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The Crossing Number of Two Cartesian Products

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Abstract: There are several known exact results on the crossing number of Cartesian products of paths, cycles, and complete graphs. In this paper, we find the crossing numbers of Cartesian products of P_n with two special 6-vertex graphs.

Keywords: Cartesian product; Crossing number.

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§1. Introduction

A *drawing* D of a graph G on a surface S consists of an immersion of G in S such that no edge has a vertex as an interior point and no point is an interior point of three edges. We say a drawing of G is a *good drawing* if the following conditions hold:

(1) no edge has a self-intersection;
(2) no two adjacent edges intersect;
(3) no two edges intersect each other more than once;
(4) each intersection of edges is a crossing rather than tangential.

The *crossing number* $cr(G)$ of a graph G is the smallest number of pairs of nonadjacent edges that intersect in a drawing of G in the plane. An *optimal drawing* of a graph G is a drawing whose number of crossings equals $cr(G)$.

Now let G_1 and G_2 be two vertex-disjoint graphs. Then the *union* of G_1 and G_2, denoted by $G_1 \bigcup G_2$, is a graph with $V(G_1 \bigcup G_2) = V(G_1) \bigcup V(G_2)$ and $E(G_1 \bigcup G_2) = E(G_1) \bigcup E(G_2)$. The *Cartesian product* $G_1 \times G_2$ of graphs G_1 and G_2 has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{\{(u_i, v_j), (u_h, v_k)\} | (u_i = u_h \text{ and } v_j v_k \in E(G_2)) \text{ or } (v_j = v_k \text{ and } u_i u_h \in E(G_1))\}$. A circuit C of a graph G is called *non-separating* if $G/V(C)$ is connected, and *induced* if the vertex-induced subgraph $G[V(C)]$ of G is C itself. A circuit is called to be an *induced non-separating circuit* if it is both induced and non-separating. For definitions not explained in this paper, readers are referred to [1]. The following result is obvious by definitions.

Lemma 1.1 *If C is an induced non-separating circuit of G, then C must be the boundary of a face in the planar embedding.*

The problem of determining the crossing number of a graph is NP-complete. As we known, the crossing number are known only for a few families of graphs, most of them are Cartesian products of special graphs. For examples,

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$cr(C_3 \times C_3) = 3$ (Harary et al, 1973, see [5]);

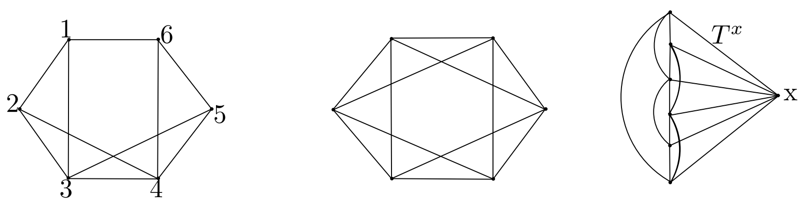
$cr(C_3 \times C_n) = n$ (Ringeisen and Beinekein, 1978, see [9]);

$cr(C_4 \times C_4) = 8$ (Dean and Richter, 1995, see [3]);

$cr(C_4 \times C_n) = 2n$, $\quad cr(K_4 \times C_n) = 3n$ (Beineke and Ringeisen, 1980, see [2])

Let S_{n-1} and P_n be the star and path with n vertices, respectively. Klesc [6] proved that $cr(S_4 \times P_n) = 2(n-2)$ and $cr(S_4 \times C_n) = 2(n-1)$. He also showed that $cr(K_{2,3} \times S_n) = 2n$ [7] and $cr(K_5 \times P_n) = 6n$ in [7]. Peng and Yiew [4] proved that $cr(P_{3,1} \times P_n) = 4(n-1)$.

In this paper, we extend these results to the product $G_j \times P_n, 1 \leq j \leq 2$ for two special graphs shown in Fig.1 following.



G_1 $\qquad$ G_2 $\qquad$ G_1^*

Fig.1

For convenience, we label these six vertices on their outer circuits of G_1 consecutively by integers $1, 2, 3, 4, 5$ and 6 in clockwise, such as those shown in Fig.1. Notice that for any graph $G_i, i = 1, 2$, $G_i \times P_n$ contains n copies of G_i, denoted by $G_i^j (1 \leq j \leq n)$ and 6 copies of P_n. We call the edges in G_i^j black and the edges in these copies of P_n red. For $j = 1, 2, \cdots n-1$, let $L(j, j+1)$ denote the subgraph of $G_i \times P_n$, induced by six red edges joining G_i^j to G_i^{j+1}. Note that $L(j, j+1)$ is homeomorphic to $6K_2$.

§2. The crossing number of $G_1 \times P_n$

By joining all 6 vertices of G_1 to a new vertex x, we obtain a new graph, denoted by G_1^*. Let T^x be the six edges incident with x, see Fig.1. We know $G_1^* = G_1 \bigcup T^x$ by definition.

Lemma 2.1 $\quad cr(G_1^*) = 2.$

Proof A good drawing of G_1^* shown in Fig.2 following enables us to get $cr(G_1^*) \leq 2$. We prove the reverse inequality by a case-by-case analysis. In any good drawing D of G_1^* , there are only three cases, i.e., $cr_D(G_1) = 0$, $cr_D(G_1) = 1$ or $cr_D(G_1) \geq 2$.

Case 1 $\quad cr_D(G_1) = 0.$

Use Euler's formula, $f = 6$ and we note that there are 6 induced non-separating circuits 1231, 2342, 3453, 4564, 12461, 13561. So there are at most 4 vertices of G_1 on each boundary.

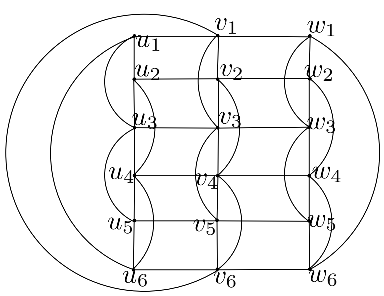
Joining all 6 vertices to x, there are 2 crossings among the edges of G_1 and the edges of T^x at least. This implies $cr(G_1^*) \geq 2$.

Case 2 $cr_D(G_1) = 1$.

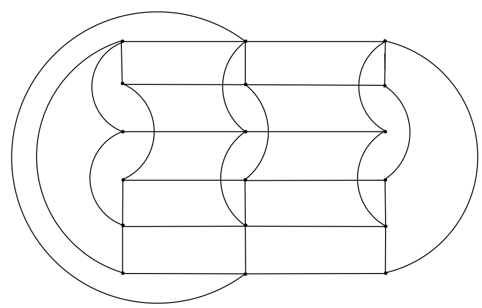
There are at most five vertices of G_1 on each boundary. Joining all 6 vertices to x, there are at least one crossing made by edges of G_1 with edges of T^x. So $cr(G_1^*) \geq 2$.

Case 3 $cr_D(G_1) \geq 2$.

Then $cr(G_1^*) \geq 2$. Whence, $cr(G_1^*) = 2$. □



$G_1 \times P_3$



$K_{2,3} \times S_2$

Fig.2

Lemma 2.2 *In any good drawing of $G_1 \times P_n$, $n \geq 2$, there are at least two crossings on the edges of G_1^i for $i = 1, 2, \cdots n$.*

Proof Let w_i denote the number of crossings on the edges of G_1^i for $i = 1, 2, \cdots n$ and $H_i = \langle V(G_1^i) \bigcup V(G_1^{i+1}) \rangle_{G_1 \times P_n}$ for $i = 1, 2, \cdots n-1$. First, we prove that $w_n \geq 2$. Let T' be a graph obtained by contracting the edges of G_1^{n-1} in H_{n-1} resulting in a graph homeomorphic to G_1^*.

By the proof of Lemma 2.1, $w_n \geq cr(T') = cr(G_1^*) = 2$. For $i = 1, 2, \cdots n-1$, let T_i be the graph obtained by contracting the edges of G_1^{i+1} in H_i resulting in a graph homeomorphic to G_1^*. Similarly, by Lemma 2.1, we get that $w_i \geq cr(T_i) = cr(G_1^*) = 2$ for $i = 1, 2, \cdots n-1$. □

Lemma 2.3 *If D is a good drawing of $G_1 \times P_n$ in which every copy of G_1 has at most three crossings on its edges, then D has at least $4(n-1)$ crossings.*

Proof Let D be a good drawing of $G_1 \times P_n$ in which every copy of G_1 has at most three crossings on its edges. We first show that in D no black edges of G_1^i cross any black edges of G_1^j for $i \neq j$. If not, suppose there is a black edge of G_1^i crossing with a black edge of G_1^j. Since D is a good drawing and every edge of G_1 is an edge of a cycle, there exists a cycle induced by $V(G_1^i)$ which contains a black edge crossing with at least two black edges of G_1^j. Now delete the black edges of G_1^i. The resulting graph is either

(1) homeomorphic to $G_1 \times P_{n-1}$ for $i = 2, 3, \cdots n-1$; or

(2) contains a subgraph homeomorphic to $G_1 \times P_{n-1}$ for $i = 1$ or $i = n$.

Since every copy of G_1 in $G_1 \times P_n$ has at most three crossings on its edges, the drawing of the resulting graph has at most one crossing on the edges of G_1^j. Contradicts to Lemma 2.2.

Next, we show that no black edge of G_1^i crosses with a red edge of $L(t-1,t)$ for $t \neq i$ and $t \neq i+1$. If not, suppose that in D there is a black edge of G_1^i, ($i \neq t$ or $i \neq t-1$) crossing with a red edge of $L(t-1,t)$. Then the red edge crosses at least two black edges of G_1^i, for otherwise, in D, the subdrawing $D(G_1^i)$ separates two G_1 and G_1^i is crossed by all six edges of $L(t-1,t)$, a contradiction. Therefore, the red edge crosses at least two black edges of G_1^i. Thus, D contains a subdrawing of a graph homeomorphic to $G_1 \times P_2$ induced by $V(G_1^{i-1}) \bigcup V(G_1^i)$ or $V(G_1^i) \bigcup V(G_1^{i+1})$ with at most one crossing on the edges of G_1^i. Also contradicts to the Lemma 2.2.

For $i = 2, 3, \cdots n-1$, let

$$Q^i = \langle V(G_1^{i-1}) \bigcup V(G_1^i) \bigcup V(G_1^{i+1}) \rangle_{G_1 \times P_n}.$$

Thus, Q^i has six red edges in each of $L(i-1,i)$ and $L(i,i+1)$, and ten black edges in each of G_1^{i-1}, G_1^i and G_1^{i+1}. Note that Q^i is homeomorphic to $G_1 \times P_3$. See Fig.2 for details.

Denote by Q_c^i the subgraph of Q^i obtained by removing nine edges u_2u_3, u_3u_4,u_4u_6,v_2v_3, v_3v_4, v_4v_6, w_2w_3, w_3w_4 and w_4w_6. Notice that Q_c^i is homeomorphic to $K_{2,3} \times S_2$, such as shown in Fig.2.

In a good drawing of $G_1 \times P_n$, define the force $f(Q_c^i)$ of Q_c^i to be the total number of crossing types following.

(1) a crossing of a red edge in $L(i-1,i) \bigcup L(i,i+1)$ with a black edge in G_1^i;

(2) a crossing of a red edge in $L(i-1,i)$ with a red edge in $L(i,i+1)$;

(3) a self-intersection in G_1^i.

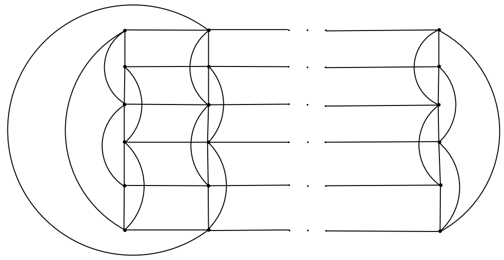
The total force of the drawing is the sum of $f(Q_c^i)$ for $i = 2, 3, \cdots n-1$. It is readily seen that a crossing contributes at most one to the total force of a drawing.

Consider now a drawing D_c^i of Q_c^i induced by D. As we have shown above, in D_c^i no two black edges of different G_1^x and G_1^y, for $x, y \in \{i-1, i, i+1\}$ cross each other, no red edge of $L(i-1,i)$ crosses a black edge of G_1^{i+1} and no red edge of $L(i,i+1)$ crosses a black edge of G_1^{i-1}. Thus, we can easily see that in any optimal drawing D_c^i of Q_c^i there are only crossing of types (i) , (ii) or (iii) above. This implies that in D, for every i, $i = 2, 3, \cdots n-1$, $f(Q_c^i) \geq cr(K_{2,3} \times S_2) = 4$ ([7]), and thus the total force of D is $\sum_{i=2}^{n-1} f(Q_c^i) \geq 4(n-2)$.

By lemma 2.2, in D there are at least two crossings on the edges of G_1^1 and at least two crossings on the edges of G_1^n. None of these crossings is counted in the total force of D. Therefore, in D there are at least $\sum_{i=2}^{n-1} f(Q_c^i) + 4 \geq 4(n-1)$ crossings. □

Theorem 2.1 *$cr(G_1 \times P_n) = 4(n-1)$, for $n \geq 1$.*

Proof The drawing in Fig.3 shows that $cr(G_1 \times P_n) \leq 4(n-1)$ for $n \geq 1$.



$G_1 \times P_n$

Fig.3

We prove the reverse inequality by the induction on n. First we have $cr(G_1 \times P_1) = 4(1-1) = 0$. So the result is true for $n = 1$. Assume it is true for $n = k$, $k \geq 1$ and suppose that there is a good drawing of $G_1 \times P_{k+1}$ with fewer than $4k$ crossings. By Lemma 2.3, some G_1^i must then be crossed at least four times. By the removal of all black edges of this G_1^i, we obtain either

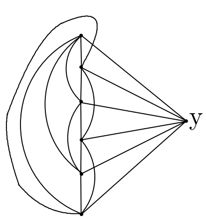
(1) a graph homeomorphic to $G_1 \times P_k$ for $i = 2, 3, \cdots n-1$; or

(2) a graph which contains the subgraph $G_1 \times P_k$ for $i = 1$ or $i = n$.

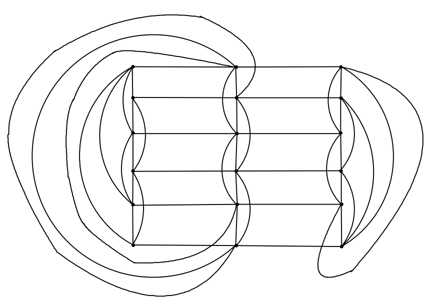
The drawing of any of these graphs has fewer than $4(k-1)$ crossings and thus contradicts the induction hypothesis. □

§3. The crossing number of $G_2 \times P_n$

By joining all 6 vertices of G_2 to a new vertex y, we obtain a new graph denoted by G_2^*.



G_2^*



$G_2 \times P_3$

Fig.4

Lemma 3.1 $cr(G_2^*) = 3$.

Proof A good drawing of G_2^* in Fig.4 shows that $cr(G_2^*) \leq 3$.$|V(G_2^*)| = 7$, $|E(G_2^*)| = 18$. Apply

$|E| \leq 3|V| - 6,$

$|E(G_2^*)| + 2 \times cr(G_2^*) \leq 3 \times (|V(G_2^*| + cr(G_2^*)) - 6,$

it follows that $cr(G_2^*) \geq 3$. Therefore $cr(G_2^*) = 3$. □

Lemma 3.2 *In any good drawing of $G_2 \times P_n$, $n \geq 2$, there are at least three crossings on the edges of G_2^i for $i = 1, 2, \cdots n$.*

Proof Using the same way as in the proof of Lemma 2.2 just instead of G_1^i by G_2^i), we can get the result. □

Lemma 3.3 *If D is a good drawing of $G_2 \times P_n$ in which every copy of G_2 has at most five crossings on its edges, then D has at least $6(n-1)$ crossings.*

Proof Let D be a good drawing of $G_2 \times P_n$ in which every copy of G_2 has at most five crossings on its edges. We first show that in D no black edges of G_2^i crosses with any black edges of G_2^j for $i \neq j$. if not, suppose there is a black edge of G_2^i crossing with a black edge of G_2^j. Since D is a good drawing and there are four disjoint paths between any two vertices in G_2, there are at least four crossings on the edges of G_2^j crossed with edges of G_2^i. Now delete the black edges of G_2^i. Then the resulting graph is either

(1) homeomorphic to $G_2 \times P_{n-1}$ for $i = 2, 3, \cdots n-1$; or

(2) contains a subgraph homeomorphic to $G_2 \times P_{n-1}$ for $i = 1$ or $i = n$.

Since every copy of G_2 in $G_2 \times P_n$ has at most five crossings on its edges, the drawing of the resulting graph has at most one crossing on the edges of G_1^j. Contradicts to Lemma 3.2.

Next, we show that no black edge of G_2^i is crossed by a red edge of $L(t-1, t)$ for $t \neq i$ and $t \neq i+1$. If not, suppose that in D there is a black edge of G_2^i, ($i \neq t$ or $i \neq t-1$) crossed by a red edge of $L(t-1, t)$. Then the red edge crosses at least four black edges of G_2^i, for otherwise, in D, the subdrawing $D(G_2^i)$ separates two G_2 and G_2^i is crossed by all six edges of $L(t-1, t)$, a contradiction. Therefore, the red edge crosses at least four black edges of G_2^i. Thus, D contains a subdrawing of a graph homeomorphic to $G_2 \times P_2$ induced by $V(G_2^{i-1}) \bigcup V(G_2^i)$ or $V(G_2^i) \bigcup V(G_1^{i+1})$ with one crossing on the edges of G_2^i at most. Contradicts to Lemma 3.2.

For $i = 2, 3, \cdots n-1$, let

$$Q^i = \langle V(G_2^{i-1}) \bigcup V(G_2^i) \bigcup V(G_2^{i+1}) \rangle_{G_2 \times P_n}.$$

Thus, Q^i has six red edges in each of $L(i-1, i)$ and $L(i, i+1)$, and twelve black edges in each of G_2^{i-1}, G_2^i, and G_2^{i+1}. Note that Q^i is homeomorphic to $G_2 \times P_3$. See Fig.4 for details.

It is easy to see that $G_2 \times P_3$ contains a subgraph homeomorphic to $G_1 \times P_3$, denoted by Q_c^i. In a good drawing of $G_2 \times P_n$, define the force $f(Q_c^i)$ of Q_c^i to be the total number of crossing types following.

(1) a crossing of a red edge in $L(i-1, i) \bigcup L(i, i+1)$ with a black edge in G_2^i;

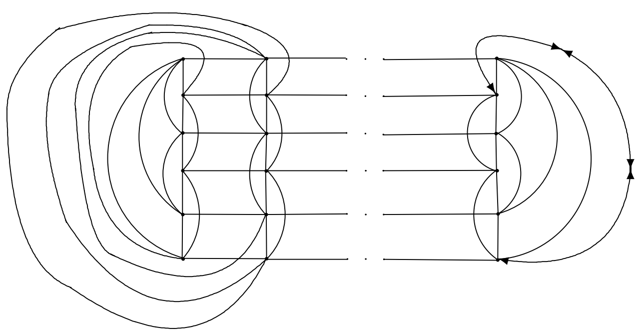
(2) a crossing of a red edge in $L(i-1, i)$ with a red edge in $L(i, i+1)$;

(3) a self-intersection in G_2^i.

The total force of the drawing is the sum of $f(Q_c^i)$ for $i = 2, 3, \cdots n-1$. It is readily seen that a crossing contributes at most one to the total force of the drawing.

Consider now a drawing D_c^i of Q_c^i induced by D. As we have shown previous, in D_c^i no two black edges of G_2^x and G_2^y, for $x, y \in \{i-1, i, i+1\}$ cross each other, no red edge of $L(i-1, i)$ crosses with a black edge of G_2^{i+1} and no red edge of $L(i, i+1)$ crosses with a black edge of G_2^{i-1}. Thus, we can easily see that in any optimal drawing D_c^i of Q_c^i there are only crossings of types (i), (ii) or (iii) above. This implies that in D, for every i, $i = 2, 3, \cdots n-1$, $f(Q_c^i) \geq cr(G_1 \times P_3) = 8$, and thus the total force of D is $\sum_{i=2}^{n-1} f(Q_c^i) \geq 8(n-2)$.

By lemma 2.2, in D there are at least three crossings on the edges of G_2^1 and at least three crossings on the edges of G_2^n. None of these crossings is counted in the total force of D. Therefore, there are at least $\sum_{i=2}^{n-1} f(Q_c^i) + 6 \geq 6(n-1)$ crossings in D. □



$G_2 \times P_n$

Fig.5

Theorem 3.1 *$cr(G_2 \times P_n) = 6(n-1)$, for $n \geq 1$.*

Proof The drawing in Fig.5 following shows that $cr(G_2 \times P_n) \leq 6(n-1)$ for $n \geq 1$. We prove the reverse inequality by the induction on n. First we have $cr(G_2 \times P_1) = 6(1-1) = 0$. So the result is true for $n = 1$. Assume it is true for $n = k$, $k \geq 1$ and suppose that there is a good drawing of $G_2 \times P_{k+1}$ with fewer than $6k$ crossings. By Lemma 2.3, some G_2^i must then be crossed at least six times. By the removal of all black edges of this G_2^i, we obtain either

(1) a graph homeomorphic to $G_2 \times P_k$ for $i = 2, 3, \cdots n-1$; or

(2) a graph which contains the subgraph $G_2 \times P_k$ for $i = 1$ or $i = n$.

The drawing of any of these graphs has fewer than $6(k-1)$ crossings and thus contradicts the induction hypothesis. □

References

[1] Bondy J. A. and Murty U.S.R., *Graph Theory with Applications*, New York: Macmilan Ltd Press, 1976.

[2] L.W.Beineke and R.D.Ringeisen, On the crossing numbers of product of cycles and graphs of order four, *J.Graph Theory*, 4 (1980), 145-155.

[3] A.M.Dean, R.B.Richter, The crossing number of $C_4 \times C_4$, *J.Graph Theory*, 19(1995), 125-129.

[4] Y.H.Peng, Y.C.Yiew, The crossing number of $P(3,1)\times P_n$, *Discrete Mathematics*, 306(2006), 1941-1946.

[5] F.Harary, P.C.Kainen, A.J.Schwenk, Toriodal graphs with arbitratily high crossing numbers, *Nanta Math.*, 6(1973), 58-67.

[6] M.Klesc, On the crossing numbers of Cartesian products of paths and stars or cycles, *Math. Slovaca*, 41(1991), 113-120.

[7] M.Klesc, On the crossing numbers $K_{2,3} \times S_n$, *Tatra Mountains Math Publ.*, 9(1996), 51-56.

[8] M.Klesc, The crossing numbers of $K_5 \times P_n$, *Tatra Mountains.Publ.*, 18(1999), 63-68.

[9] R.D.Ringeisen, L.W.Beineke, The crossing number of $C_3 \times C_n$, *J.Combin.Theory*, Ser.B, 24(1978), 134-136.