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Semi-Symmetric Metric Connection on a 3-Dimensional Trans-Sasakian Manifold

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Abstract: The object of the present paper is to study the nature of curvature tensor, Ricci tensor, scalar curvature and Weyl conformal curvature tensors with respect to a semi-symmetric metric connection on a 3-dimensional trans-Sasakian manifold.We have given an example regarding it.

Key Words: α -Sasakian manifold, β-Kenmotsu manifold, cosymplectic manifold, Levi-Civita connection, semi-symmetric connection, Weyl conformal curvature tensor.

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§1. Introduction

The notion of locally φ-symmetric Sasakian manifold was introduced by T. Takahashi [14] in 1977. Also J.A. Oubina in 1985 introduced a new class of almost contact metric structures which was a generalization of Sasakian [13], α-Sasakian [11], Kenmotsu [11], β-Kenmotsu [11] and cosymplectic [11] manifolds, which was called trans-Sasakian manifold [12]. After him many authors [4],[5],[10],[12] have studied various type of properties in trans-Sasakian manifold.

In this paper we have obtained the curvature tensor and also the first Bianchi identity with respect to a semi-symmetric connection on a 3-dimensional trans-Sasakian manifold. We also find out the condition of Ricci tensor to be symmetric under this connection. We have shown that the Riemannian Weyl conformal curvature tensor is equal to the Weyl conformal curvature tensor with respect to semi-symmetric connection and also equal to the curvature tensor with respect to semi-symmetric connection when the Ricci tensor under this connection vanishes.

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§2. **Preliminaries**

Let M^n be an n-dimensional (n is odd) almost contact C^∞ manifold with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric.

Then the manifold satisfies the following relations ([3]):

(2.1) $\phi^2(X) = -X + \eta(X)\xi, \quad \eta \circ \phi = 0;$
(2.2) $\eta(X) = g(X, \xi), \quad \eta(\xi) = 1;$
(2.3) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$

Now an almost contact manifold is called trans-Sasakian manifold if it satisfies ([13]):

(2.4) $(\nabla_X \phi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X].$

From (2.4) it follows

(2.5) $(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta[g(X, Y) - \eta(X)\eta(Y)], \forall\ X, Y \in \chi(M)$

where $\alpha, \beta \in F(M)$ and ∇ be the Levi-Civita connection on M^n.

A linear connection $\bar{\nabla}$ on M^n is said to be semi-symmetric [1] if the torsion tensor $\bar{T}$ of the connection $\bar{\nabla}$ satisfies

(2.6) $\bar{T}(X, Y) = \pi(Y)X - \pi(X)Y,$

where π is a 1-form on M^n with U as associated vector field, i.e,

(2.7) $\pi(X) = g(X, U)$

for any differentiable vector field X on M^n.

A semi-symmetric connection $\bar{\nabla}$ is called semi-symmetric metric connection [2] if it further satisfies

(2.8) $\bar{\nabla} g = 0.$

In [2] Sharfuddin and Hussain defined a semi-symmetric metric connection in an almost contact manifold by identifying the 1-form π of [1] with the contact 1-form η i.e., by setting

(2.9) $T(X, Y) = \eta(Y)X - \eta(X)Y.$

The relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ of (M^n, g) has been obtained by K.Yano [9], which is given by

(2.10) $\bar{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X, Y)U.$

Further, a relation between the curvature tensor R and $\bar{R}$ of type $(1,3)$ of the connections ∇ and $\bar{\nabla}$ respectively are given by [7],[8],[9]

(2.11) $\bar{R}(X, Y)Z = R(X, Y)Z + \hat{\alpha}(X, Z)Y - \hat{\alpha}(Y, Z)X - g(Y, Z)LX + g(X, Z)LY,$

where,

(2.12) $\hat{\alpha}(Y, Z) = g(LY, Z) = (\nabla_Y \pi)(Z) - \pi(Y)\pi(Z) + \frac{1}{2}\pi(U)g(Y, Z).$

The Weyl conformal curvature tensor of type $(1,3)$ of the manifold is defined by

(2.13) $C(X, Y)Z = R(X, Y)Z + \lambda(Y, Z)X - \lambda(X, Z)Y + g(Y, Z)QX - g(X, Z)QY,$

where,

(2.14) $\lambda(Y,Z) = g(QY,Z) = -\frac{1}{n-2}S(Y,Z) + \frac{r}{2(n-1)(n-2)}g(Y,Z)$,

where S and r denote respectively the $(0,2)$ Ricci tensor and scalar curvature of the manifold.

We shall use these results in the next sections for a 3-dimensional trans-Sasakian manifold with semi-symmetric metric connection.

§3. Curvature tensors with Respect to the Semi-Symmetric Metric Connection On a 3-Dimensional Trans-Sasakian Manifold

From (2.5), (2.9) and (2.12) we have

(3.1) $\hat{\alpha}(Y,Z) = -\alpha g(\phi Y,Z) - (\beta+1)\eta(Y)\eta(Z) + (\beta+\frac{1}{2})g(Y,Z)$.

Using (2.12), we get from (3.1)

(3.2) $LY = -\alpha\phi Y - (\beta+1)\eta(Y)\xi + (\beta+\frac{1}{2})Y$.

Now using (3.1) and (3.2), we get from (2.11) after some calculations

$$\begin{aligned}(3.3)\ \bar{R}(X,Y)Z &= R(X,Y)Z - \alpha[g(\phi X,Z)Y - g(\phi Y,Z)X] \\ &\quad -\alpha[g(X,Z)\phi Y - g(Y,Z)\phi X] + (2\beta+1)[g(X,Z)Y - g(Y,Z)X] \\ &\quad -(\beta+1)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \\ &\quad -(\beta+1)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi.\end{aligned}$$

Thus we can state

Theorem 3.1 *The curvature tensor with respect to $\bar{\nabla}$ on a 3-dimensional trans-Sasakian manifold is of the form* (3.3).

From (3.3) it is seen that

(3.4) $\bar{R}(Y,X)Z = -\bar{R}(X,Y)Z$.

We now define a tensor $\bar{R}'$ of type $(0,4)$ by

(3.5) $\bar{R}'(X,Y,Z,V) = g(\bar{R}(X,Y)Z,V)$.

From (3.4) and (3.5) it follows that

(3.6) $\bar{R}'(Y,X,Z,V) = -\bar{R}'(X,Y,Z,V)$.

Combining (3.6) and (3.4) we can see that

(3.7) $\bar{R}'(X,Y,Z,V) = \bar{R}'(Y,X,V,Z)$.

Again from (3.3) exchanging X,Y,Z cyclically and adding them, we get

(3.8) $\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 2\alpha[g(\phi X,Y)Z + g(\phi Y,Z)X + g(\phi Z,X)Y]$.

This is the first Bianchi identity with respect to $\bar{\nabla}$. Thus we state

Theorem 3.2 *The first Bianchi identity with respect to* $\bar{\nabla}$ *on a* 3*-dimensional trans-Sasakian manifold is of the form* (3.8)*.*

Let $\bar{S}$ and S denote respectively the Ricci tensor of the manifold with respect to $\bar{\nabla}$ and ∇. From (3.3) we get by contracting X,

(3.11) $\bar{S}(Y,Z) = S(Y,Z) + \alpha g(\phi Y, Z) - (3\beta+1)g(Y,Z) + (\beta+1)\eta(Y)\eta(Z)$.

In (3.11) we put $Y = Z = e_i, 1 \leq i \leq 3$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold. Then summing over i, we get

(3.12) $\bar{r} = r - 2(4\beta+1)$.

From (3.11), we get

(3.13) $\bar{S}(Y,Z) - \bar{S}(Z,Y) = \alpha(g(\phi Y, Z) - g(\phi Z, Y)) = 2\alpha g(\phi Y, Z)$.

But $g(\phi Y, Z)$ is not identically zero. So $\bar{S}(Y,Z)$ is not symmetric. Thus we state

Theorem 3.3 *The Ricci tensor of a* 3*-dimensional trans-Sasakian manifold with respect to the semi-symmetric metric connection is not symmetric.*

The Weyl conformal curvature tensor of type $(1,3)$ of the 3-dimensional trans-sasakian manifold with respect to the semi-symmmtric metric connection $\bar{\nabla}$ is defined by

(3.14) $\bar{C}(X,Y)Z = \bar{R}(X,Y)Z + \bar{\lambda}(Y,Z)X - \bar{\lambda}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y$,

where,

(3.15) $\bar{\lambda}(Y,Z) = g(\bar{Q}Y,Z) = -\frac{1}{2}\bar{S}(Y,Z) + \frac{\bar{r}}{4}g(Y,Z)$.

Putting the values of $\bar{S}$ and $\bar{r}$ from (3.11) and (3.12) respectively in (3.15) we get

(3.16) $\bar{\lambda}(Y,Z) = g(\bar{Q}Y,Z) = \lambda(Y,Z) - \alpha g(\bar{Y},Z) + \frac{2\beta+1}{2}g(Y,Z) - (\beta+1)\eta(Y)\eta(Z)$.

and,

(3.17) $\bar{Q}Y = QY - \alpha\bar{Y} + \frac{2\beta+1}{2}Y - (\beta+1)\eta(Y)\xi$.

Using (3.3),(3.16) and (3.17), we get from (3.14) after a brief calculations

(3.18) $\bar{C}(X,Y)Z = C(X,Y)Z$.

Thus we can state

Theorem 3.4 *The Weyl conformal curvature tensors of the* 3*-dimensional trans-sasakian manifold with respect to the Levi-Civita connection and the semi-symmetric metric connection are equal.*

If in particular $\bar{S} = 0$, then $\bar{r} = 0$, so from (3.15) we get

(3.19) $\bar{\lambda}(Y,Z) = 0$.

From (3.19) and (3.14) we get

(3.20) $\bar{C}(X,Y)Z = \bar{R}(X,Y)Z$.

From (3.18) and (3.20) we have

(3.21) $C(X,Y)Z = \bar{R}(X,Y)Z.$

Corollary 3.5 *If the Ricci tensor of a 3-dimensional trans-Sasakian manifold with respect to the semi-symmetric metric connection vanishes, the Weyl conformal curvature tensor of the manifold is equal to the curvature tensor of the manifold with respect to the semi-symmetric metric connection.*

§4. Example of a 3-Dimensional Trans-Sasakian Manifold Admitting A Semi-Symmetric Metric Connection

Let the 3-dim. C^∞ real manifold $M = \{(x,y,z) : (x,y,z) \in R^3, z \neq 0\}$ with the basis $\{e_1, e_2, e_3\}$, where $e_1 = z\frac{\partial}{\partial x},\ e_2 = z\frac{\partial}{\partial y},\ e_3 = z\frac{\partial}{\partial z}$.

We consider the Riemannian metric g defined by

$$g(e_i, e_j) = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j. \end{cases}$$

Now we define a $(1,1)$ tensor field ϕ by $\phi(e_1) = -e_2,\ \phi(e_2) = e_1$ and $\phi(e_3) = 0$, and choose the vector field $\xi = e_3$ and define a 1-form η by $\eta(X) = g(X, e_3), \forall\ X \in \chi(M)$. Then $\eta(e_1) = \eta(e_2) = 0$ and $\eta(e_3) = 1$.

From the above construction we can easily show that

$$\begin{aligned} &\phi^2(X) = -X + \eta(X)\xi, \quad \eta \circ \phi = 0 \\ , \qquad &\eta(X) = g(X,\xi), \quad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \end{aligned}$$

Thus M is a 3-dim. almost contact C^∞ manifold with the almost contact structure (ϕ, ξ, η, g).

We also obtain $[e_1, e_2] = 0, [e_2, e_3] = -e_2$ and $[e_1, e_3] = -e_1$. By Koszul's formula we get

$$\begin{aligned} &\nabla_{e_1} e_1 = e_3, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_3} e_1 = 0, \\ &\nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_3} e_2 = 0, \\ &\nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_2} e_3 = -e_2, \quad \nabla_{e_3} e_3 = 0. \end{aligned}$$

Then it can be shown that M is a trans-Sasakian manifold of type $(0,-1)$.

Now we define a linear connection $\bar{\nabla}$ such that

$$\bar{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + \eta(e_j)e_i - g(e_i, e_j)e_3, \forall\ i,j = 1,2,3.$$

Then we get

$$\begin{aligned} &\bar{\nabla}_{e_1} e_1 = 0, \quad \bar{\nabla}_{e_2} e_1 = 0, \quad \bar{\nabla}_{e_3} e_1 = 0, \\ &\bar{\nabla}_{e_1} e_2 = 0, \quad \bar{\nabla}_{e_2} e_2 = 0, \quad \bar{\nabla}_{e_3} e_2 = 0, \\ &\bar{\nabla}_{e_1} e_3 = 0, \quad \bar{\nabla}_{e_2} e_3 = 0, \quad \bar{\nabla}_{e_3} e_3 = 0. \end{aligned}$$

If $\bar{T}$ is the torsion tensor of the connection $\bar{\nabla}$, then we have

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y \text{ and } (\bar{\nabla}_X g)(Y,Z) = 0,$$

which implies that $\bar{\nabla}$ is a semi-symmetric metric connection on M.

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