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ON $\omega'\beta$ *Sets and* $-\omega'\beta$ CONTINUOUS FUNCTIONS

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ABSTRACT

The aim of this paper is to investigate   class of continuity named $\omega'\beta$ continuity. Some characterizations and preservation theorems are investigated.   Relationship between lindelof space and $\omega'\beta continuity$ is studied. Furthermore some basic properties of $\omega'\beta - \text{open and closed sets}$ are investigated.

KEYWORDS: $\omega'\beta - \text{open}$, $\omega'\beta$ irresolute, β-Lindelof.

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INTRODUCTION

In both pure and applied domains, General topology has great significance.it plays a very significant role in data mining [15].If one has to produce knowledge from data in any real life field.Information systems can prove very useful. As a matter of fact, topological structure on the collection of data are quite suitable. The influence of general topological spaces can be observed in computer science.Apart from that we see its use in computational topology for geometric and molecular design[13].

Many Mathematicians have researched and studied continuity on topological spaces,as significant and fundamental subject in the study of topology. Mathematicians have introduced various forms of continuity. These continuities involve different kinds of generalized sets such as

b-open$[3]$, $\beta - \text{open}[1]$, $\omega - open[6]$, $\omega' - open[4] \text{ sets}$ and many more.

Hdeib [6] explored the concept of ω-closed sets in 1982 and ω-continuous functions in 1989.later on H Aljarrah and M Noorani[2] investigated $\omega\beta - \text{continuous functions using } \omega\beta - \text{open sets}$. T.Noiri, A. Al-Omari and M.S.M. Noorani[16] introduced $\omega b - \text{open sets.}$

The purpose of the paper is to investigate class of $\omega'\beta - \text{continuous functions}$ using $\omega'\beta$ –open sets.

Throughout the present paper, a space means topological space on which there are no separation axioms assumed. Exceptions are explicitly stated. Let A be a subset of a space (X,τ). The closure of A and interior of A in (X,τ) are denoted by $Int(A)$ and $cl(A)$, respectively.

Definition1.1 A subset A of a space (X,τ) is said to be

(1) $b - \text{open}$ [3], $A \subset Int\ (cl(A))\ \cup cl\big(Int(A)\big)$

(2) $\beta - \text{open}$ [1] if , $A \subset cl(Int(cl\ (A)).$

(3) $\omega - \text{open}$ [6]. set if for every $x \in A$ there exists an $-$ open set U containing x such that $U - A$ is countable.

(4) $\omega\beta - \text{open}$ [2] set if for every $x \in A$ there exists an $\beta - $ open set U containing x such that $U - A$ is countable.

We use $\omega\beta O(X,\tau)$ $(resp.,\ \beta O(X,\tau),\ \omega O(X,\tau),\ bO(X,\tau))$ to denote the family of all $\omega\beta - open, (resp.\ \beta - open, \omega - open, b - open)$ subsets of (X,τ).

Definition 1.2 $A\ function\ f:(X,\tau) \to (Y,\sigma)$ is called $\omega - continuous$ [7] if for every $x\ \in X$ and each open set V in (Y,σ), containing $f(x)$ there exists an $\omega O\ (X,\tau)$ $set\ U$ containing x such that $f\ (U) \subset V$.

Definition1.3 $\omega'\beta - $ open set if for every $x\ \epsilon A$ there exists an $\beta - $ open set U containing x such that $U- cl(A)$

is countable.

Lemma 1.4 Let (X,τ) be a topological space:

   i. The arbitrary union $of\ \omega'\beta O(X,\tau)\ sets\ is\ \omega'\beta O(X,\tau).$

   ii. The intersection of an $\omega'\beta O(X,\tau)$ set and open set is $\omega'\beta O(X,\tau).$

Theorem 1.5 Let (Y,τ_Y) be a subspace of $(X,\tau).which$ is $\beta O(X,\tau).Let\ A \subset Y, then$ $A\ \in \omega'\beta O(X,\tau)$ if and only if $A\ \epsilon\omega'\beta O(Y,\tau_Y).$

Theorem 1.6 Let A be a subset of a topological space (X,τ). Then $x \in \omega'\beta cl(A)$ if and only if $A \cap U \neq \emptyset$. for every $\omega'\beta O\ (X,\tau\ )$ set U containing $x.$

Theorem 1.7 [5] if $f:(X,\tau) \to (Y,\sigma)$ is an open continuous function, then $f^{-1}\big(cl(B)\big) = cl\big(f^{-1}(B)\big)\ for\ every\ subset\ B\ of\ Y.$

$\omega'\beta -$CONTINUOUS FUNCTIONS

Definition 2.1 $A\ function\ f:(X,\tau) \to (Y,\sigma)$ is called $\omega'\beta - continuous$ at a point $x\ \in X,$ if for every open set V $\in \sigma$ containing $f(x)$ there exists an $U \in \omega'\beta O(X,\tau)$ set containing x such that $f(U) \subset V.$ If f is $\omega'\beta - continuous$ at each point of X then f is said to be $\omega'\beta - continuous\ on\ X$.

Definition 2.2 Let (X,τ) be any space, a set $A \subset X$ is said to be $\omega'\beta - neighbourhood$ of a point $x \in\ X$ if and only if there exists a $U \in \omega'\beta O(X,\tau)$ set containing x such that $U \subset A.$

Definition 2.3.. The following are equivalent for a function f $:\ (X,\tau) \to (Y,\sigma)$, where X and Y are topological space:

   i. The function f is $\omega'\beta - continuous.$

   ii. For each open set V $\subset Y, f^{-1}(V)\ \in \omega'\beta O(X,\tau).$

   iii. For each $x \in X$, the inverse of every neighborhood of $f(x)$ is an $\omega'\beta - neighbourhood$ of $x.$

   iv. For each $x\ \in X$ and each neighborhood V of $f(x)$, there is an $\omega'\beta - neighbourhood$ U of x such that $f(U) \subset\ V.$

   v. For each closed set $B\ of\ Y, f^{-1}(B)$ is $\omega'\beta - closed\ in\ X.$

   vi. For each subset $A\ of\ X, f(\omega'\beta cl(A)) \subset cl(f(A)).$

   vii. For each subset $B\ of\ Y, \omega'\beta cl(f^{-1}(B)) \subset (f^{-1}\big(cl(B)\big)).$

Proof. $(\mathbf{i} \to \mathbf{ii})$ Let V be open in Y and $x \in f^{-1}(V)$ then $f(x) \in V, so\ by(i)$, there exists an $\omega'\beta O(X,\tau)$ set U_x in X containing x such that $f(U_x) \subset V.$ Then $x \in U_x \subset f^{-1}(V)$ and hence

$f^{-1}(V) = \cup_{x \in f^{-1}(v)} U_x$. By Lemma 1.4(i), $f^{-1}(V) \in \omega'\beta O(X,\tau)$, Which implies that f is $\omega'\beta - continuous$.

$(ii \rightarrow iii)$ For $x \in X$, let V be the neighborhood of $f(x)$, $then\ f(x) \in W \subset V$, where W is open in Y. By (ii), $f^{-1}(W) \in \omega'\beta O(X,\tau), and\ x \in f^{-1}(W) \subset f^{-1}(V)$. Then by Definition2,2, $f^{-1}(V)\ is\ \omega'\beta - neighbourhood$ of x.

$(iii \rightarrow iv)$. For $x \in X$ and V be a neighborhood of $f(x)$. Then U=$f^{-1}(V)$ is an $\omega'\beta -$ neighborhood of x and $f(U) = f\big(f^{-1}(V)\big) \subset V$.

$(iv \rightarrow v)$. For any $x \in X - f^{-1}(B), f(x) \in Y - B$. Since B is closed, th $Y - B$ is neighborhood of $f(x)$, hence there is a $\omega'\beta - neighbourhood\ U$ of x such that $f(U) \subset Y - B$, there exists an $\omega'\beta O\ (X,\tau)$ set U_x in X containing x and $U_x \subset U \subset X - f^{-1}(B)$, take $(X - f^{-1}(B)) = \underset{x\,\in\,f^{-1}(Y - B)}{\cup\ U_x}$. By Lemma 1.4 (i), the set $(X - f^{-1}(B)) \in \omega'\beta O\ (X,\tau)$, which implies $f^{-1}(B)$ is $\omega'\beta C(X,\tau)$.

$(v \rightarrow vi)$. Let $A \subset X$, Since cl$(f(A))$ is a closed set in Y by (vi) , $f^{-1}cl(fA)))\ is\ an\ \omega'\beta C(X,\tau)$ set containing A , then $f\ (\omega'\beta cl(A)) \subset cl(f(A))$.

$(vi \rightarrow vii)$. Let $B \subset Y$. By (vi), $f(\omega'\beta cl(f^{-1}(B))) \subset cl\ (B), so\ \omega\beta cl(f^{-1}(B)) \subset f^{-1}(cl(B))$.

$(vii \rightarrow i)$. We Suppose that f is not $\omega'\beta - continuous$. So there exist $x \in X$ and $V \in \sigma$ with $f\ (x) \in V$ such that for all $\omega'\beta O(X,\tau)$ sets U with $x \in U$ and f (U) $\not\subset (V)$ i.e. f (U) $\cap (Y - V) \neq \emptyset$. Therefore, $x \in \omega\ '\beta cl(f^{-1}(Y - V))$ by Theorem 1.6, and so by (vii), $f(x) \in cl\ (Y - V)$, thus $V \cap (Y - V) \neq \emptyset$, for all open sets V in (Y,σ) containing $f(x)$, a contradiction. Therefore, f is $\omega'\beta -$ continuous.

Definition2. 4. For any subset A of a topological space (X,τ) the frontier of A, denoted by$\omega'\beta F_r\ (A)$, is defined as $\omega'\beta cl(A) \cap \omega'\beta cl(X - A)$.

Theorem2.5. Let $f : (X,\tau) \rightarrow (Y,\sigma)$ be a function.
Then $X - \omega'\beta c(f) = \cup \{\omega'\beta F_r\ (f^{-1}(V)) : V \in \sigma, f(x) \in V, x \in X\}\ where\ \omega'\beta c\ (f)$ denotes the set of points at which f is $\omega'\beta -$ continuous.

Proof. Let $x \in X - \omega'\beta c(f)$. Then for every $\omega'\beta O\ (X,\tau)$ set U containing x there exists open sets V in (Y,σ) containing $f(x)$ such $f(U) \not\subset V, Hence\ U \cap (X - f^{-1}(V)) \neq \phi$ for every $\omega'\beta O\ (X,\tau)$ set U containing x . Therefore, $x \in \omega'\beta cl(X - f^{-1}(V))$by Theorem 1.6. Then $x \in f^{-1}\ (V) \cap \omega'\beta cl\ 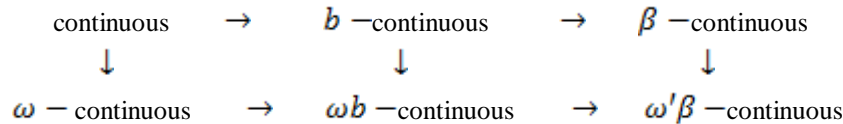(X - f^{-1}(V)) \subset \omega'\beta F_r\ (f^{-1}(V))$. Hence, $X - \omega'\beta c(f) \subset \cup \{\omega'\beta F_r(f^{-1}(V)), V \in \sigma, f(x) \in V, x \in X\}$. Conversely, let $x \notin X - \omega'\beta c(f)$. Then for each open set V in (Y,σ) containing $f\ (x), (f^{-1}(V)$ is $\omega'\beta O\ (X,\tau)$ containing x, thus for every $V \in \sigma$ containing $f(x), x \in \omega'\beta Int(f^{-1}(V))$ and hence $x \notin \omega'\beta F_r\ (f^{-1}(V))$. So $\cup \{\omega'\beta F_r\ (f^{-1}(V)) : V \in \sigma, f(x) \in V, x \in X\} \subset X - \omega'\beta c(f)$.

Corollary 2.6 *A* function $f : (X,\tau) \rightarrow (Y,\sigma)$ is $\omega'\beta - continuous$ if and only if f^{-1} (int (G)) $\subset \omega'\beta$ int $(f^{-1}(G))$, for any subset G *of Y*.

Proof. Let $G \subset Y$. Since f is $\omega'\beta - continuous$, $f^{-1}(int(G)) \in \omega'\beta O(X,\tau)$. As $f^{-1}(int(G)) \subset f^{-1}(G)$, so $f^{-1}(int(G)) \subset \omega'\beta$ int $(f^{-1}(G))$.

Now . if $x \in X$ and V $\in \sigma$ with $f(x) \in V$. Then $x \in f^{-1}(V)$ and so $x \in \omega'\beta Int(f^{-1}(V))$. There exists U$\in \omega'\beta O(X,\tau)$ such that $x \in U \subset f^{-1}(V)$. Hence $f(x) \in f(U) \subset V$ and hence the result .

Further if X is a countable set then every function f : $(X,\tau) \to (Y,\sigma)$ is $\omega'\beta - continuous$. The following diagram follows immediately from the definitions in which none of the implications is reversible.

$$
\begin{array}{ccccc}
\text{continuous} & \to & b -\text{continuous} & \to & \beta -\text{continuous} \\
\downarrow & & \downarrow & & \downarrow \\
\omega - \text{continuous} & \to & \omega b -\text{continuous} & \to & \omega'\beta -\text{continuous}
\end{array}
$$

Example 2. 7 Let $X = \{1, 2, 3\}$ with the topology $\tau = \{X, \phi, \{1\}, \{2\}, \{1,2\}\}$ and $Y = \{a, b\}$ with the topology $\sigma = \{\phi, Y, \{a\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the function defined by

$$f(x) = \begin{cases} b & x = \{1, 2\} \\ a & x = 3 \end{cases}$$

Then f is not $\beta -$ continuous, but it can be easily seen that f is $\omega'\beta - continuous$.

Example 2.8 Let $X = R$ with the topology $\tau = \tau_u$ and $Y = \{a, b\}$ with the topology $\sigma = \{\phi, Y, \{a\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the function defined by

$$f(x) = \begin{cases} a & x \in [0,1) \cap \mathbb{R} - \mathbb{Q} \\ b & x \in [0,1) \cap \mathbb{Q} \end{cases}$$

Then f is $\omega'\beta -$ continuous, but it is not $\omega'b - continuous$.

Proposition2.9. If $f : (X,\tau) \to (Y,\sigma)$ is an $\omega'\beta - continuous$ function X, then the restriction $f\big|_A : (A,\tau_A) \to (Y,\sigma)$ is $\omega'\beta - continuous$ provided A is an open set in X.

Proof. Since f is an $\omega'\beta - continuous function$, for any open set V $\in \sigma$, $f^{-1}(V) \in \omega'\beta O(X,\tau)$. Hence by Lemma 1.4(ii), $f^{-1}(V) \cap A \in \omega'\beta O(X,\tau)$ since A is an open set. Therefore, by Theorem 1.5, $(f\big|_A)^{-1}(V) = f^{-1}(V) \cap A \in \omega'\beta O(A,\tau_A)$ sets, which implies that $f\big|_A$ is $\omega'\beta - continuous$ function.

Example 2.10 Let X = R with the topology $\tau = \tau_u$ and Y= $\{0,1\}$ with the topology $\sigma = \{\phi, Y, \{1\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the function defined by

$$f(x) = \begin{cases} 1 & x = \sqrt{2} \\ 0 & x \in \mathbb{Q} \end{cases}$$

It can be easily seen that f is $\omega'\beta - continuous$. We take A = $\mathbb{R} - \mathbb{Q}$. Then A $\in \omega' O(X,\tau)$ and $f\big|_A$ is not $\omega\beta - continuous$ since $f\big|_A)^{-1}$ (Y) = $\{\sqrt{2}\} \notin \omega'\beta O(A,\tau_A)$.

Definition 2.11 A cover U= $\{U_\alpha : \alpha \in \Delta\}$ of subset of X is called a $\beta O(X,\tau)$ cover if U_α is $\beta O(X,\tau)$ for each $\alpha \in \Delta$.

Proposition 2.12　　Let $f : (X,\tau) \to (Y,\sigma)$ be any function and A ={ $A_\alpha : \alpha \in \Delta$} be a　cover of X $by\ \beta O(X,\tau)$. If the restriction, $f|_{A_\alpha} : (A_\alpha, \tau_{A_\alpha}) \to (Y,\sigma)\ of\ f\ is\ \omega'\beta - continuous$ for each $\alpha \in \Delta$, then f is $\omega'\beta - continuous$.

Proof.　Let　$V \in Y$　Since　$f|_{A_\alpha}$　is $\omega'\beta - continuous,$　then　for　each　$\alpha \in \Delta,$　we　have $(f|_A)^{-1}(V) = f^{-1}(V) \cap A_\alpha \in \omega'\beta O(A_\alpha, \tau_{A_\alpha}).$ So by Theorem 1.5, $f^{-1}(V) \cap A_\alpha$ $\in \omega'\ \beta O(X,\tau)$ for each $\alpha \in \Delta.$ Take $f^{-1}(V) = \underset{\alpha \in \Delta}{\cup}\ f^{-1}(V) \cap A_\alpha.$ By Lemma 1.4 (i) $f^{-1}(V)$ $\in \omega'\ \beta O(X,\tau).$

Corollary 2.13 Let $f : (X,\tau) \to (Y,\sigma)$ be any function and U={ $A_\alpha : \alpha \in \Delta$} a open cover of $X.$ If the restriction, $f|_{A_\alpha} : (A_\alpha, \tau_{A_\alpha}) \to (Y,\sigma)$　is　$\omega'\beta - continuous$　for　each　$\alpha \in \Delta,$　then　f　is $\omega'\beta - continuous$.

Remark:2.14　　The composition g o $f : (X,\tau) \to (Z,\rho)$ of a continuous function $f : (X,\tau) \to (Y,\sigma)$ and an $\omega'\beta - continuous$ function g : $\to (Y,\sigma) \to (Z,\rho)$ is not necessarily $\omega'\beta - continuous$ function as the following example shows. Thus, the composition of $\omega'\beta - continuous$ functions need not be $\omega'\beta - continuous$.

Example 2.15. Let　$X = \mathbb{R}$　with the topology $\tau = \{R, \phi, R - Q,\},$　$Y = \{1,2\}$ with the topology $\sigma = \{\phi, Y, \{1\}\}$　and $Z = \{a,b\}$ with the topology $\rho = \{\phi, Z, \{a\}\}.$ Let $f : (X,\tau) \to (Y,\sigma)$ be the function defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{R} - \mathbb{Q} \\ 2 & x \in \mathbb{Q} \end{cases}$$

and　g : $(X,\sigma) \to (Y,\rho)$ be the function defined by

$$g(x) = \begin{cases} a & x = 2 \\ b & x = 1 \end{cases}$$

Then f is continuous (hence $\omega'\beta - continuous)\ and$ g is $\omega'\beta - continuous.$ However g o f is not $\omega'\beta - continuous,$ because (g o $f)^{-1}$ ({a} = $\mathbb{Q} \notin \omega'\beta O(X,\tau).$

Proposition 2.16. The composition g o $f : (X,\tau) \to) \to (Z,\rho)\ is\ \omega'\beta - continuous.$

　If $f : (X,\tau) \to (Y,\sigma)\ is\ \omega'\beta - continuous\ and$ g : $\to (Y,\sigma) \to (Z,\rho)$ is continuous.

　　Proof. Let $x \in X$　and $V \in \rho$ with (g o f) $(x) \in V$, *since* g is continuous, there exists open sets $W \in \sigma$　with $f(x) \in W$ and g (W) $\subset V.$ *Morever* f is $\omega'\beta - continuous,$ *there exists open* $U \in \omega'\beta O\ (X,\tau)$ *say containing* x *such that* $f(U) \subset W.$ Now (g o f) (U) $\subset$ g (W) $\subset V$. hence the result.

We note that this result fails if g is assumed to be only $\omega - continuous\ or\ \beta - continuous$ *as it is shown in the next example.*

Example2.17. Consider $X = \mathbb{R}$ with the topology $\tau = \{R, \phi, R - Q,\}, Y = \{a,b,c\}$ with the topology σ $= \{\phi, Y, \{a\}, \{b\}, \{a,b\}\} and\ Z = \{1,2,3,4\}\ with\ the\ topology\ \rho = \{\phi, Z, \{1\}, \{1,2\}, \{1,2,3\}\}.$ Let $f : (X,\tau) \to (Y,\sigma)$ *be the function define by*

$$f(x) = \begin{cases} a & x \in \mathbb{R} - \mathbb{Q} \\ b & x \in \mathbb{Q} \end{cases}$$

and g : $(Y, \sigma) \to (Z, \rho)$ be the function define by

$$g(x) = \begin{cases} 1 & x = a \\ 3 & x = b \\ 2 & x = c \end{cases}$$

Then f is $\omega'\beta - continuous$, g is $\omega'\beta - continuous$ and $\beta - continuous$ functions but g o f is not $\omega'\beta - continuous$ since $(g \circ f)^{-1}(\{3\} = \mathbb{Q} \notin \omega'\beta O(X, \tau)$.

Corollary 2.18 *If* $f : (X, \tau) \to \prod_{\alpha \in \Delta} X_\alpha$ *is an* $\omega'\beta - continuous$ *function from a space* (X, τ) *into a product space* $\prod_{\alpha \in \Delta} X_\alpha$ *then* P_α o f *is* $\omega'\beta - continuous$ *for each* $\alpha \in \Delta$, *where* P_α *is the projection function from the product space* $\prod_{\alpha \in \Delta} X_\alpha$ *onto the space* X_α *for each* $\alpha \in \Delta$.

Theorem 2.19. Let X and Y be a topological spaces, let $f : (X, \tau) \to (Y, \sigma)$ be a function and g : $(X, \tau) \to (X \times Y, \tau \times \sigma)$ be the graph function of f *given by* g $(x) = (x, f(x))$ *for every point* $x \in X$. Then g is $\omega'\beta - continuous$ if is $\omega'\beta - continuous$.

Proof. Suppose that g is $\omega'\beta - continuous$. Now $f = P_Y$ o g where $P_Y : X \times Y \to Y$, then f is $\omega'\beta - continuous$ by Corollary 2.18. Conversely, assume that f is $\omega'\beta - continuous$. Let $x \in X$ and W be any open set in X $\times Y$ containing g (x). Then there exist open sets $U \subset X$ and $V \subset Y$ *such that* g $(x) \in U \times V \subset W$. Since f is $\omega'\beta - continuous$, there exists $U_1 \in \omega'\beta O(X, \tau)$ containing x and $f(U_1) \subset V$. Take $H = U \cap U_1$. Then $H \in \omega'\beta O(X, \tau)$ by lemma1.4 (ii), such that $x \in H$ and $f(H) \subset V$

Therefore we have g $(H) \subset U \times V \subset W$. Thus g is $\omega'\beta - continuous$.

Definition 2.20. [14] A function $f : (X, \tau) \to (Y, \sigma)$ is called pre-semi-preopen if the image of each semi-preopen set in X is a semi-preopen set in Y.

Theorem 2.21. If g o $f : (X, \tau) \to (Z, \rho)$ is $\omega\beta - continuous$ and $f : (X, \tau) \to (Y, \sigma)$ is pre-semi-preopen surjection, then g: $\to (Y, \sigma) \to (Z, \rho)$ is $\omega\beta - continuous$.

Proof. we first prove that if $f : (X, \tau) \to (Y, \sigma)$ is an pre-semi-preopen function and $U \in \omega'\beta O(X, \tau)$, then f(U) $\epsilon \omega'\beta O(Y, \sigma)$. So let U $\in \omega'\beta O(X, \tau)$ then for all $x \in U$ there exists $\beta O(X, \tau)$ sets U_1 in (X, τ) containing x and $U_1 - cl(U) \subset C$ where C is a countable set. Thus $f(U_1) - cl(f(U)) \subset f(C)$ where $f(C)$ is a countable set. This implies $f(U) \in \omega'\beta O(Y, \sigma)$. Now, Let $y \epsilon Y$ and let V $\in \rho$ with g $(y) \in V$. Choose $x \in X$ such that $f(x) = y$. Since g o f is $\omega\beta - continuous$ there exists $U \in \omega\beta O(X, \tau)$ with $x \in U$ and g $(f(U)) \subset V$. But f is pre-semi-preopen function therefore, by assumption, $f(U) \in \omega\beta O(Y, \sigma)$ with $f(x) \in f(U)$. So we get the result.

Corollary 2.22. Let $f_\alpha : (X_\alpha, \tau_\alpha) \to (Y_\alpha, \tau_\alpha)$ be a function for each $\alpha \epsilon \Delta$. If the product function $f = \prod_{\alpha \in \Delta} f_\alpha : \prod_{\alpha \in \Delta} X_\alpha \to \prod_{\alpha \in \Delta} Y_\alpha$ is $\omega'\beta - continuous$, then f_α is $\omega'\beta - continuous$.

Proof. We first prove that any projection function is pre-semi-preopen function . Let $U \in \beta O\ (X, \tau)$ hence $f(U) \subset f(cl(int\ (cl(U))))$, by using the assumption that f is open and continuous surjective, $f\ (U) \subset cl$ (int (cl (f (U)))). Thus $f\ (\ U\ ) \in \beta O\ (Y, \sigma)$. Now For each $\beta\ \in\ \Delta$, let $P_\beta :\ \prod_{\alpha\ \epsilon\ \Delta} X_\alpha\ \to\ X_\beta$ and

$q_\beta :\ \prod_{\alpha\ \epsilon\ \Delta} Y_\alpha\ \to\ Y_\beta$ be the projections, then we have $q_\beta\ o\ f = f_\beta\ o\ p_\beta$ for each $\beta\ \epsilon\ \Delta$. Now

f is $\omega'\beta -$ continuous and q_β is continuous, $q_\beta\ o\ f\ is\ \omega\beta -$ continuous by Proposition 2.16 and hence $f_\beta\ o\ P_\beta$ is $\omega\beta -$ continuous function. Since P_β is pre-semi-preopen function it follows from Theorem 2.21, that f_β is $\omega\beta -$ continuous function.

Theorem 2.23. For any space X, the following properties are equivalent :

   i. X is $\beta -$ Lindelőf.

   ii. Every $\omega'\ \beta O(X, \tau)$ cover of X has a countable subcover.

Proposition 2.24. If $f :\ (X, \tau) \to (\ Y, \sigma)$ be an $\omega'\beta -$ continuous surjective function. And X is $\beta -$ Lindelőf , then Y is Lindelőf.

Proof. Let { $V_\alpha: \alpha\ \in\ \Delta$} be an open cover of Y. Then $\{f^{-1}(\ V_\alpha): \alpha\ \in\ \Delta\}$ is $\omega'\ \beta O(X, \tau)$ cover of X, as f is $\omega'\beta -$ continuous. Since X is $\beta -$ Lindelőf, by Theorem 2.23, X has a countable subcover, say $f^{-1}(\ V_{\alpha_1}), f^{-1}(\ V_{\alpha_2}), \dots, f^{-1}(\ V_{\alpha_n}),\ \dots$, Thus $V_{\alpha_1}, V_{\alpha_2}, \dots, V_{\alpha_n},\ \dots$, is a subcover of { $V_\alpha: \alpha\ \epsilon\ \Delta$} of Y. It follows that Y is Lindelőf.

Corollary2.2 5. Let $f :\ (X, \tau) \to (\ Y, \sigma)$ be a $\beta -$ continuous (or $\omega' -$ continuous) surjective function. And X is $\beta -$ Lindelőf, then Y is Lindelőf.

$\omega'\beta -$IRRESOLUTE FUNCTIONS

Definition 3.1 A function $f :\ (X, \tau) \to (\ Y, \sigma)$ is called $\omega'\beta -$Irresolute if the inverse image of each $\omega'\beta\sigma\ (\ Y, \sigma)$ set is an $\omega'\beta\sigma(X, \tau)$set.

 Remark: We observe that every $\omega'\beta -$Irresolute function is $\omega'\beta -$ continuous but the converse is not true, which is shown by the following example.

Example 3.2. Let $X = \mathbb{R}$ with the topologies $\tau = \{R, \phi, R - Q, \},\ and\ Y = \{1, 2\}$ with the topology $\sigma = \{\ \phi, Y, \{2\}\}$. Let $f :\ (X, \tau) \to (\ Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 1 & \mathbb{R} - \mathbb{Q} \\ 2 & x \in \mathbb{Q} \end{cases}$$

Then f is $\omega'\beta -$ continuous but not $\omega'\beta -$Irresolute since $f^{-1}(\{1\}) = \mathbb{Q} \notin \omega'\beta O\ (X, \tau)$.

Theorem 3.3 The following conditions are equivalent for a function $f :\ (X, \tau) \to (\ Y, \sigma)$.

   i.   *The function f is $\omega'\beta -$Irresolute.*

   ii. *For each $x \in X$ and $V \in \omega'\beta O(Y, \sigma)$ containing $f(x)$, there exists $U \in \omega'\beta\sigma(X, \tau)$ containing x such that $f(U)) \subset V$.*

   iii. *For each $x \in X$, the inverse of every $\omega'\beta -$ neighbourhood of $f(x)$ is $\omega'\beta -$ neighbourhood of x .*

(iv)For each $x \in X$ *and* $\omega'\beta - neighbourhood\ V\ of\ f(x)$, *there exists* $\omega'\beta - neighbourhood$ U *of* x *such that* $f(U)) \subset V$.

Proof. $(i \rightarrow ii\ )$ Suppose that $x \in X\ and\ V \in \omega'\beta\ O(Y,\sigma)$ *containing* $f(x)$, *since* $f\ is\ \omega'\beta - Irresolute$ *then* $f^{-1}(V) \in \omega'\beta O(X,\tau)$ containing x. It follows that $f(f^{-1}(V)) \subset V$.

$(ii \rightarrow iii\ )$ Suppose that $x \in U\ and\ V is\ \omega'\beta - neighbourhood$ of $f(x)$, by Definition 2.2 there exists $V_1 \in \omega\beta O(Y,\sigma)$ such that $f(x) \in V_1 \subset V$, there exists $U \in \omega'\beta O(X,\tau)$ containing x such that $f(U) \subset V_1$. $so, x \in U \subset f^{-1}(V_1) \subset f^{-1}(V)$. Hence, $f^{-1}(V)$ is $\omega\beta - neighbourhood$ of x.

$(iii \rightarrow iv\ )$ If V is $\omega'\beta - neighbourhood$ of $f(x)$, $f^{-1}(V)$ is $\omega'\beta - neighbourhood$ of x by (iii) .and $f(f^{-1}(V)) \subset V$.

$(iv \rightarrow i)$ For each $x \in X, let\ V \in \omega'\beta O(Y,\sigma)$ containing $f(x)$. Take $A = f^{-1}(V)$, if $x \in A$. Then $f(x) \in V$. Since $V \in \omega'\beta O(Y,\sigma)$ so V is a $\omega'\beta - neighbourhood$ of $f(x)$. So A $= f^{-1}(V)$ is $\omega'\beta - neighbourhood$ of x. From which It follows that there exists $A_x \in \omega\beta O(X,\tau)$ such that $x \in A_x \subset A$. Thus, by Lemma 1.4(i) A $= \underset{x \in A}{\cup} A_x$ is $\omega'\beta O(X,\tau)$

Set. Hence, f is $\omega'\beta - Irresolute$.

Theorem 3.4. The following conditions are equivalent for a function $: (X,\tau) \rightarrow (Y,\sigma):$

   i. f is $\omega'\beta - Irresolute$
   ii. For each $\omega'\ \beta\ C\ (Y,\sigma)$ subset $F\ of\ Y$, f^{-1}(F) is $\omega'\ \beta\ C\ (X,\tau)$.
   iii. For each subset A of X, $f(\omega'\beta cl(A)) \subset \omega'\beta cl(f(A))$.
   iv. For each subset B of Y, $\omega'\beta cl(f^{-1}(B)) \subset f^{-1}(\omega'\beta cl(B))$

Proof. $(i \rightarrow ii\ )$ If $F \in \omega'\ \beta\ C\ (Y,\sigma)$ subset of Y. Then $X - f^{-1}(F) \in \omega'\beta O(X,\tau)$, which implies that f^{-1}(F) $\in \omega'\ \beta\ C\ (X,\tau)$.

$(ii \rightarrow iii\ )$ Let A be a subset of X. since $A \subset f^{-1} f(A))$, we have $A \subset f^{-1}(\omega'\beta cl(f(A)))$. Now $f^{-1}(\omega'\beta cl(f(A))) \in \omega'\ \beta\ C\ (X,\tau)$ set containing A by (ii), then $\omega'\beta cl(A) \subset f^{-1}(\omega'\beta cl(f(A)))$, It follows that $f(\omega'\beta cl(f(A)) \subset \omega'\beta cl(f(A))$.

$(iii \rightarrow iv\ )$ Let B $\subset$ Y , by (iii) $f(\omega\beta cl(f^{-1}(B))) \subset \omega\beta cl(f(f^{-1}(B))) \subset \omega\beta cl(B)$, hence $\omega\beta cl(f^{-1}(B)) \subset f^{-1}(\omega\beta cl(B))$.

$(iv \rightarrow i\ )$ Suppose f is not $\omega'\beta - Irresolute$. So there exist $x \in X$ and $V \in \omega\beta O(Y,\sigma)$ with $f(x) \in V$ such that for all U $\in \omega\beta O(X,\tau)$ with $x \in U$ and $f(U) \not\subset (V)$ i.e $f(U) \cap (Y - V) \neq \phi$. Therefore . $x \in f^{-1}(\omega\beta cl(Y - V))$. So by Theorem 1.6, $f(x) \in \omega\beta cl(Y - V)$. Thus for all $V \in \omega'\beta O(Y,\sigma)$ containing $f(x)$, we have $V \cap (Y - V) \neq \phi$, a contradiction. Therefore, f is $\omega'\beta - Irresolute$.

Theorem 3.5. Let $f : (X,\tau) \rightarrow (Y,\sigma)$ be a function. Then f is $\omega\beta - Irresolute\ if\ and\ only\ if$ $f^{-1}(\omega'\beta Int(B)) \subset \omega'\beta Int(f^{-1}(B)\ for\ every\ B \subset Y$.

Proof.   First suppose f is $\omega'\beta -Irresolute$. Let $B \subset$ Y.   Since f is $\omega'\beta -$Irresolute, we have $f^{-1}(\omega'\beta Int(B))$ is $\omega'\beta O(X,\tau)$ set.   As f^{-1} $(\omega'\beta Int(B))$ $\subset$ f^{-1} $(\omega'\beta Int(B))$ $\subset \omega'\beta Int\ (f^{-1}(B))$.

Conversely, Let $x \in X$ and $V \in \omega'\beta O\ (Y,\sigma)$ with $f(x) \in V$. Then $x \in\ f^{-1}(V)$ and so by assumption $x \in \omega'\beta Int\left(f^{-1}(V)\right)$. There exists an $U \in \omega'\beta O(X,\tau)$ sets such that $x \in U \subset\ f^{-1}(V)$.

Hence $f(x) \in f(U) \subset V$ and hence the result.

Proposition 3.7 g o f is $\omega'\beta -$continuous, if $f : (X,\tau) \to (Y,\sigma)$ is $\omega'\beta - i$rresolute and g $: \to (Y,\sigma) \to$ (Z,ρ) is $\omega\beta -$continuous.

Proof. Let $x \in X$ and let V be any open set in (Z,ρ) containing g $(f(x))$. Since g is $\omega\beta -$continuous, there exists an $\omega\beta O(Y,\sigma)$ set W containing $f(x)$ such that g $(W) \subset V$. Put g $(f(U)) \subset$ g $(W) \subset V$. Hence g o f is $\omega'\beta -$continuous.

Corollary 3.8. If $f : (X,\tau) \to (Y,\sigma)$ is $\omega'\beta -$irresolute and g $: (Y,\sigma) \to (Z,\rho)$ is $\omega'\beta -$continuous, then g o f is $\omega'\beta -$continuous.

Recall that a function $f : (X,\tau) \to (Y,\sigma)$ is said to be $\omega' -$irresolute [4] if the inverse image of each $\omega'O(Y,\sigma)$ set is an $\omega'O(X,\tau)$.

Proposition3.9. Let $f : (X,\tau) \to (Y,\sigma)$ be an open continuous function and every $\omega'O(Y,\sigma)$ is closed in the space (Y,σ) then f is $\omega'\beta - irresolute.$

Proof. Let $U \in \omega'\beta O(Y,\sigma)$, $\omega\beta cl(f^{-1}(U)) \subset cl\left(f^{-1}(U)\right) = (f^{-1}(cl(U)) \subset f^{-1}(\omega\beta cl(U))$, by Theorem 1.7,   hence f is $\omega'\beta - irresolute,$ by Theorem 3.4

Definition: A space X is $\omega'\beta - T_2$ [4] as if for each two distinct point $x,y \in X$,

there exists $U, V \in \omega'\beta O(X,\tau)$ such that $x \in U, y \in V$ and $U \cap V = \phi$.

Theorem 3.10   If $f : (X,\tau) \to (Y,\sigma)$ is an $\omega'\beta - irresolute$ injective function and the space Y is $\omega'\beta - T_2$, then X is $\omega'\beta - T_2$.

Proof. Let x_1 and x_2 be two distinct points of X. Since f is injective and Y is $\omega'\beta - T_2$, so,there exist V_1 , V_2 $\in \omega'\beta O(Y,\sigma)$ such that $f(x_1) \in V_1, f(x_2) \in V_2$ and $V_1 \cap V_2 = \phi$. Now $x_1 \in f^{-1}(V_1)$, $x_2 \in f^{-1}(V_2)$ and $f^{-1}(V_1 \cap V_2) = f^{-1}(V_1), \cap f^{-1}(V_2) = \phi$. Since f is $\omega'\beta - irresolute$ then $f^{-1}(V_1), f^{-1}(V_2)$ is $\omega'\beta O(X,\tau)$. Hence X is $\omega'\beta - T_2$.

Definition 3.11 *A space X* is said to be $\omega'\beta - connected\ if\ there\ exist\ disjoint\ \omega'\beta O\ (X,\tau)$ sets *A and B such that A $\cup B = X$.*

Proposition 3.12   *If* $f : (X,\tau) \to (Y,\sigma)$ *is an* $\omega'\beta - irresolute$ *surjective function and X is* $\omega'\beta - connected$, *then Y is* $\omega'\beta - connected.$

Proof.   Assume that Y is not $\omega'\beta - connected.$ Then there exist disjoint $\omega'\beta O(Y,\sigma)$ sets *A* and *B* such that $A \cup B = Y$. *Since* f *is* $\omega'\beta - irresolute$ surjective, $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty $\omega'\beta O(X,\tau)$ sets. Further $f^{-1}(A) \cup f^{-1}(B) = X$. It follows that (X,τ) is not $\omega'\beta - connected,$ which is a contradiction. Hence (Y,σ) is $\omega'\beta - connected.$

$\omega'\beta -$ **Open and** $\omega'\beta -$ **Closed Functions**

Definition 4.1   A function $f : (X, \tau) \to (Y, \sigma)$ is called $\omega'\beta - open\ (resp.\omega'\beta - closed)$ if the image of each open (resp. closed) set in (X, τ) is an $\omega'\beta O\ (Y, \sigma)$ (resp. $\omega'\beta C(Y, \sigma))$.

Remark: We observe that every open (closed) function is $\omega'\beta - open$ (resp. $\omega'\beta - closed)$ function, but the converse is not true, which is shown by the following example.

Example 4.2. Let X = {a,b} with the topology $\tau = \{\ \phi, X, \{a\}\}\ and\ Y = \{1,2,3\}$ with the topology $\sigma = \{\ \phi, X, \{1\}, \{2\}, \{1,2\ \}\}.$ Let $f : (X, \tau) \to (Y, \sigma)$ be   the   function   define   by $f(x) = 3\ for\ all\ x \in X.$ Then f is $\omega\beta - open$ and $\omega\beta - closed\ function,$ but it is neither open nor closed function.

Proposition 4.3 *A function* $f : (X, \tau) \to (Y, \sigma)$ *is* $\omega\beta - open$ *if and only if for each* $x \in X$ *and each open set* $U \in \tau$ *containing* x *, there exists* $W \in \omega\beta O(Y, \sigma)$ *set containing* $f(x)$ *such that* $W \subset f(U)$.

Theorem 4.4   Let $f : (X, \tau) \to (Y, \sigma)$ be a function from space (X, τ) into a space (Y, σ). Then f is $\omega'\beta - closed$ if and only if $\omega'\beta cl(f(A)) \subset f\ (\omega'\beta cl(A))$ for each set A subset of (X, τ).

Proof. Let $f\ \ be\ \omega'\beta - closed$ function and *A* any subset of *X*. Then $f(A) \subset f\ (\omega'\beta cl(A)) \in\ \omega'\beta C(\ Y, \sigma)$, therefore $\omega'\beta cl(f(A)) \subset f\ (\omega'\beta cl(A))$. Conversely, suppose that $B \in \omega'\beta C(X, \tau)$.   Then $\omega'\beta cl\ (f(B)) \subset\ f(\omega'\beta cl\ (f(B)) = f\ (B)$.   Thus we obtain that $\omega'\beta cl(f(B)) = f(B),\ it\ follows\ that\ f\ is\ \omega'\beta - closed\ function.$

Proposition 4.5 Let $f : (X, \tau) \to (Y, \sigma)$ be a containing surjection function and let
g : $(Y, \sigma) \to (Z, \rho)$ be such that   g o f : $(X, \tau) \to (Z, \rho)$ is $\omega'\beta - open$ function,   then g is $\omega'\beta - open.$

     Proof. Let $y\ \in Y$ and let V $\in \sigma$ with g $(y) \in$ V. Choose $x \in$ X such that $f(x) = y.$ Since   g o f is $\omega'\beta - open$ function, then g (V)= g o $f(f^{-1}(V)) \in\ \omega''\beta O(Z, \rho)$. It follows that g is $\omega'\beta - open.$

     The   following   examples   show   that   the $\omega'\beta - open$ function   is   independent   with $\omega'\beta - irresolute$ and $\omega'\beta - continuous\ function.$

Example 4.6 Let $X = \mathbb{R}$ with the topologies $\tau = \{R, \phi, R - Q, \},$ and let Y = {2, 3 } with the topology $\rho = \{\phi, Y, \{3\}\}.$ Let $f : (X, \tau) \to (Y, \rho)$ be the function defined by

$$f(x) = \begin{cases} 3 & x \in Q \\ 2 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Then f is not $\omega'\beta - continuous,$ but it can easily seen that $f\ (x)$ is $\omega'\beta - open\ function.$

Example 4.7 Let   X = {1, 2} with the topology $\tau = \{\phi, X, \{1\}\}$ and let $Y = \mathbb{R}$ with the topologies $\sigma = \tau_{coc}.$ Let $f : (X, \tau) \to (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} \mathbb{R} - \mathbb{Q} & x = 2 \\ \mathbb{Q} & x = 1 \end{cases}$$

Then f is not $\omega\beta - open,$ but it can be easily seen that f is $\omega'\beta - continuous$ and $\omega'\beta - irresolute$ funciton.

Example 4.8 Consider the function f in the Example 8 which is $\omega'\beta - open$, but not $\omega'\beta - irresolute$.

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