# On the Basis Number of the Strong Product of Theta Graphs with Cycles 

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#### Abstract

A basis $\mathcal{B}$ for the cycle space $\mathcal{C}(G)$ of a graph $G$ is called a $d$-fold if each edge of $G$ occurs in at most $d$ of the cycles in the basis $\mathcal{B}$. A basis $\mathcal{B}$ for the cycle space $\mathcal{C}(G)$ of a graph $G$ is Smarandachely if each edge of $G$ occurs in at least 2 of the cycles in $\mathcal{B}$. The basis number, $b(G)$, of a graph $G$ is defined to be the least integer $d$ such that there is a $d$-fold basis of the cycle space of $G$. MacLane [20] made a connection between the the number of occurrence of edges of a graph in its cycle bases and the planarity of a graph, which is related with parallel bundles on planar map geometries, a kind of Smarandache geometries. In fact, he proved that a graph $G$ is planar if and only if $b(G) \leq 2$. Jaradat [10] gave an upper bound of the basis number of the strong product of a graph with a bipartite graph in terms of the factors. In this work, we show that the basis number of the strong product of a theta graph with a cycle is either 3 or 4 . Our result, improves Jaradat's upper bound in the case of specializing the factors by a theta graph and a cycle.


Key Words: Cycle space, cycle basis, Smarandache basis, basis number, strong product.
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## §1. Introduction

In graph theory, there are many numbers that give rise to a better understanding and interpretation of the geometric properties of a given graph such as the crossing number, the thickness, the genus, the basis number, etc.. The basis number of a graph is of a particular importance because MacLane, in [20], made a connection between the number of occurrences of edges of a graph in its cycle bases and the planarity of a graph; in fact, he proved that a graph is planar if and only if its basis number is at most 2. For the completeness, it should be mentioned that a basis $\mathcal{B}$ of the cycle space $\mathcal{C}(G)$ of a graph $G$ is Smarandachely if each edge of $G$ occurs in at least 2 of the cycles in $\mathcal{B}$

Product of graphs occur naturally in discrete mathematics as tools in combinatorial constructions. They give rise to important classes of graphs and deep structure problems. There are many graph products in the literature, such as, Cartesian product, strong product, lexicographic product, semi-strong product and semi-composition product. The extensive literature

[^0]on products that has evolved over the years presents a wealth of profound and beautiful results. This led Imrich and Klavzar to write a whole book on graph products [7].

The main purpose of this paper is to investigate the basis number of the strong product of a theta graph with a cycle. Our result improves the upper bounds that expected from applying Jaradat's theorems.

## §2. Definitions and preliminaries

Unless otherwise specified, the graphs considered in this paper are finite, undirected, simple and connected. For a given graph $G$, we denote the vertex set of $G$ by $V(G)$ and the edge set by $E(G)$.

For a given graph $G$, the set $\mathcal{E}$ of all subsets of $E(G)$ forms an $|E(G)|$-dimensional vector space over $Z_{2}$ with vector addition $X \oplus Y=(X \backslash Y) \cup(Y \backslash X)$ and scalar multiplication $1 \cdot X=X$ and $0 \cdot X=\emptyset$ for all $X, Y \in \mathcal{E}$. The cycle space, $\mathcal{C}(G)$, of a graph $G$ is the vector subspace of $(\mathcal{E}, \oplus, \cdot)$ spanned by the cycles of $G$. Note that the non-zero elements of $\mathcal{C}(G)$ are cycles and edge disjoint union of cycles. It is known that for a connected graph $G$ the dimension of the cycle space is the cyclomatic number or the first Betti number

$$
\begin{equation*}
\operatorname{dim} \mathcal{C}(G)=|E(G)|-|V(G)|+r \tag{1}
\end{equation*}
$$

where $r$ is the number of components in $G$.
The first important use of the basis number dates back to MacLane [20] when he made the connection between the basis number of a graph and the planarity. There after, in 1981, E. Schmeichel [21] formalized the definition of the basis number of a graph as follows: A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a cycle basis of $G$. A cycle basis $\mathcal{B}$ of $G$ is called a $d$-fold if each edge of $G$ occurs in at most $d$ of the cycles in $\mathcal{B}$. The basis number, $b(G)$, of $G$ is the least non-negative integer $d$ such that $\mathcal{C}(G)$ has a $d$-fold basis.

Latter on, Schmeichel [21] investigate the basis number of the known classes of graphs such as the complete graphs $K_{n}$ and the complete bipartite graphs $K_{n, m}$. In fact, he proved that $b\left(K_{n}\right)=3$, for $n \geq 5$ and $b\left(K_{n, m}\right)=4$ for all $n, m \geq 5$ except a few numbers of graphs. Also, he proved that for any positive integer $r$, there exists a graph $G$ with $b(G) \geq r$. After that, he joined Banks to prove that the basis number of $n$-cube is 4 for all $n \geq 7$ (see [6])

Since 1992, many researchers were attracted to study the basis number of graph products. The Cartesian product, $\square$, was studied by Ali and Marougi [3] when they gave the following result:

Theorem 2.1 (Ali and Marougi) If $G$ and $H$ are two connected disjoint graphs, then $\mathrm{b}(G \square H) \leq$ $\max \left\{\mathrm{b}(G)+\triangle\left(T_{H}\right), \mathrm{b}(H)+\triangle\left(T_{G}\right)\right\}$ where $T_{H}$ and $T_{G}$ are spanning trees of $H$ and $G$, respectively, such that the maximum degrees $\triangle\left(T_{H}\right)$ and $\Delta\left(T_{G}\right)$ are minimum with respect to all spanning trees of $H$ and $G$.

Also, Alsardary and Wojciechowski [4] proved that for every $d \geq 1$ and $n \geq 2, b\left(K_{n}^{d}\right) \leq 9$ where $K_{n}^{d}$ is a $d$ times Cartesian product of the complete graph $K_{n}$.

Upper bounds of the strong product, $\boxtimes$, were obtained by Jaradat [11], [14] and [15] when he gave the following results:

Theorem 2.2(Jaradat) Let $G$ be a bipartite graph and $H$ be a graph. Then $b(H \boxtimes G) \leq$ $\max \left\{b(G)+1,2 \Delta(G)+b(H)-1,\left\lfloor\frac{3 \Delta\left(T_{H}\right)+1}{2}\right\rfloor, b(H)+2\right\}$.

Theorem 2.3(Jaradat) Let $G$ be a bipartite graph and $C$ be a cycle. Then $b(G \boxtimes C) \leq 4+b(G)$.
The lexicographic product of two graphs $G$ and $H, G[H]$, was studied by Jaradat and Al-zoubi [17] and Jaradat [13]. They obtained the following results:

Theorem 2.4 (Jaradat and Al-Zoubi) For each two connected graphs $G$ and $H, b(G[H]) \leq$ $\operatorname{Max}\{4,2 \Delta(G)+b(H), 2+b(G)\}$.

Theorem 2.5(Jaradat Let $G, T_{1}$ and $T_{2}$ be a graph, a spanning tree of $G$ and a tree, respectively. Then, $b\left(G\left[T_{2}\right]\right) \leq b(G[H]) \leq \max \left\{5,4+2 \Delta\left(T_{\min }^{G}\right)+b(H), 2+b(G)\right\}$ where $T^{G}$ stands for the complement graph of a spanning tree $T$ in $G$ and $T_{\min }$ stands for a spanning tree for $G$ such that $\Delta\left(T_{\mathrm{min}}^{G}\right)=\min \left\{\Delta\left(T^{G}\right) \mid T\right.$ is a spanning tree of $\left.G\right\}$.

Ali [1], [2] gave an upper bound for the basis number of the semi-strong product, •, and the direct product, $\times$, of some special graphs when he proved that $b\left(K_{m} \bullet K_{n}\right) \leq 9$ for any integers $m, n$ and $b\left(C_{n} \times C_{m}\right)=3$ for any two cycles $C_{n}$ and $C_{m}$ with $n, m \geq 3$. Also the following upper bound (among other results) were obtained by Jaradat [8], [9], [14] and [18]:

Theorem 2.6(Jaradat) For each bipartite graphs $G$ and $H, b(G \times H) \leq 5+b(G)+b(H)$.
Theorem 2.7 (Jaradat) For each bipartite graphs $G$ and $H$,
$b(G \bullet H) \leq \max \left\{b(G)+b(H)+\left\{\begin{array}{ll}3, & \text { if both of } T_{G} \text { and } T_{H} \text { are paths, } \\ 4, & \text { if } T_{H} \text { is a path, } \\ 5, & \text { if } T_{G} \text { is a path, } \\ 6, & \text { if both of } T_{G} \text { and } T_{H} \text { are not paths. }\end{array}\right\}, \Delta\left(T_{G}\right)+b(H)\right\}$

The wreath product, was studied by Jaradat and Al-Qeyyam (See [5], [12] and [16]).
For completeness, we recall that for any two graphs $G$ and $H$, the strong product $G \boxtimes$ $H$ is the graph with the vertex set $V(G \boxtimes H)=V(G) \times V(H)$ and the edge set $E(G \boxtimes$ $H)=\left\{\left(u_{1}, u_{2}\right)\left(v_{1}, v_{2}\right) \mid u_{1} v_{1} \in E(G)\right.$ and $u_{2}=v_{2}$ or $u_{1}=v_{1}$ and $u_{2} v_{2} \in E(H)$ or $u_{1} v_{1} \in$ $E(G)$ and $\left.u_{2} v_{2} \in E(H)\right\}$. The Cartesian product $G \square H$ is the graph with the vertex set $V(G \square H)=V(G) \times V(H)$ and the edge set $E(G \square H)=\left\{\left(u_{1}, u_{2}\right)\left(v_{1}, v_{2}\right) \mid u_{1} v_{1} \in E(G)\right.$ and $u_{2}=$ $v_{2}$ or $u_{1}=v_{1}$ and $\left.u_{2} v_{2} \in E(H)\right\}$. Also, the direct product $G \times H$ is the graph with the vertex set $V(G \times H)=V(G) \times V(H)$ and the edge set $E(G \times H)=\left\{\left(u_{1}, u_{2}\right)\left(v_{1}, v_{2}\right) \mid u_{1} v_{1} \in\right.$ $E(G)$ and $\left.u_{2} v_{2} \in E(H)\right\}$.

In the rest of this paper, $f_{B}(e)$ stand for the number of elements of $B$ containing the edge $e$ where $B \subseteq \mathcal{C}(G)$.

## §3. The basis number of $\theta_{n} \boxtimes C_{m}$

In this section we investigate the basis number of the strong product of theta graphs and cycles. In fact we show that $3 \leq b\left(\theta_{n} \boxtimes C_{m}\right) \leq 4$. Throughout this section we assume that $1,2, \ldots, n$ and $1,2 \ldots, m$ to be the vertices of $\theta_{n}$ and $C_{m}$, respectively.

Definition 3.1 A theta graph $\theta_{n}$ is defined to be a cycle to which we add a new edge that joins two non-adjacent vertices. We may assume 1 and $\delta$ are the two vertices of $\theta_{n}$ of degree 3 .

Applying Theorem 2.2 for the case $H=\theta_{n}$ and $G$ to be a cycle of even length $C_{m}$, we get $b\left(\theta_{n} \boxtimes C_{m}\right) \leq \max \{2,5,3,4\}=5$. Also, applying the same theorem by considering $H=C_{m}$ and $G$ to be a theta graph that contains no odd cycles $\theta_{n}$, we get $b\left(C_{m} \boxtimes \theta_{n}\right) \leq \max \{3,6,3,4\}=6$. Moreover, by specializing $G$ in Theorem 2.3 to $\theta_{n}$ that contains no odd cycle, then $b\left(\theta_{n} \boxtimes C_{m}\right) \leq$ 6. However, these upper bounds will be reduced to 4 as we will see in Theorem 3.6. Now, for this purpose, we consider the following cycles: For each $j=1,2, \ldots, m-2$, set

$$
\begin{aligned}
\mathcal{A}_{1}^{(j)} & =(1, j)(2, j+1)(1, j+2)(\delta, j+1)(1, j) \\
\mathcal{A}_{2}^{(j)} & =(\delta, j)(\delta-1, j+1)(\delta, j+2)(1, j+1)(\delta, j)
\end{aligned}
$$

and let

$$
\mathcal{A}_{1}=\bigcup_{j=1}^{m-2} \mathcal{A}_{1}^{(j)} \quad \text { and } \quad \mathcal{A}_{2}^{(i)}=\bigcup_{j=1}^{m-2} \mathcal{A}_{2}^{(j)}
$$

The following result will be useful in our main result.

Lemma 3.2 Every linear combination of cycles of $\mathcal{A}_{1} \cup \mathcal{A}_{2}$ contains at least one edge of $\{(1, j)(\delta, j+1),(1, j+1)(\delta, j) \mid 1 \leq j \leq m-2\}$.

Proof Consider $\mathcal{O}$ to be a linear combinations of cycles of $\mathcal{A}_{1} \cup \mathcal{A}_{2}^{(i)}$. Then

$$
\mathcal{O}=\bigoplus_{j=1}^{s_{1}} \mathcal{A}_{1}^{\left(1_{j}\right)} \oplus \bigoplus_{j=1}^{s_{2}} \mathcal{A}_{2}^{\left(2_{j}\right)}
$$

where $\mathcal{A}_{1}^{\left(1_{j}\right)} \in \mathcal{A}_{1}, \mathcal{A}_{2}^{\left(2_{j}\right)} \in \mathcal{A}_{2}, 1_{1}<1_{2}<\cdots<1_{s_{1}}$ and $2_{1}<2_{2}<\cdots<2_{s_{2}}$. Now, let $t_{1}=\min \left\{1_{1}, 2_{1}\right\}$. We now consider the following two cases.
Case 1. $\quad t_{1}=1_{1}$. Then by the definition of $\mathcal{A}_{1}, \mathcal{A}_{1}^{\left(1_{1}\right)}$ contains the edge $\left(1,1_{1}\right)\left(\delta, 1_{1}+1\right)$ where $1_{1} \leq m-2$. Since $E\left(\mathcal{A}_{1}^{(j)}\right) \cap E\left(\mathcal{A}_{1}^{(i)}\right)=\varnothing,\left(1,1_{1}\right)\left(\delta, 1_{1}+1\right) \notin \mathcal{A}_{1}^{\left(1_{j}\right)}$ for each $1 \leq$ $j \leq s_{1}$. Also, since $1_{1} \leq 2_{1},\left(1,1_{1}\right)\left(\delta, 1_{1}+1\right) \notin \mathcal{A}_{2}^{\left(2_{j}\right)}$ for each $1 \leq j \leq s_{2}$. Therefore, $\left(1,1_{1}\right)\left(\delta,, 1_{1}+1\right) \in \mathcal{O}$.

Case 2. $\quad t_{1}=2_{1}$. Then we argue more or less as in Case 1 , to have that $\left(1,2_{1}+1\right)\left(\delta, 2_{1}\right) \in \mathcal{O}$ where $2_{1} \leq m-2$.

Now, for $j=1,2, \ldots, m-1$, consider the following set of cycles:

$$
\mathcal{K}_{j}=(1, j)(\delta, j)(1, j+1)(\delta, j+1)(1, j)
$$

and let

$$
\mathcal{K}=\bigcup_{j=1}^{m-1} \mathcal{K}_{j} .
$$

Lemma 3.3 Every linear combination of cycles of $\mathcal{K}$ contains at least one edge of $\{(1, j)(\delta, j) \mid 1 \leq$ $j \leq m-1\}$.

Proof Let

$$
\mathcal{O}=\sum_{i=1}^{s} \mathcal{K}_{j_{i}}(\bmod 2)
$$

where $\mathcal{K}_{j_{i}} \in \mathcal{K}$ and $j_{1}<j_{2}<\cdots<j_{s} \leq m-1$. Then by the definition of $\mathcal{K}$,

$$
E\left(\mathcal{K}_{j_{1}}\right) \cap E\left(\cup_{i=2}^{S} \mathcal{K}_{j_{i}}\right) \subseteq\left\{\left(1, j_{1}+1\right)\left(\delta, j_{1}+1\right)\right\} .
$$

But, $\left(1, j_{1}\right)\left(\delta, j_{1}\right) \in E\left(\mathcal{K}_{j_{1}}\right)$. Hence, $\left(1, j_{1}\right)\left(\delta, j_{1}\right) \in \mathcal{O}$.
Lemma 3.4 Let $\theta_{n}$ be a graph of order $n \geq 4$ and $C_{m}$ be a cycle of order $m \geq 3$. Then $b\left(\theta_{n} \boxtimes C_{m}\right) \geq 3$.

Proof Assume that $\theta_{n} \boxtimes C_{m}$ has a 2-fold basis $\mathcal{B}$. Since the girth of $\theta_{n} \boxtimes C_{m}$ is 3 , we have that

$$
\begin{aligned}
3|\mathcal{B}| & \leq 2\left|E\left(\theta_{n} \boxtimes C_{m}\right)\right| \\
3(3 m(n+1)+1) & \leq 2(3 m(n+1)+n m) \\
9 m n+9 m+3 & \leq 6 m n+6 m+2 n m \\
m n+3 m+3 & \leq 0 \\
m(n+3)+3 & \leq 0
\end{aligned}
$$

which is a contradiction. Hence $\mathcal{B}\left(\theta_{n} \boxtimes C_{m}\right)$ is a 3 -fold basis.
The following result of Jaradat and et al. will be needed in our coming result:

Proposition 3.5 (Jaradat and et al) Let $A$ and $B$ be two linearly independent sets of cycles such that $E(A) \cap E(B)$ subset of an edge set of a forest or an empty set. Then $A \cup B$ is linearly independent.

The following cycles which were introduced in [11] will be used frequently in the coming results.

$$
\begin{aligned}
\mathcal{L}_{a b}= & \left\{\mathcal{L}^{(j)}=\left(a, v_{j}\right)\left(b, v_{j+1}\right)\left(a, v_{j+1}\right)\left(a, v_{j}\right) \mid j=1,2,3, \cdots, m-1\right\} \\
& \cup\left\{\mathcal{L}^{(n)}=\left(a, v_{n}\right)\left(b, v_{1}\right)\left(a, v_{1}\right)\left(a, v_{n}\right)\right\} \\
\mathcal{T}_{a b}= & \left\{\mathcal{T}^{(j)}=\left(a, v_{j}\right)\left(a, v_{j+1}\right)\left(b, v_{j}\right)\left(a, v_{j}\right) \mid j=1,2,3, \cdots, m-1\right\} \\
& \cup\left\{\mathcal{T}^{(n)}=\left(a, v_{n}\right)\left(a, v_{1}\right)\left(b, v_{n}\right)\left(a, v_{n}\right)\right\} \\
\mathcal{S}_{a b}= & \left\{\mathcal{S}^{(j)}=\left(a, v_{j+1}\right)\left(b, v_{j}\right)\left(b, v_{j+1}\right)\left(a, v_{j+1}\right) \mid j=1,2,3, \cdots, m-1\right\} \\
& \cup\left\{\mathcal{S}^{(n)}=\left(a, v_{1}\right)\left(b, v_{n}\right)\left(b, v_{1}\right)\left(a, v_{1}\right)\right\}
\end{aligned}
$$

Also

$$
\mathcal{F}_{n}=\left\{\begin{array}{c}
\left(a, v_{1}\right)\left(b, v_{2}\right)\left(a, v_{3}\right)\left(b, v_{4}\right) \ldots\left(a, v_{n-1}\right)\left(b, v_{n}\right)\left(a, v_{1}\right) \text { if } m \text { is even } \\
\left(a, v_{1}\right)\left(b, v_{1}\right)\left(a, v_{2}\right)\left(b, v_{3}\right) \ldots\left(a, v_{n-1}\right)\left(b, v_{n}\right)\left(a, v_{1}\right) \text { if } m \text { is odd }
\end{array}\right.
$$

Let

$$
\mathcal{B}_{a b}=\mathcal{L}_{a b} \cup \mathcal{T}_{a b} \cup \mathcal{S}_{a b} \text { and } \mathcal{B}_{a b}^{*}=\mathcal{B}_{a b}-\left\{\mathcal{S}^{(m)}\right\} \cup\left\{\mathcal{F}_{m}\right\}
$$

Moreover, by Theorem 2.6 of [11], we have that

$$
\begin{equation*}
\operatorname{dim} \mathcal{C}\left(C_{n} \boxtimes C_{m}\right)=3 m n+1 \tag{2}
\end{equation*}
$$

Note that $\theta_{n} \boxtimes C_{m}$ is decomposable into $\left(C_{n} \boxtimes C_{m}\right) \cup\left(1 \alpha \square N_{m}\right) \cup\left(1 \alpha \times C_{m}\right)$ where $N_{m}$ is the null graph with vertex set $V\left(C_{m}\right)$. Thus,

$$
\begin{align*}
\operatorname{dim} \mathcal{C}\left(\theta_{n} \boxtimes C_{m}\right) & =\operatorname{dim} \mathcal{C}\left(C_{n} \boxtimes C_{m}\right)+m+2 m,  \tag{3}\\
& =3 m n+3 m+1 . \tag{4}
\end{align*}
$$

Now, we state and prove our main result.
Theorem 3.6 For any graph $\theta_{n}$ of order $n \geq 4$ and cycle $C_{m}$ of order $m \geq 3$, we have $3 \leq b\left(\theta_{n} \boxtimes C_{m}\right) \leq 4$.

Proof By Lemma 3.4, it is sufficient to exhibit a 4 -fold basis, $\mathcal{B}$, for $\mathcal{C}\left(\theta_{n} \boxtimes C_{m}\right)$. According to the parity of $m, n$ and $\delta$ (odd or even), we consider the following cases.

Case 1. $m$ and $n$ are even and $\delta$ is odd. Then define

$$
\mathcal{B}\left(\theta_{n} \boxtimes C_{m}\right)=\left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_{i} a_{i+1}}\right) \cup \mathcal{B}_{a_{n} a_{1}}^{*} \cup \mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K} \cup\{C\} \cup\left\{C_{1}, C_{2}, C_{3}, C_{4}, C_{5}\right\}
$$

where $B_{a_{i} a_{i+1}}$ and $B_{a_{n} a_{1}}^{*}$ are as in above and

$$
C=(1,1)(2,2)(3,1)(4,2) \ldots(n-1,1)(n, 2)(1,1) .
$$

Also,

$$
\begin{aligned}
& C_{1}=(1, m-1)(2, m)(3, m)(4, m) \ldots(\delta, m)(1, m-1) . \\
& C_{2}=(1,1)(1, m)(\delta, m)(1,1) . \\
& C_{3}=(\delta, 1)(\delta+1,2)(\delta, 3) \ldots(\delta, m-1)(1, m)(\delta, 1) . \\
& C_{4}=(1,1)(2, m)(3,1)(4, m) \ldots(\delta, 1)(1, m)(2,1)(3, m) \ldots(\delta, m)(1,1) . \\
& C_{5}=(1, m)(2,1)(3, m)(4,1) \ldots(\delta, m)(1, m) .
\end{aligned}
$$

Let $\mathcal{B}_{1}=\bigcup_{i=1}^{n-1} \mathcal{B}_{a_{i} a_{i+1}} \cup \mathcal{B}_{a_{n} a_{1}}^{*} \cup\{C\}$. Note that $\mathcal{B}_{1}=\mathrm{B}\left(C_{n} \boxtimes C_{m}\right)$ is a basis for $\mathcal{C}\left(C_{n} \boxtimes C_{m}\right)$ (see Theorem 2.6, Case 1 of [11]). Thus, $\mathcal{B}_{1}$ is linearly independent. Note that $C_{5}$ contains the edge $(\delta, m)(1, m)$ which does not appear in any cycle of $\mathcal{B}_{1}$. Hence, $\mathcal{B}_{1} \cup\left\{C_{5}\right\}$ is linearly independent. Now, $C_{2}$ contains the edge $(\delta, m)(1,1)$ which does not appear in any cycle of $\mathcal{B}_{1} \cup\left\{C_{5}\right\}$. So, $\mathcal{B}_{1} \cup\left\{C_{5}, C_{2}\right\}$ is linearly independent. Similarly, the cycle $C_{4}$ contains the edge $(\delta, 1)(1, m)$ which does not appear in any cycle of $\mathcal{B}_{1} \cup \cup\left\{C_{5}, C_{2}\right\}$. Thus, $\mathcal{B}_{1} \cup\left\{C_{2}, C_{4}, C_{5}\right\}$ is linearly independent. Also, $C_{3}$ contains the edge $(\delta, m-1)(1, m)$ which does not appear in any cycle of $\mathcal{B}_{1} \cup\left\{C_{2}, C_{4}, C_{5}\right\}$. Therefore, $\mathcal{B}_{1} \cup\left\{C_{2}, C_{3}, C_{4}, C_{5}\right\}$ is linearly independent. Finally, $C_{1}$ contains the edge $(1, m-1)(\delta, m)$ which does not appear in any cycle of $\mathcal{B}_{1} \cup\left\{C_{2}, C_{3}, C_{4}, C_{5}\right\}$. Thus, $\mathcal{B}_{1} \cup\left\{C_{1}, C_{2}, C_{3}, C_{4}, C_{5}\right\}$ is linearly independent. By Lemma 3.2, any linear combination of cycles of $\mathcal{A}_{1} \cup \mathcal{A}_{2}$ contains at least one edge of $\{(1, j)(\delta, j+1),(1, j+1)(\delta, j) \mid 1 \leq j \leq m-2\}$ which does not occur in any cycle of $\mathcal{B}_{1} \cup\left\{C_{1}, C_{2}, C_{3}, C_{4}, C_{5}\right\}$. Thus, $\mathcal{B}_{1} \cup \mathcal{A}_{1} \cup \mathcal{A}_{2} \cup\left\{C_{1}, C_{2}, C_{3}, C_{4}, C_{5}\right\}$ is linearly independent. Similarly, by Lemma 3.2, any linear combination of cycles of $\mathcal{K}$ contains at least one edge of $\{(1, j)(\delta, j) \mid 1 \leq j \leq m-1\}$, which does not occur in any cycle of $\mathcal{B}_{1} \cup \mathcal{A}_{1} \cup$ $\mathcal{A}_{2} \cup\left\{C_{1}, C_{2}, C_{3}, C_{4}, C_{5}\right\}$. Therefore, $\mathcal{B}\left(\theta_{n} \boxtimes C_{m}\right)$ is linearly independent. Note that

$$
\begin{aligned}
\left|\mathcal{B}\left(\theta_{n} \boxtimes C_{m}\right)\right| & =\left|\mathcal{B}_{1}\right|+|\mathcal{K}|+\left|\mathcal{A}_{1}\right|+\left|\mathcal{A}_{2}\right|+\sum_{i=1}^{5}\left|C_{i}\right| \\
& =3 m n+1+|\mathcal{K}|+\left|\mathcal{A}_{1}\right|+\left|\mathcal{A}_{2}\right|+\sum_{i=1}^{5}\left|C_{i}\right| \\
& =3 m n+1+(m-1)+(m-2)+(m-2)+5 \\
& =3 m n+3 m+1 \\
& =3 m(n+1)+1 \\
& =\operatorname{dim\mathcal {C}}\left(\theta_{n} \boxtimes C_{m}\right)
\end{aligned}
$$

where the last equality follows from equation (4). Therefore, $\mathrm{B}\left(\theta_{n} \boxtimes C_{m}\right)$ is a basis for $\mathcal{C}\left(\theta_{n} \boxtimes\right.$ $\left.C_{m}\right)$. To complete the proof of this case, we show that $\mathrm{B}\left(\theta_{n} \boxtimes C_{m}\right)$ is a 3-fold basis. Let $e \in E\left(\theta_{n} \boxtimes C_{m}\right)$. Then 1) if $e=(1, m-1)(2, m)$, then $f_{\mathcal{B}_{1}}(e)=1, f_{\mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K}}(e)=0$ and $\left.f_{\left\{C_{i}\right\}_{i=1}^{5}}(e)=1.2\right)$ If $e \in\{(i, m)(i+1, m) \mid i=2,3, \ldots, m-1\}$, then $f_{\mathcal{B}_{1}}(e)=2, f_{\mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K}}(e)=0$ and $f_{\left\{C_{i}\right\}_{i=1}^{5}}(e)=1$. 3) If $e=(1, m)(\delta, m)$ or $(1,1)(\delta, 1)$, then $f_{\mathcal{B}_{1}}(e)=0, f_{\mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K}}(e)=1$ and $\left.f_{\left\{C_{i}\right\}}(e) \leq 2.4\right)$ If $e=(1,1)(1, m)$, then $f_{\mathcal{B}_{1}}(e)=2, f_{\mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K}}(e)=0$ and $f_{\left\{C_{i}\right\}_{i=1}^{5}}(e)=1$. 5) If $e \in\{(i, 1)(i+1, m),(i+1,1)(i, m) \mid i=1,2, \ldots, n-1\} \cup\{(1,1)(n, m),(1, m)(\delta, 1)\}$, then $f_{\mathcal{B}_{1}}(e)=$ $0, f_{\mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K}}(e)=0$ and $f_{\left\{C_{i}\right\}_{i=1}^{5}}(e) \leq 2$. 6) If $e \in\{(1, j)(2, j+1) \mid j=1,2, \ldots, m-2\} \cup\{(\delta-$ $1, j)(\delta, j+1) \mid j=1,2, \ldots, m-1\}$, then $f_{\mathcal{B}_{1}}(e)=1, f_{\mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K}}(e)=1$ and $\left.f_{\left\{C_{i}\right\}_{i=1}^{5}}(e)=0.7\right)$ If
$e \in\{(1, j+1)(2, j),(\delta-1, j+1)(\delta, j) \mid j=1,2, \ldots, m-1\}$, then $f_{\mathcal{B}_{1}}(e)=2, f_{\mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K}}(e)=1$ and $\left.f_{\left\{C_{i}\right\}_{i=1}^{5}}(e)=0.8\right)$ If $e=(1, m-1)(\delta, m)$ or $(1, m)(\delta, m-1)$, then $f_{\mathcal{B}_{1}}(e)=0, f_{\mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K}}(e)=1$ and $f_{\left\{C_{i}\right\}_{i=1}^{5}}(e)=1$. 9) If $e \in\{(1, j)(\delta, j+1),(1, j+1)(\delta, j) \mid j=1,2, \ldots, m-2\}$, then $f_{\mathcal{B}_{1}}(e)=0$, $f_{\mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K}}(e) \leq 2$ and $f_{\left\{C_{i}\right\}_{i=1}^{5}}(e)=0$. 10) If $e \in\{(\delta, j)(\delta+1, j+1),\{(\delta, j+1)(\delta+1, j) \mid j=$ $1,2, \ldots, m-1\}$, then $f_{\mathcal{B}_{1}}(e) \leq 2, f_{\mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K}}(e)=0$ and $f_{\left\{C_{i}\right\}_{i=1}^{5}}(e) \leq 1$. if $e$ is not of the above form, then $f_{\mathcal{B}_{1}}(e) \leq 3, f_{\mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K}}(e)=0$ and $f_{\left\{C_{i}\right\}_{i=1}^{5}}(e)=0$. From all of the above, we have that $f_{\mathcal{B}\left(\theta_{n} \boxtimes C_{m}\right)}(e) \leq 3$.

Case 2. $m$ and $\delta$ are even and $n$ is odd. Then define

$$
\mathcal{B}\left(\theta_{n} \boxtimes C_{m}\right)=\left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_{i} a_{i+1}}\right) \cup \mathcal{B}_{a_{n} a_{1}}^{*} \cup \mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K} \cup\left\{C^{*}, C_{1}, C_{2}, C_{3}, C_{4}, C_{5}\right\}
$$

where

$$
C^{*}=(1,1)(2,2)(3,1)(4,2) \ldots(n, 1)(1,1)
$$

and $\mathcal{B}_{a_{i} a_{i+1}}, \mathcal{B}_{a_{n} a_{1}}^{*}, \mathcal{A}_{1}, \mathcal{A}_{2}, \mathcal{K}, C_{1}, C_{2}$ and $C_{3}$ are as defined in Case 1 and

$$
\begin{aligned}
& C_{4}=(1,1)(2, m)(3,1)(4, m) \ldots(\delta, m)(1,1) \\
& C_{5}=(1, m)(2,1)(3, m)(4,1) \ldots(\delta, 1)(1, m)
\end{aligned}
$$

By the same argument as in Case 1 of Theorem 2.6 of [11], we show that $\left(\bigcup_{i=1}^{n} \mathcal{B}_{a_{i} a_{i+1}}\right) \cup$ $\mathcal{B}_{a_{n} a_{1}}^{*} \cup\left\{C^{*}\right\}$ is linearly independent. Following, more or less, the same proof of Case 1 by replacing $C$ with $C^{*}$, we can show that $\mathcal{B}\left(\theta_{n} \boxtimes C_{m}\right)$ is a 4 -fold basis for $\mathcal{C}\left(\theta_{n} \boxtimes C_{m}\right)$.

Case 3. $m, n$ and $\delta$ are even. Then we define

$$
\mathcal{B}\left(\theta_{n} \boxtimes C_{m}\right)=\left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_{i} a_{i+1}}\right) \cup \mathcal{B}_{a_{n} a_{1}}^{*} \cup \mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K} \cup\left\{C, C_{1}, C_{2}, C_{3}, C_{4}, C_{5}\right\},
$$

where $\mathcal{B}_{a_{i} a_{i+1}}, \mathcal{B}_{a_{n} a_{1}}^{*}, \mathcal{A}_{1}, \mathcal{A}_{2}, \mathcal{K}, C_{1}, C_{2}, C_{3}, C_{4}$ and $C_{5}$ are as defined in Case 2 and $C$ is as in Case 1. By following, word by word, the proof of Case 2 after replacing $C^{*}$ by $C$ we get that $\mathcal{B}\left(\theta_{n} \boxtimes C_{m}\right)$ is a 4 -fold basis.

Case 4. $m$ is even and $\delta$ and $n$ are odd. By relabeling the vertices of $\theta_{n}$ in the opposite direction, we get a similar case to Case 2.

Case 5. $m$ is odd and $n$ and $\delta$ are even. Then we define

$$
\mathcal{B}\left(\theta_{n} \boxtimes C_{m}\right)=\left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_{i} a_{i+1}}\right) \cup \mathcal{B}_{a_{n} a_{1}}^{*} \cup \mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K} \cup\{C\} \cup\left\{C_{1}, C_{2}, C_{3}, C_{4}, C_{5}\right\}
$$

where $\mathcal{B}_{a_{i} a_{i+1}}, \mathcal{B}_{a_{n} a_{1}}^{*}$ and $C$ are as defined in Case 1. Also, $\mathcal{A}_{1}, \mathcal{A}_{2}, \mathcal{K}, C_{2}, C_{4}, C_{5}$, are as in Case 3, and

$$
\begin{aligned}
C_{1} & =(1, m)(2, m)(3, m)(4, m-1)(5, m)(6, m-1) \ldots(\delta, m-1)(1, m) \\
C_{3} & =(1, m-1)(2, m)(3, m-1)(4, m) \ldots(\delta, m)(1, m-1)
\end{aligned}
$$

Let $\mathcal{B}_{1}=\bigcup_{i=1}^{n-1} \mathcal{B}_{a_{i} a_{i+1}} \cup \mathcal{B}_{a_{n} a_{1}}^{*} \cup\{C\}$. Note that $\mathcal{B}_{1}=\mathrm{B}\left(C_{n} \boxtimes C_{m}\right)$ is a basis for $\mathcal{C}\left(C_{n} \boxtimes C_{m}\right)$ (see Theorem 2.6 Case 2 of [11]). Thus, $\mathcal{B}_{1}$ is linearly independent. Note that $E\left(\mathcal{B}_{1}\right) \cap E\left(C_{4}\right)=\{(1,1)(2, m),(2, m)(3,1), \ldots,(\delta-1,1)(\delta, m)\}$ which is an edge set of a path. Thus, by Proposition 3.5, $\mathcal{B}_{1} \cup\left\{C_{4}\right\}$ is linearly independent. Similarly, $E\left(\mathcal{B}_{1} \cup\left\{C_{4}\right\}\right) \cap E\left(C_{5}\right)=$ $\{(1, m)(2,1),(2,1)(3, m), \ldots,(\delta-1, m)(\delta, 1)\}$ which is an edge set of a path. Thus, by Proposition 3.5, $\mathcal{B}_{1} \cup\left\{C_{4}, C_{5}\right\}$ is linearly independent. Also, $E\left(\mathcal{B}_{1} \cup\left\{C_{4}, C_{5}\right\}\right) \cap E\left(C_{2}\right)=\{(1, m)(1,1)$, $(1,1)(\delta, m)\}$ which is an edge set of a forest. Thus, $\mathcal{B}_{1} \cup\left\{C_{2}, C_{4}, C_{5}\right\}$ is linearly independent. Now, since $E\left(C_{1}\right) \cap E\left(C_{3}\right)=\varnothing$ and

$$
\begin{aligned}
E\left(C_{1} \cup C_{3}\right) \cap E\left(\mathcal{B}_{1} \cup\left\{C_{2}, C_{4}, C_{5}\right\}\right) & =\{(i, m-1)(i+1, m) \mid 1 \leq i \leq \delta-1\} \cup \\
\{(i, m)(i+1, m-1) \mid 2 & \leq i \leq \delta-1\} \cup\{(1, m)(2, m),(2, m)(3, m)\}
\end{aligned}
$$

which is an edge set of a tree, we have that $\mathcal{B}_{1} \cup\left\{C_{1}, C_{2}, C_{3}, C_{4}, C_{5}\right\}$ is linearly independent. Now, by a similar argument as in Case 1 , we can show that $\mathrm{B}\left(\theta_{n} \boxtimes C_{m}\right)$ is a 4 -fold basis.

Case 6. $m$ and $n$ are odd and $\delta$ is even. Then we define

$$
\mathcal{B}\left(\theta_{n} \boxtimes C_{m}\right)=\left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_{i} a_{i+1}}\right) \cup \mathcal{B}_{a_{n} a_{1}}^{*} \cup \mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K} \cup\left\{C^{*}, C_{1}, C_{2}, C_{3}, C_{4}, C_{5}\right\}
$$

where $\mathcal{B}_{a_{i} a_{i+1}}, \mathcal{B}_{a_{n} a_{1}}^{*}, \mathcal{A}_{1}, \mathcal{A}_{2}, \mathcal{K}, C_{1}, C_{2}, C_{3}, C_{4}$ and $C_{5}$ are as defined in Case 5 and $C^{*}$ is as in Case 2. To this end, we use the same argument as in Case 5 to show that $\mathcal{B}\left(\theta_{n} \boxtimes C_{m}\right)$ is a 4 -fold basis.

Case 7. $m, n$ and $\delta$ are odd. Then we define

$$
\mathcal{B}\left(\theta_{n} \boxtimes C_{m}\right)=\left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_{i} a_{i+1}}\right) \cup \mathcal{B}_{a_{n} a_{1}}^{*} \cup \mathcal{A}_{1} \cup \mathcal{A}_{2} \cup \mathcal{K} \cup\left\{C^{*}, C_{1}, C_{2}, C_{3}, C_{4}, C_{5}\right\}
$$

where $\mathcal{B}_{a_{i} a_{i+1}}, \mathcal{B}_{a_{n} a_{1}}^{*}$ are as defined above, $\mathcal{K}, \mathcal{A}_{1}, \mathcal{A}_{2}$ ad $C_{2}$ are as in Case 5 and $C^{*}$ is as in Case 2. Also, we set,

$$
\begin{aligned}
C_{1} & =(1, m-1)(2, m)(3, m-1)(4, m) \ldots(\delta-1, m)(\delta, m)(1, m-1) \\
C_{3} & =(1, m)(2, m-1)(3, m)(4, m-1) \ldots(\delta-1, m-1)(\delta, m-1)(1, m) \\
C_{4} & =(1, m)(2, m)(3,1)(4, m)(5,1) \ldots(\delta, 1)(1, m) \\
C_{5} & =(1,1)(2,1)(3, m)(4,1)(5, m) \ldots(\delta, m)(1,1) .
\end{aligned}
$$

As in Case 2, we can see that $\mathcal{B}_{1}=\left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_{i} a_{i+1}}\right) \cup \mathcal{B}_{a_{n} \mathcal{A}_{1}^{(i)}}^{*} \cup C^{*}$ is linearly independent. Now, the cycle $C_{1}$ contains the edge $(\delta, m)(1, m-1)$ which does not appear in any cycle of $\mathcal{B}_{1}$. Hence $\mathcal{B}_{1} \cup\left\{C_{1}\right\}$ is linearly independent. Also, $C_{4}$ contains the edge $(\delta, 1)(1, m)$ which does not appear in any cycle of $\mathcal{B}_{1} \cup\left\{C_{1}\right\}$. Thus, $\mathcal{B}_{1} \cup\left\{C_{1}, C_{4}\right\}$ is linearly independent. $C_{5}$ contains the edge $(1,1)(\delta, m)$ which does not appear in any cycle of $\mathcal{B}_{1} \cup\left\{C_{1}, C_{4}\right\}$. So, $\mathcal{B}_{1} \cup\left\{C_{1}, C_{4}, C_{5}\right\}$ is linearly independent. $C_{2}$ contains the edge $(1, m)(\delta, m)$ which does not appear in any cycle of $\mathcal{B}_{1} \cup\left\{C_{1}, C_{4}, C_{5}\right\}$. Hence $\mathcal{B}_{1} \cup\left\{C_{1}, C_{2}, C_{4}, C_{5}\right\}$ is linearly independent. Finally, $C_{3}$ contains the
edge $(1, m)(\delta, m-1)$ which does not appear in any cycle of $\mathcal{B}_{1} \cup\left\{C_{1}, C_{2}, C_{4}, C_{5}\right\}$. Therefore, $\mathcal{B}_{1} \cup\left\{C_{1}, C_{2}, C_{3}, C_{4}, C_{5}\right\}$ is linearly independent. To this end, to complete this case, we use the same argue as in Case 1.

Case 8. $m$ and $\delta$ are odd and $n$ is even. By relabeling the vertices of $\theta_{n}$ in the opposite direction, we get a similar case to Case 7 .

By noting that $C_{m} \boxtimes \theta_{n}$ is isomorphic to $\theta_{n} \boxtimes C_{m}$, we get the following result:
Corollary 3.1 For any graph $\theta_{n}$ of order $n \geq 4$ and cycle $C_{m}$ of order $m \geq 3$, we have $3 \leq b\left(C_{m} \boxtimes \theta_{n}\right) \leq 4$.

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