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Around The Berge Problem And Hadwiger Conjecture

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Abstract: We say that a graph B is *berge*, if every graph $B' \in \{B, \bar{B}\}$ does not contain an induced cycle of odd length ≥ 5 [$\bar{B}$ is the complementary graph of B]. A graph G is *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$, where $\chi(G')$ is the *chromatic number* of G' and $\omega(G')$ is the *clique number* of G'. The Berge conjecture states that a graph H is *perfect* if and only if H is *berge*. Indeed, the difficult part of the Berge conjecture consists to show that $\chi(B) = \omega(B)$ for every *berge* graph B. The Hadwiger conjecture states that every graph G satisfies $\chi(G) \leq \eta(G)$ [where $\eta(G)$ is the *hadwiger number* of G (i.e., the maximum of p such that G is *contractible* to the complete graph K_p)]. The Berge conjecture (see [1] or [2] or [3] or [5] or [6] or [7] or [9] or [10] or [11]) was proved by Chudnovsky, Robertson, Seymour and Thomas in a paper of at least 140 pages (see [1]), and an elementary proof of the Berge conjecture was given by Ikorong Nemron in a detailled paper of 37 pages long (see [9]). The Hadwiger conjecture (see [4] or [5] or [7] or [8] or [10] or [11] or [12] or [13] or [15] or [16]) was proved by Ikorong Nemron in a detailled paper of 28 pages long (see [13]), by using arithmetic calculus, arithmetic congruences, elementary complex analysis, induction and reasoning by reduction to absurd. That being so, in this paper, via two simple Theorems, we rigorously show that the difficult part of the Berge conjecture (solved) and the Hadwiger conjecture (also solved), are exactly the same conjecture. The previous immediately implies that, the Hadwiger conjecture is only a non obvious special case of the Berge conjecture.

Key Words: True pal, parent, berge, the berge problem, the berge index, representative, the hadwiger index, son.

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§0. Preliminary and Some Denotations

We recall that in a graph $G = [V(G), E(G), \chi(G), \omega(G), \bar{G}]$, $V(G)$ is the set of vertices, $E(G)$ is the set of edges, $\chi(G)$ is the chromatic number, $\omega(G)$ is the clique number and $\bar{G}$ is the complementary graph of G. We say that a graph B is *berge* if every $B' \in \{B, \bar{B}\}$ does not contain an induced cycle of odd length ≥ 5. A graph G is *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. The Berge conjecture states that a graph H is *perfect* if

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and only if H is berge. Indeed the difficult part of the Berge conjecture consists to show that $\chi(B) = \omega(B)$ for every berge graph B. Briefly, the difficult part of the Berge conjecture will be called the Berge problem. In this topic, we rigorously show that the Berge problem and the Hadwiger conjecture are exactly the same problem [the Hadwiger conjecture states that every graph G is $\eta(G)$ colorable (i.e. we can color all vertices of G with $\eta(G)$ colors such that two adjacent vertices do not receive the same color). $\eta(G)$ is the hadwiger number of G and is the maximum of p such that G is contractible to the complete graph K_p]. That being so, this paper is divided into six simple Sections. In Section 1, we present briefly some standard definitions known in Graph Theory. In Section 2 , we introduce definitions that are not standard, and some elementary properties. In Section 3 we define a graph parameter denoted by β (β is called the berge index) and we give some obvious properties of this parameter. In Section 4 we introduce another graph parameter denoted by τ (τ is called the *hadwiger index*) and we present elementary properties of this parameter. In Section 5 , using the couple (β, τ), we show two simple Theorems which are equivalent to the Hadwiger conjecture and the Berge problem. In Section 6, using the two simple Theorems stated and proved in Section 5, we immediately deduce that the Berge problem and the Hadwiger conjecture are exactly the same problem, and therefore, the Hadwiger conjecture is only a non obvious special case of the Berge conjecture. In this paper, all results are simple, and every graph is finite, is simple and is undirected. We start.

§1. Standard Definitions Known in Graph Theory

Recall (see [2] or [14]) that in a graph $G = [V(G), E(G)]$, $V(G)$ is the set of vertices and $E(G)$ is the set of edges. $\bar{G}$ is the complementary graph of G (recall $\bar{G}$ is the *complementary* graph of G, if $V(G) = V(\bar{G})$ and two vertices are adjacent in G if and only if they are not adjacent in $\bar{G}$). A graph F is a *subgraph* of G, if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. We say that a graph F is an *induced subgraph* of G by Z, if F is a subgraph of G such that $V(F) = Z$, $Z \subseteq V(G)$, and two vertices of F are adjacent in F, if and only if they are adjacent in G. For $X \subseteq V(G)$, $G \setminus X$ denotes the *subgraph* of G induced by $V(G) \setminus X$. A *clique* of G is a subgraph of G that is complete; such a subgraph is necessarily an induced subgraph (recall that a graph K is complete if every pair of vertices of K is an edge of K); $\omega(G)$ is the size of a largest clique of G, and $\omega(G)$ is called the *clique number* of G. A **stable set** of a graph G is a set of vertices of G that induces a subgraph with no edges; $\alpha(G)$ is the size of a largest stable set, and $\alpha(G)$ is called the *stability number* of G. The *chromatic number* of G (denoted by $\chi(G)$) is the smallest number of colors needed to color all vertices of G such that two adjacent vertices do not receive the same color. It is easy to see:

Assertion 1.0 *Let G be a graph. Then* $\omega(G) \leq \chi(G)$

The *hadwiger number* of a graph G (denoted by $\eta(G)$), is the maximum of p such that G is contractible to the complete graph K_p. Recall that, if e is an edge of G incident to x and y, we can obtain a new graph from G by removing the edge e and identifying x and y so that the resulting vertex is incident to all those edges (other than e) originally incident to x or to y. This is called *contracting* the edge e. If a graph F can be obtained from G by a succession of such edge-contractions, then, G is *contractible* to F. The maximum of p such that G is contractible to the complete graph K_p is the hadwiger number of G, and is denoted by $\eta(G)$. The Hadwiger conjecture states that $\chi(G) \leq \eta(G)$ for every graph G. Clearly we have:

Assertion 1.1 *Let G be a graph, and let F be a subgraph of G. Then $\eta(F) \leq \eta(G)$.*

§2. Non-Standard Definitions and Some Elementary Properties

In this section, we introduce definitions that are not standard. These definitions are crucial for the two theorems which we will use in Section 6 to show that the Berge problem and the Hadwiger conjecture are exactly the same problem. We say that a graph B is *berge*, if every $B' \in \{B, \bar{B}\}$ does not contain an induced cycle of odd length ≥ 5. A graph G is *perfect*, if every induced subgraph G' of G is $\omega(G')$-colorable. The Berge conjecture states that a graph G is perfect if and only if G is berge. Indeed, the Berge problem (i.e. the difficult part of the Berge conjecture, see Preliminary of this paper) consists to show that $\chi(B) = \omega(B)$, for every berge graph B. We will see in Section6 that the Berge problem and the Hadwiger conjecture are exactly the same problem.

We say that a graph G is a *true pal* of a graph F, if F is a subgraph of G and $\chi(F) = \chi(G)$; *trpl*(F) denotes the set of all true pals of F (so, $G \in$ *trpl*(F) means G is a *true pal* of F).

Recall that a set X is a *stable* subset of a graph G, if $X \subseteq V(G)$ and if the subgraph of G induced by X has no edges. A graph G is a *complete $\omega(G)$-partite* graph (or a *complete multipartite* graph), if there exists a partition $\Xi(G) = \{Y_1, \cdots, Y_{\omega(G)}\}$ of $V(G)$ into $\omega(G)$ stable sets such that $x \in Y_j \in \Xi(G)$, $y \in Y_k \in \Xi(G)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in G. *It is immediate that $\chi(G) = \omega(G)$, for every complete $\omega(G)$-partite graph.* Ω denotes the set of graphs G which are complete $\omega(G)$-partite. So, $G \in \Omega$ means G is a complete $\omega(G)$-partite graph. Using the definition of Ω, then the following Assertion becomes immediate.

Assertion 2.0 *Let $H \in \Omega$ and let F be a graph. Then we have the following two properties.*

(2.0.0) $\chi(H) = \omega(H)$;

(2.0.1) There exists a graph $P \in \Omega$ such that P is a true pal of F.

Proof Property (2.0.0) is immediate (use definition of Ω and note $H \in \Omega$). Property (2.0.1) is also immediate. Indeed, let F be graph and let $\Xi(F) = \{Y_1, \cdots, Y_{\chi(F)}\}$ be a partition of $V(F)$ into $\chi(F)$ stable sets (it is immediate that such a partition $\Xi(F)$ exists). Now let Q be a graph defined as follows: (i) $V(Q) = V(F)$; (ii) $\Xi(Q) = \{Y_1, ..., Y_{\chi(F)}\}$ is a partition of $V(Q)$ into $\chi(F)$ stable sets such that $x \in Y_j \in \Xi(Q)$, $y \in Y_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in Q. Clearly $Q \in \Omega$, $\chi(Q) = \omega(Q) = \chi(F)$, and F is visibly a subgraph of Q; in particular Q is a true pal of F such that $Q \in \Omega$ (because F is a subgraph of Q and $\chi(Q) = \chi(F)$ and $Q \in \Omega$). Now putting $Q = P$, the property (2.0.1) follows. $\square$

So, we say that a graph P is a *parent* of a graph F, if $P \in \Omega \bigcap$ *trpl*(F). In other words, P is a *parent* of F, if P is a complete $\omega(P)$-partite graph and P is also a true pal of F (observe that such a P exists, via property (2.0.1) of Assertion 2.0). $parent(F)$ denotes the set of all parents of a graph F (so, $P \in parent(F)$ means P is a parent of F). Using the definition of a parent, then the following Assertion is immediate.

Assertion 2.1 *Let F be a graph and let $P \in parent(F)$. We have the folowing two properties.*

(2.1.0) Suppose that $F \in \Omega$. Then $\chi(F) = \omega(F) = \omega(P) = \chi(P)$;
(2.1.1) Suppose that $F \notin \Omega$. Then $\chi(F) = \omega(P) = \chi(P)$.

§3. The Berge Index of a Graph

In this section, we define a graph parameter called the berge index and we define a representative of a graph; we also give some elementary properties concerning the berge index. We recall that a graph B is berge, if every $B' \in \{B, \bar{B}\}$ does not contain an induced cycle of odd length ≥ 5. A graph G is perfect, if every induced subgraph G' of G is $\omega(G')$-colorable. The Berge conjecture states that a graph G is perfect if and only if G is berge. Indeed the Berge problem, consists to show that $\chi(B) = \omega(B)$ for every berge graph B. Using the definition of a berge graph and the definition of Ω the following assertion becomes immediate.

Assertion 3.0 *Let $G \in \Omega$. Then, G is berge.*

Assertion 3.0 says that the set Ω is an obvious example of berge graphs. Now, we define the berge index of a graph G. Let G be a graph. Then the berge index of G (denoted by $\beta(G)$) is defined in the following two cases (namely case where $G \in \Omega$ and case where $G \notin \Omega$).

First, we define the berge index of G in the case where $G \in \Omega$.

Case *i* Suppose that $G \in \Omega$, and put $\mathcal{B}(G) = \{F; G \in parent(F) \text{ and } F \textit{ is berge}\}$; clearly $\mathcal{B}(G)$ is the set of graphs F such that G is a parent of F and F is berge. Then, $\beta(G) = \min_{F \in \mathcal{B}(G)} \omega(F)$. In other words, $\beta(G) = \omega(F'')$, where $F'' \in \mathcal{B}(G)$, and $\omega(F'')$ is *minimum* for this property.

We prove that such a β clearly exists via the following remark.

Remark *i* Suppose that $G \in \Omega$. Then, the berge index $\beta(G)$ exists. Indeed put $\mathcal{B}(G) = \{F; G \in parent(F) \text{ and } F \textit{ is berge}\}$. Recall $G \in \Omega$, so G is berge (use Assertion 3.0); clearly $G \in \mathcal{B}(G)$, so $\min_{F \in \mathcal{B}(G)} \omega(F)$ exists, and the previous clearly says that $\beta(G)$ exists.

Now, we define the berge index of G, in the case where $G \notin \Omega$.

Case *ii* Suppose that $G \notin \Omega$ and let $parent(G)$ be the set of all parents of G. Then, $\beta(G) = \min_{P \in parent(G)} \beta(P)$. In other words, $\beta(G) = \beta(P'')$, where $P'' \in parent(G)$, and $\beta(P'')$ is *minimum* for this property.

We prove that such a β clearly exists, via the following remark.

Remark *ii* Suppose that $G \notin \Omega$. Then, the berge index $\beta(G)$ exists. Indeed, let $P \in \Omega$ such that P is a true pal of G [such a P exists (use property (2.0.1) of Assertion 2.0)], clearly $P \in parent(G)$; note $P \in \Omega$, and Remark.*(i)* implies that $\beta(P)$ exists. So $\min_{P \in parent(G)} \beta(P)$ exists, and clearly $\beta(G)$ also exists.

Remark *iii* Let G be a graph. Then the berge index $\beta(G)$ exists. In fact, applying Remark i if $G \in \Omega$, and Remark ii if $G \notin \Omega$, we get the conclusion.

To conclude, note that the berge index of a graph G is $\beta(G)$, where $\beta(G)$ is defined as follows.

$\beta(G) = \min_{F \in \mathcal{B}(G)} \omega(F)$ if $G \in \Omega$; and $\beta(G) = \min_{P \in parent(G)} \beta(P)$ if $G \notin \Omega$. Recall $\mathcal{B}(G) = \{F; G \in parent(F) \text{ and } F \textit{ is berge}\}$, and $parent(G)$ is the set of all parents of G.

We recall (see Section 1) that $\eta(G)$ is the hadwiger number of G, and we clearly have.

Proposition 3.1 *Let K be a complete graph and let $G \in \Omega$. Then, we have the following three properties.*

(3.1.0) *If $\omega(G) \leq 1$, then $\beta(G) = \omega(G) = \chi(G) = \eta(G)$;*
(3.1.1) *$\beta(K) = \omega(K) = \chi(K) = \eta(K)$;*
(3.1.2) *$\omega(G) \geq \beta(G)$.*

Proof Property (3.1.0) is immediate. We prove property (3.1.1). Indeed let $\mathcal{B}(K) = \{F; K \in parent(F) \text{ and } F \textit{ is berge}\}$, recall K is complete, and clearly $\mathcal{B}(K) = \{K\}$; observe $K \in \Omega$, so $\beta(K) = \min_{F \in \mathcal{B}(K)} \omega(F)$ (use definition of parameter β and note $K \in \Omega$), and we easily deduce that $\beta(K) = \omega(K) = \chi(K)$. Note $\eta(K) = \chi(K)$ (since K is complete), and using the previous, we clearly have $\beta(K) = \omega(K) = \chi(K) = \eta(K)$. Property (3.1.1) follows.

Now we prove property (3.1.2). Indeed, let $\mathcal{B}(G) = \{F; G \in parent(F) \text{ and } F \textit{ is berge}\}$, recall $G \in \Omega$, and so $\beta(G) = \min_{F \in \mathcal{B}(G)} \omega(F)$ (use definition of parameter β and note $G \in \Omega$); observe G is berge (use Assertion 3.0), so $G \in \mathcal{B}(G)$, and the previous equality implies that $\omega(G) \geq \beta(G)$. □

Using the definition of the berge index, then we clearly have:

Proposition 3.2 *Let B be berge, and let $P \in parent(B)$. Then, $\beta(P) \leq \omega(B)$.*

Proof Let $\mathcal{B}(P) = \{F; P \in parent(F) \text{ and } F \textit{ is berge}\}$, clearly $B \in \mathcal{B}(P)$; observe $P \in \Omega$, so $\beta(P) = \min_{F \in \mathcal{B}(P)} \omega(F)$, and we immediately deduce that $\beta(P) \leq \omega(B)$. □

Now, we define a representative of a graph. Let G be a graph and let $\beta(G)$ be the berge index of G [observe $\beta(G)$ exists, by using Remark *iii*]; we say that a graph S is a *representative* of G if S is defined in the following two cases (namely case where $G \in \Omega$ and case where $G \notin \Omega$.

First, we define a *representative* of G in the case where $G \in \Omega$.

Case i' Suppose that $G \in \Omega$. Put $\mathcal{B}(G) = \{F; G \in parent(F) \text{ and } F \textit{ is berge}\}$. Then S is a *representative* of G, if $S \in \mathcal{B}(G)$ and $\omega(S) = \beta(G)$. In other words, S is a representative of G, if S is berge and $G \in parent(S)$ and $\omega(S) = \beta(G)$. In other terms again, S is a representative of G if S is berge, $G \in parent(S)$, and $\omega(S)$ is minimum for this property. Via Remarks i' and $i'.0$, we prove that such a S exists, and we have $\chi(S) = \chi(G) = \omega(G)$.

Remark i' Suppose that $G \in \Omega$. Then, there exists a graph S such that S is a representative of

G. Indeed, let $\beta(G)$ be the berge index of G, recalling that $G \in \Omega$, clearly $\beta(G) = \min_{F \in \mathcal{B}(G)} \omega(F)$, where $\mathcal{B}(G) = \{F; G \in parent(F) \text{ and } F \textit{ is berge}\}$ (use definition of parameter β and note $G \in \Omega$) ; now let $B \in \mathcal{B}(G)$ such that $\omega(B) = \beta(G)$ (such a B exists, since $\beta(G)$ exists (use Remark *iii*), clearly B is a representative of G. Now put $B = S$; then Remark i' clearly follows.

Remark $i'.0$ Suppose that $G \in \Omega$. Now let S be a representative of G (such a S exists, by using Remark i'). Then, $\chi(S) = \chi(G) = \omega(G)$. Indeed, let $\mathcal{B}(G) = \{F; G \in parent(F) \text{ and } F \textit{ is berge}\}$, and let S be a representative of G. Recall $G \in \Omega$, and clearly $S \in \mathcal{B}(G)$ (use definition of a representative and note $G \in \Omega$) ; so $G \in parent(S)$, and clearly $\chi(S) = \chi(G)$. Note $\chi(G) = \omega(G)$ (since $G \in \Omega$), and the last two equalities immediately imply that $\chi(S) = \chi(G) = \omega(G)$. Remark $i'.0$ follows.

Now, we define a representative of G, in the case where $G \notin \Omega$.

Case ii' Suppose that $G \notin \Omega$. Now let $parent(G)$ be the set of all parents of G, and let $P' \in parent(G)$ such that $\beta(P') = \beta(G)$ (observe that such a P' exists, since $G \notin \Omega$, and by using the definition of $\beta(G)$); put $\mathcal{B}(P') = \{F'; P' \in parent(F') \text{ and } F' \textit{ is berge}\}$. Then S is a *representative* of G if $S \in \mathcal{B}(P')$ and $\omega(S) = \beta(P') = \beta(G)$. In other words, S is a representative of G (recall $G \notin \Omega$), if S is berge and $P' \in parent(S)$ and $\omega(S) = \beta(P') = \beta(G)$ [where $P' \in parent(G)$ and $\beta(P') = \beta(G)$]. Via Remarks ii' and Remark $ii'.0$, we prove that such a S exists, and we have $\chi(S) = \chi(G)$.

Remark ii' Suppose that $G \notin \Omega$. Then, there exists a graph S such that S is a representative of G. Indeed, let $\beta(G)$ be the berge index of G, recalling that $G \notin \Omega$, clearly $\beta(G) = \min_{P \in parent(G)} \beta(P)$. Now, let $P' \in parent(G)$ such that $\beta(P') = \beta(G)$ [observe that such a P' exists, since $G \notin \Omega$, and by using the definition of $\beta(G)$]; note $P' \in \Omega$, and clearly $\beta(P') = \min_{F' \in \mathcal{B}(P')} \omega(F')$ (note $\mathcal{B}(P') = \{F'; P' \in parent(F') \text{ and } F' \textit{ is berge}\}$). Now, let $B' \in \mathcal{B}(P')$ such that $\omega(B') = \beta(P')$. Clearly B' is berge and $\omega(B') = \beta(P') = \beta(G)$. It is easy to see that B' is a representative of G. Now put $S = B'$, then Remark ii' follows.

Remark $ii'.0$ Suppose that $G \notin \Omega$. Now let S be a representative of G (such a S exists by using Remark ii'). Then $\chi(S) = \chi(G)$. Indeed, let S be a representative of G, and consider $P' \in parent(G)$ such that P' is a parent of S and $\beta(P') = \beta(G)$ (such a P' clearly exists, by observing that S be a representative of G, $G \notin \Omega$ and by using the definition of a representative of G), clearly $\chi(S) = \omega(P') = \chi(P') = \chi(G)$ (since P' is a parent of G and S). So $\chi(S) = \chi(G)$, and Remark.*(ii'.0)* follows.

Remark iii' Let G be a graph. Then, there exists a graph S such that S is a representative of G. Applying Remark i' if $G \in \Omega$ and applying Remark ii' if $G \notin \Omega$, we get the conclusion.

Remark iv Let G be a graph and let S be a representative of G (such a S exists, by using Remark iii'). Then, $\chi(G) = \chi(S)$. Applying Remark $i'.0$ if $G \in \Omega$, and Remark $ii'.0$ if $G \notin \Omega$, the conclusion follows.

It is clear that a representative of a graph G is not necessarily unique, and in all the cases, we have $\chi(G) = \chi(S)$ for every representative S of G [use Remark *iv*].

To conclude, note that a graph S is a representative of a graph G if S is defined in the following two cases.

Case 1. Suppose that $G \in \Omega$. Then S is a representative of G, if and only if S is berge and $G \in parent(S)$ and $\omega(S) = \beta(G)$.

Case 2. Suppose that $G \notin \Omega$. Now let $P \in parent(G)$ such that $\beta(P) = \beta(G)$. Then S is a representative of G if and only if S is berge and $P \in parent(S)$ and $\omega(S) = \beta(P) = \beta(G)$; in other words, S is a representative of G if and only if S is a representative of P, where $P \in parent(G)$ and $\beta(P) = \beta(G)$.

We will see in Section 5 that the berge index and a representative help to obtain an original reformulation of the Berge problem, and this original reformulation of the Berge problem is crucial for the result of Section 6 which clearly implies that the Hadwiger conjecture is only a non obvious special case of the Berge conjecture.

§4. The Hadwiger Index of a Graph

Here, we define the hadwiger index of a graph and a son of a graph, and we also give some elementary properties related to the hadwiger index. Using the definition of a true pal, the following assertion is immediate.

Assertion 4.0 *Let G be a graph. Then, there exists a graph S such that G is a true pal of S and $\eta(S)$ is minimum for this property.*

Now we define the hadwiger index and a son. Let G be a graph and put $\mathcal{A}(G) = \{H; G \in trpl(H)\}$; clearly $\mathcal{A}(G)$ is the set of all graphs H, such that G is a true pal of H. The *hadwiger index* of G is denoted by $\tau(G)$, where $\tau(G) = \min\limits_{F \in \mathcal{A}(G)} \eta(F)$. In other words, $\tau(G) = \eta(F'')$, where $F'' \in \mathcal{A}(G)$, and $\eta(F'')$ is minimum for this property . We say that a graph S is a *son* of G if $G \in trpl(S)$ and $\eta(S) = \tau(G)$. In other words, a graph S is a son of G, if $S \in \mathcal{A}(G)$ and $\eta(S) = \tau(G)$. In other terms again, a graph S is a son of G, if G is a true pal of S and $\eta(S)$ is *minimum* for this property. Observe that such a son exists, via Assertion 4.0. It is immediate that, if S is a son of a graph G, then $\chi(S) = \chi(G)$ and $\eta(S) \leq \eta(G)$.

We recall that $\beta(G)$ is the berge index of G, and we clearly have.

Proposition 4.1 *Let K be a complete graph and let $G \in \Omega$. We have the following three properties.*

(4.1.0) *If $\omega(G) \leq 1$, then $\beta(G) = \omega(G) = \chi(G) = \eta(G) = \tau(G)$;*
(4.1.1) *$\beta(K) = \omega(K) = \chi(K) = \eta(K) = \tau(K)$;*
(4.1.2) *$\omega(G) \geq \tau(G)$.*

Proof Properties (4.1.0) and (4.1.1) are immediate. Now we show property (4.1.2). Indeed, recall $G \in \Omega$, and clearly $\chi(G) = \omega(G)$. Now, put $\mathcal{A}(G) = \{H; G \in trpl(H)\}$ and let K' be a complete graph such that $\omega(K') = \omega(G)$ and $V(K') \subseteq V(G)$; clearly K' is a subgraph of G and

$$\chi(G) = \omega(G) = \chi(K') = \omega(K') = \eta(K') = \tau(K') \qquad (4.1.2.0).$$

In particular K' is a subgraph of G with $\chi(G) = \chi(K')$, and therefore, G is a true pal of K'. So $K' \in \mathcal{A}(G)$ and clearly

$$\tau(G) \leq \eta(K') \qquad (4.1.2.1).$$

Note $\omega(G) = \eta(K')$ (use *(4.1.2.0)*), and inequality *(4.1.2.1)* immediately becomes $\tau(G) \leq \omega(G)$. □

Observe Proposition 4.1 resembles to Proposition 3.1. Using the definition of τ, the following proposition becomes immediate.

Proposition 4.2 *Let F be a graph and let $G \in trpl(F)$. Then $\tau(G) \leq \tau(F)$.*

Proof Put $\mathcal{A}(G) = \{H; G \in trpl(H)\}$, and let S be a son of F, recalling that $G \in trpl(F)$, clearly $G \in trpl(S)$; so $S \in \mathcal{A}(G)$ and clearly $\tau(G) \leq \eta(S)$. Now, observe $\eta(S) = \tau(F)$ (because S is a son of F), and the previous inequality immediately becomes $\tau(G) \leq \tau(F)$. □

Corollary 4.3 *Let F be a graph and let $P \in parent(F)$. Then $\tau(P) \leq \tau(F)$.*

Proof Observe that $P \in trpl(F)$ and apply Proposition 4.2. □

We will see in Section 5 that the hadwiger index and a son help to obtain an original reformulation of the Hadwiger conjecture, and this original reformulation of the Hadwiger conjecture is also crucial for the result of Section 6 which clearly implies that the Hadwiger conjecture is only a non obvious special case of the Berge conjecture.

§5. An Original Reformulation of the Berge Problem and the Hadwiger Conjecture

In this section, we prove two simple Theorems which are equivalent to the Berge problem and the Hadwiger conjecture. These original reformulations will help in Section 6 to show that the Berge problem and the Hadwiger conjecture are exactly the same problem. That being so, using the berge index β, then the following first simple Theorem is an original reformulation of the Berge problem.

Theorem 5.1 *The following are equivalent.*

(1) The Berge problem is true (i.e. $\chi(B) = \omega(B)$ for every berge graph B).
(2) $\chi(F) = \beta(F)$, for every graph F.
(3) $\omega(G) = \beta(G)$, for every $G \in \Omega$.

Proof (2) ⇒ (3) Let $G \in \Omega$, in particular G is a graph, and so $\chi(G) = \beta(G)$; observe $\chi(G) = \omega(G)$ (since $G \in \Omega$), and the last two equalities imply that $\omega(G) = \beta(G)$. So (2) ⇒ (3)].

(3) ⇒ (1) Let B be berge and let $P \in parent(B)$; Proposition 3.2 implies that $\beta(P) \leq \omega(B)$. Note $\beta(P) = \omega(P)$ (because $P \in \Omega$), and the previous inequality becomes $\omega(P) \leq \omega(B)$. It is immediate that $\chi(B) = \chi(P) = \omega(P)$ [since $P \in parent(B)$], and the last inequality becomes $\chi(B) \leq \omega(B)$; observe $\chi(B) \geq \omega(B)$, and the previous two inequalities imply that

$\chi(B) = \omega(B)$. So (3) $\Rightarrow$ (1)].

(1) $\Rightarrow$ (2) Let F be a graph and let S be a representative of F, in particular S is berge (because S is a representative of F) and clearly $\chi(S) = \omega(S)$, now, observing that $\omega(S) = \beta(F)$ (because S is a representative of F), then the previous two equalities imply that $\chi(S) = \beta(F)$; note $\chi(S) = \chi(F)$ (by observing that S is a representative of F and by using Remark *iv* of Section 3), and the last two equalities immediately become $\chi(F) = \beta(F)$. So (1) $\Rightarrow$ (2)], and Theorem 5.1 follows. □

We recall that the Hadwiger conjecture states that $\chi(G) \leq \eta(G)$ for every graph G. Using the hadwiger index τ, then the following is a corresponding original reformulation of the Hadwiger conjecture.

Theorem 5.2 *The following are equivalent.*

(1) The Hadwiger conjecture is true, i.e., $\chi(H) \leq \eta(H)$ *for every graph* H*;*
(2) $\chi(F) \leq \tau(F)$*, for every graph* F*;*
(3) $\omega(G) = \tau(G)$*, for every* $G \in \Omega$*.*

Proof (2) $\Rightarrow$ (3) Let $G \in \Omega$, clearly G is a graph and so $\chi(G) \leq \tau(G)$. Note $\chi(G) = \omega(G)$ (since $G \in \Omega$), and the previous inequality becomes $\omega(G) \leq \tau(G)$; now, using property (4.1.2) of Proposition 4.1, we have $\omega(G) \geq \tau(G)$, and the last two inequalities imply that $\omega(G) = \tau(G)$.

(3) $\Rightarrow$ (1) Let H be a graph and let $P \in parent(H)$, then $\tau(P) \leq \tau(H)$ (use Corollary 4.3); observe $P \in \Omega$ (since $P \in parent(H)$), clearly $\omega(P) = \tau(P)$ (since $P \in \Omega$), and $\chi(H) = \chi(P) = \omega(P)$ (since $P \in parent(H)$). Clearly $\tau(P) = \chi(H)$ and the previous inequality becomes $\chi(H) \leq \tau(H)$. Recall $\tau(H) \leq \eta(H)$, and the last two inequalities become $\chi(H) \leq \tau(H) \leq \eta(H)$. So $\chi(H) \leq \eta(H)$, and clearly (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Indeed, let F be a graph and let S be a son of F, clearly $\chi(S) \leq \eta(S)$; now observing that $\chi(S) = \chi(F)$ (since $F \in trpl(S)$) and $\eta(S) = \tau(F)$ (because S is a son of F), then the previous inequality immediately becomes $\chi(F) \leq \tau(F)$. So (1) $\Rightarrow$ (2)] and Theorem 5.2 follows. □

Theorems 5.1 and 5.2 immediately imply that the Berge problem and the Hadwiger conjecture are exactly the same problem, and therefore, the Hadwiger conjecture is only a special non-obvious case of the Berge conjecture.

§6. Conclusion

Indeed, the following two theorems follow immediately from Theorems 5.1 and 5.2.

Theorem 6.1 *The following are equivalent.*

(i) The Berge problem is true;
(ii) $\omega(G) = \beta(G)$*, for every* $G \in \Omega$*.*

Proof Indeed, it is an immediate consequence of Theorem 5.1. □

Theorem 6.2 *The following are equivalent.*

(i) The Hadwiger conjecture is true;
(ii) $\omega(G) = \tau(G)$ *for every* $G \in \Omega$.

Proof Indeed, it is an immediate consequence of Theorem 5.2. □

Using Theorems 6.1 and 6.2, the following Theorem becomes immediate.

Theorem 6.3 *The Berge problem and the Hadwiger conjecture are exactly the same problem.*

Proof Indeed observing that the Berge conjecture is true (see [1] or see [9]), then in particular the Berge problem is true.Now using Theorem 6.1 and the previous, then it becomes immediate to deduce that

$$\omega(G) = \beta(G), \ for\ every\ G \in \Omega \qquad (6.3.1).$$

That being so, noticing that the Hadwiger conjecture is true (see [13]) and using Theorem 6.2, then it becomes immediate to deduce that

$$\omega(G) = \tau(G), \ for\ every\ G \in \Omega \qquad (6.3.2).$$

(6.3.1) and *(6.3.2)* clearly say that the Berge problem and the Hadwiger conjecture are exactly the same problem. □

From Theorem 6.3, then it comes:

Theorem 6.4(Tribute to Claude Berge) *The Hadwiger conjecture is a special case of the Berge conjecture.*

Proof It is immediate to see that

$$the\ Berge\ conjecture\ implies\ the\ Berge\ problem \qquad (6.4.1).$$

Now by Theorem 6.3

$$the\ Berge\ problem\ and\ the\ Hadwiger\ conjecture\ are\ exactly\ the\ same\ problem \qquad (6.4.2).$$

That being so, using (6.4.1) and (6.4.2) , then it becomes immediate to deduce that the Hadwiger conjecture is a special case of the Berge conjecture. □

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