Reply to Martino's comments on "The normal, the natural, and the harmonic"

I am indebted to Joseph Martino for bringing to my attention Asimov's article "The Abnormality of Being Normal" that I was unaware of when I wrote my article. It has not been easy obtaining a copy of Asimov's work but with perseverance I succeeded [1]. In it Asimov gives a short course in probability theory to demonstrate that what psychologists call a "normal" person has practically zero probability of occurring in real life. However, Asimov does not push his thesis as far as redefining the concept of a "normal" person the way I do in "The Normal, the Natural, and the Harmonic".

In his criticism Martino tends to reason as a mathematician purist, but in so doing he risks missing the forest for the trees. He points out, for example, that I am using the term "normal" improperly because "it makes no sense to call a single item 'normal' in a statistical sense." Granted! But only for mathematician purists, because psychologists, sociologists, anthropologists, doctors, lawyers, judges, Asimov himself, and just about everyone else uses the term in this way.

More serious is the fact that Martino fails to recognize that the chaos equation is mathematically equivalent to the logistic difference equation intimately linked to the logistic differential equation. Instead he makes unfounded criticism about a "common mistake" people make misinterpreting some "similarity" between the logistic and the chaos equations. There is no mistake or fortuitous similarity; there is only a direct relationship:

$$\frac{dX}{dt} = aX(M-X) \quad \text{Logistic differential equation} \qquad (1)$$

According to the definition of a derivative:

$$\frac{dX}{dt} \equiv \frac{\Delta X}{\Delta t}\Big|_{\Delta t \to 0}$$

if Δt is small but does not tend to 0, we can replace $\frac{dX}{dt}$ in Eq. (1) by $\frac{\Delta X}{\Delta t}$ and obtain

$$(\Delta X)_n = aX_n(M-X_n)\Delta t \quad \text{Logistic difference equation} \qquad \text{Martino's Equation} \quad (2^*)$$

and because $(\Delta X)_n \equiv X_{n+1} - X_n$ we obtain

$$X_{n+1} = X_n + (\Delta X)_n \Delta t = X_n + aX_n(M-X_n)\Delta t \qquad \text{Martino's Equation} \quad (2^{*\prime})$$

Now $\Delta t=1$ corresponds to time steps of size 1 and concords with incrementing n by 1 at a time. A trivial algebraic manipulation then yields:

$$X_{n+1} = aX_n(C-X_n) \quad \text{where} \quad C = \left(\frac{1+aM}{a}\right)$$

which is the *chaos equation*, otherwise known as the *logistic discrete equation*. It may not refer to the exact same S-shaped pattern as the logistic Eq. (1), but it is a bonafide chaos equation and it can give rise to chaotic as well as to S-shaped patterns.

There is absolutely no reason to perform a cumulative sum of Eq. (2*') as Martino suggests. The solution of Eq. (2*') – and consequently of Eq. (3) – is simply the value of X at time n, which is given in terms of the preceding term X_{n-1} and therefore an iteration is the appropriate procedure to follow.

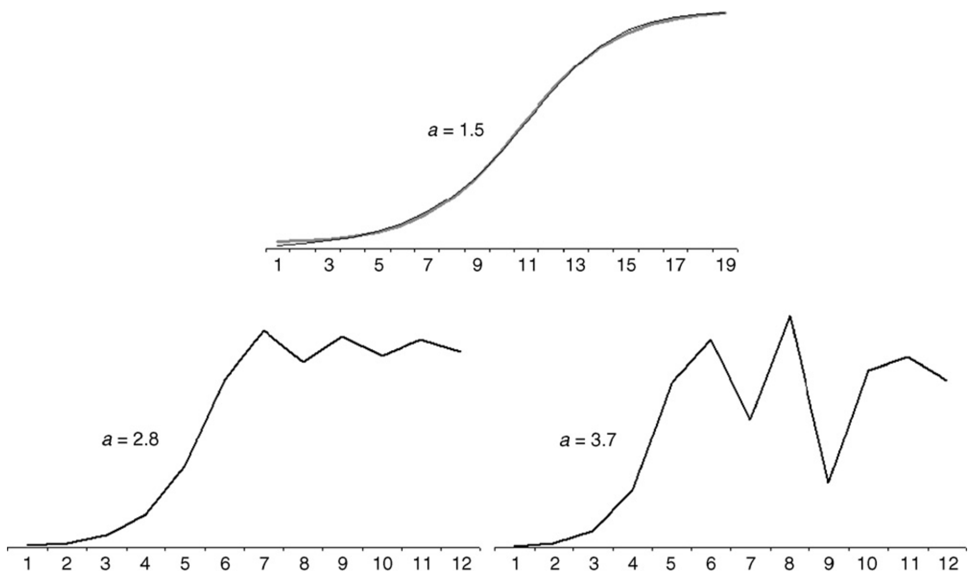
Martino's argument about a fundamental difference between the logistic and the chaos equations is based on the fact that we integrate the former whereas we iterate the latter. But the method used to solve an equation is irrelevant. What counts is finding the solution, namely X as a function of time; among the various techniques used to solve differential and difference equations, integration of the former and iteration of the latter are the most common ones.

Alternatively, one may choose to be more pragmatic and forget about semantics, definitions, nomenclature, and mathematical derivations. Let us just look directly at the solutions of the two equations, namely X as a function of time as given by the integration of the logistic and the iteration of the chaos equation.

The top graph on the drawing below shows the solution $X(t)$ of the logistic equation, Eq. (1), (gray line) as obtained by integration. The superimposed thin black line is the solution X_n of the chaos equation, Eq. (3), for $a=1.5$ as obtained by iteration. The two curves are practically indistinguishable.

The two graphs at the bottom of the drawing show solutions of the chaos equation for $a=2.8$ and $a=3.7$ respectively, the former depicting the onset of damped deviations from an S-shaped pattern and the latter depicting the onset of real chaos.

The S-shaped patterns are not the same in the three graphs, but it is worth noting that deviations occur only as we approach the ceiling of an S-shaped pattern, another obvious fact disputed by Martino.



The reader is invited to verify the plotting of the above graphs. The logistic equation (gray line) has parameters $M=0.328$, $a=0.519$, and $t_o=9.777$. The chaos equation has parameters $X_0=0.005$, $C=1$, and a as indicated.

Martino is right about one thing: chaotic behavior is linked to discretization. In real life everything is discrete; continuous analytic functions are only approximations of reality. In fact, with Alain Debecker we have demonstrated, as I mentioned in the article, that even the smooth S-shaped analytic solution of the logistic equation will produce chaos-like fluctuations if it is subsequently cast in a discrete form; in this case too the irregularities appear as we approach the ceiling but also before we enter the steep rise of the pattern [2].

I will not go further addressing Martino's other criticism that consists mostly of beliefs and anecdotes, and does not render itself to scientific debate. He is entitled, for example, to believe that a quantitative recipe for a harmonic introduction of product replacements is of no added value to a practicing marketer. After all, marketers have survived up to now without it.

References

[1] I. Asimov, The abnormality of being normal, Astounding Science Fiction, May 1956.
[2] T. Modis, A. Debecker, Chaoslike states can be expected before and after logistic growth, Technol. Forecast. Soc. Change 41 (2) (1992).

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