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Some Results on Pair Sum Labeling of Graphs

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Abstract: Let G be a (p,q) graph. An injective map $f : V(G) \to \{\pm 1, \pm 2, \cdots, \pm\ p\}$ is called a pair sum labeling if the induced edge function, $f_e : E(G) \to Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm\ k_1, \pm\ k_2, \cdots, \pm\ k_{\frac{q}{2}}\}$ or $\{\pm\ k_1, \pm\ k_2, \ldots, \pm\ k_{\frac{q-1}{2}}\} \cup \{k_{\frac{q+1}{2}}\}$ according as q is even or odd. Here we study about the pair sum labeling of some standard graphs.

Key Words: Path, cycle, star, ladder, quadrilateral snake, Smarandachely pair sum V-labeling.

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§1. Introduction

The graphs considered here will be finite, undirected and simple. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. p and q denote respectively the number of vertices and edges of G. The Union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. If P_n denotes a path on n vertices, the graph $L_n = P_2 \times P_n$ is called a ladder. Let G be a graph. A 1-1 mapping $f : V(G) \to \{\pm\ 1, \pm\ 2, \ldots, \pm\ |V|\}$ is said to be a Smarandachely pair sum V-labeling if the induced edge function, $f_e : E(G) \to Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ for $uv \in E(G)$ is also one-one and $f_e(E(G))$ is either of the form $\{\pm\ k_1, \pm\ k_2, \ldots \pm\ k_{(|E|2}\}$ if $|E| \equiv 0(mod\ 2)$ or $\{\pm\ k_1, \pm\ k_2, \ \pm\ k_{\frac{|E|-1)}{2}}\} \cup \{k_{\frac{(|E|+1)}{2}}\}$ if $|E| \equiv 1(mod\ 2)$. Particularly we abbreviate a Smarandachely pair sum V-labeling to a pair sum labeling and define a graph with a pair sum labeling to be a pair sum graph. The notion of pair sum labeling has been introduced in [4]. In [4] we investigate the pair sum labeling behavior of complete graph, cycle, path, bistar etc. Here we study pair sum labeling of union of some standard graphs and we find the maximum size of a pair sum graph. Terms not defined here are used in the sense of Gary Chartrand [2] and Harary [3].

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§2. Pair Sum Labeling

Definition 2.1 *Let G be a (p,q) graph. A one - one map $f: V(G) \to \{\pm 1, \pm 2, \ldots, \pm p\}$ is said to be a pair sum labeling if the induced edge function $f_e: E(G) \to Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q}{2}}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q-1}{2}}\} \cup \{k_{\frac{q+1}{2}}\}$ according as q is even or odd. A graph with a pair sum labeling defined on it is called a pair sum graph.*

Notation 2.2 Let G be a pair sum graph with pair sum labeling f. We denote $M = Max\{f(u) : u \in V(G)\}$ and $m = Min\{f(u) : u \in V(G)\}$.

Observation 2.3

(a) *If G is an even size pair sum graph then $G - e$ is also a pair sum graph for every edge e.*

(b) *Let G be an odd size pair sum graph with - $f_e(e) \notin f_e(E)$. Then $G - e$ is a pair sum graph.*

Proof These results follow from Definition 2.1. □

Observation 2.4 *Let G be a pair sum graph with even size and let f be a pair sum labeling of G with $f(u) = M$. Then the graph G^* with $V(G^*) = V(G) \cup \{v\}$ and $E(G^*) = E(G) \cup \{uv\}$ is also a pair sum graph.*

Proof Define $f^* : V(G^*) \to \{\pm 1, \pm 2, , \pm (p+1)\}$ by $f^*(w) = f(w)$ for all $w \in V(G)$ and $f^*(v) = p+1$. Then $f_e(E(G^*)) = f_e(E(G)) \cup \{M + p + 1\}$. Hence f is a pair sum labeling. □

S.M. Lee and W. Wei define super vertex-graceful labeling of a graph [1].

Definition 2.5 *A (p,q) graph is said to be super vertex-graceful if there is a bijection f from V to $\{0, \pm 1, \pm 2, \ldots, \pm \frac{p-1)}{2}\}$ when p is odd and from V to $\{\pm 1, \pm 2, \ldots, \pm \frac{p}{2}\}$ when p is even such that the induced edge labeling f^+ defined by $f^+(uv) = f(u) + f(v)$ over all edges uv is a bijection from E to $\{0, \pm 1, \pm 2, \ldots, \pm \frac{q-1}{2}\}$ when q is odd and from E to $\{\pm 1, \pm 2, \ldots, \pm \frac{q}{2}\}$ when q is even.*

Observation 2.6 *Let G be an even order and even size graph. If G is super vertex graceful then G is a pair sum graph.*

Remark. *K_4 is a pair sum graph but not super vertex graceful graph.*

Theorem 2.7 *If G is a (p,q) pair sum graph then $q \leq 4p - 2$.*

Proof Let f be a pair sum labeling of G. Obviously $-2p+1 \leq f_e(uv) \leq 2p-1, f_e(uv) \neq 0$ for all uv. This forces $q \leq 4p-2$. □

We know that $K_{1,n}$ and $K_{2,n}$ are pair sum graph [4]. Now we have

Corollary 2.8 *If $m, n \geq 8$, then $K_{m,n}$ is not a pair sum graph.*

Proof This result follows from the inequality $(m-4)(n-4) \leq 14$ and the condition $m \geq 8, n \geq 8$. □

§3. Pair Sum Labeling of Union of Graphs

Theorem 3.1 *$K_{1,n} \cup K_{1,m}$ is a pair sum graph.*

Proof Let $u, u_1, u_2, \ldots u_n$ be the vertices of $K_{1,n}$ and $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. Let $v, v_1, v_2, \ldots, v_m$ be the vertices of $K_{1,m}$ and $E(K_{1,m}) = \{vv_i : 1 \leq i \leq n\}$.

Case 1 $m = n$.

Define

$$\begin{aligned}
&f(u) = 1, && \\
&f(u_i) = i+1, && 1 \leq i \leq m \\
&f(v) = -1, && \\
&f(v_i) = -(i+1), && 1 \leq i \leq m
\end{aligned}$$

Case 2 $m > n$.

Define

$$\begin{aligned}
&f(u) = 1, && \\
&f(u_i) = i+1, && 1 \leq i \leq n, \\
&f(v) = -1, && \\
&f(v_i) = -(i+1), && 1 \leq i \leq n, \\
&f(v_{n+2i-1}) = -(n+1+i), && 1 \leq i \leq \frac{m-n}{2} \text{ if } m-n \text{ is even or} \\
& && 1 \leq i \leq \frac{m-n-1}{2} \text{ if } m-n \text{ is odd,} \\
&f(v_{n+2i}) = n+i+3, && 1 \leq i \leq \frac{m-n}{2} \text{ if } m-n \text{ is even or} \\
& && 1 \leq i \leq \frac{m-n-1}{2} \text{ if } m-n \text{ is odd.}
\end{aligned}$$

Then clearly f is a pair sum labeling. □

Theorem 3.2 *$P_m \cup K_{1,n}$ is a pair sum graph.*

Proof Let $u_1, u_2, \ldots u_m$ be the path P_m. Let $V(K_{1,n}) = \{v, v_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{vv_i : 1 \leq i \leq n\}$.

Case 1 $m = n$.

Define

$$f(u) = 1, \qquad 1 \le i \le m,$$
$$f(v) = -1,$$
$$f(v_i) = -2i, \qquad 1 \le i \le m,$$

Case 2 $n > m$.

Define

$$f(u_i) = i, \qquad 1 \le i \le m,$$
$$f(v) = -1,$$
$$f(v_i) = -2i, \qquad 1 \le i \le m-1,$$
$$f(v_{m+2i-1}) = 2m+i, \qquad 1 \le i \le \frac{n-m+1}{2} \text{ if } n-m \text{ is odd or } 1 \le i \le \frac{n-m}{2} \text{ if } n-m \text{ is even,}$$
$$f(v_{m+2i-2}) = -(2m+i-2), \qquad 1 \le i \le \frac{n-m+1}{2} \text{ if } n-m \text{ is odd or } 1 \le i \le \frac{n-m}{2}+1 \text{ if } n-m \text{ is even.}$$

Then f is a pair sum labeling. □

Theorem 3.3 *If $m = n$, then $C_m \cup C_n$ is a pair sum graph.*

Proof Let $u_1u_2, \ldots u_nu_1$ be the first copy of the cycle in $C_n \cup C_n$ and $v_1v_2 \ldots v_nv_1$ be the second copy of the cycle in $C_n \cup C_n$.

Case 1 $m = n = 4k$.

Define

$$f(u_i) = i, \qquad 1 \le i \le 2k-1,$$
$$f(u_{2k}) = 2k+1,$$
$$f(u_{2k+i}) = -i, \qquad 1 \le i \le 2k-1,$$
$$f(u_n) = -2k-1,$$
$$f(v_i) = 2k+2i, \qquad 1 \le i \le 2k,$$
$$f(v_{2k+i}) = -2k-2i, \qquad 1 \le i \le 2k.$$

Case 2 $m = n = 4k+2$.

Define

$$f(u_i) = i, \qquad 1 \le i \le 2k+1,$$
$$f(u_{2k+1+i}) = -i, \qquad 1 \le i \le 2k+1,$$
$$f(v_i) = 2k+2i, \qquad 1 \le i \le 2k+1,$$
$$f(v_{2k+1+i}) = -2k-2i, \qquad 1 \le i \le 2k+1.$$

Case 3 $m = n = 2k + 1$.

Assigning $-i$ to u_i and i to v_i, we get a pair sum labeling. □

Remark. *mG denotes the union of m copies of G.*

Theorem 3.4 *If $n \leq 4$, then mK_n is a pair sum graph.*

Proof If $n = 1$, the result is obvious.

Case 1 $n = 2$.

Assign the label i and $i+1$ to the vertices of i^{th} copy of K_2 for all odd i. For even values of i, label the vertices of the i^{th} copy of K_2 by $-i+1$ and $-i$.

Case 2 $n = 3$.

Subcase 1 m is even.

Label the vertices of first $\frac{m}{2}$ copies by $3i-2, 3i-1, 3i (1 \leq i \leq m/2)$. Remaining $\frac{m}{2}$ copies are labeled by $-3i+2, -3i+1, -3i$.

Subcase 2 m is odd.

Label the vertices of first $(m-1)$ copies as in Subcase (a). In the last copy label the vertices by $\frac{3(m-1)}{2}+1, \frac{-3(m-1)}{2}-2, \frac{3(m-1)}{2}+3$ respectively.

Case 3 $n = 4$.

Subcase 1 m is even.

Label the vertices of first $\frac{m}{2}$ copies by $4i-3, 4i-2, 4i-1, 4i \ \ (1 \leq i \leq \frac{m}{2})$. Remaining $\frac{m}{2}$ copies are labeled by $-4i+3, -4i+2, -4i+1, -4i$.

Subcase 2 m is odd.

Label the vertices of first $(m-1)$ copies as in Sub case (a). In the last copy label the vertices by $-2m, 2m+1, 2m+2$ and $-2m-3$ respectively. □

Theorem 3.5 *If $n \geq 9$, then mK_n is not a pair sum graph.*

Proof Suppose mK_n is a pair sum graph. By Theorem 2.7, we know that $\dfrac{mn(n-1)}{2} \leq 4mn-2$, i.e., $mn(n-1) \leq 8mn-4$. That is $8mn - mn^2 + mn - 4 \geq 0$. Whence, $9mn(9-n)-4 \geq 0$, a contradiction. □

§4. Pair Sum Labeling on Standard Graphs

Theorem 4.1 *Any ladder L_n is a pair sum graph.*

Proof Let $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_iv_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\}$.

Case 1 n is odd.

Let $n = 2m+1$. Define $f : V(L_n) \to \{\pm\, 1, \pm\, 2, \ldots, \pm\, (4m+2)\}$ by

$$\begin{aligned}
&f(u_i) = -4(m+1) + 2i, && 1 \le i \le m,\\
&f(u_{m+1}) = -(2m+1), &&\\
&f(u_{m+1+i}) = 2m + 2i + 2, && 1 \le i \le m,\\
&f(v_i) = -4m - 3 + 2i, && 1 \le i \le m,\\
&f(v_{m+1}) = 2m + 2 &&\\
&f(v_{m+1+i}) = 2m + 2i + 1, && 1 \le i \le m.
\end{aligned}$$

Case 2 n is even.

Let $n = 2m$. Define $f : V(L_n) \to \{\pm\, 1, \pm\, 2, \ldots, \pm\, (4m+2)\}$ by

$$\begin{aligned}
&f(u_{m+1-i}) = -2i, && 1 \le i \le m,\\
&f(u_{m+i}) = 2i - 1, && 1 \le i \le m,\\
&f(u_{m+i}) = 2i, && 1 \le i \le m,\\
&f(u_{m+1-i}) = -(2i-1), && 1 \le i \le m.
\end{aligned}$$

Then obviously f is a pair sum labeling. □

Notation 4.2 We denote the vertices and edges of the Quadrilateral Snake Q_n as follows:

$$V(Q_n) = \{u_i, v_j, w_j : 1 \le i \le n+1, 1 \le j \le n\}$$
$$E(Q_n) = \{u_i v_i, v_i w_i, u_i u_{i+1}, u_{i+1} w_i : 1 \le i \le n\}.$$

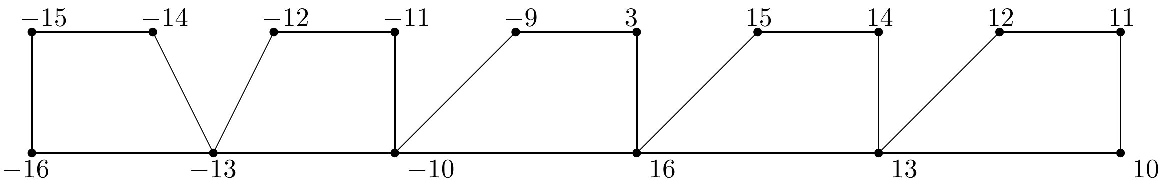
Theorem 4.3 *The quadrilateral snake Q_n is a pair sum graph if n is odd.*

Proof Let $n = 2m+1$. Define $f : V(G) \to \{\pm\, 1, \pm\, 2, \ldots, \pm\, (6m+4)\}$ by

$$\begin{aligned}
&f(u_i) = -3n + 3i - 4, && 1 \le i \le m+1,\\
&f(u_{m+i}) = 3n - 3i + 4, && 1 \le i \le m+1,\\
&f(v_i) = -3n + 3i - 3, && 1 \le i \le m+1,\\
&f(v_{m+1+i}) = 3n - 3i + 3, && 1 \le i \le m,\\
&f(w_i) = -3n + 3i - 2, && 1 \le i \le m,\\
&f(w_{m+1}) = 3, &&\\
&f(w_{m+i+1}) = 3n - 3i + 2, && 1 \le i \le m,
\end{aligned}$$

Then f is a pair sum labeling. □

Example 4.4 A pair sum labeling of Q_5 is shown in the following figure.



Notation 4.5 We denote the vertices and edges of the triangular snake T_n as follows:

$$V(T_n) = \{u_i, v_j : 1 \le i \le n+1, 1 \le j \le n\},$$
$$E(T_n) = \{u_i u_{i+1}, u_i v_j, v_i v_{j+1} : 1 \le i \le n, 1 \le j \le n-1\}.$$

Theorem 4.6 *Any triangular snake T_n is a pair sum graph.*

Proof The proof is divided into three cases following.

Case 1 $n = 4m - 1$.

Define

$$\begin{aligned}
&f(u_i) = 2i - 1, && 1 \le i \le 2m,\\
&f(u_{2m+i}) = -2i + 1, && 1 \le i \le 2m,\\
&f(v_i) = 2i, && 1 \le i \le 2m-1,\\
&f(v_{2m}) = -8m + 3, &&\\
&f(v_{2m+i}) = -2i, && 1 \le i \le 2m-1.
\end{aligned}$$

Case 2 $n = 4m + 1$.

Define

$$\begin{aligned}
&f(u_i) = -8m - 3 + 2(i-1), && 1 \le i \le 2m+1,\\
&f(u_{2m+1+i}) = 8m + 3 - 2(i-1), && 1 \le i \le 2m+1,\\
&f(v_i) = -2 + 2(i-1), && 1 \le i \le 2m,\\
&f(v_{2m+1}) = 3, &&\\
&f(v_{2m+i+1}) = 8m + 2 - 2(i-1), && 1 \le i \le 2m.
\end{aligned}$$

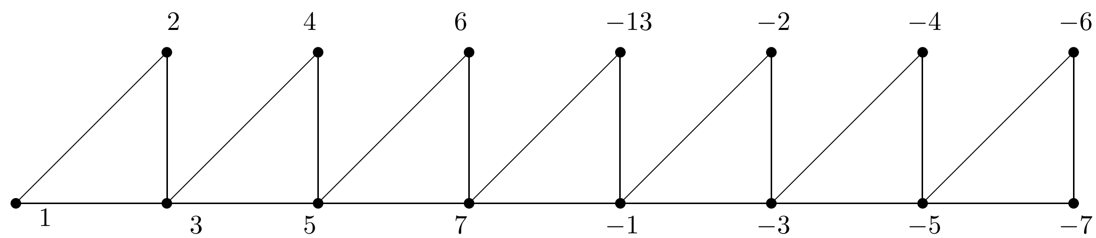
Case 3 $n = 2m$.

Define

$$
\begin{aligned}
&f(u_{m+1}) = 1, \\
&f(u_{m+1+i}) = 2i, && 1 \le i \le m, \\
&f(u_{m+1-i}) = -2i, && 1 \le i \le m, \\
&f(v_m) = 3, \\
&f(v_{m+1}) = -5, \\
&f(v_{m+1+i}) = 5 + 2i, && 1 \le i \le m-1, \\
&f(v_{m-i}) = -(5+2i), && 1 \le i \le m-1.
\end{aligned}
$$

Clearly f is a pair sum labeling. □

Example 4.7 A pair sum labeling of T_7 is shown in the following figure.



Theorem 4.8 *The crown $C_n \odot K_1$ is a pair sum graph.*

Proof Let C_n be the cycle given by $u_1u_2, \ldots, u_nu_1$ and let $v_1, v_2, \ldots, v_n$ be the pendent vertices adjacent to $u_1, u_2, \ldots, u_n$ respectively.

Case 1 n is even.

Subcase (a) $n = 4m$.

Define

$$
\begin{aligned}
&f(u_i) = 2i - 1, && 1 \le i \le 2m, \\
&f(u_{2m+i}) = -2i + 1, && 1 \le i \le 2m, \\
&f(v_i) = 4m + (2i-1), && 1 \le i \le 2m, \\
&f(v_{2m+i}) = -(4m + 2i - 1), && 1 \le i \le 2m, .
\end{aligned}
$$

Subcase (b) $n = 4m + 2$.

Define

$$
\begin{aligned}
&f(u_i) = i, && 1 \le i \le 2m+1,\\
&f(u_{2m+1+i}) = -i, && 1 \le i \le 2m+1,\\
&f(v_i) = 4m+i, && 1 \le i \le 2m+1,\\
&f(v_{2m+1+i}) = -(4m+i), && 1 \le i \le 2m+1.
\end{aligned}
$$

obviously f is a pair sum labeling.

Case 2 $n = 2m+1$.

Define

$$
\begin{aligned}
&f(u_1) = m-1, &&\\
&f(u_i) = 2m+2i+1, && 2 \le i \le m+1,\\
&f(u_{m+1+i}) = -(2m+2i+1), && 1 \le i \le m,\\
&f(v_1) = -3m+3, &&\\
&f(v_i) = f(u_i)+1, && 2 \le i \le m+1,\\
&f(v_{m+1+i}) = f(u_{m+1+i})-1, && 1 \le i \le m.
\end{aligned}
$$

Clearly f is a pair sum labeling. □

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