ON THE MODE SYNTHESIS IN THE SYNCHROSQUEEZING METHOD

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ABSTRACT

The synchrosqueezing is a powerful tool to analyse and represent multicomponent signals in the time-frequency plane, but the method does not provide a complete and accurate mode decomposition. This paper investigates the modes synthesis issue for a multicomponent signal from selected coefficients of its wavelet transform. We first discuss the reconstruction method proposed by Brevdo et al. to find the wavelet coefficients of a mode and propose an alternative which appears to be more stable. We then remark that the classical synthesis formula from selected wavelet coefficients is not mathematically sound, therefore we propose a new one based on the projection of the coefficients onto some reproducing kernel space. Numerical experiments show the efficiency of our approach for a large range of applications.

Index Terms— Analytic Wavelet Transform, Empirical Mode Decomposition, Synchrosqueezing, Intrinsic Mode Functions

1. INTRODUCTION

In the last two decades, many methods based on the wavelet transform, the short-time Fourier transform [1, 2, 3, 4], or the so-called Empirical Mode Decomposition (EMD) [5, 6], have been proposed to analyse narrowband signals. While the first kind of methods are mathematically sound and allow a better and more robust separation [7], the EMD-based strategies are fully adaptive, and can handle non-harmonic waves (e.g. triangular signals). Recently, Daubechies et al. showed interesting results on what they called "Synchrosqueezing" to represent ideal multicomponent signals, namely signals which are the sum of frequency and amplitude modulated waves [8, 9]. Even if this method gives an accurate time-frequency representation, the associated technique for mode reconstruction appears to be seriously flawed when signals contaminated by noise or containing interferences are considered.

After briefly recalling the synchrosqueezing method, we discuss the issue of the identification of relevant wavelet coefficients for mode synthesis. We propose a new technique based on the projection onto some reproducing kernel space prior to reconstruction, whose good behaviour is illustrated by numerical experiments.

2. DEFINITIONS AND NOTATIONS

2.1. Wavelet Transform

In the following, we denote by $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ the space of real integrable and square-integrable functions respectively, and by $\mathcal{S}(\mathbb{R})$ the Schwartz class. χ_X stands for the characteristic function of the set X, $\bar{z}$ is the complex conjugate of z, and we denote by $f(a\,\cdot)$ the function $x \mapsto f(ax)$. Given a signal $s \in L^1(\mathbb{R})$, we define its Fourier transform as:

$$\hat{s}(\xi) = \int_{\mathbb{R}} s(t)\, e^{-2i\pi\xi t}\, dt. \quad (1)$$

Taking an admissible wavelet $\psi \in \mathcal{S}(\mathbb{R})$ (such that $C_\Psi = \int_0^\infty \frac{|\hat{\psi}(\xi)|^2}{\xi}\, d\xi < \infty$) and letting $\psi_{a,b}(t) = \frac{1}{a}\psi\left(\frac{t-b}{a}\right)$, we define the continuous wavelet transform of the signal s by:

$$\begin{aligned} W_s(a,b) &= \langle s, \psi_{a,b}\rangle \\ &= \frac{1}{a}\int_{\mathbb{R}} s(t)\,\overline{\psi\left(\frac{t-b}{a}\right)}\, dt. \end{aligned} \quad (2)$$

The wavelet transform maps $L^2(\mathbb{R})$ onto the space $WL^2 \subset L^2(\mathbb{R}_+ \times \mathbb{R})$. Note that this normalisation is not the usual one, but allows for a better analysis of multicomponent signals, as any monochromatic wave has the same wavelet transform magnitude, whatever its frequency centre. We are particularly interested in *analytic wavelets* ψ which live in the Hardy space $\mathcal{H}(\mathbb{R})$ (that is $\mathrm{Supp}(\hat{\psi}) \in [0,\infty[$), because of the following property.

Proposition 2.1. *If $\psi \in \mathcal{S}(\mathbb{R})$ is an analytic admissible wavelet, then the wavelet transform W_s of a real signal s is the half of the wavelet transform of its analytic signal $s_a = s + i\mathbf{H}(s)$, where $\mathbf{H}$ stands for the Hilbert transform (see [10] for example).*

The classical synthesis formula then writes:

$$s_a(t) = \frac{1}{C_\psi}\int_0^\infty \int_{\mathbb{R}} W_s(a,b)\psi\left(\frac{t-b}{a}\right) db\,\frac{da}{a^2}. \quad (3)$$

and we also recall the Morlet formula (obtained by taking a Dirac for synthesis, see [10] for instance):

$$s_a(t) = \frac{1}{C'_\psi} \int_0^\infty W_s(a,t)\, \frac{da}{a}, \tag{4}$$

where $C'_\psi = \int_0^\infty \overline{\hat{\psi}(\xi)} \frac{d\xi}{\xi}$. The real signal is easily obtained by $s = \frac{1}{2}\mathcal{R}e(s_a)$.

2.2. Mother wavelets

In what follows, we will use mother wavelets having a unique peak frequency. For such a wavelet ψ, let us denote by ξ_ψ its frequency centre:

$$\xi_\psi = \arg\max_\xi |\hat{\psi}(\xi)|. \tag{5}$$

For a compactly-supported analytic wavelet ψ, we will denote by Δ_ψ the minimum quantity such that $\hat{\psi}$ is compactly supported on $[\xi_\psi - \Delta_\psi, \xi_\psi + \Delta_\psi]$. A typical example is the *Bump* wavelet defined in the Fourier domain by $\hat{\psi}_{Bump}(\xi) = e^{1-\frac{1}{1-\left(\frac{\xi-\mu}{\sigma}\right)^2}} \chi_{[\mu-\sigma,\mu+\sigma]}$, which admits the peak frequency $\xi_\psi = \mu$ and the width $\Delta_\psi = \sigma$. Other analytic mother wavelets include the complex Shannon wavelet (which is badly time-localised and does not belong to $\mathcal{S}(\mathbb{R})$), the complex Meyer wavelet, the Bessel wavelet, or the complex Morlet wavelet (which is strictly speaking neither analytic nor admissible, but numerically suitable), we refer to [3, 11] for more details.

3. SYNCHROSQUEEZING IN A NUTSHELL

3.1. A reassignment method

Let us denote by *Intrinsic Mode Function (IMF)*, or simply *mode*, any signal of the form $h(t) = A(t)\cos(2\pi\phi(t))$, with $A(t) > 0$, $\phi'(t) > 0$ and where functions $|A'|$ and $|\phi''|$ are small compared to ϕ'. The wavelet transform of such an IMF is a ridge centred around the scale $a(b) = \xi_\psi/\phi'(b)$, whose width is proportional to the bandwidth of $\psi_a = \psi(\cdot/a)$. By linearity of the transform, for a multicomponent signal of the form $s(t) = \sum_k A_k(t)\cos(2\pi\phi_k(t))$, the following approximation holds ([3]):

$$W_s(a,b) = \frac{1}{2}\sum_k A_k(b) e^{2i\pi\phi_k(b)} \overline{\hat{\psi}(a\phi'_k(b))} + O(|A'_k|, |\phi'_k(k)|). \tag{6}$$

Provided the IMFs are sufficiently well separated (i.e. $|\phi'_k(t) - \phi'_l(t)|$ is large enough for each $k \neq l$ and t), each ridge can be identified and computed. The idea of the Synchrosqueezing, originally introduced in [2], is to automatically reassign the wavelet transform according to the candidate instantaneous frequency defined by:

$$\omega(a,b) = \frac{1}{2i\pi}\frac{\partial_b W_s(a,b)}{W_s(a,b)}, \tag{7}$$

by computing

$$T_s(\xi,b) = \int_{\{a \text{ s.t. } \omega(a,b)=\xi \text{ and } |W_s(a,b)|>\gamma\}} W(a,b)\frac{da}{a}. \tag{8}$$

The motivation for computing $\omega(a,b)$ is that it gives an interesting frequency information near the ridges: for a well-separated multicomponent signal, if (a,b) satisfies $1 - \Delta_\psi \leq a\phi'_k(b) \leq 1 + \Delta_\psi$, then $\omega(a,b) \approx \phi'_k(b)$ [8]. Imposing some regularity assumptions on the phase and amplitude of the studied multicomponent signals, Daubechies et al. showed in [8] that the Synchrosqueezing is able to detect and extract each mode with good accuracy.

3.2. Extraction of ridges and reconstruction

The modulus of the synchrosqueezed transform $|T_s|$ corresponds to a concentrated time-frequency analysis of the signal, and this representation can be sufficient for some applications. But in many cases, we need quantitative measures for each modes, namely the instantaneous frequencies $\phi'_k(t)$. To this end, one can introduce a variational approach to extract one "ridge" from T_s by finding the curve with the largest energy subject to some penalisation term. An appropriate energy of a curve $c(t)$ in the synchrosqueezing plane was found to be [9]:

$$E_s(c) = \int \log|T_s(c(t),t)|\, dt, \tag{9}$$

while to measure the regularity of c, authors in [9] chose the norm $\|c'\|^2$:

$$J(c) = \int |c'(t)|^2\, dt. \tag{10}$$

The ridges are then extracted successively by minimising

$$E(c) = J(c) - \lambda E_s(c), \tag{11}$$

the parameter λ monitoring the trade-off between the two terms. Note that the whole approach works conversely to the EMD: knowing the time-frequency representation, one extracts the instantaneous frequencies and finally computes the modes. On the contrary, the EMD computes directly the modes to get the so-called Hilbert-Huang spectrum. Note also that only an approximation of the solution is computed, using a greedy algorithm as in [9]. Improving the formulation and the algorithm for the ridge computation should be the topic of a future work.

Once the curve $c(t)$ is known, one needs to compute the associate mode h. For that, authors in [8] propose to sum up the Synchrosqueezed transform near that curve: at time t, defining the set $X_t = \{a \text{ s.t. } \omega(a,t) \approx c(t)\}$, they propose the following:

$$h(t) = \int_{\{\xi\approx c(t)\}} T_s(\xi,t)\, d\xi = \int_{X_t} W_s(a,t)\,\frac{da}{a}. \tag{12}$$

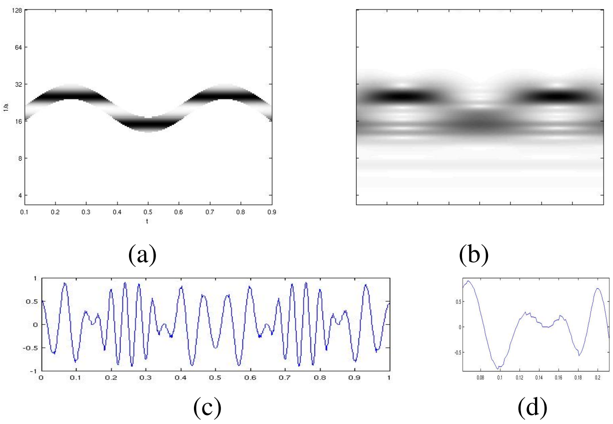

Fig. 1. (a) : The truncated wavelet transform W of the sum of 2 cosines. (b) : The wavelet transform of the pseudo-mode obtained by applying equation (4) on W. (c) and (d): The pseudo-mode, and a zoom on the singularities.

In fact, this is the Morlet formula (4) where one only selects the wavelet coefficients in the set X_t. There are however serious flaws with this method:

- If some measures of the candidate frequency $\omega(a,t)$ are irrelevant, which is the case for noisy or interfering signals, the synthesis formula becomes unstable.
- Formally, denoting by X the set $\bigcup_t X_t$, one computes the truncated wavelet transform $W = W_s \chi_X$, and then inverses it using the Morlet formula. But W is not the wavelet transform of a signal (i.e. it does not belong to the space WL^2).
- Even if the set X is "narrow", there is no reason for the reconstructed mode to be narrowband. Figure 1 illustrates this drawback: we select a fine smooth band of the wavelet transform of a simple 2-wave signal which we truncate as shown in Figure 1 (a), then we reconstruct a pseudo-mode using equation (4), and finally display the corresponding wavelet transform. Figures 1 (b) and (c) show clearly that the result is not narrowband, while Figure 1 (d) shows small singularities which appeared during the reconstruction process.

To summarise, the original extraction and reconstruction method can work for an ideal multicomponent signal, but as soon as the modes are not strictly speaking IMFs, or if the signal is contaminated by noise, or if the sets X_t have not been perfectly identified, the reconstruction becomes unstable.

4. THE RECONSTRUCTION ISSUE

4.1. Selecting a ridge near the frequency peak

Here we still suppose that the curve $c(t)$ is known. Instead of using ω again to reconstruct the mode, one can select the coefficients of the wavelet transform in a band in the vicinity of the ridge, with the appropriate width. This amounts to defining the set of coefficients

$$Y_t = \left\{ a \text{ s.t. } \frac{\xi_\psi - \Delta_\psi}{c(t)} \leq a \leq \frac{\xi_\psi + \Delta_\psi}{c(t)} \right\}, \qquad (13)$$

and then to reconstructing the mode by applying equation (4) on $W_s \chi_Y$, with $Y = \bigcup_t Y_t$. One verifies easily that the width of this strip is about the bandwidth of the function $\psi_a = \psi(a\,\cdot)$. Figure 2 shows the interest of using this set of coefficients Y instead of the set X and formula (12). Figures 2 (a) and (b) display the wavelet transform of a multicomponent strongly modulated signal and its synchrosqueezed transform. The coefficients sets X and Y are then displayed on Figures 2 (c) and (d) for each mode, which suggests that the three modes have been correctly identified. One notices that Y looks smoother than X, and that it has a constant width. Finally, the second modes synthesised from $W_s \chi_X$ and $W_s \chi_Y$ are displayed on Figures 2 (e) and (f) respectively, showing that the set Y leads to much better results. Let us add that for a noisy signal, reconstructing the modes with equation (12) would be worse, whereas our reconstruction method remains stable, provided the curve c has been correctly computed. To show this, we display on Figure 3 the signal-to-noise ratio (SNR) after denoising for different noise levels, for the same second mode.

4.2. Projection onto WL^2

The image WL^2 of $L^2(\mathbb{R})$ by the continuous wavelet transform is a Reproducing Kernel Hilbert Space (RKHS), i.e. each element $W_s \in WL^2$ satisfies [10]:

$$W_s(a,b) = \frac{1}{C_\psi} \int_0^\infty \int_{\mathbb{R}} K_{a,b}(a',b') W_s(a',b')\, db' \, \frac{da'}{a'}, \qquad (14)$$

where the function $K_{a,b}$ is called the reproducing kernel and is defined by $K_{a,b}(a',b') = \langle \psi_{a,b}, \psi_{a',b'} \rangle$. Using the Plancherel theorem one can rewrite this kernel in the fre-

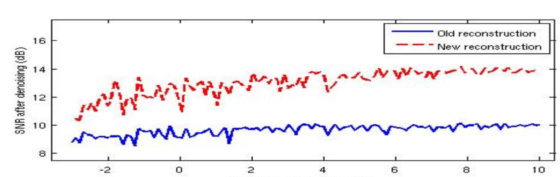

Fig. 3. Extraction of the low-frequency mode of the multicomponent test-signal. The SNR after denoising is displayed for different SNRs. We compare the methods based on the set X (solid line) and the one based on the set Y (dashed line).

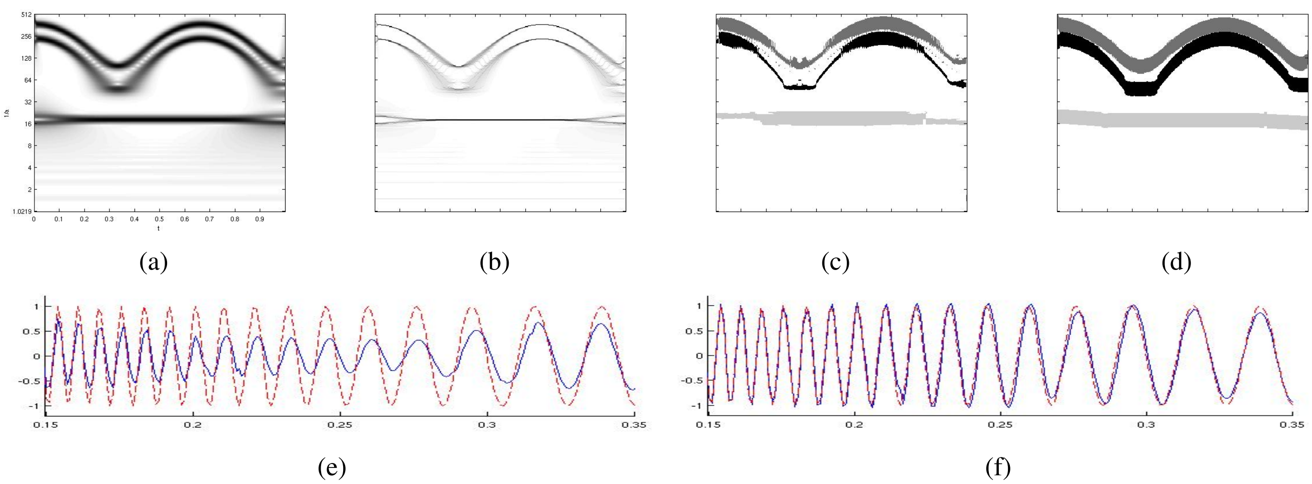

Fig. 2. (a) : Wavelet transform of a multicomponent signal. (b) : Synchrosqueezed transform. (c) and (d) : Sets of coefficients X and Y respectively. Each grey level corresponds to the coefficients selected for reconstructing one mode. The coefficients of the second IMF are in black. (e) and (f) : Corresponding reconstructed second mode (solid line) and the desired one (dashed line).

quency domain, which is often more convenient:

$$\begin{aligned} K_{ab}(a', b') &= \left\langle \hat{\psi}_{a,b}, \hat{\psi}_{a',b'} \right\rangle \\ &= \int_0^\infty \hat{\psi}(a\xi)\overline{\hat{\psi}(a'\xi)} e^{2i\pi\xi(b-b')}\, d\xi. \end{aligned} \tag{15}$$

Now, given any $W \in L^2(\mathbb{R}_+ \times \mathbb{R})$, we can define a unique projection $PW \in WL^2$:

$$PW(a,b) = \frac{1}{C_\psi} \int_0^\infty \int_{\mathbb{R}} K_{a,b}(a', b') W(a', b')\, db' \, \frac{da'}{a'}. \tag{16}$$

Our method for mode reconstruction is then straightforward: once the coefficient set Y has been identified, one computes $W = W_s\chi_Y$, and then its projection PW. One then uses the synthesis formula (4) on PW to get the mode. Figure 4 illustrates the benefit of using such a projection on the test signal of Figure 2 contaminated by white Gaussian noise such that the SNR equals $2dB$. We artificially construct a modified curve c for the second mode, which simulates what happens in practice when the signal contains noise (see Figure 4 (b)). Then, Figures 4 (c) and (d) show the truncated wavelet transform $W = W_s\chi_Y$ for the second mode, and its projection PW. The corresponding reconstructed modes are displayed on Figures 4 (e) and (f) respectively, together with the expected mode. One can see how the projection regularises the solution by suppressing the discontinuities. This projection has a major role when reconstructing a mode, and suggests a lot of working perspectives. For instance, the IMFs of any signal s could be characterised in the time-scale plane by the coefficient set Y, so that the RKHS projection $P(W_s\chi_Y)$ is sufficiently narrow and regular.

4.3. Implementation

The implementation of the continuous wavelet transform and the synchrosqueezing is made as in [9]: the scale parameter a is discretized into n_v values per octave. In all our tests, we took $n_v = 32$, and we used the Morlet complex wavelet defined by:

$$\hat{\psi}(\xi) = \frac{1}{\sqrt{\sigma}} e^{-\pi\frac{(\xi-\xi_\psi)^2}{\sigma}}, \tag{17}$$

with $\sigma = 0.1$ and $\xi_\psi = 1$. As $\hat{\psi}$ has not a compact support, we set $\Delta_\psi = 2\sigma$ so that $\hat{\psi}$ is negligible outside $[\xi_\psi - \Delta_\psi, \xi_\psi + \Delta_\psi]$. The synchrosqueezing threshold γ (see [8]) is set to 0.01, while we choose $\lambda = 100$ for the ridge extraction (see section 3.2). Note that concerning the computation of the curve c, the minimisation of the energy E given by equation (11) is carried out by means of the following greedy algorithm: for a time index k, one defines $c(k) = \arg\max_a |W_s(a,k)|$ and minimises the energy at time indices $k+1, k+2, \cdots$ assuming it has been minimised for all previous time indices (forward computation). One then uses a backward step to compute $k-1, k-2, \cdots$ in the same way, and one finally keeps the best result among different initialisations. The Matlab code of all algorithms and the scripts used to create the figures of this paper can be downloaded from [12].

5. CONCLUSION

This paper discussed the issue of the IMF reconstruction from the Synchrosqueezing transform, or more generally from a "strip" of the wavelet transform. We proposed an alternative

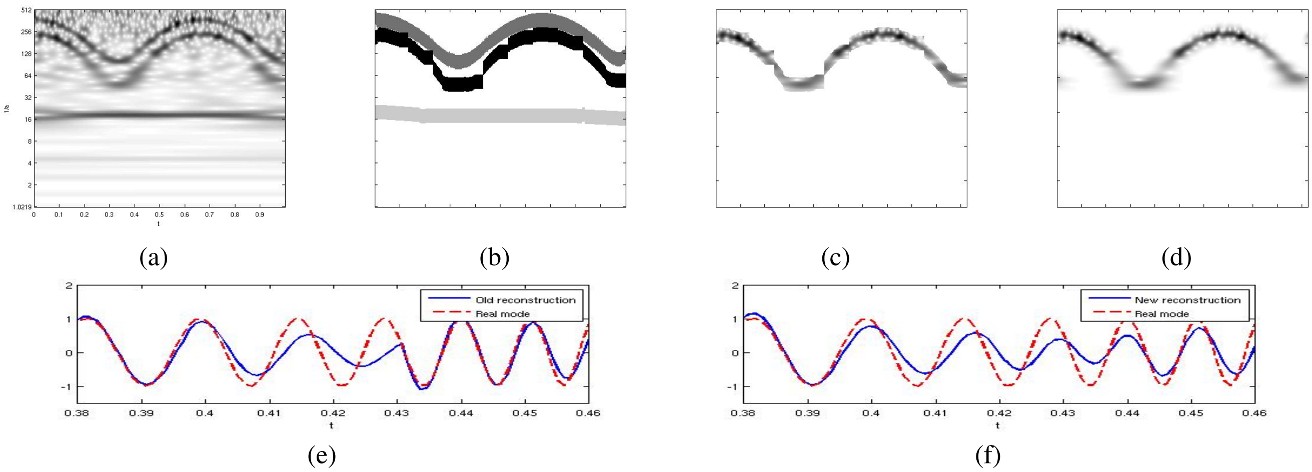

Fig. 4. (a) : Wavelet transform of a noisy multicomponent signal. (b) : Mask of the wavelet transform computed from the set Y of equation (13). The black area represents coefficients of mode 2. (c) and (d) : Selected coefficients for the reconstruction of the second mode, without and with projection onto WL^2, respectively. (e) and (f) : Corresponding reconstructed second mode (solid line) and the desired one (dashed line), zoom around a singularity.

method for modes reconstruction which appears to be more stable than the original formulation. We also introduced a projection method for modes reconstruction from truncated wavelet transforms, which is mathematically grounded and exhibits a better behaviour than the original method. The variational formulation for the ridge extraction as well as its implementation remain a key point, which should be further investigated.

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