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Formation and dynamics of shock waves in the Degasperis–Procesi equation

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Abstract

Solutions of the Degasperis–Procesi nonlinear wave equation may develop discontinuities in finite time. As shown by Coclite and Karlsen, there is a uniquely determined entropy weak solution which provides a natural continuation of the solution past such a point. Here we study this phenomenon in detail for solutions involving interacting peakons and antipeakons. We show that a jump discontinuity forms when a peakon collides with an antipeakon, and that the entropy weak solution in this case is described by a "shockpeakon" ansatz reducing the PDE to a system of ODEs for positions, momenta, and shock strengths.

1 Introduction

The Degasperis–Procesi (DP) equation

$$m_t + m_x u + 3 m u_x = 0, \qquad m = u - u_{xx} \tag{1.1}$$

was isolated by Degasperis and Procesi [11] as one of three equations in the family

$$u_t - \alpha^2 u_{xxt} + \gamma u_{xxx} + c_0 u_x = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x \tag{1.2}$$

satisfying "asymptotic integrability to third order", a necessary condition for complete integrability. The other two cases are the KdV equation $u_t + u u_x + u_{xxx} = 0$ and the Camassa–Holm (CH) shallow water equation [5]

$$m_t + m_x u + 2 m u_x = 0, \qquad m = u - u_{xx}, \tag{1.3}$$

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both well known to be integrable. Degasperis, Holm and Hone [10] later showed that the DP equation is integrable as well, by deriving a Lax pair and a bi-Hamiltonian structure for it. Although the DP equation is similar to the CH equation in several aspects, there are also important differences, as we shall see.

Peakons

One of the features that has made the CH equation (1.3) famous is that it admits a class of weak solutions known as *peakons*. A peakon (peaked soliton) is a wave of the form $u = m_0\,G(x - x_0)$ where

$$G(x) = e^{-|x|}. \tag{1.4}$$

This wave profile corresponds to the momentum $m = u - u_{xx} = 2m_0\,\delta(x - x_0)$ being a discrete measure (δ is the Dirac delta distribution). Such a single peakon is a travelling wave solution if m_0 is constant and $x_0(t) = x_0(0) + m_0 t$, and it moves to the right if $m_0 > 0$. A peakon with $m_0 < 0$ moves to the left and is called an *antipeakon*. The n-peakon solution of the CH equation is a superposition of interacting peakons,

$$\begin{aligned} u(x,t) &= \sum_{i=1}^{n} m_i(t)\,G(x - x_i(t)), \\ m(x,t) &= 2\sum_{i=1}^{n} m_i(t)\,\delta(x - x_i(t)), \end{aligned} \tag{1.5}$$

whose positions $x_1(t), \ldots, x_n(t)$ and momenta $m_1(t), \ldots, m_n(t)$ are governed by the system of ODEs

$$\dot{x}_k = \sum_{i=1}^{n} m_i\,G(x_k - x_i), \qquad \dot{m}_k = -\sum_{i=1}^{n} m_k m_i\,G'(x_k - x_i), \tag{1.6}$$

which is a canonical Hamiltonian system with $H = \frac{1}{2}\sum_{i,j} m_i m_j G(x_i - x_j)$. Here the value zero is assigned to the otherwise undetermined derivative $G'(0)$, so that

$$G'(x) := -\operatorname{sgn}(x)e^{-|x|} = \begin{cases} e^{x}, & x < 0, \\ 0, & x = 0, \\ -e^{-x}, & x > 0. \end{cases} \tag{1.7}$$

For us this is just a notational convention which simplifies the statement of the peakon ODEs (1.6) and some other equations to appear later on. However, it can also be used to provide meaning to the term mu_x in the PDE (1.3), where the function u_x otherwise would be undefined exactly where the Dirac deltas in the distribution m are situated. We refer to [2] for a discussion of why this is justified, since another way to make sense of weak solutions will be described below.

Degasperis, Holm and Hone [10] showed that the DP equation also has peakon solutions. In fact, they showed that the ansatz (1.5) satisfies the more general peakon PDE

$$m_t + m_x u + bmu_x = 0, \qquad m = u - u_{xx}, \tag{1.8}$$

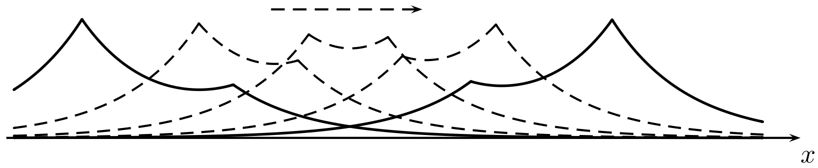

Figure 1: Two-peakon solution of the Camassa–Holm equation, computed from the exact solution formulas (3.2) in the pure peakon case (m_1 and m_2 both positive). The wave profile $u(x,t)$ is shown at some evenly sampled times. Both peakons move to the right and no collision (in the sense $x_1 = x_2$) occurs. Before the interaction, the left peakon is higher and faster than the right peakon (left solid curve). When the peakons get close, some of the momentum is transferred (dashed curves). Afterwards it is the peakon on the right that is higher and faster (right solid curve). The two-peakon solution of the Degasperis–Procesi equation, given by (3.6), has the same qualitative behaviour in the pure peakon case.

if and only if the positions and momenta satisfy the ODEs

$$\dot{x}_k = u(x_k), \qquad \dot{m}_k = -(b-1)\, m_k\, u_x(x_k), \tag{1.9}$$

where the shorthand $u(x_k)$ denotes $u(x_k(t), t)$ as obtained by letting $x = x_k(t)$ in (1.5), and similarly for $u_x(x_k)$. The integrable CH and DP cases correspond to $b = 2$ and $b = 3$, respectively. For $b = 2$ this coincides with (1.6) above, while $b = 3$ gives the DP peakon ODEs

$$\dot{x}_k = \sum_{i=1}^{n} m_i\, G(x_k - x_i), \qquad \dot{m}_k = -2 \sum_{i=1}^{n} m_k m_i\, G'(x_k - x_i). \tag{1.10}$$

The CH and DP peakon ODEs (1.6) and (1.10) can be solved explicitly in terms of elementary functions. The case $n = 1$ is trivial, while the two-peakon solutions were obtained using sum-and-difference variables $x_1 \pm x_2$ and $m_1 \pm m_2$ in the original papers by Camassa and Holm [5], and by Degasperis, Holm and Hone [10]. See Figure 1. The general n-peakon solutions were derived using inverse scattering techniques by Beals, Sattinger and Szmigielski for CH peakons [3, 2] and by Lundmark and Szmigielski for DP peakons [18, 19]. Equations (3.2) and (3.6) below show the two-peakon formulas, in the form that they appear when using that method of solution. The formulas for $n \geq 3$ are of the same flavour but more involved; we omit them here since writing them down efficiently requires defining a fair amount of notation.

We will always assume that all m_k's are nonzero, since any vanishing m_k remains identically zero and never enters the solution. Without loss of generality we will restrict our attention to solutions satisfying

$$x_1(t) < x_2(t) < \cdots < x_n(t). \tag{1.11}$$

When peakons and antipeakons are present simultaneously, it may happen that some $x_k = x_{k+1}$ after finite time, which we refer to as a collision.[1] It is not

[1] Sometimes the word 'collision' is used also for the type of interaction shown in Figure 1, but not so in this paper.

obvious what happens to the solution $u(x,t)$ of the PDE at (or after) a collision. The CH case is well understood by now [2, 22, 4, 12], while the DP case is the subject of this paper. We will return to this in Section 3.

Let us mention in passing that a term αu_x can be added to the left-hand side of either the CH equation (1.3) or the DP equation (1.1) without destroying integrability. Then smooth solitons are obtained, which converge to peakons as $\alpha \to 0^+$. In the DP case this has recently been studied by Matsuno [20, 21].

We also remark that the long paper by Holm and Staley [14] contains, among many other things, a large number of numerically computed solutions of the peakon PDE (1.8), both in integrable and nonintegrable cases.

Weak solutions in general

As mentioned above, the sense in which the peakons are weak solutions can be specified by enforcing the convention $G'(0) = 0$, but a more appealing approach is to write the equation as

$$\begin{aligned} 0 &= m_t + m_x u + bmu_x \\ &= (u - u_{xx})_t + (b+1)uu_x - bu_x u_{xx} - uu_{xxx} \\ &= (1-\partial_x^2)[u_t + (\tfrac{1}{2}u^2)_x] + b(\tfrac{1}{2}u^2)_x + (3-b)(\tfrac{1}{2}u_x^2)_x \end{aligned} \tag{1.12}$$

and then apply the formal inverse of $1 - \partial_x^2$, which is $(1-\partial_x^2)^{-1} f = \frac{1}{2} G * f = \int_{\mathbf{R}} \frac{1}{2} e^{-|y|} f(x-y)\, dy$. This gives

$$0 = u_t + \partial_x \left[\tfrac{1}{2}u^2 + \tfrac{1}{2}G * (\tfrac{b}{2}u^2 + \tfrac{3-b}{2}u_x^2) \right], \tag{1.13}$$

so that the CH equation ($b = 2$) becomes [9]

$$u_t + \partial_x \left[\tfrac{1}{2}u^2 + \tfrac{1}{2}G * \left(u^2 + \tfrac{1}{2}u_x^2\right) \right] = 0 \tag{1.14}$$

and the DP equation ($b = 3$) becomes [24]

$$u_t + \partial_x \left[\tfrac{1}{2}u^2 + \tfrac{1}{2}G * \tfrac{3}{2}u^2 \right] = 0. \tag{1.15}$$

Weak solutions (not only peakons) are then defined as functions which satisfy this conservation law (with nonlocal flux term) in the usual distributional sense.

For the CH equation (1.14) it is natural to impose at least H^1 regularity (and hence continuity) with respect to x in the definition of weak solution, because of the term u_x^2 in the equation. Breakdown of smooth initial data can occur, but then it is only u_x that develops singularities; the solution u itself remains continuous (see for example McKean [22]). The DP equation (1.15) has been studied from a similar point of view, for example by Yin who has given existence and well-posedness results for strong solutions with initial data $u_0 \in H^s(\mathbf{R})$ with $s > 3/2$ (hence with u_{0x} continuous) [24], and for weak solutions with $u_0 \in H^1(\mathbf{R}) \cap L^3(\mathbf{R})$ [25]. A detailed study of the blow-up of u_x in the H^s ($s > 3/2$) setting is undertaken in a forthcoming paper by Liu and Yin [17].

However, since the DP equation (1.15) does not involve u_x explicitly one can also consider less regular solutions, as shown recently by Coclite and Karlsen [7] who proved various existence and uniqueness results in spaces of discontinuous functions. They defined a *weak solution* of the DP equation as a function $u \in L^\infty(\mathbf{R}^+; L^2(\mathbf{R}))$ which satisfies (1.15) in the sense of distributions for $(x,t) \in \mathbf{R} \times [0,\infty)$, and an *entropy weak solution* of the DP equation as a weak solution which in addition belongs to $L^\infty(0,T;BV(\mathbf{R}))$ for all $T > 0$ and satisfies the Kružkov-type entropy condition

$$\partial_t \eta(u) + \partial_x q(u) + \eta'(u) \partial_x \left[\tfrac{1}{2} G * \tfrac{3}{2} u^2 \right] \leq 0 \tag{1.16}$$

in the distributional sense, for all convex C^2 entropies $\eta : \mathbf{R} \to \mathbf{R}$ with the corresponding entropy flux $q : \mathbf{R} \to \mathbf{R}$ defined by $q'(u) = \eta'(u) u$. Then they showed that for initial data $u_0 \in L^1(\mathbf{R}) \cap BV(\mathbf{R})$ there exists a unique entropy weak solution of the DP equation, which is obtained as the limit (as $\varepsilon \to 0^+$) of smooth functions u_ε satisfying the viscous regularization

$$\partial_t u_\varepsilon + \partial_x \left[\tfrac{1}{2} u_\varepsilon^2 + \tfrac{1}{2} G * \tfrac{3}{2} u_\varepsilon^2 \right] = \varepsilon \, \partial_x^2 u_\varepsilon. \tag{1.17}$$

A forthcoming paper by Coclite, Karlsen and Risebro [8] deals with numerical methods for computing such entropy weak solutions of the DP equation, and another forthcoming paper by Coclite and Karlsen [6] studies weak solution satisfying an Oleĭnik-type entropy condition instead of (1.16).

Outline of the paper

Our purpose here is to provide concrete examples of entropy weak solutions of the DP equation, in the sense of Coclite and Karlsen [7]. We will consider a class of solutions where $u(x,t)$ at each instant t consists of a finite number of smooth segments, each a linear combination of e^x and e^{-x}, just like for peakons. The new feature here is that these segments are not required to join to form a continuous function, since u is allowed to have jump discontinuities. We will begin by presenting this generalized "shockpeakon" ansatz and the ODEs that it gives rise to. Then we will show how this type of solution forms when a peakon collides with an antipeakon in the DP equation, and contrast this with the behaviour of the CH equation. Finally, we will study the solutions of the shockpeakon ODEs in some particular cases.

2 Shockpeakon solutions of the DP equation

Let $G(x)$ and $G'(x)$ be given by (1.4) and (1.7) as before; note in particular the convention $G'(0) = 0$. We will look for solutions of the DP equation of the form

$$u(x,t) = \sum_{i=1}^{n} m_k(t) \, G(x - x_k(t)) + \sum_{i=1}^{n} s_k(t) \, G'(x - x_k(t)), \tag{2.1}$$

that is, a superposition of n "shockpeakons", each shaped like

$$m \, G(x) + s \, G'(x) = \begin{cases} (m+s) \, e^x, & x < 0, \\ m, & x = 0, \\ (m-s) \, e^{-x}, & x > 0. \end{cases} \tag{2.2}$$

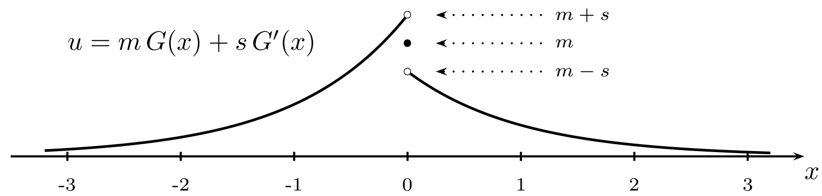

Figure 2: Shockpeakon with momentum $m = 1$ and shock strength $s = \frac{1}{4}$.

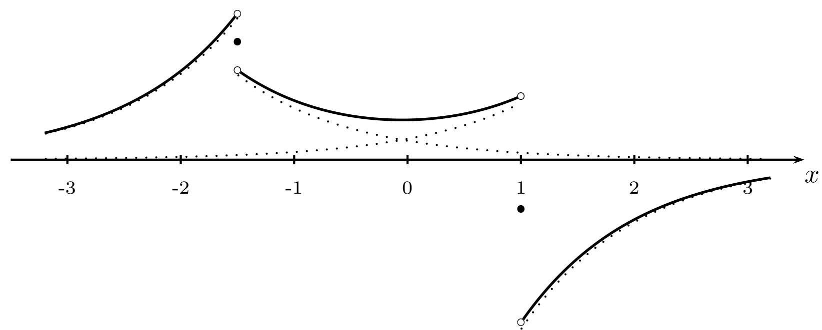

Figure 3: Superposition (solid curve) of two shockpeakons (dotted curves) with $x_1 = -\frac{3}{2}$, $m_1 = 1$, $s_1 = \frac{1}{4}$ and $x_2 = 1$, $m_2 = -\frac{1}{2}$, $s_2 = 1$.

At $x = x_k$, the function $u(x,t)$ has a jump of $-2m_k$ in the derivative u_x just like for peakons, but also a jump of $-2s_k$ in the function u itself. (Mnemonic: m stands for momentum and s for shock strength.) See Figures 2 and 3.

If one tries to substitute (2.1) into the usual form (1.1) of the DP equation, then $m = u - u_{xx}$ will be a linear combination of δ and δ' distributions, while u_x will contain δ, so the equation $m_t + m_x u + 3mu_x = 0$ will contain meaningless terms involving products of Dirac deltas. Through the *ad hoc* procedure of simply neglecting all such terms, one obtains (2.3) below, but of course rigour requires that we verify the result using the proper weak formulation (1.15).

Theorem 2.1. *The shockpeakon ansatz* (2.1) *satisfies the DP equation in the weak form* (1.15) *if and only if*

$$\begin{aligned} \dot{x}_k &= u(x_k), \\ \dot{m}_k &= 2s_k u(x_k) - 2m_k\{u_x(x_k)\}, \\ \dot{s}_k &= -s_k\{u_x(x_k)\}, \end{aligned} \tag{2.3}$$

where curly brackets denote the nonsingular part, i.e.,

$$u(x_k) = \sum_{i=1}^{n} m_i\,G(x_k - x_i) + \sum_{i=1}^{n} s_i\,G'(x_k - x_i) \tag{2.4}$$

and

$$\{u_x(x_k)\} := \sum_{i=1}^{n} m_i\,G'(x_k - x_i) + \sum_{i=1}^{n} s_i\,G(x_k - x_i). \tag{2.5}$$

Proof. The proof is a straighforward but lengthy computation; see the Appendix for details. □

Remark 2.2. The condition $s_k = 0$ is preserved by the equations. If some $s_k = 0$, then we assume that the corresponding m_k is nonzero, since otherwise $m_k = s_k = 0$ identically. When all $s_k = 0$, the shockpeakon ODEs (2.3) reduce to the ordinary DP peakon ODEs (1.10) as they should.

Theorem 2.3. *The solution described in Theorem 2.1 satisfies the entropy condition* (1.16) *if and only if $s_k \geq 0$ for all k. In other words, all shocks must satisfy $u(x_k^-) \geq u(x_k^+)$.*

This result is due to Coclite and Karlsen [6], but we sketch a proof here for completeness.

Proof. Since the nonlocal term $P = \frac{1}{2} G * \frac{3}{2} u^2$ is twice differentiable even when u is discontinuous, the jump conditions for the DP equation $u_t + \partial_x\big(\frac{1}{2}u^2 + P\big) = 0$ will be the same as for the inviscid Burgers equation $u_t + \partial_x(\frac{1}{2}u^2) = 0$. More precisely, let u be a strong solution piecewise, and consider an isolated discontinuity along a curve $x = x_0(t)$, with left and right limits $u_l(t)$ and $u_r(t)$. Then for a test function $\phi \geq 0$ with support contained in a small neighbourhood D of a point on the curve, the entropy condition requires that

$$0 \leq \iint_D \Big(\eta(u)\phi_t + q(u)\phi_x - \eta'(u) P_x \phi \Big)\, dx\, dt.$$

Let D_1 and D_2 be the parts of D to the left and right of the curve. The integral over each D_i equals

$$\begin{aligned} &\iint_{D_i} \Big((\eta(u)\phi)_t + (q(u)\phi)_x \Big)\, dx\, dt - \iint_{D_i} \Big(\eta(u)_t + q(u)_x + \eta'(u) P_x \Big) \phi\, dx\, dt \\ &= \oint_{\partial D_i} \Big(-\eta(u)\phi\, dx + q(u)\phi\, dt \Big) - \iint_{D_i} \Big(u_t + u u_x + P_x \Big) \eta'(u) \phi\, dx\, dt, \end{aligned}$$

where the second term vanishes since u is a strong solution in each D_i. What remains does not contain P, and the standard arguments for entropy weak solutions of the inviscid Burgers equation [13] show that $u_l \geq u_r$. (It also follows that the usual Rankine–Hugoniot relation $\dot{x}_0 = \frac{1}{2}(u_l + u_r)$ must hold. This agrees with the equation $\dot{x}_k = u(x_k)$ in (2.3), since $u(x_k)$ is nothing but an abbreviation for $\frac{1}{2}\big(u(x_k^-) + u(x_k^+)\big)$ because of the convention $G'(0) = 0$.) □

Letting $n = 1$ in (2.3) we see that the dynamics of a single shockpeakon is described by the trivial equations

$$\dot{x}_1 = m_1, \qquad \dot{m}_1 = 0, \qquad \dot{s}_1 = -s_1^2. \tag{2.6}$$

Consequently the solitary shockpeakon moves at constant speed m_1; in particular, it does not move at all if $m_1 = 0$. As for the shock strength, the equation for s_1 is equivalent to $s_1 \equiv 0$ or $\frac{d}{dt}(1/s_1) = 1$, hence

$$s_1(t) = \frac{s_1(t_0)}{1 + (t - t_0)\, s_1(t_0)}, \tag{2.7}$$

so that the shock strength decays like $1/t$ as $t \to +\infty$, assuming that the entropy condition $s_1(t_0) > 0$ is satisfied. If $s_1(t_0) < 0$ then $s_1(t)$ blows up after finite time, but in this case our ansatz does not yield the entropy weak solution (which is a rarefaction wave instead; see Remark 3.7). As we will see in section 3, shockpeakons which form at peakon-antipeakon collisions automatically satisfy $s_k > 0$.

The case $n = 2$ is already quite complicated. Assuming $x_1 < x_2$ and using the abbreviation $R = \exp(x_1 - x_2)$, the ODEs (2.3) take the form

$$\begin{aligned}
\dot{x}_1 &= m_1 + (m_2 + s_2)R, \\
\dot{x}_2 &= m_2 + (m_1 - s_1)R, \\
\dot{m}_1 &= -2(m_1 - s_1)(m_2 + s_2)R, \\
\dot{m}_2 &= +2(m_1 - s_1)(m_2 + s_2)R, \\
\dot{s}_1 &= -s_1^2 - s_1(m_2 + s_2)R, \\
\dot{s}_2 &= -s_2^2 + s_2(m_1 - s_1)R.
\end{aligned} \tag{2.8}$$

It is clear that $m_1 + m_2$ is conserved, but we have not found any other constants of motion, so it is still an open question whether the shockpeakon ODEs (2.3) are integrable in any sense. We will study the reduction $0 = x_1 + x_2 = m_1 + m_2 = s_1 - s_2$ in section 4. The results there indicate that if the shockpeakon ODEs are integrable, the constants of motion must take a considerably more complicated form than in the shockless case (where they are polynomials in $m_1, \dots, m_n$ with coefficients depending rationally on $e^{x_1}, \dots, e^{x_n}$).

3 Peakons, antipeakons, and shock formation

As mentioned earlier, an *antipeakon* is a peakon with $m_k < 0$. Since $\dot{x}_k = \sum_i m_i \, e^{-|x_k - x_i|} \approx m_k$ when the x_i's are well separated, peakons generally speaking move to the right and antipeakons to the left.

There is a qualitative difference between "pure" peakon solutions where all m_k's are positive (or all negative), and mixed peakon-antipeakon solutions where both signs occur. This can be seen by considering the functions

$$M_1 = \sum_{k=1}^{n} m_k \qquad \text{and} \qquad M_n = \left(\prod_{k=1}^{n} m_k \right) \prod_{k=1}^{n-1} \left(1 - e^{x_k - x_{k+1}} \right)^{b-1} \tag{3.1}$$

which are constants of motion for the peakon ODEs (1.9) under the usual ordering assumption $x_1 < \cdots < x_n$. (Recall that $b = 2$ for CH and $b = 3$ for DP.)

In the pure peakon case we have $0 < M_n / M_1^{n-1} < m_k < M_1$ for all k, and then no $x_k - x_{k+1}$ can become zero since this would violate $M_n > 0$. In other words, the peakons never collide and the ordering assumption is preserved for all t.

In the peakon-antipeakon case a collision can occur after finite time. The only way to keep $M_n \neq 0$ constant if $x_k - x_{x+1} \to 0$ is for at least one m_i to go to infinity, and to keep M_1 constant this has to be cancelled by some other m_j going to minus infinity. In fact, what will happen is that $m_{k+1} \to +\infty$ and $m_k \to -\infty$ (this follows from the explicit solution formulas). Since the ODEs break down, the question arises if and how the PDE solution $u(x,t)$ can be continued past the collision. As we show below, the DP equation and the CH equation behave completely differently with regard to this.

The CH case

The following facts are well known, and collected here mainly to facilitate comparison to the DP case. The general solution of the CH peakon ODEs (1.6) in the case $n = 2$ is

$$
\begin{aligned}
x_1(t) &= \log \frac{(\lambda_1-\lambda_2)^2 b_1 b_2}{\lambda_1^2 b_1 + \lambda_2^2 b_2}, \qquad & x_2(t) &= \log(b_1+b_2), \\
m_1(t) &= \frac{\lambda_1^2 b_1 + \lambda_2^2 b_2}{\lambda_1\lambda_2(\lambda_1 b_1 + \lambda_2 b_2)}, \qquad & m_2(t) &= \frac{b_1+b_2}{\lambda_1 b_1 + \lambda_2 b_2},
\end{aligned}
\tag{3.2}
$$

where $b_k = b_k(t) = b_k(0)\, e^{t/\lambda_k}$; the real, nonzero, distinct constants λ_1, λ_2, and the positive constants $b_1(0)$, $b_2(0)$ are determined from the initial conditions through the relations

$$
1 - (m_1+m_2)\, z + m_1 m_2 \left(1 - \frac{e^{x_1}}{e^{x_2}}\right) z^2 = \big(1 - \frac{z}{\lambda_1}\big)\big(1 - \frac{z}{\lambda_2}\big) \tag{3.3}
$$

(which implies that the number of positive/negative λ_k's equals the number of positive/negative m_k's) and

$$
b_1 + b_2 = e^{x_2}, \qquad \frac{b_1}{\lambda_1} + \frac{b_2}{\lambda_2} = m_1 e^{x_1} + m_2 e^{x_2}. \tag{3.4}
$$

The solution given by (3.2) automatically satisfies $x_1(t) \le x_2(t)$ for all t since

$$
e^{x_2} - e^{x_1} = \frac{(\lambda_1 b_1 + \lambda_2 b_2)^2}{\lambda_1^2 b_1 + \lambda_2^2 b_2} \ge 0.
$$

Equality $x_1(t) = x_2(t)$ holds exactly when the denominator $\lambda_1 b_1 + \lambda_2 b_2$ in m_1 and m_2 vanishes, which only happens in the peakon-antipeakon case $\lambda_1\lambda_2 < 0$, and then for exactly one $t = t_0$. As $t \to t_0^{\pm}$, the momenta m_1 and m_2 blow up in such a way that the derivative $u_x(x,t)$ tends to $\pm\infty$ on the shrinking interval $x_1(t) < x < x_2(t)$, but at the same time $u(x_1(t),t) - u(x_2(t),t) \to 0$ so that the peaks meet and the wave profile $u(x,t) = m_1\, e^{-|x-x_1|} + m_2\, e^{-|x-x_2|}$ converges uniformly to

$$
u(x,t_0) = \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) e^{-|x-x_1(t_0)|}. \tag{3.5}
$$

See Figure 4. Defining $u(x,t)$ by (1.5) and (3.2) for $t \ne t_0$ and by (3.5) for $t = t_0$ provides a global solution of the CH equation. This continuation of u past the collision is not unique, but it is distinguished by the desirable property that the total energy $\int_{\mathbf{R}} (u^2 + u_x^2)\, dx$ is preserved for all t except at the instant of collision, where the discrepancy can be accounted for by an "invisible" Dirac delta contribution from the term u_x^2. See [4, 12] for detailed discussions.

The solution for general n depends on $2n$ parameters $\{\lambda_k, b_k\}_{k=1}^n$ which arise as eigenvalues and Weyl function residues of the "discrete string", a certain spectral problem related to the CH Lax pair. The eigenvalues λ_k are real,

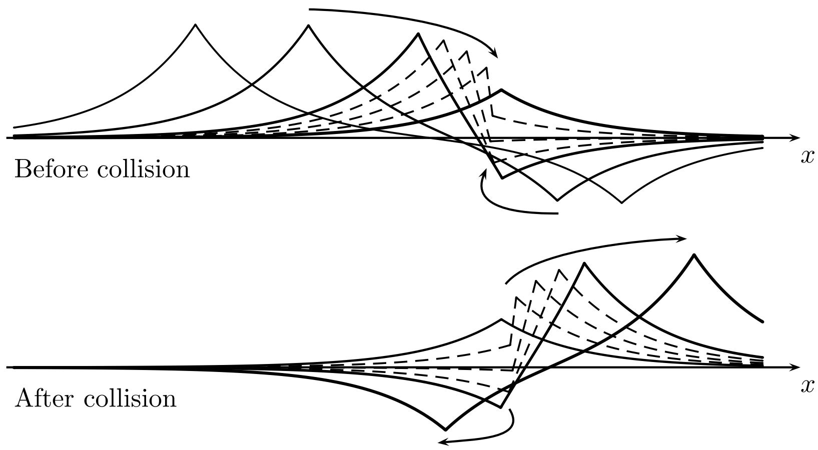

Figure 4: Camassa–Holm peakon-antipeakon interaction, computed from the exact solution formulas (3.2) with asymptotic speeds $0 < -\lambda_2^{-1} < \lambda_1^{-1}$, the case where the peakon is stronger than the antipeakon. Solid curves show $u(x,t)$ at evenly sampled times, with some additional samples close to the collision shown by dashed curves; notice how the slope u_x steepens between the peaks. The arrows indicate roughly the motion of the peaks, which coalesce into a single peak at the instant of collision and then reemerge. In the symmetric case $\lambda_1 + \lambda_2 = 0$ (not shown here), the peakon and the antipeakon cancel out exactly so that $u = 0$ at the instant of collision.

nonzero, distinct constants which we number such that $\lambda_n^{-1} < \cdots < \lambda_1^{-1}$ for definiteness. The number of positive/negative eigenvalues equals the number of peakons/antipeakons. The residues b_k are positive and evolve as $b_k(t) = b_k(0)\, e^{t/\lambda_k}$. As $t \to \pm\infty$ the peakons behave like asymptotically free particles with distinct speeds; $\dot{x}_k \sim m_k \sim \lambda_k^{-1}$ as $t \to -\infty$ and $\dot{x}_k \sim m_k \sim \lambda_{n+1-k}^{-1}$ as $t \to +\infty$. The explicit solution formulas define $x_k(t)$ and $m_k(t)$ for all t, except that $m_k(t)$ may be undefined for finitely many values of t where collisions occur (always in distinct peakon-antipeakon pairs, no triple collisions). Away from the times of collision $x_1(t) < \cdots < x_n(t)$ holds. The peakon and antipeakon involved in a collision behave just like in the case $n = 2$ described above; in particular, the corresponding terms $m_k\, e^{-|x-x_k|} + m_{k+1}\, e^{-|x-x_{k+1}|}$ tend to a well-defined limit $\lim_{t\to t_0} [m_k(t) + m_{k+1}(t)]\, e^{-|x-x_k(t_0)|}$ at the instant of collision, so that $u(x, t_0)$ is defined even though $m_k(t_0)$ and $m_{k+1}(t_0)$ are not. The derivation and analysis of the solution formulas uses concepts and identities related to the classical moment problem, such as Stieltjes continued fractions and the theory of orthogonal polynomials [1, 2]).

Remark 3.1. Reference [2] uses the opposite sign convention for the eigenvalues λ_k, and also a different normalization of the CH equation, which produces some additional factors of 2 in the solution formulas. The notation here is chosen to be similar to that used for the DP case below and in [19].

Remark 3.2. A similar phenomenon with u_x blowing up but u remaing continuous occurs for piecewise linear solutions of the Hunter–Saxton equation $(u_t + uu_x)_{xx} = u_x u_{xx}$, where it has been called a "zero-strength shock" by D.

Holm, according to [15]. The Hunter–Saxton equation can be obtained from the CH equation $(u-u_{xx})_t+3uu_x = uu_{xxx}+2u_xu_{xx}$ by substituting $(x,t) \mapsto (\varepsilon x, \varepsilon t)$ and letting $\varepsilon \to 0$. The same high-frequency limit applied to the DP equation yields the "derivative Burgers equation" $(u_t+uu_x)_{xx} = 0$. (This was first stated in a footnote in the preprint version of [10], but that was removed in the published version. It has also been pointed out elsewhere, for example in [14].) Since the inviscid Burgers equation $u_t + uu_x = 0$ is the prototype equation for studying shock formation, this provides some intuition for why shock waves form in the DP equation. It is interesting that the derivative Burgers equation inherits the integrable structure of the DP equation. Like the Hunter–Saxton equation, it has piecewise linear solutions $u(x,t) = \sum_{k=1}^n m_k(t)\,|x-x_k(t)|$ which can be computed using inverse scattering [16]. These solutions do not satisfy the ordinary Burgers equation $u_t + uu_x = 0$ unless $\sum m_k = 0$, so the additional x derivatives do make a difference. The derivative Burgers equation belongs to an integrable hierarchy described by Qiao and Li [23].

The DP case

Solutions of the DP equation may develop discontinuities in finite time, as we will show below using the explicit solution formulas.

The general two-peakon solution of the DP equation, except for the case $\lambda_1 + \lambda_2 = 0$ which is treated separately below, is

$$\begin{aligned} x_1(t) &= \log \frac{\frac{(\lambda_1-\lambda_2)^2}{\lambda_1+\lambda_2} b_1 b_2}{\lambda_1 b_1 + \lambda_2 b_2}, \qquad x_2(t) = \log(b_1+b_2), \\ m_1(t) &= \frac{(\lambda_1 b_1 + \lambda_2 b_2)^2}{\lambda_1\lambda_2 \left(\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2} b_1 b_2\right)}, \\ m_2(t) &= \frac{(b_1+b_2)^2}{\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2} b_1 b_2}, \end{aligned} \tag{3.6}$$

with $b_k(t) = b_k(0)\,e^{t/\lambda_k}$. Here the constants λ_1 and λ_2, which we will number such that $\lambda_2^{-1} < \lambda_1^{-1}$, are the real, nonzero, disctinct zeros of the invariant polynomial[2]

$$1 - (m_1+m_2)\,z + m_1 m_2 \left(1 - \frac{e^{x_1}}{e^{x_2}}\right)^2 z^2 = \left(1 - \frac{z}{\lambda_1}\right)\left(1 - \frac{z}{\lambda_2}\right), \tag{3.7}$$

while b_1 and b_2 are again given by (3.4).

The n-peakon ODEs are fully understood in the pure peakon case where all m_k's are positive. Then the general solution is given in terms of eigenvalues and Weyl function residues $\{\lambda_k, b_k\}_{k=1}^n$ of the "discrete cubic string", a third order nonselfadjoint spectral problem related to the Lax pair of the DP equation [19]. The eigenvalues λ_k are positive and distinct, the residues b_k are positive, the peakons behave like free particles with distinct speeds $\lambda_n^{-1} < \cdots < \lambda_1^{-1}$ as $t \to \pm\infty$, and no collisions occur.

In the general case $n > 2$ with both peakons and antipeakons present, not much is really known for sure, although reasonable conjectures can be made.

[2] Note that the coefficient of z^2 in (3.7) is slightly different from the CH case (3.3).

The crucial point is whether the eigenvalues must be real and distinct, with the same number of positive (negative) eigenvalues as the number of positive (negative) m_k's, like in the CH case. This is true for $n = 2$, as can be verified directly from (3.7), but has not been proved for $n > 2$. Unlike the CH case, the residues b_k need not always be positive (see the proof of Theorem 3.5). For initial data such that the eigenvalues are indeed real and distinct, with no $\lambda_i + \lambda_j$ equal to zero, the explicit formulas for $x_k(t)$ and $m_k(t)$ provide solutions which are valid locally in t but may have singularities after finite time. As we will see already for $n = 2$, it is not possible to extend these solutions past the singularities without taking shock formation into account.

Leaving the complete analysis of the general case as an open problem for future research, we will concentrate here on the case $n = 2$. We begin with the completely symmetric peakon-antipeakon case $m_1 + m_2 = 0$, which by (3.7) is exactly the exceptional case $\lambda_1 + \lambda_2 = 0$ not covered by (3.6). Since the solutions will not be globally defined in t, we focus on the inital value problem. By shifting the x axis we can assume $x_1(0)+x_2(0) = 0$ without loss of generality. See Figure 5 for an illustration of the result.

Theorem 3.3. *The solution of the $n = 2$ DP peakon ODEs* (1.10) *in the symmetric peakon-antipeakon case $m_1 + m_2 = 0$ with $-x_1(0) = x_2(0) > 0$ is given by*

$$\begin{aligned} -x_1(t) = x_2(t) &= x_2(0) - \frac{t}{\lambda}, \\ m_1(t) = -m_2(t) &= \frac{1}{\lambda\big(1 - e^{-2x_2(t)}\big)}, \end{aligned} \tag{3.8}$$

where $\lambda = \big(m_1(0)(1 - e^{-2x_2(0)})\big)^{-1}$.

- *If $m_1(0) < 0 < m_2(0)$, then $\lambda < 0$ and the solution* (3.8) *is valid for $t > t_{\min}$, where $t_{\min} = \lambda\, x_2(0) < 0$. In particular, $u = \sum_1^2 m_k\, e^{-|x-x_k|}$ provides a solution of the initial value problem which is valid for all $t \geq 0$.*
- *If $m_1(0) > 0 > m_2(0)$, then $\lambda > 0$ and a collision occurs at $x = 0$ for $t = t_0 = \lambda\, x_2(0) > 0$. The function $u = \sum_1^2 m_k\, e^{-|x-x_k|}$ only satisfies the DP equation* (1.15) *for $t < t_0$. The unique continuation of $u(x, t)$ into an entropy weak solution is given by the stationary decaying shockpeakon*

$$u(x,t) = \frac{-\operatorname{sgn}(x)\, e^{-|x|}}{\lambda + (t - t_0)} \qquad \text{for } t \geq t_0. \tag{3.9}$$

Proof. Direct substitution shows that (3.8) satisfies the peakon ODEs, and that

$$u(x_1(t), t) = -u(x_2(t), t) = \frac{1}{\lambda}, \tag{3.10}$$

independently of t. Hence, unlike the CH case where $u \to 0$ uniformly, the peakon and the antipeakon do not cancel out completely at a collision. Instead $u(x, t)$ converges as $t \to t_0^-$ to the discontinuous function

$$u(x, t_0) = -\frac{1}{\lambda} \operatorname{sgn}(x)\, e^{-|x|} = \frac{1}{\lambda} G'(x). \tag{3.11}$$

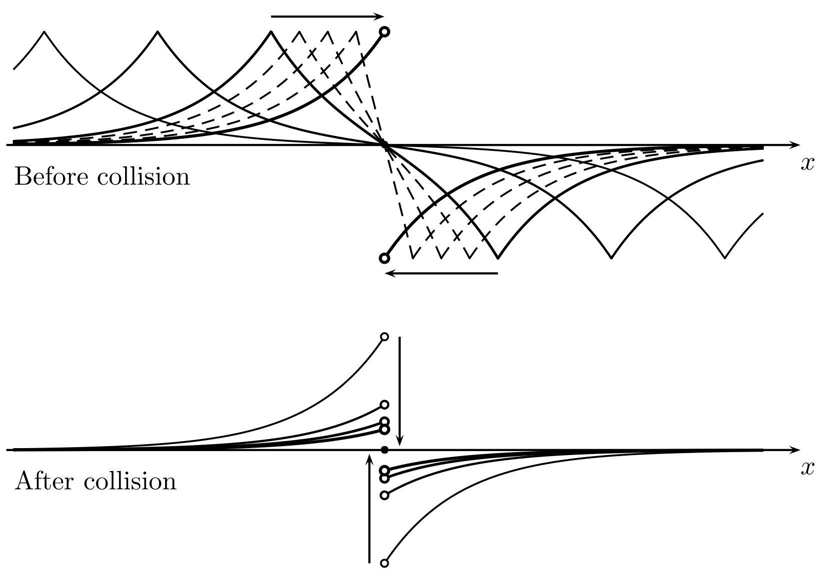

Figure 5: Degasperis–Procesi peakon-antipeakon collision in the symmetric case $\lambda_1 + \lambda_2 = 0$, computed from the exact solution formulas (3.8) and (3.9). Solid curves show $u(x,t)$ at evenly sampled times, with some additional samples close to the collision shown by dashed curves. As the arrows indicate, the peaks approach each other with constant speed and height, and form a stationary shockpeakon whose shock strength decays like $1/t$ as $t \to +\infty$.

(The convergence is uniform on any interval not containing $x = 0$). In other words, a shock forms at the collision. The unique entropy weak solution with (3.11) as initial data at $t = t_0$ is given by the $n = 1$ shockpeakon ODEs (2.6) with $x_1(t_0) = m_1(t_0) = 0$ and $s_1(t_0) = 1/\lambda$. This proves (3.9). ☐

Remark 3.4. Degasperis, Holm and Hone [10] give the formula

$$u(x,t) = \frac{c}{1 - e^{-2c|t|}} \left[e^{-|x+c|t||} - e^{-|x-c|t||} \right], \tag{3.12}$$

which (if $c > 0$) satisfies the DP equation in the interval $t < 0$, but not for $t > 0$ since the formula is even in t so that the peakon and antipeakon incorrectly would move "backwards" for $t > 0$. This can easily be remedied by multiplying (3.12) by $-\operatorname{sgn}(t)$, which gives the solution that the authors probably had in mind. However, that solution suffers from the problem that the wave profile flips abruptly from $u(x, 0^-) = -c \operatorname{sgn}(x)\, e^{-|x|}$ to $u(x, 0^+) = c \operatorname{sgn}(x)\, e^{-|x|}$, so it is doubtful whether this can be considered a valid global solution, even though it is fine in the intervals $t < 0$ and $t > 0$ separately (where it agrees with (3.8) up to a translation of t).

Similarly, Holm and Staley [14, Sect. 5.4], arguing using explicit solution formulas for the two-peakon ODEs, state the following for symmetric ($m_1+m_2 = 0$, $x_1+x_2 = 0$) peakon-antipeakon collisions in any equation from the "b family" (1.8) of which the CH and DP equations are the members $b = 2$ and $b = 3$:

> As the separation $q \to 0$, the positive and negative peaks "bounce", thereby reversing polarity, after which they separate in opposite directions.

This can also be questioned from the point of view of making global sense of the PDE. If $b < 3$ then $u(x, t) \to 0$ uniformly at the collision, as can be seen from the explicit formula $u(x_1) = -u(x_2) = \text{const.} \times (1 - e^{x_1-x_2})^{(3-b)/2}$. In this case the statement above is true. (An alternative description of the same thing is that the peakon and the antipeakon appear to pass through each other.) But for the DP equation ($b = 3$) we have just seen that it is instead the shock scenario of Theorem 3.3 that gives the correct entropy weak solution after a collision. And for $b > 3$, $u(x_1) = -u(x_2) \to +\infty$ at the collision, so it does not seem reasonable to try to continue the solution of the PDE past the collision at all in this case. These different behaviours are related to the sign of the term $\frac{3-b}{2}\, u_x^2$ in (1.13). (The significance of this sign can also be seen in the proof of the "peakon steepening lemma" later in the same paper [14, Prop. 6.1], where the authors do impose the condition $b \leq 3$.)

Next we describe the solution of the DP initial value problem for the generic case $m_1 + m_2 \neq 0$. See Figure 6 for an illustration of the third case in the theorem.

Theorem 3.5. *The solution of the $n = 2$ DP peakon ODEs* (1.10) *with $x_1(0) < x_2(0)$ and $m_1 + m_2 \neq 0$ is given by* (3.6)*, where λ_k and $b_k(0)$ are determined from the initial conditions.*

- *If $m_1(0)$ and $m_2(0)$ have the same sign, then $u = \sum_1^2 m_k\, e^{-|x-x_k|}$ together with* (3.6) *defines a global solution of the DP equation* (1.15)*. In particular, as a solution of the initial value problem it is valid for all $t \geq 0$.*

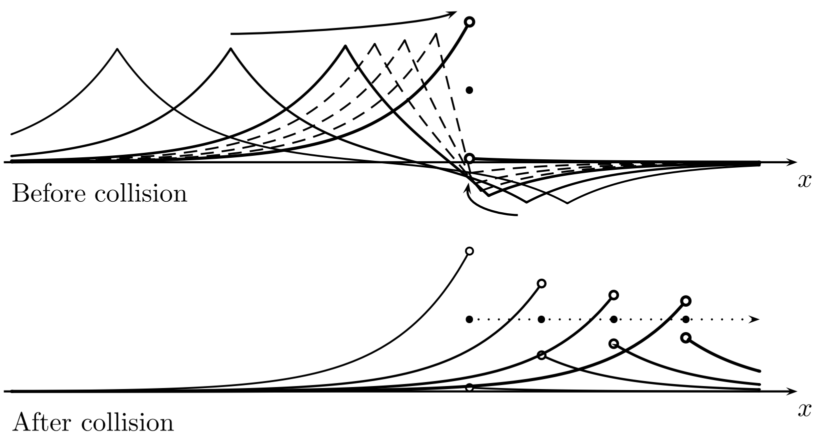

Figure 6: Nonsymmetric Degasperis–Procesi peakon-antipeakon collision, computed from the exact solution formulas (3.6) and (3.15) with $0 < -\lambda_2^{-1} < \lambda_1^{-1}$, the case where the peakon is stronger than the antipeakon, so that the resulting shockpeakon moves to the right. Solid curves show $u(x,t)$ at evenly sampled times, with some additional samples close to the collision shown by dashed curves. The shock strength decays like $1/t$ as $t \to +\infty$.

- *If* $m_1(0) < 0 < m_2(0)$, *then* (3.6) *gives a valid solution of the DP equation for* $t > t_{\min}$, *where*

$$t_{\min} = \frac{-1}{\lambda_1^{-1} - \lambda_2^{-1}} \log\left(\frac{1-\kappa}{\kappa(1+\kappa)} \frac{b_1(0)}{b_2(0)}\right) < 0 \qquad (\kappa = \sqrt{-\lambda_2/\lambda_1}). \quad (3.13)$$

  In particular, as a solution of the initial value problem it is valid for all $t \geq 0$.

- *If* $m_1(0) > 0 > m_2(0)$, *then* (3.6) *gives a valid solution of the DP equation for* $t < t_0$, *where the time of collision* t_0 *is*

$$t_0 = \frac{1}{\lambda_1^{-1} - \lambda_2^{-1}} \log\left(\frac{\kappa(\kappa-1)}{1+\kappa} \frac{b_2(0)}{b_1(0)}\right) > 0 \qquad (\kappa = \sqrt{-\lambda_2/\lambda_1}). \quad (3.14)$$

  The continuation of $u(x,t)$ *into the unique entropy weak solution of the initial value problem is for* $t \geq t_0$ *given by the moving shockpeakon*

$$u(x,t) = \Big(\widetilde{m}_1 - \operatorname{sgn}\big(x - \widetilde{x}_1(t)\big)\, \widetilde{s}_1(t)\Big)\, e^{-|x-\widetilde{x}_1(t)|}, \qquad (3.15)$$

  where $\widetilde{m}_1 = m_1 + m_2 = \lambda_1^{-1} + \lambda_2^{-1}$ *is constant,* $\widetilde{x}_1(t) = (t - t_0)\widetilde{m}_1 + \widetilde{x}_1(t_0)$ *with* $\widetilde{x}_1(t_0) = x_1(t_0) = x_2(t_0)$ *the point of collision, and* $\widetilde{s}_1(t) = \big(t - t_0 + \widetilde{s}_1(t_0)^{-1}\big)^{-1}$ *with* $\widetilde{s}_1(t_0) = \sqrt{-\lambda_1^{-1}\lambda_2^{-1}} > 0$.

Proof. It can be verified by substitution that (3.6) satisfies (1.10) for any values of λ_1, λ_2, $b_1(0)$, $b_2(0)$, and t such that all expressions make sense and $x_1(t) <$

$x_2(t)$. (This restriction is needed to get rid of the absolute value signs in the ODEs (1.10).) See [19] for the derivation of these formulas.

In what follows, we will use the abbreviations $R = e^{x_1-x_2} \in (0,1)$ and $Q = 1-R \in (0,1)$. From (3.7), the eigenvalues λ_k are given by

$$\lambda_{1,2}^{-1} = \tfrac{1}{2}\big(m_1 + m_2 \pm \sqrt{(m_1+m_2)^2 - 4m_1 m_2 Q^2}\big), \tag{3.16}$$

where, because of our convention $\lambda_2^{-1} < \lambda_1^{-1}$, the plus sign refers to λ_1^{-1}. This implies that $0 < \lambda_2^{-1} < \lambda_1^{-1}$ if m_1 and m_2 are positive, $\lambda_2^{-1} < \lambda_1^{-1} < 0$ if m_1 and m_2 are negative, and $\lambda_2^{-1} < 0 < \lambda_1^{-1}$ if $m_1 m_2 < 0$. Solving for the residues b_k from (3.4) then yields

$$b_{1,2} = \tfrac{1}{2} e^{x_2}\big(1 \mp \operatorname{sgn}(m_1) f(m_2/m_1)\big), \tag{3.17}$$

where the minus sign refers to b_1, and[3]

$$f(s) = \frac{-1+2Q-s}{\sqrt{(1+s)^2 - 4sQ^2}}. \tag{3.18}$$

Simple calculus shows that f increases from its limit 1 at $s=-\infty$ to $f(-1-2Q) = \sqrt{2/(1+Q)} \in (1, \sqrt{2})$, and then decreases to its limit -1 at $s=+\infty$, passing $f(-1) = 1$ and $f(-1+2Q) = 0$ on its way down. Thus $f(s) > 1$ for $s < -1$, so by (3.17) b_1 and b_2 are both positive if $-1 < m_2/m_1 < 0$ or $0 < m_2/m_1$, and have opposite signs if $m_2/m_1 < -1$. The case $m_2/m_1 = -1$, where one of the b_k's is zero, will be excluded here since it is the completely symmetric peakon-antipeakon case $\lambda_1 + \lambda_2 = 0$ already treated in Theorem 3.3.

From (3.6) we have

$$e^{x_2} - e^{x_1} = \frac{W}{\lambda_1 b_1 + \lambda_2 b_2}, \tag{3.19}$$

where

$$W = \lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2} b_1 b_2 \tag{3.20}$$

is the expression also occuring in the denominator of m_1 and m_2. Recall that the b_k's evolve according to $b_k(t) = b_k(0)\, e^{t/\lambda_k}$ and consequently do not change sign. If $m_1(0)m_2(0) > 0$, then $\lambda_1\lambda_2 > 0$, while $b_1(0)$ and $b_2(0)$ are positive. Hence W never changes sign, and neither does any of the other expressions involved in (3.6). Hence, in this case $e^{x_2} - e^{x_1} > 0$ for all t and the solution is valid globally.

In the peakon-antipeakon case $m_1(0)m_2(0) < 0$ the situation is more complicated since W, as well as the expressions $U = b_1 + b_2$ and $V = \lambda_1 b_1 + \lambda_2 b_2$, might become zero for some $t \neq 0$, causing the solution (3.6) to blow up. To analyze this it is convenient to define

$$v_1 = \lambda_1^{-1} > 0, \qquad v_2 = -\lambda_2^{-1} > 0, \tag{3.21}$$

and

$$\beta = \frac{b_2}{b_1}, \qquad \kappa = \sqrt{\frac{-\lambda_2}{\lambda_1}} = \sqrt{\frac{v_1}{v_2}} \neq 1. \tag{3.22}$$

[3]The CH case is identical except that Q^2 is replaced by Q in (3.16) and in the denominator of (3.18). This change makes $|f(s)| < 1$ for all s, so that b_1 and b_2 are always positive in that case.

The time evolution of β is

$$\beta(t) = \frac{b_2(0)\, e^{t/\lambda_2}}{b_1(0)\, e^{t/\lambda_1}} = \beta(0)\, e^{-(v_1+v_2)t},$$

so $|\beta|$ decays exponentially to zero. Note that U becomes zero if $\beta = -1$, while V becomes zero if $\beta = 1/\kappa^2$. Moreover $W = \lambda_2 b_1^2 P(\beta)$, where

$$P(\beta) = \beta^2 + \frac{4\beta}{1-\kappa^2} - \frac{1}{\kappa^2} = \left(\beta - \frac{1-\kappa}{\kappa(1+\kappa)}\right)\left(\beta + \frac{1+\kappa}{\kappa(1-\kappa)}\right), \tag{3.23}$$

so W vanishes for one positive and one negative value of β. There are four different cases to consider, depending on whether $m_1 < 0 < m_2$ or $m_1 > 0 > m_2$, and whether $m_1 + m_2 = v_1 - v_2$ is positive ($\kappa > 1$) or negative ($0 < \kappa < 1$).

We show the details for the case $m_1 > 0 > m_2$, $\kappa > 1$, with a peakon on the left and a (weaker) antipeakon on the right, as in Figure 6. In this case $-1 < m_2(0)/m_1(0) < 0$, so that $b_1(0) > 0$ and $b_2(0) > 0$. Then the following chart describes the signs of U, V, W as functions of β:

β		-1		$-\frac{\kappa-1}{\kappa(1+\kappa)}$	0	$\frac{1}{\kappa^2}$		$\frac{1+\kappa}{\kappa(\kappa-1)}$	
U	$-$	0	$+$		$+$		$+$		$+$
V	$+$		$+$		$+$	0	$-$		$-$
W	$-$		$-$	0	$+$		$+$	0	$-$

Since at $t = 0$ we have $m_2 = U^2/W$ negative and $W/V = e^{x_2} - e^{x_1}$ positive, W and V must both be negative. This shows that $\beta(0)$ must be in the rightmost part of the chart, $\beta(0) > (1+\kappa)/\kappa(\kappa-1) > 0$. As time passes and β decreases towards zero, W therefore vanishes before V does. This happens at $t = t_0 > 0$, where

$$\beta(t_0) = \beta(0)\, e^{-(v_1+v_2)t_0} = \frac{1+\kappa}{\kappa(\kappa-1)},$$

which proves (3.14) in this case. Since $e^{x_2} - e^{x_1} = W/V$ becomes negative after t passes t_0, the solution formulas (3.6) do not provide a valid solution for $t > t_0$.

Next we investigate what happens to the wave profile $u(x,t)$ as $t \to t_0^-$. Since $m_1(t) + m_2(t) = \lambda_1^{-1} + \lambda_2^{-1} = v_1 - v_2$ and $x_1(t) - x_2(t) \to 0$, we have

$$u(x_1(t),t) + u(x_2(t),t) = (m_1 + m_2)(1 + e^{x_1 - x_2}) \to 2(v_1 - v_2). \tag{3.24}$$

Moreover,

$$\frac{1}{\lambda_1\lambda_2} = m_1 m_2 Q^2 = \left(\frac{m_1+m_2}{2}Q\right)^2 - \left(\frac{m_1-m_2}{2}Q\right)^2,$$

where the first term tends to zero since m_1+m_2 is constant and $Q = 1-e^{x_1-x_2} \to 0$, so that

$$u(x_1(t),t) - u(x_2(t),t) = (m_1 - m_2)Q \to 2\sqrt{\frac{-1}{\lambda_1\lambda_2}} = 2\sqrt{v_1 v_2}. \tag{3.25}$$

(The sign of the square root is determined since $m_1 > 0 > m_2$.) Consequently $u(x,t)$ converges to a single shockpeakon with position $\widetilde{x}_1 = x_1(t_0) = x_2(t_0)$, momentum $\widetilde{m}_1 = v_1 - v_2$, and shock strength $\widetilde{s}_1 = \sqrt{v_1 v_2} > 0$. The continuation of $u(x,t)$ into an entropy weak solution for $t \geq t_0$ is given by the $n = 1$ shockpeakon dynamics (2.6) with these values as initial data at $t = t_0$, resulting in (3.15).

The other three cases are analyzed in a similar way. We omit the details. □

Remark 3.6. If, in the case $m_1(0) > 0 > m_2(0)$ of Theorem 3.5, the right-moving peakon is the stronger one ($v_1 > v_2$) we get a shock moving to the right, while if the left-moving antipeakon dominates ($v_1 < v_2$) we get a shock moving to the left. Note that in both cases the jump in u is from high at the left to low at right; it is the average value of u at the jump that determines the direction of motion.

If the shock strength is less than the resulting momentum ($\sqrt{v_1 v_2} < |v_1 - v_2|$), then the weaker peak is pulled over to the opposite side of the x axis before collision, so that both peaks actually travel in the same direction for a while. For example, if the peakon dominates sufficiently[4] over the antipeakon, then the antipeakon changes its velocity $\dot{x}_2 = u(x_2)$ from negative to positive a while before the collision, but it still cannot escape being "run over" by the faster peakon catching up from the left.

Remark 3.7. In the case $m_1(0) < 0 < m_2(0)$ of Theorem 3.5 (or Theorem 3.3), the solution describes an antipeakon to the left and a peakon to the right, drifting apart. The time $t = t_{\min}$ is when they would collide if time was running backwards, and the solution defined for $t > t_{\min}$ is a rarefaction wave solution to the initial value problem with initial data at $t = t_{\min}$ consisting of a single shockpeakon with a negative shock strength $\widetilde{s}_1 = -(-\lambda_1^{-1}\lambda_2^{-1})^{1/2}$ (or $\widetilde{s}_1 = -1/|\lambda|$ in the symmetric situation of Theorem 3.3).

For the general peakon ODEs with $n > 2$, we also expect a shock to form whenever a peakon and an antipeakon collide, but we cannot at present exclude that Camassa–Holm style zero-strength shocks might also be possible. As times passes, further collisions may occur. The limiting wave profile at each collision must piecewise be a linear combination of e^x and e^{-x}, with decay at infinity, and therefore describable by the shockpeakon ansatz (2.1). The dynamics between collisions are then described by the appropriate shockpeakon ODEs (2.3) with n decreasing after each collision, as the colliding (shock)peakons merge.

As shown in [7], weak solutions of the DP equation satisfy the following one-sided Lipschitz estimate for any $T > 0$:

$$u_x(t,x) \leq \frac{1}{t} + K_T \qquad \text{for a.e. } (x,t) \in \mathbf{R} \times (0,T),$$

where K_T is a constant depending on T and on the $L^2 \cap BV$ norm of $u(x,0)$. This implies that u_x cannot tend to $+\infty$ in finite time, and consequently any shocks that form must jump downwards, from high at the left to low at the right. So whenever a shockpeakon forms at a collision, it will automatically satisfy the entropy condition $s_k > 0$.

[4]That is, if $\sqrt{v_1 v_2} < v_1 - v_2$, which is the same as $\kappa > (1 + \sqrt{5})/2$. In Figure 6, where $\kappa \approx 1.66$, this condition is just barely satisfied.

4 Shock dynamics

In this section we will study properties of solutions of the shockpeakon ODEs (2.3), repeated here for convenience:

$$\begin{aligned} \dot{x}_k &= u(x_k), \\ \dot{m}_k &= 2s_k\, u(x_k) - 2m_k\{u_x(x_k)\}, \\ \dot{s}_k &= -s_k\{u_x(x_k)\}. \end{aligned} \tag{2.3}$$

Proposition 4.1. *$M = \sum_{k=1}^{n} m_k$ is a constant of motion for* (2.3).

Proof. This can be checked directly from (2.3), or one can use that the DP equation (1.15) is in the form of a conservation law $u_t + (\frac{1}{2}u^2 + P)_x = 0$, where $P = \frac{1}{2}G * \frac{3}{2}u^2$. Thus $\int_{\mathbf{R}} u\,dx$ is conserved, and for the shockpeakon ansatz (2.1) we find that $\int_{\mathbf{R}} u\,dx = \sum m_k$. □

Remark 4.2. The other (infinitely many) conservation laws of the DP equation [10] do not produce additional constants of motion, since their derivation relies on the chain rule, which is not valid for discontinuous functions. For example, using that $P - P_{xx} = \frac{3}{2}u^2$ by definition, one derives a conservation law for $\int_{\mathbf{R}} u^3\,dx$:

$$\begin{aligned} -(u^3)_t = -3u^2u_t = 3u^2\big((\tfrac{1}{2}u^2)_x + P_x\big) &= 3\big((\tfrac{1}{2}u^2)^2\big)_x + 2(P - P_{xx})P_x \\ \implies \quad (u^3)_t + \left(\frac{3u^4}{4} + P^2 - P_x^2\right)_x &= 0. \end{aligned}$$

But for shockpeakons it can be checked that

$$\frac{\partial}{\partial t}\left(\int_{\mathbf{R}} u^3\,dx\right) = -4\sum_{k=1}^{n} u(x_k)s_k^3, \tag{4.1}$$

so that $\int_{\mathbf{R}} u^3\,dx$ is in fact not conserved if there are shocks.

We note that if the shockpeakons are well separated, then $u(x_k) \approx m_k$ and $\{u_x(x_k)\} \approx s_k$, so that each shockpeakon behaves almost like in the case $n = 1$:

$$\dot{x}_k \approx m_k, \qquad \dot{m}_k \approx 0, \qquad \dot{s}_k \approx -s_k^2.$$

Consequently, if the separation is large enough at $t = 0$, then the shocks (which decay like $1/t$) will be very small by the time two shockpeakons come close. At least in the case with all $m_k > 0$ and all shocks sufficiently small, one expects the interaction to be virtually undistinguishable from normal peakon dynamics. This is verified by numerical experiments.

On the other hand, interactions where s_k is large compared to m_k can behave very differently. For example, the property $m_k = 0$ is not preserved by the ODEs, hence there is nothing to stop the m_k's from changing sign. In other words, the distinction between peakons and antipeakons is not so clear when shocks are present.

We have not been able to prove much about the shockpeakon ODEs in general, even for the $n = 2$ case (2.8). Studying collisions numerically is a nontrivial problem since the solutions can be rather badly behaved, so here we will restrict ourselves to the some simple cases where progress can be made analytically.

Two shockpeakons, symmetric case

Consider the reduction of the $n = 2$ shockpeakon ODEs (2.8) obtained by choosing the variables so that $u(-x,t) = -u(x,t)$:

$$-x_1 = x_2 =: \xi > 0, \qquad -m_1 = m_2 =: \mu, \qquad s_1 = s_2 =: \sigma > 0. \tag{4.2}$$

Inserting this into (2.8) yields

$$\begin{aligned}
\dot{\xi} &= \mu(1-R) - \sigma R, \\
\dot{\mu} &= -2(\mu+\sigma)^2 R, \\
\dot{\sigma} &= -\sigma^2(1+R) - \mu\sigma R,
\end{aligned} \tag{4.3}$$

where

$$R = e^{x_1 - x_2} = e^{-2\xi} \in (0,1). \tag{4.4}$$

We impose $\sigma > 0$ because of the entropy condition (cf. Theorem 2.3), and this is clearly preserved by the equations.

If the shock strength σ is small enough not to be influential, one expects the shockpeakons to collide at $x = 0$ if $\mu(0) < 0$, and to drift apart to $\pm\infty$ with asymptotically constant speed if $\mu(0) > 0$. The question is whether something else can happen if σ is large.

Now it turns out that the reduced system (4.3) admits the constant of motion

$$K = \mu(1-R) - 2\sigma R. \tag{4.5}$$

(We have not been able to find any corresponding constant of motion for the non-reduced system (2.8).) If $K > 0$ we can think of it as being the asymptotic speed, since if indeed $\xi \to \infty$ as $t \to \infty$, then $R \to 0$ and $\mu \to K$, provided σ is bounded. In that case, $\dot{\xi} \approx K$ for large t. However, if $K < 0$ this would contradict $\xi \to \infty$, which leads one to suspect that $K = 0$ is the breaking point where σ is large enough (compared to μ) to change the dynamics qualitatively.

To investigate this closer, we eliminate μ from (4.3) using (4.5). This yields

$$\dot{\xi} = K + \sigma R, \qquad \dot{\sigma} = -\frac{\sigma^2(1+R^2) + K\sigma R}{1-R} \qquad (R = e^{-2\xi}). \tag{4.6}$$

We consider three separate cases.

The case $K = 0$

This case represents a delicate balance between $\mu(0)$ and $\sigma(0)$, and can only happen if $\mu(0) > 0$. The equations become separable,

$$\frac{d\sigma}{d\xi} = \frac{\dot{\sigma}}{\dot{\xi}} = -\sigma\,\frac{1+R^2}{R(1-R)},$$

which gives

$$\sigma = L\exp(-\tfrac{1}{2}e^{2\xi})\frac{e^{-\xi}}{1-e^{-2\xi}}, \tag{4.7}$$

with $L > 0$ a constant of integration. (Expressed differently, $L = \sigma \exp(\frac{1}{2}e^{2\xi})(e^{\xi} - e^{-\xi})$ is a constant of motion when $K = 0$.) Now $r = e^{\xi} = R^{-1/2}$ satisfies $\dot{r} = r\dot{\xi} = r\sigma r^{-2} = L \exp(-r^2/2)/(r^2 - 1)$, so $\xi(t)$ is given implicitly by

$$Lt = \int_{e^{\xi(0)}}^{e^{\xi(t)}} (r^2 - 1) e^{r^2/2}\, dr. \tag{4.8}$$

This result shatters any hope for explicit solution formulas (or constants of motion) as simple as those for the shockless case. It follows that $\xi(t) \to \infty$ (although very slowly since $\dot{\xi} \to K = 0$), $\mu(t) \to 0$, and $\sigma(t) \to 0$ (at least as fast as $1/t$, since $(1/\sigma)^{\cdot} = (1 + R^2)/(1 - R) \to 1^+$).

The case $K > 0$

For $K \neq 0$ we have not been able to integrate (4.6). But when $K > 0$, which happens if $\mu(0) > 0$ and $\sigma(0)$ is relatively small, it is immediately seen that $\dot{\xi} > K > 0$ and $\dot{\sigma} < 0$, so that σ is bounded and the scenario outlined after (4.5) indeed takes place: $\xi(t) \to \infty$ with asymptotically constant speed $\dot{x} \to K$, $\mu(t) \to K$, and $\sigma(t) \to 0$ (at least as fast as $1/t$).

The case $K < 0$

This case happens if $\mu(0) \leq 0$, or (more interestingly) if $\mu(0) > 0$ and $\sigma(0)$ is relatively large. Since the equations are singular along the axis $\xi = 0$, we consider first the nonsingular auxiliary system obtained by multiplying the right-hand side of (4.6) by $1 - R$,

$$\dot{\xi} = (K + \sigma R)(1 - R), \qquad \dot{\sigma} = -\sigma^2(1 + R^2) - K\sigma R \qquad (R = e^{-2\xi}). \tag{4.9}$$

The phase portrait of (4.9) is shown in Figure 7. It is clear that all orbits of (4.9) starting in the first quadrant approach the stable equilibrium $(\xi, \sigma) = (0, |K|/2)$ as $t \to \infty$. Since $1 - R > 0$ in the first quadrant, our original system (4.6) follows the orbits of (4.9) with the same direction (but different speed). Hence the orbits of (4.6) starting in the first quadrant also tend to $(\xi, \sigma) = (0, |K|/2)$, with the difference that now this point is reached in finite time. Indeed, if $\sigma \geq 3|K|/4$ and $\xi > 0$, then

$$\dot{\sigma} = -\frac{1 + R^2}{1 - R}\left(\sigma - \frac{|K| R}{1 + R^2}\right)\sigma < -1 \cdot \frac{|K|}{4} \cdot \frac{3|K|}{4},$$

which implies that eventually $\sigma < 3|K|/4$, and from then on $\dot{\xi} = \sigma R - |K| < -|K|/4$, which drives ξ to zero in finite time.

In other words, the two shockpeakons collide at $x = 0$ after finite time, merging into a single shockpeakon, and after that the entropy weak solution is given by the $n = 1$ shockpeakon equations (2.6). Since $u(x_2) = \dot{x}_2 = \dot{\xi} = K + \sigma R \to K/2$, we have $u(x_1^-) = -u(x_2^+) = -u(x_2) + \sigma \to -K$, so the shockpeakon that forms has momentum $\widetilde{m}_1 = 0$ and shock strenght $\widetilde{s}_1 = |K|$.

Finally, by (4.5) $\mu(t)$ is given by

$$\mu = \frac{K + 2\sigma R}{1 - R} = -2\sigma + \frac{K + 2\sigma}{1 - e^{-2\xi}} = -2\sigma + \frac{\sigma - (-K/2)}{\xi + O(\xi^2)},$$

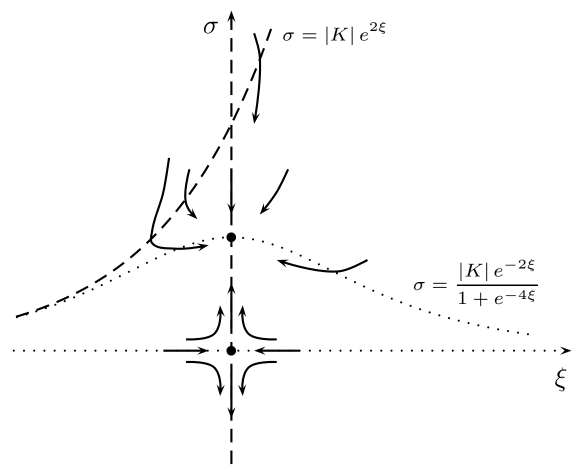

Figure 7: Phase portrait for the auxiliary system (4.9) in the case $K < 0$. The nullclines $\dot{\xi} = 0$ and $\dot{\sigma} = 0$ are drawn as dashed and dotted curves, respectively. There are two equilibria: the origin is a saddle point with Jacobian $J = \operatorname{diag}(2K, -K)$, while $(0, |K|/2)$ is an attracting star node with $J = \operatorname{diag}(K, K)$.

where both terms have finite limits at the collision. The limit of the second term equals the slope with which the orbit approaches the star node $(\xi, \sigma) = (0, |K|/2)$, and so depends on the initial conditions in some complicated manner, but the actual value does not affect the shape of the resulting shockpeakon, and is consequently of little interest here. We note that $\mu > 0$ corresponds to the region above the curve $\sigma = \frac{1}{2}|K|e^{2\xi}$ in the phase portrait, so that μ can be positive at the time of collision, but it is also possible for μ to change sign from positive to negative before the collision. (But not from negative to positive of course, since $\dot{\mu} \leq 0$ by (4.3).)

Remark 4.3. From the $n = 2$ shockpeakon ODEs (2.8) one sees that a positive s_2 increases the velocity of x_1 compared to the shockless case, while a positive s_1 slows x_2 down. Hence the shocks tend to draw the peakons together, at least in the short run. Over longer time scales, the effect of the shocks on the evolution of m_1 and m_2 makes the overall effect harder to predict. Nevertheless, this simple argument provides some explanation of the phenomenon observed above where shockpeakons initially moving apart are pulled back and collide provided that the shocks are large enough.

Remark 4.4. In the shockless case m_1 and m_2 blow up at a collision, as we know from Section 3. In the symmetric collision with shocks (the case $K < 0$), $\mu = -m_1 = m_2$ and $\sigma = s_1 = s_2$ both have finite limits, so this collision is less dramatic from the point of view of the ODEs. Nevertheless, we learn that the variables can in some cases change extremely quickly close to a collision, so numerical experiments must be performed with caution. For example, if one starts in the lower right of the first quadrant in the phase portrait in Figure 7, then σ will be small until the last moment, when the trajectory makes an abrupt turn and very rapidly approaches $(0, |K|/2)$ almost straight from below.

A triple collision

Another interesting reduction, due to Coclite, Karlsen and Risebro [8], is obtained by placing a stationary shockpeakon between a peakon and an antipeakon of equal strength:

$$\begin{aligned} -x_1 = x_3 =: \xi > 0, && x_2 = 0, \\ -m_1 = m_3 =: \mu, && m_2 = 0, \\ s_1 = s_3 = 0, && s_2 =: \sigma > 0. \end{aligned}$$

From the $n = 3$ shockpeakon ODEs one then obtains

$$\begin{aligned} \dot{\xi} &= \mu(1 - R^2) - \sigma R, \\ \dot{\mu} &= -2(\mu^2 R^2 + \mu\sigma R), \\ \dot{\sigma} &= -\sigma^2 - 2\mu\sigma R, \end{aligned} \tag{4.10}$$

where

$$R = e^{x_1 - x_2} = e^{x_2 - x_3} = e^{-\xi} \in (0, 1). \tag{4.11}$$

In [8] this is used as a test case with $\xi(0) = 5$, $\mu(0) = -1$, $\sigma(0) = 1$. For these initial data the system (4.10) is easily integrated numerically, which reveals that ξ decreases with nearly constant speed[5] and reaches zero at $t = t_0 \approx 5.32$, at which time $\mu \approx -1.10$ and $\sigma \approx 0.80$. After this triple collision, the entropy weak solution is given by a single decaying shockpeakon at $x = 0$, with initial shock strenght $\widetilde{s}_1(t_0) = \sigma(t_0) \approx 0.80$. (Since $\mu(t_0)$ is finite, the peakon and the antipeakon cancel out and do not affect the strength of the resulting shockpeakon.) Note that the collision would have taken place exactly at $t = 5$ without the shock. Thus it takes longer for the peakons to collide when the shock is present, which serves as a warning that one cannot predict global dynamics based on the attracting effect of shocks described in Remark 4.3. However, the phenomenon with shockpeakons moving apart and being pulled back again can be (numerically) observed here as well, if one takes $\mu(0) > 0$ and starts with a sufficiently large shock $\sigma(0)$.

5 Concluding remarks

The continued study of the Degasperis–Procesi equation should provide interesting insights into the interplay between complete integrability and the theory of weak solutions of conservation laws. In this initial work we have used very elementary methods to investigate the formation and dynamics of shocks in simple cases. Clearly, more refined tools are required to understand these phenomena in greater generality.

For example, the Camassa–Holm peakon-antipeakon solutions can be analyzed for any n using the highly developed machinery of orthogonal polynomials, because the CH Lax pair is related to the discrete string, Padé approximation, Stieltjes continued fractions, the classical moment problem, etc. [2]. Something similar would be needed in order to fully analyze the Degasperis–Procesi

[5] Note that $\ddot{\xi} = -\sigma\mu R(1 - R)^2 + \sigma^2 R(1 - R)$ is small if σ is small or if ξ is either large or small.

solution formulas, but the discrete cubic string and the associated Padé-like approximation problem appearing in the solution of the inverse spectral problem [19] are much more involved, and the corresponding theory is only in its infancy.

The shockpeakon ODEs (2.1) introduced in this paper also need to be better understood. The DP Lax pair due to Degasperis–Holm–Hone, which was the key to solving the usual DP peakon ODEs, involves $m = u - u_{xx}$ and is compatible with the DP equation in the form (1.1). Hence it does not seem to be of much help in understanding the shockpeakons, for which (1.1) does not make sense and the weak formulation (1.15) must be used instead. A systematic numerical study of the shockpeakon ODEs might be useful, but we have not attempted that here since it is a nontrivial problem to handle the collisions, where the variables can behave badly and the number of shockpeakons change.

The numerical experiments in [8] give indications that shockpeakons are stable, but this has yet to be proved.

Finally, it is of course of interest to study other weak solutions than shockpeakons. We hope that the examples presented here can provide useful intuition about what kind of blowup behaviour one can expect from the DP equation in general.

A Appendix: Proof of Theorem 2.1

Proof. Equation (1.15) reads $u_t + \frac{1}{2}(u^2)_x + P_x = 0$, where $P = \frac{1}{2}G * \frac{3}{2}u^2$ and thus $P_x = \frac{1}{2}G' * \frac{3}{2}u^2$. This is to be satisfied in the space of distributions $\mathcal{D}'(\mathbf{R} \times [0,\infty))$, which means that

$$\int_0^\infty \int_{\mathbf{R}} \big(u\,\phi_t + \tfrac{1}{2}u^2\phi_x - P_x\phi\big)\,dx\,dt + \int_{\mathbf{R}} u_0(x)\,\phi(x,0)\,dx = 0 \tag{A.1}$$

for any test function $\phi(x,t) \in C_c^\infty(\mathbf{R}\times[0,\infty))$. As explained in [7, Remark 3.3], one can interpret the initial conditions in the L^1 sense, $\|u(\cdot,t) - u_0\|_{L^1} \to 0$ as $t \to 0^+$, and then simplify by restricting the choice of test functions to those satisfying $\phi(x,0) = 0$.

Moreover, here we will deal with functions of the form $u(x,t) = f\big(x - x_0(t)\big)$, with $f(x)$ arbitrary and $x_0(t)$ differentiable. For such functions, the distributional partial derivative with respect to t is given by

$$\begin{aligned}
\langle u_t, \phi\rangle &= -\int_0^\infty \int_{\mathbf{R}} f\big(x - x_0(t)\big)\,\phi_t(x,t)\,dx\,dt \\
&= -\int_0^\infty \int_{\mathbf{R}} f(x)\,\phi_t\big(x + x_0(t), t\big)\,dx\,dt \\
&= -\int_0^\infty \int_{\mathbf{R}} f(x)\left(\frac{d}{dt}\phi\big(x + x_0(t), t\big) - \dot{x}_0(t)\phi_x\big(x + x_0(t), t\big)\right)dx\,dt \\
&= \int_0^\infty \left(\dot{x}_0(t)\int_{\mathbf{R}} f\big(x - x_0(t)\big)\,\phi_x(x,t)\,dx\right)dt.
\end{aligned}$$

The inner integral can be viewed as the action of the distribution $-f'\big(x - x_0(t)\big) \in \mathcal{D}'(\mathbf{R})$ on a test function $\phi(\cdot,t) \in C_c^\infty(\mathbf{R})$ which depends only parametrically on t. Here of course $f'(x)$ means the distributional derivative of $f(x)$ in $\mathcal{D}'(\mathbf{R})$. With this interpretation of our functions being distributions with respect to x for each fixed t, we will identify u_t with $-\dot{x}_0(t)f'\big(x - x_0(t)\big)$. Similary, for functions of the form $u(x,t) = m_0(t)\, f\big(x - x_0(t)\big)$ it can be verified that the chain rule holds and thus the identification $u_t = \dot m_0 f(x-x_0) - m_0 \dot x_0 f'(x-x_0)$ can be made.

This means that we can simplify the calculations by computing $u_t + \frac{1}{2}(u^2)_x + P_x$ as a distribution in the variable x with the rules of distributional calculus, without having to involve test functions and double integrals explicitly. Abbreviating $G(x - x_k(t))$ as G_k, the ansatz (2.1) reads $u = \sum_{k=1}^n (m_k G_k + s_k G_k')$, so according to the remarks above we have at once

$$u_t = \sum_{k=1}^n \Big(\dot m_k G_k - m_k \dot x_k G_k' + \dot s_k G_k' - s_k \dot x_k (G_k - 2\delta_k) \Big), \tag{A.2}$$

where $\delta_k = \delta(x - x_k(t))$. Furthermore,

$$u^2 = \sum_{k,l=1}^n \Big(m_k m_l G_k G_l + s_k s_l G_k' G_l' + m_k s_l G_k G_l' + s_k m_l G_k' G_l \Big), \tag{A.3}$$

hence

$$\begin{aligned} (u^2)_x = \sum_{k,l=1}^n \Big(&(m_k m_l + s_k s_l)(G_k G_l' + G_k' G_l) \\ &+ (m_k s_l + s_k m_l)(G_k G_l + G_k' G_l') \\ &- 2\big[s_k s_l (G_k' \delta_l + G_l' \delta_k) + m_k s_l G_k \delta_l + m_l s_k G_l \delta_k \big] \Big). \end{aligned} \tag{A.4}$$

We also need to compute $G' * u^2$, which is more tedious. When dealing with a term like $G_k G_l$ it is natural to view the the real line as split into three intervals by the points x_k and x_l (or two intervals if $x_k = x_l$), and write the term as a sum of functions with support in the respective intervals. Let $\chi_I(x)$ be 1 if $x \in I$ and 0 otherwise, and let $i \le j$ be the unique indices such that $\min(x_k, x_l) = x_i$ and $\max(x_k, x_l) = x_j$. Then

$$\begin{aligned} G_k G_l &= a_{kl} + b_{kl} + c_{kl}, \\ G_k' G_l' &= a_{kl} - b_{kl} + c_{kl}, \\ G_i G_j' &= a_{kl} + b_{kl} - c_{kl}, \\ G_i' G_j &= a_{kl} - b_{kl} - c_{kl}, \end{aligned} \tag{A.5}$$

with

$$\begin{aligned} a_{kl}(x) &= \frac{G_k G_l + G_k' G_l' + G_k G_l' + G_k' G_l}{4} = e^{2x - x_k - x_l} \chi_{(-\infty, x_i)}(x), \\ b_{kl}(x) &= \frac{G_k G_l - G_k' G_l'}{2} = \frac{G_i G_j' - G_i' G_j}{2} = e^{-|x_k - x_l|} \chi_{(x_i, x_j)}(x), \\ c_{kl}(x) &= \frac{G_k G_l + G_k' G_l' - G_k G_l' - G_k' G_l}{4} = e^{-2x + x_k + x_l} \chi_{(x_j, \infty)}(x), \end{aligned}$$

(These equalities hold pointwise except maybe at the points x_k and x_l, hence they hold in the distributional sense.) Now,

$$(G' * a_{kl})(x) = \int_{\mathbf{R}} G'(x-y) e^{2y - x_i - x_j} \chi_{(-\infty, x_i)}(y)\, dy$$

equals

$$e^{-x_i-x_j}\left(\int_{-\infty}^{x}(-e^{y-x})e^{2y}\,dy+\int_{x}^{x_i}e^{x-y}e^{2y}\,dy\right)$$

if $x<x_i$, and

$$e^{-x_i-x_j}\int_{-\infty}^{x_i}(-e^{y-x})e^{2y}\,dy$$

if $x_i<x$. We evaluate the integrals and collect the results to obtain

$$\begin{aligned}(G'*a_{kl})(x)&=(-\tfrac{4}{3}e^{2x-x_i-x_j}+e^{x-x_j})\chi_{(-\infty,x_i)}(x)-\tfrac{1}{3}e^{-x+2x_i-x_j}\chi_{(x_i,\infty)}(x)\\&=-\tfrac{4}{3}a_{kl}+e^{x_i-x_j}\frac{G_i+G_i'}{2}-\tfrac{1}{3}e^{x_i-x_j}\frac{G_i-G_i'}{2}.\end{aligned}$$

Similar computations apply to $G'*b_{kl}$ and $G'*c_{kl}$. The results are

$$\begin{aligned}G'*a_{kl}&=\tfrac{1}{3}e^{-|x_k-x_l|}(G_i+2G_i')-\tfrac{4}{3}a_{kl},\\G'*b_{kl}&=e^{-|x_k-x_l|}(G_i-G_j),\\G'*c_{kl}&=\tfrac{1}{3}e^{-|x_k-x_l|}(-G_j+2G_j')+\tfrac{4}{3}c_{kl}.\end{aligned}\tag{A.6}$$

It follows from (A.5) and (A.6) that

$$\begin{aligned}G'*(G_kG_l)&=\tfrac{2}{3}e^{-|x_k-x_l|}(2G_i-2G_j+G_i'+G_j')-\tfrac{4}{3}(a_{kl}-c_{kl}),\\G'*(G_k'G_l')&=\tfrac{2}{3}e^{-|x_k-x_l|}(-G_i+G_j+G_i'+G_j')-\tfrac{4}{3}(a_{kl}-c_{kl}),\\G'*(G_iG_j')&=\tfrac{2}{3}e^{-|x_k-x_l|}(2G_i-G_j+G_i'-G_j')-\tfrac{4}{3}(a_{kl}+c_{kl}),\\G'*(G_i'G_j)&=\tfrac{2}{3}e^{-|x_k-x_l|}(-G_i+2G_j+G_i'-G_j')-\tfrac{4}{3}(a_{kl}+c_{kl}),\end{aligned}$$

and consequently

$$\begin{aligned}G'*(G_kG_l)&=\tfrac{2}{3}e^{-|x_k-x_l|}(2\operatorname{sgn}(x_l-x_k)(G_k-G_l)+G_k'+G_l')\\&\quad-\tfrac{2}{3}(G_kG_l'+G_k'G_l),\\G'*(G_k'G_l')&=\tfrac{2}{3}e^{-|x_k-x_l|}(-\operatorname{sgn}(x_l-x_k)(G_k-G_l)+G_k'+G_l')\\&\quad-\tfrac{2}{3}(G_kG_l'+G_k'G_l),\\G'*(G_kG_l')&=\tfrac{2}{3}e^{-|x_k-x_l|}\big(2G_k-G_l+\operatorname{sgn}(x_l-x_k)(G_k'-G_l')\big)\\&\quad-\tfrac{2}{3}(G_kG_l+G_k'G_l'),\\G'*(G_k'G_l)&=\tfrac{2}{3}e^{-|x_k-x_l|}\big(-G_k+2G_l+\operatorname{sgn}(x_l-x_k)(G_k'-G_l')\big)\\&\quad-\tfrac{2}{3}(G_kG_l+G_k'G_l').\end{aligned}$$

Applying this to (A.3) we obtain

$$
\begin{aligned}
2P_x = G' * \tfrac{3}{2}u^2 = \sum_{k,l=1}^{n} \Big(&-(m_k m_l + s_k s_l)(G_k G_l' + G_k' G_l) \\
&- (m_k s_l + s_k m_l)(G_k G_l + G_k' G_l') \Big) \\
+ \sum_{k,l=1}^{n} e^{-|x_k - x_l|} \Big(&(m_k m_l + s_k s_l)(G_k' + G_l') \\
&+ m_k s_l (2G_k - G_l) + s_k m_l (-G_k + 2G_l) \Big) \\
+ \sum_{k,l=1}^{n} \operatorname{sgn}(x_l - x_k) e^{-|x_k - x_l|} \Big(&(2m_k m_l - s_k s_l)(G_k - G_l) \\
&+ (m_k s_l + s_k m_l)(G_k' - G_l') \Big).
\end{aligned}
\tag{A.7}
$$

When adding (A.4) and (A.7), all terms of type GG, GG' and $G'G'$ cancel out. We add on (A.2) as well, and swap the labels k and l in some terms in order to collect the remaining x-dependent ingredients $\delta_k(x)$, $G_k(x)$, and $G_k'(x)$:

$$
\begin{aligned}
&2u_t + (u^2)_x + 2P_x \\
&= 2\sum_{k=1}^{n} \Big((\dot m_k - s_k \dot x_k) G_k + (\dot s_k - m_k \dot x_k) G_k' + 2 s_k \dot x_k \delta_k \Big) \\
&- 4\sum_{k=1}^{n} \Big(\sum_{l=1}^{n} s_l G_l'(x_k) + \sum_{l=1}^{n} m_l G_l(x_k) \Big) s_k \delta_k \\
&+ \sum_{k=1}^{n} \Big(2m_k \sum_{l=1}^{n} m_l e^{-|x_k - x_l|} + 2s_k \sum_{l=1}^{n} s_l e^{-|x_k - x_l|} \Big) G_k' \\
&+ \sum_{k=1}^{n} \Big(4m_k \sum_{l=1}^{n} s_l e^{-|x_k - x_l|} - 2s_k \sum_{l=1}^{n} m_l e^{-|x_k - x_l|} \Big) G_k \\
&+ \sum_{k=1}^{n} \Big(4m_k \sum_{l=1}^{n} m_l \operatorname{sgn}(x_l - x_k) e^{-|x_k - x_l|} \\
&\qquad - 2s_k \sum_{l=1}^{n} s_l \operatorname{sgn}(x_l - x_k) e^{-|x_k - x_l|} \Big) G_k \\
&+ \sum_{k=1}^{n} \Big(2m_k \sum_{l=1}^{n} s_l \operatorname{sgn}(x_l - x_k) e^{-|x_k - x_l|} \\
&\qquad + 2s_k \sum_{l=1}^{n} m_l \operatorname{sgn}(x_l - x_k) e^{-|x_k - x_l|} \Big) G_k'.
\end{aligned}
$$

The DP equation (1.15) requires the above to equal zero, that is

$$\begin{aligned}
0 &= 4\sum_{k=1}^{n} s_k\Big(\dot{x}_k - u(x_k)\Big)\delta_k \\
&+ 2\sum_{k=1}^{n}\Big(\dot{m}_k + 2m_k\{u_x(x_k)\} - 2s_k\, u(x_k) - s_k\big(\dot{x}_k - u(x_k)\big)\Big)G_k \\
&+ 2\sum_{k=1}^{n}\Big(\dot{s}_k + s_k\{u_x(x_k)\} - m_k\big(\dot{x}_k - u(x_k)\big)\Big)G'_k,
\end{aligned}$$

which is equivalent to (2.3) since $\{\delta_k, G_k, G'_k\}_{k=1}^{n}$ is a linearly independent set.
□

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