A sufficient condition for a map to be cobordant to an embedding

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A sufficient condition for a map to be cobordant to an embedding

Yoshiyuki Kuramoto*

(Received 13 October, 2000)

Abstract

In this paper we give a proof of the theorem in [6] which asserts a sufficient condition for a map \( f : M^n \rightarrow N^{2n-k} \) between compact manifolds without boundary to be cobordant to an embedding, since we did not give the details for the general case in [6].

1 Introduction

Throughout this paper, \( n \)-manifolds mean compact differentiable manifolds of dimension \( n \). The (co-)homology is understood to have \( \mathbb{Z}_2 \) for coefficients.

For a map \( f : M^n \rightarrow N^{2n-k} \) between compact manifolds without boundary, let \( w_i(f) \) be the \( i \)-th Stiefel-Whitney class of \( f \) and \( f_! : H^i(M) \rightarrow H^{i+n-k}(N) \) the transfer homomorphism (or Umkehr homomorphism) of \( f \). Further let

\[
\theta(f) = f^*f!(1) - w_{n-k}(f).
\]

Then by [5, Lemma 2], \( M \times \theta(f) \) is the \( H^n(M) \times H^{n-k}(M) \)-component of \( U_M(1 \times w_{n-k}(f)) + (f \times f)^*U_N \), where \( U_V \in H^{\dim V}(V \times V) \) denotes the \( \mathbb{Z}_2 \)-Thom class (or the \( \mathbb{Z}_2 \)-diagonal class) of a manifold \( V \). Therefore, A. Haefliger [Theorem 5.2] implies that

**Theorem (Haefliger)** If \( f \) is homotopic to an embedding, then
\[
\theta(f) = 0 \quad \text{and} \quad w_{n-i}(f) = 0 \quad \text{for} \quad i < k. \tag{1.1}
\]

The inverse of this theorem may be hard to study. So we will study whether \( f \) is cobordant to an embedding in the sense of Stong [9] if the condition (1.1) in the above Theorem is satis-

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2000 Mathematics Subject Classification: Primary 57R40; Secondary 57R20, 57R90.

Key words and phrases: Embeddings, Stiefel-Whitney classes of a map, cobordism of maps.
fied. Here a map \( f_1 : M^n_i \rightarrow N_1^{n+k} \) is said to be cobordant to \( f_2 : M^n_2 \rightarrow N_2^{n+k} \) if there exist two cobordisms \((W,M^n_i,M^n_2)\), \((V,N^{n+k}_i,N^{n+k}_2)\) and a map \( F : W \rightarrow V \) such that \( F|W_i = f_i (i = 1,2) \). M. A. Aguilar and G. Pastor [1] determined the necessary and sufficient condition that a map \( f : M^n \rightarrow N^{2n-k}, (k = 1,2) \) is cobordant to an embedding. In [6] we have considered cases when \( k \geq 3 \) and obtained following results:

**Corollary 1.3 in [6]** Let \( f : M^n \rightarrow N^{2n-k}, (k = 3,4) \) be a map. If \( w_{n-i}(f) = 0 \) for \( 0 < i < k \) and \( \theta(f) = 0 \), then \( f \) is cobordant to an embedding.

Moreover we have stated the following theorem:

**Theorem (Theorem 5.1' in [6])** Let \( n > 2k > 0 \). Then a map \( f : M^n \rightarrow N^{2n-k} \) is cobordant to an embedding if

1. \( w_{n-i}(f) = 0 \) for \( 1 \leq i < k \),
2. \( \theta(f) = 0 \) and
3. \( w_i(M) \in f^*H^i(N) \) for \( 4i < k \).

From this theorem we obtained the following corollaries:

**Corollary 1** If \( 1 \leq k \leq 4, n \geq 2k + 1 \), a map \( f : M^n \rightarrow N^{2n-k} \) is cobordant to an embedding if \( w_{n-i}(f) = 0 \) for \( 1 \leq i < k \) and \( \theta(f) = 0 \).

**Corollary 2** If \( 5 \leq k \leq 8, n \geq 2k + 1 \), a map \( f : M^n \rightarrow N^{2n-k} \) is cobordant to an embedding if

1. \( w_{n-i}(f) = 0 \) for \( 1 \leq i < k \),
2. \( \theta(f) = 0 \) and
3. \( w_i(M) = 0 \) or \( w_i(f) = 0 \).

Since in [6] we have omitted details of the proof of the above theorem for the general case, we will give the proof in this paper.

This paper is organized as follows: In \( \S 2 \), we recall the Stiefel-Whitney class \( w(f) \) and the transfer homomorphism \( f_\ast \) of a map \( f : M^n \rightarrow N^{2n-k} \) and prepare some lemmas concerning \( f_\ast, w_i(f) \)'s, and the Steenrod squaring operations \( Sq^i \)'s. In \( \S 3 \), we give the proof of the
2 Preliminaries

We adopt same notations and symbols as in [6]. For a manifold $V$, we denote by $w(V)$ and $\overline{w}(V) = w(V)^{-1}$ the total Stiefel-Whitney class and the total normal Stiefel-Whitney class of $V$, respectively. For a map $f : M^n \to N^{2n-k}$, the total Stiefel-Whitney class of $f$, $w(f) = \sum_{i \geq 0} w_i(f)$, is defined by the equation

$$w(f) = \overline{w}(M)f^*(w(N)),$$

and the transfer homomorphism $f_\ast : H^i(M) \to H^{i+n-k}(N)$ is defined by

$$f_\ast(x) = D_N f_\ast(x \cap [M]),$$

where $D_N$ is the Poincaré duality and $[M] \in H_n(M)$ denotes the fundamental class of $M$. For $\mu = (i_1, i_2, \ldots, i_p)$, let $w_\mu(V) = w_{i_1}(V)w_{i_2}(V)\cdots w_{i_p}(V)$ and $|\mu| = \sum_{1 \leq j \leq p} i_j$. Then R. L. W. Brown's theorem [2, p. 247] implies that

**Theorem (Brown)** Let $n > 2k > 0$. Then a map $f : M^n \to N^{2n-k}$ is cobordant to an embedding if and only if the following conditions (1) and (2) are satisfied:

1. $(w_\mu(M)w_\lambda(f), [M]) = 0$ if $|\mu| + |\lambda| = n$ and $\lambda$ has a component with $> n - k$, and
2. $(f^*(w_\lambda(N))w_\mu(M)f_\ast(w_\nu(M))) - f^*(w_\lambda(N))w_\mu(M)w_\nu(M)w_{n-k}(f), [M]) = 0$

for all $\lambda, \mu$ and $\nu$ with $|\lambda| + |\mu| + |\nu| = k$.

We denote by $w(M) = \sum_{i \geq 0} w_i(M)$ the total Wu class of $M$. The following relations are well-known:

$$Sq(w(M)) = w(M),$$

$$Sq^i x_{n-i} = w_i(M)$$

for all $x_{n-i} \in H^{n-i}(M)$,

$$Sq^j w_j(\xi) = \sum_{0 \leq t \leq j} \binom{j - i + t - 1}{t} w_{i+t}(\xi)w_{j+t}(\xi).$$
In the following lemmas, we list some relations among \( f_* \), the Steenrod operations \( Sq^i \) and the Stiefel-Whitney classes, the first of which is seen in eg. [3] (cf. [1]), while the last follows from the definition of \( f_* \), (cf. [2]):

**Lemma 1**  For a map \( f: M^n \to N^{2n-k} \) there are relations

1. \( f_*(f^*(x)y) = xf(y) \) for \( x \in H^*(N), y \in H^*(M) \),
2. \( Sq^i(x) = f_*\left(Sq(x)w^i(f)\right) \),
3. \( \langle xf(y), [N] \rangle = \langle f^*(x)y, [M] \rangle \) if \( \dim x + \dim y = n \),
4. \( \langle f^*(x)yf^*f(z), [M] \rangle = \langle f^*(x)zf^*f(y), [M] \rangle \) if \( \dim x + \dim y + \dim z = k \).

In particular \( \langle f^*(x)f^*f(1), [M] \rangle = \langle f^*(x)f^*f(1), [M] \rangle \).

Further, we have the following

**Lemma 2**  Let \( f: M^n \to N^{2n-k} \) be a map. Then

1. \( f^*(x_i)w_{n-i}(f) = 0 \) for \( x_i \in H^i(N),(0 \leq i < k) \).
2. \( f^*(y)xf^*f_i(x) = f^*(y)\sum_{r=0}^{i-1} Sq^r(x)w_{n-k+i-r}(f) + f^*(y)x^2w_{n-k}(f) \)
   for \( x \in H^i(M), y \in H^{k-2i}(N),(0 \leq 2i \leq k) \).
3. In particular \( f^*(y)f^*f(1) - f^*(y)w_{n-k}(f) = 0 \) for \( y \in H^k(N) \).

*Proof.* See the proof of Lemma 2.2[6].

### 3 Proof of the Theorem

The following lemmas are consequences of the definition of the Wu class and the Wu's formula (2.4).

**Lemma 3**  Let \( t \) be an integer such that \( 1 \leq t \leq k/2 \). Let \( l \geq 0 \) and \( r \geq 1 \) be the integers defined by \( t = 2^l + 2^{r-1} \). Assume that \( l \geq 1 \). Then

\[
  w_t(M) = Sq^{2^{r-1}}w_{2^{r-1}}(M) + \sum_{s=1}^{2^{r-1}} a_s w_s(M) w_{l-s}(M), \tag{3.1}
\]
where \( a_s \in \mathbb{Z}_2 \).

**Lemma 4** If \( 2 \leq 2^{-l} \leq n/2 \) then

\[
u_{2^{-l}}(M) = w_{2^{-l}}(M) + w_{2^{-l}}(M)^2 + \sum_{s=1}^{2^{-l}-1} b_s w_s(M) w_{\lambda(s)}(M),
\]

where \( b_s \in \mathbb{Z}_2, |\lambda(s)| = 2^{-l} - s \).

We postpone the proof of the above two lemmas and prove the Theorem using them. By virtue of Lemma 1(4), to prove the Theorem we have only to prove

\[
\left( E_{\lambda, \mu, \nu} \right) w_{\lambda}(M) f^* f_i(\nu_{\mu}(M) f^* w_v(N)) = w_{\lambda}(M) w_{\mu}(M) f^* w_v(N) w_{n-k}(f),
\]

for \(|\lambda| + |\mu| + |\nu| = k\) and \(|\lambda| \leq k/2\), under the assumptions (1)(2)(3) of the Theorem.

**Proof of the Theorem:**

We prove by induction on \(|\lambda| = t\) that \( \left( E_{\lambda, \mu, \nu} \right) \) holds for \(|\lambda| + |\mu| + |\nu| = k\) and \(|\lambda| = t \leq k/2\),

By the assumption \( \theta(f) = 0, \left( E_{(0), \mu, \nu} \right) \) holds. Let \(|\lambda| = t \geq 1\). Suppose that \( \left( E_{\lambda, \mu, \nu} \right) \) holds for \(|\lambda| \leq t-1\).

**Case 1: \( \lambda = (t) \)**

First we consider the case \( t = 2^l t_{2^{-l}} \) for \( l \geq 1 \) and \( r \geq 1 \). Then since \( 2^{-l} < t/2 \leq k/4 \) we have \( w_i(M) \in f^* H^l(N) \) for \( i \leq 2^{-l} \). Hence by the assumption for \( 1 \leq s \leq 2^{-l} \) and \(|\mu| + |\nu| = k - t\) we have

\[
w_s(M) w_{t-s}(M) \left( f^* f_i(\nu_{\mu}(M) f^* w_v(N)) - w_{\mu}(M) f^* w_v(N) w_{n-k}(f) \right) = 0.
\]

Thus denoting \( w_{\mu}(M) f^* w_v(N) = x \) we have by Lemma 3

\[
w_i(M) \left( f^* f_i x - x w_{n-k}(f) \right) = Sq^{2^{-l}} w_{2^{-l}}(M) \left( f^* f_i x - x w_{n-k}(f) \right)
\]

\[
= Sq^{2^{-l}} \left( w_{2^{-l}}(M) \left( f^* f_i x - x w_{n-k}(f) \right) \right)
\]

\[
+ \sum_{s=0}^{2^{-l}-1} Sq^s w_{2^{-l}}(M) Sq^{2^{-l}-s} \left( f^* f_i x - x w_{n-k}(f) \right)
\]

\[
= \nu_{2^{-l}}(M) w_{2^{-l}}(M) \left( f^* f_i x - x w_{n-k}(f) \right) + \sum_{|\lambda| \leq t} w_{\lambda}(M) \left( f^* f_i x_{\lambda'} - x_{\lambda'} w_{n-k}(f) \right)
\]

\[
+ \sum_{i=1}^k y_i w_{n-k+i}(f),
\]
where $x_{\lambda} \in H^{t-|\lambda|}(M), y_{i} \in H^{k-|i|}(M)$ and they are expressed as $\sum_{p,t} w_{\lambda}(M) f^{*} w_{\mu}(N)$.

Since $w_{2,*}(M)(f^{*} f_{x_{\lambda}} - x_{\lambda} w_{n-k}(f)) = 0$ for $|\lambda| < t$ by the induction hypothesis and $y_{i} w_{n-k+i}(f) = 0$ for $1 \leq i \leq k$ by the assumption, we have

$$w_{i}(M)(f^{*} f_{x} - x w_{n-k}(f)) = \nu_{2,-t}(M)w_{2,*}(M)(f^{*} f_{x} - x w_{n-k}(f))$$

$$= w_{2,*}(M)(f^{*} f_{i}(\nu_{2,-t}(M)x) - \nu_{2,-t}(M) x w_{n-k}(f)),$$

since $\nu_{2,-t}(M) \in f^{*} H^{k}(N)$. Hence by the induction hypothesis we have

$$w_{i}(M)(f^{*} f_{x} - x w_{n-k}(f)) = 0.$$  

Next we consider the case $t = 2r-1$. If $2t < k$ then $2^{r-2} < k/4$ and $w_{i}(M) \in f^{*} H^{k}(N)$ for $i \leq 2^{r-2}$ hence by Lemma 4

$$w_{2,r-1}(M)(f^{*} f_{x} - x w_{n-k}(f))$$

$$= \nu_{2,r-1}(M)(f^{*} f_{x} - x w_{n-k}(f)) + \sum_{s=1}^{2^{r-2}} b_{s} w_{s}(M) w_{\lambda}(M)(f^{*} f_{x} - x w_{n-k}(f))$$

$$= S q^{2^{r-1}}(f^{*} f_{x} - x w_{n-k}(f)) + \sum_{s=1}^{2^{r-2}} b_{s} w_{s}(M)(f^{*} f_{i}(w_{s}(M)x) - w_{s}(M)x w_{n-k}(f))$$

$$= S q^{2^{r-1}}(f^{*} f_{x} - x w_{n-k}(f))$$

$$= \theta(f) \sum_{s=0}^{2^{r-1}} S q^{s} w_{2,-r,s}(f) + \sum_{i=1}^{k} y_{i} w_{n-k+i}(f) \quad \text{(where $y_{i} \in H^{k-i}(M)$)}$$

$$= 0.$$  

Now we consider the remaining case $t = 2s = 2r-1 = k/2$. In this case $\nu = (0)$ and $|\lambda| = |\mu|$. Hence if $\mu \neq (t),(s,s)$ then $w_{\mu}(M) = w_{p}(M)w_{\mu'}(M)$ for some $p, \mu'$ such that $1 \leq p < s = k/4$ and $|\mu'| < t$. Then since $w_{p}(M) \in f^{*} H^{p}(N)$ we have by the induction hypothesis

$$w_{i}(M)(f^{*} f_{i}(w_{p}(M)w_{\mu'}(M))) = w_{i}(M)w_{p}(M)(f^{*} f_{i}(w_{\mu'}(M)))$$

$$= w_{\mu'}(M)(f^{*} f_{i}(w_{i}(M)w_{p}(M))) = w_{\mu'}(M)w_{i}(M)w_{p}(M)w_{n-k}(f)$$

$$= w_{i}(M)w_{\mu}(M)w_{n-k}(f).$$

If $\mu = (t) = \lambda$, then $\left(E_{\lambda,\mu,(0)}\right)$ holds by Lemma 2 (2).

If $\mu = (s,s)$, then we have by Lemma 4, Lemma 2 (2), the assumption and the induction
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hypothesis

\[ w_s(M)f^*f(w_s(M)^2) - w_s(M)w_s(M)^2 w_{n-k}(f) \]

\[ = Sq\left( f^*f(w_s(M)^2) - w_s(M)^2 w_{n-k}(f) \right) + w_s(M)^2\left( f^*f(w_s(M)^2) - w_s(M)^2 w_{n-k}(f) \right) \]

\[ + \sum_{s=1}^{2s-1} b_s w_s(M)w_{\lambda(s)}(M)\left( f^*f(w_s(M)^2) - w_s(M)^2 w_{n-k}(f) \right) \]

\[ = 0. \]

Case 2: \( \lambda \neq (t) \). In this case we have

\[ w_\lambda(M) = w_s(M)w_{\lambda}(M), \quad 1 \leq s \leq t/2 \quad \text{and} \quad |\lambda'| = t-s < t. \]

First we consider the case \( \lambda \neq (t/2,t/2) \). Then we may assume that \( s < t/2 \) and by the assumption (3) we have \( w_s(M) \subset f^*H^s(N) \). Therefore we have by the induction hypothesis

\[ w_\lambda(M)f^*f(w_\mu(M)f^*w_v(N)) = w_s(M)w_\lambda(M)f^*f(w_\mu(M)f^*w_v(N)) \]

\[ = w_\lambda(M)f^*f(w_s(M)w_\mu(M)f^*w_v(N)) \]

\[ = w_\lambda(M)w_s(M)f^*w_v(N)w_{n-k}(f) \]

\[ = w_\lambda(M)w_\mu(M)f^*w_v(N)w_{n-k}(f). \]

Now we consider the case \( w_\lambda(M) = w_s(M)^2, s = t/2 \). If \( t < k/2 \) then \( w_s(M) \subset f^*H^s(N) \) by the assumption (3), therefore \( (E_{\lambda,\mu,v}) \) holds. Hence we may assume \( 4s = 2t = k. \)

Then \( V = (0) \) since we assume \( |\lambda'| \leq |\mu'| \). Hence if \( \mu \neq (t),(s,s) \) then \( w_\mu(M) = w_p(M)w_{\mu'}(M) \) for some \( p, \mu' \) such that \( 1 \leq p < s \) and \( |\mu'| < t. \) Then since \( w_p(M) \subset f^*H^p(N) \) we have by the induction hypothesis

\[ w_\lambda(M)^2 f^*f(w_p(M)w_{\mu'}(M)) = w_s(M)^2 w_p(M)f^*f(w_\mu(M)) \]

\[ = w_\mu(M)f^*f(w_s(M)^2 w_p(M)) = w_\mu(M)w_s(M)^2 w_p(M)w_{n-k}(f) \]

\[ = w_s(M)^2 w_\mu(M)w_{n-k}(f). \]

If \( \mu = (t) \) then \( (E_{(s,s),(t),(0)}) \) holds since \( (E_{(t),(s,s),(0)}) \) holds by Case 1. If \( \mu = (s,s) = \lambda \) then \( (E_{(s,s),(s,s),(0)}) \) holds by Lemma 2 (2). Thus we complete the proof.

Now we prove Lemma 3 and 4.

Proof of Lemma 3: By Wu's formula (2.4) we have
Since 
\[ (2^r - 1)^{2r - 1} \equiv 1 \mod 2 \]
we have 
\[ S^{2r-1} \omega_{2r-1}(M) = \sum_{s=0}^{2r-1} \binom{2^r - 1}{s} \omega_{2r-1-s}(M) \omega_{2r+s}(M). \]

Proof of Lemma 4: For an integer \( k \geq 1 \) we define \( \mathbb{Z}_2 \)-submodules \( A_k, B_k \) of \( H^*(M) \) as follows:

\[
A_k := \sum_{p \geq 1} Z_2 W_p(M) \cdot W_{k-p}(M), \quad B_k := \sum_{l \geq k} A_l.
\]

Moreover, we denote \( A'_k := \sum_{p \neq q} Z_2 W_p(M) \cdot W_q(M). \)

Note that \( S^{q} B_k \subseteq B_k \). To prove Lemma 4 it suffices to prove the following

**Lemma 5** For an integer \( t \geq 2 \), let \( r, l \) be the integers defined by 
\( t = 2^r + l \), \( 1 \leq r, 0 \leq l < 2^r \). Then

\[
(*) \left\{ \begin{array}{ll}
& \text{if } l > 0, \text{ then } v_t(M) \in B_2, \text{ and } \\
& \text{if } l = 0, \text{ then } v_t(M) \in w_{2r}(M) + w_{2r-1}(M)^2 + A'_2 + B_3.
\end{array} \right.
\]

**Proof.** We prove the lemma by induction on \( t \).

For \( t = 2 \), we have \( v_2(M) = w_2(M) + w_{2r-2}(M)^2 \) and \((*)_2\), holds.

Suppose that \((*)_s\) holds for \( s < t \).

If \( t = 2^r + l, 0 \leq l < 2^r \), then
\[
v_t(M) = w_t(M) + \sum_{1 \leq s \leq l} S^{q} v_{t-s}(M)
\]
\[
= w_t(M) + S^{q} w_{2r}(M) + \sum_{s \neq l} S^{q} v_{t-s}(M).
\]

Since \( \binom{2^r - 1}{l} \equiv 1 \mod 2 \) we have from (2.4) \( w_t(M) + S^{q} w_{2r}(M) \in B_2 \). On the other hand \( v_{t-s}(M) \in B_2 \) for \( s \neq l \) by the induction hypothesis. Hence we have \( v_t(M) \in B_2 \).

If \( t = 2^r \) then
\[
v_t(M) = w_t(M) + \sum_{1 \leq s \leq 2^-1} S^{q} v_{t-s}(M)
\]
\[
= w_{2r}(M) + w_{2r-1}(M)^2 + \sum_{1 \leq s \leq 2r-1} S^{q} v_{2r-1-s}(M).
\]

We have \( v_{2r-s}(M) \in B_2 \) for \( 1 \leq s \leq 2r-2 \) by the induction hypothesis. Let \( s_i (i = 1, 2) \) be integers such that \( 0 \leq s_i < 2r-2 \). Since \( \binom{2^-1 - s_i - 1}{s_i} \equiv 0 \mod 2 \) for \( 1 \leq s_i < 2r-2 \) we
have from (2.4) \( Sq^{s_{i}}w_{2r-1-s_{i}}(M) \in B_{2} \) for \( 1 \leq s_{i} \leq 2^{r-2} \). Therefore if \( 1 \leq s_{1} + s_{2} \) then \( Sq^{s_{1}}w_{2r-1-s_{1}}(M) Sq^{s_{2}}w_{2r-1-s_{2}}(M) \) does not contain \( w_{2r-1}(M)^{2} \). Hence \( Sq^{s}u(M) \in A_{2} + B_{3} \) for \( 1 \leq s \leq 2^{r-1} - 1 \). Thus we have \( u_{i}(M) \in w_{2r}(M) + w_{2r-1}(M)^{2} + A_{2} + B_{3} \).

References


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