COMPARISON OF NON-STATIONARY SINUSOID ESTIMATION METHODS USING REASSIGNMENT AND DERIVATIVES

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ABSTRACT

In this paper, three state of the art non-stationary sinusoidal analysis methods based on Fourier transform (FT) are compared - the derivative method, reassignment and generalized reassignment. [1] The derivative method and reassignment were designed to analyze linear log-AM/linear FM sinusoids. Generalized reassignment can analyze sinusoids containing arbitrary order modulations, however the discussion will be limited to linear log-AM/linear FM in order to compare it objectively to reassignment and the derivative method. In this paper, the equivalence of reassignment and the derivative method is shown to hold for arbitray order modulation estimation and theoretical comparison with generalized reassignment is presented. The results of tests conducted on two different frequency ranges, full range (frequencies up to Nyquist) and reduced range (frequencies up to 3/4 Nyquist) frequency range, are compared to the Cramer-Rao bounds (CRBs).

1. INTRODUCTION

Sinusoidal modeling of sound signals is used in many audio analysis/synthesis applications [1],[2],[3]. Several analysis methods for estimating sinusoidal model parameters based on Short Time Fourier Transform (STFT) assume that the underlying sinusoid is quasi-stationary inside a selected time frame [4],[5],[6]. Since real world sounds often violate this assumption, the analysis methods able to detect first order polynomial modulations have received much attention [7],[8],[9],[10],[11]. It is straightforward to see [12] that the reassignment method could be theoretically generalized to detect higher order polynomial modulations for both log-AM and FM. As it was shown in [8] and [13] that the reassignment and the derivative methods are theoretically identical, then the same must hold

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[1] The latter method was not explicitly named in [14] where it was first presented, therefore the authors decided on the name *generalized reassignment*, because the method exhibits similarity with the original reassignment and can successfully analyze a *generalized sinusoid* (a sinusoid with arbitrary order polynomial log-AM and FM function), a term adopted from [15]. The method exhibits close relation to the derivative method as well, however the name *generalized derivative method* could cause ambiguity with the method described in [8].

for the derivative method. However, expressions for estimating higher order modulations get very complex in both cases and its derivations are not straightforward. The theoretical equality of both methods can be exploited to construct the generalized reassignment method [14], which estimates parameters of signal modulations up to an arbitrary degree, although some restrictions concerning window function apply. In [15] the generalized reassignment is shown to work for any linear transform, which includes the STFT, the wavelet transform or even a combination of them.

The derivative method, reassignment and generalized reassignment are all based on the same theoretical background, yet perform quite different in practice. Therefore, a detailed comparison of its internal mechanics is performed in the present document. In section 2, the common theoretical background including general equations for parameter estimations is derived. The differences between reassignment and the derivative method are described and method-specific parameter estimate expressions for the two are derived from the general ones, providing a mathematically identical proof already given in [8] and [13], extended to an arbitrary modulation degree. In section 3 the theoretical differences between original and generalized reassignment are outlined. Section 4 describes the test environment and summarizes accuracies collected in tests of all three estimators and compares them to CRBs. Conclusions and future work suggestions are given in section 5.

2. REASSIGNMENT AND THE DERIVATIVE METHOD

This section demonstrates that reassignment and the derivative method are two versions of the same algorithm. This fact was already pointed out in [8] and [13]. Present derivations prove that the derivative method is essentially a reassignment method with a slightly modified STFT definition or vice versa. In the following expression, the STFT definitions for the derivative method (D) and reassignment (R) are given respectively:

$$STFT\big(s(t), w(t); t, \omega\big) = {}^{\text{D}}S_w(t, \omega) =$$
$$\int_{-\infty}^{\infty} s(\tau + t)w(\tau)e^{-j\omega\tau}d\tau \tag{1}$$

$$STFT\big(s(t), w(t); t, \omega\big) = \overset{\text{\tiny R}}{S}_w(t, \omega) =$$
$$\int_{t-\infty}^{t+\infty} s(\bar{\tau}) w(\bar{\tau} - t) e^{-j\omega(\bar{\tau}-t)} d\bar{\tau}, \tag{2}$$

where $s(t), w(t)$ are signal and window functions respectively. It is obvious, that the second definition can be derived from the first by substituting the variables: $\tau = \bar{\tau} - t$. Essentially $\overset{\text{\tiny R}}{S}_w = \overset{\text{\tiny D}}{S}_w$; however, its the time/frequency derivatives yield different expressions (see Appendix A), yet a common notation S_w will be used for both $\overset{\text{\tiny R}}{S}_w$ and $\overset{\text{\tiny D}}{S}_w$ from this point on.

S_w is a function of time and frequency which can be written in polar form as:

$$S_w(t, \omega) = \exp\big(a(t, \omega) + j\phi(t, \omega)\big). \tag{3}$$

From the above representation, amplitude and phase functions of time and frequency can be written as:

$$a(t, \omega) = \Re\Big(\log \big(S_w(t, \omega)\big) \Big) \tag{4}$$

$$\phi(t, \omega) = \Im\Big(\log \big(S_w(t, \omega)\big) \Big). \tag{5}$$

Reassigned frequency and time equations can be expressed for this general case as:

$$\hat{\omega}(t, \omega) = \frac{\partial}{\partial t}\phi(t, \omega) = \Im\left(\frac{\frac{\partial S_w}{\partial t}}{S_w} \right) \tag{6}$$

$$\hat{t}(t, \omega) = t - \frac{\partial}{\partial \omega}\phi(t, \omega) = t - \Im\left(\frac{\frac{\partial S_w}{\partial \omega}}{S_w} \right). \tag{7}$$

Linear log-AM/linear FM are commonly defined in the following form:

$$s(t) = \exp\big(\lambda_0 + \mu_0 t + j(\phi_0 + \omega_0 t + \frac{\psi_0}{2}t^2)\big). \tag{8}$$

As pointed out in [8] and [10], general log-AM and FM expressions can be written as:

$$\hat{\mu}(t, \omega) = \frac{\partial}{\partial t}a(t, \omega) = \Re\left(\frac{\frac{\partial S_w}{\partial t}}{S_w} \right) \tag{9}$$

$$\hat{\psi}(t, \omega) = \frac{\partial \hat{\omega}}{\partial \hat{t}} = \frac{\partial \hat{\omega}}{\partial t} \Big/ \frac{\partial \hat{t}}{\partial t} \tag{10}$$

$$\frac{\partial \hat{\omega}}{\partial t} = \Im\left(\frac{\frac{\partial^2 S_w}{\partial t^2} S_w - \left(\frac{\partial S_w}{\partial t}\right)^2}{\left(S_w\right)^2} \right) \tag{11}$$

$$\frac{\partial \hat{t}}{\partial t} = 1 - \Im\left(\frac{\frac{\partial^2 S_w}{\partial \omega \partial t} S_w - \frac{\partial S_w}{\partial \omega} \frac{\partial S_w}{\partial t}}{\left(S_w\right)^2} \right). \tag{12}$$

The above equations provide estimate espressions independent of the method used and thus hold for both reassignment and the derivative method. Once linear log-AM, frequency and linear FM parameters are estimated, the following expressions can be used to obtain accurate esti-

mates for the two static parameters [7],[8]:

$$\Gamma_w(\omega, \mu, \psi) = \int_{-\infty}^{\infty} w(t) \exp\left(\mu t + j\left(\omega t + \frac{\psi}{2}t^2 \right) \right) dt \tag{13}$$

$$\hat{a}_0 = \left| \frac{S_w}{\Gamma_w(\omega_\Delta, \hat{\mu}_0, \hat{\psi}_0)} \right| \tag{14}$$

$$\hat{\phi}_0 = \angle\left(\frac{S_w}{\Gamma_w(\omega_\Delta, \hat{\mu}_0, \hat{\psi}_0)} \right). \tag{15}$$

In order to obtain the above expressions for the two methods, partial frequency and time derivatives of S_w should be computed for reassignment and the derivative method. In the following formulas, $S_{w'}$ and S_{tw} represent the STFT of a signal, but the window derivative and time-ramped window functions are used instead of the original ones respectively, while S_w' represents the STFT of the time derivative of a signal. For reassignment, the following expressions with some restrictions (see Appendix A) apply:

$$\frac{\partial}{\partial t}S_w = -S_{w'} + j\omega S_w \tag{16}$$

$$\frac{\partial}{\partial \omega}S_w = -jS_{tw} \tag{17}$$

$$\frac{\partial^2}{\partial \omega \partial t}S_w = jS_{tw'} + jS_w + \omega S_{tw} \tag{18}$$

$$\frac{\partial^2}{\partial t^2}S_w = S_{w''} - 2j\omega S_{w'} - \omega^2 S_w. \tag{19}$$

Detailed derivations of 16 and 17 can be found in Appendix A (see equations 39, 40 and 43). Equation 18 is derived in detail in Appendix A (see equation 41), however the generalized rule (see equation 42) for time derivatives can be used as well. Equation 19 can be derived by using equation 42 twice. For the derivative method, slightly simpler expressions hold:

$$\frac{\partial}{\partial t}S_w = S_w' \tag{20}$$

$$\frac{\partial}{\partial \omega}S_w = -jS_{tw} \tag{21}$$

$$\frac{\partial^2}{\partial \omega \partial t}S_w = -jS_{tw}' \tag{22}$$

$$\frac{\partial^2}{\partial t^2}S_w = S_w''. \tag{23}$$

Substituting reassignment STFT expressions 16-19 into general equations for parameter estimations 6-12 yields:

$$\overset{\text{\tiny R}}{\hat{\omega}}(t, \omega) = \omega - \Im\left(\frac{S_{w'}}{S_w} \right) \tag{24}$$

$$\overset{\text{\tiny R}}{\hat{\mu}}(t, \omega) = -\Re\left(\frac{S_{w'}}{S_w} \right) \tag{25}$$

$$\overset{\text{\tiny R}}{\hat{\psi}}(t, \omega) = \frac{\Im\left(\frac{S_w S_{w''} - (S_{w'})^2}{(S_w)^2} \right)}{\Re\left(\frac{S_{tw'} S_w - S_{tw} S_w}{(S_w)^2} \right)}, \tag{26}$$

which are well known reassignment expressions for estimating parameters of log-AM/FM sinusoids. Analogously,

substituting derivative method STFT expressions 20-23 into same equations results in:

$$\overset{\scriptscriptstyle D}{\hat\omega}(t,\omega) = \Im\left(\frac{S'_w}{S_w}\right) \tag{27}$$

$$\overset{\scriptscriptstyle D}{\hat\mu}(t,\omega) = \Re\left(\frac{S'_w}{S_w}\right) \tag{28}$$

$$\overset{\scriptscriptstyle D}{\hat\psi}(t,\omega) = \frac{\Im\left(\frac{S''_w}{S_w}\right) - 2\overset{\scriptscriptstyle D}{\hat\mu}(t,\omega)\overset{\scriptscriptstyle D}{\hat\omega}(t,\omega)}{1 + \Re\left(\frac{S'_{tw}S_w - S_{tw}S'_w}{(S_w)^2}\right)}, \tag{29}$$

which are the derivative method expressions as given in [8] and [13]. Since expressions 27 and 28 are straightforward, only detailed derivations of equation 29 can be found in Appendix B (see equations 50, 51).

This section has clearly demonstrated that reassignment and the derivative method are in fact analogous methods, derived from the same general linear log-AM/linear FM equations. The only difference is the definition of STFT, which results in quite different expressions for parameter estimates. Mathematically identical proof was already given in [8] and [13], however it was given for each parameter of linear log-AM/linear FM sinusoids separately and thus does not prove the equivalence of the two methods for arbitrarly modulated sinusoids. In order to prove equivalence of the methods in such a general case, arbitrary order time derivatives of general linear FM parameter expressions (equation 10) should be considered: $\frac{\partial^n \hat\omega}{\partial \hat t^n} = \frac{\partial^n \hat\omega}{\partial t^n} / \frac{\partial^n \hat t}{\partial t^n}$. Such expressions would contain STFTs of the form $\frac{\partial^{k+l}S_w}{\partial t^k \partial \omega^l}$. By using the rules 42, 43, 44, 45 it is possible to transform the general expressions into reassignment ones, containing STFTs of the form $S_{w^{(k)}t^l}$ and analogously into the derivative method ones, containing STFTs of the form $S_{t^l}^{(k)}$. It is straightforward that reassignment and corresponding derivative method expressions are identical for all modulation degrees. The same procedure can be performed for log-AM, concluding the proof of equality of the two methods for an arbitrary modulated sinusoid.

The derivative method requires computation of signal time-derivatives, as opposed to reassignment, which requires computation of the window time-derivatives. In practice, it is impossible to avoid errors computing time derivative of the signal in time domain. For that purpose, a derivation filter is used, however unacceptable errors occur at high frequencies [8]. Further, using such filter increases the frame length requirements of STFT and raises computational complexity. When performing STFT, analytical expression for window function is known in most cases, therefore exact analytical expression for its time derivatives can generally be computed before performing STFT, which does not add any computational complexity. It can be concluded that lower computational complexity and higher accuracy is expected from the reassignment estimates compared to the derivative method ones. However, tests have shown that in the reduced frequency range (up to 3/4 Nyquist), methods perform comparably [8].

3. REASSIGNMENT AND GENERALIZED REASSIGNMENT

Generalized reassignment is the latest method based on the same backround as reassignment and the derivative method. The method is essentially based on the derivative method, as it uses signal derivatives for estimating the parameters. However, integration *per-partes* is used to transform expressions containing signal derivatives to expressions containing ramped window derivatives [14]. Final expressions resemble much more those of reassignment than those of the derivative method. Although it is designed to estimate arbitrary order log-AM/FM modulations, the discussion will be restricted to linear log-AM/FM signals. The following equations apply in this context (from [14]):

$$(\overset{\scriptscriptstyle GR}{\hat\mu}(t,\omega) + j\overset{\scriptscriptstyle GR}{\hat\omega}(t,\omega))S_w + j\overset{\scriptscriptstyle GR}{\hat\psi}(t,\omega)S_{tw} = \\ - S_{w'} + j\omega S_w \tag{30}$$

$$S_{w''} - j2\omega S_{w'} - S_w\omega^2 = \\ (\overset{\scriptscriptstyle GR}{\hat\mu}(t,\omega) + j\overset{\scriptscriptstyle GR}{\hat\omega}(t,\omega))(-S_{w'} + j\omega S_w) - \overset{\scriptscriptstyle GR}{\hat\psi}(t,\omega)S_{tw}. \tag{31}$$

From above equations, the following estimates can be expressed [14]:

$$\overset{\scriptscriptstyle GR}{\hat\psi}(t,\omega) = \frac{\Im\left(\frac{S_w S_{w''} - (S_{w'})^2}{(S_w)^2}\right)}{\Re\left(\frac{S_{w'}S_{tw} - S_{tw}S_w}{(S_w)^2}\right)} \tag{32}$$

$$\overset{\scriptscriptstyle GR}{\hat\mu}(t,\omega) + j\overset{\scriptscriptstyle GR}{\hat\omega}(t,\omega) = j\omega - \frac{S_{w'}}{S_w} - j\overset{\scriptscriptstyle GR}{\hat\psi}(t,\omega)\frac{S_{tw}}{S_w} \Rightarrow \tag{33}$$

$$\overset{\scriptscriptstyle GR}{\hat\omega}(t,\omega) = \overbrace{\omega - \Im\left(\frac{S_{w'}}{S_w}\right)}^{\overset{\scriptscriptstyle R}{\hat\omega}} - \overset{\scriptscriptstyle GR}{\hat\psi}(t,\omega)\overbrace{\Re\left(\frac{S_{tw}}{S_w}\right)}^{t_\Delta = \hat t - t} \tag{34}$$

$$\overset{\scriptscriptstyle GR}{\hat\mu}(t,\omega) = \overbrace{-\Re\left(\frac{S_{w'}}{S_w}\right)}^{\overset{\scriptscriptstyle R}{\hat\mu}} - \overset{\scriptscriptstyle GR}{\hat\psi}(t,\omega)\Im\left(\frac{S_{tw}}{S_w}\right). \tag{35}$$

The FM estimate expression is identical to that of original reassignment. The frequency and log-AM estimates on the other hand, contain additional terms compared to those of original reassignment. In the case of the frequency estimate, the additional term is ψt_Δ. The original frequency reassignment gives a frequency estimate at a reassigned time $\hat t$, so the time shift $t_\Delta = \hat t - t = \Re\left(\frac{S_{tw}}{S_w}\right)$ can be used to correct the frequency estimate (e.g.: *move* frequency estimate back to desired time), once FM is estimated. In the case of a log-AM estimate, no such time correction is present, as the log-AM is modeled to be constant. However, another term $\hat\psi\Im\left(\frac{S_{tw}}{S_w}\right)$ with no straightforward interpretation is present. It can be thought of as a correction of an error that presence of FM causes on AM estimation.

It will be shown in tests that this additional terms improve frequency and AM estimate significantly in high signal-to-noise (SNR) ratios.

4. TESTS AND RESULTS

For easier comparison, the test parameter set was choosen identical to the one in [8], with exception of a frequency range. The test frequencies (100 in total) were linearly distributed over two different frequency ranges: reduced frequency range, 20Hz-16538Hz (3/4 of Nyquist) and full frequency range, 20Hz-22050Hz (Nyquist), the results for each range are plotted separately. All other test parameters were linearly distributed in the following ranges: 7 different phase values in range $[-\pi, \pi]$, 5 log-AM values values in $[-100, 100]$ range and 5 different FM values in $[-10000, 10000]$. Hanning window of length 511 samples and sampling frequency of 44100Hz was used. The error variance was calculated using all parameter combinations. In the case of the derivative method, the derivation filter as described in [8] of length 1023 was used and the frame length was extended to 1533 samples ($511 + 2\frac{1023-1}{2}$). After convolution with the filter, only the middle 511 samples out of 1533 were kept to avoid edge effect.

In order to perform parameter estimations, all the algorithms require an initial frequency estimate, which is a consequence of the fact that STFT is a function of frequency and time. The initial frequency estimate is commonly acquired by taking the bin frequency of the magnitude spectrum peak. Once a frequency estimate is made (using the bin frequency of the peak), this estimate itself can be used to estimate other parameters, even to *re-estimate* the frequency itself. Such a procedure can be performed many times, thus iteratively obtaining presumably more accurate estimations in each iteration. In [8], the derivative method was tested in the following setting: the bin frequency of the spectrum peak is used to obtain a frequency estimate, which is in turn used to estimate log-AM/FM and finally static log-amplitude/phase, but the frequency itself is not reestimated. On the other hand, reassignment was tested by using the bin frequency of spectrum peak for all estimates. In the presented test results, all algorithms use a frequency estimate (not the inital bin frequency of the peak) to obtain all subsequent higher order, as well as static parameter estimates. For that, it is reasonable to expect that reassignment will achieve better results as reported in [8]. Further, the FM estimates of the derivative method as defined in [8] did not take into account the group delay, which was pointed out in [13], thus the improvement of the FM estimate of the derivative method is also expected.

The estimation errors of the derivative method are identical for log-amplitude, log-AM, phase and frequency to those presented in [8], tested in 20Hz-16538Hz (3/4 Nyquist) frequency range. The FM estimate improved as expected. Reassignment performs significantly better than reported in [8], which renders the accuracy difference between the two in the reduced frequency range negligible. Generalized reassignment performs superior for all parameter estimations except FM in this frequency range, where all three methods perform roughly the same. Using the full frequency range, estimate errors of the derivative method rise significantly, while the original reassignment performs even better. Generalized reassignment again performs significantly better in all cases and seems to be unaffected by frequency range changes. This suggests, that generalized reassignment achieves identical accuracy for all frequencies. It is important to note, that a significant accuracy difference between reassignment and generalized reassignment occurs in higher SNR region. For each parameter estimation, it is possible to define a SNR value, above which the accuracies of reassignment and generalized reassignment differ significantly. Below such SNR value however, the two methods perform equally.
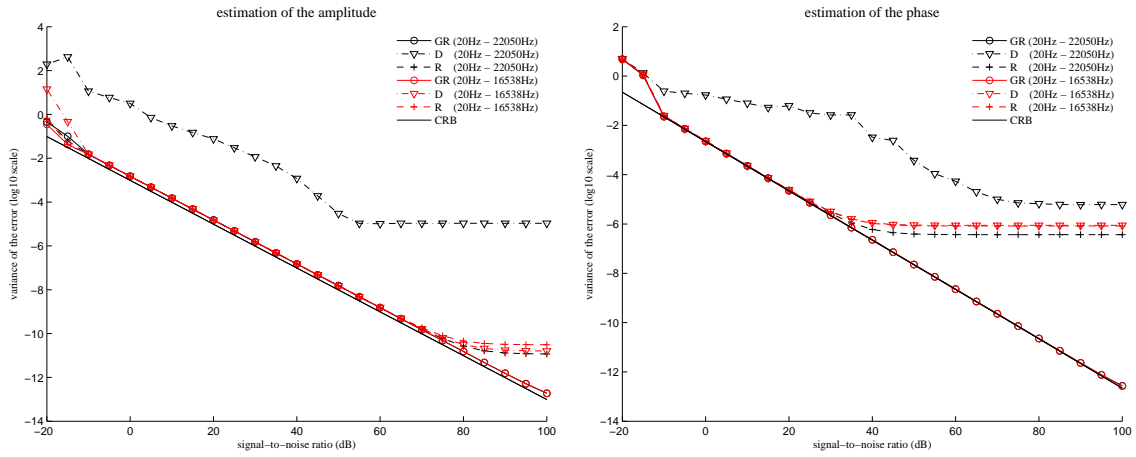
All three algorithms were implemented in Octave programming language and the tests were conducted with Octave 2.9.9 (x86-64 bit Debian distribution) on a Sun Grid Engine 6.2 (SGE) cluster.

5. CONCLUSION

This document analyzes and compares three state of the art methods for non-stationary analysis and compares them to CRBs. It has been shown that generalized reassignment is the most appropriate method for analyzing complex linear log-AM/FM signals. Generalized reassignment is able to detect an arbitrary log-AM/FM degree of a sinusoid and achieves superior accuracy in the linear log-AM/linear FM case and should therefore be the topic of further research concerning the analysis of real world signals: both multi-component and real. In this case, the degree of modulation of each sinusoid under study is unknown and determining its exact degree is crucial, as errors rise significantly when the modeled modulation degree is set either too high or too low [13]. Furthermore, the usual window functions like Hanning cannot be used when analyzing higher order modulations, as its second time derivative does not reach 0 at the start and end of the frame (required by the algorithm). Some higher order window functions were proposed in [13], however its exact accuracy currently remains unknown; therefore a more detailed study of window functions satisfying the restrictions of the algorithm should be researched. It is reasonable to expect, that such window functions would exhibit less desirable time-frequency trade offs. To avoid accuracy deterioration, limiting the analysis algorithm to a certain degree of modulation is crucial and could be balanced with a multi resolution method of some sort, for example a wavelet transform based one, a similar idea already pointed out in [15].
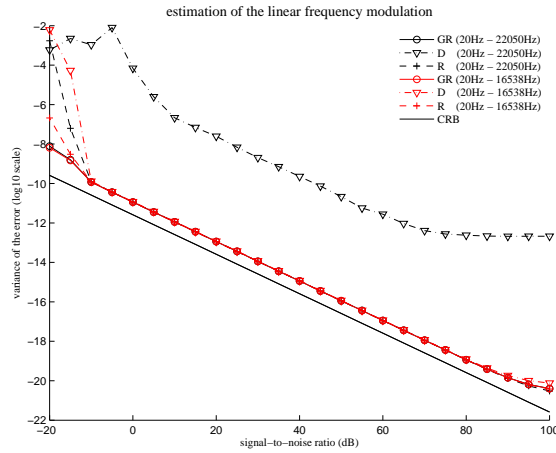
6. ACKNOWLEDGEMENTS

(a) Amplitude estimation error for linear log-AM/linear FM sinusoid in comparison with CRB

(b) Phase estimation error for linear log-AM/linear FM sinusoid in comparison with CRB
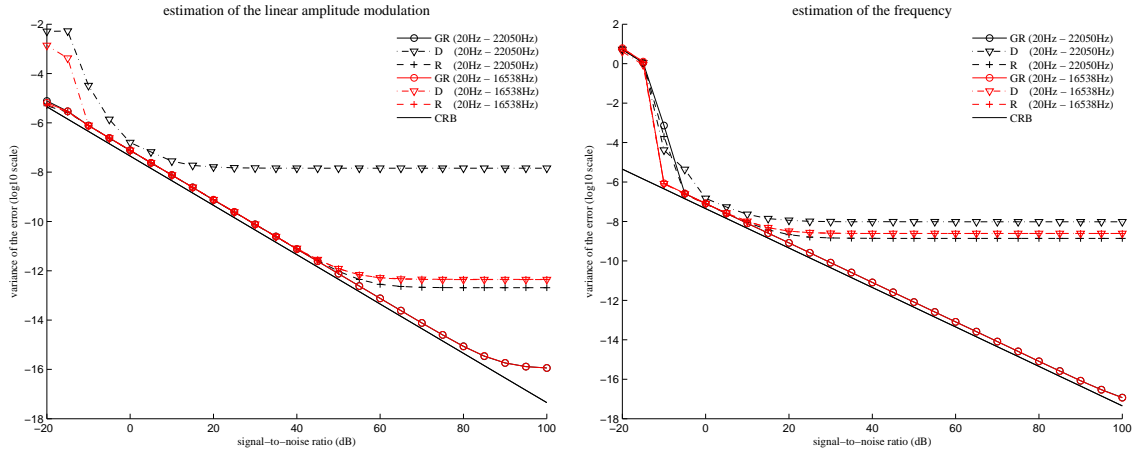
(c) Log-AM estimation error for linear log-AM/linear FM sinusoid in comparison with CRB

(d) Frequency estimation error for linear log-AM/linear FM sinusoid in comparison with CRB

(e) FM estimation error for linear log-AM/linear FM sinusoid in comparison with CRB

Figure 1: Parameter estimation errors for reassignment, the derivative method and generalized reassignment.

7. APPENDIX A

As already briefly mentioned in section 1, the equality ${}^{\text{R}}S_w = {}^{\text{D}}S_w$ holds for all t and ω, yet each equation yields a different time derivative expression. When the window function is assumed to be of finite support, and consequently the infinite integral bounds are replaced by finite ones, the expressions for ${}^{\text{R}}S_w$ and ${}^{\text{D}}S_w$ change to:

$$
{}^{\text{D}}S_w(t,\omega) = \int_{-\frac{T}{2}}^{\frac{T}{2}} s(\tau + t)w(\tau)e^{-j\omega\tau}\, d\tau \qquad (36)
$$

$$
{}^{\text{R}}S_w(t,\omega) = \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} s(\bar\tau)w(\bar\tau - t)e^{-j\omega(\bar\tau - t)}\, d\bar\tau, \qquad (37)
$$

where the window function time support is assumed to be T. The partial time derivative of integral expression 37 should be taken with care, as integral limits depend on time and the time derivative operator cannot simply be brought inside the integral. Using the Leibniz's integral rule on ${}^{\text{R}}S_w$ yields:

$$
\frac{\partial}{\partial t}{}^{\text{R}}S_w(t,\omega) =
$$
$$
\frac{\partial}{\partial t}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} s(\bar\tau)w(\bar\tau - t)e^{-j\omega(\bar\tau - t)}\, d\bar\tau =
$$
$$
s(t + \tfrac{T}{2})\overbrace{w(\tfrac{T}{2})}^{=0} e^{-j\omega\frac{T}{2}} - s(t - \tfrac{T}{2})\overbrace{w(-\tfrac{T}{2})}^{=0} e^{j\omega\frac{T}{2}} +
$$
$$
\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} s(\bar\tau)\frac{\partial}{\partial t}\left[w(\bar\tau - t)e^{-j\omega(\bar\tau - t)}\right] d\bar\tau.
$$
$$(38)$$

With the additional restriction that the window function reaches 0 at both its edges, e.g.: $w(\frac{T}{2}) = w(-\frac{T}{2}) = 0$, the first two terms of the above expression can be neglected. The restriction modifies equation 38 in a way, as if the integral boundaries wouldn't depend on the time variable and thus the time derivative operator can simply be brought inside the integral. Throughout this appendix it will be assumed, that the window function and its arbitrary time derivative reach 0 at both edges, eg: $w^{(k)}(\frac{T}{2}) = w^{(k)}(-\frac{T}{2}) = 0$ for all k and consequently the time derivation operator can always be brought inside the integral. Note, that the partial frequency derivative operators can be brought inside the integral for both ${}^{\text{R}}S_w$ and ${}^{\text{D}}S_w$, as well as the partial time derivative in the case of ${}^{\text{D}}S_w$ without any additional restrictions. In the following derivations the integral bounds will be omitted for the sake of clarity. For reassignment the following holds:

$$
\frac{\partial}{\partial t}{}^{\text{R}}S_w = \int s(\tau)\frac{\partial}{\partial t}\left[w(\tau - t)e^{-j\omega(\tau - t)}\right] d\tau =
$$
$$
\int s(\tau)\overbrace{\frac{\partial}{\partial t}[w(\tau - t)]}^{-w'(t)} e^{-j\omega(\tau - t)}d\tau +
$$
$$
\int s(\tau)w(\tau - t)\overbrace{\frac{\partial}{\partial t}\left[e^{-j\omega(\tau - t)}\right]}^{j\omega e^{-j\omega(\tau - t)}} d\tau =
$$
$$
- S_{w'} + j\omega S_w
$$
$$(39)$$

$$
\frac{\partial}{\partial \omega}{}^{\text{R}}S_w = \int s(\tau)w(\tau - t)\overbrace{\frac{\partial}{\partial \omega}\left[e^{-j\omega(\tau - t)}\right]}^{-j(\tau - t)e^{-j\omega(\tau - t)}} d\tau = \quad (40)
$$
$$
- jS_{tw}
$$

$$
\frac{\partial}{\partial t}{}^{\text{R}}S_{tw} =
$$
$$
\int s(\tau)\frac{\partial}{\partial t}\left[w(\tau - t)(\tau - t)e^{-j\omega(\tau - t)}\right] d\tau =
$$
$$
\int s(\tau)\overbrace{\frac{\partial}{\partial t}[w(\tau - t)]}^{-w'(\tau - t)}(\tau - t)e^{-j\omega(\tau - t)}d\tau +
$$
$$
\int s(\tau)w(\tau - t)\overbrace{\frac{\partial}{\partial t}[\tau - t]}^{-1} e^{-j\omega(\tau - t)}d\tau +
$$
$$
\int s(\tau)w(\tau - t)(\tau - t)\overbrace{\frac{\partial}{\partial t}\left[e^{-j\omega(\tau - t)}\right]}^{j\omega e^{-j\omega(\tau - t)}} d\tau =
$$
$$
- S_{w'} - S_w + j\omega S_w.
$$
$$(41)$$

It is informative to generalize the time and frequency derivative expressions for the reassignment STFT using arbitrary window time derivative, ramped with an arbitrary polynomial:

$$
\frac{\partial}{\partial t}{}^{\text{R}}S_{t^n w^{(k)}} =
$$
$$
\int s(\tau)\frac{\partial}{\partial t}\left[w^{(k)}(\tau - t)(\tau - t)^n e^{-j\omega(\tau - t)}\right] d\tau =
$$
$$
\int s(\tau)\overbrace{\frac{\partial}{\partial t}[w^{(k)}(\tau - t)]}^{-w^{(k+1)}(\tau - t)}(\tau - t)^n e^{-j\omega(\tau - t)}d\tau +
$$
$$
\int s(\tau)w^{(k)}(\tau - t)\overbrace{\frac{\partial}{\partial t}[(\tau - t)^n]}^{-n(\tau - t)^{n-1}} e^{-j\omega(\tau - t)}d\tau +
$$
$$
\int s(\tau)w^{(k)}(\tau - t)(\tau - t)^n \overbrace{\frac{\partial}{\partial t}\left[e^{-j\omega(\tau - t)}\right]}^{j\omega e^{-j\omega(\tau - t)}} d\tau =
$$
$$
- S_{t^n w^{(k+1)}} - S_{t^{n-1}w^{(k)}}n + j\omega S_{t^n w^{(k)}}
$$
$$(42)$$

$$\frac{\partial}{\partial \omega}\,{}^{\text{R}}S_{t^n w^{(k)}} =$$

$$\int s(\tau)w^{(k)}(\tau-t)(\tau-t)^n \overbrace{\frac{\partial}{\partial \omega}\left[e^{-j\omega(\tau-t)}\right]}^{-j(\tau-t)e^{-j\omega(\tau-t)}} d\tau = \qquad (43)$$

$$- jS_{t^{n+1}w^{(k)}}.$$

Equation 42 corresponds to equations 39 and 41 for the values of $k = 0, n = 0$ and $k = 0, n = 1$ respectively and equation 43 corresponds to equation 40 for the values of $k = 0, n = 0$.

Analogously, for the derivative method STFTs, time and frequency derivatives can be generalized for an arbitray signal time derivative, using a window ramped with an arbitrary polynomial:

$$\frac{\partial}{\partial t}\,{}^{\text{D}}S_{t^n w}^{(k)} =$$

$$\int \frac{\partial}{\partial t}\left[s^{(k)}(\tau+t)\right]w(\tau)\tau^n e^{-j\omega\tau}d\tau = \qquad (44)$$

$$S_{t^n w}^{(k+1)}$$

$$\frac{\partial}{\partial \omega}\,{}^{\text{D}}S_{t^n w}^{(k)} =$$

$$\int_{-\infty}^{\infty} s(\tau+t)w(\tau)\tau^n \overbrace{\frac{\partial}{\partial \omega}\left[e^{-j\omega\tau}\right]}^{-j\tau e^{-j\omega\tau}} d\tau = \qquad (45)$$

$$- jS_{t^{n+1}w}^{(k)}.$$

8. APPENDIX B

Substituting the reassignment STFT expressions 16-19 into general equations for frequency, log-AM and FM estima-

tions defined in 6,9,10 yields:

$$\begin{aligned}
{}^{\text{R}}\hat{\omega}(t,\omega) &= \Im\left(\frac{-S_{w'} + j\omega S_w}{S_w}\right) \\
&= \Im\left(j\omega - \frac{S_{w'}}{S_w}\right) \qquad (46) \\
&= \omega - \Im\left(\frac{S_{w'}}{S_w}\right)
\end{aligned}$$

$$\begin{aligned}
{}^{\text{R}}\hat{\mu}(t,\omega) &= \Re\left(\frac{-S_{w'} + j\omega S_w}{S_w}\right) \\
&= \Re\left(j\omega - \frac{S_{w'}}{S_w}\right) \qquad (47) \\
&= -\Re\left(\frac{S_{w'}}{S_w}\right)
\end{aligned}$$

$$\begin{aligned}
{}^{\text{R}}\hat{\psi}(t,\omega) &= \frac{\partial^{\text{R}}\hat{\omega}}{\partial t} \Big/ \frac{\partial^{\text{R}}\hat{t}}{\partial t} \\
\frac{\partial^{\text{R}}\hat{\omega}}{\partial t} &= \Im\left(\frac{(S_{w''} - 2j\omega S_{w'} - \omega^2 S_w)S_w}{(S_w)^2}\right) \\
&\quad - \Im\left(\frac{(-S_{w'} + j\omega S_w)^2}{(S_w)^2}\right) \qquad (48) \\
&= \Im\left(\frac{S_{w''}S_w - (S_{w'})^2}{(S_w)^2}\right) \\
\frac{\partial^{\text{R}}\hat{t}}{\partial t} &= 1 - \Im\left(\frac{(-S_{w'} - S_w + j\omega S_w)S_w)}{(S_w)^2}\right) \\
&\quad - \Im\left(\frac{jS_w(-S_{w'} + j\omega S_w)}{(S_w)^2}\right) \qquad (49) \\
&= 1 - \Im\left(j\frac{(S_w)^2 + S_{tw'}S_w - S_{tw}S_w}{(S_w)^2}\right) \\
&= \Re\left(\frac{S_{tw'}S_w - S_{tw}S_w}{(S_w)^2}\right).
\end{aligned}$$

Substituting the derivative method STFT expressions 20-23 into general equations for frequency and log-AM estimations defined in 6,9 yield straightforward expressions, however the expression for FM estimate 10 deserves more attention:

$$\begin{aligned}
{}^{\text{D}}\hat{\psi}(t,\omega) &= \frac{\partial^{\text{D}}\hat{\omega}}{\partial t} \Big/ \frac{\partial^{\text{D}}\hat{t}}{\partial t} \\
\frac{\partial^{\text{D}}\hat{\omega}}{\partial t} &= \Im\left(\frac{S_w''S_w - (S_w')^2}{(S_w)^2}\right) \\
&= \Im\left(\frac{S_w''}{S_w}\right) - \overbrace{\Im\left(\left(\frac{S_w'}{S_w}\right)^2\right)}^{2\Re\left(\frac{S_w'}{S_w}\right)\Im\left(\frac{S_w'}{S_w}\right)} \qquad (50) \\
&= \Im\left(\frac{S_w''}{S_w}\right) - 2\,{}^{\text{D}}\hat{\mu}(t,\omega)\,{}^{\text{D}}\hat{\psi}(t,\omega) \\
\frac{\partial^{\text{D}}\hat{t}}{\partial t} &= 1 - \Im\left(\frac{-jS_{tw}'S_w + jS_{tw}S_w'}{(S_w)^2}\right) \\
&= 1 + \Re\left(\frac{S_{tw}'S_w - S_{tw}S_w'}{(S_w)^2}\right). \qquad (51)
\end{aligned}$$

9. REFERENCES

[1] X. Serra, *A System for Sound Analysis/Transformation/Synthesis based on a Deterministic plus Stochastic Decomposition*. PhD thesis, CCRMA, Department of Music, Stanford University, 1989.

[2] R. McAulay and T. Quatieri, "Speech analysis/synthesis based on a sinusoidal representation," *Acoustics, Speech and Signal Processing, IEEE Transactions on*, vol. 34, pp. 744–754, August 1986.

[3] G. Peeters and X. Rodet, "Sinola: A new analysis/synthesis method using spectrum peak shape distortion, phase and reassigned spectrum," in *Proc. of 25th International Computer Music Conference*, (Beijing, China), pp. 153–156, October 1999.

[4] M. Desainte-Catherine and S. Marchand, "High precision fourier analysis of sounds using signal derivatives," *Journal of the Audio Engineering Society*, vol. 48, no. 7/8, pp. 654–667, 1998.

[5] F. Keiler and S. Marchand, "Survey on extraction of sinusoids in stationary sounds," in *Proc. of 5th Int. Conference on Digital Audio Effects (DAFx-02)*, (Barcelona, Spain), September 2002.

[6] S. Marchand, "Improving spectral analysis precision with an enhanced phase vocoder using signal derivatives," in *Proc. of Digital Audio Effects Workshop (DAFX-98)*, (Barcelona, Spain), pp. 114–118, November 1998.

[7] B. David, G. Richard, M. Betser, and P. Collen, "Estimation of Frequency for AM/FM Models Using the Phase Vocoder Framework," *Signal Processing, IEEE Transactions on*, vol. 56, pp. 505–517, 2008.

[8] P. Depalle and S. Marchand, "Generalization of the derivative analysis method to non-stationary sinusoidal modeling," in *Proc. of the 11th Int. Conference on Digital Audio Effects (DAFx-08)*, (Espoo, Finland), September 2008.

[9] J. Smith and M. Abe, "AM/FM rate estimation for time-varying sinusoidal modeling," in *Proc. of the IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP'05)*, vol. 3, (Philadelphia, PA, USA), pp. 201–204, March 2005.

[10] A. Röbel, "Estimating partial frequency and frequency slope using reassignment operators," in *Proc. of the International Computer Music Conference (ICMC'02)*, (Gothenburg, Sweden), pp. 122–125, September 2002.

[11] F. Auger and P. Flandrin, "Improving the readability of time-frequency and time-scale representations by the reassignment method," *Signal Processing, IEEE Transactions on*, vol. 43, no. 5, pp. 1068–1089, 1995.

[12] S. W. Hainsworth, *Techniques for the Automated Analysis of Musical Audio*. PhD thesis, Department of Engineering, University of Cambridge, Cambridge, 2004.

[13] B. Hamilton, P. Depalle, and S. Marchand, "Theoretical and practical comparisons of the reassignment method and the derivative method for the estimation of the frequency slope," in *IEEE Workshop on Applications of Signal Processing to Audio and Acoustics*, (New Paltz, NY), October 2009.

[14] W. Xau and M. Sandler, "Notes on model-based non-stationary sinusoid estimation methods using derivative," in *Proc. of the 12th Int. Conference on Digital Audio Effects (DAFx-09)*, (Como, Italy), September 2009.

[15] M. Betser, "Sinusoidal polynomial parameter estimation using the distribution derivative," *Signal Processing, IEEE Transactions on*, vol. 57, pp. 4633–4645, December 2009.

[16] F. Auger and P. Flandrin, "Generalization of the reassignment method to all bilinear time-frequency and time-scale representations," in *Proc. of the IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP '94)*, vol. iv, (Washington, DC, USA), pp. 317–320, 1994.