### Title

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Enumerating embeddings of homologically $(k-1)$-connected $n$-manifolds in Euclidean $(2n-k)$-space

Dedicated to Professor Nobuo Shimada on his 60th birthday

By Tsutomu YASUI

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§ 1. Introduction.

Throughout this paper, an $n$-manifold and an embedding mean a closed connected differentiable manifold of dimension $n$ and a differentiable embedding, respectively. Let $[M \subset \mathbb{R}^m]$ denote the set of isotopy classes of embeddings of $M$ in Euclidean $m$-space $\mathbb{R}^m$. In [5] (cf. [6]), Haefliger has proved the following theorem:

**Theorem (Haefliger).** If $k \leq (n-4)/2$ and if $M$ is an orientable homologically $k$-connected $n$-manifold, then $[M \subset \mathbb{R}^{2n-k}]$ is equivalent to $H_{k+1}(M; \mathbb{Z})$ or $H_{k+1}(M; \mathbb{Z}_2)$ according as $n-k$ is odd or even.

Here a space $X$ is called homologically $k$-connected if it satisfies the condition $\tilde{H}_i(M; \mathbb{Z})=0$ for $i \leq k$. A $k$-connected path connected space is clearly homologically $k$-connected.

The purpose of this paper is to prove the following theorem, which is an extension of the above theorem:

**Main Theorem.** If $2 \leq k \leq (n-4)/2$ and if $M$ is a homologically $(k-1)$-connected $n$-manifold whose $(n-k)$-th normal Stiefel-Whitney class vanishes, then the set $[M \subset \mathbb{R}^{2n-k}]$ is given as follows:

(i) if $k=2$ and $M$ is not a spin manifold, then

$$[M \subset \mathbb{R}^{2n-2}] = H^{n-3}(M; \mathbb{Z}_2)$$

for $n \equiv 0(4)$,

$$= H^{n-3}(M; \mathbb{Z}_2) \times \mathbb{Z}_2$$

for $n \equiv 2(4)$,

$$= H^{n-3}(M; \mathbb{Z}) \times H^{n-2}(M; \mathbb{Z}_2)$$

for $n \equiv 1(4), w_3 \neq 0$,

$$= H^{n-3}(M; \mathbb{Z}) \times H^{n-2}(M; \mathbb{Z}_2) \times \mathbb{Z}_2$$

for $n \equiv 1(4), w_3 = 0$, or $n \equiv 3(4)$;
(ii) if \( k \geq 3 \) or \( M \) is a spin manifold, then

\[
[M \subset R^{2n-k}] = H^{n-k-1}(M; \mathbb{Z}_2) \quad n-k \equiv 0 \ (4),
\]

\[
\times H^{n-k}(M; \mathbb{Z}) \cup H^{n-k}(M; \mathbb{Z}_2) / Sq^2 \rho_2 H^{n-k-2}(M; \mathbb{Z}) \quad n-k \equiv 1 \ (4),
\]

\[
= H^{n-k-1}(M; \mathbb{Z}_2) \times H^{n-k}(M; \mathbb{Z}_2) / Sq^2 H^{n-k-2}(M; \mathbb{Z}_2) \quad n-k \equiv 2 \ (4),
\]

\[
= H^{n-k-1}(M; \mathbb{Z}) \times H^{n-k}(M; \mathbb{Z}_2) \cup H^{n-k}(M; \mathbb{Z}_2) / (Sq^1 H^{n-k-1}(M; \mathbb{Z}_2) + Sq^2 \rho_2 H^{n-k-2}(M; \mathbb{Z})) \quad n-k \equiv 3 \ (4).
\]

In this theorem, the \((n-k)\)-th normal Stiefel-Whitney class \( \overline{w}_{n-k} \) of an orientable \( n \)-manifold \( M \) is defined by \( \overline{w}_{n-k} \) or \( \beta_2 \overline{w}_{n-k-1} \in H^{n-k}(M; \mathbb{Z}) \) according as \( n-k \) is even or odd, where \( \overline{w}_i \) is the \( i \)-th mod 2 normal Stiefel-Whitney class of \( M \) and \( \beta_2 \) is the Bockstein operator, and moreover \( \overline{W}_{n-k} \) is the unique obstruction to embedding a homologically \((k-1)\)-connected \( n \)-manifold in \( R^{2n-k} \) by the theorem in [5, §1.3] (cf. [6, Theorem (2.3)]).

The remainder of this paper is organized as follows: In §2, we shall state a method of computing \([M \subset R^{2n-k}]\) of a homologically \((k-1)\)-connected \( n \)-manifold \( M \) (Theorem 2.5). In §3, we state the cohomology group of the reduced symmetric product \( M^* = (M \times M - \Delta M)/\mathbb{Z}_2 \) of \( M \) (Theorem 3.3), postponing the proof till §5, the last section. §4 is devoted to proving the main theorem.

### §2. The method of computing \([M \subset R^{2n-k}]\).

We begin this section by explaining notations. Let \( X \) be the product \( X \times X \) of a space \( X \) and let \( \Delta X \) be the diagonal in \( X^2 \). The cyclic group of order 2, \( \mathbb{Z}_2 \), acts on \( X^2 \) via the map \( t: X^2 \to X^2 \) defined by \( t(x, y) = (y, x) \). Then \( \Delta X \) is the fixed point set of this action. The quotient space

\[
X^* = (X^2 - \Delta X)/\mathbb{Z}_2
\]

is called the reduced symmetric product of \( X \). Here the projection \( p: X^* - \Delta X \to X^* \) is a double covering, whose classifying map we denote by

\[
\xi: X^* \to P^m.
\]

For a fibration \( \pi: E \to B \) and a map \( f: Y \to B \), let

\[
Y \times_B E \to Y \quad \text{and} \quad [Y, E; f]
\]

be the pull-back of \( \pi \) along \( f \) and the homotopy set of liftings of \( f \) to \( E \).

Notice that the sphere bundle \( \pi: S^m \times_{\mathbb{Z}_2} S^m \to P^m \) is homotopically equivalent to the natural inclusion \( P^m \to P^m \) of the real projective \( m \)-space \( P^m \). Hence we regard them as identical. Using the above notations, we deduce the following
theorem from Haefliger's theorem [4, Théorème 1'] (cf. Yasui [18, § 1]):

**THEOREM 2.1 (Haefliger).** For an n-manifold $M$, there is a bijection

$$[M \subset \mathbb{R}^{2n-k}] \cong [M^*, P^{2n-k-1}; \xi] \quad \text{if} \quad k \leq (n-4)/2.$$  

For any abelian group $G$ and a homomorphism $\phi : \pi_1(P^\infty) = \mathbb{Z}_2 \rightarrow \text{Aut}(G)$, let $G_\phi$ be the sheaf over $P^\infty$, locally isomorphic to $G$, defined by $\phi$, i.e., the local system associated with $\phi$. This homomorphism $\phi$ gives an action of $\mathbb{Z}_2$ on $(K(G, m), *)$. Hence we have a fibration

$$q : \Phi(G, m) = S^\infty \times_{Z_2} K(G, m) \rightarrow P^\infty$$

with fiber $K(G, m)$ and a canonical cross section $s$. It has been established (see, for example, G. W. Whitehead [17, Chap. VI, (6.13)]) that there exists a unique fundamental class $\xi \in H^n(\Phi(G, m), P^\infty; q^*G_\phi)$ whose restriction to $K(G, m)$ is the ordinary one ($\xi$ is equal to $\delta(sq, 1)$ up to sign in [17]), and that given $\xi : X \rightarrow P^\infty$, the correspondence $f \mapsto f^* \xi$ leads to a bijection

$$[X, L_\phi(G, m); \xi] \cong H^n(X; \xi G_\phi).$$

Further, if $\xi$ has a lifting $\tilde{\xi}$ to $P^{2n-k-1}$, then there is a bijection

$$[X, P^{2n-k-1} \times_{P}\Phi(G, m); \tilde{\xi}] \cong [X, L_\phi(G, m); \tilde{\xi}]$$

by [8, Theorem 3.1] and hence we have a bijection

$$[X, P^{2n-k-1} \times_{P}\Phi(G, m); \tilde{\xi}] \cong H^n(X; \tilde{\xi} G_\phi).$$

Let

$$G_j = \pi_{2n-k-1+j}(S^{2n-k-1}).$$

Since the sphere bundle $\pi : P^{2n-k-1} \rightarrow P^\infty$ is the one associated with $(2n-k)\gamma$, $\gamma$ being the universal real line bundle over $P^\infty$, the action of $\pi_1(P^\infty) = \mathbb{Z}_2$ on $G_j$ is given by the homomorphism

$$\phi : Z_2(= \{1, a\}) \rightarrow \text{Aut}(G_j)$$

defined by

$$\phi(a)(x) = (-1)^{2n-k}x \quad \text{for} \quad x \in G_j$$

and moreover the sheaf $(G_j)_\phi$ is given by

$$(G_j)_\phi = \begin{cases} G_j & \text{if} \ k \text{ is even}, \\ G_j[u] & \text{if} \ k \text{ is odd}, \end{cases}$$

where $G_j[u]$ is the sheaf over $P^\infty$, locally isomorphic to $G_j$, twisted by $u(\neq 0) \in H^1(P^\infty; Z_2) = Z_2$. For $\xi : M^* \rightarrow P^\infty$, let

$$G_j = \xi^*(G_j)_\phi = \begin{cases} G_j & \text{if} \ k \text{ is even}, \\ G_j[v] (v = \xi * u) & \text{if} \ k \text{ is odd}, \end{cases}$$

where $G_j[v]$ is the sheaf over $P^\infty$, locally isomorphic to $G_j$, twisted by $v(\neq 0) \in H^1(P^\infty; Z_2) = Z_2$. For $\xi : M^* \rightarrow P^\infty$, let

$$G_j = \xi^*(G_j)_\phi = \begin{cases} G_j & \text{if} \ k \text{ is even}, \\ G_j[v] (v = \xi * u) & \text{if} \ k \text{ is odd}, \end{cases}$$

where $G_j[v]$ is the sheaf over $P^\infty$, locally isomorphic to $G_j$, twisted by $v(\neq 0) \in H^1(P^\infty; Z_2) = Z_2$. For $\xi : M^* \rightarrow P^\infty$, let

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where $G_j[v]$ is the sheaf over $P^\infty$, locally isomorphic to $G_j$, twisted by $v(\neq 0) \in H^1(P^\infty; Z_2) = Z_2$. For $\xi : M^* \rightarrow P^\infty$, let

$$G_j = \xi^*(G_j)_\phi = \begin{cases} G_j & \text{if} \ k \text{ is even}, \\ G_j[v] (v = \xi * u) & \text{if} \ k \text{ is odd}, \end{cases}$$

where $G_j[v]$ is the sheaf over $P^\infty$, locally isomorphic to $G_j
and let
\[ \overline{\rho}_2: H^i(M^*; \mathbb{Z}) \rightarrow H^i(M^*; \mathbb{Z}_2), \]
\[ \overline{\beta}_2: H^{i-1}(M^*; \mathbb{Z}_2) \rightarrow H^i(M^*; \mathbb{Z}) \]
be the ordinary reduction mod 2 and Bockstein operator or the ones twisted by \( v \) according as \( k \) is even or odd. Then

\[ (2.4) \quad \overline{\rho}_2 \overline{\beta}_2 = \begin{cases} Sq^1 & \text{if } k \text{ is even}, \\ Sq^1 + v & \text{if } k \text{ is odd}, \end{cases} \]

by [2], [14]. With the above notations, we shall prove

**Theorem 2.5.** Let \( 2 \leq k \leq (n-4)/2 \) and let \( M \) be a homologically \((k-1)\)-connected \( n \)-manifold. If \( M \) can be embedded in \( \mathbb{R}^{2n-k} \), then there exists a bijection

\[ [M \subset \mathbb{R}^{2n-k}] = H^{2n-k-1}(M^*; \mathbb{Z}) \times \text{Coker} \Theta \]

where
\[ \Theta = (Sq^1 + \binom{2n-k}{2} v^2) \overline{\rho}_2 : H^{2n-k-2}(M^*; \mathbb{Z}) \rightarrow H^{2n-k}(M^*; \mathbb{Z}_2). \]

In order to prove this, it is sufficient, by Theorem 2.1, to show that

\[ [M^*, P^{2n-k-1}; \xi] = H^{2n-k-1}(M^*; \mathbb{Z}) \times \text{Coker} \Theta. \]

Let \( P = P^{2n-k-1} \) and let \( \pi': P' \rightarrow P \) be the pull-back of \( \pi: P \rightarrow P^\infty \) along \( \pi \). If \( M \) can be embedded in \( \mathbb{R}^{2n-k} \), then \( \xi \) has a lifting \( \xi': M^* \rightarrow P \) by the first half of [4, Théorème 1'] and so

\[ (2.6) \quad [M^*, P'; \xi'] \cong [M^*, P; \xi] \]

by [8, Theorem 3.1]. Since \( \pi: P \rightarrow P^\infty \) is the sphere bundle associated with \( (2n-k)\gamma \), the Postnikov tower of \( \pi': P' \rightarrow P \) is given as follows:

\[ \cdots \]
\[ E_{j+1} \]
\[ \downarrow_{h_j} \]
\[ P' \]
\[ \overrightarrow{\downarrow} \]
\[ E_j \]
\[ \overset{h_j}{\rightarrow} \]
\[ E_j \]
\[ \overset{k_j}{\rightarrow} \]
\[ P \times_{P^\infty} L_{\phi}(G_j, 2n-k+j) \]
\[ \downarrow \]
\[ \cdots \]
\[ E_{g} \]
\[ \downarrow_{k_g} \]
\[ P \times K(Z_2, 2n-k+2) \]
\[ \downarrow \]
\[ E_{1} \]
\[ \overset{k_1}{\rightarrow} \]
\[ P \times K(Z_2, 2n-k+1) \]
\[ \downarrow \]
\[ P^{2n-k-1} = P \]
where $h_j$ is a $(2n-k-1+j)$-equivalence, $p_j : E_{j+1} \rightarrow E_j$ is a $P$-principal fibration with classifying map $k_j$ in the category $TP$ of $P$-sectioned spaces and maps. By [10, Part IV, Theorem 1], for $\xi' : M^* \rightarrow P$, $p_j : E_{j+1} \rightarrow E_j$ induces an exact sequence

$$(\Omega P k_j)_* : \ldots \rightarrow [M^*, P \times \prod L_j^c(G_j, 2n-k-1+j); \xi'] \rightarrow [M^*, E_{j+1}; \xi'] \rightarrow \cdots$$

with the help of (2.2), (2.3), this is converted into the exact sequence

$$(\Omega P k_j)_* : \ldots \rightarrow H^{2n-k-1+j}(M^*; \xi') \rightarrow [M^*, E_{j+1}; \xi'] \rightarrow \cdots$$

where $(\Omega P k_j)_*$ is the map of loops associated with $k_j$ in $TP$. With the help of (2.2), (2.3), this is converted into the exact sequence

$$(\Omega P k_1)_* : \ldots \rightarrow \Theta = H^{2n-k-1}(M^*; Z) \rightarrow [M^*, E_1; \xi'] \rightarrow \cdots$$

where $\Theta = (Sq^2 + w_2((2n-k)\gamma))\overline{\rho}_2$. From the above argument, it is clear that there exists a short exact sequence

$$0 \rightarrow H^{2n-k}(M^*; Z) \rightarrow [M^*, P'; \xi'] \rightarrow H^{2n-k-1}(M^*; Z) \rightarrow 0.$$
$H^n(M; Z_2) = Z_2 \langle M \rangle$,

and let

$\sigma = 1 + t^* : H^*(M^2; Z_2) \rightarrow H^*(M^2; Z_2)$.

**Lemma 3.1.** Assume that $M$ is a homologically $(k-1)$-connected $n$-manifold $(k \geq 2)$. Then

(i) $H^i(M^*; Z_2) = 0$ if $i > 2n - k$,

(ii) $H^{2n-k}(M^*; Z_2) = \{ \rho \sigma(M \otimes x) | x \in H^{n-k}(M; Z_2) \}$

(iii) $H^{2n-k-1}(M^*; Z_2) = \{ \rho (u^{k-1} \otimes x^2) | x \in H^{n-k}(M; Z_2) \}$

(iv) $H^{2n-k-2}(M^*; Z_2) = \{ \rho (u^{k} \otimes x^2) | x \in H^{n-k-1}(M; Z_2) \}$

where the term in the square brackets is present only when $k=2$.

**Proof.** (i), (ii) are given by Thomas [16, Proposition 2.9]. By [19, Proposition 2.6], there are two relations:

$\rho(u^{k+1} \otimes x^2) = \rho(U(1 \otimes x) + u^{k-1} \otimes (S^1 x)^2)$

if $x \in H^{n-k-1}(M; Z_2)$,

$\rho(u^{k+2} \otimes x^2) = \rho(U(1 \otimes x) + u^{k} \otimes (S^1 x)^2 + u^{k-2} \otimes ((S^2 + w_2) x)^2)$

if $x \in H^{n-k-2}(M; Z_2)$.

Moreover $U(1 \otimes x)$ is expressed in the form

$U(1 \otimes x) = \sigma(M \otimes x) + \sum x' \otimes x''$, \quad dim $x'$, dim $x'' < n$.

Applying [16, Proposition 2.9], we can prove (iii), (iv) immediately.

The actions of $\nu \in H^1(M^*; Z_2)$ and the square operation $S^q$ $(i=1, 2)$ on $H^*(M^*; Z_2)$ are given by Thomas [16, Corollary 2.10] and Bausum [1, Lemmas 11 and 24] as follows:

**Lemma 3.2.** There are the following relations in $H^*(M^*; Z_2)$:

(i) $\nu \rho \sigma(x \otimes y) = 0$, $\nu \rho(u^i \otimes x^2) = \rho(u^{i+1} \otimes x^2)$;

(ii) if $x \in H^i(M; Z_2)$, then

$S^q \rho(u^i \otimes x^2) = \begin{cases} (i+r) \rho(u^{i+1} \otimes x^2) & i > 0, \\ r \rho(u^i \otimes x^2) + \rho \sigma(S^1 x \otimes x) & i = 0 \end{cases}$.
For a homologically \((k-1)\)-connected \(n\)-manifold \(M (k \geq 2)\), the cohomology groups \(H^i(M^*; \mathbb{Z})\) for \(2n-k-2 \leq i \leq 2n-k\) are given in the following theorem, postponing the proof till §5:

**Theorem 3.3.** Assume that \(M\) is a homologically \((k-1)\)-connected \(n\)-manifold \((k \geq 2)\). Then

(i) \(H^{2n-k}(M^*; \mathbb{Z}) \cong \{H^{n-k}(M, \mathbb{Z}_2) \cup H^n(M, \mathbb{Z})\} \) if \(n-k\) is even,

(ii) \(H^{2n-k-1}(M^*; \mathbb{Z}) \cong \{H^{n-k-1}(M, \mathbb{Z}_2) \cup H^{n-k}(M, \mathbb{Z})\} \) if \(n-k\) is odd;

(iii) \(\overline{\rho}_{2}H^{2n-k-2}(M^*; \mathbb{Z}) = \{p(u^{k-2} \otimes x^2) \mid x \in H^{n-k}(M; \mathbb{Z}_2)\} \cup \{\rho(u^{k+2} \otimes x^2) \mid x \in H^{n-k-2}(M; \mathbb{Z}_2)\} \) if \(n-k\) is even,

\[\cup \{\rho\sigma(x \otimes y) \mid x, y \in H^{n-2}(M; \mathbb{Z}_2), x \neq y\}\] if \(n-k\) is odd,

where the terms in the square brackets \([\ ]\) are present only when \(k=2\).

§ 4. Proof of the main theorem.

In this section, let \(M\) be a homologically \((k-1)\)-connected \(n\)-manifold \((k \geq 2)\). If its \((n-k)\)-th normal Stiefel-Whitney class vanishes, then \(M\) can be embedded in Euclidean \((2n-k)\)-space by Haefliger [5, §1] and there is a bijection

\([M \subset \mathbb{R}^{2n-k}] = H^{2n-k-1}(M^*; \mathbb{Z}) \times \text{Coker } \Theta\]

where

\[\Theta = (Sq^n + \binom{2n-k}{2}v^2)\overline{\rho}_2 : H^{2n-k-1}(M^*; \mathbb{Z}) \rightarrow H^{2n-k}(M^*; \mathbb{Z})\]

by Theorem 2.5. Since \(H^{2n-k-1}(M^*; \mathbb{Z})\) is given in Theorem 3.3 (ii), we shall concentrate on calculating \(\text{Coker } \Theta\). Notice that there are an isomorphism

\[(4.1) \quad \chi : H^{n-k}(M; \mathbb{Z}_2) \rightarrow H^{n-k}(M^*; \mathbb{Z}_2) \quad (\chi(x) = \rho\sigma(M \otimes x)),\]

and equalities

\[(4.2) \quad \rho(u^{k} \otimes x^2) = \rho(1 \otimes x) = \rho\sigma(M \otimes x) \quad \text{for } x \in H^{n-k}(M; \mathbb{Z}_2),\]
which follow from [19, Proposition 2.6] and (*)& in §3.

Case I: \(n-k\) is even. See Theorem 3.3 (iii) for the group \(\overline{\rho}_{2}H^{n-k-2}(M^{*}; Z)\).

If \(x \in H^{n-k-2}(M; Z_{2})\), then

\[
(Sq^{2}+\binom{2n-k}{2}v^{2})\rho(u^{k+2}\otimes x^{2})
\]

\[
=\left(\binom{n}{2}+\binom{2n-k}{2}\right)\rho(u^{k+4}\otimes x^{2})+\rho(u^{k+2}\otimes (Sq^{1}x)^{2})
\]

by Lemma 3.2,

\[
=\left(\binom{n}{2}+\binom{2n-k}{2}\right)\rho(u^{k}\otimes ((Sq^{2}+w_{2})x)^{2})
\]

because there are two relations

\[
\rho(u^{k+4}\otimes x^{2}+u^{k+2}\otimes (Sq^{1}x)^{2}+u^{k}\otimes ((Sq^{2}+w_{2})x)^{2})=0,
\]

\[
\rho(u^{k+2}\otimes (Sq^{1}x)^{2})=0,
\]

which are easily proved by using [19, (2.5) and Proposition 2.6]. Therefore, by (4.2), we have

\[
\rho\left((\binom{n}{2}+\binom{2n-k}{2})\rho(u^{k}\otimes ((Sq^{2}+w_{2})x)^{2})\right)
\]

\[
\lambda\rho\sigma(M\otimes (Sq^{2}+w_{2})x)
\]

for \(x \in H^{n-k-2}(M; Z_{2})\),

where

\[
\lambda=\left\{\begin{array}{ll}
0 & \text{for } n-k\equiv 0(4), \\
1 & \text{for } n-k\equiv 2(4).
\end{array}\right.
\]

Similarly, we have a relation

\[
(Sq^{2}+\binom{2n-k}{2}v^{2})\rho(u^{k-2}\otimes x^{2})
\]

\[
=(1-\lambda)\rho\sigma(M\otimes x)+[\rho\sigma(w_{2}x\otimes x)]
\]

for \(x \in H^{n-k}(M; Z_{2})\).

Moreover the relation

\[
(Sq^{2}+\binom{2n-k}{2}v^{2})\rho\sigma(x\otimes y)=\rho\sigma(w_{2}x\otimes y+w_{2}y\otimes x)
\]

for \(x, y \in H^{n-k}(M; Z_{2})\) with \(x \neq y\)

follows from Lemma 3.2. Therefore, if \(k \geq 3\) or \(w_{2}=0\), then

\[
\ker\Theta=\binom{(1-\lambda)\rho\sigma(M\otimes x)}{\rho\sigma(w_{2}x\otimes y+w_{2}y\otimes x)}
\]

for \(x, y \in H^{n-k}(M; Z_{2})\) with \(x \neq y\)

and so

\[
\ker\Theta=\left\{\begin{array}{ll}
0 & \text{for } n-k\equiv 0(4), \\
\rho\sigma(w_{2}x\otimes y+w_{2}y\otimes x) & \text{for } n-k\equiv 2(4),
\end{array}\right.
\]
by (4.1), (4.2). Next, consider the case $k=2$ and $w_{2}\neq 0$. In general, for a simply connected $n$-manifold $M$ with non-trivial second Stiefel-Whitney class $w_{2}$, the group $H^{n-k}(M; Z_{2})$ can be expressed, by using Poincaré duality, in the form

\begin{equation}
H^{n-k}(M; Z_{2}) = \sum_{i=1}^{a} Z_{2}\langle z_{i} \rangle, \quad w_{2}z_{i} = \begin{cases} M & \text{if } i=1, \\ 0 & \text{if } 2 \leq i \leq a. \end{cases}
\end{equation}

Then a simple calculation yields that

\[ \text{Im} \Theta = \begin{cases} \sum_{i=1}^{a} Z_{2}\langle \rho \sigma(M \otimes z_{i}) \rangle & \text{if } n-2 \equiv 0 \pmod{4}, \\ H^{n-k}(M^{*}; Z_{2}) & \text{if } n-2 \equiv 2 \pmod{4}, \end{cases} \]

and hence

\begin{equation}
\text{Coker} \Theta = \begin{cases} Z_{2} & \text{if } k=2, w_{2}\neq 0 \text{ and } n \equiv 2 \pmod{4}, \\ 0 & \text{if } k=2, w_{2}\neq 0 \text{ and } n \equiv 0 \pmod{4}. \end{cases}
\end{equation}

by (4.1). Thus we deduce the main theorem in case $n-k$ is even, from (4.6), (4.8) and Theorems 2.5, 3.3 (ii).

Case II: $n-k$ is odd. See also Theorem 3.3 (iii) for the group $\bar{\rho}_{2}H^{n-k-2}(M^{*}; Z)$. In the same way as in the case when $n-k$ is even, we have the following relations:

\begin{align*}
\left( Sq^{2} + \binom{2n-k}{2} v^{2} \right) \rho(u^{k} \otimes x^{2}) &= \mu \rho \sigma(M \otimes Sq^{1}x), \quad \mu = \begin{cases} 0 & \text{for } n-k \equiv 1 \pmod{4}, \\ 1 & \text{for } n-k \equiv 3 \pmod{4}, \end{cases} \quad \text{if } x \in H^{n-k-1}(M; Z_{2}); \\
\left( Sq^{2} + \binom{2n-k}{2} v^{2} \right) \rho \sigma(M \otimes \rho_{2}x) &= \rho \sigma(M \otimes Sq^{2}\rho_{2}x) \quad \text{if } x \in H^{n-k-2}(M; Z_{2}); \\
\left( Sq^{2} + \binom{2n-k}{2} v^{2} \right) \rho \sigma(x \otimes y) &= \rho \sigma(w_{2}x \otimes y + w_{2}y \otimes x) \quad \text{if } x, y \in H^{n-2}(M; Z_{2}).
\end{align*}

If $w_{2}=0$, then (4.1) and the above relations (4.9) lead at once to the relation

\begin{equation}
\text{Im} \Theta \cong \begin{cases} S_{k}^{a} \rho_{2}H^{n-k-1}(M; Z) & \text{for } n-k \equiv 1 \pmod{4}, \\ S_{k}^{a} \rho_{2}H^{n-k-1}(M; Z) + Sq^{1}H^{n-k-1}(M; Z_{2}) & \text{for } n-k \equiv 3 \pmod{4}. \end{cases}
\end{equation}

If $w_{2}\neq 0$, it is easily verified, in the same way as in the case when $n-k$ is even, that the subgroup of $\text{Im} \Theta$ determined by the last relation of (4.9) is equal to $\sum_{i=1}^{a} Z_{2}\langle \rho \sigma(M \otimes z_{i}) \rangle$. On the other hand, the following relations hold:

\begin{align*}
w_{2}S_{k}^{a}\rho_{2}x = S_{k}^{a}S_{k}^{a}\rho_{2}x &= S_{k}^{a}S_{k}^{1}\rho_{2}x = 0 \quad \text{for } x \in H^{n-k-1}(M; Z), \\
w_{2}S_{k}^{1}x = S_{k}^{1}(w_{2}x) + (S_{k}^{1}w_{2})x &= w_{2}x \quad \text{for } x \in H^{n-3}(M; Z_{2}).
\end{align*}

Therefore, it is shown immediately that $\rho \sigma(M \otimes z_{1}) \in \text{Im} \Theta$ if and only if $n-2 \equiv 3 \pmod{4}$ and $w_{2}\neq 0$, and hence
(4.11) \[ \text{Coker} \Theta = \begin{cases} 0 & \text{if } n \equiv 1 \mod{4} \text{ and } w_3 \neq 0, \\ Z_2 & \text{if } n \equiv 1 \mod{4}, w_3 = 0, \text{ or if } n \equiv 3 \mod{4}. \end{cases} \]

Thus (4.10), (4.11), together with Theorems 2.5, 3.3(ii), deduce the main theorem in case \( n-k \) is odd.

§ 5. Proof of Theorem 3.3.

Throughout this section, we assume that \( M \) is a homologically \((k-1)\)-connected \( n \)-manifold \((k \geq 2)\) and we compute \( H^{2n-k-i}(M^*; \underline{Z}) \) for \( 0 \leq i \leq 2 \), where \( \underline{Z} = Z \) or \( Z[v] \) according as \( k \) is even or odd.

Case I: \( n-k \) is even. First we consider the odd torsion subgroup of \( H^{2n-k-i}(M^*; Z) \) for \( i = 0, 1 \). Considering the cohomology spectral sequence (cf. [11, Theorem 1.1]) for a fibration \( M^2-\Delta M \to S^\infty \times_{Z_2} (M^2-\Delta M) \to P^\infty \), which is homotopically equivalent to \( M^2-\Delta M \to M^* \to P^\infty \), we see that the odd torsion subgroup of \( H^{2n-k-i}(M^*; Z) \) is isomorphic, by \( p^* \), to that of

\[ \{ x \in H^{2n-k-i}(M^2-\Delta M; Z) | t^*x = (-1)^{n}x \} = H^{2n-k-i}(M^2-\Delta M; Z)^{(-1)^{n}t} \]

Since \( M \) is orientable, there is a short exact sequence

\[ 0 \to H^i(M; Z) \to H^{n+i}(M^2; Z) \to H^{n+i}(M^2-\Delta M; Z) \to 0, \]

where \( \phi_1(x) = U(1 \otimes x) \) for \( x \in H^i(M; Z) \), \( U \in H^n(M^2; Z) \) is called the Thom class or the diagonal cohomology class of \( M \), e.g. by [12], and \( i \) is the natural inclusion. Therefore, \( \tilde{i}^* \) induces an isomorphism

\[ (H^{2n-k-i}(M^*; Z)/\phi_1H^{n-k-i}(M; Z))^{(-1)^{n}t} \cong H^{2n-k-i}(M^2-\Delta M; Z)^{(-1)^{n}t}. \]

Here \( \phi_1H^{n-k-i}(M; Z) \cong H^{2n-k-i}(M^2; Z)^{(-1)^{n}t} \) by [15, p. 305]. On the other hand, it is easily verified that \( H^{2n-k-i}(M^2; Z)^{(-1)^{n}t} \) is isomorphic to \( H^{n-k-i}(M; Z) \) for \( i = 0, 1 \). Therefore, \( H^{2n-k-i}(M^2-\Delta M; Z)^{(-1)^{n}t} \) has no odd torsion subgroup and hence

\[ H^{2n-k-i}(M^*; Z) \] has no odd torsion for \( i = 0, 1 \).

In order to study \( H^{2n-k-i}(M^*; Z) \), consider the Bockstein exact sequence associated with \( 0 \to Z_{2} \to Z_{2} \to Z_{2} \to 0 \),

\[ \cdots \to H^{i-1}(M^*; Z_{2}) \xrightarrow{\overline{p}_{2}} H^{i}(M^*; Z_{2}) \xrightarrow{\times 2} H^{i}(M^*; Z) \to H^{i}(M^*; Z_{2}) \to \cdots \]

By using the relations in (2.4) and Lemma 3.2, we have the following relations:
\[ \overline{\rho}_{2}\overline{\beta}_{2}\rho(x \otimes y) = 0 \quad \text{if} \quad k = 2 \quad \text{and} \quad x, y \in H^{n-k}(M; Z), \]
\[ \overline{\rho}_{2}\overline{\beta}_{2}\rho(u^{i} \otimes x^{2}) = \rho(u^{i+1} \otimes x^{2}) \]
for \((i, \dim x) = (k-1, n-k), (k, n-k-1), (k-3, n-k), (k+1, n-k-2)\).

These relations, (4.2), (5.1) and the exact sequence (5.2), together with Lemma 3.1, lead to Theorem 3.3 in case \(n-k\) is even.

**Case II: \(n-k\) is odd.** The group \(Z_{2}\) acts on \(SM\), the tangent sphere bundle over \(M\), via the antipodal map on each fibre \(S^{n-1}\). Let
\[ PM = SM/Z_{2}, \quad (A^{2}M, \Delta M) = (M^{2}/Z_{2}, \Delta M/Z_{2}), \]
\[ i : M^{*} = A^{2}M - \Delta M \subset (A^{2}M, \Delta M), \]
and let
\[ j : PM \longrightarrow M^{*} \]
be the embedding such that \(j^{*}v\) is the first Stiefel-Whitney class of the double covering \(SM \rightarrow PM\). We write \(j^{*}v\) as \(v \in H^{1}(PM; Z_{2})\) if no confusion can arise. Then there exists a long exact sequence, cf. [19, Lemma 1.3],
\[ \delta j^{*}j^{*} \rightarrow H^{i-1}(PM; Z) \rightarrow H^{i}(A^{2}M, \Delta M; Z) \rightarrow H^{i}(M^{*}; Z) \rightarrow H^{i}(PM; Z) \rightarrow \cdots \]
The cohomology of \(PM\) has been given by Rigdon [13, § 9] as follows:

**Lemma 5.4 (Rigdon).** Assume that \(M\) is a homologically \((k-1)\)-connected \(n\)-manifold \((k \geq 2)\) and that \(n-k\) is odd. Then

(i) \[ H^{n-k}(PM; Z) = \begin{cases} 0 & \text{if } k \text{ is even}, \\ Z_{2} \langle \tilde{\beta}_{2}(v^{n-k-1}M) \rangle & \text{if } k \text{ is odd}; \end{cases} \]

(ii) \[ H^{n-k-1}(PM; Z) = \begin{cases} \{ \beta_{2}(v^{n-k}x + v^{n-k-2}Sq^{k}x) | x \in H^{n-k}(M; Z_{2}) \} + Z_{2} \langle \tilde{\beta}_{2}(v^{n-k-2}M) \rangle & \text{if } k \text{ is even}, \\ Z_{2} \langle \tilde{\beta}_{2}(v^{n-k-2}M) \rangle & \text{if } k \text{ is odd}; \end{cases} \]

(iii) \[ H^{n-k-2}(PM; Z) = \begin{cases} \{ \beta_{2}(v^{n-k}x + v^{n-k-3}Sq^{k+1}x) | x \in H^{n-k-2}(M; Z_{2}) \} + Z_{2} \langle \tilde{\beta}_{2}(v^{n-k-3}M) \rangle & \text{if } k \text{ is even}, \\ \{ \beta_{2}(v^{n-k}x) | x \in H^{n-k-2}(M; Z_{2}) \} + Z_{2} \langle \tilde{\beta}_{2}(v^{n-k-2}M) \rangle & \text{if } k \text{ is odd}. \end{cases} \]

In the above lemma, and also from now on, \(\tilde{\beta}_{2}\) denotes the Bockstein operator twisted by \(v\).

The cohomology of \((A^{2}M, \Delta M)\) has been investigated by Larmore [9].

**Lemma 5.5 (Larmore).** Assume that \(M\) is a homologically \((k-1)\)-connected \(n\)-manifold \((k \geq 2)\) and that \(n-k\) is odd. Then
(i) \[ H^{2n-k}(A^2 M, \Delta M; Z) \cong \begin{cases} H^{n-k}(M; Z) & \text{if } k \text{ is even}, \\ H^{n-k-1}(M; Z) & \text{if } k \text{ is odd}; \end{cases} \]

(ii) \[ H^{2n-k-1}(A^2 M, \Delta M; Z) \cong \begin{cases} H^{n-k-1}(M; Z) & \text{if } k \text{ is even}, \\ H^{n-k-2}(M; Z) + Z_2\langle \tilde{\beta}_2(v^{n-k-2}M) \rangle & \text{if } k \text{ is odd}; \end{cases} \]

(iii) \[ i^*\overline{\rho}_2 H^{2n-k-2}(A^2 M, \Delta M; Z) = \{ \rho \sigma(\rho_2 x \otimes M) | x \in H^{n-k-2}(M; Z) \} + \{ \rho \sigma(x \otimes y) | x, y \in H^{n-2}(M; Z_2), x \neq y \}, \]

where the term in the square brackets is present only when \( k=2 \).

**Proof.** The cohomology groups \( H^{2n-k-i}(A^2 M, \Delta M; Z) \) for \( i=0, 1, 2 \) are given directly by [9, Theorem 20]. Their \( i^*\overline{\rho}_2 \)-images are easily obtained by using the relations

\[(5.6) \quad \delta(v^i x) = v^{i+1} \Lambda x, \quad i^*(Ax A y) = \rho \sigma(x \otimes y) + \rho \sigma(xy \otimes 1),\]

in [18, Lemma 1.5], [19, Lemma 3.3] and the two congruences mod \( \text{Im} \delta \)

\[ \tilde{\rho}_2 \tilde{\beta}_2(\Lambda x) \equiv \Lambda(\rho_2 \beta_2 x) \quad \text{if } x \in H^*(M; Z_r), \]

\[ \tilde{\rho}_2 \beta_2(\Delta x, \rho_2 y) \equiv \tilde{\rho}_2 \Delta(\beta_2 x, y) \quad \text{if } x \in H^*(M; Z_r), y \in H^*(M; Z), \]

which are easily proved.

**Remark.** The author has proved this lemma in the same way as he proved the propositions in [18, §5], i.e., by using the results on pp. 908–915 in [9]. He thinks that the expression “\( r \) is a power of 2 or” in I (iv), II (v) of [9, Theorem 20] should be omitted.

Using the first relation of (5.6) and the relation

\[ j^*\rho(u^r \otimes x) = \sum_{0 \leq q \leq r} v^r v^{r-q} Sq^{i} x \quad \text{if } x \in H^q(M; Z_2), \]

in [16, §2], we have the following relations:

\[ \tilde{\rho}_2 \tilde{\beta}_2(\delta(v^{n-k-1}M)) = \delta(v^{n-k}M) = v^{n-k+1} \Lambda M \neq 0 \quad \text{if } k \text{ is odd}, \]

\[ \delta \tilde{\beta}_2(v^{n-k-3}M) = \tilde{\beta}_2(v^{n-k-1}M) \quad \text{if } k \text{ is even}, \]

\[ \delta \tilde{\beta}_2(v^{n-k-2}M) = \tilde{\beta}_2(v^{n-k-3}M) \quad \text{if } k \text{ is odd}, \]

\[ j^*\tilde{\rho}_2(\rho(u^{k-3} \otimes x^2)) \]

\[ = \begin{cases} \beta_2(v^{n-2} x + v^{n-2-k} Sq^{k} x) & \text{if } k \text{ is even and } \dim x = n-k, \\ \tilde{\beta}_2(v^{n-2} x) & \text{if } k \text{ is odd and } \dim x = n-k, \end{cases} \]

\[ j^*\tilde{\rho}_2(\rho(u^{k-1} \otimes x^2)) \]

\[ = \begin{cases} \beta_2(v^{n-2} x) & \text{if } k \text{ is even and } \dim x = n-k-1, \\ \tilde{\beta}_2(v^{n-2} x + v^{n-k-3} Sq^{k+1} x) & \text{if } k \text{ is odd and } \dim x = n-k-1. \end{cases} \]
On considering the exact sequence (5.3), it follows, from Lemmas 5.4, 5.5 and the above relations, that $j^*: H^{2n-k-i}(M*; \underline{Z}) \rightarrow \text{Im } j^*$ is a split epimorphism for $i=1, 2$. Further, the relation

$$p_2\bar{\beta}_2\rho(u^{k-1} \otimes x^2) = \rho(u^k \otimes x^2) \quad \text{for } x \in H^{n-k-1}(M; \underline{Z})$$

follows from Lemma 3.2. Hence, the theorem is established in case $n-k$ is odd.

References

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