

Norm of Bethe Wave Function as a Determinant

Vladimir Korepin

Steklov Mathematical Institute of Academy of Sciences, Sankt Petersburg, Russia

(Dated: Feb 26, 1981)

This is a historical note. Bethe Ansatz solvable models are considered, like XXZ Heisenberg anti-ferromagnet and Bose gas with delta interaction. Periodic boundary conditions lead to Bethe equation. The square of the norm of Bethe wave function is equal to a determinant of linearized system of Bethe equations (determinant of matrix of second derivatives of Yang action). The proof was first published in *Communications in Mathematical Physics*, vol 86, page 391 in 1982. Also **domain wall boundary conditions for 6 vertex model** were discovered in the same paper [see Appendix D]. These play an important role for algebraic combinatorics: alternating sign matrices, domino tiling and plane partition.

INTRODUCTION

Many two dimensional models have been solved by means of the Bethe Ansatz, see for example [1–5] [30]. Quantum inverse scattering method (QISM) [6, 7, 26] discloses algebraic nature of these solutions. Michael Gaudin studied norms of Bethe wave functions for the quantum nonlinear Schroedinger equation [31] [8] and suggested a remarkable conjecture that the norm of the eigenfunction is equal to a Jacobian. In [25] these formulae are proved and a more general result is obtained. The norms are calculated for any exactly solvable models with the R matrix either of the X X X model or of the X X Z Heisenberg models. This note gives an idea about an approach of the paper.

NOTATIONS

First of all let us remind the reader of some notations of the QISM. Eigenfunctions of the Hamiltonian of the physical system are constructed by means of the monodromy matrix of an auxiliary linear problem $T(\lambda)$. In our case $T(\lambda)$ is a 2×2 matrix, its matrix elements being quantum operators, which depend on the spectral parameter λ :

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

Commutation relations of these entries are given by the formula:

$$R(\lambda, \mu) (T(\lambda) \otimes I) (I \otimes T(\mu)) = (I \otimes T(\mu)) (T(\lambda) \otimes I) R(\lambda, \mu) \quad (1)$$

Here I is the unit 2×2 matrix, $R(\lambda, \mu)$ is a 4×4 matrix with c-number elements. Another way to write this equation is:

$$R(\lambda, \mu)_{\alpha, \beta} T(\lambda)_{\alpha} T(\mu)_{\beta} = T(\mu)_{\beta} T(\lambda)_{\alpha} R(\lambda, \mu)_{\alpha, \beta} \quad (2)$$

This means that $T(\lambda)_{\alpha}$ is a 2×2 matrix which acts on the 2 dimensional space with index α and $T(\mu)_{\beta}$ is a matrix which acts on space with index β . We assume that $\alpha \neq \beta$. The matrix $R(\lambda, \mu)_{\alpha, \beta}$, acts in the tensor product of these two spaces. We shall deal with R matrices of the following form:

$$R(\lambda, \mu) = \begin{pmatrix} f(\mu, \lambda) & 0 & 0 & 0 \\ 0 & 1 & g(\mu, \lambda) & 0 \\ 0 & g(\mu, \lambda) & 1 & 0 \\ 0 & 0 & 0 & f(\mu, \lambda) \end{pmatrix}$$

For models of the XXZ type:

$$f(\lambda, \mu) = \frac{\sinh(\lambda - \mu + 2i\eta)}{\sinh(\lambda - \mu)}, \quad g(\lambda, \mu) = \frac{i \sin(2\eta)}{\sinh(\lambda - \mu)} \quad (3)$$

For models of XXX type:

$$f(\lambda, \mu) = \frac{(\lambda - \mu + 2i\kappa)}{(\lambda - \mu)}, \quad g(\lambda, \mu) = \frac{i\kappa}{(\lambda - \mu)} \quad (4)$$

Here λ and μ are spectral parameters and η and κ are coupling constants. Note that XXX is a limiting case of XXZ. We can also write the R matrix using notations of formula (2).

$$R(\lambda, \mu)_{\alpha, \beta} = \cos(\eta) \frac{\sinh(\mu - \lambda + i\eta)}{\sinh(\mu - \lambda)} I_{\alpha} I_{\beta} + i \sin(\eta) \frac{\cosh(\mu - \lambda + i\eta)}{\sinh(\mu - \lambda)} \sigma_{\alpha}^3 \sigma_{\beta}^3 \quad (5)$$

$$+ i \frac{\sin(2\eta)}{\sinh(\mu - \lambda)} (\sigma_{\alpha}^+ \sigma_{\beta}^- + \sigma_{\alpha}^- \sigma_{\beta}^+) \quad (6)$$

Here σ are the standard Pauli matrices and $2\sigma^{\pm} = \sigma^1 \pm \sigma^2$. Let us write down some commutation relations of (1) explicitly

$$[B(\lambda), B(\mu)] = 0, \quad [C(\lambda), C(\mu)] = 0, \quad [A(\lambda) + D(\lambda), A(\mu) + D(\mu)] = 0 \quad (7)$$

$$A(\mu)B(\lambda) = f(\mu, \lambda)B(\lambda)A(\mu) + g(\lambda, \mu)B(\mu)A(\lambda) \quad (8)$$

$$D(\mu)B(\lambda) = f(\lambda, \mu)B(\lambda)D(\mu) + g(\mu, \lambda)B(\mu)D(\lambda) \quad (9)$$

$$[C(\lambda^c), B(\lambda^b)] = g(\lambda^c, \lambda^b) \{A(\lambda^c)D(\lambda^b) - A(\lambda^b)D(\lambda^c)\} \quad (10)$$

Other commutation relations can be found in the book [26]. Pseudo-vacuum $|0\rangle$ and dual pseudo-vacuum $\langle 0|$ are important:

$$C(\lambda)|0\rangle = 0, \quad A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle \quad (11)$$

$$\langle 0|B(\lambda) = 0, \quad \langle 0|A(\lambda) = \langle 0|a(\lambda), \quad \langle 0|D(\lambda) = \langle 0|d(\lambda) \quad (12)$$

Here $a(\lambda)$ and $d(\lambda)$ are complex valued functions. The space in which the operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ act is constructed in [26]. The norms and scalar products in question are functionals of these $a(\lambda)$ and $d(\lambda)$. We shall vary $a(\lambda)$ and $d(\lambda)$ and study the dependence of scalar products on these functional arguments. The Hamiltonian of the physical system in question is expressed in terms of the transfer matrix $t(\mu) = A(\lambda) + D(\lambda)$. The Hamiltonian and the transfer matrix have common eigenfunctions which are constructed as follows. Put

$$\psi_N(\{\lambda_j\}) = B(\lambda_1) \dots B(\lambda_N)|0\rangle \quad (13)$$

and suppose that λ_j satisfy the system of the transcendental equations (TE) :

$$\frac{a(\lambda_n)}{d(\lambda_n)} \prod_{j=1}^N \frac{f(\lambda_n, \lambda_j)}{f(\lambda_j, \lambda_n)} = 1 \quad \text{here} \quad j \neq n \quad (14)$$

Then $\psi_N(\{\lambda_j\})$ is an eigenfunction of $t(\lambda)$ with the eigenvalue:

$$\theta(\mu) = a(\mu) \prod_{j=1}^N f(\mu, \lambda_j) + d(\mu) \prod_{j=1}^N f(\lambda_j, \mu) \quad (15)$$

Here N is called the number of particles. Note that

$$\overline{\psi_N(\{\lambda_j\})} = \langle 0|C(\lambda_1) \dots C(\lambda_N) \quad (16)$$

is a dual eigenfunction for $t(\mu)$ with the same eigenvalue. Pauli principle was proved in the original publication [25], see also [26]. We take all λ_j to be different. Finally let us present two remarks. First of all new variables

$$\phi_k = i \ln \frac{a(\lambda_k)}{d(\lambda_k)} + i \sum_{j=1, j \neq k}^N \ln \frac{f(\lambda_k, \lambda_j)}{f(\lambda_j, \lambda_k)}, \quad k = 1, \dots, N \quad (17)$$

are convenient. For example (14) can be written as

$$\phi_k = 0 \pmod{2\pi} \quad (18)$$

EXPRESSION FOR THE NORM IN THE XXX TYPE MODELS

For models with an R matrix (4) the scalar product of an eigenfunction and dual eigenfunction is equal to

$$\begin{aligned} \langle 0|C(\lambda_1)\dots C(\lambda_N)B(\lambda_1)\dots B(\lambda_N)|0\rangle &= \kappa^N \left(\prod_{j=1}^N a(\lambda_j)d(\lambda_j) \right) \langle 0|0\rangle \\ & \left(\prod_{j=1}^N \prod_{k=1, j\neq k}^N f(\lambda_j, \lambda_k) \right) \det_N \left(\frac{\partial \phi_j}{\partial \lambda_k} \right) \end{aligned} \quad (19)$$

Here the ϕ_k are the variables (17) and the set of the λ_j is a solution of the system (18), (14). The derivatives can be written in the explicit form:

$$\frac{\partial \phi_k}{\partial \lambda_j} = \delta_{k,j} \left(i \frac{\partial}{\partial \lambda_k} \ln \frac{a(\lambda_k)}{d(\lambda_k)} + \sum_{p=1}^N \frac{2\kappa}{(\lambda_k - \lambda_p)^2 + \kappa^2} \right) - \frac{2\kappa}{(\lambda_k - \lambda_j)^2 + \kappa^2} \quad (20)$$

The formula (19) is useful for calculation of norms if the set of λ_j invariant under complex conjugation $\{\lambda_j\} = \overline{\{\lambda_j\}}$ and $B^\dagger(\lambda) = \pm C(\bar{\lambda})$. Examples of applications can be useful. Let consider nonlinear Schroedinger equation, it is also known as Bose gas with delta interaction. It has a Hamiltonian

$$\begin{aligned} H &= \int dx \left(\partial_x \psi^\dagger \partial_x \psi + \kappa \psi^\dagger \psi^\dagger \psi \psi \right), \\ & [\psi(x), \psi^\dagger(y) = \delta(x - y)] \end{aligned} \quad (21)$$

The monodromy matrix has the following property:

$$\sigma^1 T^\dagger(\bar{\lambda}) \sigma^1 = T(\lambda), \quad B^\dagger(\lambda) = C(\bar{\lambda}) \quad (22)$$

This is *XXX* case. The vacuum eigenvalues are equal to:

$$a(\lambda) = \exp(-iL\lambda/2), \quad d(\lambda) = \exp(iL\lambda/2)$$

This proves Gaudin conjecture [8]. This determinant formula for norm also applicable for Heisenberg *XXX* spin chain.

EXPRESSION FOR THE NORM IN THE XXZ TYPE MODELS

For solvable models with *XXZ* R matrix (3) the scalar product of the eigenfunction and dual eigenfunction is equal

$$\begin{aligned} \langle 0|C(\lambda_1)\dots C(\lambda_N)B(\lambda_1)\dots B(\lambda_N)|0\rangle &= (\sin 2\eta)^N \left(\prod_{j=1}^N a(\lambda_j)d(\lambda_j) \right) \langle 0|0\rangle \\ & \left(\prod_{j=1}^N \prod_{k=1, j\neq k}^N f(\lambda_j, \lambda_k) \right) \det_N \left(\frac{\partial \phi_j}{\partial \lambda_k} \right) \end{aligned} \quad (23)$$

Here λ_j has to satisfy equation (14). The Jacobi matrix can be written down in the explicit form:

$$\frac{\partial \phi_k}{\partial \lambda_j} = \delta_{k,j} \left(i \frac{\partial \ln a(\lambda_k)/d(\lambda_k)}{\partial \lambda_k} + \sum_{p=1}^N \frac{\sin(4\eta)}{\sinh(\lambda_k - \lambda_p + 2i\eta) \sinh(\lambda_k - \lambda_p - 2i\eta)} \right) + \frac{-\sin(4\eta)}{\sinh(\lambda_k - \lambda_j + 2i\eta) \sinh(\lambda_k - \lambda_j - 2i\eta)} \quad (24)$$

The whole paper [25] is devoted to the derivation of this formula. Let us calculate the norms for *XXZ* model. The Hamiltonian of the model is

$$H = \sum_{k=1}^M \sigma_k^1 \sigma_{k+1}^1 + \sigma_k^2 \sigma_{k+1}^2 + \cos(2\eta) (\sigma_k^3 \sigma_{k+1}^3 - 1) \quad (25)$$

The model was imbedded into QISM in [7]. This monodromy matrix has the following property at real η :

$$\sigma^2 T^\dagger(\bar{\lambda}) \sigma^2 = T(\lambda), \quad B^\dagger(\lambda) = -C(\bar{\lambda}) \quad (26)$$

Pseudo-vacuum is the ferromagnetic state $|0\rangle = \prod_{j=1}^M \uparrow_j = |0\rangle$. The vacuum eigenvalues are:

$$a(\lambda) = \sinh^M(\lambda - i\eta), \quad d(\lambda) = \sinh^M(\lambda + i\eta)$$

In order to write down the square of the norm it is convenient to introduce:

$$\chi(\lambda, \eta) = \frac{\sin(2\eta)}{\sinh(\lambda - i\eta) \sinh(\lambda + i\eta)}$$

The formula for the square of the norm is:

$$\begin{aligned} \langle 0|B^\dagger(\lambda_N) \dots B^\dagger(\lambda_1)B(\lambda_1) \dots B(\lambda_N)|0\rangle &= \sin^N(2\eta) \left(\prod_{j=1}^N \sinh^M(\lambda_j - i\eta) \sinh^M(\lambda_j + i\eta) \right) \\ &\left(\prod_{k>j=1}^N \frac{\sinh(\lambda_j - \lambda_k - 2i\eta) \sinh(\lambda_j - \lambda_k + 2i\eta)}{\sinh^2(\lambda_j - \lambda_k)} \right) \det_N \left[\delta_{kj} \left(M\chi(\lambda_k, \eta) - \sum_{l=1}^N \chi(\lambda_k - \lambda_l, 2\eta) \right) + \chi(\lambda_k - \lambda_j, 2\eta) \right] \end{aligned} \quad (27)$$

The formula was presented in [9] and verifies for $N = 2$ and $N = 3$. The proof for arbitrary N was first published in [25]. The formula (23) also describes norms in Sine-Gordon and lattice Sine-Gordon [18, 19].

THE IDEA OF THE PROOF

In order to prove determinant formula for the norm of Bethe wave function let us introduce an object:

$$\frac{\langle 0|C(\lambda_1) \dots C(\lambda_N)B(\lambda_1) \dots B(\lambda_N)|0\rangle}{(\sin 2\eta)^N \left(\prod_{j=1}^N a(\lambda_j)d(\lambda_j) \right) \langle 0|0\rangle \left(\prod_{j=1}^N \prod_{k=1, j \neq k}^N f(\lambda_j, \lambda_k) \right)} = |\lambda_1 \dots \lambda_N| \quad (28)$$

We have to prove that

$$|\lambda_1 \dots \lambda_N| = \det_N \left(\frac{\partial \phi_j}{\partial \lambda_k} \right) \quad (29)$$

We will assume that $\langle 0|0\rangle = 1$. The author of [25] proved that $a(\lambda)$ and $d(\lambda)$ can be considered an arbitrary functions. So we can consider the variables

$$X_p = i \frac{\partial}{\partial \lambda_p} \ln \frac{a(\lambda_p)}{d(\lambda_p)}$$

as independent of λ_j , see [25]. The following theorem is proved in [25].

In order to prove (29) it is enough to prove the following five properties of $|\lambda_1 \dots \lambda_N|$:

- *It is invariant under simultaneous replacement*

$$\lambda_j \leftrightarrow \lambda_k, \quad X_j \leftrightarrow X_k \quad (30)$$

- *It is linear function of X_1*

$$|\lambda_1 \dots \lambda_N| = U_1 X_1 + V_1 \quad (31)$$

- *The coefficient U_1 is*

$$U_1 = |\lambda_2 \dots \lambda_N|^{\text{modified}} \quad (32)$$

The right hand side is given by formula (28) with λ_1 removed and functions $a(\lambda)$ and $d(\lambda)$ replaced by:

$$a^{\text{modified}}(\lambda) = a(\lambda)f(\lambda, \lambda_1), \quad d^{\text{modified}}(\lambda) = d(\lambda)f(\lambda_1, \lambda) \quad (33)$$

- *It vanish if all $X_p = 0$*

$$|\lambda_1 \dots \lambda_N| = 0, \quad \text{if all } X_p = 0, \quad \text{at } p = 1, \dots, N \quad (34)$$

- *For $N = 1$*

$$|\lambda_1| = X_1$$

The paper [25] proves that the right hand side of (28) has all five properties listed above. The proof is reduced to analysis of six vertex model with domain wall boundary conditions.

SIX VERTEX MODEL

The six vertex model is an important 'counterexample' of statistical mechanics: the bulk free energy depends on the boundary conditions even in thermodynamic limit, see [11].

Let us start the formal presentation: L operator in site number k is

$$L_k(\lambda - \nu_k) = \begin{pmatrix} \sinh(\lambda - \nu_k - i\eta\sigma_k^3) & -i\sigma_k^- \sin(2\eta) \\ -i\sigma_k^+ \sin(2\eta) & \sinh(\lambda - \nu_k + i\eta\sigma_k^3) \end{pmatrix}$$

It also can be written as:

$$L_k(\lambda - \nu_k) = \cos(\eta) \sinh(\lambda - \nu_k) - i \sin(\eta) \sigma_k^3 \cosh(\lambda - \nu_k) - i \sin(2\eta) (\sigma_k^+ \sigma_k^- + \sigma_k^- \sigma_k^+) \quad (35)$$

The monodromy matrix:

$$T(\lambda) = L_M(\lambda - \nu_M) \dots L_1(\lambda - \nu_1) \quad (36)$$

obeys the commutation relations (1) with R matrix (3). Pseudo-vacuum is the ferromagnetic state

$$|0\rangle = \prod_{j=1}^M \downarrow_j = |0\rangle$$

The vacuum eigenvalues are now equal to

$$a(\lambda) = \prod_{j=1}^M \sinh(\lambda - \nu_j - i\eta), \quad d(\lambda) = \prod_{j=1}^M \sinh(\lambda - \nu_j + i\eta)$$

Such an inhomogeneous generalization was used for example in [20–22]. Let us consider a special state

$$B(\lambda_1) \dots B(\lambda_M) |0\rangle \quad (37)$$

Here the number of the $B(\lambda)$ is equal to the number of the sites in the lattice $N = M$. All spins are looking down in this state:

$$B(\lambda_1) \dots B(\lambda_M) |0\rangle = Z_N \Omega \quad (38)$$

Here Z_N is a complex number and

$$\Omega = \prod_{j=1}^M \downarrow_j \quad (39)$$

The definition of Z_N is

$$Z_N = \Omega B(\lambda_1) \dots B(\lambda_M) |0\rangle$$

The paper [25] proves that Z_N is the partition function of six vertex model with domain wall boundary conditions, see Appendix D. Actually these boundary conditions were introduced in this paper, including the name. Explicit description of six vertex model with domain wall boundary conditions also can be found in [28].

The following recursion relations were discovered in [25]: If $\nu_1 = \lambda_1 + i\eta$ then Z_N reduces to Z_{N-1} with λ_1 and ν_1 removed.

$$Z_N(\{\lambda_\alpha\}, \{\nu_j\})|_{\nu_1 - \lambda_1 = i\eta} = -i \sin(2\eta) \left(\prod_{k=2}^N \sinh(\lambda_1 - \nu_k - i\eta) \right) \left(\prod_{\alpha=2}^N \sinh(\lambda_\alpha - \nu_1 - i\eta) \right) Z_{N-1}(\{\lambda_{\alpha \neq 1}\}, \{\nu_{j \neq 1}\})$$

The derivation of this recursion relations also can be found in section 2 of the paper [28].

CONCLUSION

There was a lot of progress since the paper [25]. The partition function of six vertex model with domain wall boundary conditions has a lot of applications [32] and generalizations, see for example [29]. Still there are open problems. For example the determinant formula for Bethe wave function in Hubbard still not proven, see [27].

-
- [1] Bethe, H.: Z. Phys. 71, 205-226 (1931)
 - [2] Yang, C.N., Yang, C.P.: Phys. Rev. 150, 321-327 (1966)
 - [3] Lieb, E.H.: Phys. Rev. Lett. 18, 692-694 (1967)
 - [4] Berezin, F.A., Pokhil, C.P., Finkelberg, V.M.: Vestnik Mosk. Gos. Univ. Ser. 1.1, 21-28 (1964)
 - [5] McGuire, J., B. J. Math. Phys. 5, 622-636 (1964)
Brezin, E., Zinn-Justin, J.: C.R. Acad. Sci. Paris, 263, 670-613 (1966)
Gaudin, M.: J. Math. Phys. 12, 1674-1680, (1971)
 - [6] Faddeev, L.D.: Sov. Sci. Rev. Math. Phys. C. 1, 107-160 (1981)
 - [7] Faddeev, L.D., Takhtajan, L.A.: Usp. Mat. Nauk 34, 13-63 (1979)
 - [8] Gaudin, M.: Preprint Centre d'Etudes Nucleaire de Saclay CEA-N-1559 (1), (1972)
 - [9] Gaudin, M., McCoy, B.M., Wu, T.T.: Phys. Rev. D, vol 23, 417 (1981)
 - [10] Baxter, R.J.: J. Stat. Phys. 9, 145-182 (1973)
 - [11] Eloranta, K., J. Stat. Phys. 96 (1999), 1091.
 - [12] Izergin, A.G., Korepin, V.E.: Lett. Math. Phys. (to be published)
 - [13] Faddeev, L.D., Sklyanin, E.K.: Dokl. Akad. Nauk SSSR 243, 1430-1433 (1978)
 - [14] Sklyanin, E.K.: Dokl. Akad. Nauk SSSR 244, 1337-1341 (1978)
 - [15] Izergin, A.G., Korepin, V.E., Smirnov, F.A.: Teor. Mat. Fiz. 48, 319-323 (1981)
 - [16] Sklyanin, E.K.: Zap. Nauchn. Seminarov LOMI, 95, 55-128 (1980)
 - [17] Izergin, A.G., Korepin, V.E.: Dokl. Akad. Nauk SSSR 259, 76-79 (1981)
 - [18] Izergin, A.G., Korepin, V.E.: Nucl. Phys. B. Field Theory and Statistical Systems B205 [FS5] 401-413 (1982)
 - [19] Izergin, A.G., Korepin, V.E.: Lett. Math. Phys. 5, 199-205 (1981)
 - [20] Baxter, R.J.: Stud. Appl. Math. L50, 51-67 (1971)
 - [21] Belavin, A.A.: Phys. Lett. B87, 117-121 (1980)
 - [22] Kulish, P.P.: Physica D3, 246-257 (1981)
 - [23] Faddeev, L.D., Takhtajan, L.A.: Zap. Nauchn. Seminarov LOMI 109, 134-178 (1981)
 - [24] Baxter, R.J.: Philos. Trans. Soc. London, Ser. A 289, 315-346 (1978)
 - [25] Korepin, V.E., Communications in Mathematical Physics, vol 86, page 391 in 1982. A link to PDF file can be found in the following way: go to <http://insti.physics.sunysb.edu/physics/forms/profilesearch.cgi?lastname=korepin> click on PERSONAL HOME PAGE [to the right] and go to the fifth bullet.
 - [26] V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, a book Quantum Inverse Scattering Method and Correlation Functions, Cambridge University Press, 1993, <http://www.cambridge.org/catalogue/catalogue.asp?isbn=9780521586467>
 - [27] F. Goehmann, V. E. Korepin, Phys. Lett. A263 (1999) 293-298, arXiv:cond-mat/9908114
 - [28] V. Korepin, P. Zinn-Justin, arXiv:cond-mat/0004250, J. Phys. A 33 No. 40 (2000), 7053
 - [29] Hjalmar Rosengren <http://arxiv.org/abs/0911.0561>
 - [30] Experts believe that each universality class in 2D contain at list one solvable model.
 - [31] the model is also known as Bose gas with delta interaction
 - [32] Google finds thousands of publications for: Six vertex model with domain wall boundary conditions