International J.Math. Combin. Vol.3(2015), 01-32

A Calculus and Algebra Derived from Directed Graph Algebras

Kh.Shahbazpour and Mahdihe Nouri

(Department of Mathematics, University of Urmia, Urmia, I.R.Iran, P.O.BOX, 57135-165)

E-mail: shahbazpour@hotmail.com

Abstract: Shallon invented a means of deriving algebras from graphs, yielding numerous examples of so-called graph algebras with interesting equational properties. Here we study directed graph algebras, derived from directed graphs in the same way that Shallon's undirected graph algebras are derived from graphs. Also we will define a new map, that obtained by Cartesian product of two simple graphs p_n, that we will say from now the mah-graph. Next we will discuss algebraic operations on mah-graphs. Finally we suggest a new algebra, the mah-graph algebra (Mah-Algebra), which is derived from directed graph algebras.

Key Words: Direct product, directed graph, Mah-graph, Shallon algebra, kM-algebra

AMS(2010): 08B15, 08B05.

§1. Introduction

Graph theory is one of the most practical branches in mathematics. This branch of mathematics has a lot of use in other fields of studies and engineering, and has competency in solving lots of problems in mathematics. The Cartesian product of two graphs are mentioned in[18]. We can define a graph plane with the use of mentioned product, that can be considered as isomorphic with the plane $\mathbb{Z}^+ * \mathbb{Z}^+$.

Our basic idea is originated from the nature. Rivers of one area always acts as unilateral courses and at last, they finished in the sea/ocean with different sources. All blood vessels from different part of the body flew to heart of beings. The air-lines that took off from different part of world and all landed in the same airport. The staff of an office that worked out of home and go to the same place, called Office, and lots of other examples give us a new idea of directed graphs. If we consider directed graphs with one or more primary point and just one conclusive point, in fact we could define new shape of structures.

A km-map would be defined on a graph plane, made from Cartesian product of two simple graphs $p_n * p_n$. The purpose of this paper is to define the kh-graph and study of a new structure that could be mentioned with the definition of operations on these maps.

The km-graph could be used in the computer logic, hardware construction in smaller size with higher speed, in debate of crowded terminals, traffics and automations. Finally by rewriting

[1]Received January 1, 2015, Accepted August 6, 2015.

the km-graphs into mathematical formulas and identities, we will have interesting structures similar to Shallon's algebra (graph algebra).

At the first part of paper, we will study some preliminary and essential definitions. In section 3 directed graphs and directed graph algebras are studied. The graph plane and mentioned km-graph and its different planes and structures would be studied in section 4.

§2. Basic Definitions and Structures

In this section we provide the basic definitions and theorems for some of the basic structures and ideas that we shall use in the pages ahead. For more details, see [18], [23], [12].

Let A be a set and n be a positive integer. We define A^n to be the set of all n-tuples of A, and $A^0 = \emptyset$. The natural number n is called the *rank* of the operation, if we call a map $\phi : A^n \to A$ an $n - ary$ *operation* on A. Operations of rank 1 and 2 are usually called unary and binary operations, respectively. Also for all intends and purposes, nullary operations (as those of rank 0 are often called) are just the elements of A. They are frequently called constants.

Definition 2.1 *An algebra is a pair* $< A, F >$ *in which A is a nonempty set and* $F =< f_i : i \in I >$ *is a sequence of operations on A, indexed by some set I. The set A is the universe of the algebra, and the* f_i*'s are the fundamental or basic operations.*

For our present discussion, we will limit ourselves to finite algebras, that is, those whose universes are sets of finite cardinality . The *equational theory* of an algebra is the set consisting of all equations true in that algebra. In the case of groups, one such equation might be the associative identity. If there is a finite list of equations true in an algebra from which all equations true in the algebra can be deduced, we say the algebra is *finitely based*. For example in the class of one-element algebras, each of those is finitely based and the base is simply the equation $x \approx y$. We typically write A to indicate the algebra $< A, F >$ expect when doing so causes confusion. For each algebra A, we define a map $\rho : I \to \omega$ by letting $\rho(i) = rank(F_i)$ for every $i \in I$. The set I is called the *set of operation symbols*. The map ρ is known as the *signature* of the algebra A and it simply assigns to each operation symbol the natural number which is its rank. When a set of algebras share the same signature, we say that they are *similar* or simply state their shared signature.

If κ is a class of similar algebras, we will use the following notations:

$H(\kappa)$ represents the class of all homomorphic images of members of κ;

$S(\kappa)$ represents the class of all isomorphic images of sub algebras of members of κ;

$P(\kappa)$ represents the class of all direct products of system of algebras belonging to κ.

Definition 2.2 *The class* ν *of similar algebras is a variety provided it is closed with respect to the formation of homomorphic images, sub algebras and direct products.*

According to a result of Birkhoff (Theorem 2.1), it turns out that ν is a variety precisely if it is of the form $HSP(\kappa)$ for some class κ of similar algebras. We use $HSP(A)$ to denote the variety generated by an algebra A. The equational theory of an algebra is the set consisting of all equations true in that algebra. In order to introduce the notion of equational theory, we begin by defining the set of terms.

Let $T(X)$ be the set of all terms over the alphabet $X = \{x_0, x_1, \cdots\}$ using juxtaposition and the symbol ∞. $T(X)$ is defined inductively as follows:

(i) every $x_i, (i = 0, 1, 2, \cdots)$ (also called variables) and ∞ is a term;

(ii) if t and t' are terms, then (tt') is a term;

(iii) $T(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.

The *left most variable* of a term t is denoted by $Left(t)$. A term in which the symbol ∞ occurs is called trivial. Let $T'(X)$ be the set of all non-trivial terms. To every non-trivial term t we assign a directed graph $G(t) = (V(t), R(t))$ where $V(t)$ is the set of all variables and $R(t)$ is defined inductively by $R(t) = \emptyset$ if $t \in X$ and $R(tt') = R(t) \cup R(t') \cup \{Left(t), Left(t')\}$. Note that $G(t)$ always a connected graph.

An *equation* is just an ordered pair of terms. We will denote the equation (s, t) by $s \approx t$. We say an equation $s \approx t$ is true in an algebra A provided s and t have the same signature. In this case, we also say that A is a model of $s \approx t$, which we will denote by $A \models s \approx t$.

Let κ be a class of similar algebras and let Σ be any set of equations of the same similarity type as κ. We say that κ is a class of models Σ(or that Σ is true in κ) provided $A \models s \approx t$) for all algebras A found in κ and for all equations $s \approx t$ found in Σ. We use $\kappa \models \Sigma$ to denote this, and we use $Mod\Sigma$ to denote the class of all models of Σ.

The set of all equations true in a variety ν (or an algebra A) is known as the *equational theory* of ν (respectively, A). If Σ is a set of equations from which we can derive the equation $s \approx t$, we write $\Sigma \vdash s \approx t$ and we say $s \approx t$ is *derivable* from Σ. In 1935, Garrett Birkhoff proved the following theorem:

Theorem 2.1 (Bikhoffs HSP Theorem) *Let ν be a class of similar algebras. Then ν is a variety if and only if there is a set σ of equations and a class κ of similar algebras so that $\nu = HSP(\kappa) = Mod\Sigma$.*

From this theorem, we have a clear link between the algebraic structures of the variety ν and its equational theory.

Definition 2.3 *A set Σ of equations is a base for the variety ν (respectively, the algebra A) provided ν (respectively, HSP(A)) is the class of all models of* Σ.

Thus an algebra A is finitely based provided there exists a finite set Σ of equations such that any equation true in A can be derived from Σ. That is, if $A \models s \approx t$ and Σ is a finite base of the equational theory of A, then $\Sigma \vdash s \approx t$. If a variety or an algebra does not have a finite base, we say that it fails to finitely based and we call it *non-finitely based*.

We say an algebra is *locally finite* provided each of its finitely generated sub algebras is finite, and we say a variety is locally finite if each of its algebras is locally finite.

A useful fact is the following:

Theorem 2.2 *Every variety generated by a finite algebra is locally finite.*

Thus if **A** is an inherently nonfinitely based finite algebra that is a subset of **B**, where **B** is also finite, then **B** must also be inherently nonfinitely based. It is in this way that the property of being inherently nonfinitely based is contagious. In an analogous way, we define an inherently non-finitely based variety ϑ as one in which the following conditions occur:

(i) ϑ has a finite signature;
(ii) ϑ is locally finite;
(iii) ϑ is not included in any finitely based locally finite variety.

Let ν be a variety and let $n \in \omega$. The class $\nu^{(n)}$ of algebras is defined by the following condition:

An algebra B is found in $\nu^{(n)}$ if and only if every sub algebra of B with n or fewer generators belongs to ν. Equivalently, we might think of $\nu^{(n)}$ as the variety defined by the equations true in $\nu(n)$ that have n or fewer variables. Notice that $(\nu \subseteq \cdots \subseteq \nu^{(n+1)} \subseteq \nu^{(n)} \subseteq \nu^{(n-1)} \subseteq \cdots)$ and $\nu = \bigcap_{n\in\omega}^{nu^{(n)}}$.

A nonfinitely based algebra A might be nonfinitely based in a more infectious manner:

It might turn out that if A is found in $HSP(B)$, where B is a finite algebra, then B is also nonfinitely based. This leads us to a stronger non finite basis concept.

Definition 2.4 *An algebra A is inherently non-finitely based provided:*

(i) A has only finitely many basic operations;
(ii) A belongs to some locally finite operations;
(iii) A belong to no locally finite variety which is finitely based.

In an analogous way, we say that a locally finite variety v of finite signature is inherently nonfinitely based provided it is not included in any finitely based locally finite variety. In [2], Birkhoff observed that $v^{(n)}$ is finitely based whenever v is a locally finite variety of finite signature. As an example, let v be a locally finite variety of finite signature. If the only basic operations of v are either of rank 0 or rank 1, then every equation true in v can have at most two variables. In other words, $v = v^{(2)}$ and so v is finitely based. From Birkhoff's observation, we have the following as pointed out by McNulty in [15]:

Theorem 2.3 *Let v be locally finite variety with a finite signature. Then the following conditions are equivalent:*

(i) v is inherently non-finitely based;
(ii) The variety $v^{(n)}$ is not locally finite for any natural number n;
(iii) For arbitrary large natural numbers N, there exists a non-locally finite algebra B_N whose N-generated sub algebras belong to v.

Thus to show that a locally finite variety v of finite signature is inherently nonfinitely based, it is enough to construct a family of algebras B_n (for each $n \in \omega$) so that each B_n fails to be locally finite and is found inside $v^{(n)}$.

In 1995, Jezek and McNulty produced a five element commutative directoid and showed that while it is nonfinitely based, it fails to be inherently based [17]. This resolved the original question of jezek and Quackenbush but did not answer the following:

Is there a finite commutative directed that is inherently non-finitely based?

In 1996, E.Hajilarov produced a six-element commutative directoid and asserted that it is inherently based [5]. We will discuss an unresolved issue about Hajilarov's directoid that reopens its finite basis question. We also provide a partial answer to the modified question of jezek and Quackenbush by constructing a locally finite variety of commutative directoids that is inherently nonfinitely based.

A *sub direct representation* of an algebra $\mathbf{A}$ is a system $< h_i : i \in I >$ of homomorphisms, all with domain A, that separates the points of A: that is, if a and b are distinct elements of A, ten there is at least one $i \in I$ so that $h_i(a) \neq h_i(b)$. The algebra $h_i(A)$ are called **subdirectfactors** of the representation. Starting with a complicated algebra $\mathbf{A}$, one way to better understand its structure is to analyze a system $< h_i(A)\ i \in I >$ of potentially less complicated homomorphic image, and a sub direct representation of $\mathbf{A}$ provides such a system.

The *residual bound* of a variety ϑ is the least cardinal κ (should one exist) such that for every algebra $\mathbf{A} \in \vartheta$ there is a sub direct representation $< h_i i \in I >$ of $\mathbf{A}$ such that each sub direct factor has fewer than κ elements. If a variety ν of finite signature has a finite residual bound, it also satisfies the following condition: there is a finite set $\mathbf{S}$ of finite algebras belonging to ϑ so that every algebra in ϑ has a sub direct representation using only sub direct factors from $\mathbf{s}$.

According to a result of Robert Quackenbush, if a variety generated by a finite algebra has an infinite sub directedly irreducible member, it must also have arbitrarily large finite once [20]. In 1981, Wieslaw Dziobiak improved this result by showing that the same holds in any locally finite variety [3]. A Problem of Quackenbush asks whether there exists a finite algebra such that the variety it generates contains infinitely many distinct (up to isomorphism) sub directly irreducible members but no infinite ones. One of the thing Ralph McKenzie did in [13] was to provide an example of 4-element algebra of countable signature that generates a variety with this property. Whether or not this is possible with an algebra with only finitely many basic operations is not yet know.

If all of the sub directly irreducible algebra in a variety are finite, we say that the variety is *residual finite*. Starting with a finite algebra, there is no guarantee that the variety it generates is residually finite, nor that the algebra is finitely based. The relationship between these three finiteness conditions led to the posing of the following problem in 1976:

Is every finite algebra of finite signature that generates a variety with a finite a finite residual bound finitely based?

Bjarni Jonsson posed this problem at a meeting at a meeting at the Mathematical Research Institute in Oberwolfach while Robert Park offered it as a conjecture in his PH.D. dissertation [19]. At the time this problem was framed, essentially only five nonfinitely based finite algebras were known. Park established that none of these five algebras could be a counterexample. In the ensuing years our supply of nonfinitely based finite algebras has become infinite and varied, yet no counterexample is known. Indeed, Ros Willard [26] has offered a 50 euro reward for the first published example of such an algebra. In chapter 2, we show that a wide class of algebras known to be nonfinitely based will not supply such an example. It is still an open problem whether some of the nonfinitely based finite algebras known today generate varieties with finite residual bound. while the condition of generating a variety with a finite residual bound could well be sufficient to ensure that a finite algebra is finitely based, it is in its own right a very subtle property of finite algebras. Indeed, Ralph McKenzie has shown that there is no algorithm for recognizing when a finite algebra has this property [13]. The last question motivating our research was originally formulated in 1976 by Eilenberg and Schutzenberger [4]. In their investigation of pseudo varieties, they ask, If ϑ is a variety generated by a finite algebra, W is a finitely based variety, and ϑ and W share the same finite algebras, must ϑ be finitely based?

To answer this question in the negative, one would need to supply a finite, nonfinitely based algebra to generate ϑ and a finitely based variety W so that ϑ and W have the same finite algebras. McNulty, Szekely, and Willard have proven that no counterexample can be found among a wide class of finite, non finitely based algebras [26]; furthermore they noted that this property also cannot be recognized by any algorithm.We show that the locally finite, inherently based variety of commutative directoids we construct will fail to yield a counterexample if it is shown to be generated by a finite algebra.

§3. Directed Graphs and Directed Graph Algebras

Definition 3.1 *A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each two vertexes called its conclusive points.*

Definition 3.2 *A path in the directed graph (v, E) is an ordered $(n+1)-tuple$ $(x_1, \cdots, x_{n+1})$ such that $(x_i, x_{i+1}) \in E$ for all $i = 1, \cdots, n$. The path $(x_1, ..., x_{n+1})$ has length n. A cycle is a path from some vertex to itself. Given a graph G and $x \in V_G$, $[x\rangle_G$, (or just $[x\rangle$, when G is clear), is the set of $y \in V_G$ such that there is a path from x to y in G. A directed graph is acyclic if it contains no cycles. A directed graph (V, E) is loop-free if there is no cycle of length 1, that is if there is no $x \in V$ such that $(x, x) \in E$. A directed graph (V, E) is looped if there is a loop at every vertex, that is if $(x, x) \in E$ for every $x \in V$.*

Definition 3.3 *A directed graph or digraph $G =< V, E >$ is a triple consisting of a nonempty set $V(G)$ of elements called vertexes, together with a set $E(G)$ of ordered pairs from $V \times V \to V$, called edges, and a map that assigning to each edge an ordered pair of vertexes.*

Thus our directed graphs do not allow multiple edges, but they do allow edges of the form (x, x) (that is, we allow vertexes to be looped). Given a directed graph G, we can refer to the vertex set and edge set of G as V_G and E_G, respectively. Let us say that $\acute{G}$ is a subgraph of G if $V_{\acute{G}} \subseteq V_G$ and $E_{\acute{G}} = E_G \bigcap (V_{\acute{G}} \times V_{\acute{G}})$. When we draw a directed graph, we generally draw the edge (x, y) as an arrow from vertex x to y. When drawing an undirected graph, we simply draw the edge (x, y) as a line from x to y; since we know that (y, x) must also be in the graph edge set, there is no question of which way the edge goes. Let us call any variety generated by directed graph algebras a directed graph variety. As noted in [25], any directed graph variety ν contains $A(G)$ for all G that are direct products, subgraphs, disjoint unions, directed unions and homomorphic images of directed graphs underlying algebra in ν. More generally, any variety is closed under homomorphisms, sub algebras and direct products. We shall need this more general fact to obtain the results in section. The following definition is from [25].

Definition 3.4([25]) *Let $G = (V, E)$ be a directed graph. The directed graph algebra $A(G)$ is the algebra with underlying set $\{v \bigcup \infty\}$, where $\infty \notin V$, and two basic operations: one nullary operation, also denoted by ∞, which has value ∞, and one binary operation, sometimes called multiplication, denoted by juxtaposition, which is given by*

$$(u, v) = \begin{cases} u & if (u, v) \in E \\ \infty & otherwise \end{cases}$$

Let $G =< V, E >$ be a complete undirected graph. We define a tournament T as an algebra with universe $V \cup \{\infty\}$ (where $\infty \notin V$) and binary relation $\twoheadrightarrow$ so that for distinct $x, y \in V$ exactly one of $x \twoheadrightarrow y$ and $y \twoheadrightarrow x$ is true. We make each edge directed using the relation $\twoheadrightarrow$, that is, we make the edge between x and y directed toward y if and only if $x \twoheadrightarrow y$. We can then make $\twoheadrightarrow$ into a binary operation as follows:

$$x.y = y.x = \begin{cases} x & if x \twoheadrightarrow y \\ \infty & otherwise \end{cases}$$

The relation $x \twoheadrightarrow y$ is generally read as "x defeats y" or " y loses to x" in the tournament T. Notice that in tournament we require a directed edge between any two distinct vertexes. When we draw a tournament, we will represent the ordered edge between any two distinct vertexes. When we draw a tournament, we will represent the ordered pair $(x, y) \in \twoheadrightarrow$ as vertexes joined by a double-headed arrow pointing away from $x.y$.

A semi-tournament T is simply a tournament in which we relax the restriction that the underlying graph G be complete. In this case, if x and y are distinct vertexes in v and neither $x \twoheadrightarrow y$ nor $y \twoheadrightarrow x$, we define $x.y = y.x = \infty$. In this way the element ∞ acts as a default element. The table of Park's semi-tournament P with three element is shown in Figure 2.

O	t	s	r	∞
t	t	t	∞	∞
s	t	s	s	∞
r	∞	s	r	∞
∞	∞	∞	∞	∞

Let v be the variety generated by p. We obtain graph algebras in exactly the same way, except that in a graph algebra the underlying G is an undirected graph . one difference between our terminology and the terminology of the authors of [25] is that they refer to algebras $A(G)$ defined as graph algebras, while we refer to such algebras as directed graph algebra. Our purpose in doing so is to avoid confusion with the (undirected) graph algebra of [25]. Let $T(X)$ be the set of all terms over a set X of variables in the type of directed graph algebras. We shall make frequent use of the following definition and lemma from Kiss-poschel-prohle [25].

Definition 3.5([25]) *For $t \in T(X)$, the term graph $G(t) = (V(t), E(t))$ is the directed graph defined as follows. $v(t)$ is the set of variables that appear in t. $E(t)$ is defined inductively as follows:*

$E(t) = \phi$ if t is a variable, and $E(ts) = E(t) \cup E(S) \bigcup (L(t), L(s))$, where $L(t)$ is the leftmost variable that appears in t. The rooted graph derived from t is $(G(t), L(t))$.

As an example, consider the term $t = (x(y((zx)y)))x$, the term graph $G(t)$ is pictured in figure(1). Different terms can have the same term graph, another term that has the term graph pictured in Figure 1 is $(z((xx)y))(yz)$.

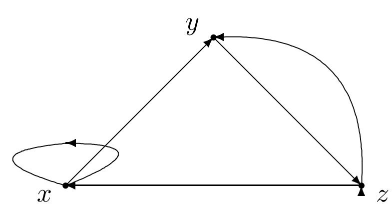


Figure 1 The term graph of $(x(y((zx)y)))x$

Following [25], we call a term trivial if ∞ occurs in it.

Lemma 3.1([25]) *Let $G = (V, E)$ be a graph , $t, s \in T(X)$, and $h : X \to A(G)$ an evaluation of the variable. Let the same h denote the unique extension of this evaluation to the algebra $T(x)$ of all terms.*

1. *If $t \in T(X)$ is nontrivial, then $(G(t), L(t))$ is a finite rooted graph. Conversely, for every finite rooted graph (G, ν) there exists $t \in T(V_G)$ with $G(t) = G$ and $L(t) = \nu$.*

2. *If t is a trivial term, or if h takes the value ∞ on some element of $V(t)$, then $h(t) = \infty$. Otherwise,if $h : G(t) \to G$ is a homomorphism of directed graphs, then $h(t) = h(L(t))$, and if h is not a homomorphism of directed graphs, then $h(t) = \infty$.*

3. *The identity $s \approx t$ is true in every graph algebra if and only if either both s and t are trivial terms, or neither of them is trivial, $G(s) = G(t)$, and $L(s) = L(t)$.*

Of course, when specifying an evaluation h of a particular term t, it is enough to define h on $V(t)$ rather than on all of X, and this is what we typically do. For efficiency, we typically say *graph homomorphism* when what we really mean is *directed graph homomorphism.*

A law $s \approx t$ is regular if $V(s) = V(t)$.

Lemma 3.2 *If D is a directed graph algebra that contains a loop, then all nontrivial laws of D are regular.*

Proof Suppose $s \approx t$ is a nontrivial law of D that is not regular, without loss of generality, we may take $x \in V(s) \setminus V(t)$. Let h be the graph homomorphism that maps x to ∞ and maps everything in $(V(S) \bigcup V(t)) \setminus x$ to some looped element a. Then $h(t)$ is a, but $h(s)$ is ∞, whence $s \approx t$ is not a law of D, contrary to our assumption.

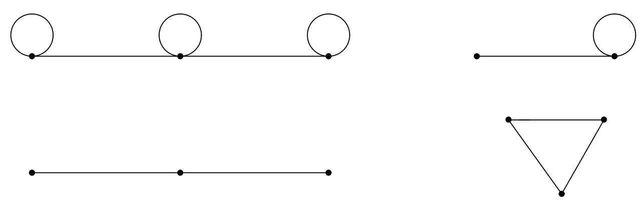


Figure 2 Graphs of the four minimal INFB graph algebras

In [1] it is shown that:

Definition 3.6 *A locally finite variety ν is finitely based, or FB, if there is a finite basis for the equations of ν. ν is inherently nonfinitely based, or $INFB$, if ν is not contained in any other FB locally finite variety. We say that the algebra A is FB if $\nu(A)$ is FB, which is the case exactly when there is a finite basis for $Eq(A)$. We say that A is INFB if ν is $INFB$. Note that if A is INFB then A is not FB; otherwise $\nu(A)$ would be contained in an FB locally finite variety, namely $\nu(A)$ itself.*

Theorem 3.1 *A graph algebra A is FB if and only if its underlying graph G_A has no subgraph isomorphic to one of the graphs in Figure 2*

This theorem gives a complete classification of the FB graph algebras. since every graph algebra is also a directed graph algebra, the above theorem will be of some use to us as we work to classify the FB directed algebras.

Our work falls under the heading of universal algebra, so we use the language and notation of that subject. Our algebras can be viewed as models in the sense of Model Theory, so we sometimes borrow from the notation of that subject as well. For example, we use $A \models \sigma$ to mean that the sentence σ is true in the algebra A, and we use $\Gamma \vdash \sigma$ to mean that there is a derivation of σ from the sentences in Γ.

We shall distinguish carefully between the symbols $=$ and $\approx$. We shall use $=$ only for exact equality; if we say , for example, $s = t$, then we mean that s and t are identical. We shall use $\approx$ when writing down laws. Thus we shall say things like $A \models s \approx t$ and $\Gamma \vdash s \models t$. (of course, it is true but uninteresting that $A \models s = s$ and $\Gamma \vdash s = s$ for every A, s, and Γ.)

When writing down a law λ, we shall use λ^L to refer to the term on the left-hand side of λ and λ^R to refer to the term on the right-hand side.

Given a term t, $l(t)$ is the length of t, defined by the following recursion:

$$l(t) = \begin{cases} 1 & if \text{ t } is\ a\ single\ variable \\ l(r) + l(s) & if\ t = rs. \end{cases}$$

Thus $l(t)$ is the number of places at which variables occur in t.

When dealing with terms, it is expeditious to avoid writing down as many parentheses as possible. Toward this end, we adopt the convention that sub terms will be grouped from the left. Thus, for example, when we say

$$xy_1y_2...y_n$$

we mean that

$$(\cdots((xy_1)y_2)\cdots)y_n.$$

When giving a derivation, we justify the steps as follows. When a step is justified by a numbered entity, such as an equation or proposition, the number appears underneath the $\approx$ or $=$ on that step's line in the proof. When the justification is something that does not have a number, the justification appears in square brackets at the right end of the line.

In general, we use u, v, w, x, y, and z for variables, s, t, and lowercase Greek letters for terms and sub terms, lower case Greek letters for laws, and uppercase Greek letters for sets of laws.

§4. **km-Graphs**

In this section we will define graph plane and then we will construct km-graphs. Next by using the language and notation of universal algebra, we will define a new algebra derived from directed graph algebras.

Definition 4.1 *Some of the structures with regular configurations can be expressed as the Cartesian product of two or more graphs. After the formation of the nodes of such a graph according to the nodes of the generators, a member should be added between two typical nodes* (u_i, v_j) *and* (u_k, v_l)*, as show in Figure 3, if the following conditions are satisfied*

$$[(u_i = u_k,\ v_l\ adj\ v_j)\ or\ (v_i = v_l,\ u_i\ adj\ u_k)].$$

Some other structures with regular configurations can be expressed as the strong Cartesian product of two or more graphs. After the formation of the nodes of such a graph a member should be added between two typical nodes (u_k, v_l) *and* (u_i, v_j) *if the following conditions are satisfied:*

$$[(u_i = u_k\ and\ v_l\ adj\ v_j)\ or\ (v_j = v_l,\ u_i\ adj\ u_k)]\ or\ u_i\ adj\ u_k\ and\ v_l\ adj\ v_j.$$

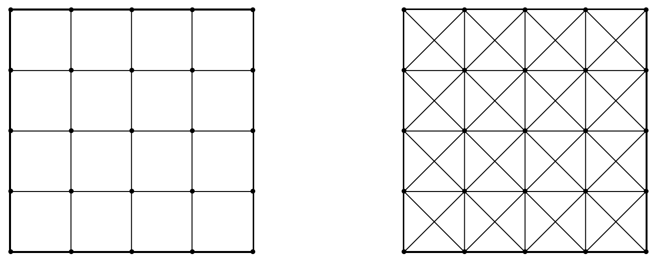


Figure 3 Cartesian product and strongly Cartesian product of p_n

Let p_n be a simple graph. By Cartesian product of $p_n * p_n$ we have a plane that from now we will call this plane the **graphic plane**. The graphic plane from right and up is infinite and from down and left is bounded. This plane can be considered isomorphic with the plane produced by $\mathbb{Z}^+ * \mathbb{Z}^+$, but in our plane the symmetric axis are defined in an other ways. Samples of graphic plane from natural are the chess plane, the factories producer line, train road of a country, air lines, electric cables in a city or home. The importance of our idea is to find simpler maps for relations between natural phenomena such that natural factors less often injured in connection together. First of all we will consider some axioms in our graphic plane:

Definition 4.2 *we will call the segment crossing from every node (u_i, v_i) for every $i \in N$ the symmetric axis of the graphic plane.*

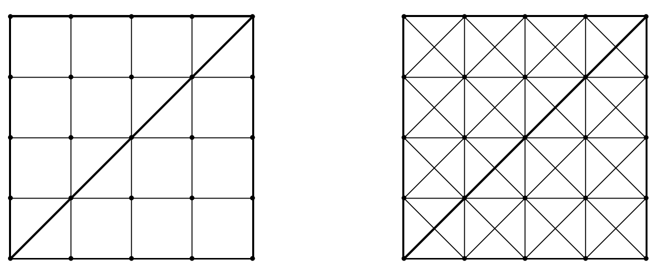


Figure 4 Symmetric axis of Cartesian product and strongly Cartesian product of $p_n \times p_n$

The symmetric axis splits the graphic plane into two half graphic plane that we will show the upper half graphic plane by positive and the lower half graphic plane by negative notations.

By the above definitions we will consider the nodes from the positive half graph plane with positive notations and the nodes from the negative half graph plane with negative notations. In this manner the all nodes on symmetric axis are without notations.

Definition 4.3 *A km-graph G_F consists of a vertex set $\{V(G_F) \bigcup \infty\}$ and an edge set $E(G_F)$,* *where for every* $x, y \in V$*.*

$$G_F(x) = \begin{cases} y & if x \twoheadrightarrow y \\ \infty & otherwise \end{cases}$$

that satisfies the following conditions:

(i) There exist only one and unique conclusive point, that we will denote it by M*;*

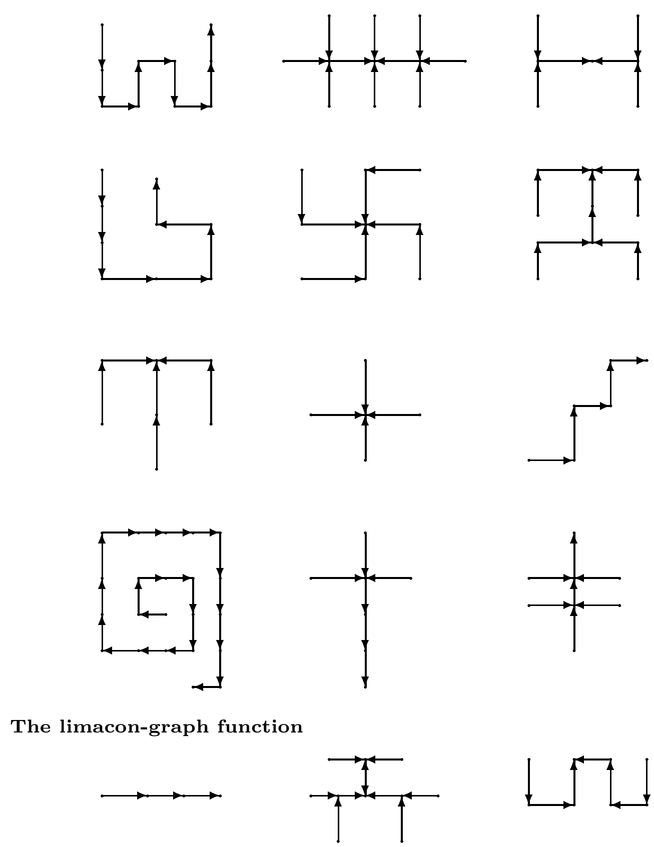
(ii) From every inception point in every path there is at least one tournament to M*. Therefore a km-graph may have many inception point;*

(iii) Each vertex is loop less;

(iv) There is no path of a vertex to itself. That is, there is no sequence of vertexes such that

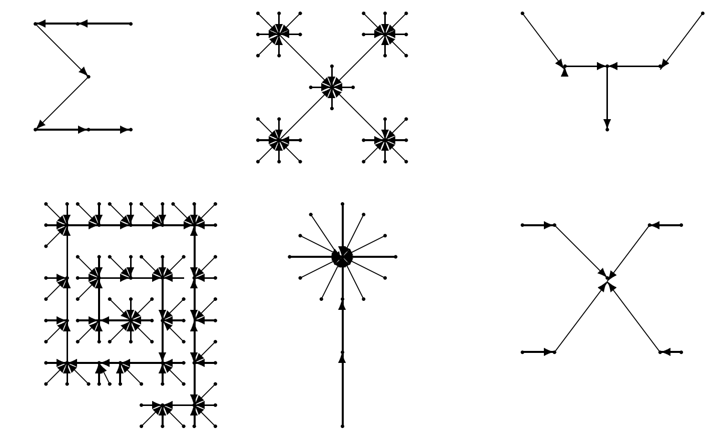
$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow \cdots \longrightarrow v_n \longrightarrow v_1.$$

Some of km-graphs are shown below:

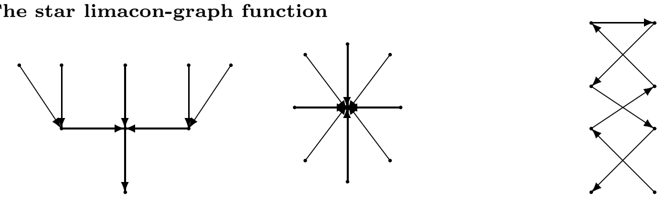


(a) Graph function in the graph plane

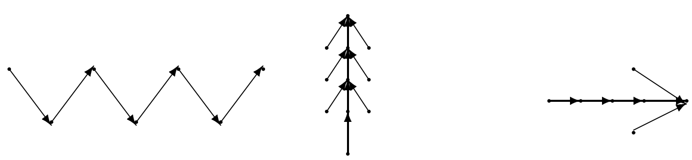
Figure 5 Samples of km-graphs



The star limacon-graph function

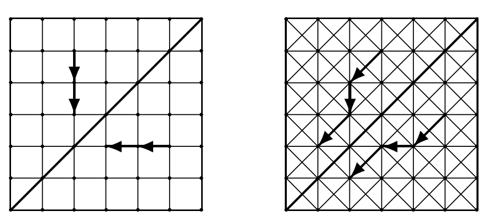


The star like-graph function



(b) Graph functions in the strong graph plane

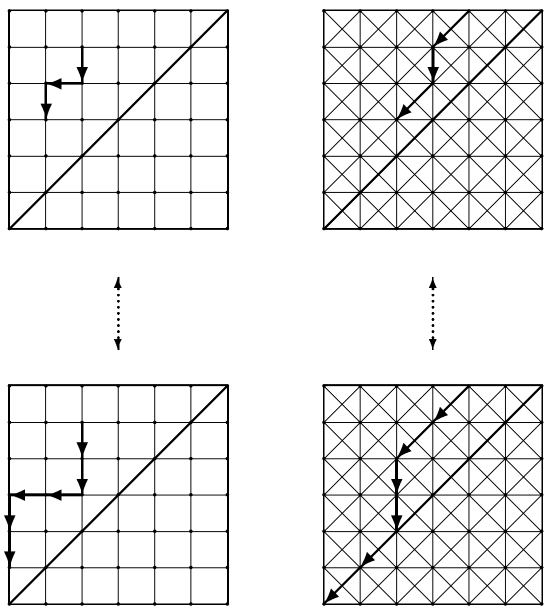
Figure 6 Samples of km-graphs



Symetry of a graph function

Figure 7 Sample of Symmetry for km-graphs

It is obvious that we can define the positive scalar multiplication on km-graphs. If let $k \in \mathbb{Z}$ be a integer number and $G_F(x)$ be a arbitrary km-graphs., then $kG_F(x)$ is a km-graph that every vertex of $kG_F(x)$ is k times as much of $G_F(x)$. Also one can consider inversee km-graph of km-graph $G_F(x)$, denoted by $(G_F(x))^{-1}$, as a km-graph such that the direction of any vertex will be denoted by inversee direction. Therefore it is not true that the inversee of a km-graph is usually a km-graph. The only case that we have inversee km-graph is the km-graphs with only one inception point. On the other hand the symmetry of a km-graph $G_F(x)$ is the km-graph $G'_F(x)$ such that every vertex $v'_i \in G'_F(x)$ is symmetric by a vertex $v'_i \in G_F(x)$, with respect to symmetric axis of graph plane.



Scalar multiplication of a graph function

Figure 8 Sample for Scalar Multiplication of km-graphs for k = 2

Also we can define inverse km-graph as follows:

Definition 4.4 *The inversee of a km-graph G_F, that denoted by G_F^{-1}, obtained by change of path direction.*

Note that only km-graphs with one inception point and one conclusive point have inversee. That is, in general km-graphs don't have corresponding inversee km-graph.

Definition 4.5 *Two km-graphs G_{F_1} and G_{F_2} are equal if and only if they have the same vertex set and the same input set for a vertex. Although this a perfectly reasonable definition, for most purposes the module of relationship is not essentially changed if G_{F_2} is obtained from G_{F_1} just by renaming the vertex set.*

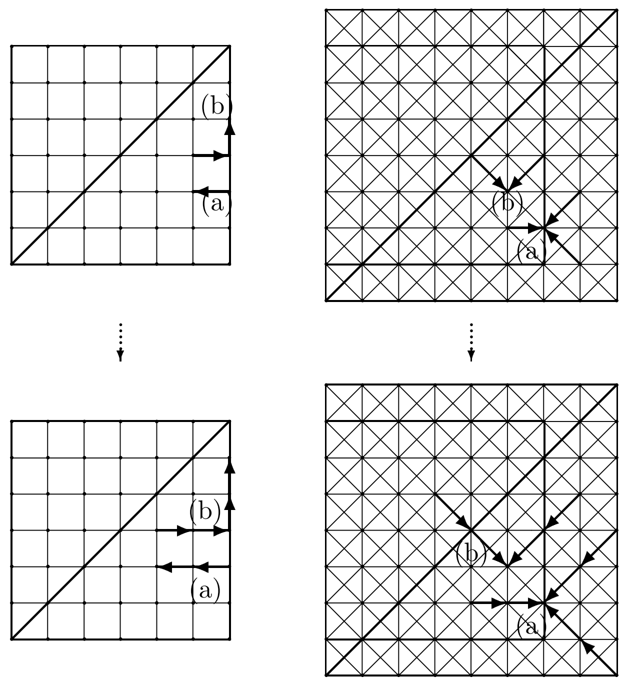


Figure 9 Samples for Scalar Multiplication km-graphs for K=2

For arbitrary two km-graphs G_{F_1} and G_{F_2}, we can define union and intersection of km-graphs.

Definition 4.6 *The union of two km-graph G_{F_1} and G_{F_2} is just obtained by superposition of conclusive points of G_{F_1} and G_{F_2}. There fore if G_{F_1} have V_{F_1} as vertex set and E_{F_1} as edge set and G_{F_2} have G_{F_1} have V_{F_2} as vertex set and E_{F_2} as edge set, then $G_{F_1} \cup G_{F_2}$ is defined by $V_{F_1} \cup V_{F_2}$ as vertex set and $E_{F_1} \cup E_{F_2}$ as edge set.*

It is obvious that union of two km-graph is not a km-graph in general. However we will define sum of two km-graph by similar definition without closed path in result. Before do it, in the following we have definition of intersection.

Definition 4.7 *Intersection of two km-graph G_{F_1} and G_{F_2} is just obtained by superposition of conclusive points of G_{F_1} and G_{F_2}, such that $G_{F_1} \cap G_{F_2}$ is defined by $V_{F_1} \cap V_{F_2}$ as vertex set and $E_{F_1} \cap E_{F_2}$ as edge set. The intersection of two km-graph may have two path that have similar vertexes but the edges are not in same direction. In this type the two edges will be deleted and we have the discrete km-graph.*

There fore we see that the intersection of two km-graph is not necessary to be a km-graph.

Definition 4.8 *A km-graph with at least one singular (isolated) vertex is called discrete km-graph. We will used the name $D - GF_i$ for discrete km-graphs.*

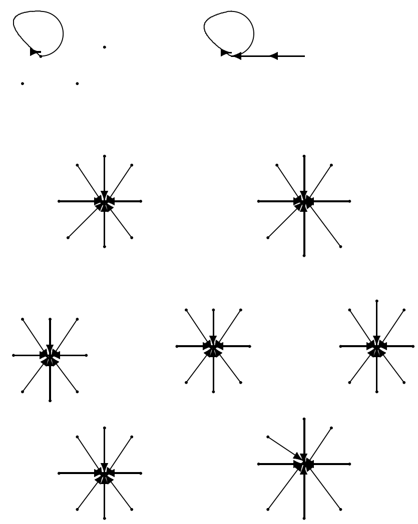


Figure 10 Sample of M_4,M_3,M_7-km-graph

Thus every single loop is a looped km-graph.

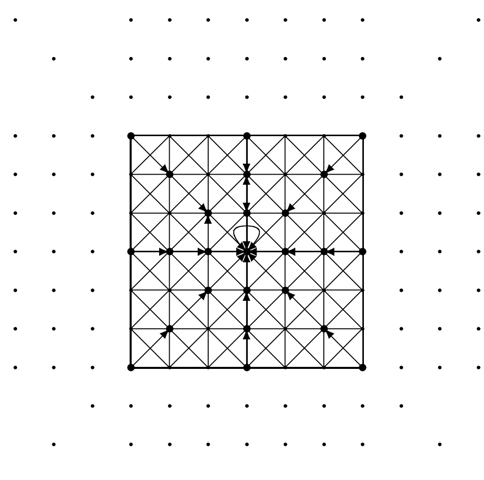


Figure 11

By the mentioned definition we can consider a new class of km-graphs that we will call them M_n*-graph maps*. Here the n is the union of singular vertexes with conclusive point. We can think on km-graphs by a loop in the conclusive point. The idea of such km-graphs come back to a square in city, when we consider the traffic problem, or the river in a sea/ocean, when we consider all rivers that have a sea/ocean as conclusive point. There fore we have the following definition:

Definition 4.9 *We will call a km-graph looped graph map if the conclusive point have a loop.*

Definition 4.10 *The sum of two km-graph G_{F_1} and G_{F_2} is a km-graph G_F that just obtained by superposition of conclusive points of G_{F_1} and G_{F_2} such that there is no sequence of vertexes such that*

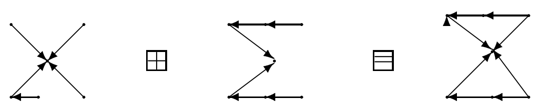
$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow \cdots \longrightarrow v_n \longrightarrow v_1,$$

or there is no close path such that

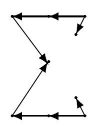
$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow \cdots \longrightarrow v_n = v_1 \longrightarrow v_{2'} \longrightarrow v_{3'} \longrightarrow \cdots \longrightarrow v_n,$$

Otherwise, by deleting one edge that have smallest number of input vertex we can obtain a km-graph.

One can see that for every two km-graph G_{F_1} and G_{F_2}, the difference between sum of G_{F_1} with G_{F_2} and G_{F_2} with G_{F_1} is only in the place of M. Such that in sum processing, we do it by superposition of conclusive point of second km-graph on conclusive point of first km-graph.



(a)



(b)

Figure 12

In the following one can find sum of two km-graph with restriction law:

Also if in sum of two km-graph we have two direction that are inverse, then we can use from this fact that, by definition, we have not closed path, the result form (a) can be considered as picture (b). That is we delete such paths. This law was called restricted law.

On the other hand we will see that the set of km-graphs with binary operation $\boxplus$ is semi group. Now if we consider the km-graph M, that is the km-graph with only one node and whit out vertex, the identity element of the set of all km-graphs, then we have a monoid that defined on set of all km-graphs with the binary operation $\boxplus$ because we can see that

$$(G_F1 \boxplus G_F2) \boxplus G_F3 = G_F1 \boxplus (G_F2 \boxplus G_F3)$$

$$G_F \boxplus M = M \boxplus G_F = G_F$$

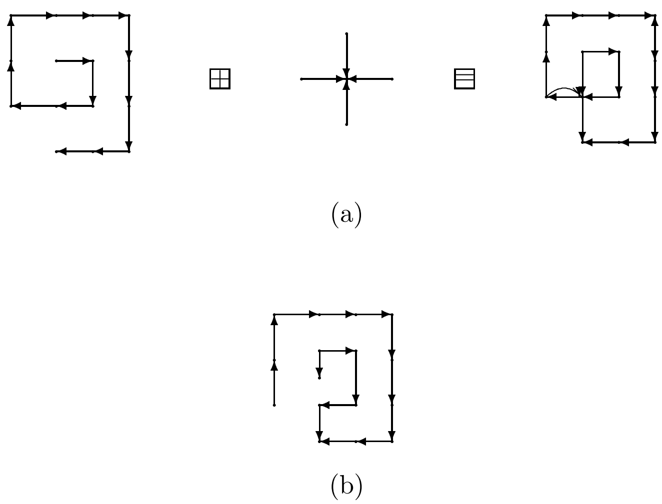


Figure 13

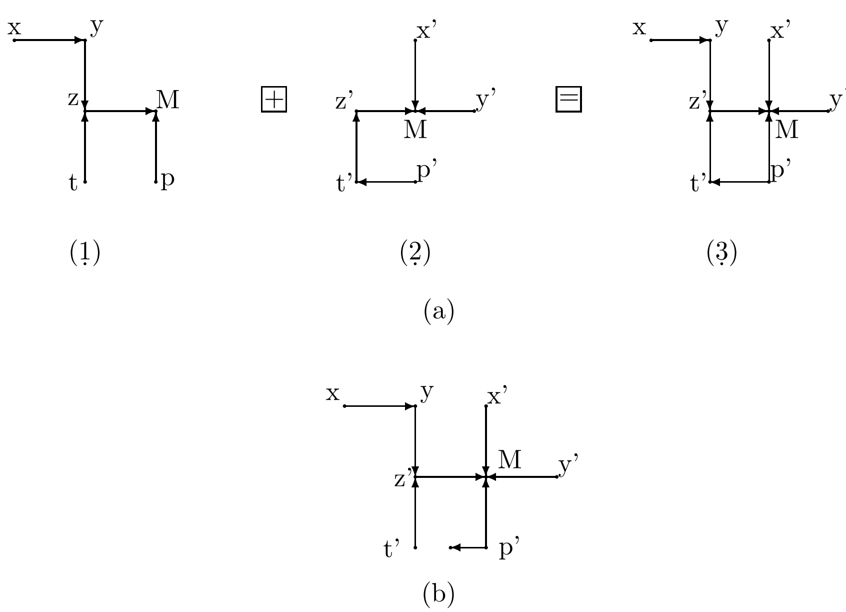


Figure 14 Samples for sum of two km-graphs

Also one can defined sum of looped km-graphs analogically.

Definition 4.11 *The sum of two looped km-graph G_{F_1} and G_{F_2} is a looped km-graph G_F that just obtained by superposition of conclusive points of G_{F_1} and G_{F_2} such that there is no sequence of vertexes such that*

$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow ... \longrightarrow v_n \longrightarrow v_1,$$

or there is no close path such that

$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow ... \longrightarrow v_n = v_1 \longrightarrow v_{2'} \longrightarrow v_{3'} \longrightarrow ... \longrightarrow v_n,$$

Otherwise, by deleting one edge that have smallest number of input vertex we can obtain a km-graph.

In the following we consider some samples of sum of two looped km-graphs.It is obvious that we can do this definition for km-graph from one hand and looped km-graph from other hand.

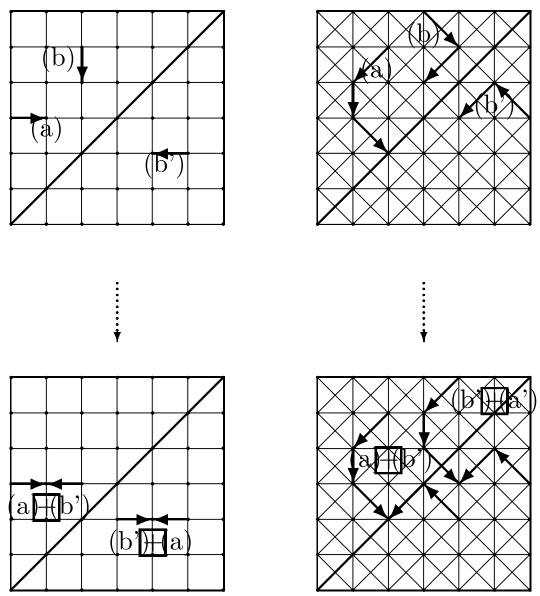
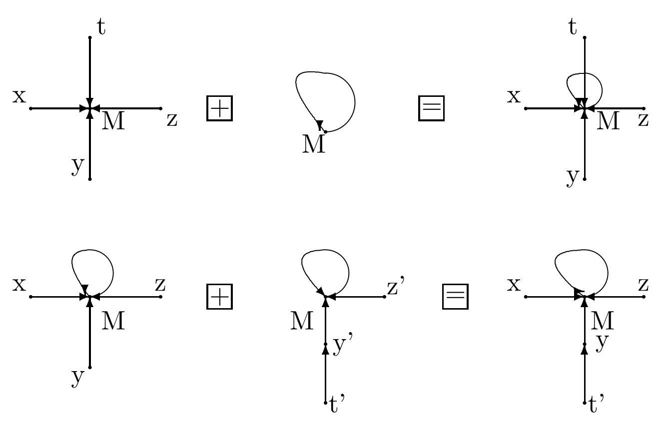


Figure 16 Samples for sum of two km-graphs



Sum of two looped graph function

Figure 17 Samples for sum of two graph function

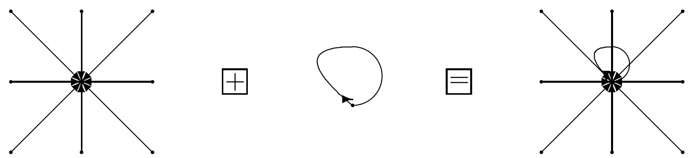
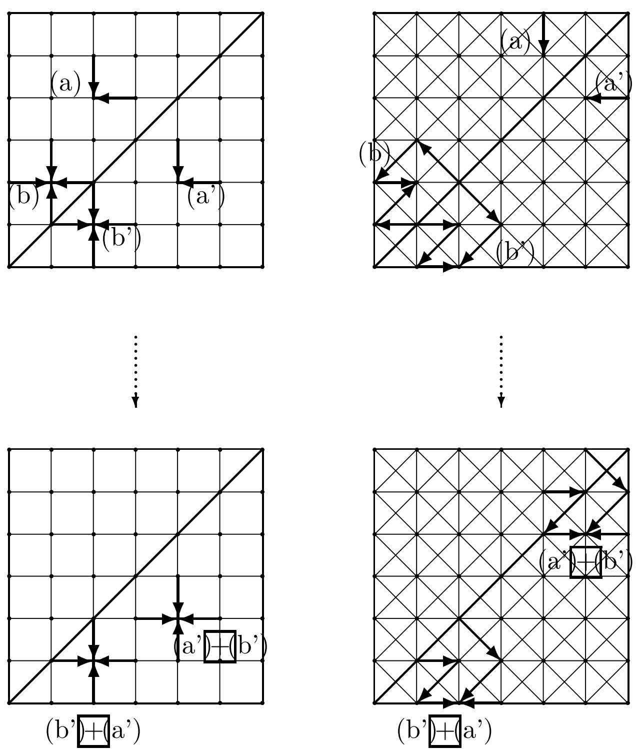


Figure 18 Samples for sum of two graph function

We can define $-G_F$ as a km-graph that is in negative part of graph plane. There fore we have $G_{F_1} \boxminus G_{F_2} = G_{F_1} \boxplus (-G_{F_2})$. On the other hand for $-G_{F_1} - GF_2$ we can consider $-(G_{F_1} \boxplus (-G_{F_2}))$ that is equal with $-(GF_1 \boxplus G_{F_2})$. In the following we have some samples for combining of such km-graphs:



Sum of two negative graph function

Figure 19 Samples for sum of two km-graphs

Therefore if we observe positive scalar multiplication and the negative part of an km-graph, then we can define negative scalar multiplication of a km-graph analogously. In this manner we can first multiple any scalar k in km-graph G_F and then found the negative position of this map. On the other hand we can first found for every km-graph G_F the negative position of G_F and then multiple $(-G_F)$ in k, for every arbitrary scalar $k \in \mathbb{Z}$.

4.1 Algebraic Properties of km-Graphs

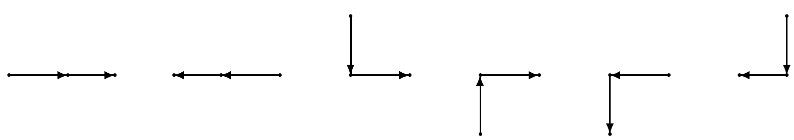
First of all we will consider associativity and commutativity properties for sum of km-graphs. Sum of two km-graphs are defined in previous section. The drawing km-graph for sum of G_{F_1} with G_{F_2} and G_{F_2} with G_{F_1} are similar, but are in different place of graph plane. There fore one can conclude that the sum of two km-graph is approx-commutative. Also one can see that the approx-associativity property hold for sum of three km-graphs.

On the other hand sum of every km-graph G_{F_1} with km-graph G_{F_M}, is G_{F_1} (where G_{F_M} is the km-graph with only one vertex and no edges). Thus, one can conclude that the class of all km-graphs considerable as a approx-commutative monoid (the approx-commutative semi-group with identity element).

On the other hand if we consider the km-graph as objects and the binary operation between them be the sum of two km-graphs, then can we have a category?

In this direction one can discussed that which properties of categories are satisfies in mentioned category and vise verse. It is obvious that by definition of km-graph and drawing them we have the following lemmas:

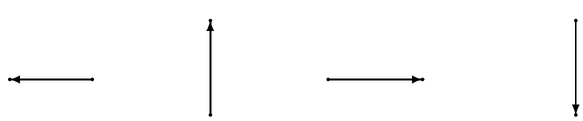
Lemma 4.1 *There are only one isomorphic km-graph with two nods in the $p_n * p_n$ plane.*



(a)

(b)

The classes of 3-vertex graph function



The classes of 2-vertex graph function

Figure 19

Lemma 4.2 *There are only two isomorphic class of km-graphs with three nods in the $p_n * p_n$ plane.*

Lemma 4.3 *There are only four class of km-graphs with four nods in the $p_n * p_n$ plane.*

4.2 km-Algebras

Definition 4.12 *We call an algebra defined on km-graphs a* $\mathbf{KM-Algebra}$*, if the following laws holds:*

$$xy \approx y \tag{1}$$

$$yx \approx \infty \tag{2}$$

$$MM \approx \infty \tag{3}$$

$$yM \approx \infty \tag{4}$$

$$My \approx \infty \tag{5}$$

$$M\infty \approx \infty \tag{6}$$

$$\infty M \approx \infty \tag{7}$$

$$y\infty \approx \infty \tag{8}$$

$$\infty y \approx \infty \tag{9}$$

$$y(yM) \approx yM \tag{10}$$

$$M(yM) \approx MM \approx \infty \tag{11}$$

$$(xy)z \approx (xy)(yz) \approx yz \tag{12}$$

$$x_1x_2x_3\cdots x_ny \approx y_1y_2y_3\cdots y_my \tag{13}$$

Equational theory on km-algebras can be discussed for identities and hyperidentities in nontrivial terms.

In [1], Baker, McNulty, and Werner give a method, the Shift-Automorphism Theorem, for showing that certain algebras are INFB. This method is particularly useful in the case of graph algebras; it is an essential ingredient in the classification in [1] of the FB graph algebras. The shift-automorphism theorem 4.5 can be used to show that many directed graph algebras are INFB as well, and we shall use it to obtain several such results in this chapter.

The form of the shift-automorphism theorem that we shall use is Theorem in [1], and it appears below as our . In an algebra that has an absorbing element ∞, as do all directed graph algebras, the proper elements are the elements other than ∞. Given a $\mathbf{Z}-$sequence α, the translates of α are the sequences α^i for $i \in \mathbf{Z}$, where α^i is α shifted i places to the right. (Thus if $i < 0$, then we shift to the left, and if $i = 0$ then we do not shift at all.)

Theorem 4.1(Shift-Automorphism Theorem) *Let B be a finite algebra of finite type, with an absorbing element ∞. Suppose that a sequence α of proper elements of B can be found with these properties:*

(1) *in $B^{\mathbf{z}}$, any fundamental operation f applied to translates of α yields as a value either a translate of α or a sequence containing ∞;*

(2) *there are only finitely many equations* $f(\alpha^{i_1}, \cdots, \alpha^{i_n(f)}) = \alpha^{(j)}$ *in which* f *is a fundamental operation and some argument is a* α *itself;*

(3) *there is at least one equation* $f(\alpha^{i_1}, \cdots, \alpha^{i_n(f)}) = \alpha^{(1)}$ *in which some argument is a itself, in an entry on which* f *actually depends,*

then B *is INFB.*

Proof See [23]. □

The general idea of this version of the Shift-Automorphism Theorem, as applied specifically to a directed graph algebra B, is that we want to use α to create an infinite directed graph with certain properties. The elements of this directed graph are the translates of α, and the edges are the natural ones inherited from G_B. We pool all sequences that contain ∞ into an equivalence class, which acts as the ∞ element for resulting infinite directed graph algebra. Condition (a) of the theorem ensures that this view makes sense. Condition (b) requires there to be only finitely many edges into and out of α (and therefore into and out of any $\alpha^{(i)}$); another way to say this is that there must be an N such that if $n \geq N$, then $\alpha\alpha^{(n)}$ and $\alpha^{(n)}\alpha$ must both contain an occurrence of ∞. (Here $\alpha^{(i)}\alpha^{(j)}$ is understood to be the result of applying our binary operation coordinate wise to $\alpha^{(i)}$ and $\alpha^{(j)}$.) Condition (c) tells us that there has to be edge from $\alpha^{(1)}$ to α.

Because the multiplication in our km-algebras has the property that uv is either v or ∞, if B is a KM-Algebra and α is any sequence of proper elements of B, then it is clear that condition (a) of the theorem is satisfied; in each coordinate of α, the product will either be the right-hand operand or ∞. Hence we do not need to mention condition (a) again when dealing with km-algebra. Note also that if α is an infinite path through G_B, as it will be in all of the cases we consider, then condition (c) must hold.

The original version of the shift automorphism theorem, as formulated in 1989 by Baker, McNulty, and Werner [1], stated that any shift automorphism algebra is inherently nonfinitely based. In 2008, McNulty, Szekly, and Willard were able to show every shift automorphism algebra must be inherently nonfinitely based in the finite sense [16]. It is the contribution of this author that every shift automorphism variety has countably infinite sub directly irreducible members.

For some interesting examples of finite algebra proven to be inherently nonfinitely based with help of the Shift Automorphism Theorem, see [1, 6, 10, 23]. As an example of the shift automorphism theorem, let us consider the looped star-like km-algebra; it is based on the km-graph pictured in figure(10). We let $\alpha = \cdots aaaaabcccc \cdots$. Now, α is an infinite path through looped star-like km-algebra, so we simply need to show that condition (b) of the Shift-Automorphism Theorem holds for this algebra and α.

To begin, let us observe that if $i \notin \{j, j+1\}$, then $\alpha^{(i)}\alpha^{(j)}$ will be a sequence that contains ∞, since there will be at least one coordinate at which the entry is ac, ba, cb, or ca, and all of these are ∞. If $i \in \{j, j+1\}$, then $\alpha^{(i)}\alpha^{(j)}$, then $\alpha^{(i)}\alpha^{(j)} = \alpha^{(i)}$. Thus α meets condition (b) of the theorem; the only equations that hold here of the kind mentioned in (b) are $\alpha^{(0)}\alpha^{(0)} = \alpha^{(0)}$, $\alpha^{(0)}\alpha^{(-1)} = \alpha^{(0)}$ and $\alpha^{(1)}\alpha^{(0)} = \alpha^{(1)}$.

Thus we have proven the following:

Lemma 4.4 *For $n \geq 3$ every n-Starlike km-algebra is INFB.*

Also for the looped n-starlike km-algebra the above lemma is true. The infinite looped star-like km-algebra that α yields is pictured in Figure 10.

Note that the same α would have worked even if the km-algebra in question had either (or both) of the edges (b, a) and (c, b). Thus the km-algebras in Figure 10 give rise to INFB km-algebras as well.

Also the nullary looped km-algebra (with one vertex and one edges) is just a single looped element, so is FB. Thus we have now completely classified the looped star-like km-algebras.

To use the Shift-Automorphism Method, consider $\mathbf{B}$, a finite algebra of finite signature. We will consider the $\mathbb{Z}-$tuple $(\cdots, b_{-2}, b_{-1}, b_0, b_1, b_2, \cdots)$ in $\mathbf{B}^{\mathbb{Z}}$ as a sequence $\alpha = \cdots b_{-2}b_{-1}b_0b_1b_2\cdots$ where each b_j is a proper element of $\mathbf{B}$ for every $j \in \mathbb{Z}$. We define α_i as the i^{th} translate of α, that is, as α shifted $i$positions to the right (if $i > 0$), to the left (if $i < 0$), or not at all (if $i = 0$). Note that the shift by 1 position is an automorphism of $\mathbf{B}^{\mathbb{Z}}$. If σ gives only one infinite orbit, then we can summarize Theorem 4.1 as the following, also found on [1].

Theorem 4.2 *Let $\mathbf{B}$ be a finite algebra of finite signature with absorbing element 0. Suppose that a sequence α of proper elements of $\mathbf{B}$ can be found with these properties:*

(1) *in $\mathbf{B}^{\mathbb{Z}}$, any fundamental operation F applied translates of α yields as a value either a translate of α or a sequence containing 0;*

(2) *there are only finitely many equations $(\alpha_{i_0}, \cdots, \alpha_{i_{r-1}}) = \alpha_j$ in which F is a fundamental operation of rank r and some argument is α itself;*

(3) *there is at least one equation $F(\alpha_{i_0}, \cdots, \alpha_{i_{r-1}}) = \alpha_1$ in which some argument is α itself, in an entry on which F actually depends.*

We apply Theorems 4.1 and 4.2 to various algebras to show that they are inherently nonfinitely based.

4.3 Walter's Looped Directed Graphs

Walter gives the following adapted version of Theorem 4.2.

Theorem 4.3([23]) *Let G be a directed graph, and let α be a $\mathbb{Z}-$sequence that is a path of $\mathbf{G}$. If there is an N such that $n > N$ implies that $\alpha_n.\alpha$ and $\alpha.\alpha_n$ both contain an occurrence of ∞, then the graph algebra of $\mathbf{G}$ in inherently nonfinitely based.*

Now for our km-algebras we have another version of Theorem 4.2.

Theorem 4.4 *Let $\mathbf{G}$ be a KM- algebra, and α be a $\mathbb{Z}-$sequence that is a path of $\mathbf{G}$. If there is an N such that $n > N$ implies that $\alpha_n.\alpha$ and $\alpha.\alpha_n$ both contain an occurrence of ∞, then the km-algebra G is inherently nonfinitely based.*

Proof We will show the connection between Theorems 4.1 and 4.3. Let μ be a km-algebra and let α be a $\mathbb{Z}$-sequence that is a path through $\mathbf{B}$. We will check the conditions of Theorem 4.1.

The first condition of Theorem 4.1 requires that any fundamental operation applied to any translate of α results in another translate of α, or a sequence containing ∞. As our operation is inherited from a graph algebra, the operation. works as follows:

$$u.v = \begin{cases} v & if(u,v) \in E \\ \infty & otherwise \end{cases}$$

Thus for the coordinate wise product of two $Z-$sequence α_j, α_k we get

$$\alpha_j, \alpha_k \begin{cases} \alpha_k & \\ \gamma & where \gamma is a sequence containing \infty \end{cases}$$

since the result of applying. Coordinate wise gives either right input or ∞.

The second condition of Theorem 4.1 requires that there be only finitely many proper equations using the fundamental operation and the translates of α. In terms of our new infinite km-algebra, this conditions there to only finitely many edge into and out of each α_i. This is equivalent to having the existence of a number N so that for all $n > N$ we have $\alpha_n.\alpha$ and $\alpha.\alpha_n$ each contain an occurrence of ∞.

Since α_1 is a path through the km-algebra, $\alpha_1.\alpha_0 = \alpha_0$. To see this, consider a string $\cdots x_{-3}x_{-2}x_{-1}x_0x_1x_2x_3\cdots$ in α_1 where each x_i is a vertex in the km-graph related to km-algebra. Then α_1 has the same string, just shifted one place rightward. Since α_1 gives a path, we know that must be an edge between x_i and x_{i+1} for $i = -1, ...4$. Thus the product $x_i.x_{i+1}$ results x_{i+1}. Hence we get the following

$$\alpha_1 : \cdots x_{-1}x_0x_1x_2x_3\cdots$$

$$\alpha_0 : \cdots x_0x_1x_2x_3x_4\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_0 : \cdots x_0x_1x_2x_3x_4\cdots$$

This gives the last condition of Theorem 4.1. □

§5. Basic Laws for Directed M_n-km-Graphs

Definition 5.1

$$M_1M_1 \approx \infty \tag{14}$$

$$M_1x \approx \infty \tag{15}$$

$$xM_1 \approx M_1 \tag{16}$$

$$x\infty \approx \infty \tag{17}$$

$$\infty x \approx \infty \tag{18}$$

$$xx \approx \infty \tag{19}$$

$$xy \approx y \tag{20}$$

$$(xM_1)M_1 \approx M_1M_1 \approx \infty \tag{21}$$

$$x_1x_2\cdots x_nM_1 \approx y_1y_2\cdots y_mM_1 \tag{22}$$

For discrete points, $M_iM_j \approx \infty$.

Lemma 5.1 *The identity* $(xM_1)M_1 \approx M_1(xM_1)$ *holds.*

Proof $(xM_1)M_1 \approx$ by (16), $(xM_1)(xM_1) \approx$ by (16) and then $\approx M_1(xM_1)$. □

§6. Basic Laws for Directed Looped M_n-km-Graphs

Definition 6.1

$$M_1M_1 \approx M_1 \tag{23}$$

$$M_1x \approx \infty \tag{24}$$

$$xM_1 \approx M_1 \tag{25}$$

$$x\infty \approx \infty \tag{26}$$

$$\infty x \approx \infty \tag{27}$$

$$xx \approx \infty \tag{29}$$

$$xy \approx y \tag{30}$$

$$(xy)y \approx yy \approx \infty \tag{31}$$

$$x_1x_2\cdots x_nM \approx y_1y_2\cdots y_mM \tag{32}$$

For discrete points, $M_iM_j \approx \infty$.

Lemma 6.1 *The identity* $(xy)y \approx y(xy)$ *holds.*

Proof $(xy)y \approx$ by (29), $yy \approx$ by (29) and then $\approx y(xy)$. □

§7. Basic Laws for Looped km-Algebra

Definition 7.1

$$MM \approx M \tag{33}$$

$$yM \approx M \tag{34}$$

$$My \approx \infty \tag{35}$$

$$M\infty \approx \infty \tag{36}$$

$$\infty M \approx \infty \tag{37}$$

$$y(yM) \approx yM \approx M \tag{38}$$

$$(xy)M \approx yM \approx M \tag{39}$$

$$(yM)M \approx MM \approx \infty \tag{40}$$

$$M(yM) \approx MM \approx \infty \tag{41}$$

$$x_1 x_2 \cdots x_n M \approx y_1 y_2 \cdots y_m M \tag{42}$$

Lemma 7.1 *The identity $(yM)M \approx M(yM)$ holds.*

Proof $(yM)M \approx$ by (34), $MM \approx$ by (34) and then $\approx M(yM)$. □

It turns out that the theory of avoidable words will be quite useful. We will begin with a bunch of definitions.

Definition 7.1 *An alphabet* $\mathbf{\Sigma}$ *is a set of letters and a letter is a member of some alphabet. A word is a finite string of letters from some alphabet. The* **empty word** *is the word of length* 0. *To denote the set of all nonempty words over an alphabet Σ, we use Σ^+. Another formulation of Σ^+ is that it is the semi group freely generated by Σ together with the binary operation of concatenation.*

Definition 7.2 *A word w is an instance of a word u provided that w can be obtained from u by substituting nonempty words for the letters of u.*

For instance, the word baabaabaabaa is an instance of the word $xxxx$ obtained by substituting the word baa for the letter x. We say that a word u is a sub word of the word w if there are (potentially) words x and y so that $w = xuy$.

Definition 7.3 *The word w encounters the word u means that some instance of u is a sub word of w. If no instance of u is a sub word of w, then we say w avoids u.*

To generalize this notion, we say that the word u is *avoidable the alphabet* Σ provided that infinitely many words in Σ^+ avoid u. Note that since two alphabets of the same size avoid the same words, only the cardinality of Σ is important. If the cardinality of Σ is n and u is avoidable on Σ, then we say that u is *n-avoidable*. Lastly, the word u is *avoidable* if and only if there is some natural number n for which u is *n-avoidable*. If no such n exists, we call u *unavoidable*.

Definition 7.4 *The Zimin words Z_n (where n is a natural number) are defined recursively by:*

(i) $Z_0 = x_0$*;*

(ii) Given Z_n, define $Z_{n+1} = Z_n x_{n+1} Z_n$ for each natural number n.

Thus the first three Zimin words are $x_0, x_0x_1x_0$, and $x_0x_1x_0x_2x_0x_1x_0$. If S is a semi group and w is a word in which the letters of w are regarded as variables, we say w is an *isoterm of* S when u and w are identical whenever $\mathbf{S} \models w \approx u$. See [26] for a development of the theory of avoidable words.

Definition 7.5 *An algebra* $\mathbf{B} =< B, . >$ *is a* **semigroup** *provided the associative law holds, i.e. for all* $a, b,\ c \in B,\ (a.b).c = a.(b.c)$.

Parkins in [19] constructs the following semigroup, denoted by $\mathbf{B_2^1}$:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We denoted these matrices by **O**, **I**, **A**, **B**, **C**, and **D**, respectively. The semi group structure of this algebra is given by Table 3.

Perkins's semi group is inherently nonfinitely based, as shown by Sapir in [7, 8]. To show this, we need a theorem of Sapir from [7, 8].

Theorem 7.1 *Let* **S** *be a finite semi group. If every unavoidable word is an isoterm of* **S***, then* **S** *is inherently nonfinitely based.*

.	O	I	A	B	C	D
O	O	O	O	O	O	O
I	O	I	A	B	C	D
A	O	A	A	B	O	O
B	O	B	O	O	A	B
C	O	C	C	D	O	O
D	O	D	O	O	C	D

TABLE (*) of the Semi group table of $\mathbf{B_2^1}$

We note that Sapir also showed the more difficult converse, that if **S** is inherently nonfinitely based then every unavoidable word is an isoterm of **S**. For its proof, see [9].

Corollary 7.1 *The semigroup* $\mathbf{B_2^1}$ *is inherently nonfinitely based.*

Proof See [9]. □

Theorem 7.2 *The set of six km-algebra(shown below) with binary Operation, is isomorphic by Perkin's semi group. Therefore we have inherently nonfinitely based algebra for mentioned six km-algebra.*

Proof The result obtained from Theorem 4.1 immediately. Correspondence of these six km-algebra and Parkin's semi group shown in the following figure. □

By similar method we will see that many of known $INFB$ algebras are isomorphic with some subsets of M and parkin's semigroup shown in the following. Therefore,

Theorem 7.3 *The variety of all km-algebras is inherently nonfinitely based.*

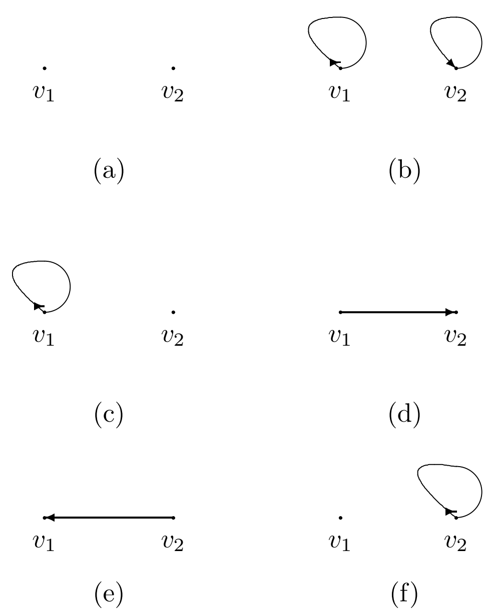


Figure 20

O	v_1	v_2
v_1	0	0
v_2	0	0

Table a

v_1	v_2	
v_1	1	0
v_2	0	1

Table b

O	v_1	v_2
v_1	1	0
v_2	0	0

Table c

O	v_1	v_2
v_1	0	1
v_2	0	0

Table d

O	v_1	v_2
v_1	0	0
v_2	1	0

Table e

O	v_1	v_2
v_1	0	0
v_2	0	1

Table f

We will see that (by definition of operation $\boxplus$) sum of two M_n km-graph can be considered as one of the following pictures:

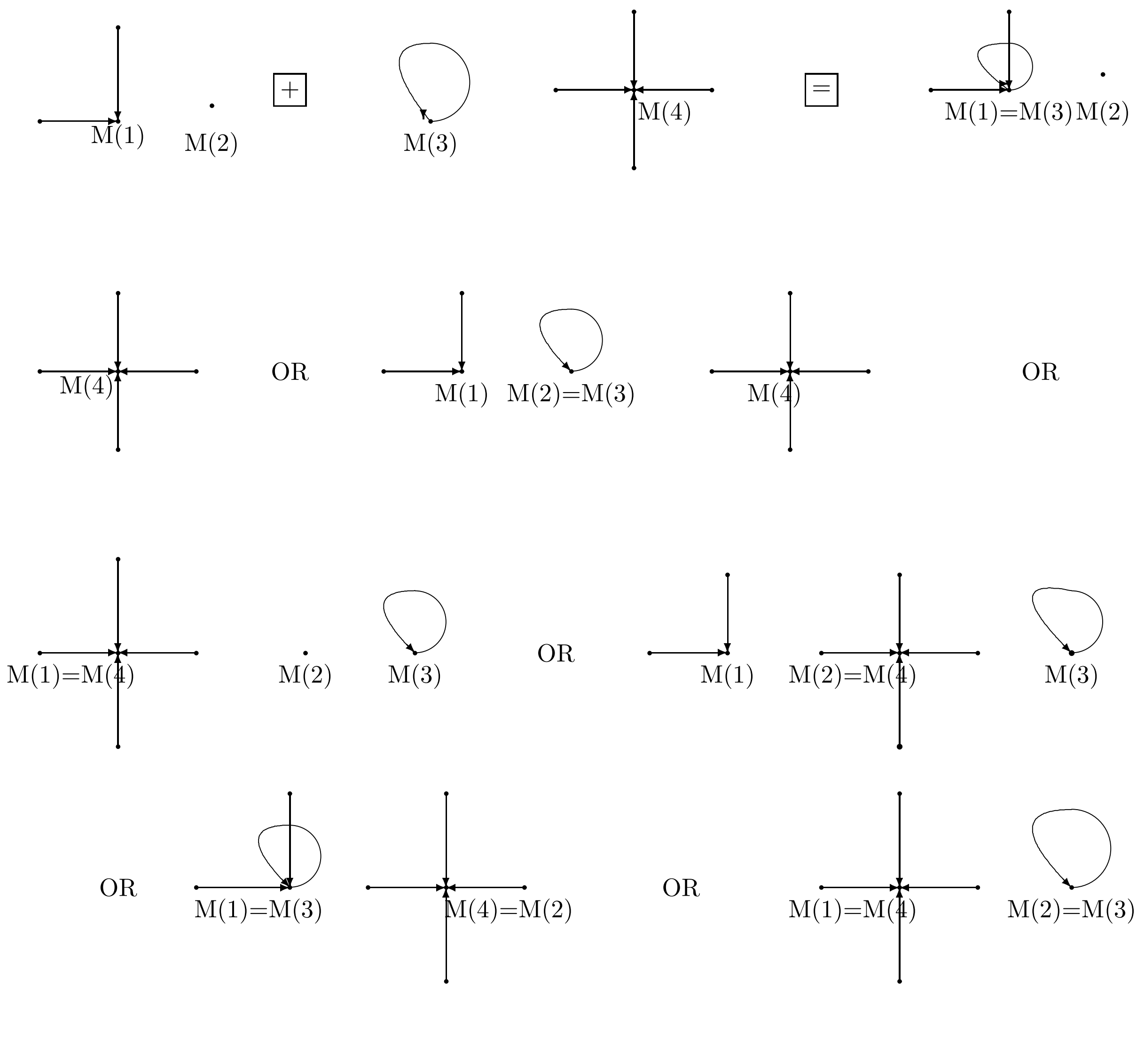


Figure 21 Sum of two $M_n$km-graphs

References

[1] Kirby A. Baker, G. F. McNulty, H. Werner, The finitely based varieties of graph algebras, *Acta Scient. Math.*, 51: 3-15, (1987).

[2] Garrett Birkhoff, On Structure of Abstract Algebras, *Proc. Cambridge Philos. Soc.*, 31 (1935), 433-454.

[3] Wieslaw Dziobiak, On infinite vsubdirectly irreducible algebras in locally finite equational classes, *Algebra Universelis*, 13 (1981), No. 3, 393-394.

[4] Samuel Eilenberg, M. P. Schutzenberger, On pseudovarieties, *Advances in Math.*, 19 (1976), No. 3, 413-418.

[5] E. Hajilarov, An inherently nonfinitely based commutative directoid, *Algebra Universelis*, 36 (1996), No.4, 431-435.

[6] Keith Kearnes and Ross Willard, Inherently nonfinitely based solvable algebras, *Canad. Math. Bull.*, 37 (1994), No.4, 514-521.

[7] M. V. Sapir, Inherently nonfinitely based finite semigroups, *Mat. Sb. (N.S.)*, 133 (175) (1987), no. 2, 154-166.

[8] M. V. Sapir, Problems of Burnside type and the finite basis property in varieties of semigroups, *Izv. Akad. Nauk SSSR Ser. Mat.*, 51 (1987), no. 2, 319-340.

[9] Kathryn Hope Scott, *A Catalogue of Finite Algebras with Nonfinitely Axiomatizable Equational theories*, Ph.D. Thesis, University of south Carolina, 2007.

[10] Agnes Szendrei, Non-finitely based finite groupoids generating minimal varieties, *Acta Sci. Math.* (Szeged), 57 (1993), No. 1-4, 593-600.

[11] W.Kiss, R. Poschel, P.prohle, Subvarieties of varieties generated by graph algebras, *Acta Scient. Math.*,54(1-2):57-57,(1990).

[12] Ralph McKenzie, George F. McNulty, Walter F. Taylor, *Algebras, Lattices, Varieties.* Vol I, The Wads worth and Brooks/cole Mathematics Series, Wads worth and Brooks/cole Advanced Books and Software, Monterey, CA, 1987.

[13] Ralph McKenzie,The residual bounds of finite algebras, *Internat. J. Algebra Comput.*, 6 (1996), No.1, 1-28.

[14] Ralph McKenzie, Tarski's finite basis problem is undecidable, *Internat. J. Algebra Comput.*, 6 (1996), No. 1, 49-104.

[15] Georg F. McNulty, Residual finiteness and finite equational bases: undecidable properties of finite algebras, *Lectures on Some Recent Works of Ralph McKenzie and Ross Willard.*

[16] Georg F. McNulty, Zoltan Szekely, Ross Willard, Equational complexity of the finite algebra membership problem, *IJAC*, 18 (2008), 1283-1319.

[17] J. Jezek, G. F. McNulty, Finite axiomatizablity of congruence rich varieties, *Algebra Universelis*, 34 (1995), no.2, 191-213.

[18] A.Kaveh, M.Nouri, Weighted graph products for configuration processing of planer and space structures, *International Journal of Space Structure*, Vol.24, Number 1,(2009).

[19] Robert E. Park, *Equational classes on non-associative ordered algebras*, Ph.D. thesis, University of California, Los Angeles, 1976.

[20] Robert W. Quackenbush, Equational classes generated by finite algebras, *Algebra Universelis*, 1 (1971/72), 265-266.

[21] Kathryn Scott Owens, *On Inherently Nonfinitely Based Varieties.* Ph.D. Thesis, University of south Carolina, 2009.

[22] L. Wald, *Minimal inherently nonfinitely based varieties of groupoids*, Ph.D. thesis, University of California, Los Angeles, 1998.

[23] Brin L.Walter, Finite Equntional Bases for Directed Graph Algebras. Ph.D. Thesis,University of California, 2002.

[24] Brian L.Walter, The finitely based varieties of looped directed graph algebras, *Acta Sci. Math* (Szeged), 72(2006), No.3-4, 421-458.

[25] Emil W. Kiss, R. Poschel, P. Prohle, Subvarieties of varieties generated by graph algebras, *Acta Scient. Math.*, 54(1-2): 57-75, (1990).

[26] Ross Willard, The finite basis problem, *Contributions to General Algebra*, 15, Heyn, Klagenfurt, 2004, pp. 199-206.