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Star Edge Coloring of Corona Product of Path with Some Graphs

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Abstract: A star edge coloring of a graph G is a proper edge coloring of G, such that any path of length 4 in G is not bicolored, denoted by $\chi'_{st}(G)$, is the smallest integer k for which G admits a star edge coloring with k colors. In this paper, we obtain the star edge chromatic number of $P_m \circ P_n$, $P_m \circ S_n$, $P_m \circ K_{1,n,n}$ and $P_m \circ K_{m,n}$.

Key Words: Star edge coloring, Smarandachely subgraph edge coloring, corona product, path, sunlet graph, double star and complete bipartite.

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§1. Introduction

All graphs considered in this paper are finite and simple, i.e., undirected, loopless and without multiple edges.

The corona of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.

The $n-$sunlet graph on $2n$ vertices is obtained by attaching n pendant edges to the cycle C_n and is denoted by S_n.

Double star $K_{1,n,n}$ is a tree obtained from the star $K_{1,n}$ by adding a new pendant edge of the existing n pendant vertices. It has $2n + 1$ vertices and $2n$ edges.

A star edge coloring of a graph G is a proper edge coloring where at least three distinct colors are used on the edges of every path and cycle of length four, i.e., there is neither bichro-

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matic path nor cycle of length four. The minimum number of colors for which G admits a star edge coloring is called the star edge chromatic index and it is denoted by $\chi'_{st}(G)$. Generally, a Smarandachely subgraphs edge coloring of G for $H_1, H_2, \cdots, H_m \prec G$ is such a proper edge coloring on G with at least three distinct colors on edges of each subgraph H_i, where $1 \leq i \leq m$.

The star edge coloring was initiated in 2008 by Liu and Deng [8], motivated by the vertex version (see [1, 3, 4, 6, 7, 10]). Dvořák, Mohar and Šámal [5] determined upper and lower bounds for complete graphs. Additional graph theory terminology used in this paper can be found in [2].

§2. Preliminaries

Theorem 2.1([5]) *The star chromatic index of the complete graph K_n satisfies*

$$2n\left(1+\mathcal{O}(n)\right) \leq \chi'_{st}(K_n) \leq n\frac{2^{2\sqrt{2}(1+\mathcal{O}(1))\sqrt{\log n}}}{\left(\log n^{\frac{1}{4}}\right)}$$

In particular, for every $\in > 0$ there exists a constant c such that $\chi'_{st}(K_n) \leq cn^{1+c}$ for every $n \geq 1$.

They asked what is true order of magnitude of $\chi'_{st}(K_n)$, in particular, if $\chi'_{st}(K_n) = \mathcal{O}(n)$. From Theorem 2.1, they also derived the following near-linear upper bound in terms of the maximum degree Δ for general graphs.

Theorem 2.2([5]) *Let G be an arbitrary graph of maximum degree Δ. Then*

$$\chi'_{st}(G) \leq \chi'_{st}(K_{n+1}) \cdot \mathcal{O}\left(\frac{\log \Delta}{\log\log \Delta}\right)^2$$

and therefore $\chi'_{st}(G) \leq \Delta \cdot 2^{\mathcal{O}(1)\sqrt{\log \Delta}}$.

Theorem 2.3([5])

(a) If G is a subcubic graph, then $\chi'_{st}(G) \leq 7$.

(b) If G is a simple cubic graph, then $\chi'_{st}(G) \geq 4$, and the equality holds if and only if G covers the graph of the 3-cube.

A graph G covers a graph H if there is a locally bijective graph homomorphism from G to H. While there exist cubic graphs with the star chromatic index equal to 6. e.g., $K_{3,3}$ or Heawood graph, no example of a subcubic graph that would require 7 colors is known. Thus, Dvořák et al. proposed the following conjecture.

Conjecture 2.4([5]) *If G is a subcubic graph, then $\chi'_{st}(G) \leq 6$.*

Theorem 2.5([9]) *Let T be a tree with maximum degree Δ. Then*

$$\chi'_{st}(T) \leq \left\lfloor \frac{3}{2}\Delta \right\rfloor.$$

Moreover, the bound is tight.

Theorem 2.6([9]) *Let G be an outerplaner graph with maximum degree Δ. Then*

$$\chi'_{st}(G) \leq \left\lfloor \frac{3}{2}\Delta \right\rfloor + 12.$$

Lemma 2.7([9]) *Every outerplanar embedding of a light cactus graph admits a proper 4-edge coloring such that no bichromatic 4-path exists on the boundary of the outer face.*

Theorem 2.8([9]) *Let G be an subcubic outerplaner graph. Then,*

$$\chi'_{st}(G) \leq 5.$$

Conjecture 2.9([9]) *Let G be an outerplaner graph with maximum degree $\Delta \geq 3$. Then*

$$\chi'_{st}(G) \leq \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1.$$

For graphs with maximum degree $\Delta = 2$, i.e. for paths and cycles, there exist star edge coloring with at most 3 colors except for C_5 which requires 4 colors. In case of subcubic outerplanar graphs the conjecture is confirmed by Theorem 2.8.

§3. Main Results

Theorem 3.1 *For any positive integer m and n, then*

$$\chi'_{st}(P_m \circ P_n) = \begin{cases} n & \text{if } m = 1 \\ n+1 & \text{if } m = 2 \\ n+2 & \text{if } m \geq 3 \end{cases}$$

Proof Let $V(P_m) = \{u_i : i = 1, 2, \cdots, m\}$ and $V(P_n) = \{v_j : j = 1, 2, \cdots, n\}$. Let $E(P_m) = \{u_i u_{i+1} : i = 1, 2, \cdots, m-1\}$ and $E(P_n) = \{v_j v_{j+1} : j = 1, 2, \cdots, n-1\}$. By the definition of corona product,

$$\begin{aligned} V(P_m \circ P_n) &= V(P_m) \bigcup_{i=1}^{m} V\left(P_n^i\right), \\ E(P_m \circ P_n) &= E(P_m) \bigcup_{i=1}^{m} E\left(P_n^i\right) \bigcup_{i=1}^{m} \{u_i v_{i,j} : 1 \leq j \leq n\}. \end{aligned}$$

Let σ be a mapping from $E(P_m \circ P_n)$ as follows:

Case 1. For $m = 1$,

$$\begin{cases} \sigma(u_i v_{i,j}) = i + j - 2 \,(\text{mod } n), 1 \le j \le n; \\ \sigma(v_{i,j} v_{i,j+1}) = i + j \,(\text{mod } n), 1 \le j \le n-1; \end{cases}$$

Case 2. For $m = 2$,

$$\begin{cases} \text{For } i = 1, 2, \\ \sigma(u_i v_{i,j}) = i + j - 2 \,(\text{mod } n+1), 1 \le j \le n; \\ \sigma(v_{i,j} v_{i,j+1}) = i + j \,(\text{mod } n+1), 1 \le j \le n-1; \\ \sigma(u_1 u_2) = n; \end{cases}$$

Case 3 For $m \ge 3$, $\sigma(u_i u_{i+1}) = n + 2 \,(\text{mod } n+3), 1 \le i \le m-1$;

$$\begin{cases} \text{For } 1 \le i \le m, \\ \sigma(u_i v_{i,j}) = i + j - 2 \,(\text{mod } n+3), 1 \le j \le n; \\ \sigma(v_{i,j} v_{i,j+1}) = i + j \,(\text{mod } n+1), 1 \le j \le n-1; \end{cases}$$

It is easy to see that σ is satisfied length of path-4 are not bicolored. To prove

$$\chi'_{st}(P_m \circ P_n) \le \begin{cases} n & \text{if } m = 1 \\ n+1 & \text{if } m = 2 \\ n+2 & \text{if } m \ge 3. \end{cases}$$

we have

$$\chi'_{st}(P_m \circ P_n) \ge \chi'(P_m \circ P_n) \ge \Delta(P_m \circ P_n) \ge \begin{cases} n & \text{if } m = 1 \\ n+1 & \text{if } m = 2 \\ n+2 & \text{if } m \ge 3. \end{cases}$$

Thus the conclusion is true. □

Theorem 3.2 *For any positive integer m and n, then*

$$\chi'_{st}(P_m \circ S_n) = \begin{cases} 2n & \textit{if } m = 1 \\ 2n+1 & \textit{if } m = 2 \\ 2n+2 & \textit{if } m \ge 3. \end{cases}$$

Proof Let $V(P_m) = \{u_i : i = 1, 2, \cdots, m\}$ and $V(S_n) = \{v_j : j = 1, 2, \cdots, n\} \cup \{v_{n+j} : j = 1, 2, \cdots, n\}$. Let $E(P_m) = \{u_i u_{i+1} : i = 1, 2, \cdots, m-1\}$ and $E(S_n) = \{v_j v_{j+1} : j = 1, 2,$

$\cdots, n-1\} \cup \{v_{n-1}v_n\} \cup \{v_j v_{n+j} : j = 1, 2, \cdots, n\}$, where v_{n+j}'s are pendent edges of v_j. By the definition of corona product,

$$\begin{aligned} V(P_m \circ S_n) &= V(P_m) \bigcup_{i=1}^{m} V\left(S_n^i\right), \\ E(P_m \circ S_n) &= E(P_m) \bigcup_{i=1}^{m} E\left(S_n^i\right) \bigcup_{i=1}^{m} \{u_i v_{i,j} : 1 \leq j \leq 2n\} \end{aligned}$$

Let σ be a mapping from $E(P_m \circ S_n)$ as follows:

Case 1. For $m = 1$,

$$\begin{cases} \sigma(u_i v_{i,j}) = j - 1 \,(\text{mod } 2n), 1 \leq j \leq 2n; \\ \sigma(v_{i,j} v_{i,j+1}) = i + j \,(\text{mod } 2n), 1 \leq j \leq n-1; \\ \sigma(v_{i,j} v_{i,n+j}) = n + i + j \,(\text{mod } 2n), 1 \leq j \leq n; \\ \sigma(v_{i,n-1} v_{i,n}) = n + 1; \end{cases} \tag{1}$$

Case 2. For $m = 2$,

$f(u_1 u_2) = 2n$ and using Equation (1).

Case 3. For $m \geq 3$, $\sigma(u_i u_{i+1}) = 2n + i \,(\text{mod } 2n+2), 1 \leq i \leq m-1$;

$$\begin{cases} \text{For } 1 \leq i \leq m, \\ \sigma(u_i v_{i,j}) = i + j - 2 \,(\text{mod } 2n+2), 1 \leq j \leq 2n; \\ \sigma(v_{i,j} v_{i,j+1}) = i + j \,(\text{mod } 2n+2), 1 \leq j \leq n-1; \\ \sigma(v_{i,j} v_{i,n+j}) = n + i + j \,(\text{mod } 2n+2), 1 \leq j \leq n; \\ \sigma(v_{i,n-1} v_{i,n}) = n + i \,(\text{mod } 2n+2); \end{cases}$$

It is easy to see that σ is satisfied length of path-4 are not bicolored. To prove

$$\chi'_{st}(P_m \circ S_n) \leq \begin{cases} 2n & \text{if } m = 1 \\ 2n+1 & \text{if } m = 2 \\ 2n+2 & \text{if } m \geq 3. \end{cases}$$

we have

$$\chi'_{st}(P_m \circ S_n) \geq \chi'(P_m \circ S_n) \geq \Delta(P_m \circ S_n) \geq \begin{cases} 2n & \text{if } m = 1 \\ 2n+1 & \text{if } m = 2 \\ 2n+2 & \text{if } m \geq 3. \end{cases}$$

Thus the conclusion is true. □

Theorem 3.3 *For any positive integer m and n, then*

$$\chi'_{st}\left(P_m \circ K_{1,n,n}\right) = \begin{cases} 2n+1 & if \;\; m=1 \\ 2n+2 & if \;\; m=2 \\ 2n+3 & if \;\; m\geq 3 \end{cases}$$

Proof Let $V\left(P_m\right) = \{u_i : i = 1, 2, \cdots, m\}$ and $V\left(K_{1,n,n}\right) = \{v_0\} \cup \{v_{2j-1} : j = 1, 2, \cdots, n\}$ $\cup \{v_{2j} : j = 1, 2, \cdots, n\}$. Let $E\left(P_m\right) = \{u_i u_{i+1} : i = 1, 2, \cdots, m-1\}$, $E\left(K_{1,n,n}\right) = \{v_0 v_{2j-1} : j = 1, 2, \cdots, n\} \cup \{v_{2j-1} v_{2j} : j = 1, 2, \cdots, n\}$, where v_0 is adjacent to v_{2j-1} and v_{2j} are pendent vertices of v_{2j-1}. By the definition of corona product,

$$\begin{aligned} V\left(P_m \circ K_{1,n,n}\right) &= V\left(P_m\right) \bigcup_{i=1}^{m} V\left(K^i_{1,n,n}\right), \\ E\left(P_m \circ K_{1,n,n}\right) &= E\left(P_m\right) \bigcup_{i=1}^{m} E\left(K^i_{1,n,n}\right) \bigcup_{i=1}^{m} \{u_i v_{i,j} : 0 \leq j \leq 2n\} \end{aligned}$$

Let σ be a mapping from $E\left(P_m \circ K_{1,n,n}\right)$ as follows:

Case 1. For $m = 1$,

$$\begin{cases} \sigma\left(u_i v_{i,j}\right) = j \bmod 2n, 0 \leq j \leq 2n; \\ \sigma\left(v_{i,0} v_{i,2j-1}\right) = 2j+2 \pmod{2n+1}, 1 \leq j \leq n; \\ \sigma\left(v_{i,2j-1} v_{i,2j}\right) = 2j+3 \pmod{2n+1}, 1 \leq j \leq n; \end{cases} \tag{2}$$

Case 2. For $m = 2$,

$\sigma\left(u_1 u_2\right) = 2n + 1$; and using Equation (2).

Case 3. For $m \geq 3$,

$\sigma\left(u_i u_{i+1}\right) = 2n + i \pmod{2n+3}, 1 \leq i \leq m-1$;

$$\begin{cases} \text{For } 1 \leq i \leq m, \\ \sigma\left(u_i v_{i,j}\right) = i + j - 1 \pmod{2n+3}, 0 \leq j \leq 2n; \\ \sigma\left(v_{i,0} v_{i,2j-1}\right) = i + 2j - 1 \pmod{2n+3}, 1 \leq j \leq n; \\ \sigma\left(v_{i,2j-1} v_{i,2j}\right) = i + 2j \pmod{2n+3}, 1 \leq j \leq n; \end{cases}$$

It is easy to see that σ is satisfied length of path-4 are not bicolored. To prove

$$\chi'_{st}\left(P_m \circ K_{1,n,n}\right) \leq \begin{cases} 2n+1 & \text{if} \;\; m=1 \\ 2n+2 & \text{if} \;\; m=2 \\ 2n+3 & \text{if} \;\; m\geq 3. \end{cases}$$

we have

$$\chi'_{st}\left(P_m \circ K_{1,n,n}\right) \geq \chi'\left(P_m \circ K_{1,n,n}\right) \geq \Delta\left(P_m \circ K_{1,n,n}\right) \geq \begin{cases} 2n+1 & \text{if } m=1 \\ 2n+2 & \text{if } m=2 \\ 2n+3 & \text{if } m \geq 3. \end{cases}$$

So the conclusion is true. □

Theorem 3.4 *For any positive integer $l \geq 3$, $m \geq 3$ and $n \geq 3$, then*

$$\chi'_{st}\left(P_l \circ K_{m,n}\right) = m+n+2.$$

Proof Let $V\left(P_l\right) = \{u_i : 1 \leq i \leq l\}$ and $V\left(K_{m,n}\right) = \{v_j : 1 \leq j \leq m\} \cup \{v'_k : 1 \leq k \leq n\}$. Let $E\left(P_l\right) = \{u_i u_{i+1} : 1 \leq i \leq l-1\}$ and $E\left(K_{m,n}\right) = \bigcup_{j=1}^{m} \{v_j v'_k : 1 \leq k \leq n\}$. By the definition of corona product,

$$\begin{aligned} V\left(P_l \circ K_{m,n}\right) &= V\left(P_l\right) \bigcup_{i=1}^{l} \{v_{ij} : 1 \leq j \leq m\} \bigcup_{i=1}^{l} \{v'_{ik} : 1 \leq k \leq n\}, \\ E\left(P_l \circ K_{m,n}\right) &= E\left(P_l\right) \bigcup_{i=1}^{l} E\left(K^i_{m,n}\right) \bigcup_{i=1}^{l} \{u_i v_{ij} : 1 \leq j \leq m\} \bigcup_{i=1}^{l} \{u_i v'_{ik} : 1 \leq k \leq n\}. \end{aligned}$$

Let σ be a mapping from $P_l \circ K_{m,n}$ as follows:

$\sigma\left(u_{2i-1} u_{2i}\right) = n-1, 1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor$; $\sigma\left(u_{2i} u_{2i+1}\right) = n, 1 \leq i \leq \left\lceil \frac{l}{2} \right\rceil$ and

$$\begin{cases} \text{For } 1 \leq i \leq l, \\ \sigma\left(v_{ij} v'_{ik}\right) = j+k-1, 1 \leq j \leq m, 1 \leq k \leq n; \\ \sigma\left(u_i v_{ij}\right) = n+j, 1 \leq j \leq m; \\ \sigma\left(u_i v'_{ik+2}\right) = k, 1 \leq k \leq n-2; \\ \sigma\left(u_i v'_{i1}\right) = m+n+1; \\ \sigma\left(u_i v'_{i2}\right) = m+n+2. \end{cases}$$

Clearly above color partitions are satisfied length of path-4 are not bicolored. We assume that $\chi'_{st}\left(P_m \circ K_{m,n}\right) \leq m+n+2$. We know that $\chi'_{st}\left(P_m \circ K_{m,n}\right) \geq \chi'\left(P_m \circ K_{m,n}\right) \geq m+n+2$, since $\chi'_{st}\left(P_m \circ K_{m,n}\right) \geq m+n+2$. Therefore $\chi'_{st}\left(P_m \circ K_{m,n}\right) = m+n+2$. □

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