International J.Math. Combin. Vol.2(2017), 60-69

On the Spacelike Parallel Ruled Surfaces with Darboux Frame

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Abstract: In this study, the spacelike parallel ruled surfaces with Darboux frame are introduced in Minkowski 3-space. Then some characteristic properties of the spacelike parallel ruled surfaces with Darboux frame such as developability, the striction point and the distribution parameter are obtained in Minkowski 3-space.

Key Words: Ruled surface, parallel surface, Darboux frame.

AMS(2010): 53A05, 53A15.

§1. Introduction

In differential geometry, ruled surface is a special type of surface which can be defined by choosing a curve and a line along that curve. The ruled surfaces are one of the easiest of all surfaces to parametrize. That surface was found and investigated by Gaspard Monge who established the partial differential equation that satisfies all ruled surface. V. Hlavaty [9] also investigated ruled surfaces which are formed by one parameter set of lines.

A surface and another surface which have constant distance with the reference surface along its surface normal have a relationship between their parametric representations. Such surfaces are called parallel surface [8]. By this definition, it is convenient to carry the points of a surface to the points of another surface. Since the curves are set of points, then the curves lying fully on a reference surface can be carry to another surface.

Another one of the most important subjects of the differential geometry is the Darboux frame which is a natural moving frame constructed on a surface. It is the version of the Frenet frame as applied to surface geometry. A Darboux frame exists on a surface in a Euclidean or non-Euclidean spaces. It is named after the French mathematician Jean Gaston Darboux, in the four volume collection of studies published between 1887 and 1896. Since that time, there have been many important repercussions of Darboux frame, having been examined for example in [2], [3].

[1]Received September 26, 2016, Accepted May 20, 2017.

The Minkowski space E_1^3 is the Euclidean space E^3 provided with the Lorentzian inner product

$$\langle u, v\rangle = u_1v_1 + u_2v_2 - u_3v_3,$$

where

$$u = (u_1, u_2, u_3),\ v = (v_1, v_2, v_3) \in E^3.$$

We say that a vector u in E_1^3 is spacelike, lightlike or timelike if

$$\langle u, u\rangle > 0,\ \ \langle u, u\rangle = 0 \text{ or } \langle u, u\rangle < 0,$$

respectively. The norm of the vector $u \in E_1^3$ is defined by

$$\|u\| = \sqrt{|\langle u, u\rangle|}.$$

G.Y. Senturk and S. Yucel have taken into a consideration about the ruled surface with Darboux frame in E^3 ([4]).

By making use of the paper of Unluturk et al. [10], we describe our general approach to compute the spacelike parallel ruled surfaces with Darboux frame, and give some theorems for these kinds of surfaces.

§2. Preliminaries

A ruled surface M in $\mathbb{R}^3$ is generated by a one-parameter family of straight lines which are called the rulings. The equation of the ruled surface can be written as

$$\varphi(s, v) = \alpha(s) + vX(s),$$

where (α) is curve which is called the base curve of the ruled surface and (X) is called the unit direction vector of the straight line.

An unit direction vector of straight line X is stretched by the system $\{T, g\}$. So it can be written as

$$X = T\sin\phi + g\cos\phi,$$

where ϕ is the angle between T and X vectors ([10]).

The striction point on the ruled surface is the foot of the common perpendicular line successive rulings on the main ruling. The set of the striction points of the ruled surface generates its striction curve. It is given by ([10])

$$c(s) = \alpha(s) - \frac{\langle \alpha_s, X_s\rangle}{\langle X_s, X_s\rangle}X(s).$$

Theorem 2.1([7]) *If successive rulings intersect, the ruled surface is called developable. The unit tangent vector of the striction curve of a developable ruled surface is the unit vector with direction X.*

The distribution of the ruled surface is identified by

$$P_X = \frac{\det(\alpha_s, X, X_s)}{\langle X_s, X_s\rangle}.$$

Theorem 2.2([7]) *The ruled surface is developable if and only if* $P_X = 0$.

The ruled surface is said to be a noncylindrical ruled surface provided that ([1])

$$\langle X_s, X_s\rangle \neq 0$$

Theorem 2.3([1]) *Let M be a noncylindrical ruled surface and defined by its striction curve. The Gaussian curvature of M is given by its distribution parameter by*

$$K = -\frac{P_X^2}{(P_X^2+v^2)^2}$$

.

Definition 2.1([5]) *Let M and $\overline{M}$ be two surfaces in Euclidean space. The function*

$$\begin{array}{rlll} f: & M & \to & \overline{M} \\ & P & \to & f(P) = P + rN_P \end{array} \tag{2.1}$$

is called the parallelization function between M and $\overline{M}$ and furthermore $\overline{M}$ is called parallel surface to M, where N is the unit normal vector field on M and r is a given real number.

Theorem 2.4([5]) *Let M and $\overline{M}$ be two parallel surfaces in Euclidean space and*

$$f: \quad M \quad \to \quad \overline{M}$$

be the parallelization function. Then for $X \in \chi(M)$

1. $f_*(X) = X - rS(X)$
2. $S^r\left(f_*(X)\right) = S(X)$
3. *f preserves principal directions of curvature, that is*

$$S^r(f_*(X)) = \frac{\kappa}{1-r\kappa} f_*(X),$$

where S^r is the shape operator on $\overline{M}$, and κ is a principal curvature of M at P in direction X.

Theorem 2.5([11]) *Let Darboux frame of curve β at $f(\alpha(t_0)) = f(P)$ on $\overline{M}$ be $\{\overline{T}, \overline{g}, \overline{N}\}$, then*

$$\begin{aligned} \overline{T} &= \frac{1}{v}\left[(1-r\kappa_n)T - r\tau_g g\right], \\ \overline{g} &= \frac{1}{v}\left[(1-r\kappa_n)g + r\tau_g T\right], \ \cdot \\ \overline{N} &= N \end{aligned} \tag{2.2}$$

Definition 2.2([2]) *Let x be a spacelike vector and y be a timelike vector in E_1^3. Then there is an unique real number $\theta \geq 0$ such that $\langle x, y\rangle = |x|\,|y| \sinh\theta$. This number is called the Lorentzian timelike angle between the vectors x and y.*

Definition 2.3([2]) *Let x and y be future pointing (or past pointing) timelike vectors in E_1^3. Then there is an unique real number $\theta \geq 0$ such that $\langle x, y\rangle =$-$|x|\,|y| \cosh\theta$. This number is called the hyperbolic angle between the vectors x and y.*

We denote by $\{T, n, b\}$ the moving Frenet frame along the unit speed curve $\alpha(s)$ in the Minkowski space E_1^3, the following Frenet formulae are given

$$\frac{d}{ds}\begin{bmatrix} T \\ n \\ b \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\varepsilon\kappa_g & 0 & \tau_g \\ \kappa_n & \tau_g & 0 \end{bmatrix}\begin{bmatrix} T \\ n \\ b \end{bmatrix}, \tag{2.3}$$

in [6], where $\langle T, T\rangle = 1,\ \langle n, n\rangle = \varepsilon = \pm 1,\ \langle b, b\rangle = -1$.

If the surface M is a spacelike surface, then the curve $\alpha(s)$ lying on surface M is a spacelike curve. So, the relations between the frames can be given as follows ([6]):

$$\begin{bmatrix} T \\ g \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix}\begin{bmatrix} T \\ n \\ b \end{bmatrix}, \tag{2.4}$$

where $\{T, g, N\}$ is Darboux frame.

Besides, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by ([6])

$$\frac{d}{ds}\begin{bmatrix} T \\ g \\ N \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ \kappa_n & \tau_g & 0 \end{bmatrix}\begin{bmatrix} T \\ g \\ N \end{bmatrix}, \tag{2.5}$$

where $\langle T, T\rangle = \langle g, g\rangle = 1$ and $\langle N, N\rangle = -1$.

Theorem 2.6([2]) *Let $\alpha(s)$ be non-unit speed curve on surface M in E^3. The Darboux frame of curve $\alpha(s)$ where $\|\alpha'(s)\| = v$, is $\{T, g, N\}$. Geodesic, normal curvatures and geodesic torsion of this curve-surface pair which are denoted by κ_g, κ_n, τ_g respectively are defined as follows:*

$$\begin{aligned} \kappa_g &= \frac{1}{v^2}\langle \alpha'', g\rangle, \\ \kappa_n &= \frac{1}{v^2}\langle \alpha'', N\rangle, \ . \\ \tau_g &= -\frac{1}{v}\langle N', g\rangle \end{aligned} \tag{2.6}$$

§3. On the Spacelike Parallel Ruled Surfaces with Darboux Frame

A spacelike parallel ruled surface can be given as in the following parametrization:

$$\overline{\varphi}(s,\overline{v}) = \overline{\alpha}(s) + \overline{v}\overline{X}(s), \tag{3.1}$$

where the curve $\overline{\alpha}(s)$ is lying on $\overline{\varphi}(s,\overline{v})$ is a spacelike curve.

Darboux frame is obtained by rotating Frenet frame around $\overline{T}$ as far as $\overline{\theta} = \overline{\theta}(s)$ while the hyperbolic angle $\overline{\theta}$ is between the timelike unit vector $\overline{b}$ and the timelike normal vector field $\overline{N}$ of $\overline{\varphi}(s,\overline{v})$.

By Definitions 2.2 and 2.3, the spacelike unit vector $\overline{g}$ and the timelike vector $\overline{N}$ are written, in terms of $\overline{\theta}$, as:

$$\begin{aligned} \overline{g} &= \overline{n}\cosh\overline{\theta} + \overline{b}\sinh\overline{\theta}, \\ \overline{N} &= \overline{n}\sinh\overline{\theta} + \overline{b}\cosh\overline{\theta}. \end{aligned} \tag{3.2}$$

From the expression (3.2), the spacelike unit vector $\overline{n}$ and timelike unit binormal vector $\overline{b}$ are obtained

$$\begin{aligned} \overline{n} &= \overline{g}\cosh\overline{\theta} - \overline{N}\sinh\overline{\theta}, \\ \overline{b} &= -\overline{g}\sinh\overline{\theta} + \overline{N}\cosh\overline{\theta}. \end{aligned} \tag{3.3}$$

Taking into the expressions (3.2) and (3.3), Darboux derivative formulae of the spacelike parallel ruled surface can be found as

$$\frac{d}{ds}\begin{bmatrix} \overline{T} \\ \overline{g} \\ \overline{N} \end{bmatrix} = \begin{bmatrix} 0 & \overline{\kappa}_g & \overline{\kappa}_n \\ -\overline{\kappa}_g & 0 & \overline{\tau}_g \\ \overline{\kappa}_n & \overline{\tau}_g & 0 \end{bmatrix}\begin{bmatrix} \overline{T} \\ \overline{g} \\ \overline{N} \end{bmatrix}, \tag{3.4}$$

where

$$\overline{\kappa}_g = \overline{\kappa}\cosh\overline{\theta},\ \overline{\kappa}_n = -\overline{\kappa}\sinh\overline{\theta} \text{ and } \overline{\tau}_g = \overline{\tau} + \overline{\theta}'. \tag{3.5}$$

Differentiating the expression (3.3) and using the matrix representation (3.4), we have the Frenet derivative formulae of the spacelike parallel ruled surface as:

$$\begin{aligned} \overline{T}' &= \overline{\kappa n}, \\ \overline{n}' &= -\overline{\kappa T} + \overline{\tau b}, \\ \overline{b}' &= \overline{\tau n}. \end{aligned} \tag{3.6}$$

Theorem 3.1 *Let $\overline{M}$ be a spacelike parallel ruled surface and Darboux frame of curve β be*

$\{\overline{T},\overline{g},\overline{N}\}$ *at* $f(\alpha(t_0))=f(P)$ *on* $\overline{M}$*, then we have*

$$\begin{aligned}\overline{T} &= \frac{1}{v}\left[(1+r\kappa_n)T+r\tau_g g\right],\\ \overline{g} &= \frac{1}{v}\left[-(1+r\kappa_n)g+r\tau_g T\right],\\ \overline{N} &= N.\end{aligned} \tag{3.7}$$

Proof From Theorem 2.4 and the matrix representation (2.5), tangent vector of $(f\circ\alpha)=\beta$ of the curve at $f(\alpha(t_0))$ on the spacelike parallel ruled surface $\overline{M}$ is

$$\begin{aligned}\beta' = \overline{T} &= \frac{1}{v}\left[T+r(\kappa_n T+\tau_g g)\right]\\ &= \frac{1}{v}\left[(1+r\kappa_n)T+r\tau_g g\right],\end{aligned} \tag{3.8}$$

where the norm of β' is

$$\|\beta'\| = \sqrt{(1+r\kappa_n)^2+r^2\tau_g^2}=v. \tag{3.9}$$

Same as, we find $\overline{g}$. And also there is the equation $\overline{N}=N$ between normal vectors of surfaces M and $\overline{M}$. □

Theorem 3.2 *Let* α *be a regular curve on the surface* M*. Then the geodesic curvature, the normal curvature and the geodesic torsion of the curve* $(f\circ\alpha)=\beta$ *are respectively;*

$$\begin{aligned}\overline{\kappa}_g &= -\frac{\kappa_g}{v^3}+\frac{r}{v^3}\left[((1+r\kappa_n)\tau_g'+r\tau_g\kappa_n')\right],\\ \overline{\kappa}_n &= \frac{1}{v^2}\left[\kappa_n+r(\kappa_n^2+\tau_g^2)\right],\\ \overline{\tau}_g &= \frac{\tau_g}{v^2},\end{aligned} \tag{3.10}$$

at the point $f(\alpha(t_0))$ *on the spacelike parallel ruled surface* $\overline{M}$*.*

Proof Because β spacelike curve is a non-unit speed curve, we use the Theorems 2.6, 2.4 and the equation (3.8), hence the following equation is obtained

$$\beta''=(r\kappa_n'-r\tau_g\kappa_g)T+(\kappa_g(1+r\kappa_n)+r\tau_g')g+(\kappa_n(1+r\kappa_n)+r\tau_g^2)N \tag{3.11}$$

Using the expressions (2.6), (3.7) and (3.11), we find $\overline{\kappa}_g$, $\overline{\kappa}_n$, $\overline{\tau}_g$ for spacelike parallel ruled surface $\overline{M}$. □

Theorem 3.3 *Let Frenet frame of the curve* $(f\circ\alpha)=\beta$ *be* $\{\overline{T},\overline{n},\overline{b}\}$ *at* $f(\alpha(t_0))=f(P)$ *on* *the surface $\overline{M}$, then Frenet frame of the spacelike parallel ruled surface is as follows:*

$$\overline{T} = \frac{1}{v}\left[(1+r\overline{\kappa}_n)T + r\overline{\tau}_g g\right], \tag{3.12}$$

$$\overline{n} = \frac{1}{v\sqrt{(\overline{\kappa}_n)^2 - (\overline{\kappa}_g)^2}}\left[-(r\overline{\tau}_g\overline{\kappa}_g)T + (1+r\overline{\kappa}_n)\overline{\kappa}_g g + (v\overline{\kappa}_n)N\right],$$

$$\overline{b} = \frac{1}{v^3\sqrt{(\overline{\kappa}_n)^2 - (\overline{\kappa}_g)^2}}\left[(v^2 r\overline{\tau}_g\overline{\kappa}_n)T - (v^2(1+r\overline{\kappa}_n)\overline{\kappa}_n)g - (v^3\overline{\kappa}_g)N\right].$$

Proof If we use the equations (3.8) and (3.11), the following equation is obtained

$$\beta' \wedge \beta'' = \left(r\overline{\tau}_g\overline{\kappa}_n v^2\right)T - \left((1+r\overline{\kappa}_n)\overline{\kappa}_n v^2\right)g - (\overline{\kappa}_g v^3)N. \tag{3.13}$$

The norm of the equation (3.13) is

$$\|\beta' \wedge \beta''\| = v^3\sqrt{(\overline{\kappa}_n)^2 - (\overline{\kappa}_g)^2}. \tag{3.14}$$

By the equations (3.13) and (3.14), we obtain the expression (3.12). □

Let $\overline{\phi}$ be the angle between direction vector $\overline{X}$ and tangent vector $\overline{T}$ at $\overline{\alpha} \in \overline{\varphi}(s,\overline{v})$. If we choose the direction vector $\overline{X}$, then we get

$$\overline{X} = \overline{T}\cos\overline{\phi} + \overline{g}\sin\overline{\phi}, \tag{3.15}$$

where $\|\overline{X}\| = 1$.

Differentiating (3.15) and using (3.4), we find

$$\overline{X}' = -(\overline{\phi}' + \overline{\kappa}_g)\sin\overline{\phi}\overline{T} + (\overline{\phi}' + \overline{\kappa}_g)\cos\overline{\phi}\overline{g} + (\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi})\overline{N}. \tag{3.16}$$

Holding $\overline{v}$ =constant, we obtain the curve on the spacelike parallel ruled surface whose vector field as follows:

$$\begin{aligned} \overline{T}^* &= \overline{T} + \overline{v}\overline{X}' \\ &= (1-\overline{v}(\overline{\phi}' + \overline{\kappa}_g)\sin\overline{\phi}\overline{T} + (\overline{\phi}' + \overline{\kappa}_g)\cos\overline{\phi}\overline{g} + (\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi})\overline{N}. \end{aligned} \tag{3.17}$$

Substituting the expression (2.6) into (3.5), we have

$$\begin{aligned} \overline{T}^* = {} & \left(1 - \frac{\overline{v}}{v^3}(-\kappa_g + r((r\kappa_n+1)\tau_g)' + v^3\overline{\phi}')\sin\overline{\phi}\right)\overline{T} \\ & + \frac{\overline{v}}{v^3}\left((-\kappa_g + r((r\kappa_n+1)\tau_g)' + v^3\overline{\phi}')(\kappa_n + r(\kappa_n^2+\tau_g^2) + v^2\overline{\phi}')\cos\overline{\phi}\right)\overline{g} \\ & + \frac{\overline{v}}{v^2}\left((\kappa_n + r(\kappa_n^2+\tau_g^2))\cos\overline{\phi} + \tau_g\sin\overline{\phi}\right)\overline{N}. \end{aligned}$$

The distribution parameter of the spacelike parallel ruled surface is defined by

$$\overline{P}_{\overline{X}}=\frac{-\sin\overline{\phi}(\overline{\kappa}_n\cos\overline{\phi}+\overline{\tau}_g\sin\overline{\phi})}{(\overline{\phi}'+\overline{\kappa}_g)^2-(\overline{\kappa}_n\cos\overline{\phi}+\overline{\tau}_g\sin\overline{\phi})^2}. \tag{3.18}$$

By the expression (3.10), we obtain $\overline{P}_{\overline{X}}$ for spacelike parallel ruled surface as follows:

$$\overline{P}_{\overline{X}}=\frac{-v^4\sin\overline{\phi}\left((\kappa_n+r(\kappa_n^2+\tau_g^2))\cos\overline{\phi}+\tau_g\sin\overline{\phi})\right)}{(-\kappa_g+r((r\kappa_n+1)\tau_g)'+v^3\overline{\phi}')^2+v^2\left((\kappa_n+r(\kappa_n^2+\tau_g^2))\cos\overline{\phi}+\tau_g\sin\overline{\phi}\right)^2}.$$

Theorem 3.4 *The spacelike parallel ruled surface with Darboux frame is developable surface if and only if*

$$\sin\overline{\phi}(\overline{\kappa}_n\cos\overline{\phi}+\overline{\tau}_g\sin\overline{\phi})=0. \tag{3.19}$$

Proof Supposing that the spacelike parallel ruled surface with Darboux frame is developable surface, then $\overline{P}_{\overline{X}}=0$. In this case, let us study the following subcases related to the equation (3.19) vanishing:

(1) If $\sin\overline{\phi}=0$, then from (3.15), we obtain $\overline{X}=\overline{T}\cos\overline{\phi}$. So $\overline{T}^*=\overline{T}$. It means that tangent plane is constant along the main ruling on the spacelike parallel ruled surface.

(2) If $(\overline{\kappa}_n\cos\overline{\phi}+\overline{\tau}_g\sin\overline{\phi})=0$, then from the equation (3.17), it is seen that the tangent plane and the normal vector of spacelike parallel ruled surface with Darboux frame are orthogonal vectors. Therefore the spacelike parallel ruled surface with Darboux frame is developable surface.

Conversely, if

$$\sin\overline{\phi}(\overline{\kappa}_n\cos\overline{\phi}+\overline{\tau}_g\sin\overline{\phi})=0,$$

then from (3.18), $\overline{P}_{\overline{X}}=0$.

The striction curve of the spacelike parallel ruled surface with Darboux frame is calculated as follows:

$$\overline{c}(s)=\overline{\alpha}(s)+\frac{\sin\overline{\phi}(\overline{\phi}'+\overline{\kappa}_g)}{(\overline{\phi}'+\overline{\kappa}_g)^2+(\overline{\kappa}_n\cos\overline{\phi}+\overline{\tau}_g\sin\overline{\phi})^2}\overline{X}$$

Using the expression (3.10), we find $\overline{c}(s)$ for the spacelike parallel ruled surface as:

$$\overline{c}(s)=\overline{\alpha}(s)+\frac{-v^4\sin\overline{\phi}(-\kappa_g+r((r\kappa_n+1)\tau_g)'+v^3\overline{\phi}')}{(-\kappa_g+r((r\kappa_n+1)\tau_g)'+v^3\overline{\phi}')^2-v^2\left((\kappa_n+r(\kappa_n^2+\tau_g^2))\cos\overline{\phi}+\tau_g\sin\overline{\phi}\right)^2}\overline{X} \tag{3.20}$$

□

Theorem 3.5 *Let $\overline{M}$ be a spacelike parallel ruled surface with Darboux frame as in (3.1). Then the shortest distance between the rulings of $\overline{M}$ along the orthogonal trajectories is the distance measured equaled to the value:*

$$\overline{v}=\frac{-v^4\sin\overline{\phi}(-\kappa_g+r((r\kappa_n+1)\tau_g)'+v^3\overline{\phi}')}{(-\kappa_g+r((r\kappa_n+1)\tau_g)'+v^3\overline{\phi}')^2-v^2\left((\kappa_n+r(\kappa_n^2+\tau_g^2))\cos\overline{\phi}+\tau_g\sin\overline{\phi}\right)^2}. \tag{3.21}$$

Proof From (3.17), we have

$$\overline{J}(\overline{v})=\int_{s_1}^{s_2}\Big(1\text{-}2\overline{v}(\overline{\phi}'+\overline{\kappa}_g)\sin\overline{\phi}+\overline{v}^2(\overline{\phi}'+\overline{\kappa}_g)^2+\overline{v}^2(\overline{\kappa}_n\cos\overline{\phi}+\overline{\tau}_g\sin\overline{\phi})^2\Big)^{\frac{1}{2}}\,ds, \tag{3.22}$$

where $s_1 < s_2$.

Differentiating the expression (3.22) according to the parameter $\overline{v}$ which gives the minimal value of $\overline{J}(\overline{v})$, we get

$$\overline{v}=\frac{\sin\overline{\phi}(\overline{\phi}'+\overline{\kappa}_g)}{(\overline{\phi}'+\overline{\kappa}_g)^2+(\overline{\kappa}_n\cos\overline{\phi}+\overline{\tau}_g\sin\overline{\phi})^2}. \tag{3.23}$$

Using the expression (3.10) in (3.23), the parameter $\overline{v}$ turns into the equation (3.21). □

Theorem 3.6 *Let $\overline{M}$ be a spacelike parallel ruled surface with Darboux frame as in (3.1), the absolute value of Gauss curvature K of the spacelike parallel ruled surface $\overline{M}$ along a ruling takes the maximum value at the striction point on that ruling.*

Proof Calculating Gauss curvature of the spacelike parallel ruled surface with Darboux frame, we get

$$K(s,\overline{v})=\frac{(\overline{\kappa}_n\cos\overline{\phi}+\overline{\tau}_g\sin\overline{\phi})^2\sin^2\overline{\phi}}{\Big(1\text{-}2\overline{v}(\overline{\phi}'+\overline{\kappa}_g)\sin\overline{\phi}+\overline{v}^2(\overline{\phi}'+\overline{\kappa}_g)^2+\overline{v}^2(\overline{\kappa}_n\cos\overline{\phi}+\overline{\tau}_g\sin\overline{\phi})^2-\cos^2\overline{\phi}\Big)^2}. \tag{3.24}$$

Differentiating the equation (3.24) with respect to $\overline{v}$, we have

$$\overline{v}=\frac{\sin\overline{\phi}(\overline{\phi}'+\overline{\kappa}_g)}{(\overline{\phi}'+\overline{\kappa}_g)^2+(\overline{\kappa}_n\cos\overline{\phi}+\overline{\tau}_g\sin\overline{\phi})^2}.$$

Therefore, the absolute value of Gauss curvature K of the spacelike parallel ruled surface $\overline{M}$ along a ruling takes the maximum value at the striction point on that ruling. □

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