(G)$ defined in Theorem 3.1. We prove this result in different cases as follows.

Case 1. $n = r + 3$.

We have two subcases.

Subcase 1.1 $n = 4k$, where $k = 2, 3, \cdots$.

Here $diam(G) = \frac{n+8}{4}$. Clearly $rc(G) \geq diam(G)$. If we assign $diam(G)$-colors to the edges of G, we fail to obtain a rainbow path between a pair of vertices in G. This holds up to $\frac{n}{2}$ colors. Hence, to prove this we consider an edge coloring $c : E(G) \to \{1, 2, \cdots, \frac{n}{2}\}$ to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & \text{if } \frac{n}{2} + 1 \leq i \leq n \\ i - n & \text{if } n + 1 \leq i \leq \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if } \frac{3n}{2} + 1 \leq i \leq 2n \end{cases}$$

and

$$c(e_i^{'}) = \frac{n}{2} \text{ if } 1 \leq i \leq n$$

This coloring will not yield a rainbow $v_0 - v_{n+1}$ path $\forall n$, which is illustrated in Figure 9.

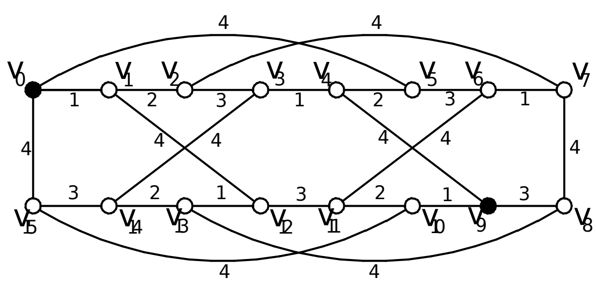

Figure 9 Assignment of colors in $C(16, 1, 5)$.

Subcase 1.2 $n = 5k$, where $k = 2, 3, \cdots$.

Here $diam(G) = \frac{n+6}{4}$. Clearly $rc(G) \geq diam(G)$. Similar to subcase 1 we fail to obtain a rainbow path between a pair of vertices in G up to $\frac{n}{2}$-colors. From Subcase 1 and Subcase 2, we have $rc(G) \geq \frac{n}{2} + 1$. Accordingly, we construct an edge coloring $c : E(G) \to \{1, 2, \cdots, \frac{n}{2} + 1\}$ and assign the colors to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & \text{if } \frac{n}{2} + 1 \leq i \leq n \\ i - n & \text{if } n + 1 \leq i \leq \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if } \frac{3n}{2} + 1 \leq i \leq 2n \end{cases}$$

and

$$c(e_i^{'}) = \frac{n}{2} + 1 \quad \text{if} \quad 1 \le i \le n$$

From the above assignment, it is clear that for any two vertices $x, y \in V(G)$, there exists a rainbow $x - y$ path in G and hence $rc(G) \le \frac{n}{2} + 1$. This proves $rc(G) = \frac{n}{2} + 1$. An illustration for the assignment of colors in $C(20, 1, 5)$ is provided in Figure 10.

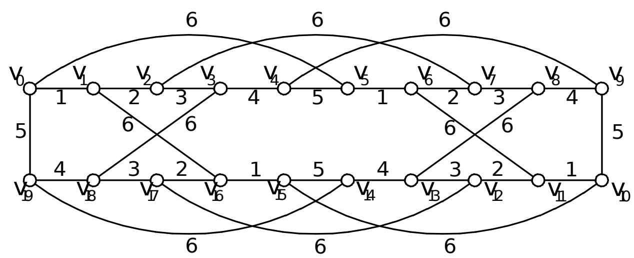

Figure 10 Assignment of colors in $C(20, 1, 5)$.

Case 2. $r + 4 \le n \le 2r$.

We consider the following subcases.

Subcase 2.1 n is even.

The proof is similar to the proof provided in Subcase 1.1 of Case 1.

If we consider assignment of $\frac{n}{2}$ colors in any combination to the edges of G, we fail to obtain a rainbow $v_0 - v_{n+1}$ path $\forall n$ (as illustrated in Figure 9).

This shows $rc(G) \ge \frac{n}{2} + 1$. It remains to show that $rc(G) \le \frac{n}{2} + 1$. Define a coloring $c : E(G) \to \{1, 2, \cdots \frac{n}{2} + 1\}$ and consider the assignment of colors to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if} \ \ 1 \le i \le \frac{n}{2} \\ i - \frac{n}{2} & \text{if} \ \ \frac{n}{2} + 1 \le i \le n \\ i - n & \text{if} \ \ n + 1 \le i \le \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if} \ \ \frac{3n}{2} + 1 \le i \le 2n \end{cases}$$

and

$$c(e_i^{'}) = \frac{n}{2} + 1 \quad \text{if} \quad 1 \le i \le n$$

From the above assignment, it is clear that for any two vertices $x, y \in V(G)$, there exists a rainbow $x - y$ path with coloring c. Hence $rc(G) \le \frac{n}{2} + 1$. An illustration for the assignment of colors in $C(20, 1, 5)$ is provided in Figure 10.

Subcase 2.2 n is odd.

If we assign $diam(G)$ colors to the edges of G, we fail to obtain a rainbow path between a pair of vertices in G. This holds up to $\lceil \frac{n}{2} \rceil$ colors. Hence, to prove this we consider an edge coloring $c : E(G) \to \{1, 2, \cdots, \lceil \frac{n}{2} \rceil\}$ to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if } \ 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ i - \lfloor \frac{n}{2} \rfloor & \text{if } \ \lceil \frac{n}{2} \rceil \le i \le n-1 \\ i - (n-1) & \text{if } \ n \le i \le \lfloor \frac{3n}{2} \rfloor - 1 \\ i - (\lfloor \frac{3n}{2} \rfloor - 1) & \text{if } \ \lceil \frac{3n}{2} \rceil \le i \le 2n-2 \\ i - (2n-2) & \text{if } \ 2n-1 \le i \le 2n \end{cases}$$

and

$$c(e_i^{'}) = \lceil \tfrac{n}{2} \rceil \ \text{ if } \ \ 1 \le i \le n$$

This coloring will not yield a rainbow $v_2 - v_{\frac{3n-1}{2}}$ path $\forall n$, which is illustrated in Figure 11.

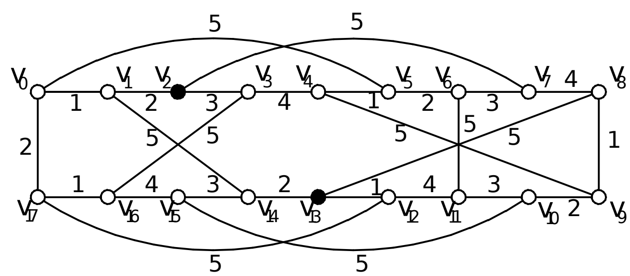

Figure 11 Assignment of colors in $C(18, 1, 5)$.

This shows $rc(G) \ge \lceil \frac{n}{2} \rceil + 1$. It remains to show that $rc(G) \le \lceil \frac{n}{2} \rceil + 1$. We construct an edge coloring $c : E(G) \to \{1, 2, \cdots, \lceil \frac{n}{2} \rceil + 1\}$ and assign the colors to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if } \ 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ i - \lfloor \frac{n}{2} \rfloor & \text{if } \ \lfloor \frac{n}{2} \rfloor + 1 \le i \le n-1 \\ \lceil \frac{n}{2} \rceil & \text{if } \ i = n \ \ and \ \ i = 2n \\ i - n & \text{if } \ n+1 \le i \le \lfloor \frac{3n}{2} \rfloor \\ i - \lfloor \frac{3n}{2} \rfloor & \text{if } \ \lfloor \frac{3n}{2} \rfloor + 1 \le i \le 2n-1 \end{cases}$$

and

$$c(e_i^{'}) = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & \text{if } \ 1 \le i \le \frac{n-r}{2} \ \ and \ \ \frac{n+3}{2} \le i \le \frac{2n-r+1}{2} \\ \lceil \frac{n}{2} \rceil & \text{if } \ \frac{n-r+2}{2} \le i \le \frac{n+1}{2} \ \ and \ \ \frac{2n-r+3}{2} \le i \le n \end{cases}$$

From the above assignment, it is clear that for any two vertices $x, y \in V(G)$, there exists a rainbow $x - y$ path. Hence $rc(G) \le \lceil \frac{n}{2} \rceil + 1$.

Combining both the Subcases, we have $rc(G) \le \lceil \frac{n}{2} \rceil + 1$, $\forall n$ such that $r + 4 \le n \le 2r$.

This proves $rc(G) = \lceil \frac{n}{2} \rceil + 1$. An illustration for the assignment of colors in $C(18, 1, 5)$ is provided in Figure 12.

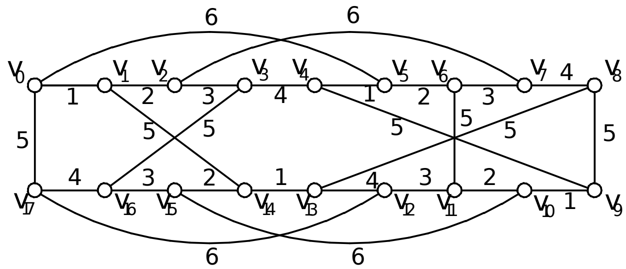

Figure 12 Assignment of colors in $C(18, 1, 5)$.

Case 3. For $(2r+1)+xr \le n \le 3r+xr$ and $(3r+1)+xr \le n \le 4r+xr$ where $x = 0, 2, 4, \cdots$.

Let

$$h = \begin{cases} (n-r+1) - x\lfloor \frac{r}{2} \rfloor & \text{for } \ (2r+1)+xr \le n \le 3r+xr \\ (2r+2) + x(\lfloor \frac{r}{2} \rfloor + 1) & \text{for } \ (3r+1)+xr \le n \le 4r+xr \\ & \text{where } \ x = 0, 2, 4, \cdots \end{cases}$$

As in Case 1 and Case 2 we again fail to obtain a rainbow path between a pair of vertices in G up to $h-1$ colors. This shows $rc(G) \ge h$.

To show that $rc(G) \le h$, we construct a rainbow coloring $c : E(G) \to \{1, 2, \cdots, (n-r+1) - x\lfloor \frac{r}{2} \rfloor\}$ for $(2r+1)+xr \le n \le 3r+xr$ and $c : E(G) \to \{1, 2, \cdots, (2r+2) + x(\lfloor \frac{r}{2} \rfloor + 1)\}$ for $(3r+1)+xr \le n \le 4r+xr$ and assign the colors to the edges of G. Before assignment of colors, we split $E(G)$ as, e_i where $1 \le i \le n-1$, $i = n$, $i = 2n$ and $n+1 \le i \le 2n-1$ (cycle edges) and $e_i^{'}$ where $1 \le i \le n$ (brick edges). we assign the colors to the cycle edges from e_1 to e_{n-1} as

$$c(e_i) = \begin{cases} i & \text{for } \ 1 \le i \le r \\ i - kr & \text{for } \ (2k-1)r+1 \le i \le l \end{cases}$$

where

$$l = \begin{cases} (2k+1)r & \text{for each } \ k = 1, 2, \cdots, \lceil \frac{n-r}{2r} \rceil - 1 \\ n-1 & \text{for } \ k = \lceil \frac{n-r}{2r} \rceil \end{cases}$$

and assign the colors to the cycle edge e_n and e_{2n} as

$$c(e_n) = c(e_{2n}) = \begin{cases} [(n-r) - x\lfloor \frac{r}{2} \rfloor] - \frac{x}{2} & \text{for } \ (2r+1)+xr \le n \le 3r+xr \\ 2r + x[\lfloor \frac{r}{2} \rfloor + 1] - \frac{x}{2} & \text{for } \ (3r+1)+xr \le n \le 4r+xr \\ & \text{where } \ x = 0, 2, \cdots \end{cases}$$

Also, we assign the colors to the cycle edges from e_{n+1} to e_{2n-1} as

$$c(e_i) = \begin{cases} i - n & \text{for } \ n+1 \le i \le n+r \\ i - (n+kr) & \text{for } \ n+(2k-1)r+1 \le i \le m \end{cases}$$

where

$$m = \begin{cases} n + (2k+1)r & \text{for each } \ k = 1, 2, \cdots, \lceil \frac{n-r}{2r} \rceil - 1 \\ 2n - 1 & \text{for } \ k = \lceil \frac{n-r}{2r} \rceil \end{cases}$$

and assign the colors to the brick edges from $e_1^{'}$ to $e_n^{'}$ as

$$c(e_i^{'}) = \begin{cases} c(e_n) + k & \text{for } \ (k-1)r + 1 \leq i \leq l \\ c(e_n) & \text{for } \ \lfloor \frac{n-4}{2} \rfloor + 1 \leq i \leq \lceil \frac{n}{2} \rceil \\ c(e_n) + k & \text{for } \ \lceil \frac{n}{2} \rceil + (k-1)r + 1 \leq i \leq m \\ c(e_n) & \text{for } \ (n-1) \leq i \leq n \end{cases}$$

where

$$l = \begin{cases} kr & \text{for each } \ k = 1, 2, \cdots, \lceil \frac{n-r}{2r} \rceil - 1 \\ \lfloor \frac{n-4}{2} \rfloor & \text{for } \ k = \lceil \frac{n-r}{2r} \rceil \end{cases}$$

and

$$m = \begin{cases} \lceil \frac{n}{2} \rceil + kr & \text{for each } \ k = 1, 2, \cdots, \lceil \frac{n-r}{2r} \rceil - 1 \\ n - 2 & \text{for } \ k = \lceil \frac{n-r}{2r} \rceil \end{cases}$$

From the above assignment, it is clear that for any two vertices $x, y \in V(G)$, there exists a rainbow $x - y$ path and hence

$$rc(G) = \begin{cases} (n - r + 1) - x\lfloor \frac{r}{2} \rfloor & \text{for } \ (2r+1) + xr \leq n \leq 3r + xr \\ (2r + 2) + x(\lfloor \frac{r}{2} \rfloor + 1) & \text{for } \ (3r+1) + xr \leq n \leq 4r + xr \\ & \text{where } \ x = 0, 2, 4, \cdots \end{cases}$$

Hence the proof. An illustration for the assignment of colors in $C(38, 1, 5)$ is provided in Figure 13. $\square$

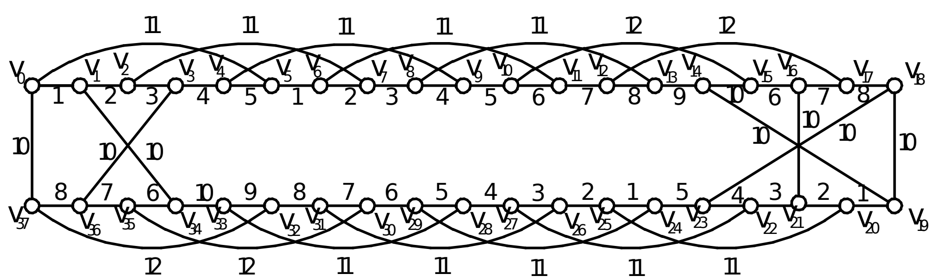

Figure 13 Assignment of colors in $C(38, 1, 5)$.

References

[1] B. Alspach, B. Alspach, C.C.Chen, Kevin McAvaney, On a class of Hamiltonian laceable 3-regular graphs, *Discrete Mathematics*, 151, (1996), pp. 19-38.

[2] Chartrand G, Johns GL, McKeon KA, Zhang P., Rainbow connection in graphs, *Math Bohem*, 133, (2008), pp. 85-98.

[3] Li X, Shi Y, Sun Y., Rainbow connection of graphs: a survey, *Graphs and Combin.*, 29(1), (2013), pp. 1-38.

[4] Li X, Sun Y., *Rainbow Connection of Graphs*, New York: Springer-Verlag, 2012.

[5] K.Srinivasa Rao, R.Murali., Rainbow critical graphs, *Int. J. of Comp. Application*, 4(4), (2014), pp. 252-259.

[6] K.Srinivasa Rao, R.Murali, S.K.Rajendra, Rainbow and strong rainbow criticalness of some standard graphs, *Int. J. of Mathematics and Computer Research*, 3(1), (2015), pp. 829-836

[7] K. Srinivasa Rao, R. Murali and S.K.Rajendra, Rainbow connection number in brick product graphs, *Bulletin of International Mathematical Virtual Institute*, 8(1)(2018), 55-66.