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The Crossing Number of the Circulant Graph $C(3k-1;\{1,k\})$

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Abstract: A Smarandache drawing of a graph G is a drawing of G on the plane with minimal intersections for its each component and a circulant graph $C(n;S)$ is the graph with vertex set $V(C(n;S)) = \{v_i|0 \leqslant i \leqslant n-1\}$ and edge set $E(C(n;S)) = \{v_iv_j|0 \leq i \neq j \leq n-1,\ (i-j) \bmod n \in S\}$, $S \subseteq \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$. In this paper, we investigate the crossing number of the circulant graph $C(3k-1;\{1,k\})$ and get the result that $k \leq cr(C(3k-1;\{1,k\})) \leq k+1$ for $k \geqslant 3$.

Key Words: Graph, Smarandache drawing, crossing number, circulant graph.

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§1. Introduction

A *graph* $G = (V, E)$ is a set V of vertices and a subset E of unordered pairs of vertices, called edges. A Smarandache drawing of a graph G is a drawing of G on the plane with minimal intersections for its each component. Certainly, we only need to consider Smarandache drawing of connected graphs. The *crossing number* $cr(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. It is well known that the crossing number of a graph is attained only in *good drawings* of the graph, which are those drawings where no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. Let D be a good drawing of the graph G, we denote the number of crossings in D by $cr(D)$.

The *circulant graph* $C(n;S)$ is the graph with vertex set $V(C(n;S)) = \{v_i|0 \leqslant i \leqslant n-1\}$ and edge set $E(C(n;S)) = \{v_iv_j|0 \leq i \neq j \leq n-1,\ (i-j) \bmod n \in S\}$, $S \subseteq \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$.

Calculating the crossing number of a given graph is NP-complete [1]. Only the crossing number of very few families of graphs are known exactly, some of which are the crossing number of circulant graph.

Yang and Lin, etc. researched on the crossing number of circulant graphs. In [2] they showed that

$$cr(C(n;\{1,3\})) = \lfloor \frac{n}{3} \rfloor + n \bmod 3 \quad (n \geq 8).$$

In [3], they gave an upper bound of $C(mk;\{1,k\})$ for $m \geq 3$, $k \geq 3$, proved that

$$cr(C(3k;\{1,k\})) = k \qquad (k \geq 3),$$

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and in [4], they obtained that the crossing number of $C(n;\{1,\lfloor\frac{n}{2}\rfloor-1\})$ is $n/2$ for even $n\geq 8$, for odd $n\geq 13$, they showed that

$$cr(C(n;\{1,\lfloor\frac{n}{2}\rfloor-1\}))\leqslant\begin{cases}4h+2, & n=8h+1,\quad h\geqslant 2,\\ 4h+2, & n=8h+3,\quad h\geqslant 2,\\ 4h+3, & n=8h+5,\quad h\geqslant 1,\\ 4h+5, & n=8h+7,\quad h\geqslant 1.\end{cases}$$

In 2005, Ma, et al. determined that the crossing number of $C(2m+2;\{1,m\})$ is $m+1$ for $m\geq 3$, see [5].

P.T.Ho [6] investigated the crossing number of the circulant graph $C(3k+1;\{1,k\})$ and proved that $cr(C(3k+1;\{1,k\}))=k+1$ for $k\geqslant 3$.

In this paper, we study the crossing number of the circulant graph $C(3k-1;\{1,k\})$ and get the main result that

$$k\leq cr(C(3k-1;\{1,k\}))\leq k+1\quad\text{for } k\geqslant 3.$$

§2. Some lemmas and the main result

Let A and B be two disjoint subsets of E. In a drawing D, the number of crossings made by an edge in A and another edge in B is denoted by $cr_D(A,B)$. The number of crossings made by two edges in A is denoted by $cr_D(A)$, then $cr(D)=cr_D(E)$. By counting the number of crossings in D, we have Lemma 2.1.

Lemma 2.1 *Let A,B,C be mutually disjoint subsets of E. Then*

$$cr_D(A\cup B)=cr_D(A)+cr_D(B)+cr_D(A,B);$$
$$cr_D(A\cup B,C)=cr_D(A,C)+cr_D(B,C).$$

Let $E_i=\{v_iv_{i+1},v_iv_{k+i},v_{k+i}v_{2k+i},v_{i+1}v_{2k+i},v_{k+i-1}v_{k+i},v_{2k+i-1}v_{2k+i}\}$ for $0\leqslant i\leqslant k-2$, and let $E_{k-1}=\{v_{k-1}v_{2k-1},v_{2k-1}v_0,v_{2k-2}v_{2k-1},v_{3k-2}v_0\}$, see Fig.1. Then it is not difficult to observe that

$$E(C(3k-1;\{1,k\}))=\bigcup_{i=0}^{k-1}E_i$$
$$E_i\cap E_j=\emptyset,\quad 0\leq i\neq j\leq k-1$$

We define $f_D(E_i)$ $(0\leq i\leq k-1)$ to be a function counting the number of crossings related to E_i in a drawing D as follows:

$$f_D(E_i)=cr_D(E_i)+\sum_{0\leq j\leq k-1,\, j\neq i}cr_D(E_i,E_j)/2.$$

With Lemma 2.1 and the above notations, we can get

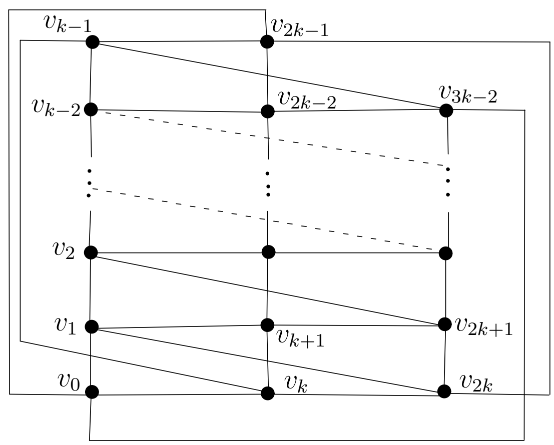

Figure 1: A good drawing of $C(3k-1;\{1,k\})$

Lemma 2.2 $cr(D)=\sum\limits_{i=0}^{k-1} f_D(E_i)$.

In a drawing D, if an edge is not crossed by any other edge, we say that it is *clean* in D; if it is crossed by at least one edge, we say that it is *crossed* in D. The following lemma is a trivial observation.

Lemma 2.3 *If there exists a crossed edge e in a drawing D and deleting it results in a new drawing D^*, then $cr(D)\geqslant cr(D^*)+1$.*

Lemma 2.4 $cr(C(3k-1;\{1,k\}))\geqslant k$ for $k\geqslant 3$.

Proof We will prove it by induction on k. For $k=3$, from [2], we have $cr(C(8;\{1,3\}))=4\geqslant 3$. Now suppose that for $k\geqslant 4$, $cr(C(3(k-1)-1;\{1,k-1\}))\geqslant k-1$, let D be a good drawing of $C(3k-1;\{1,k\})$.

Since $C(3k-1;\{1,k\})$ is non-planar, one of the edges in D must be crossed, that is to say, v_iv_{i+1} or v_iv_{k+i} is crossed for some i where $0\leqslant i\leqslant 3k-2$. If v_iv_{i+1} is crossed for some i, we may assume that $i=3k-2$. If v_iv_{k+i} is crossed for some i, we may assume that $i=k-1$. By these assumptions, we have

$$f_D(E_{k-1})\geqslant 0.5$$

We assert that

$$f_D(E_i)\geq 1 \text{ for } 0\leqslant i\leqslant k-2 \text{ or } cr(D)\geq k \tag{1}$$

Therefore, if $cr(D)<k$, we have $f_D(E_i)\geqslant 1$ for all $i=0,1,\cdots,k-2$ by (1), combining this with $f_D(E_{k-1})\geqslant 0.5$, by Lemma 2.2, we have $k>cr(D)\geqslant k-1+0.5$, which is impossible since $cr(D)$ must be an integer.

So, it suffices to verify that (1) is true. Suppose by contradiction that there exists i ($0\leqslant i\leqslant k-2$) such that $f_D(E_i)<1$. From the definition of f_D, we get that $cr_D(E_i)=0$. Furthermore, there are only two possible drawings of E_i, which are shown in Figure 2.

Figure 2: Two possible drawings of E_i

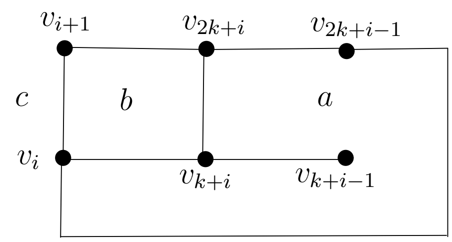

Figure 3: $E_i \cup v_i v_{2k+i-1}$

We can claim that E_i must be drawn as in the left hand side of Figure 2 in D. Suppose that E_i is drawn as in the right hand side of Figure 2. Since vertex v_{k+i-1} and vertex v_{2k+i-1} lie in different regions, so both the edge $v_{k+i-1}v_{2k+i-1}$ and the path $v_{k+i-1}v_{k+i-2}v_{2k+i-2}v_{2k+i-1}$ must cross the edges in E_i, and we have $f_D(E_i) \geqslant 1$, a contradiction to our assumption that $f_D(E_i) < 1$.

Case 1. Suppose that $f_D(E_i) > 0$. Since $f_D(E_i) < 1$, from the definition of f_D, exactly one of the edges in E_i is crossed.

First we consider that $v_i v_{2k+i-1}$ is clean. Then $E_i \cup v_i v_{2k+i-1}$ must be drawn as in Figure 3. Denote the regions by a, b and c as in Fig.3. We can assert that vertex v_{k+i-2} must lie in the same region in which vertex v_{k+i-1} lies. Or else, both the edge $v_{k+i-2}v_{k+i-1}$ and the path $v_{k+i-2}v_{i-2}v_{i-1}v_{k+i-1}$ must cross the edges on the boundary of a except $v_i v_{2k+i-1}$, so we have $f_D(E_i) \geqslant 1$, which is a contradiction. Furthermore, we can also get that v_{2k+i-2} must lie in the region a : if v_{2k+i-2} lies in the region b, then both the edge $v_{2k+i-1}v_{2k+i-2}$ and the edge $v_{k+i-2}v_{2k+i-2}$ must cross the edges on the boundary of b, which is a contradiction; if v_{2k+i-2} lies in the region c, then both the edge $v_{k+i-2}v_{2k+i-2}$ and the path $v_{k+i}v_{k+i+1} \cdots v_{2k+i-3}v_{2k+i-2}$ must cross the edges in E_i, which is also a contradiction. Since both vertex v_{k+i-2} and vertex v_{2k+i-2} lie in the region a, the pathes $v_{i+1}v_{i+2} \cdots v_{k+i-3}v_{k+i-2}$ and $v_{i+1}v_{k+i+1}v_{k+i+2} \cdots v_{2k+i-3}v_{2k+i-2}$ must cross the boundary of a, respectively, and we can have $f_D(E_i) \geqslant 1$, which is impossible.

Now consider that $v_i v_{2k+i-1}$ is crossed.

Case 1.1. Suppose that the edges $v_{i+1}v_{2k+i}$ and $v_{2k+i-1}v_{2k+i}$ are clean. We will produce from D a drawing D^*, which is constructed by drawing a new edge connecting vertex v_{i+1} to vertex v_{2k+i-1} close enough to the edges $v_{i+1}v_{2k+i}$ and $v_{2k+i-1}v_{2k+i}$, and by deleting the edges $v_i v_{2k+i-1}$, $v_i v_{k+i}$, $v_{k+i}v_{2k+i}$ and $v_{i+1}v_{2k+i}$, see Figure 4(1). Since the edges $v_{i+1}v_{2k+i}$ and $v_{2k+i-1}v_{2k+i}$ are clean, one can observe that the new edge $v_{i+1}v_{2k+i-1}$ doesn't produce any additional crossing. And because the crossed edge $v_i v_{2k+i-1}$ in D is removed from D,

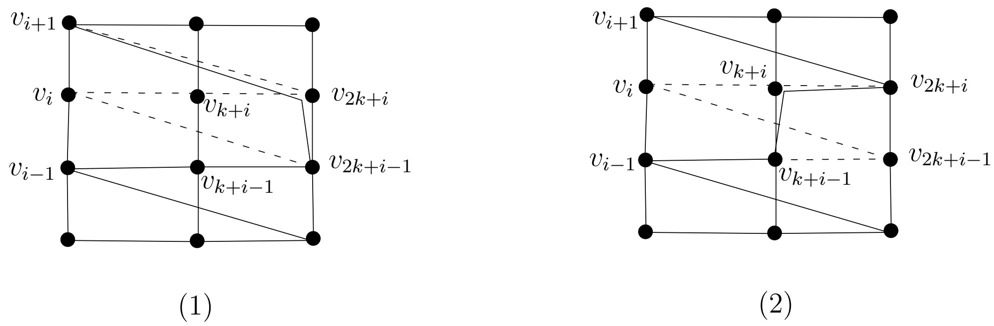

Figure 4: New drawing D^* produced from drawing D

we can get that $cr(D) \geqslant cr(D^*)+1$ by Lemma 2.3. D^* is a drawing of the subdivision of $C(3(k-1)-1;\{1,k-1\})$, so we have $cr(D) \geqslant cr(C(3(k-1)-1;\{1,k-1\}))+1 \geqslant k$.

Case 1.2. Suppose that one of the edges $v_{i+1}v_{2k+i}$ or $v_{2k+i-1}v_{2k+i}$ is crossed. Analogously, by drawing a new edge connecting vertex v_{k+i-1} to vertex v_{2k+i} quite close to the edges $v_{k+i-1}v_{k+i}$ and $v_{k+i}v_{2k+i}$, and by deleting the edges v_iv_{k+i}, $v_{k+i}v_{2k+i}$, v_iv_{2k+i-1} and $v_{k+i-1}v_{2k+i-1}$, a new drawing D^* can be produced from D, see Figure 4(2). One can easily see that the new edge $v_{k+i-1}v_{2k+i}$ doesn't produce any additional crossing since the edges $v_{k+i-1}v_{k+i}$ and $v_{k+i}v_{2k+i}$ are all clean. Since the crossed edge v_iv_{2k+i-1} in D is removed from D, by Lemma 2.3, we can obtain that $cr(D) \geqslant cr(D^*)+1$. D^* is a drawing of the subdivision of $C(3(k-1)-1;\{1,k-1\})$ as well. These facts imply that $cr(D) \geqslant cr(C(3(k-1)-1;\{1,k-1\}))+1 \geqslant k$.

Case 2. Suppose that $f_D(E_i)=0$. Since the edges in E_i are all clean, v_iv_{2k+i-1} doesn't cross any edge in E_i, then $E_i \cup v_iv_{2k+i-1}$ is drawn as in Figure 3. If v_iv_{2k+i-1} is clean, then the boundary of a is clean, we follow the analogous arguments presented in Case 1. If v_iv_{2k+i-1} is crossed, we can follow the same arguments presented in Case 1.1.

From all the above cases, we have shown that (1) is true. □

Theorem 2.5 $k \leqslant cr(C(3k-1;\{1,k\})) \leqslant k+1$ for $k \geqslant 3$.

Proof A good drawing of $C(3k-1;\{1,k\})$ in Fig.1 shows that $cr(C(3k-1;\{1,k\})) \leqslant k+1$ for $k \geqslant 3$. This together with Lemma 2.4 immediately indicate that $k \leqslant cr(C(3k-1;\{1,k\})) \leqslant k+1$ for $k \geqslant 3$. □

We end this paper with the following conjecture.

Conjecture $cr(C(3k-1;\{1,k\})) = k+1$ for $k \geqslant 3$.

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