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SOME APPROXIMATION THEOREMS VIA STATISTICAL CONVERGENCE

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Preface

Present thesis entitled 'SOME APPROXIMATION THEOREMS VIA STATISTICAL CONVERGENCE' contains the research work done by me under the constant supervision of Dr. Mursaleen, Professor, Department of Mathematics, Aligarh Muslim University, Aligarh. In the present work, we study approximation properties of different positive linear operators using Korovkin theorem via statistical convergence and its various generalizations.

The present thesis comprises of six chapters and each chapter is further divided into sections. The definitions, examples, remarks, theorems etc. have been specified with the double decimal numbers. The first figure denotes the number of the chapter, second represents the section in the chapter and third points out the number of the definition, the example, or the theorem as the case may be in a particular chapter. For example, Theorem 3.4.2 refers to the second theorem appearing in the fourth section of the third chapter.

Chapter 1 contains preliminary notions, basic definitions, examples and some important well known results related to our study which are required for the development of the subject in subsequent chapters. This chapter is an attempt to make this thesis as self contained as possible.

Chapter 2 deals with the study of statistical approximation properties of modified $q$-Stancu-Beta operators.

In Chapter 3, we study statistical approximation properties of $q$-Bernstein-Schurer operators. We establish some direct theorems. With the help of Matlab, we compute error estimate using modulus of continuity and give its algorithm. We also show graphically the convergence of the $q$-Bernstein-Schurer operators to various functions.

Chapter 4 is devoted to some positive linear operators constructed by means of $q$-Lagrange polynomials and some approximation results via $A$-statistical convergence are studied.

In chapter 5, we define $\lambda$-equi-statistical convergence and we apply our new notion to prove a Korovkin type approximation theorem and we show that our
theorem is a non-trivial extension of some well-known Korovkin type approximation theorems. We also prove a Voronovskaja type approximation theorem via the concept of $\lambda$-equi-statistical convergence.

Chapter 6 is devoted to study of 'Approximation for periodic functions via weighted statistical convergence'.

Finally, a bibliography is given which by no means is exhaustive one but lists only those books and papers which have been referred to in this thesis.
Chapter 1

Introduction and Preliminaries

1.1 Introduction

In this chapter we give basic concepts, preliminary definitions and some fundamental results. Of course, the elementary knowledge of concepts such as function, sequence, convergence etc. has been pre-assumed and no attempt has been made to discuss them here. Some key results and classical theorems related to our subject matter are also incorporated as remarks at suitable places. Most of the material included in this chapter occur in standard literature like 'Linear operator and approximation theory'(1960), Hindustan publishing carp. (India) by P.P. Korovkin [57], and survey on 'Korovkin-type theorems and approximation by positive linear operators' by Francesco Altomare (2010) [10], H.Fast [48], Fridy [50] etc.

Korovkin-type theorems furnish simple and useful tools for ascertaining whether a given sequence of positive linear operators, acting on some function space is an approximation process or, equivalently, converges strongly to the identity operator. Roughly speaking, these theorems exhibit a variety of test subsets of functions which guarantee that the approximation (or the convergence) property holds on the whole space provided it holds on them. The custom of calling these kinds of results as Korovkin-type theorems refers to P. P. Korovkin [64], who in 1953 discovered such a property for the functions $1$, $x$ and $x^2$ in the space $C[0,1]$ of all continuous functions on the real interval $[0,1]$ as well as for the functions $1$, $cos$ and $sin$ in the space of all continuous $2\pi-$ periodic functions on the real line [57].
1.2 Historical Note

Statistical convergence was introduced in connection with problems of series summation. The main idea of statistical convergence of a sequence \((x_n)\), where \(n \in \mathbb{N}\) is that the majority, in a certain sense, of its elements converge and we do not care what happens with other elements. At the same time, it is known that sequences that come from real life sources are not convergent in the strictly mathematical sense. This way, the advantage of replacing the uniform convergence by statistical convergence consists in the fact that the second convergence models improves the technique of approximation in different functions spaces. H. Fast [48], in 1951, introduced an extension of the usual concept of sequential limit which he called statistical convergence.

The study of the statistical convergence for sequences of linear positive operators was attempted in the year 2002 by A.D. Gadjiev and C. Orhan [25]. The research field was proved to be extremely fertile.

Our interest is to construct different classes of linear positive operators and to study their statistical approximation properties. We know that any convergent sequence is statistically convergent but the converse need not be true. The aim is to construct such sequences of operators that approximate the functions in the statistical sense, but not in the classical sense.

1.3 Basic definitions and examples

In the present section, we give a brief exposition of some important terminology of approximation theory.

Definition 1.3.1. Let \(\mathbb{N}\) denote the set of all natural numbers. Let \(K \subseteq \mathbb{N}\) and \(K_n = \{k \leq n : k \in K\}\). Then the natural density of \(K\) is defined by \(\delta(K) = \lim_n n^{-1}|K_n|\) if the limit exists, where \(|K_n|\) denotes the cardinality of the set \(K_n\).

Definition 1.3.2. A sequence \(x = (x_k)\) of real numbers is said to be statistically convergent to \(L\) (cf. Fast [48]) provided that for every \(\epsilon > 0\) the set \(\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}\) has natural density zero, i.e. for each \(\epsilon > 0\),

\[
\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : |x_k - L| \geq \epsilon \right\} \right| = 0.
\]
In this case, we write \( st - \lim_{k} x_k = L \). Note that every convergent sequence is statistically convergent but not conversely.

Let \( A = (a_{nk}) \), where \( n, k = 1, 2, 3... \) be an infinite matrix. For a given sequence \( x = (x_k) \), the \( A \)-transform of \( x \) is defined by \( Ax = ((Ax)_n) \), where \( (Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k \), provided the series converges for each \( n \). We say that \( A \) is regular if \( \lim_{n}(Ax)_n = L = \lim x \).

Let \( A \) be a regular matrix. We say that a sequence \( x = (x_k) \) is \( A \)-statistically convergent to a number \( L \) (cf. Kolk [39]) if for every \( \epsilon > 0 \),

\[
\lim_{n} \sum_{k \mid |x_k - L| > \epsilon} a_{nk} = 0.
\]

In this case, we denote this limit by \( st_A - \lim_{n} x_n = L \).

Note that for \( A = C_1 \), the Cesaro matrix of order 1, \( A \)-statistical convergence reduces to the statistical convergence.

The concept of \( \lambda \)-statistical convergence was introduced recently in [51] as follows.

Let \( \lambda = (\lambda_n) \) be a non-decreasing sequence of positive numbers tending to \( \infty \) such that

\[
\lambda_{n+1} \leq \lambda_n + 1 \quad \text{and} \quad \lambda_1 = 1.
\]

Also let

\[
I_n = [n - \lambda_n + 1, n] \quad \text{and} \quad K \subseteq \mathbb{N}.
\]

Then the \( \lambda \)-density of \( K \) is defined by

\[
\delta_{\lambda}(K) = \lim_{n \to \infty} \frac{1}{\lambda_n} \left| \{ j : n - \lambda_n + 1 \leq j \leq n \quad \text{and} \quad j \in K \} \right|.
\]

Clearly, in the special case when \( \lambda_n = n \), the \( \lambda \)-density reduces to the above-defined natural density.

The number sequence \( x = (x_j) \) is said to be \( \lambda \)-statistically convergent to the number \( L \) if, for each \( \epsilon > 0 \),

\[
\delta_{\lambda}(K_{\epsilon}) = 0,
\]
where

\[ K_\varepsilon = \{ j : j \in I_n \quad \text{and} \quad |x_j - L| > \varepsilon \}, \]

that is, if, for each \( \varepsilon > 0 \),

\[ \lim_{n \to \infty} \frac{1}{\lambda_n^{n}|\{ j : j \in I_n \quad \text{and} \quad |x_j - L| > \varepsilon \}| = 0. \]

In this case, we write

\( \text{st}_{\lambda^*} \lim_{n \to \infty} x_n = L \)

and we denote the set of all \( \lambda \)-statistically convergent sequences by \( S_\lambda \).

Definition 1.3.3. Let \( C[a, b] \) be the linear space of all real valued continuous functions \( f \) on \([a, b]\) and let \( T \) be a linear operator which maps \( C[a, b] \) into itself. We say that \( T \) is positive if for every non-negative \( f \in C[a, b] \), we have \( T(f, x) \geq 0 \) for all \( x \in [a, b] \).

Definition 1.3.4. Let \( f(x) \) be a function continuous in the interval \([a, b]\). We put

\[ w(\delta) = w(f, \delta) = \max_{|x-y| \leq \delta} |f(x) - f(y)|, \quad a \leq x, y \leq b. \]

The quantity \( w(\delta) \) is called modulus of continuity of the function \( f(x) \).

1.4 q-Calculus

Let us recall certain notations of q-integers. For each nonnegative integer \( k \), the q-integer \( [k]_q \) is defined by

\[ [k]_q := \begin{cases} \frac{(1-q)^k}{1-q}, & q \neq 1 \\ k, & q = 1 \end{cases} \]

\[ [k]_q! := \begin{cases} [k]_q[k-1]_q...[1]_q, & k \geq 1, \\ 1, & k = 0; \end{cases} \]

\[ \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}. \]

For a real or complex number \( q (|q| < 1) \), the number \( (\lambda; q)_n \) is defined by

\[ (\lambda; q)_n := \frac{(\lambda; q)_n}{(\lambda q^n; q)_n}. \]
where
\[
(\lambda; q)_\infty := \prod_{k=1}^{\infty} (1 - \lambda q^k),
\]
and
\[
(\lambda; q)_n = \begin{cases} 
1 & \text{if } n = 0, \\
(1 - \lambda)(1 - \lambda q) \cdots (1 - \lambda q^{n-1}) & \text{if } n = 1, 2, 3, \ldots
\end{cases}
\]

1.5 Korovkin Type Approximation Theorem

Let \( C[a, b] \) be the linear space of all real-valued continuous functions \( f \) on \([a, b]\) and let \( T \) be a linear operator which maps \( C[a, b] \) into itself. We say that \( T \) is positive if, for every non-negative \( f \in C[a, b] \), we have
\[
T(f, x) \geq 0 \quad (x \in [a, b]).
\]

We know that \( C[a, b] \) is a Banach space with the norm given by
\[
\|f\|_{C[a, b]} := \sup_{x \in [a, b]} |f(x)| \quad (f \in C[a, b]).
\]

The classical Korovkin approximation theorem states as follows [8]:

Let \( \{T_n\} \) be a sequence of positive linear operators from \( C[a, b] \) into \( C[a, b] \). Then
\[
\lim_n \|T_n(f, x) - f(x)\|_{C[a, b]} = 0 \quad (f \in C[a, b])
\]
\[
\iff \lim_n \|T_n(f_i, x) - f_i(x)\|_{C[a, b]} = 0 \quad (i = 0, 1, 2),
\]

where
\[
f_0(x) = 1, \quad f_1(x) = x \quad \text{and} \quad f_2(x) = x^2.
\]
Chapter 2

Statistical approximation properties of modified q-Stancu-Beta operators

2.1 Introduction

In this chapter we define the modified q-Stancu-Beta operators and study the weighted statistical approximation by these operators with the help of the Korovkin type approximation theorem. We also establish the rates of statistical convergence by means of the modulus of continuity and the Lipschitz type maximal function. Our results show that rates of convergence of our operators are at least as fast as classical Stancu-Beta operators.

2.2 Construction of the operators

We define modified q-Stancu-Beta operators as follows:

\[
L_n^*(f; q, x) = q \frac{K(A, [n]_q x)}{B_q([n]_q x, [n]_q + 1)} \int_0^{\infty / A} \frac{u^{[n]_q x - 1}}{(1 + u)_q^{[n]_q x + [n]_q + 1}} f(q^{[n]_q x} u) d_q u
\]

(2.2.1)

where \( x > 0 \) and \( 0 < q \leq 1 \). It is easy to verify that if \( q = 1 \), these operators turns into the classical Stancu-Beta operators.

Remark 2.2.1. Note that \( L_n^*(f; q, x) = L_n^0(f; x) \) and from the Lemma 1 of Aral and Gupta [6], we have \( L_n^0(1; x) = 1, L_n^0(t; x) = x, L_n^0(t^2; x) = \frac{(n)_{q} x + 1}{(n)_{q} - 1} \). Hence for \( x \geq 0, 0 < q \leq 1 \), we have

\[
L_n^*(1; q, x) = q, L_n^*(t; q, x) = q x \quad \text{and} \quad L_n^*(t^2; q, x) = \frac{(n)_{q} x + 1}{(n)_{q} - 1}.
\]

(2.2.2)
Remark 2.2.2. Let \( q \in (0, 1) \) then for \( x \in [0, \infty) \), we have
\[
L_n^* (t - x; x) = 0
\]
and
\[
L_n^* ((t - x)^2; x) = \frac{([n]q - q[n]q + q)x^2 + x}{([n]q - 1)}.
\]

2.3 Weighted statistical approximation of Korovkin type

In this section we obtain the Korovkin type weighted statistical approximation by the operators defined in (2.2.1).

Now, we consider a sequence \( q = (q_n) \), \( q_n \in (0, 1) \), such that
\[
st \lim_{n \to \infty} q_n = 1. \tag{2.3.1}
\]

In [18], Doğru gave some examples so that \((q_n)\) is statistically convergent to 1 but it may not convergent to 1 in the ordinary case.

Now we are ready to prove our main result as follows:

Theorem 2.3.1. Let \( (L_n^*) \) be the sequence of the operators (2.2.1) and the sequence \( q = (q_n) \) satisfies (2.3.1). Then for any function \( f \in C_B[0, \infty) \),
\[
st \lim_{n \to \infty} \|L_n^* (f; q_n; \cdot) - f\|_{p_0} = 0.
\]

2.4 Rates of statistical convergence

In this section, we give the rates of statistical convergence of the operators (2.2.1) by means of modulus of continuity and Lipschitz type maximal functions. The modulus of continuity for the functions \( f \in C_B[0, \infty) \) is defined as
\[
w(f; \delta)_{p_0} = \sup_{x, t \geq 0, |t - x| < \delta} \frac{|f(t) - f(x)|}{1 + x^{2+\lambda}}
\]
where \( w(f; \delta)_{\rho_0} \) for \( \delta > 0, \lambda \geq 0 \) satisfies the following conditions: for every \( f \in C_B[0, \infty) \)

(i) \( \lim_{\delta \to \infty} w(f; \delta)_{\rho_0} = 0 \)

(ii) \( |f(t) - f(x)| \leq w(f; \delta)_{\rho_0} \left( \frac{|t - x|}{\delta} + 1 \right) \)  \hspace{1cm} (2.4.1)

**Theorem 2.4.1.** Let the sequence \( q = (q_n) \) satisfies the condition in (2.3.1) and \( 0 < q_n < 1 \). Then we have

\[ |L_n^*(f; q_n; x) - f(x)| \leq w(f; \sqrt{\delta_n(x)})_{\rho_0} (1 + q_n), \]

where

\[ \delta_n(x) = \|e_2\|_{\rho_0} q_n \left( \frac{[n]_q}{[n]_q - 1} - \frac{q_n [n]_q}{[n]_q - 1} + \frac{q_n}{[n]_q - 1} \right) + \|e_1\|_{\rho_0} \frac{q_n}{[n]_q - 1}. \]  \hspace{1cm} (2.4.2)

**Theorem 2.4.2.** If \( L_n^* \) be defined by (2.2.1), then for all \( f \in \mathcal{W}_{\alpha,B} \)

\[ |L_n^*(f; q_n; x) - f(x)| \leq M(\eta_n^{2+q} q_n^{2+q} + q_n d(x, E)), \]  \hspace{1cm} (2.4.3)

where

\[ \eta_n = \|e_2\|_{\rho_0} \left( \frac{[n]_q}{[n]_q - 1} - \frac{q_n [n]_q}{[n]_q - 1} + \frac{q_n}{[n]_q - 1} \right) + \|e_1\|_{\rho_0} \frac{1}{[n]_q - 1}. \]  \hspace{1cm} (2.4.4)

### 2.5 Concluding remarks

(i) Note that

\[ st - \lim_{n \to \infty} \delta_n = 0. \]

By (2.4.1) we have

\[ st - \lim_{n \to \infty} w(f; \delta_n)_{\rho_0} = 0, \]

which gives us the pointwise rate of statistical convergence of the operator \( L_n^*(f; q_n; x) \) to \( f(x) \).

From the definition of the \( q \)-calculus, it can be proved that

\[ \sup_{x \geq 0} \delta_n(x) \leq q_n \left( \frac{[n]_q}{[n]_q - 1} - \frac{q_n [n]_q}{[n]_q - 1} + \frac{q_n}{[n]_q - 1} \right). \]
In classical case for \( q = 1 \), we have

\[
\sup_{x} \delta_n(x) \leq \frac{1}{n-1} = O\left(\frac{1}{n}\right)
\]

Thus, for every choice of \( q_n \), the rate of convergence of (2.2.1) to the function \( f \) is better than the Stancu-Beta operators.

(ii) If we take \( E = [0, \infty) \) in Theorem 2.4.2, since \( d(x, E) = 0 \), then we obtain the following result:

For every \( f \in \tilde{W}_{\alpha,[0,\infty)} \)

\[
|L^*_n(f; q_n; x) - f(x)| \leq M \eta_n^{\frac{q}{n} \frac{2+q}{3}}
\]

where \( \eta_n \) is defined as in (2.4.4)

(iii) It is easy to verify that

\[
st \lim_{n \to \infty} \eta_n = 0.
\]

That is, the rate of statistical convergence of operators (2.2.1) to the function \( f \) are estimated by means of Lipschitz type maximal functions.
Chapter 3

Generalized $q$-Bernstein-Schurer operators and some approximation theorems

3.1 Introduction

In this chapter, we study statistical approximation properties of $q$-Bernstein-Schurer operators. We establish some direct theorems. We compute error estimation by using modulus of continuity with the help of Matlab and give its algorithm. Furthermore, we show graphically the convergence of the $q$-Bernstein-Schurer operators to various functions.

Muraru [34] introduced the following operators known as the generalized $q$-Bernstein-Schurer operators. For any $m \in \mathbb{N}$, $p$ a fixed positive integer and $f \in C[0,p+1],$

\[
L^*_m(f; q; x) = \sum_{k=0}^{m+p} \binom{m+p}{k} x^k \prod_{i=0}^{m+p-k-1} (1-q^i x) f \left( \frac{[k]_q}{[m]_q} \right), \quad x \in [0,1]
\]

We note the following properties as in [34] for $L^*_m(f; q; x)$.

Lemma 3.1.1. For $x \in [0,1], 0 < q \leq 1$

\[
L^*_m(1; q; x) = 1.
\]

Lemma 3.1.2. For $x \in [0,1], 0 < q \leq 1$

\[
L^*_m(t; q; x) = x \frac{[m+p]_q}{[m]_q}.
\]

\(^a\)The contents of this chapter has been accepted for publication in Journal of Function Spaces and Applications.
Lemma 3.1.3. For $x \in [0, 1]$, $0 < q \leq 1$

$$L_{m,p}^*(t^2; q; x) = \frac{[m+p]_q}{[m]_q^2} ([m+q]_q x^2 + x(1-x)).$$

Lemma 3.1.4. Let $q \in (0, 1)$. Then for $x \in [0, 1]$

$$L_{m,p}^*(t-x; q; x) = x \left( \frac{[m+p]_q}{[m]_q} - 1 \right).$$

Lemma 3.1.5. Let $q \in (0, 1)$. Then for $x \in [0, 1]$

$$L_{m,p}^*((t-x)^2; q; x) = x^2 \left( \frac{([m+p]_q - [m]_q)^2 - [m+p]_q}{[m]_q^2} \right) + x \frac{[m+p]_q}{[m]_q^2}.$$

3.2 Statistical approximation

Note that every convergent sequence is statistically convergent but not conversely, even unbounded sequence may be statistically convergent. For example, let $u = (u_m)$ be defined by

$$u_m = \begin{cases} 
1, & \text{if } k = m^2 \\
0, & \text{otherwise.} 
\end{cases}$$ \hspace{1cm} (3.2.1)

Then $st\text{-}\lim u_m = 0$ but $u$ is not convergent.

Let $C_{B}[0, p+1]$ be the space of all bounded and continuous functions on $[0, p+1]$. Then $C_{B}[0, p+1]$ is a normed linear space with $\|f\| = \sup_{x \geq 0} |f(x)|$. Let $w$ be a function of the type of modulus of continuity. The principal properties of the function are the following:

(i) $w$ is a non-negative increasing function on $[0, p+1]$,

(ii) $\lim_{\delta \to 0} w(\delta) = 0$.

Let $C_{b}[0, p+1]$ be the space of all-real valued functions $f$ defined on $[0, p+1]$ satisfying the following condition

$$|f(x) - f(y)| \leq w(|x - y|)$$
for any \( x, y \in [0, p + 1] \).

We consider a sequence \( q = (q_n), q_n \in (0, 1) \), such that

\[
\lim_{n \to \infty} q_n = 1. \tag{3.2.2}
\]

The condition (3.2.2) guarantees that \([n]q_n \to \infty\) as \( n \to \infty \).

Now our first result is as follows:

**Theorem 3.2.1.** Let \((L^{s}_{m,p})\) be the sequence of the operators (3.1.1) and the sequence \( q = (q_n) \) satisfies (3.2.2). Then for any function \( f \in C_B[0, p + 1] \),

\[
st - \lim_{m \to \infty} ||L^{s}_{m,p}(f; q_n; x) - f|| = 0.
\]

**Remark 3.2.1.** In the following example, we demonstrate that the statistical version is stronger than the ordinary approximation. Let us write \( T^{s}_{m,p}(f; q_n; x) = (1 + u_m)L^{s}_{m,p}(f; q_n; x) \) where the sequence \((u_m)\) is defined by (3.2.1). Then under the hypothesis of the above theorem, we have

\[
st - \lim_{m \to \infty} ||T^{s}_{m,p}(f; q_n; x) - f|| = 0.
\]

However, \( \lim_{m \to \infty} ||T^{s}_{m,p}(f; q_n; x) - f|| \) does not exist, since \((u_m)\) is statistically convergent but not convergent.

### 3.3 Direct Theorems

The Peetre's \( K \)-functional is defined by

\[
K_2(f, \delta) = \inf\{||f - g|| + \delta||g''|| : g \in W_2^p\},
\]

where

\[
W_2^p = \{g \in C_B[0, p + 1] : g', g'' \in C_B[0, p + 1]\}.
\]

By [19], there exists a positive constant \( C > 0 \) such that \( K_2(f, \delta) \leq Cw_2(f, \delta^{\frac{1}{2}}), \delta > 0 \); where the second order modulus of continuity is given by

\[
w_2(f, \delta^{\frac{1}{2}}) = \sup_{0 < h \leq \delta^{\frac{1}{2}}} \sup_{x \in [0, p + 1]} |f(x + 2h) - 2f(x + h) + f(x)|.
\]
Also for \( f \in C_B[0,p+1] \) the usual modulus of continuity is given by

\[
w(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0,p+1]} |f(x + h) - f(x)|.
\]

**Theorem 3.3.1.** Let \( f \in C_B[0,p+1] \) and \( 0 < q_m < 1 \) such that \( q_m \to 1 \) (\( m \to \infty \)). Then for all \( n \in \mathbb{N} \) and \( p \) fixed, there exists an absolute constant \( C > 0 \) such that

\[
|L_{m,p}^*(f; q_m; x) - f(x)| \leq Cw_2(f, \delta_m(x)),
\]

where

\[
\delta_m^2(x) = x^2 \left( \frac{([m+p]_q - [m]_q^2) - [m+p]_q}{[m]_q^2} \right) + x \frac{[m+p]_q}{[m]_q^2}.
\]

**Theorem 3.3.2.** Let \( f \in C_B^2[0,p+1] \) be such that \( f', f'' \in C_B[0,p+1] \) and the sequence \( (q_m) \) satisfies (3.2.2). Then the following equality holds

\[
\lim_{m \to \infty} [m]_{q_m} (L_{m,p}^*(f; q; x) - f(x)) = \frac{x(1-x)}{2} f''
\]

uniformly on \([0,p+1]\).

### 3.4 Graphical analysis and error bound computation using Matlab

In this section, we compute error estimation [70] by using modulus of continuity with the help of Matlab and give its algorithm. We also show graphically the convergence of the \( q \)-Bernstein-Schurer operators to various functions.

**Example 3.4.1.** Let us take \( f(x) = 1 + \sin(-6.5x^2) \). We compute error estimation by using modulus of continuity for operators 3.1.1 to the function \( f(x) = 1 + \sin(-6.5x^2) \) shown in the following Table.
Error estimation table:

<table>
<thead>
<tr>
<th>m (for p=30, q=0.9)</th>
<th>error bound at x=0.2</th>
<th>error bound at x=0.5</th>
<th>error bound at x=0.</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.2309</td>
<td>1.5709</td>
<td>2.7170</td>
</tr>
<tr>
<td>30</td>
<td>0.1893</td>
<td>1.1865</td>
<td>1.5829</td>
</tr>
<tr>
<td>50</td>
<td>0.1750</td>
<td>1.0835</td>
<td>1.3792</td>
</tr>
</tbody>
</table>

Example 3.4.2. For m = 20, 30, 50; the convergence of operators 3.1.1 to function

\[ f(x) = 1 + \sin(-6.5x^2) \]

is illustrated in figure 3.1.
Example 3.4.3. Similarly, Approximation by generalized $q$-Bernstein-Schurer operators for the function $f = (x - \frac{1}{3})(x - \frac{1}{4})(x - \frac{1}{5})$ for different values of $m$, keeping $q$ and $p$ fixed is shown in figure 3.2.

Figure 3.2:
Example 3.4.4. Approximation by operators 3.1.1 for the function

\[ f(x) = (x - \frac{1}{3})(x - \frac{1}{2})(x - \frac{3}{4}) \]

for different values of 'q', taking 'm'=10 and 'p'=2 is shown in figure 3.3.

![Figure 3.3:](image-url)
Example 3.4.5. Approximation by operators 3.1.1 for the function

\[ f(x) = (x - \frac{1}{3})(x - \frac{1}{2})(x - \frac{3}{4}) \]

for different values of 'q', taking 'm'=50 and 'p'=2 is shown in figure 3.4.
Chapter 4

Operators constructed by means of \( q \)-Lagrange polynomials and \( A \)-Statistical approximation

4.1 Introduction

In this chapter, we construct a new family of operators with the help of \( q \)-analogue of Chan-Chyan-Srivastava polynomials, and study the statistical approximation properties via \( A \)-statistical convergence. We also study some approximation properties for the rate of \( A \)-statistical convergence with the help of modulus of continuity and Lipschitz class.

4.2 Construction of a new operator and its properties

Recently, Altı̈n, Erkus and Taşdelen [8] introduced \( g_{m,q}^{(c_1, \ldots, c_r)}(x_1, \ldots, x_r) \) the \( q \)-analogue generated by

\[
\prod_{i=1}^{r} \frac{1}{(x_i t; q)_{a_i}} = \prod_{i=1}^{r} (1 - t x_i q^{k_i})^{-a_i} = \sum_{m=0}^{\infty} g_{m,q}^{(c_1, \ldots, c_r)}(x_1, \ldots, x_r) t^m
\]

\[
= \sum_{m=0}^{\infty} \left\{ \sum_{k_1+k_2+\ldots+k_r=m} (q^{a_{k_1}}, q)_{k_1} (q^{a_{k_2}}, q)_{k_2} \cdots (q^{a_{k_r}}, q)_{k_r} \frac{x_1^{k_1}}{(q, q)_{k_1}} \cdots \frac{x_r^{k_r}}{(q, q)_{k_r}} \right\} t^m
\]

where

\[ |t| < \min\{|x_1|, |x_2|\}. \]

The contents of this chapter has been published in *Applied Mathematics and Computation.*
We define the following family of positive linear operators on $C[0, 1]$ as:

$$L_{n,q}^{u^{(1)}, \ldots, u^{(r)}}(f; x) = \left\{ \prod_{j=1}^{r}(1 - xu^{(j)}_{n,q}) \right\} \sum_{m=0}^{\infty} \sum_{k_1+k_2+\ldots+k_r=m} (q^{a_1}, q)_{k_1} \cdots (q^{a_r}, q)_{k_r}$$

$$\times f\left(\frac{(q, q)_{k_r}}{n + (q, q)_{k_r} - 1}\right) \left(\frac{u^{(1)}_{n,k_1}}{(q, q)_{k_1}}\right) \cdots \left(\frac{u^{(r)}_{n,k_r}}{(q, q)_{k_r}}\right)x^m. \quad (4.2.2)$$

In this section, we investigate some basic properties of the positive linear operators $L_{n,q}^{u^{(1)}, \ldots, u^{(r)}}(f; x)$ given by (4.2.2) via the concept of $A$-statistical convergence.

We will first consider the case $r = 2$ in (4.2.2). In this case, we have

$$L_{n,q}^{u^{(1)}, u^{(2)}}(f; x) = \left\{ \prod_{j=1}^{2}(1 - xu^{(j)}_{n,q}) \right\} \sum_{m=0}^{\infty} \sum_{k_1+k_2=m} (q^{a_1}, q)_{k_1} (q^{a_2}, q)_{k_2}$$

$$\times f\left(\frac{(q, q)_{k_2}}{n + (q, q)_{k_2} - 1}\right) \left(\frac{u^{(1)}_{n,k_1}}{(q, q)_{k_1}}\right) \left(\frac{u^{(2)}_{n,k_2}}{(q, q)_{k_2}}\right)x^m. \quad (4.2.3)$$

Then we have the following preliminary results.

**Lemma 4.2.1.** For each $x \in [0, 1]$ and $n \in \mathbb{N}$,

$$L_{n,q}^{u^{(1)}, u^{(2)}}(f_0; x) = 1 \quad (f_0(x) = 1).$$

**Lemma 4.2.2.** For each $x \in [0, 1]$ and $n \in \mathbb{N}$,

$$L_{n,q}^{u^{(1)}, u^{(2)}}(f_1; x) = xu^{(2)}_{n} \quad (f_1(x) = x).$$

**Lemma 4.2.3.** For each $x \in [0, 1]$ and $n \in \mathbb{N}$,

$$|L_{n,q}^{u^{(1)}, u^{(2)}}(f_2; x)| \leq 2x^2 - 1 + \frac{xu^2_{n}}{n} \quad (f_2(x) = x^2).$$

### 4.3 A-statistical approximation

We know that $C[a, b]$ is a Banach space with norm

$$\|f\|_{C[a,b]} = \sup_{x \in [a, b]} |f(x)|, \quad f \in C[a, b].$$

For typographical convenience, we will write $\|\|$ in place of $\|\|_{C[a,b]}$ if no confusion arises.
Theorem 4.3.1. Let $A = (a_{jn})$ be a non-negative regular summability matrix. Then
\[
st_A - \lim_n u_n^{(2)} = 1, \tag{4.3.1}
\]
if and only if, for all $f \in C[0,1]$
\[
st_A - \lim_n \left\| L_{n,q}^{u^{(1)}} u^{(2)}(f) - f \right\| = 0. \tag{4.3.2}
\]

Theorem 4.3.2. Let $A = (a_{jn})$ be a non-negative regular summability matrix. Then
\[
st_A - \lim_n u_n^{(r)} = 1, \tag{4.3.3}
\]
if and only if, for all $f \in C[0,1]$
\[
st_A - \lim_n \left\| L_{n,q}^{u^{(1)}u^{(2)}...u^{(r)}}(f) - f \right\| = 0.
\]

Remark 4.3.1. If, in Theorem 4.3.2, we replace $A = (a_{jn})$ by the identity matrix, we immediately get the following theorem which is a classical case of Theorem 4.3.2.

Theorem 4.3.3. $\lim_n u_n^{(r)} = 1$, if and only if, for all $f \in C[0,1]$, the sequence $L_{n,q}^{u^{(1)}u^{(2)}...u^{(r)}}(f)$ is uniformly convergent to $f$ on $[0,1]$.

4.4 Rate of $A$-statistical convergence

In this section, we compute the rates of $A$-statistical convergence of our sequence of positive linear operators in Theorem 4.3.2 by means of the modulus of continuity and the elements of Lipschitz class. Let $f \in C[0,1]$. The modulus of continuity of $f$ denoted by $w(f, \delta)$ is defined by
\[
w(f, \delta) = \sup_{\|y-x\|<\delta, x,y \in [0,1]} | f(y) - f(x) |.
\]
It is well known that the modulus of continuity of the function $f \in C[0,1]$ provides the maximum oscillation of $f$ in any interval of length not exceeding $\varepsilon > 0$. A necessary and sufficient condition for a function $f \in C[0,1]$ is that
\[
\lim_{\delta \to 0} w(f, \delta) = 0.
\]
It is also well known that, for any $\delta > 0$ and for each $x, y \in [0,1]$,
\[ |f(y) - f(x)| \leq \left( \frac{|y-x|}{\delta} + 1 \right) w(f, \delta). \] (4.4.1)

Now we prove the following result.

**Theorem 4.4.1.** For each \( n \in \mathbb{N} \) and for all \( f \in C[0, 1] \),
\[ \| L_n^{u_1, \ldots, u_{r-1}}(f) - f \| \leq 2w(f, \delta_n), \] (4.4.2)
where
\[ \delta_n = \left\{ \frac{u_n^{(r)}}{n} + 4(1 - u_n^{(r)}) \right\}. \] (4.4.3)

We now turn to our investigation of the rate of convergence of this positive linear operators \( L_n^{u_1, \ldots, u_{r-1}}(f; x) \) by means of elements of Lipschitz class \( \text{Lip}_M(\alpha) \) \((0 < \alpha \leq 1)\). Recall that a function \( f \in C[0, 1] \) belongs to the Lipschitz class \( \text{Lip}_M(\alpha) \) \((0 < \alpha \leq 1)\) if the following inequality holds:
\[ |f(y) - f(x)| \leq M |y - x|^\alpha \quad (x, y \in [0, 1]). \] (4.4.4)

**Theorem 4.4.2.** For each \( n \in \mathbb{N} \) and for all \( f \in \text{Lip}_M(\alpha) \),
\[ |L_n^{u_1, \ldots, u_{r-1}}(f) - f| \leq 2\delta_n^\alpha, \] (4.4.5)
where \( \delta_n \) is same as in Theorem 4.4.1.

### 4.5 Concluding remark

**Remark 4.5.1.** For a non-negative summability matrix \( A = (a_{jn}) \), if we take \( s_A - \lim n a_n^{(r)} = 1 \) then \( s_A - \lim n \delta_n = 0 \). It also implies that \( s_A - \lim n w(f, \delta_n) = 0 \), \((f \in C[0, 1]). \) Thus clearly Theorems 4.4.1 and 4.4.2 give us the rate of A-statistical convergence in Theorem 4.3.2. If we take the matrix \( A = I \), the identity matrix, then we get the corresponding rates of ordinary convergence.
Chapter 5

Generalized Equi-Statistical Convergence of Positive Linear Operators and Associated Approximation Theorems

5.1 Introduction

The concepts of equi-statistical convergence, statistical pointwise convergence and statistical uniform convergence for sequences of functions were introduced recently by M. Balcerzak, K. Dems and A. Komisarski [Statistical convergence and ideal convergence for sequences of functions, J. Math. Anal. Appl. 328 (2007), 715–729]. In this chapter, we use the notion of A-statistical convergence in order to generalize these concepts. We establish some inclusion relations between them. We apply our new notion of A-equistechnical convergence to prove a Korovkin type approximation theorem and we show that our theorem is a non-trivial extension of some well-known Korovkin type approximation theorems. Finally, we prove a Voronovskaja type approximation theorem via the concept of A-equistechnical convergence. Some interesting examples are also displayed here in support of our definitions and results.

The concept of \(\lambda\)-statistical convergence was introduced recently in [51] as follows. Let \(\lambda = \{\lambda_n\}\) be a non-decreasing sequence of positive numbers tending to \(\infty\) such that

\[
\lambda_{n+1} \leq \lambda_n + 1 \quad \text{and} \quad \lambda_1 = 1.
\]

Also let

\[I_n = [n - \lambda_n + 1, n] \quad \text{and} \quad K \subseteq \mathbb{N}.
\]

\(^{0}\)The contents of this chapter has been published in Mathematical and Computer Modelling.
Then the $\lambda$-density of $K$ is defined by

$$\delta_\lambda(K) = \lim_{n \to \infty} \frac{1}{\lambda_n} |\{j : n - \lambda_n + 1 \leq j \leq n \text{ and } j \in K\}|.$$ 

Clearly, in the special case when $\lambda_n = n$, the $\lambda$-density reduces to the above-defined natural density.

The number sequence $x = (x_j)$ is said to be $\lambda$-statistically convergent to the number $L$ if, for each $\epsilon > 0$,

$$\delta_\lambda(K_\epsilon) = 0,$$

where

$$K_\epsilon = \{j : j \in I_n \text{ and } |x_j - L| > \epsilon\},$$

that is, if, for each $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{\lambda_n} |\{j : j \in I_n \text{ and } |x_j - L| > \epsilon\}| = 0.$$

In this case, we write

$$\text{st}_\lambda- \lim_{n \to \infty} x_n = L$$

and we denote the set of all $\lambda$-statistically convergent sequences by $S_\lambda$.

Example 5.1.1. We know that the total number of numbers from 1 to $n$ that are divisible by 2 is $[n/2]$, where $[v]$ is the greatest integer in $v \in \mathbb{R}$. Similarly, the total number of numbers from 1 to $n$ that are divisible by 3 is $[n/3]$. In general, the total number of numbers from 1 to $n$ that are divisible by $k$ is $[n/k] (k \in \mathbb{N})$. Now define a sequence $x = (x_n)$ by

$$x_n = \begin{cases} 
\frac{1}{2} & \text{ (n is divisible by } k := \sqrt{n}) \\
0 & \text{ (otherwise)}. 
\end{cases}$$

Then the total number of numbers for which $x_n = \frac{1}{2}$ will be

$$\left[\frac{n}{\sqrt{n}}\right] = \left[\frac{n}{\sqrt{n}}\right] \leq \sqrt{n},$$

it will be zero otherwise. We thus find that

$$|\{k : k \leq n \text{ and } x_k \neq 0\}| \leq \sqrt{n},$$

which implies that

$$\text{st}_\lambda- \lim_{n \to \infty} x_n = 0.$$
Example 5.1.2. If \( n = m^2 \), then there are exactly \( 2k + 1 \) divisors, out of which exactly \( k \) are less than \( m \). We take \( n = 36 = 6^2 \), then total number of such divisors will be nine \((1, 2, 3, 4, 6, 9, 12, 18, 36)\), out of which four are exactly less than 6. We can thus define a sequence \( x = (x_n) \) by

\[
x_n = \begin{cases} 
\frac{1}{k} & (k < n = m^2) \\
\frac{e^{-\frac{1}{n}}}{n} & \text{(otherwise)}, 
\end{cases}
\]

where \( k \) is the number of divisors. Clearly, we have

\[
\text{st}_{A}\lim_{n \to \infty} x_n = 0.
\]

The concept of equi-statistical convergence was introduced by Balcerzak et al. [52] and was subsequently applied for deriving approximation theorems in [1], [53], [54] and [55] (see also the closely-related recent works [7], [38] and [30]). In this chapter, we introduce the concept of \( \lambda \)-equi-statistical convergence, \( \lambda \)-statistical pointwise convergence and \( \lambda \)-statistical uniform convergence for a sequence of real-valued functions and show that the \( \lambda \)-equi-statistical convergence lies between the \( \lambda \)-statistical pointwise and the \( \lambda \)-statistical uniform convergence. Inclusion relation between equi-statistical and \( \lambda \)-equi-statistical convergence is established and it is proved that, under some conditions, the \( \lambda \)-equi-statistical convergence and the equi-statistical convergence are equivalent to each other. We apply our new notion of \( \lambda \)-equi-statistical convergence to prove a Korovkin type approximation theorem. We also prove a Voronovskaja type approximation theorem via \( \lambda \)-equi-statistical convergence. Finally, we study the rate of \( \lambda \)-equi-statistical convergence of a sequence of positive linear operators defined on \( C(X) \) (cf. [59]).

5.2 \( \lambda \)-Equi-Statistical Convergence

We define the following concepts by using \( \lambda \)-statistical convergence. Let \( f \) and \( f_n \) \((n \in \mathbb{N})\) be real-valued functions defined on a subset \( X \) of the set \( \mathbb{N} \) of positive integers.
Definition 5.2.1. A sequence \((f_n)\) of real-valued functions is said to be \(\lambda\)-equi-statistically convergent to \(f\) on \(X\) if, for every \(\epsilon > 0\), the sequence \((S_n(\epsilon, x))_{n \in \mathbb{N}}\) of real-valued functions converges uniformly to the zero function on \(X\), that is, if, for every \(\epsilon > 0\), we have
\[
\lim_{n \to \infty} \|S_n(\epsilon, x)\|_{C(X)} = 0,
\]
where
\[
S_n(\epsilon, x) := \frac{1}{\lambda_n} \left| \left\{ k : k \in I_n \text{ and } |f_k(x) - f(x)| \geq \epsilon \right\} \right| = 0,
\]
and \(C(X)\) denotes the space of all continuous functions on \(X\). In this case, we write \(f_n \rightarrow f\) (\(\lambda\)-equi-stat).

Definition 5.2.2. A sequence \((f_n)\) is said to be \(\lambda\)-statistically pointwise convergent to \(f\) on \(X\) if, for every \(\epsilon > 0\) and for each \(x \in X\), we have
\[
\lim_{n \to \infty} \frac{1}{\lambda_n} \left| \left\{ k : k \in I_n \text{ and } |f_k(x) - f(x)| \geq \epsilon \right\} \right| = 0.
\]
In this case, we write \(f_n \rightarrow f\) (\(\lambda\)-stat).

Definition 5.2.3. A sequence \((f_n)\) is said to be \(\lambda\)-statistically uniform convergent to \(f\) on \(X\) if (for every \(\epsilon > 0\)), we have
\[
\lim_{n \to \infty} \frac{1}{\lambda_n} \left| \left\{ k : k \in I_n \text{ and } \|f_k - f\|_{C(X)} \geq \epsilon \right\} \right| = 0.
\]
In this case, we write \(f_n \rightarrow f\) (\(\lambda\)-stat).

Definition 5.2.4. (see [55]). A sequence \((f_n)\) of real-valued functions is said to be equi-statistically convergent to \(f\) on \(X\) if, for every \(\epsilon > 0\), the sequence \((P_{n, \epsilon}(x))_{n \in \mathbb{N}}\) of real-valued functions converges uniformly to the zero function on \(X\), that is, if (for every \(\epsilon > 0\)) we have
\[
\lim_{n \to \infty} \|P_{n, \epsilon}(x)\|_{C(X)} = 0,
\]
where
\[
P_{n, \epsilon}(x) = \frac{1}{n} \left| \left\{ k : k \leq n \text{ and } |f_k(x) - f(x)| \geq \epsilon \right\} \right| = 0.
\]
In this case, we write \(f_n \rightarrow f\) (equi-stat).

The following implications of the above definitions and concepts are trivial.
Lemma 5.2.1. Each of the following implications holds true:

\[ f_n \xrightarrow{\lambda-stat} f \xrightarrow{\lambda-equi-stat} f_n \prec f \xrightarrow{\lambda-equi-stat} f_n \rightarrow f \xrightarrow{\lambda-stat}. \]

Furthermore, in general, the reverse implications do not hold true.

Lemma 5.2.2. Let \( \hat{S} \) and \( \hat{S}_\lambda \) be the sets of all equi-statistically convergent and \( \lambda \)-equi-statistically convergent sequences, respectively. Then

\[ \hat{S} \subseteq \hat{S}_\lambda \iff \liminf_{n \to \infty} \frac{\lambda_n}{n} > 0. \]

(5.2.1)

Remark 5.2.1. Obviously, since

\[ \frac{\lambda_n}{n} \leq 1 \quad (n \in \mathbb{N}), \]

we have \( \hat{S}_\lambda \subseteq \hat{S} \).

5.3 Korovkin Type Approximation Theorem

In this section, we extend the result of Karakus et al. [55] by using the notion of \( \lambda \)-equi-statistical convergence.

Theorem 5.3.1. Let \( X \) be a compact subset of the set \( \mathbb{R} \) of real numbers. Also let \( \{L_n\} \) be a sequence of positive linear operators from \( C(X) \) into itself. Then, for all \( f \in C(X) \),

\[ L_n(f) \xrightarrow{\lambda-equi-stat} f \] on \( X \)

\[ \iff L_n(e_i) \xrightarrow{\lambda-equi-stat} e_i \] on \( X \),

(5.3.1)

with

\[ e_i(x) = x^i \quad (i = 0, 1, 2). \]

(5.3.2)

Example 5.3.1. Let \( X = [0, 1] \) and consider the operator \( x(1 + xD) \) \( (D = \frac{d}{dx}) \), which was used by (for example) Al-Salam [56] (see also a more recent investigation by Viskov and Srivastava [58, p. 9, Equation (53)]). Here we use this operator over the classical Bernstein polynomials \( B_n(f; x) \) on \( C[0, 1] \) and introduce the following family of linear operators on \( C[0, 1] \):

\[ \theta_n(f; x) = [1 + f_n(x)]x(1 + xD)B_n(f; x) \]

(5.3.3)

\[ \left( x \in [0, 1]; f \in C[0, 1]; D = \frac{d}{dx} \right), \]

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where \( \{f_n(x)\} \) is a sequence of functions. Then we have
\[
\theta_n(e_0; x) = (1 + f_n(x))x(1 + xD)e_0(x) = (1 + f_n(x))x,
\]
\[
\theta_n(e_1; x) = (1 + f_n(x))x(1 + xD)e_1(x) = (1 + f_n(x))e_1(x)(1 + e_1(x))
\]
and
\[
\theta_n(e_2; x) = (1 + f_n(x))x(1 + xD)\left\{e_2(x) + \frac{x(1-x)}{n}\right\}
\]
\[
= (1 + f_n(x))\left\{e_2(x) \left(2 - \frac{3e_1(x)}{n}\right)\right\}.
\]
Since
\[
f_n \rightsquigarrow f = 0 \text{ (\(\lambda\)-equi-stat)}
\]
on \([0, 1]\) for \(f_n\), we conclude that
\[
\theta_n(e_i) \rightsquigarrow (e_i) \text{ (\(\lambda\)-equi-stat)}
\]
on \([0, 1]\) for each \(i = 0, 1, 2\). Therefore, by Theorem 5.3.1, we see that
\[
\theta_n(f) \rightsquigarrow (f) \text{ (\(\lambda\)-equi-stat)}
\]
on \([0, 1]\) for all \(f \in C[0,1]\). However, since \(f_n\) is not statistically uniformly convergent to the function \(f = 0\) on \([0, 1]\), we conclude that Theorem 2.1 of the earlier work [55] does not work for the operators defined by (5.3.3). Furthermore, since \(f_n\) is not uniformly convergent (in the ordinary sense) to the function \(f = 0\) on \([0, 1]\), the classical theorem also does not work here.

Remark 5.3.1. In connection with the operator \(x(1+xD) \quad (D = \frac{d}{dx})\) used in (5.3.3), it may be of interest to observe that such much more general families of operators as the two-parameter operator \(\Theta_{x,\kappa,\eta}\) defined by
\[
\Theta_{x,\kappa,\eta} := x^\kappa(xD + \eta) \quad \left(D = \frac{d}{dx}\right) \quad (5.3.4)
\]
has appeared in the literature rather extensively (see, for details, [61, p. 447, Problem 16]; see also the recent works [17], [60] and [58]).

5.4 A Voronovskaja Type Theorem

In this section, we will show that the positive linear operators \(\theta_n\) defined by (5.3.3) satisfy a Voronovskaja type property in the \(\lambda\)-equi-statistical sense. We first present the following lemma.
Lemma 5.4.1. Let \( x \in [0,1] \) and \( \phi(y) := y - x \). Then
\[
 n^2 \theta_n(\phi^4) \overset{\text{e}}{\to} 3e_2e_3(e_2 - 8e_1 + 6e_0) \quad (\lambda\text{-equi-stat}) \quad \text{on } [0,1].
\]

We establish the following Voronovskaja type result for the operators \( \theta_n \) given by (5.3.3).

Theorem 5.4.1. For every \( f \in C[0,1] \) such that \( f', f'' \in C[0,1] \),
\[
 n\{\theta_n f - f\} \overset{\text{e}}{\to} \frac{e_1(1 - 2e_2)}{2} f'' \quad (\lambda\text{-equi-stat})
\]
on \([0,1]\).

Remark 5.4.1. Since the function sequence \( \{f_n\} \) is not uniformly convergent to the function \( f = 0 \) on the interval \([0,1]\), we observe that our operator \( \theta_n \) defined by (5.3.3) does not satisfy Voronovskaja type property in the usual sense.

5.5 Rate of the \( \lambda \)-Equi-Statistical Convergence

In this section, we study the rate of the \( \lambda \)-equi-statistical convergence of a sequence of positive linear operators defined on \( C(X) \). We begin by presenting the following definition.

Definition 5.5.1. Let \( (a_n) \) be a positive non-increasing sequence. A sequence \( (f_n) \) is equi-statistically convergent to a function \( f \) with the rate \( o(a_n) \) if, for every \( \epsilon > 0 \),
\[
 \lim_{n \to \infty} \frac{A_n(x, \epsilon)}{a_n} = 0
\]
uniformly with respect to \( x \in X \) or, equivalently, if (for every \( \epsilon > 0 \)),
\[
 \lim_{n \to \infty} \frac{\|A_n(\cdot, \epsilon)\|_{C(X)}}{a_n} = 0,
\]
where
\[
 A_n(x, \epsilon) := \frac{1}{\lambda_n} \{k : k \in I_n \quad \text{and} \quad |f_k(x) - f(x)| \leq \epsilon\}. \]
In this case, it is denoted as follows:
\[
 f_n - f = o(a_n) \quad (\lambda\text{-equi-stat}) \quad \text{on } X.
\]

We now prove the following basic lemma.
Lemma 5.5.1. Let \((f_n)\) and \((g_n)\) be sequences of functions belonging to \(C(X)\). Suppose also that
\[ f_n - f = o(a_n) \ (\lambda\text{-equi-stat}) \ \text{on} \ X \]
and
\[ g_n - g = o(b_n) \ (\lambda\text{-equi-stat}) \ \text{on} \ X. \]
If we let
\[ c_n = \max\{a_n, b_n\}, \]
then each of the following statements holds true:
(i) \((f_n + g_n) - (f + g) = o(c_n) \ (\lambda\text{-equi-stat}) \ \text{on} \ X;\)
(ii) \((f_n - f)(g_n - g) = o(a_nb_n) \ (\lambda\text{-equi-stat}) \ \text{on} \ X;\)
(iii) \(\mu(f_n - f) = o(a_n) \ (\lambda\text{-equi-stat}) \ \text{on} \ X \quad (\mu \in \mathbb{R});\)
(iv) \(\sqrt{|f_n - f|} = o(a_n) \ (\lambda\text{-equi-stat}) \ \text{on} \ X.\)

We next recall that the modulous of continuity of a function \(f \in C(X)\) is defined by
\[ w(f, \delta) = \sup_{|y-x| \leq \delta; x,y \in X} |f(y) - f(x)| \quad (\delta > 0). \]

We now state the following result.

Theorem 5.5.1. Let \(X\) be a compact subset of the real numbers. Also let \(L_n\) be a sequence of positive linear operators acting from \(C(X)\) into itself. Assume that each of the following conditions holds true:
(a) \(L_n(e_0) - (e_0) = o(a_n) \ (\lambda\text{-equi-stat}) \ \text{on} \ X;\)
(b) \(w(f, \delta_n) = o(b_n) \ (\lambda\text{-equi-stat}) \ \text{on} \ X; \) where
\[ \delta_n(x) = \sqrt{L_n(\phi^2; x)} \] and \(\phi(y) = y - x.\)

Then, for all \(f \in C(X),\) the following statement holds true:
\[ L_n(f) - f = o(c_n) \ (\lambda\text{-equi-stat}) \ \text{on} \ X, \quad (5.5.1) \]
where
\[ c_n = \max\{a_n, b_n\}. \]

5.6 Concluding Remarks and Observations

In this concluding section of our investigation, we present several further remarks and observations concerning the various results which we have proved here.
Remark 5.6.1. Let \((f_r)_{r \in \mathbb{N}}\) be the sequence of functions. Then, since
\[
f_r \sim 0 \quad (\lambda\text{-equi-stat}),
\]
we conclude that
\[
\theta_n(e_i) \sim e_i \quad (\lambda\text{-equi-stat}) \quad (i = 0, 1, 2).
\] (5.6.1)
Therefore, by Theorem 5.3.1, we can see that
\[
\theta_n(f; x) \sim f \quad (\lambda\text{-equi-stat})
\] (5.6.2)
on \([0, 1]\) for all \(f \in [0, 1]\). However, since \(f_r\) is not uniformly \(\lambda\)-equi-statistically convergent and not uniformly convergent (in the ordinary sense) to the function \(f = 0\), the classical Korovkin theorem does not work for the operators defined by (5.3.3). Consequently, Theorem 5.3.1 is a non-trivial extension of the classical Korovkin Theorem.

Remark 5.6.2. Let \((f_r)_{r \in \mathbb{N}}\) be the sequence of functions given by (5.2.1) with \(X = [0, 1]\) and let \(\lambda_n = [\sqrt{n}]\). Since
\[
f_r \sim 0 \quad (\lambda\text{-equi-stat}),
\]
we have (5.6.1). By applying (5.6.1) and Theorem 5.3.1, we have (5.6.2). But, since \(f_r\) does not converge to \(f = 0\) (equi-stat), Theorem 2.1 of Karakuş et al. [55] does not work. Therefore, Theorem 5.3.1 is also a non-trivial extension of Theorem 2.1 of Karakuş et al.

Remark 5.6.3. Suppose that we replace the conditions (a) and (b) in Theorem 5.5.1 by the following condition:
\[
L_n(e_i) - e_i = o(a_{n_0}) \quad (\lambda\text{-equi-stat}) \quad \text{on} \quad X \quad (i = 0, 1, 2).
\] (5.6.3)
Then, since
\[
L_n(\phi^2; x) = L_n(e_2; x) - 2xL_n(e_1; x) + x^2L_n(e_0; x),
\]
we may write
\[
L_n(\phi^2; x) \leq K \sum_{i=0}^{2} |(L_n(e_i; x) - e_i(x))|,
\] (5.6.4)
where
\[
K = 1 + 2\|e_1\|_c(x) + \|e_2\|_c(x).
\]
Now it follows from (5.6.3), (5.6.4) and Lemma 5.5.1 that
\[
\delta_n = \sqrt{L_n(\phi^2)} = o(d_n) \quad (\lambda\text{-equi-stat}) \quad \text{on} \quad X,
\] (5.6.5)
where
\[
d_n = \max \{a_{n_0}, a_{n_1}, a_{n_2}\}.
\]
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Hence
\[ w(f, \delta_n) = o(d_n) \text{ (} \lambda\text{-equi-stat} \text{)} \text{ on } X. \]

Using (5.6.5) in Theorem 5.5.1, we can thus see that, for all
\[ f \in C(X), L_n(f) - f = o(d_n) \text{ (} \lambda\text{-equi-stat} \text{)} \text{ on } X. \] (5.6.6)

Therefore, if we use the condition (5.6.3) in Theorem 5.5.1 instead of the conditions (a) and (b), then we obtain the rates of \( \lambda\)-equi-statistical convergence of the sequence of positive linear operators in Theorem 5.3.1.
Chapter 6

Approximation for periodic functions via weighted statistical convergence

6.1 Introduction

Korovkin type approximation theorems are useful tools to check whether a given sequence \((L_n)_{n \geq 1}\) of positive linear operators on \(C[0, 1]\) of all continuous functions on the real interval \([0, 1]\) is an approximation process. That is, these theorems exhibit a variety of test functions which assure that the approximation property holds on the whole space if it holds for them. Such a property was discovered by Korovkin in 1953 for the functions 1, \(x\) and \(x^2\) in the space \(C[0, 1]\) as well as for the functions 1, \(\cos\) and \(\sin\) in the space of all continuous \(2\pi\)-periodic functions on the real line. In this chapter, we use the notion of weighted statistical convergence to prove the Korovkin approximation theorem for the functions 1, \(\cos\) and \(\sin\) in the space of all continuous \(2\pi\)-periodic functions on the real line and show that our result is stronger. We also study the rate of weighted statistical convergence.

Let \(p = (p_k)_{k=0}^{\infty}\) be a sequence of non-negative numbers such that \(p_0 > 0\) and \(P_n = \sum_{k=0}^{n} p_k \rightarrow \infty\) as \(n \rightarrow \infty\). Set \(t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k x_k\), \(n = 0, 1, 2, \ldots\).

We define the weighted density of \(K\) by \(\delta_N(K) = \lim_n \frac{1}{P_n} |K_n|\) if the limit exists. We say that the sequence \(x = (x_k)\) is weighted statistically convergent (or \(S_N\)-convergent) to \(L\) if for every \(\epsilon > 0\), the set \(\{k \in N : p_k |x_k - L| \geq \epsilon\}\) has weighted density zero, i.e.

\[\lim_n \frac{1}{P_n} \{k \leq P_n : p_k |x_k - L| \geq \epsilon\} = 0.\]

In this case, we write \(L = S_N\)-lim \(x\).

\(^9\)The contents of this chapter has been published in Applied Mathematics and Computation.
Let \( x = (x_k) \) be a sequence defined by

\[
x_k = \begin{cases}
\sqrt{k} & \text{if } k = n^2, \ n \in \mathbb{N}, \\
0 & \text{if } k \neq n^2.
\end{cases}
\]  

That is \((x_k) = (1, 0, 0, 2, 0, 0, 0, 3, 0, ..., 0, 4, 0, 0, ...)\). Let \( p_k = k \). Then \( p_k x_k = (1, 0, 0, 8, 0, 0, 0, 27, 0, ..., 0, 64, 0, 0, ...)\). Since

\[
\lim_n \frac{1}{P_n} |\{k \leq P_n : p_k |x_k - 0| \geq \varepsilon\}| = 0 \leq \lim_n \frac{1}{P_n} \sqrt{P_n} \rightarrow 0,
\]

\((x_k)\) is weighted statistical convergent to 0 but not convergent.

Let \( F(\mathbb{R}) \) denote the linear space of all real-valued functions defined on \( \mathbb{R} \). Let \( C(\mathbb{R}) \) be the space of all functions \( f \) continuous and bounded on \( \mathbb{R} \) with the norm

\[
\|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)|, \ f \in C(\mathbb{R}).
\]

We denote by \( C_{2\pi}(\mathbb{R}) \) the space of all \( 2\pi \)-periodic functions \( f \in C(\mathbb{R}) \) equipped with the norm

\[
\|f\|_{2\pi} = \sup_{t \in \mathbb{R}} |f(t)|.
\]

The classical Korovkin first and second theorems state as follows \[64],[57]:

**Theorem 6.1.1.** Let \((T_n)\) be a sequence of positive linear operators from \( C[0,1] \) into \( F[0,1] \). Then \( \lim_n \|T_n(f,x) - f(x)\|_\infty = 0 \), for all \( f \in C[0,1] \) if and only if \( \lim_n \|T_n(f_i,x) - f_i(x)\|_\infty = 0 \), for \( i = 0, 1, 2 \), where \( f_0(x) = 1, f_1(x) = x \) and \( f_2(x) = x^2 \).

**Theorem 6.1.2.** Let \((T_n)\) be a sequence of positive linear operators from \( C_{2\pi}(\mathbb{R}) \) into \( F(\mathbb{R}) \). Then \( \lim_n \|T_n(f,x) - f(x)\|_\infty = 0 \), for all \( f \in C_{2\pi}(\mathbb{R}) \) if and only if \( \lim_n \|T_n(f_i,x) - f_i(x)\|_\infty = 0 \), for \( i = 0, 1, 2 \), where \( f_0(x) = 1, f_1(x) = \cos x \) and \( f_2(x) = \sin x \).

### 6.2 Main result

We write \( L_n(f; x) \) for \( L_n(f(s); x) \); and we say that \( L \) is a positive operator if \( L(f; x) \geq 0 \) for all \( f(x) \geq 0 \).
Theorem 6.2.1. Let \((T_k)\) be a sequence of positive linear operators from \(C_{2\pi}(\mathbb{R})\) into \(C_{2\pi}(\mathbb{R})\). Then for all \(f \in C_{2\pi}(\mathbb{R})\)

\[
S_N \lim_{k \to \infty} \left\| T_k(f; x) - f(x) \right\|_{2\pi} = 0
\]

if and only if

\[
S_N \lim_{k \to \infty} \left\| T_k(1; x) - 1 \right\|_{2\pi} = 0,
\]

\[
S_N \lim_{k \to \infty} \left\| T_k(\cos t; x) - \cos x \right\|_{2\pi} = 0,
\]

\[
S_N \lim_{k \to \infty} \left\| T_k(\sin t; x) - \sin x \right\|_{2\pi} = 0.
\]

6.3 Rate of weighted statistical convergence

In this section, we study the rate of weighted statistical convergence of a sequence of positive linear operators defined from \(C_{2\pi}(\mathbb{R})\) into \(C_{2\pi}(\mathbb{R})\).

Definition 6.3.1. Let \((a_n)\) be a positive non-increasing sequence. We say that the sequence \(x = (x_k)\) is weighted statistically convergent to the number \(L\) with the rate \(o(a_n)\) if for every \(\varepsilon > 0\),

\[
\lim_{n} \frac{1}{a_n P_n} \left\{ k : P_k |x_k - L| \geq \varepsilon \right\} = 0.
\]

In this case, we write \(x_k - L = S_N-o(a_n)\).

As usual we have the following auxiliary result whose proof is standard.

Lemma 6.3.1. Let \((a_n)\) and \((b_n)\) be two positive non-increasing sequences. Let \(x = (x_k)\) and \(y = (y_k)\) be two sequences such that \(x_k - L_1 = S_N-o(a_n)\) and \(y_k - L_2 = S_N-o(b_n)\). Then

(i) \(\alpha(x_k - L_1) = S_N-o(a_n)\), for any scalar \(\alpha\),

(ii) \((x_k - L_1) \pm (y_k - L_2) = S_N-o(c_n)\),

(iii) \((x_k - L_1)(y_k - L_2) = S_N-o(a_nb_n)\),

where \(c_n = \max\{a_n, b_n\}\).

Now, we recall the notion of modulus of continuity. The modulus of continuity of \(f \in C_{2\pi}(\mathbb{R})\), denoted by \(\omega(f, \delta)\) is defined by

\[
\omega(f, \delta) = \sup_{|x-y|<\delta} |f(x) - f(y)|.
\]
It is well known that
\[
|f(x) - f(y)| \leq \omega(f, \delta)(\frac{|x-y|}{\delta} + 1).
\] (6.3.1)

Then we have the following result.

**Theorem 6.3.1.** Let \((T_k)\) be a sequence of positive linear operators from \(C_{2\pi}(\mathbb{R})\) into \(C_{2\pi}(\mathbb{R})\). Suppose that

(i) \(\|T_k(1; x) - 1\|_{2\pi} = S_N - o(a_n)\),
(ii) \(\omega(f, \lambda_k) = S_N - o(b_n)\), where \(\lambda_k = \sqrt{T_k(\varphi_k; x)}\) and \(\varphi_k(y) = \sin^2(\frac{y-\pi}{2})\).

Then for all \(f \in C_{2\pi}(\mathbb{R})\), we have
\[
||T_k(f; x) - f(x)||_{2\pi} = S_N - o(c_n),
\]
where \(c_n = \max\{a_n, b_n\}\).

### 6.4 Example and the concluding remark

Finally, we construct an example of a sequence of positive linear operators satisfying the conditions of Theorem 6.2.1 but does not satisfy the conditions of Theorem 6.1.2.

For any \(n \in \mathbb{N}\), denote by \(S_n(f)\) the \(n\)-th partial sum of the Fourier series of \(f\), i.e.
\[
S_n(f)(x) = \frac{1}{2}a_0(f) + \sum_{k=1}^{n} a_k(f) \cos kx + b_k(f) \sin kx.
\]

For any \(n \in \mathbb{N}\), write
\[
F_n(f) := \frac{1}{n+1} \sum_{k=0}^{n} S_k(f).
\]

A standard calculation gives that for every \(t \in \mathbb{R}\)
\[
F_n(f; x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1}{n+1} \sum_{k=0}^{n} \frac{\sin((2k+1)(x-t)/2)}{\sin((x-t)/2)} dt
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1}{n+1} \sum_{k=0}^{n} \frac{\sin^2((n+1)(x-t)/2)}{\sin^2((x-t)/2)} dt
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \varphi_n(x-t) dt,
\]

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where
\[ \varphi_n(x) := \begin{cases} \frac{\sin^2((n+1)(x-\theta)/2)}{\sin^2((x-\theta)/2)} & \text{if } x \text{ is not a multiple of } 2\pi, \\ n + 1 & \text{if } x \text{ is a multiple of } 2\pi. \end{cases} \]

The sequence \((\varphi_n)_{n \in \mathbb{N}}\) is a positive kernel which is called the \textit{Fejér kernel}, and the corresponding operators \(F_n, n \geq 1\), are called the \textit{Fejér convolution operators}.

Note that the Theorem 6.1.2 is satisfied for the sequence \((F_n)\). In fact, we have, for every \(f \in C_{2\pi}(\mathbb{R})\),
\[ \lim_{n \to \infty} F_n(f) = f. \]

Let \(L_k : C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})\) be defined by
\[ L_k(f; x) = (1 + x_k)F_k(f; x), \quad (6.4.1) \]
where the sequence \(x = (x_k)\) is defined by (6.1.1). Now
\[ L_n(1; x) = 1, \]
\[ L_n(\cos t; x) = \frac{n}{n + 1} \cos x, \]
\[ L_n(\sin t; x) = \frac{n}{n + 1} \sin x. \]

So that we have
\[ S_{N-} \lim_{n \to \infty} \|L_n(1; x) - 1\|_{2\pi} = 0, \]
\[ S_{N-} \lim_{n \to \infty} \|L_n(\cos t; x) - \cos x\|_{2\pi} = 0, \]
\[ S_{N-} \lim_{n \to \infty} \|L_n(\sin t; x) - \sin x\|_{2\pi} = 0, \]
that is, the sequence \((L_n)\) satisfies the conditions (6.2.2), (6.2.3) and (6.2.4). Hence by Theorem 6.2.1, we have
\[ S_{N-} \lim_{n \to \infty} \|L_n(f) - f\|_{2\pi} = 0, \]
i.e. our theorem holds. But on the other hand, Theorem 6.1.2 does not hold for our operator defined by (6.3.1), since the sequence \((L_n)\) is not convergent.

Hence our Theorem 6.1.1 is stronger than that of 6.1.2.
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SOME APPROXIMATION THEOREMS VIA STATISTICAL CONVERGENCE

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are only possible due to their love and sacrifices

All of these and will accomplish

Our gratitude to Han and Qainat Khanam

Chyo aspiring Paras

This thesis is dedicated
Certificate

This is to certify that the contents of this thesis entitled "SOME APPROXIMATION THEOREMS VIA STATISTICAL CONVERGENCE" is an original research work of Mr. Asif Khan carried out under my supervision. He has fulfilled the prescribed conditions given in the ordinances and regulations of Aligarh Muslim University, Aligarh.

I further certify that the work has not been submitted either partly or fully to any other University or Institution for the award of any other degree.

Prof. Mursaleen
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Asif Khan

(Asif Khan)
Preface

Present thesis entitled 'SOME APPROXIMATION THEOREMS VIA STATISTICAL CONVERGENCE' contains the research work done by me under the constant supervision of Dr. Mursaleen, Professor, Department of Mathematics, Aligarh Muslim University, Aligarh. In the present work, we study approximation properties of different positive linear operators using Korovkin theorem via statistical convergence and its various generalizations.

The present thesis comprises of six chapters and each chapter is further divided into sections. The definitions, examples, remarks, theorems etc. have been specified with the double decimal numbers. The first figure denotes the number of the chapter, second represents the section in the chapter and third points out the number of the definition, the example or the theorem as the case may be in a particular chapter. For example, Theorem 3.4.2 refers to the second theorem appearing in the fourth section of the third chapter.

Chapter 1 contains preliminary notions, basic definitions, examples and some important well known results related to our study which are required for the development of the subject in subsequent chapters. This chapter is an attempt to make this thesis as self contained as possible.

Chapter 2 deals with the study of statistical approximation properties of modified $q$-Stancu-Beta operators.

In Chapter 3, we study statistical approximation properties of $q$-Bernstein-Schurer operators and establish some direct theorems. With the help of Matlab, we compute error estimate using modulus of continuity and give its algorithm. We also show graphically the convergence of the $q$-Bernstein-Schurer operators to various functions.

Chapter 4 is devoted to some positive linear operators constructed by means of $q$-Lagrange polynomials and some approximation results via A-statistical convergence are studied.

In chapter 5, we define $\lambda$-equi-statistical convergence and we apply our new notion to prove a Korovkin type approximation theorem and we show that our theorem is a non-trivial extension of some well-known Korovkin type
approximation theorems. We also prove a Voronovskaja type approximation theorem via the concept of \( \lambda \)-equi-statistical convergence.

Chapter 6 is devoted to study approximation for periodic functions via weighted statistical convergence.

Finally at the end, a bibliography is given which by no means is exhaustive one but lists only those books and papers which have been referred to in this thesis.
Chapter 1

Introduction and Preliminaries

1.1 Introduction

In this chapter, we give basic concepts, preliminary definitions and some fundamental results. Of course, the elementary knowledge of concepts such as function, sequence, convergence etc. has been pre-assumed and no attempt has been made to discuss them here. Some key results and classical theorems related to our subject matter are also incorporated as remarks at suitable places. Most of the material included in this chapter occur in standard literature like 'Linear Operator And Approximation Theory'(1960), Hindustan Publishing Corp. (India) by P.P. Korovkin [36], and a survey ‘Korovkin type theorems and approximation by positive linear operators’ by Francesco Altomare (2010) [7], H. Fast [25], Fridy [26] etc.

Korovkin type theorems furnish simple and useful tools for ascertaining whether a given sequence of positive linear operators, acting on some function space is an approximation process or, equivalently, converges strongly to the identity operator. Roughly speaking, these theorems exhibit a variety of subsets of test functions which guarantee that the approximation (or the convergence) property holds on the whole space provided it holds on them. The custom of calling these kinds of results as Korovkin type theorems refers to P. P. Korovkin [35], who in 1953 discovered such a property for the functions 1, $x$ and $x^2$ in the space $C[0, 1]$ of all continuous functions on the real interval $[0, 1]$ as well as for the functions 1, $\cos$ and $\sin$ in the space of all continuous $2\pi$-- periodic functions on the real line [36].

After this discovery, several mathematicians have undertaken the program of extending Korovkin’s theorems in many ways and to several settings, including function spaces, abstract Banach lattices, Banach algebras, Banach spaces and so on.
Such developments delineated a theory which is nowadays referred to as Korovkin type approximation theory. This theory has fruitful connections with real analysis, functional analysis, harmonic analysis, measure theory and probability theory, summability theory and partial differential equations. But the foremost applications are concerned with constructive approximation theory which uses it as a valuable tool. Even today, the development of Korovkin type approximation theory is far from complete.

1.2 Historical note

Statistical convergence was introduced in connection with problems of series summation. The main idea of statistical convergence of a sequence $(x_n)$, where $n \in \mathbb{N}$ is that the majority, in a certain sense, of its elements converge and we do not care what happens with other elements. At the same time, it is known that sequences that come from real life sources are not convergent in the strictly mathematical sense. This way, the advantage of replacing the uniform convergence by statistical convergence consists in the fact that the second convergence models improves the technique of approximation in different functions spaces. In 1951, H. Fast [25] introduced an extention of the usual concept of sequential limit which he called statistical convergence.

The idea of statistical convergence was introduced independently by Steinhaus [70], Fast [25] and Schoenberg [63]. Over the years, statistical convergence has been discussed in the theory of Fourier analysis [73], ergodic theory and number theory [17]. Later on, it was further investigated from the sequence spaces point of view and linked with the summability theory [27]. Also, it has been studied in connection with trigonometric series [73], measure theory [44, 45] and Banach space theory [16].

The study of the statistical convergence for sequences of linear positive operators was attempted in the year 2002 by A.D. Gadjiev and C. Orhan [28]. The research field was proved to be extremely fertile.

Recently the idea of statistical convergence has been used in proving some approximation theorems, in particular, Korovkin type approximation theorems [36] by
various authors, e.g. [28, 43] and [51] etc.; and it was shown that the statistical versions are stronger than the classical ones. Authors have used many types of classical operators and test functions to study the Korovkin type approximation theorems which further motivate to continue the study. Korovkin type approximation theory has also many useful connections, other than classical approximation theory, in other branches of mathematics (see Altomare and Campiti in [6]).

An application of statistical summability gave rise to the theory of statistical approximation (e.g.[13, 22, 47, 50, 51, 52, 54]) and [68] which has been an active area of research for the last one decade.

Statistical approximation properties have also been investigated for q-analogue of several operators. For instance, in [9] q-Butzer and Hahn operators; in [11, 56] q-analogue of Stancu-Beta operators; in [24] q-Bleimann, Butzer and Hahn operators; in [29] q-Baskakov–Kantorovich operators; in [58, 59] q-Szász–Mirakjan operators; and in [62] q-analogues of Bernstein-Kantorovich operators were defined and their statistical approximation properties were investigated.

Our interest is to construct different classes of linear positive operators and to study their statistical approximation properties. We know that any convergent sequence is statistically convergent but the converse need not be true. The aim is to construct such sequences of operators that approximate the functions in the statistical sense, but not in the classical sense.

### 1.3 Statistical convergence

In the present section, we give the definition of statistical convergence and some related concepts.

**Definition 1.3.1.** Let $\mathbb{N}$ denote the set of all natural numbers. Let $K \subseteq \mathbb{N}$ and $K_n = \{ k \leq n : k \in K \}$. Then the natural density of $K$ is defined by $\delta(K) = \lim_{n \to \infty} \frac{1}{n} |K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of the set $K_n$.

**Definition 1.3.2.** A sequence $x = (x_k)$ of real numbers is said to be statistically convergent to $L$ (cf. Fast [25]) provided that for every $\epsilon > 0$ the set $\{ k \in \mathbb{N} :$
$|x_k - L| \geq \varepsilon$ has natural density zero, i.e. for each $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |k \leq n : |x_k - L| \geq \varepsilon| = 0.$$ 

In this case, we write $st \lim_{k\to\infty} x_k = L$. Note that every convergent sequence is statistically convergent but not conversely. Moreover even unbounded sequence may be statistically convergent. For example, let $x = (x_k)$ be defined by

$$x_k = \begin{cases} k, & \text{if } k \text{ is a square} \\ 0, & \text{otherwise,} \end{cases}$$

then $x$ is statistically convergent to zero but not bounded.

Let $A = (a_{nk})$, where $n, k = 1, 2, 3...$ be an infinite matrix. For a given sequence $x = (x_k)$, the $A$-transform of $x$ is defined by $Ax = ((Ax)_n)$, where $(Ax)_n = \sum_{k=1}^{n} a_{nk}x_k$, provided the series converges for each $n$. We say that $A$ is regular if $\lim_{n \to \infty} (Ax)_n = L = \lim_{n \to \infty} x$.

Let $A$ be a regular matrix. We say that a sequence $x = (x_k)$ is $A$-statistically convergent to a number $L$ (cf. Kolk [37]) if for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{|I_n|} \sum_{k \in I_n} a_{nk}x_k = 0.$$ 

In this case, we denote this limit by $st_A \lim_{n \to \infty} x_n = L$.

Note that for $A = C_1$, the Cesàro matrix of order 1, $A$-statistical convergence reduces to the statistical convergence.

The concept of $\lambda$-statistical convergence was introduced recently in [49] as follows. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to $\infty$ such that

$$\lambda_{n+1} \leq \lambda_n + 1 \quad \text{and} \quad \lambda_1 = 1.$$ 

Also let

$$I_n = [n - \lambda_n + 1, n] \quad \text{and} \quad K \subseteq \mathbb{N}.$$
Then the \( \lambda \)-density of \( K \) is defined by
\[
\delta_{\lambda}(K) = \lim_{n \to \infty} \frac{1}{\lambda_n} \left| \{ j : n - \lambda_n + 1 \leq j \leq n \text{ and } j \in K \} \right|.
\]
Clearly, in the special case when \( \lambda_n = n \), the \( \lambda \)-density reduces to the natural density.

The number sequence \( x = (x_j) \) is said to be \( \lambda \)-statistically convergent to the number \( L \) if, for each \( \epsilon > 0 \),
\[
\delta_{\lambda}(K_{\epsilon}) = 0,
\]
where
\[
K_{\epsilon} = \{ j : j \in I_n \text{ and } |x_j - L| > \epsilon \},
\]
that is, if, for each \( \epsilon > 0 \),
\[
\lim_{n \to \infty} \frac{1}{\lambda_n} \left| \{ j : j \in I_n \text{ and } |x_j - L| > \epsilon \} \right| = 0.
\]
In this case, we write
\[
\text{st}_{\lambda} \lim_{n \to \infty} x_n = L
\]
and we denote the set of all \( \lambda \)-statistically convergent sequences by \( S_{\lambda} \).

**Example 1.3.1.** We know that the total number of numbers from 1 to \( n \) that are divisible by 2 is \([\frac{n}{2}]\), where \([\nu]\) is the greatest integer in \( \nu \in \mathbb{R} \). Similarly, the total number of numbers from 1 to \( n \) that are divisible by 3 is \([\frac{n}{3}]\). In general, the total number of numbers from 1 to \( n \) that are divisible by \( k \) is \([\frac{n}{k}]\) \((k \in \mathbb{N})\). Now define a sequence \( x = (x_n) \) by
\[
x_n = \begin{cases} 
1/2 & \text{if } n \text{ is divisible by } k = \sqrt{n} \\
0 & \text{otherwise}.
\end{cases}
\]
Then the total number of numbers for which \( x_n = \frac{1}{2} \) will be
\[
\left[ \frac{n}{k} \right] = \left[ \frac{n}{\sqrt{n}} \right] \leq \sqrt{n};
\]
it will be zero otherwise. We thus find that
\[
\left| \{ k : k \leq n \text{ and } x_k \neq 0 \} \right| \leq \sqrt{n},
\]
which implies that
\[
\text{st}_{\lambda} \lim_{n \to \infty} x_n = 0.
\]
Example 1.3.2. If \( n = m^2 \), then there are exactly \( 2k + 1 \) divisors, out of which exactly \( k \) are less than \( m \). We take \( n = 36 = 6^2 \), then total number of such divisors will be nine \((1, 2, 3, 4, 6, 9, 12, 18, 36)\), out of which four are exactly less than 6. We can thus define a sequence \( x = (x_n) \) by

\[
x_n = \begin{cases} 
\frac{1}{k} & (k < n = m^2) \\
\frac{e^{-k}}{n} & (\text{otherwise}),
\end{cases}
\]

where \( k \) is the number of divisors. Clearly, we have

\[
\text{st}_1 \lim_{n \to \infty} x_n = 0.
\]

1.4 \ q-integers

Let us recall certain notations of q-integers [71]. For each nonnegative integer \( k \), the q-integer \([k]_q\) is defined by

\[
[k]_q = \begin{cases} 
\frac{(1-q^k)}{(1-q)}, & q \neq 1 \\
k, & q = 1
\end{cases}
\]

\[
[k]_q! = \begin{cases} 
[k]_q[k-1]_q\ldots[1]_q, & k \geq 1, \\
1, & k = 0;
\end{cases}
\]

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.
\]

For a real or complex number \( q \) \((|q| < 1)\), the number \((\lambda; q)_n\) is defined by

\[
(\lambda; q)_n = \frac{(\lambda; q)_\infty}{(\lambda q^n; q)_\infty},
\]

where

\[
(\lambda; q)_\infty = \prod_{k=1}^{\infty} (1 - \lambda q^k),
\]

and
\[ (\lambda; q)_n = \begin{cases} 1 & \text{if } n = 0, \\ (1 - \lambda)(1 - \lambda q) \cdots (1 - \lambda q^{n-1}) & \text{if } n = 1, 2, 3, \ldots \end{cases} \]

The q-improper integral is defined as (see Koornwinder [38])

\[ \int_0^{\infty / A} f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} f\left( \frac{q^n}{A} \right) \frac{q^n}{A}, \quad A > 0. \]

The q-Beta integral representations are as follows

\[ B_q(t, s) = K(A, t) \int_0^{\infty / A} \frac{x^{t-1}}{(1 + x)^{t+s}} d_q x, \]

where

\[ (a + b)^n_q = \prod_{j=0}^{n-1} (a + q^j b), \]

and

\[ K(A, t + 1) = q^t K(A, t), \quad A > 0. \]

### 1.5 Korovkin type approximation theorem

**Definition 1.5.1.** Let \( C[a, b] \) be the linear space of all real valued continuous functions \( f \) on \([a, b]\) and let \( T \) be a linear operator which maps \( C[a, b] \) into itself. We say that \( T \) is positive if for every non-negative \( f \in C[a, b] \), we have \( T(f, x) \geq 0 \) for all \( x \in [a, b] \).

**Definition 1.5.2.** Let \( f(x) \) be a function continuous in the interval \([a, b]\). We put \( w(\delta) = w(f, \delta) = \max_{|x-y|} | f(x) - f(y) |, \quad a \leq x, y \leq b \). The quantity \( w(\delta) \) is called *modulus of continuity* of the function \( f(x) \).
Let $F(\mathbb{R})$ denote the linear space of all real-valued functions defined on $\mathbb{R}$. Let $C(\mathbb{R})$ be the space of all functions $f$ continuous on $\mathbb{R}$ with norm
\[
\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|, \quad f \in C(\mathbb{R}).
\]
We denote by $C_{2\pi}(\mathbb{R})$ the space of all $2\pi$-periodic functions $f \in C(\mathbb{R})$ equipped with the norm
\[
\|f\|_{2\pi} = \sup_{t \in \mathbb{T}} |f(t)|.
\]
The classical Korovkin first and second theorems state as follows (see [35, 36]):

**Theorem 1.5.1.** Let $(T_n)$ be a sequence of positive linear operators from $C[0,1]$ into $F[0,1]$. Then $\lim_n \|T_n(f, x) - f(x)\|_\infty = 0$, for all $f \in C[0,1]$ if and only if $\lim_n \|T_n(f_i, x) - f_i(x)\|_\infty = 0$, for $i = 0, 1, 2$, where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

**Theorem 1.5.2.** Let $(T_n)$ be a sequence of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $F(\mathbb{R})$. Then $\lim_n \|T_n(f, x) - f(x)\|_\infty = 0$, for all $f \in C_{2\pi}(\mathbb{R})$ if and only if $\lim_n \|T_n(f_i, x) - f_i(x)\|_\infty = 0$, for $i = 0, 1, 2$, where $f_0(x) = 1$, $f_1(x) = \cos x$ and $f_2(x) = \sin x$.

Note that the first and the second theorems of Korovkin are actually equivalent to the algebraic and the trigonometric version, respectively, of the classical Weierstrass approximation theorem [7]. Recently, the Korovkin second theorem is proved in [21] by using the concept of statistical convergence. Quite recently, Korovkin second theorem is proved by Demirci and Dirik [18] for statistical $\sigma$-convergence [53]. For some recent work on this topic, we refer to [8, 34, 46, 51, 68].
Chapter 2

Statistical Approximation Properties of Modified $q$-Stancu-Beta Operators

2.1 Introduction

In this chapter we define the modified $q$-Stancu-Beta operators and study the weighted statistical approximation by these operators with the help of the Korovkin type approximation theorem. We also establish the rates of statistical convergence by means of the modulus of continuity and the Lipschitz type maximal function. Our results show that rates of convergence of our operators are at least as fast as classical Stancu-Beta operators.

2.2 Construction of the operators

After the paper of Phillips [61] who generalized the classical Bernstein polynomials based on $q$-integers, many generalizations of well-known positive linear operators, based on $q$-integers were introduced and studied by several authors. Recently the statistical approximation properties have been investigated for $q$-analogue polynomials. For instance, in [62] $q$-analogues of Bernstein operators, in [40] $q$-analogues of Bernstein–Kantorovich operators; in [29] $q$-Baskakov–Kantorovich operators; in [58] $q$-Szász–Mirakjan operators; in [9] and [24] $q$-Bleimann, Butzer and Hahn operators; in [2] and [43] $q$-analogue of MKZ operators; in [41] $q$-analogue of Baskakov and Baskakov-Kantorovich operators; in [42] $q$-analogue of Szász Kantorovich operators; in [55] $q$-Lagrange polynomials were defined and their classical approximation

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or statistical approximation properties were investigated whereas in [11] q-analogue of Stancu-Beta operators were introduced.

In this chapter, we first introduce a new modification of the operators defined by Aral and Gupta [11] and study the weighted statistical approximation properties of the modified q-Stancu-Beta operators with the help of the Korovkin type approximation theorem. We also estimate the rate of statistical convergence of the sequence of positive linear operators to the function f.

D. D. Stancu [69] introduced Beta operators $L_n$ of second kind in order to approximate the Lebesgue integrable functions on the interval $(0, \infty)$ as follows:

$$L_n(f; x) = \frac{1}{B(nx, n+1)} \int_0^{\infty} \frac{t^{nx-1}}{(1+t)^{nx+n+1}} f(t) \, dt.$$  

Aral and Gupta [11] introduced the q-analogue of Stancu-Beta operators as follows:

$$L_n^q(f; x) = \frac{K(A, [n]_q x)}{B_q([n]_q x, [n]_q + 1)} \int_0^{\infty/A} \frac{u^{[n]_q x-1}}{(1+u)^{[n]_q x+[n]_q+1}} f(q^{[n]_q x} u) \, du.$$  

The following theorem was given by Aral and Gupta [11]:

**Theorem 2.2.1.** Let $q = (q_n)$ satisfy $0 < q_n < 1$ with $\lim_{n \to \infty} q_n = 1$. For each $f \in C_{x^2}[0, \infty)$, we have

$$\lim_{n \to \infty} \|(L_n^q(f); \cdot) - f\|_{x^2} = 0,$$

where $C_{x^2}[0, \infty)$ denotes the subspace of all continuous functions on $[0, \infty)$ such that $|f(x)| \leq M_f$, and $C_{x^2}[0, \infty)$ denotes the spaces of all $f \in C_{x^2}[0, \infty)$ such that $\lim_{x \to \infty} \frac{f(x)}{1 + x^2}$ is finite. The norm on $C_{x^2}[0, \infty)$ is given by $\|f\|_{x^2} = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1 + x^2}$.

We define modified q-Stancu-Beta operators as follows:

$$L_n^*(f; q, x) = q \frac{K(A, [n]_q x)}{B_q([n]_q x, [n]_q + 1)} \int_0^{\infty/A} \frac{u^{[n]_q x-1}}{(1+u)^{[n]_q x+[n]_q+1}} f(q^{[n]_q x} u) \, du \quad (2.2.1)$$

where $x \geq 0$ and $0 < q \leq 1$. It is easy to verify that if $q = 1$, these operators turns
into the classical Stancu-Beta operators.

Remark 2.2.1. Note that $L_n^*(f; q, x) = L_n^*(f; x)$ and from the Lemma 1 of Aral and Gupta [11], we have $L_n^*(1; x) = 1, L_n^*(t; x) = x, L_n^*(t^2; x) = \frac{([n]_q x + 1)x}{q([n]_q - 1)}$. Hence for $x \geq 0, \ 0 < q \leq 1$, we have

$$L_n^*(1; q, x) = q, L_n^*(t; q, x) = q \ x \text{ and } L_n^*(t^2; q, x) = \frac{([n]_q x + 1)x}{([n]_q - 1)}.$$  \hfill (2.2.2)

Remark 2.2.2. Let $q \in (0,1)$ then for $x \in [0, \infty)$, we have

$$L_n^*(t - x; x) = 0$$

and

$$L_n^*((t - x)^2; x) = \frac{([n]_q - q[n]_q + q)x^2 + x}{([n]_q - 1)}.$$

2.3 Weighted statistical approximation of Korovkin type

In this section, we obtain the Korovkin type weighted statistical approximation by the operators defined in (2.2.1).

A real function $\rho$ is called a weight function if it is continuous on $\mathbb{R}$ and $\lim_{|x| \to \infty} \rho(x) = \infty, \rho(x) \geq 1$ for all $x \in \mathbb{R}$.

Let by $B_\rho(\mathbb{R})$ denote the weighted space of real-valued functions $f$ defined on $\mathbb{R}$ with the property $|f(x)| \leq M_f \rho(x)$ for all $x \in \mathbb{R}$, where $M_f$ is a constant depending on the function $f$. We also consider the weighted subspace $C_\rho(\mathbb{R})$ of $B_\rho(\mathbb{R})$ given by $C_\rho(\mathbb{R}) = \{f \in B_\rho(\mathbb{R}): f \text{ continuous on } \mathbb{R}\}$. Note that $B_\rho(\mathbb{R})$ and $C_\rho(\mathbb{R})$ are Banach spaces with $\|f\|_\rho = \sup_{x \in \mathbb{R}} \frac{|f(x)|}{\rho(x)}$. In case of weight function $\rho_0 = 1 + x^2$, we have

$$\|f\|_{\rho_0} = \sup_{x \in \mathbb{R}} \frac{|f(x)|}{1 + x^2}.$$

Now, we consider a sequence $q = (q_n), q_n \in (0, 1)$, such that

$$st \lim_{n \to \infty} q_n = 1.$$  \hfill (2.3.1)
In [20], Doğru gave some examples so that \((q_n)\) is statistically convergent to 1 but it may not converge to 1 in the ordinary case.

Now we are ready to prove our main result as follows:

Theorem 2.3.1. Let \((L_n^*)\) be the sequence of the operators \((2.2.1)\) and the sequence \(q = (q_n)\) satisfies (2.3.1). Then for any function \(f \in C_B[0, \infty),\)

\[
\text{st} - \lim_{n \to \infty} \|L_n^*(f; q_n; \cdot) - f\|_{\rho_0} = 0.
\]

Proof. Let \(e_\nu = x^\nu\) where \(\nu = 0, 1, 2\). Since \(L_n^*(1; q_n; x) = q_n\). Therefore we can write

\[
\text{st} - \lim_{n \to \infty} \|L_n^*(1; q_n; x) - 1\|_{\rho_0} = \text{st} - \lim_{n \to \infty} \|c_0\|_{\rho_0} |q_n - 1|
\]

as

\[
\|L_n^*(1; q_n; x) - 1\|_{\rho_0} = \sup_{x \in \mathbb{R}} \frac{|L_n^*(1; q_n; x) - 1|}{1 + x^2} = \|c_0\|_{\rho_0} |q_n - 1| \leq |q_n - 1|.
\]

By (2.3.1), it can be observed that

\[
\text{st} - \lim_{n \to \infty} \|L_n^*(1; q_n; x) - 1\|_{\rho_0} = 0.
\]

Similarly

\[
\|L_n^*(t; q_n; x) - x\|_{\rho_0} = \|e_t\|_{\rho_0} \frac{\Gamma_q([n]_\varrho) \Gamma_q ([n]_\varrho)}{\Gamma_q([n]_\varrho) \Gamma_q ([n]_\varrho)} (q_n - 1) \leq |q_n - 1| = 1 - q_n.
\]

For a given \(\epsilon > 0\), let us define the following sets

\[
U = \{ n : \|L_n^*(t; q_n; x) - x\|_{\rho_0} \geq \epsilon \}
\]

and

\[
U' = \{ n : 1 - q_n \geq \epsilon \}.
\]
It is obvious that \( U \subset U' \) and hence

\[
\delta(\{ k \leq n : \|L_n^*(t; q_n; x) - x\|_{\rho_0} \geq \epsilon \}) \leq \delta(\{ k \leq n : 1 - q_n \geq \epsilon \}).
\]

By using (2.3.1), we get

\[
st - \lim_{n \to \infty} (1 - q_n) = 0.
\]

Therefore

\[
\delta(\{ k \leq n : 1 - q_n \geq \epsilon \}) = 0,
\]

and we have

\[
st - \lim_{n \to \infty} \|L_n^*(t; q_n; x) - x\|_{\rho_0} = 0.
\]

Lastly, we have

\[
\|L_n^*(t^2; q_n; x) - x^2\|_{\rho_0} = \|e_2\|_{\rho_0} \left( \frac{[n]_q}{[n]_q - 1} \right) + \|e_1\|_{\rho_0} \left( \frac{1}{[n]_q - 1} - 1 \right)
\]

\[
\leq \left( \frac{[n]_q + 1}{[n]_q - 1} - 1 \right).
\]

Also we have that

\[
\frac{[n]_q + 1}{[n]_q - 1} = \frac{1}{q_n^2} \left( \frac{2 + q_n}{n + 1} \right) + \frac{[n + 1]_{q_n}}{[n - 1]_{q_n}} - \frac{1 + q_n}{[n - 1]_{q_n} - 1}.
\]

Therefore by (2.3.2), we get

\[
\|L_n^*(t^2; q_n; x) - x^2\|_{\rho_0} \leq \left| \frac{1}{q_n^2} \left( \frac{[n + 1]_{q_n}}{[n - 1]_{q_n}} \right) - 1 \right| + \left| \frac{1}{q_n^2} \left( \frac{2 + q_n}{[n - 1]_{q_n}} - 1 \right) \right| + \left| \frac{1}{q_n^2} \left( \frac{1 + q_n}{[n - 1]_{q_n} - 1} \right) \right|.
\]

Now, if we choose

\[
\alpha_n = \frac{1}{q_n^2} \left( \frac{[n + 1]_{q_n}}{[n - 1]_{q_n}} \right) - 1,
\]

\[
\beta_n = \frac{1}{q_n^2} \left( \frac{2 + q_n}{[n - 1]_{q_n} - 1} \right),
\]

\[
\gamma_n = \frac{1}{q_n^2} \left( \frac{1 + q_n}{[n - 1]_{q_n} - 1} \right),
\]

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then by (2.3.1), we can write
\[ st - \lim_{n \to \infty} \alpha_n = st - \lim_{n \to \infty} \beta_n = st - \lim_{n \to \infty} \gamma_n = 0. \] (2.3.3)

Now for given \( \epsilon > 0 \), we define the following four sets

\[ D = \{ n : \| L_n^*(f; q_n; x) - x \|_{\mathcal{P}_0} \geq \epsilon \}, \]
\[ D_1 = \{ n : \alpha_n \geq \frac{\epsilon}{3} \}, \]
\[ D_2 = \{ n : \beta_n \geq \frac{\epsilon}{3} \}, \]
\[ D_3 = \{ n : \gamma_n \geq \frac{\epsilon}{3} \}. \]

It is obvious that \( D \subseteq D_1 \cup D_2 \cup D_3 \). Then we obtain
\[ \delta(\{ k \leq n : \| L_n^*(f; q_n; x) - x \|_{\mathcal{P}_0} \geq \epsilon \}) \]
\[ \leq \delta(\{ k \leq n : \alpha_n \geq \frac{\epsilon}{3} \}) + \delta(\{ k \leq n : \beta_n \geq \frac{\epsilon}{3} \}) + \delta(\{ k \leq n : \gamma_n \geq \frac{\epsilon}{3} \}). \]

Using (2.3.3), we get
\[ st - \lim_{n \to \infty} \| L_n^*(f; q_n; x) - x \|_{\mathcal{P}_0} = 0. \]

Since
\[ \| L_n^*(f; q_n; x) - f \|_{\mathcal{P}_0} \]
\[ \leq \| L_n^*(f; q_n; x) - x \|_{\mathcal{P}_0} + \| L_n^*(t; q_n; x) - x \|_{\mathcal{P}_0} + \| L_n^*(1; q_n; x) - 1 \|_{\mathcal{P}_0}, \]
we get
\[ st - \lim_{n \to \infty} \| L_n^*(f; q_n; x) - f \|_{\mathcal{P}_0} \]
\[ \leq st - \lim_{n \to \infty} \| L_n^*(f; q_n; x) - x \|_{\mathcal{P}_0} + st - \lim_{n \to \infty} \| L_n^*(t; q_n; x) - x \|_{\mathcal{P}_0} \]
\[ + st - \lim_{n \to \infty} \| L_n^*(1; q_n; x) - 1 \|_{\mathcal{P}_0}, \]
which implies that
\[ st - \lim_{n \to \infty} \| L_n^*(f; q_n; x) - f \|_{\mathcal{P}_0} = 0. \]

This completes the proof of the theorem.
2.4 Rates of statistical convergence

In this section, we give the rates of statistical convergence of the operators (2.2.1) by means of modulus of continuity and Lipschitz type maximal functions. The modulus of continuity for the functions \( f \in CB[0, \infty) \) is defined as

\[
w(f; \delta) = \sup_{x,t \geq 0, |t-x| < \delta} \frac{|f(t) - f(x)|}{1 + x^{2+\lambda}}\]

where \( w(f; \delta) \) for \( \delta > 0, \lambda \geq 0 \) satisfies the following conditions: for every \( f \in CB[0, \infty) \)

(i) \( \lim_{\delta \to \infty} w(f; \delta) = 0 \)

(ii) \( |f(t) - f(x)| \leq w(f; \delta) \left( \frac{|t-x|}{\delta} + 1 \right) \quad (2.4.1) \)

**Theorem 2.4.1.** Let the sequence \( q = (q_n) \) satisfy the condition in (2.3.1) and \( 0 < q_n < 1 \). Then we have

\[ |L_n^*(f; q_n; x) - f(x)| \leq w(f; \sqrt{\delta_n(x)}) \left( 1 + q_n \right), \]

where

\[ \delta_n(x) = \|e_2\|_{\rho_0} q_n \left( \frac{[n]_q}{[n]_q - 1} - \frac{q_n[n]_{q_n}}{[n]_{q_n} - 1} + \frac{q_n}{[n]_{q_n} - 1} \right) + \|e_1\|_{\rho_0} \frac{q_n}{[n]_{q_n} - 1}. \quad (2.4.2) \]

**Proof.** Since \( |L_n^*(f; q_n; x) - f| \leq L_n^*(|f(t) - f(x)|; q_n; x) \), by (4.1) we get

\[ |L_n^*(f; q_n; x) - f(x)| \leq w(f; \delta) \left( \{L_n^*(1; q_n; x) + \frac{1}{\delta} L_n^*(|t-x|; q_n; x)\} \right). \]

Using Cauchy-Schwartz inequality, we have

\[ |L_n^*(f; q_n; x) - f(x)| \leq w(f; \delta) \left( q_n + \frac{1}{\delta_n} ([|e_2|]_{\rho_0}) q_n \left( \frac{[n]_q}{[n]_q - 1} - \frac{q_n[n]_{q_n}}{[n]_{q_n} - 1} + \frac{q_n}{[n]_{q_n} - 1} \right) + ([e_1]_{\rho_0}) \frac{q_n}{[n]_{q_n} - 1} \right)^{1/2} \]

\[ \leq w(f; \delta) \left( q_n + \frac{1}{\delta_n} ([|e_2|]_{\rho_0}) q_n \left( \frac{[n]_q}{[n]_q - 1} - \frac{q_n[n]_{q_n}}{[n]_{q_n} - 1} + \frac{q_n}{[n]_{q_n} - 1} \right) + ([e_1]_{\rho_0}) \frac{q_n}{[n]_{q_n} - 1} \right)^{1/2} \]

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By choosing $\delta_n$ as in (2.4.2), we get the desired result.

This completes the proof of the theorem.

Now we will give an estimate concerning the rate of approximation by means of Lipschitz type maximal functions. In [39], Lenze introduced a Lipschitz type maximal function as

$$
\bar{f}_\alpha(x) = \sup_{t > 0, t \neq x} \frac{|f(t) - f(x)|}{|t - x|^{\alpha}}.
$$

In [9], the Lipschitz type maximal function space on $E \subset [0, \infty)$ is defined as follows

$$
\tilde{W}_\alpha = \{f = \sup(1 + x)^\alpha \bar{f}_\alpha(x) \leq M \frac{1}{(1 + y)\alpha}; x \geq 0 \text{ and } y \in E\},
$$

where $f$ is bounded and continuous function on $[0, \infty)$, $M$ is a positive constant and $0 < \alpha \leq 1$.

**Theorem 2.4.2.** If $L_n^*$ be defined by (2.2.1), then for all $f \in \tilde{W}_{\alpha, E}$

$$
|L_n^*(f; q_n; x) - f(x)| \leq M(\eta_n^{\frac{q}{q_n}} q_n^{\frac{2-\alpha}{q_n}} + q_n \, d(x, E)),
$$

(2.4.3)

where

$$
\eta_n = \|e_2\|_{\rho_0} \left( \frac{[n]_q}{[n]_q - 1} - \frac{q_n [n]_{q_n}}{[n]_{q_n} - 1} + \frac{q_n}{[n]_{q_n} - 1} \right) + \|e_1\|_{\rho_0} \frac{1}{[n]_{q_n} - 1},
$$

(2.4.4)

**Proof.** Let $x \geq 0$, $(x, x_0) \in [0, \infty) \times E$. Then we have

$$
|f - f(x)| \leq |f - f(x_0)| + |f(x_0) - f(x)|.
$$

Since $L_n^*$ is a positive and linear operator, $f \in \tilde{W}_{\alpha, E}$ and using the above inequality

$$
|L_n^*(f; q_n; x) - f(x)| \leq |L_n^*(|f - f(x_0)|; q_n; x) - f(x)| + |L_n^*(|f(x_0) - f(x)|; q_n; x) + |x - x_0|^{\alpha} L_n^*(1; q_n; x))
$$

$$
\leq M (L_n^*(|t - x_0|^{\alpha}; q_n; x) + |x - x_0|^{\alpha} L_n^*(1; q_n; x)).
$$

(2.4.5)

Therefore we have

$$
L_n^*(|t - x_0|^{\alpha}; q_n; x) \leq L_n^*(|t - x|^{\alpha}; q_n; x) + |x - x_0|^{\alpha} L_n^*(1; q_n; x).
$$
By using the Hölder's inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have
\[ L^*_n((t - x)^a; q_n; x) \leq L^*_n((t - x)^2; q_n; x)^{\alpha/2} \left( L^*_n(1; q_n; x) \right)^{2-\alpha/2} + |x - x_0|^\alpha L^*_n(1; q_n; x) \]
\[ = \eta_n^{\alpha/2} q_n^{2-\alpha} + |x - x_0|^\alpha q_n. \]
Substituting this in (2.4.5), we get (2.4.3).
This completes the proof of the theorem.

2.5 Concluding remarks

(i) Note that in condition (2.3.3),
\[ st - \lim_{n \to \infty} \delta_n = 0. \]
By (2.4.1) we have
\[ st - \lim_{n \to \infty} w(f; \delta_n) = 0, \]
which gives us the pointwise rate of statistical convergence of the operator $L^*_n(f; q_n; x)$ to $f(x)$.
From the definition of the $q$-integers, it can be proved that
\[ \sup_{x \geq 0} \delta_n(x) \leq q_n \left( \frac{\lfloor n \rfloor_q}{\lfloor n \rfloor_q - 1} - q_n \frac{\lfloor n \rfloor_q}{\lfloor n \rfloor_q - 1} + q_n - 1 + \frac{1}{\lfloor n \rfloor_q - 1} \right). \]
In classical case for $q = 1$, we have
\[ \sup_{x \geq 0} \delta_n(x) \leq \frac{1}{n - 1} = O\left(\frac{1}{n}\right) \]
Thus, for every choice of $q_n$, the rate of convergence of (2.2.1) to the function $f$ is better than the Stancu-Beta operators.

(ii) If we take $E = [0, \infty)$ in Theorem 2.4.2, since $d(x, E) = 0$, then we obtain the following result:
For every $f \in W_{a,[0,\infty)}$
\[ |L^*_n(f; q_n; x) - f(x)|_{q_n} \leq M \eta_n^{\alpha/2} q_n^{2-\alpha}. \]
where $\eta_n$ is defined as in (2.4.4)

(iii) By using (2.3.3), it is easy to verify that

$$st \lim_{n \to \infty} \eta_n = 0.$$

That is, the rate of statistical convergence of operators (2.2.1) to the function $f$ are estimated by means of Lipschitz type maximal functions.
Chapter 3

Generalized $q$-Bernstein-Schurer Operators and Some Approximation Theorems

3.1 Introduction

In this chapter, we study statistical approximation properties of $q$-Bernstein-Schurer operators and also establish some direct theorems. We compute error estimation by using modulus of continuity with the help of Matlab and give its algorithm. Furthermore, we show graphically the convergence of the $q$-Bernstein-Schurer operators to various functions.

In 1987, Lupaș [40] introduced the first $q$-analogue of Bernstein operator and investigated its approximating and shape-preserving properties. Another $q$-generalization of the classical Bernstein polynomials is due to Phillips [62]. After that many generalizations of well-known positive linear operators, based on $q$-integers were introduced and studied by several authors.

Schurer [64] introduced the following operators $L_{m,p} : C[0,p+1] \rightarrow C[0,1]$ defined for any $m \in \mathbb{N}$ and any function $f \in C[0,p+1]$

$$L_{m,p}(f;x) = \sum_{k=0}^{m+p} \binom{m+p}{k} x^k(1-x)^{m+p-k} f\left(\frac{k}{m}\right), \quad x \in [0,1]. \tag{3.1.1}$$

Recently, Muraru [48] introduced the $q$-analogue of these operators and investigated their approximation properties and rate of convergence using modulus of continuity.

Muraru [48] introduced the following operators known as the generalized $q$-Bernstein-Schurer operators. For any $m \in \mathbb{N}$, $p$ a fixed positive integer and $f \in$
\[ L_{m,p}^{*}(f; q; x) = \sum_{k=0}^{m+p} \left[ \begin{array}{c} m+p \\ k \end{array} \right] x^k \prod_{s=0}^{m+p-k-1} (1 - q^s x) f \left( \frac{[k]_q}{[m]_q} \right), \quad x \in [0, 1] \quad (3.1.2) \]

We note the following properties as in [48] for \( L_{m,p}^{*}(f; q; x) \).

**Lemma 3.1.1.** For \( x \in [0, 1], \ 0 < q \leq 1 \)

\[ L_{m,p}^{*}(1; q; x) = 1. \]

**Lemma 3.1.2.** For \( x \in [0, 1], \ 0 < q \leq 1 \)

\[ L_{m,p}^{*}(t; q; x) = x \frac{[m+p]_q}{[m]_q}. \]

**Lemma 3.1.3.** For \( x \in [0, 1], \ 0 < q \leq 1 \)

\[ L_{m,p}^{*}(t^2; q; x) = \frac{[m+p]_q}{[m]_q^2} ([m+q]_q x^2 + x(1 - x)). \]

**Lemma 3.1.4.** Let \( q \in (0, 1) \). Then for \( x \in [0, 1] \)

\[ L_{m,p}^{*}(t - x; q; x) = x \left( \frac{[m+p]_q}{[m]_q} - 1 \right). \]

**Lemma 3.1.5.** Let \( q \in (0, 1) \). Then for \( x \in [0, 1] \)

\[ L_{m,p}^{*}((t - x)^2; q; x) = x^2 \left( \frac{([m+p]_q - [m]_q x^2 - [m+p]_q)}{[m]_q^2} \right) + x \frac{[m+p]_q}{[m]_q^2}. \]

### 3.2 Statistical approximation

In this section, we obtain the Korovkin type weighted statistical approximation properties for these operators defined in (3.1.2).

Note that every convergent sequence is statistically convergent but not conversely, even unbounded sequence may be statistically convergent. For example, let \( u = (u_m) \) be defined by

\[ u_m = \frac{1}{m} \]
If \( k = m^2 \)
\[
\begin{align*}
    u_m &= \begin{cases} 
        1, & \text{if } k = m^2 \\
        0, & \text{otherwise.}
    \end{cases}
\end{align*}
\] (3.2.1)

Then \( st-lim u_m = 0 \) but \( u \) is not convergent.

Recently the idea of statistical convergence has been used in proving some approximation theorems, in particular, Korovkin type approximation theorems by various authors; and it was found that the statistical versions are stronger than the classical one. Authors have used many types of classical operators and test functions to study the Korovkin type approximation theorems which further motivate to continue this study. Most recently, \( A \)-statistical convergence has been used in Walsh-Fourier series [3].

Let \( C_B[0,p+1] \) be the space of all bounded and continuous functions on \([0,p+1]\). Then \( C_B[0,p+1] \) is a normed linear space with \( \| f \| = \sup_{x \geq 0} | f(x) | \). Let \( w \) be a function of the type of modulus of continuity. The principal properties of the function are the following:

(i) \( w \) is a non-negative increasing function on \([0,p+1]\),

(ii) \( \lim_{\delta \to 0} w(\delta) = 0 \).

Let \( C_B[0,p+1] \) be the space of all-real valued functions \( f \) defined on \([0,p+1]\) satisfying the following condition

\[ | f(x) - f(y) | \leq w(| x - y |) \]

for any \( x, y \in [0,p+1] \).

We consider a sequence \( q = (q_n), q_n \in (0,1) \), such that

\[ \lim_{n \to \infty} q_n = 1. \] (3.2.2)

The condition (3.2.2) guarantees that \( \lfloor n \rfloor q_n \to \infty \) as \( n \to \infty \).
Now our first result is as follows:

**Theorem 3.2.1.** Let \((L_{n,p}^*)\) be the sequence of the operators (3.1.2) and the sequence \(q = (q_m)\) satisfies (3.2.2). Then for any function \(f \in C_B[0, p + 1]\),

\[
st - \lim_{m \to \infty} \|L_{n,p}^*(f; q_m; x) - f\| = 0.
\]

**Proof.** Let \(e_\nu = x^\nu\) where \(\nu = 0, 1, 2\). Since \(L_{n,p}^*(1; q_m; x) = q_m\). Therefore we can write

\[
st - \lim_{m \to \infty} \|L_{n,p}^*(1; q_m; x) - 1\| = st - \lim_{n \to \infty} \|e_0\| |q_m - 1|
\]

as

\[
\|L_{n,p}^*(1; q_m; x) - 1\| = \|L_{n,p}^*(1; q_m; x) - 1\| = \|e_0\| |q_m - 1| \leq |q_m - 1|.
\]

By (3.2.2), it can be observed that

\[
st - \lim_{m \to \infty} \|L_{n,p}^*(1; q_m; x) - 1\| = 0.
\]

Similarly

\[
\|L_{n,p}^*(t; q_m; x) - x\| = \|e_2\| \frac{q_m[m + p]^2}{|m|_q^2 q_m} - 1
\]

\[
\leq \left|\frac{[m + p]_q}{[m]_q} - 1\right|
\]

For a given \(\epsilon > 0\), let us define the following sets

\[
U = \{m : \|L_{n,p}^*(t; q_m; x) - x\| \geq \epsilon\}
\]

and

\[
U' = \{m : \frac{[m + p]_q}{[m]_q} - 1 \geq \epsilon\}
\]

It is obvious that \(U \subseteq U'\), it can be written as

\[
\delta \left(\{k \leq m : \|L_{n,p}^*(t; q_m; x) - x\| \geq \epsilon\}\right) \leq \delta \left(\{k \leq m : \frac{[m + p]_q}{[m]_q} - 1 \geq \epsilon\}\right).
\]
By using 3.2.1, we get

\[ \text{st} - \lim_{m \to \infty} \left( \frac{[m + p]q_m}{[m]_q} - 1 \right) = 0. \]

Therefore

\[ \delta \left( \{ k \leq m : \frac{[m + p]q_m}{[m]_q} - 1 \geq \epsilon \} \right) = 0, \]

and we have

\[ \text{st} - \lim_{n \to \infty} \| L^*_{m,p}(t; q_m; x) - x \| = 0. \]

Lastly, we have

\[
\| L^*_{m,p}(t^2; q_m; x) - x^2 \| = \| \varepsilon_2 \| \left( \frac{q_m[m + p]^2}{[m]_q^2} - q_m[p][q_m] - 1 \right) + \| \varepsilon \| \left( \frac{q_m[m + p]q_m}{[m]_q^2} \right)
\]

\[ \leq \frac{q_m[m + p]^2}{[m]_q^2} - 1. \quad (3.2.3) \]

Therefore by (3.2.3), we get

\[
\| L^*_{m,p}(t^2; q_m; x) - x^2 \| \leq \frac{1}{q_m^2} \left( \frac{[m + p + 1]^2}{[m]_q^2} \right) - 1 + \frac{1}{q_m^2} \left( \frac{q_m + 2}{[m]_q^2} \right) + \frac{1}{q_m^2} \left( \frac{1 + q_m}{[m]_q^2} \right)
\]

Now, if we choose

\[ \alpha_m = \frac{1}{q_m^2} \left( \frac{[m + p + 1]^2}{[m]_q^2} \right) - 1, \]

\[ \beta_m = \frac{q_m + 2}{q_m^2} \left( \frac{[m + p + 1]^2}{[m]_q^2} \right), \]

\[ \gamma_m = \frac{1}{q_m^2} \left( \frac{1 + q_m}{[m]_q^2} \right), \]

then by (3.2.2), we can write

\[ \text{st} - \lim_{m \to \infty} \alpha_m = 0 = \text{st} - \lim_{m \to \infty} \beta_m = \text{st} - \lim_{m \to \infty} \gamma_m. \quad (3.2.4) \]

Now for given \( \epsilon > 0 \), we define the following four sets

\[ U = \{ m : \| L^*_{m,p}(t^2; q_m; x) - x^2 \| \geq \epsilon \}, \]

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\[ U_1 = \{ m : \alpha_m \geq \frac{\epsilon}{3} \}, \]
\[ U_2 = \{ m : \beta_m \geq \frac{\epsilon}{3} \}, \]
\[ U_3 = \{ m : \gamma_m \geq \frac{\epsilon}{3} \}. \]

It is obvious that \( U \subseteq U_1 \cup U_2 \cup U_3 \). Then we obtain
\[
\delta(\{ k \leq m : \| L_{m,p}(t^2; q_m; x) - x^2 \| \geq \epsilon \}) \\
\leq \delta(\{ k \leq m : \alpha_m \geq \frac{\epsilon}{3} \}) + \delta(\{ k \leq m : \beta_m \geq \frac{\epsilon}{3} \}) + \delta(\{ k \leq m : \gamma_m \geq \frac{\epsilon}{3} \}).
\]

Using (3.2.4), we get
\[
\text{st} - \lim_{m \to \infty} \| L_{m,p}^*(t^2; q_m; x) - x^2 \| = 0.
\]

Since
\[
\| L_{m,p}^*(f; q_m; x) - f \| \\
\leq \| L_{m,p}^*(t^2; q_m; x) - x^2 \| + \| L_{m,p}^*(t; q_m; x) - x \| + \| L_{m,p}^*(1; q_m; x) - 1 \|,
\]
we get
\[
\text{st} - \lim_{m \to \infty} \| L_{m,p}^*(f; q_m; x) - f \| \\
\leq \text{st} - \lim_{m \to \infty} \| L_{m,p}^*(t^2; q_m; x) - x^2 \| + \text{st} - \lim_{m \to \infty} \| L_{m,p}^*(t; q_m; x) - x \| \\
+ \text{st} - \lim_{m \to \infty} \| L_{m,p}^*(1; q_m; x) - 1 \|,
\]
which implies that
\[
\text{st} - \lim_{m \to \infty} \| L_{m,p}^*(f; q_m; x) - f \| = 0.
\]

This completes the proof of the theorem.

**Remark 3.2.1.** In the following example, we demonstrate that the statistical version is stronger than the ordinary approximation. Let us write \( T_{m,p}^*(f; q_m; x) = (1 + u_m)L_{m,p}^*(f; q_m; x) \) where the sequence \( (u_m) \) is defined by (3.2.1). Then under the hypothesis of the above theorem, we have
\[
\text{st} - \lim_{m \to \infty} \| T_{m,p}^*(f; q_m; x) - f \| = 0.
\]

However, \( \lim_{m \to \infty} \| T_{m,p}^*(f; q_m; x) - f \| \) does not exist, since \( (u_m) \) is statistically convergent but not convergent.
3.3 Direct theorems

The Peetre’s $K$-functional is defined by

$$K_2(f, \delta) = \inf \{ \| f - g \| + \delta \| g'' \| : g \in W^2_p \},$$

where

$$W^2_p = \{ g \in C_B[0, p + 1] : g', g'' \in C_B[0, p + 1] \}.$$  

By [19], there exists a positive constant $C > 0$ such that

$$K_2(f, \delta) < Cw_2(f, \delta^2), \quad \delta > 0;$$

where the second order modulus of continuity is given by

$$w_2(f, \delta^2) = \sup_{0 < h \leq \delta^2} \sup_{x \in [0, p + 1]} |f(x + 2h) - 2f(x + h) + f(x)|.$$ 

Also for $f \in C_B[0, p + 1]$ the usual modulus of continuity is given by

$$w(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0, p + 1]} |f(x + h) - f(x)|.$$ 

Theorem 3.3.1. Let $f \in C_B[0, p + 1]$ and $0 < q_m < 1$ such that $q_m \to 1$ ($m \to \infty$). Then for all $n \in \mathbb{N}$ and $p$ fixed, there exists an absolute constant $C > 0$ such that

$$|L^*_{m, p}(f; q_m; x) - f(x)| \leq Cw_2(f, \delta_m(x)),$$

where

$$\delta_m^2(x) = x^2 \left( \frac{[m + p]q - [m]q^2 - [m + p]q^2}{[m]^2} \right) + x \frac{[m + p]q}{[m]^2}.$$ 

Proof. Let $g \in W_2$. From Taylor’s expansion, we get

$$g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - u) g''(u) \, du, \quad t \in [0, A], \ A > 0,$$

and by Lemmas 3.1.1, 3.1.2 and 3.1.3, we get

$$L^*_{m, p}(g; x) = g(x) + L^*_{m, p} \left( \int_x^t (t - u) g''(u) \, du, x \right).$$

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Let $C_{d+1}$ be the space of all bounded functions for which $\lim_{n \to \infty} f(n) = f(\infty)$ is finite.

This completes the proof of the theorem.

$\left\| (x \cdot f)_{d+1} \right\|_C \geq \left| (x) f - (x \cdot f)^{d+1} T \right|$

In view of the property of $\mathcal{Y}$-function, we get

$\left\| (x \cdot f)_{d+1} \right\|_C \geq \left| (x) f - (x \cdot f)^{d+1} T \right|$

Hence taking infimum on the right hand side over all $\theta \in \mathbb{R}$ we get

$\left\| \theta \right\|_C \left\| \frac{\theta [u]}{d + u} x + \left( \left( \frac{\theta [u]}{d + u} - \frac{\theta [u]}{\theta [d] - \theta [u]} \right) x \right) \right\| \geq \left| (x) \theta - (x \cdot \theta)^{d+1} T \right| + \left| (x)(\theta - f) - (x \cdot \theta - f)^{d+1} T \right| \geq \left| (x) f - (x \cdot f)^{d+1} T \right|$

Now

$\left\| f \right\|_C \geq \left| (x \cdot f)^{d+1} T \right|$

On the other hand by the definition of $\mathcal{Y}$ we have

$\left\| \theta \right\|_C \left\| \frac{\theta [u]}{d + u} x + \left( \left( \frac{\theta [u]}{d + u} - \frac{\theta [u]}{\theta [d] - \theta [u]} \right) x \right) \right\| \geq \left| (x) \theta - (x \cdot \theta)^{d+1} T \right|$

Using Lemma 3.1, we obtain

$\left\| \theta \right\|_C \left( (x \cdot f)^{d+1} T \right) \geq \left( x \cdot \theta \left( \left( \left( \theta \right)_{d+1} \right) \right) \right)^{d+1} T \geq \left| (x) \theta - (x \cdot \theta)^{d+1} T \right|$

$\left( x \cdot \theta \left( \left( \left( \theta \right)_{d+1} \right) \right) \right)^{d+1} T \geq \left| (x) \theta - (x \cdot \theta)^{d+1} T \right|$
Theorem 3.3.2. Let \( f \in C^r_B[0, p + 1] \) be such that \( f', f'' \in C^r_B[0, p + 1] \) and the sequence \((q_m)\) satisfies (3.2.2). Then the following equality holds:

\[
\lim_{m \to \infty} [m]_{q_m} \left(L_{m,p}^*(f; q; x) - f(x) \right) = \frac{x(1-x)}{2} f''
\]

uniformly on \([0, p + 1]\).

Proof. By the Taylor's formula we may write

\[
f(t) = f(x) + f'(x)(t - x) + \frac{1}{2} f''(x)(t - x)^2 + r(t, x)(t - x)^2,
\]

where \( r(t, x) \) is the remainder term and \( \lim_{t \to x} r(t, x) = 0 \). Applying \( L_{m,p}^*(f; q; x) \) to (3.3.1), we obtain

\[
[m]_{q_m} \left(L_{m,p}^*(f; q; x) - f(x) \right) = [m]_{q_m} \left(L_{m,p}^*(t - x; q; x)f'(x) \right) + \frac{f''(x)}{2} + [m]_{q_m} L_{m,p}^*(r(t, x)(t - x)^2; q; x).
\]

By the Cauchy-Schwartz inequality, we have

\[
L_{m,p}^* \left(r(t, x)(t - x)^2; q; x\right) \leq L_{m,p}^* \left(r^2(t, x); q; x\right) \times L_{m,p}^* \left(r(t, x)(t - x)^4; q; x\right).
\]

Observe that \( r^2(x, x) = 0 \) and \( r^2(., x) \in C^r_B[0, p + 1] \), then it follows from Theorem 3.3.1, that

\[
L_{m,p}^* \left(r^2(t, x); q; x\right) = r^2(x, x) = 0.
\]

uniformly with respect to \( x \in [0, p + 1] \). Now from (3.3.2), (3.3.3) and Lemma 3.1.5, we get

\[
\lim_{m \to \infty} [m]_{q_m} L_{m,p}^* \left(r(t, x)(t - x)^2; q; x\right) = 0
\]

Finally using Lemmas 3.1.4 and 3.1.5, we get the following

\[
\lim_{m \to \infty} [m]_{q_m} \left(L_{m,p}^*(f; q; x) - f(x) \right) = \lim_{m \to \infty} [m]_{q_m} \left(L_{m,p}^*(t - x; q; x)f'(x) \right) +
\]

\[
\lim_{m \to \infty} [m]_{q_m} \left(L_{m,p}^*(t - x)^2; q; x\right) \cdot \frac{f''(x)}{2} +
\]

\[
\lim_{m \to \infty} [m]_{q_m} L_{m,p}^* \left(r(t, x)(t - x)^2; q; x\right) = \frac{x(1-x)}{2} f''(x).
\]

This completes the proof of the theorem.
3.4 Graphical analysis and error bound computation using Matlab

In this section, we compute error estimation [65] by using modulus of continuity with the help of Matlab and give its algorithm. We also show graphically the convergence of the q-Bernstein-Schurer operators to various functions.

**Example 3.4.1.** Let us take \( f(x) = 1 + \sin(-6.5x^2) \). We compute error estimation by using modulus of continuity for operators 3.1.2 to the function \( f(x) = 1 + \sin(-6.5x^2) \) shown in the following Table and its algorithm presented after the Table.

**Error estimation table:**

<table>
<thead>
<tr>
<th>m (for p=30, q=0.9)</th>
<th>error bound at x=0.2</th>
<th>error bound at x=0.5</th>
<th>error bound at x=0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.2309</td>
<td>1.5709</td>
<td>2.7170</td>
</tr>
<tr>
<td>30</td>
<td>0.1893</td>
<td>1.1865</td>
<td>1.5829</td>
</tr>
<tr>
<td>50</td>
<td>0.1750</td>
<td>1.0835</td>
<td>1.3792</td>
</tr>
</tbody>
</table>

**Error estimate algorithm:**

```matlab
syms x;
n = [20, 30, 50]; p = 30; q = .9; dat = zeros(3, 4);
for i=1:3
    m = n(i);
    errestimate = inline('x^2((qinteger(m+p,q) - qinteger(m,q))^2 - qinteger(m+p,q))
                         (qinteger(m,q))^2' + 'x(qinteger(m+p,q))
                         (qinteger(m,q))^2) * abs(diff(1 + sin(-6.5x^2), 2))');
    dat(i,1) = n(i);
    dat(i,2) = errestimate(.2);
    dat(i,3) = errestimate(.5);
    dat(i,4) = errestimate(.8);
end
```
Example 3.4.2. For \( m = 20, 30, 50; \) the convergence of operators 3.1.2 to function

\[
f(x) = 1 + \sin(-6.5x^2)
\]

is illustrated in figure 3.1 and its algorithm is presented below.

Algorithm:

\[
n = [20, 30, 50]; \quad p = 30; \quad q = .9;
\]

\[
\text{for } j = 1 : 3
\]

\[
m = n(j);
n = \mathbb{N}; \quad a = [1 : 100];
\]

\[
i = 1;
\]

\[
\text{for } x = 0 : 0.01 : 1
\]

\[
t = 0;
\]

\[
\text{for } k = 0 : m + p
\]

\[
z = 1;
\]

\[
\text{for } s = 0 : m + p - k - 1
\]

\[
z = z (1 - q^s x);
\]

\[
t = t + \left( \frac{\text{qintereg fact}(m + p, q)}{\text{qintereg fact}(k, q) \text{qintereg fact}(m + p - k, q)} \right) x^k z
\]

\[
x^k z \times (1 + \sin(-6.5(\text{qintereg}(k, q)^2)));
\]

\[
\text{end}
\]

\[
a(i) = t;
\]

\[
i = i + 1;
\]

\[
\text{end}
\]

\[
x = 0 : 0.01 : 1;
\]

\[
\text{if } (j == 1)
\]

\[
c = \text{plot}(x, a)
\]

\[
\text{set}(c, 'Color', 'blue', 'LineWidth', 2)
\]

\[
\text{elseif } (j == 2)
\]

\[
c = \text{plot}(x, a, 'g')
\]

\[
c = \text{plot}(x, a, 'g')
\]
set(c,'Color','green','LineWidth',2)
else
c = plot(x,a,'r')
set(c,'Color','red','LineWidth',2)
end
hold on
end
x = [0 : 0.01 : 1];
y = 1 + sin(-6.5x^2);
a = plot(x,y,'--k')

Figure 3.1:
Example 3.4.3. Similarly, Approximation by generalized $q$-Bernstein-Schurer operators for the function $f = (x - \frac{1}{3})(x - \frac{1}{2})(x - \frac{3}{4})$ for different values of 'm', keeping 'q' and 'p' fixed is shown in figure 3.2.

![Graph showing error estimation for different values of m.](image)

Figure 3.2:

Error estimation table: For $f = (x - \frac{1}{3})(x - \frac{1}{2})(x - \frac{3}{4})$

<table>
<thead>
<tr>
<th>m (for p=30, q=0.9)</th>
<th>error bound at x=0.2</th>
<th>error bound at x=0.5</th>
<th>error bound at x=0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.0420</td>
<td>0.0061</td>
<td>0.0521</td>
</tr>
<tr>
<td>30</td>
<td>0.0344</td>
<td>0.0046</td>
<td>0.0303</td>
</tr>
<tr>
<td>50</td>
<td>0.0318</td>
<td>0.0042</td>
<td>0.0264</td>
</tr>
</tbody>
</table>

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Example 3.4.4. Approximation by operators 3.1.2 for the function

\[ f(x) = (x - \frac{1}{3})(x - \frac{1}{2})(x - \frac{3}{4}) \]

for different values of ‘q’, taking ‘m’=10 and ‘p’=2 fixed is shown in figure 3.3.
Example 3.4.5. Approximation by operators 3.1.2 for the function

\[ f(x) = (x - \frac{1}{3})(x - \frac{1}{2})(x - \frac{3}{4}) \]

for different values of 'q', taking 'm'=50 and 'p'=2 fixed is shown in figure 3.4.

Matlab code for q-integer is as follows:

```matlab
function kq = qinteger(k, q)
    if(q == 1)
        kq = (1-q^k) / (1-q);
    else
        kq = k;
    end
end
```
Matlab code for q-integer factorial:

```matlab
function kqfact = qintegerfact(k, q)
    fact = 1;
    if(k == 0)
        kqfact = 1;
    elseif (q == 1)
        kqfact = factorial(k);
    else
        while(k 
```

---

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Chapter 4

Operators Constructed by Means of $q$-Lagrange Polynomials and $A$-Statistical Approximation

4.1 Introduction

In this chapter, we construct a new family of operators with the help of $q$-analogue of Chan-Chyan-Srivastava polynomials, and study the statistical approximation properties via $A$-statistical convergence. We also study some approximation properties for the rate of $A$-statistical convergence with the help of modulus of continuity and Lipschitz class.

4.2 Construction of a new operator and its properties

For a real or complex number $q \ (|q|<1)$, the number $(\lambda; q)_n$ is defined by

$$(\lambda; q)_n = \frac{(\lambda; q)_\infty}{(\lambda q^n; q)_\infty},$$

where

$$(\lambda; q)_\infty = \prod_{k=1}^{\infty}(1 - \lambda q^k),$$

and

$$(\lambda; q)_n = \begin{cases} 1 & \text{if } n = 0, \\ (1 - \lambda)(1 - \lambda q) \cdots (1 - \lambda q^{n-1}) & \text{if } n = 1, 2, 3, \cdots \end{cases}$$

The contents of this chapter have been published in *Applied Mathematics and Computation*, 219 (2013), 6911-6918.
In [14], Chan, Chyan and Srivastava introduced and studied the following multi-variable Lagrange polynomials:

\[ g_n^{(a_1, \ldots, a_r)}(x_1, \ldots, x_r) = \sum_{k_1+k_2+\ldots+k_r=n} (a_1)_{k_1} (a_2)_{k_2} \cdots (a_r)_{k_r} \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!}, \]

(4.2.1)

where \( (\lambda)_k = \lambda(\lambda + 1) \cdots (\lambda + k - 1) \) and \( (\lambda)_0 = 1 \).

Using the above polynomials, Erkuş, Duman and Srivastava [23] introduced the following family of positive linear operators on \( C[0, 1] \):

\[ L_n^{(u_1, \ldots, u_r)}(f; x) = \]

\[ \left\{ \prod_{j=1}^{r}(1-xu_n^{(j)})^n \right\} \sum_{m=0}^{\infty} \left\{ \sum_{k_1+k_2+\ldots+k_r=m} f \left( \frac{k_r}{n+k_r-1} \right) \frac{(u_n^{(1)})_{k_1}}{k_1!} \cdots \frac{(u_n^{(r)})_{k_r}}{k_r!} (n)_{k_1} \cdots (n)_{k_r} \right\} x^m \]

(4.2.2)

where \( u^{(j)} = (u_n^{(j)})_{n \in \mathbb{N}} \) are sequences of real numbers such that \( 0 < u_n^{(j)} < 1 \), \( (j = 1, 2, \ldots, r, n \in \mathbb{N}) \) and \( f \in C[0, 1], \ x \in [0, 1] \).

Recently, Altin, Erkuş and Taşdelen [5] introduced \( g_n^{(a_1, \ldots, a_r)}(x_1, \ldots, x_r) \) the \( q \)-analogue of (4.2.1), generated by

\[ \prod_{i=1}^{r} \frac{1}{(x_it; q)_{a_i}} = \prod_{i=1}^{r} (1-tx_iq^{k_i}-a_i) = \sum_{m=0}^{\infty} g_n^{(a_1, \ldots, a_r)}(x_1, \ldots, x_r)t^m \]

\[ = \sum_{m=0}^{\infty} \left\{ \sum_{k_1+k_2+\ldots+k_r=m} (q^{a_1}, q)_{k_1} (q^{a_2}, q)_{k_2} \cdots (q^{a_r}, q)_{k_r} \frac{x_1^{k_1}}{(q, q)_{k_1}} \cdots \frac{x_r^{k_r}}{(q, q)_{k_r}} \right\} t^m, \]

(4.2.3)

where

\[ ||t|| < \min\{|x_1|, |x_2|\}. \]

We define the following family of positive linear operators on \( C[0, 1] \) which is \( q \)-analogue of 4.2.2.

\[ L_{n,q}^{(u_1, \ldots, u_r)}(f; x) = \left\{ \prod_{j=1}^{r}(1-xu_n^{(j)}q^{n})^n \right\} \sum_{m=0}^{\infty} \left\{ \sum_{k_1+k_2+\ldots+k_r=m} (q^{a_1}, q)_{k_1} \cdots (q^{a_r}, q)_{k_r} \right\} t^m, \]

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In this section, we investigate some basic properties of the positive linear operators $L_{n,q}^{(1),\ldots,n^{(r)}}(f;x)$ given by (4.2.4) via the concept of $A$-statistical convergence.

We will first consider the case $r = 2$ in (4.2.4). In this case, we have

$$\sum_{j=1}^{2}(1-xu_n^{(j)}q^2)^2n \sum_{m=0}^{\infty} \{ \sum_{k_1+k_2=m} (q^{a_1}, q)_{k_1} (q^{a_2}, q)_{k_2}$$

$$\times f\left( \frac{(q,q)_{k_2}}{n+(q,q)_{k_2}-1} \right) \frac{(u_n^{(1)})_{k_1} (u_n^{(2)})_{k_2}}{(q,q)_{k_1} (q,q)_{k_2}} \} x^m.\tag{4.2.5}$$

Then we have the following preliminary results.

**Lemma 4.2.1.** For each $x \in [0,1]$ and $n \in \mathbb{N}$,

$$L_{n,q}^{u^{(1)},u^{(2)}}(f_0;x) = \{ f_0(x) = 1 \}.$$

**Proof.** In (4.2.4), we first set $r = 2$, and $\alpha_1 = \alpha_2 = n$. Observe that, since $0 < u_n^{(1)}, u_n^{(2)} < 1$ ($n \in \mathbb{N}$) the condition is satisfied for each $x \in [0,1]$. Then by (4.2.3), we get

$$L_{n,q}^{u^{(1)},u^{(2)}}(f_0;x) = \sum_{j=1}^{2}(1-xu_n^{(j)}q^2)^2n \sum_{m=0}^{\infty} g_{n,q}^{(n,m)}(u_n^{(1)}, u_n^{(2)}x^m = 1.$$

**Lemma 4.2.2.** For each $x \in [0,1]$ and $n \in \mathbb{N}$,

$$L_{n,q}^{u^{(1)},u^{(2)}}(f_1;x) = xu_n^{(2)} \quad (f_1(x) = x).$$

**Proof.** Let $x \in [0,1]$ be fixed. Then from (4.2.5) we get

$$L_{n,q}^{u^{(1)},u^{(2)}}(f_1;x) = \sum_{j=1}^{2}(1-xu_n^{(j)}q^2)^2n \sum_{m=0}^{\infty} \{ \sum_{k_1+k_2=m} (q^{a_1}, q)_{k_1} (q^{a_2}, q)_{k_2}$$

$$\times f\left( \frac{(q,q)_{k_2}}{n+(q,q)_{k_2}-1} \right) \frac{(u_n^{(1)})_{k_1} (u_n^{(2)})_{k_2}}{(q,q)_{k_1} (q,q)_{k_2}} \} x^m.$$
\[ L_{n,q}^{u(1),u(2)}(f_1; x) = \prod_{j=1}^{2}(1 - xu_n^{(j)}q^2)^n \sum_{m=0}^{\infty} \left\{ \sum_{k=1}^{m} (q^{a1}, q)_{m-k} (q^{a2}, q)_k \times f\left(\frac{(q, q)_k}{n + (q, q)_k - 1}\right) \frac{u_n^{(1)}}{(q, q)_{m-k}} \frac{u_n^{(2)}}{(q, q)_k} \right\} x^{m-1}. \]

Hence

\[ L_{n,q}^{u(1),u(2)}(f; x) = xu_n^{(2)} \prod_{j=1}^{2}(1 - xu_n^{(j)}q^2)^n \sum_{m=0}^{\infty} g_{m,n}^{(n,n)} (u_n^{(1)}, u_n^{(2)}) x^m = xu_n^{(2)}. \]

**Lemma 4.2.3.** For each \( x \in [0, 1] \) and \( n \in \mathbb{N}, \)

\[ |L_{n,q}^{u(1),u(2)}(f_2; x)| \leq 2x^2\{1 - (u_n^{(2)})^2\} + \frac{xu_n^{(2)}}{n} \quad (f_2(x) = x^2). \]

**Proof.** Let \( x \in [0, 1] \) be fixed. Then from (4.2.5) we get

\[ L_{n,q}^{u(1),u(2)}(f_2; x) = \prod_{j=1}^{2}(1 - xu_n^{(j)}q^2)^n \sum_{m=0}^{\infty} \left\{ \sum_{k=1}^{m} (q^{a1}, q)_{m-k} (q^{a2}, q)_k \times f\left(\frac{(q, q)_k}{n + (q, q)_k - 1}\right) \frac{u_n^{(1)}}{(q, q)_{m-k}} \frac{u_n^{(2)}}{(q, q)_k} \right\} x^{m-1}. \]

After a simple calculation, we get

\[ L_{n,q}^{u(1),u(2)}(f_2; x) = x^2(u_n^{(2)}) \prod_{j=1}^{2}(1 - xu_n^{(j)}q^2)^n \sum_{m=2}^{\infty} g_{m+2,n}^{(n,n)} x^{m-2} + x(u_n^{(2)})^2 \prod_{j=1}^{2}(1 - xu_n^{(j)}q^2)^n \sum_{m=2}^{\infty} g_{m+2,n}^{(n,n)} x^{m-2} \]

\[ L_{n,q}^{u(1),u(2)}(f_2; x) \leq x^2(u_n^{(2)})^2 + \frac{xu_n^{(2)}}{n} \quad (4.2.6) \]

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Also

\[ 0 \leq L_{n,q}^{u(1),u(2)}((y-x)^2; x) = L_{n,q}^{u(1),u(2)}(f_2; x) - 2xL_{n,q}^{u(1),u(2)}(f_1; x) + x^2. \]

It follows from Lemma 4.2.1 and 4.2.2 that

\[ L_{n,q}^{u(1),u(2)}(f_2; x) - f_2(x) \geq 2x^2(u_n^{(2)} - 1)^2. \tag{4.2.7} \]

From (4.2.6) and (4.2.7), we get

\[ |L_{n,q}^{u(1),u(2)}(f_2; x)| \leq 2x^2(1 - (u_n^{(2)})) + \frac{xu_n^{(2)}}{n}. \]

### 4.3 A-statistical approximation

We know that \( C[a, b] \) is a Banach space with norm

\[ \| f \|_{C[a,b]} = \sup_{x \in [a,b]} |f(x)|, \quad f \in C[a,b]. \]

For typographical convenience, we will write \( \| . \| \) in place of \( \| . \|_{C[a,b]} \) if no confusion arises.

**Theorem 4.3.1.** Let \( A = (a_{jn}) \) be a non-negative regular summability matrix. Then

\[ \text{st}_A \lim_n u_n^{(2)} = 1, \tag{4.3.1} \]

if and only if, for all \( f \in C[0, 1] \)

\[ \text{st}_A \lim_n \|L_{n,q}^{u(1),u(2)}(f) - f\| = 0. \tag{4.3.2} \]

**Proof.** Suppose that (4.3.2) holds for all \( f \in C[0, 1] \). Then we have

\[ \text{st}_A \lim_n \|L_{n,q}^{u(1),u(2)}(f_1) - f_1\| = 0, \tag{4.3.3} \]

since \( f_1 \in C[0, 1] \). By Lemma (4.2.2), we have

\[ \|L_{n,q}^{u(1),u(2)}(f_1) - f_1\| = 1 - u_n^{(2)}. \tag{4.3.4} \]
By (4.3.3) and (4.3.4), we immediately get

$$\text{st}_A - \lim_{n} u_n^{(2)} = 1.$$  

Conversely, suppose that (4.3.1) holds. Then from Lemma 4.2.1, we have

$$\lim_n \|L_{n,q}^{(1),u^{(2)}}(f_0) - f_0\| = 0.$$ 
Hence

$$\text{st}_A - \lim_{n} \|L_{n,q}^{(1),u^{(2)}}(f_0) - f_0\| = 0, \ (f_0(x) = 1) \quad (4.3.5)$$

Also from Lemma 4.2.2, it follows that

$$\|L_{n,q}^{(1),u^{(2)}}(f_1) - f_1\| = 1 - u_n^{(2)}.$$

Therefore by using (4.3.1), we get

$$\text{st}_A - \lim_{n} \|L_{n,q}^{(1),u^{(2)}}(f_1) - f_1\| = 0, \ (f_1(x) = x). \quad (4.3.6)$$

Now we claim that

$$\text{st}_A - \lim_{n} \|L_{n,q}^{(1),u^{(2)}}(f_2) - f_2\| = 0, \ (f_2(x) = x^2). \quad (4.3.7)$$

By Lemma 4.2.3, we have

$$\|L_{n,q}^{(1),u^{(2)}}(f_2) - f_2\| \leq 2(1 - u_n^{(2)}) + \frac{u_n^{(2)}}{n}. \quad (4.3.8)$$

Now, for a given $\epsilon > 0$, we define the following sets:

$$D = \{n : \|L_{n,q}^{(1),u^{(2)}}(f_2) - f_2\| \geq \epsilon\},$$

$$D_1 = \{n : 1 - u_n^{(2)} \geq \frac{\epsilon}{4}\},$$

$$D_2 = \{n : \frac{u_n^{(2)}}{n} \geq \frac{\epsilon}{2}\}.$$ 

From (4.3.8), it is easy to see that $D \subseteq D_1 \cup D_2$. Then, for each $j \in N$, we get

$$\sum_{n \in D} a_{jn} \leq \sum_{n \in D_1} a_{jn} + \sum_{n \in D_2} a_{jn}. \quad (4.3.9)$$

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Using (4.3.3), we get

$$\text{StA} - \lim_{n} (1 - u_{n}^{(2)}) = 0,$$

and

$$\text{StA} - \lim_{n} \frac{u_{n}^{(2)}}{n} = 0.$$

Now using the above facts and taking the limit as \( j \to \infty \) in (4.3.9), we conclude that

$$\lim_{j} \sum_{n \in D} a_{jn} = 0,$$

which gives (4.3.7). Now, combining (4.3.5)-(4.3.7), and using the statistical version of Korovkin approximation theorem (see Gadjiv and Orhan [[28], Theorem 1]), we get the desired result.

This completes the proof of the theorem.

In a similar manner, we can extend Theorem 1 to the \( r \)-dimensional case for the operators \( L_{n}^{u^{(1)}, \ldots, u^{(r)}} (f; x) \) given by (4.2.4) as follows.

**Theorem 4.3.2.** Let \( A = (a_{jn}) \) be a non-negative regular summability matrix. Then

$$\text{StA} - \lim_{n} u_{n}^{(r)} = 1,$$

if and only if, for all \( f \in C[0, 1] \)

$$\text{StA} - \lim_{n} \| L_{n, q}^{u^{(1)}, \ldots, u^{(r)}} (f) - f \| = 0.$$

**Remark 4.3.1.** If, in Theorem 4.3.2, we replace \( A = (a_{jn}) \) by the identity matrix, we immediately get the following theorem which is a classical case of Theorem 4.3.2.

**Theorem 4.3.3.** \( \lim_{n} u_{n}^{(r)} = 1 \), if and only if, for all \( f \in C[0, 1] \), the sequence \( L_{n, q}^{u^{(1)}, \ldots, u^{(r)}} (f) \) is uniformly convergent to \( f \) on \([0, 1]\).

Finally, we display an example which satisfies all hypotheses of Theorem 4.3.2, but not of Theorem 4.3.3. Therefore this indicates that our \( A \)-statistical approximation in Theorem 4.3.2 is stronger than its classical case.
Take $A = C = (c_{jn})$, the Cesaro matrix of order 1 and

$$u^{(j)} = (u^{(j)}_n)_{n \in \mathbb{N}} \quad (j = 1, ..., r - 1)$$

be sequences of real numbers defined by

$$u^{(j)}_n = \begin{cases} 
\frac{1}{3} & \text{if } n = m^2, \; (m \in \mathbb{N}); \\
\frac{n^2 - 1}{n^2 + 1} & \text{otherwise}.
\end{cases} \quad (4.3.10)$$

We then observe that

$$0 < u^{(j)}_n < 1 \quad (n \in \mathbb{N})$$

and also that

$$\text{st}_A - \lim_{n \to \infty} u^{(j)}_n = 1$$

Therefore, by Theorem 4.3.2, we have that for all $f \in C[0, 1]$

$$\text{st}_A - \lim_{n \to \infty} \| L^{(j)}_{n_1, ..., n_r} (f) - f \| = 0.$$

However, since the sequence $u^{(j)}_n$ defined by (4.3.10) is non-convergent, Theorem 4.3.3 does not hold in this case.

### 4.4 Rate of $A$-statistical convergence

In this section, we compute the rates of $A$-statistical convergence of our sequence of positive linear operators in Theorem 4.3.2 by means of the modulus of continuity and the elements of Lipschitz class. Let $f \in C[0, 1]$. The modulus of continuity of $f$ denoted by $w(f, \delta)$ is defined by

$$w(f, \delta) = \sup_{|y - x| < \delta, x, y \in [0, 1]} | f(y) - f(x) |.$$

It is well known that the modulus of continuity of the function $f \in C[0, 1]$ provides the maximum oscillation of $f$ in any interval of length not exceeding $\varepsilon > 0$. A necessary and sufficient condition for a function $f \in C[0, 1]$ is that

$$\lim_{\delta \to 0} w(f, \delta) = 0.$$
It is also well known that, for any \( \delta > 0 \) and for each \( x, y \in [0, 1] \),
\[
| f(y) - f(x) | \leq \left( \frac{|y - x|}{\delta} + 1 \right) w(f, \delta). \tag{4.4.1}
\]

Now we prove the following result.

**Theorem 4.4.1.** For each \( n \in \mathbb{N} \) and for all \( f \in C[0, 1] \),
\[
\|L_{n,q}^{(1),...,u(r)}(f) - f\| \leq 2w(f, \delta_n), \tag{4.4.2}
\]
where
\[
\delta_n = \left\{ \frac{u(r)}{n} + 4(1 - u(r)_n) \right\} \tag{4.4.3}
\]

**Proof.** Let \( f \in C[0, 1] \) and let \( x \in [0, 1] \) be fixed. By linearity and monotonicity of \( L_{n,q}^{(1),...,u(r)}(f; x) \), and using (4.4.1), we have
\[
| L_{n,q}^{(1),...,u(r)}(f; x) - f(x) | \leq L_{n,q}^{(1),...,u(r)}(f(y) - f(x); x)
\leq w(f, \delta) L_{n,q}^{(1),...,u(r)}(\frac{|y - x|}{\delta} + 1; x)
= w(f, \delta) \left\{ 1 + \frac{1}{\delta} (L_{n,q}^{(1),...,u(r)}(\phi; x))^{\frac{1}{2}} \right\}.
\]

for any \( \delta > 0 \).

Furthermore, by the Cauchy-Schwartz inequality for positive linear operators, we have
\[
\| L_{n,q}^{(1),...,u(r)}(f; x) - f(x) \| \leq w(f, \delta) \left\{ 1 + \frac{1}{\delta} (L_{n,q}^{(1),...,u(r)}(\phi; x))^{\frac{1}{2}} \right\} (\phi = \phi(y) = (y - x)^2).
\tag{4.4.4}
\]

Since
\[
L_{n,q}^{(1),...,u(r)}(\phi; x) = L_{n,q}^{(1),...,u(r)}(f_3; x) - 2x L_{n,q}^{(1),...,u(r)}(f_1; x) + x^2
\leq | L_{n,q}^{(1),...,u(r)}(f_3; x) - f_2(x) | + 2x \left| L_{n,q}^{(1),...,u(r)}(f_1; x) - f_1(x) \right|,
\]
which, by using Lemmas 4.2.2 and 4.2.3, gives
\[
L_{n,q}^{(1),...,u(r)}(\phi; x) \leq 2x^2(1 - u(r)_n) + \frac{xu(r)_n}{n} + 2x^2(1 - u(r)_n).
\]
Now, combining the last inequality with (4.4.4), we get
\[
| L_{n,q}^{u_1^{(r)}, \ldots, u_r^{(r)}}(f; x) - f(x) | \leq w(f, \delta) \left\{ 1 + \frac{1}{\delta} \left( \frac{x u_n^{(r)}}{n} + 4x^2(1 - u_n^{(r)})^{\frac{1}{r}} \right) \right\},
\]
which implies that
\[
| L_{n,q}^{u_1^{(r)}, \ldots, u_r^{(r)}}(f) - f | \leq w(f, \delta) \left\{ 1 + \frac{1}{\delta} \left( \frac{u_n^{(r)}}{n} + 4(1 - u_n^{(r)})^{\frac{1}{r}} \right) \right\}. \tag{4.4.5}
\]
If we choose \( \delta = \delta_n \) where \( \delta_n \) is given as in (4.4.3), then the assertion (4.4.2) of Theorem 4.4.1 follows immediately from (4.4.5).

We now turn to our investigation of the rate of convergence of the positive linear operators \( L_{n,q}^{u_1^{(r)}, \ldots, u_r^{(r)}}(f; x) \) given by (4.2.6) by means of elements of Lipschitz class \( \text{Lip}_M(\alpha) \) \((0 < \alpha \leq 1)\). Recall that a function \( f \in C[0,1] \) belongs to the Lipschitz class \( \text{Lip}_M(\alpha) \) \((0 < \alpha \leq 1)\) if the following inequality holds:
\[
| f(y) - f(x) | \leq M | y - x |^{\alpha} \quad (x, y \in [0,1]). \tag{4.4.6}
\]

**Theorem 4.4.2.** For each \( n \in \mathbb{N} \) and for all \( f \in \text{Lip}_M(\alpha) \),
\[
| L_{n,q}^{u_1^{(r)}, \ldots, u_r^{(r)}}(f) - f | \leq 2\delta_n^{\alpha}, \tag{4.4.7}
\]
where \( \delta_n \) is same as in Theorem 4.4.1.

**Proof.**
\[
| L_{n,q}^{u_1^{(r)}, \ldots, u_r^{(r)}}(f; x) - f(x) | \leq L_{n,q}^{u_1^{(r)}, \ldots, u_r^{(r)}}(| f(y) - f(x) |; x)
\leq M L_{n,q}^{u_1^{(r)}, \ldots, u_r^{(r)}}(| y - x |^{\alpha}; x).
\]

Now, by applying Holder's inequality for \( p = \frac{2}{\alpha} \) and \( q = \frac{2}{2 - \alpha} \), we get
\[
| L_{n,q}^{u_1^{(r)}, \ldots, u_r^{(r)}}(f; x) - f(x) | \leq M \{ L_{n,q}^{u_1^{(r)}, \ldots, u_r^{(r)}}(\phi; x) \}. \tag{4.4.8}
\]
Since
\[
L_{n,q}^{u_1^{(r)}, \ldots, u_r^{(r)}}(\phi; x) \leq \frac{x u_n^{(r)}}{n} + 4x^2(1 - u_n^{(r)}),
\]
...
from (4.4.8) we get that

\[ |L_{n,q}^{(r)}(f;x) - f(x)| \leq M \left\{ \frac{xu_n^{(r)}}{n} + 4x^2(1 - u_n^{(r)}) \right\}^{\frac{3}{2}}, \]

which immediately implies that

\[ \|L_{n,q}^{(r)}(f) - f\| \leq M \left\{ \frac{xu_n^{(r)}}{n} + 4x^2(1 - u_n^{(r)}) \right\}^{\frac{3}{2}}. \] (4.4.9)

If we take \( \delta_n \) as in (4.4.3), then the assertion (4.4.7) follows immediately.

### 4.5 Concluding remark

**Remark 4.5.1.** For a non negative summability matrix \( A = (a_{jn}) \), if we take \( st_A - \lim u_n^{(r)} = 1 \) then \( st_A - \lim \delta_n = 0 \). It also implies that \( st_A - \lim w(f, \delta_n) = 0, (f \in C[0,1]) \). Thus clearly Theorem 4.4.1 and 4.4.2 give us the rate of \( A \)-statistical convergence in Theorem 4.3.2. If we take the matrix \( A = I \), the identity matrix, then we get the corresponding rates of ordinary convergence.
Chapter 5

Generalized Equi-Statistical Convergence of Positive Linear Operators and Associated Approximation Theorems

5.1 Introduction

The concept of equi-statistical convergence, statistical pointwise convergence and statistical uniform convergence for sequences of functions were introduced by Balcerzak et al. [12] and was subsequently applied for deriving approximation theorems in [1] and ([31]-[33]) (see also the closely-related recent works [10], [22] and [51]). In this chapter, we introduce the concept of $\lambda$-equi-statistical convergence, $\lambda$-statistical pointwise convergence and $\lambda$-statistical uniform convergence for a sequence of real-valued functions and show that the $\lambda$-equi-statistical convergence lies between the $\lambda$-statistical pointwise and the $\lambda$-statistical uniform convergence. Inclusion relation between equi-statistical and $\lambda$-equi-statistical convergence is established and it is proved that, under some conditions, the $\lambda$-equi-statistical convergence and the equi-statistical convergence are equivalent to each other. We apply our new notion of $\lambda$-equi-statistical convergence to prove a Korovkin type approximation theorem. We also prove a Voronovskaja type approximation theorem via $\lambda$-equi-statistical convergence. Finally, we study the rate of $\lambda$-equi-statistical convergence of a sequence of positive linear operators defined on $C(X)$ (cf. [60]) and Some interesting examples are also displayed here in support of our definitions and results.

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5.2 $\lambda$-Equi-statistical convergence

We define the following concepts by using $\lambda$-statistical convergence. Let $f$ and $f_n (n \in \mathbb{N})$ be real-valued functions defined on a subset $X$ of the set $\mathbb{N}$ of positive integers.

**Definition 5.2.1.** A sequence $(f_n)$ of real-valued functions is said to be $\lambda$-equi-statistically convergent to $f$ on $X$ if, for every $\varepsilon > 0$, the sequence $(S_n(\varepsilon, x))_{n \in \mathbb{N}}$ of real-valued functions converges uniformly to the zero function on $X$, that is, if, for every $\varepsilon > 0$, we have

$$
\lim_{n \to \infty} \|S_n(\varepsilon, x)\|_{C(X)} = 0,
$$

where

$$
S_n(\varepsilon, x) = \frac{1}{\lambda_n} \sum_{k \in I_n} \{ k : k \in I_n \quad \text{and} \quad |f_k(x) - f(x)| \geq \varepsilon \} = 0
$$

and $C(X)$ denotes the space of all continuous functions on $X$. In this case, we write

$$
f_n \sim f \quad (\lambda\text{-equi-stat}).
$$

**Definition 5.2.2.** A sequence $(f_n)$ is said to be $\lambda$-statistically pointwise convergent to $f$ on $X$ if, for every $\varepsilon > 0$ and for each $x \in X$, we have

$$
\lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} \{ k : f_k(x) - f(x) \geq \varepsilon \} = 0.
$$

In this case, we write

$$
f_n \to f \quad (\lambda\text{-stat}).
$$

**Definition 5.2.3.** A sequence $(f_n)$ is said to be $\lambda$-statistically uniform convergent to $f$ on $X$ if for every $\varepsilon > 0$, we have

$$
\lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} \|f_k - f\|_{C(X)} = 0.
$$

In this case, we write

$$
f_n \Rightarrow f \quad (\lambda\text{-stat}).
$$

**Definition 5.2.4.** (see [31]). A sequence $(f_n)$ of real-valued functions is said to be equi-statistically convergent to $f$ on $X$ if, for every $\varepsilon > 0$, the sequence $(P_n(\varepsilon, x))_{n \in \mathbb{N}}$ of real-valued functions converges uniformly to the zero function on $X$, that is, if for every $\varepsilon > 0$ we have

$$
\lim_{n \to \infty} \|P_n(\varepsilon, x)\|_{C(X)} = 0,
$$

where

$$
P_n(\varepsilon, x) = \frac{1}{n} \sum_{k \leq n} \{ k : |f_k(x) - f(x)| \geq \varepsilon \} = 0.
$$
In this case, we write
\[ f_n \rightsquigarrow f \ (\text{equi-stat}). \]

The following implications of the above definitions and concepts are trivial.

**Lemma 5.2.1.** Each of the following implications holds true:

\[ f_n \rightarrow f \ (\lambda\text{-stat}) \Rightarrow f_n \rightsquigarrow f \ (\lambda\text{-equi-stat}) \Rightarrow f_n \rightarrow f \ (\lambda\text{-stat}). \]

Furthermore, in general, the reverse implications do not hold true.

**Example 5.2.1.** Let \( \lambda = (\lambda_n) \) be a sequence as described above and consider the sequence of continuous functions \( f_r : [0, 1] \rightarrow \mathbb{R} \ (r \in \mathbb{N}) \), defined as follows:

\[
 f_r(x) = \begin{cases} 
 \frac{(r+1)^2}{x^2 + 1} & \text{for } x \in \left[0, \frac{1}{r+1}\right] \\
 0 & \text{otherwise}.
\end{cases}
\]

(5.2.1)

Then, for every \( \epsilon > 0 \), we have

\[
 \frac{1}{\lambda_n} \left| \{ r : r \in I_n \text{ and } |f_r(x)| \geq \epsilon \} \right|
\leq \frac{1}{\lambda_n} \rightarrow 0 \quad (n \to \infty)
\]

uniformly in \( x \). This implies that

\[ f_r \rightarrow 0 \ (\lambda\text{-equi-stat}). \]

But, since

\[
 \sup_{x \in [0, 1]} |f_r(x)| = 1 \quad (r \in \mathbb{N}),
\]

we conclude that the following condition:

\[ f_r \rightarrow 0 \ (\lambda\text{-stat}) \]

does not hold true.

**Example 5.2.2.** Let \( \lambda_n = \lfloor \sqrt{n} \rfloor \) and consider the sequence of continuous functions

\[ f_r : [0, 1] \rightarrow \mathbb{R} \quad (f_r(x) = x^r \ (r \in \mathbb{N})). \]

If \( f \) is the pointwise limit of \( f_r \) (in the ordinary sense), then

\[ f_r \rightarrow f \ (\lambda\text{-stat}). \]
but the condition:  

\[ f_r \rightsquigarrow f \text{ (equi-stat).} \]

does not hold true. Let us take \( \varepsilon = \frac{1}{2} \). Then, for all \( n \in \mathbb{N} \), there exists \( r > N \) such that \[ m \in [n - \lfloor \lambda_n \rfloor + 1, n] \quad \text{and} \quad x \in \left( \frac{\sqrt{\frac{1}{2}}}{n}, 1 \right), \]
so that \[ |f_m(x)| = |x^m| \geq \left| \left( \frac{\sqrt{\frac{1}{2}}}{n} \right)^m \right| \geq \left| \left( \frac{\sqrt{\frac{1}{2}}}{2} \right)^n \right| = \frac{1}{2}. \]

Lemma 5.2.2. Let \( \hat{S} \) and \( \hat{S}_\lambda \) be the sets of all equi-statistically convergent and \( \lambda \)-equi-statistically convergent sequences, respectively. Then

\[ \hat{S} \subseteq \hat{S}_\lambda \iff \liminf_{n \to \infty} \frac{\lambda_n}{n} > 0. \quad (5.2.2) \]

Proof. For a given \( \varepsilon > 0 \), we have

\[ \{ k : k \leq n \text{ and } |f_k - f| \geq \varepsilon \} \supseteq \{ k : k \in I_n \text{ and } |f_k - f| \geq \varepsilon \}. \]

We, therefore, find that

\[ \frac{1}{n} \left| \{ k : k \leq n \text{ and } |f_k - f| \geq \varepsilon \} \right| \geq \frac{1}{n} \left| \{ k : k \in I_n \text{ and } |f_k - f| \geq \varepsilon \} \right| \]

\[ \geq \frac{\lambda_n}{n} \cdot \frac{1}{\lambda_n} \left| \{ k : k \leq n \text{ and } |f_k - f| \geq \varepsilon \} \right|, \]

which, upon proceeding to the limit as \( n \to \infty \) and using the fact that \( \liminf_{n \to \infty} \frac{\lambda_n}{n} > 0 \), shows that \( \hat{S} \subseteq \hat{S}_\lambda \).

Conversely, we suppose that

\[ \liminf_{n \to \infty} \frac{\lambda_n}{n} = 0. \]

As in [49], we can choose a subsequence \( (n(j))_{j=1}^\infty \) such that

\[ \frac{\lambda_{n(j)}}{n(j)} \leq \frac{1}{j} \quad (j \in \mathbb{N}). \]
Define a sequence \( f_i(x) \) by
\[
f_i(x) = \begin{cases} 
1 & (i \in I_n(j), j \in N) \\
0 & \text{(otherwise)}. 
\end{cases}
\]

Then this sequence is equi-statistically convergent to 0, but \( x \notin \hat{S}_\lambda \). Hence the condition (5.2.2) is necessary.

This evidently completes the proof of the theorem.

Remark 5.2.1. Obviously, since
\[
\frac{\lambda_n}{n} \leq 1 \quad (n \in N),
\]
we have \( \hat{S}_\lambda \subseteq \hat{S} \).

5.3 Korovkin type approximation theorem

In this section, we extend the result of Karakus et al. [31] by using the notion of \( \lambda \)-equi-statistical convergence.

Theorem 5.3.1. Let \( X \) be a compact subset of the set \( \mathbb{R} \) of real numbers. Also let \( \{L_n\} \) be a sequence of positive linear operators from \( C(X) \) into itself. Then, for all \( f \in C(X) \),
\[
L_n(f) \rightarrow f(\lambda\text{-equi-stat}) \quad \text{on} \quad X \\
\iff L_n(e_i) \rightarrow e_i(\lambda\text{-equi-stat}) \quad \text{on} \quad X,
\]

(5.3.1)

with
\[
e_i(x) = x^i \quad (i = 0, 1, 2).
\]

(5.3.2)

Proof. Since each \( e_i \in C(X) \) \( (i = 0, 1, 2) \), the forward implication in (5.3.1) is obvious. In order to complete the proof of the assertion (5.3.1) of Theorem 5.3.1, we first assume that the second member of implication in (5.3.1) holds true. Let \( f \in C(X) \) and \( x \in X \) be fixed. By the continuity of \( f \) at the point \( x \), we may write that, for every \( \epsilon > 0 \), there exists a number \( \delta > 0 \) such that
\[
|f(y) - f(x)| = |f(y) - f(x)|_{X \cap \{y\}} + |f(y) - f(x)|_{X \setminus X \cap \{y\}},
\]

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(\mathbf{3.3})

\begin{align*}
(x_0, (x_1)_{\omega}) \subseteq (x_1)_{\omega} + \gamma & \le \left\{(x_0, (x_1)_{\omega}) \subseteq (x_1)_{\omega} + \gamma \right\} = (\mathbf{3.3})_{\omega} \\
\text{and} & \\
\{(x_0, (x_1)_{\omega}) \subseteq (x_1)_{\omega} + \gamma \} = (\mathbf{3.3})_{\omega} & \text{by setting } \\
\text{Now, for each } \epsilon > 0, \text{ we choose } c \epsilon < \epsilon \text{ such that } c_\epsilon + \epsilon \ge 2 \epsilon \\
\text{Then, for each } t \epsilon (0, 1, 2) \\
\text{We obtain} \\
\text{Therefore, we conclude that} \\
\text{where}
\end{align*}

\begin{align*}
(x_0, (x_1)_{\omega}) \subseteq (x_1)_{\omega} + \gamma & \le \left\{ (x_0, (x_1)_{\omega}) \subseteq (x_1)_{\omega} + \gamma \right\} = (\mathbf{3.3})_{\omega} \\
\text{By the positivity and linearity of the operator } \mathbf{3.3} & \text{ we conclude that} \\
\text{where}
\end{align*}
Hence we have
\[
\frac{\|\psi_n(\cdot, r)\|_{C([x])}}{\lambda_n} \leq \sum_{i=0}^{2} \left( \frac{\|\psi_{i,n}(\cdot, r)\|_{C([x])}}{\lambda_n} \right).
\]
(5.3.4)

Now, using the above assumption about the second member of the implication in (5.3.1) and Definition (5.2.1), the right-hand side of (5.4.4) is seen to tend to zero as \( n \to \infty \). Consequently, we get

\[
\lim_{n \to \infty} \frac{\|\psi_n(\cdot, r)\|_{C([x])}}{\lambda_n} = 0 \quad (r > 0),
\]
that is, the backward implication in the assertion (5.3.1) holds true.

This completes the proof of Theorem.

Example 5.3.1. Let \( X = [0,1] \) and consider the operator \( x(1 + x D) \) \((D = \frac{d}{dx})\), which was used by (for example) Al-Salām [4] (see also a more recent investigation by Viskov and Srivastava [72, p. 9, Equation (53)]). Here we use this operator over the classical Bernstein polynomials \( B_n(f; x) \) on \( C[0,1] \) and introduce the following family of linear operators on \( C[0,1] \):

\[
\theta_n(f; x) = [1 + f_n(x)]x(1 + x D)B_n(f; x)
\]
(5.3.5)

\[
\left( x \in [0,1]; \ f \in C[0,1]; \ D = \frac{d}{dx} \right),
\]
where \( \{f_n(x)\} \) is a sequence of functions. Then we have

\[
\theta_n(e_0; x) = [1 + f_n(x)]x(1 + x D)e_0(x) = [1 + f_n(x)]x.
\]

\[
\theta_n(e_1; x) = [1 + f_n(x)]x(1 + x D)e_1(x)
\]
\[
= [1 + f_n(x)]e_1(x)[1 + e_1(x)]
\]

and

\[
\theta_n(e_2; x) = [1 + f_n(x)]x(1 + x D) \left\{ e_2(x) + \frac{x(1-x)}{n} \right\}
\]
\[
= [1 + f_n(x)] \left\{ e_2(x) \left( 2 - \frac{3e_1(x)}{n} \right) \right\}.
\]

Since
\( f_n \rightsquigarrow f = 0 \) (\( \lambda \)-equi-stat)
on \([0,1]\) for \( f_n \) defined by (5.2.1), we conclude that

\[
\theta_n(e_i) \rightsquigarrow (e_i) \ (\lambda \text{-equi-stat})
\]
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on $[0, 1]$ for each $i = 0, 1, 2$. Therefore, by Theorem 5.3.1, we see that

$$\theta_n(f) \Rightarrow (f) \ (\lambda\text{-equi-stat})$$

on $[0, 1]$ for all $f \in C[0, 1]$. However, since $f_n$ is not statistically uniformly convergent to the function $f = 0$ on $[0, 1]$, we conclude that Theorem 2.1 of the earlier work [31] does not work for the operators defined by (5.3.5). Furthermore, since $f_n$ is not uniformly convergent (in the ordinary sense) to the function $f = 0$ on $[0, 1]$, the classical theorem also does not work here.

**Remark 5.3.1.** In connection with the operator $x(1 + xD) \ (D = \frac{d}{dx})$ used in (5.3.5), it may be of interest to observe that such much more general families of operators as the two-parameter operator $\Theta_{x, \kappa, \eta}$ defined by

$$\Theta_{x, \kappa, \eta} = x^\kappa (xD + \eta) \ \left(D = \frac{d}{dx}\right) \quad (5.3.6)$$

has appeared in the literature rather extensively (see, for details, [67, p. 447, Problem 16]; see also the recent works [15], [66] and [72]).

### 5.4 A Voronovskaja type theorem

In this section, we will show that the positive linear operators $\theta_n$ defined by (5.3.5) satisfy a Voronovskaja type property in the $\lambda$-equi-statistical sense. We first prove the following lemma.

**Lemma 5.4.1.** Let $x \in [0, 1]$ and $\phi(y) = y - x$. Then

$$n^2 \theta_n(\phi^4) \Rightarrow 3e_2e_1e_2 - 8e_1 + 6e_0 \ (\lambda\text{-equi-stat}) \text{ on } [0, 1].$$

**Proof.** Using (5.3.5), a simple calculation shows that

$$n^2 \theta_n(\phi^4; x) = [1 + f_n(x)]x(1 + xD) \left\{3x^4 - 6x^3 + 3x^2 + \left(\frac{-6x^4 + 12x^3 - 7x^2 + x}{n}\right)\right\}.$$

Thus we get

$$\left|n^2 \theta_n(\phi^4; x) - 3e_2(x)e_1(x)[e_2(x) - 8e_1(x) + 6e_0(x)]\right| \leq 57f_n(x) + \frac{101[1 + f_n(x)]}{n} \quad (x \in [0, 1]). \quad (5.4.1)$$

By (5.2.1), we have

$$57f_n(x) + \frac{101[1 + f_n(x)]}{n} \Rightarrow 0 \ (\lambda \text{-equi-stat}) \quad (5.4.2)$$
on $[0, 1]$. Now, combining (5.4.1) and (5.4.1), we get

$$n^2\theta_n(\phi^4) \sim 3e_2e_1(e_2 - 8e_1 + 6e_0)$$

(\lambda\text{-equi-stat})

on $[0, 1]$.

This completes the proof of Lemma.

We establish the following Voronovskaja type result for the operators $\theta_n$ given by (5.3.5).

**Theorem 5.4.1.** For every $f \in C[0, 1]$ such that $f', f'' \in C[0, 1]$,

$$n(\theta_n f - f) \sim \frac{e_1(1 - 2e_2)}{2} f'' (\lambda\text{-equi-stat})$$

on $[0, 1]$.

**Proof.** Suppose that $f, f', f'' \in C[0, 1]$ and that $x \in [0, 1]$. Define

$$\xi_x(y) = \begin{cases} 
\frac{f(y) - f(x) - (y - x)f'(x) - \frac{1}{2}(y - x)^2f''(x)}{(y - x)^2} & (y \neq x) \\
0 & (y = x).
\end{cases}$$

Then

$$\xi_x(x) = 0 \quad \text{and} \quad \xi_x \in C[0, 1].$$

Hence, by Taylor's theorem, we get

$$f(y) = f(x) + (y - x)f'(x) + \frac{(y - x)^2}{2} f''(x) + (y - x)^2 \xi_x(y).$$

Now, operating $\theta_n$ given by (5.3.5) upon both sides of the above equation, we find that

$$\theta_n(f; x) - f(x) = f(x)f_n(x) + f'(x)\theta_n(\phi; x) + f''(x) \theta_n(\phi^2; x) + \theta_n(\phi^2\xi_x; x)
\begin{align*}
&= f(x)f_n(x) + [1 + f_n(x)]
\left\{ \frac{x(1 - x)}{n} \right\} + \theta_n(\phi^2\xi_x; x),
\end{align*}$$

which yields

$$\left| n[\theta_n(f; x) - f(x)] - \frac{e_1(1 - 2e_2(x))}{2} f''(x) \right| \leq Mn f_n(x) + n \left| \theta_n(\phi^2\xi_x; x) \right|, \quad (5.4.3)$$

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where
\[ \phi(y) = y - x \quad \text{and} \quad M = \|f\|_{C[0,1]} + \|f''\|_{C[0,1]} . \]

By applying the familiar Cauchy-Schwarz inequality for the second term on the right-hand side of (5.4.3), we obtain
\[ n \left| \theta_n(\phi^2 \xi^2; x) \right| \leq \left[ n^2 \theta_n(\phi^4; x) \right]^\frac{1}{2} \left[ \theta_n(\xi^2; x) \right]^\frac{1}{2} . \]  

Putting
\[ \eta_\epsilon(y) = \xi_\epsilon^2(y), \]
we observe that
\[ \eta_\epsilon(x) = 0 \quad \text{and} \quad \eta_\epsilon(\cdot) \in C[0,1] . \]

It follows from Theorem 5.3.1 that
\[ \theta_n(\eta_\epsilon) \xrightarrow{} 0 \quad (\text{\lambda-equistat}) \quad \text{on} \quad [0,1] . \]  

Now, from (5.4.4) and (5.4.5), and by Lemma 5.4.1, we have
\[ \theta_n(\phi^2 \xi^2; x) \xrightarrow{} 0 \quad (\text{\lambda-equistat}) \quad \text{on} \quad [0,1] . \]  

For a given \( \epsilon > 0 \), we define
\[ \psi_n(x, \epsilon) = \left\{ k : k \leq n \quad \text{and} \quad \left| k(\theta_k(f; x) - f(x)) \right| - \frac{e_1(x)\left| 1 - 2e_2(x) \right|}{2} f''(x) \right| \geq \epsilon \right\} \]
and
\[ \psi_{1,n}(x, \epsilon) = \left\{ k : k \in I_n \quad \text{and} \quad \left| k\theta_n(x) \right| \geq \frac{\epsilon}{2M} \right\} \]
and
\[ \psi_{2,n}(x, \epsilon) = \left\{ k : k \in I_n \quad \text{and} \quad \left| k\theta_n(\phi^2 \xi; x) \right| \geq \frac{\epsilon}{2} \right\} . \]

Then it follows from (5.4.3) that
\[ \frac{\psi_n(x, \epsilon)}{\lambda_n} \leq \frac{\psi_{1,n}(x, \epsilon)}{\lambda_n} + \frac{\psi_{2,n}(x, \epsilon)}{\lambda_n} , \]
which yields
\[ \frac{\|\psi_n(\cdot, \epsilon)\|_{C[0,1]}}{\lambda_n} \leq \frac{\|\psi_{1,n}(\cdot, \epsilon)\|_{C[0,1]}}{\lambda_n} + \frac{\|\psi_{2,n}(\cdot, \epsilon)\|_{C[0,1]}}{\lambda_n} . \]  

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Also, by the definition of $f_n$, we obtain

$$n f_n \rightarrow f = 0 \text{ (}$\lambda$\text{-equi-stat}) \text{ on } [0, 1]. \quad (5.4.8)$$

Now, by (5.4.6) and (5.4.8), the right-hand side of (5.4.7) is seen to tend to zero as $n \to \infty$. Therefore, we have

$$\lim_{n \to \infty} \frac{\|\psi_n(\cdot, \epsilon)\|_{C[0,1]}}{\lambda_n} = 0.$$ 

This completes the proof of 5.4.1.

Remark 5.4.1. Since the function sequence $\{f_n\}$ given by (5.2.1) is not uniformly convergent to the function $f = 0$ on the interval $[0, 1]$, we observe that our operator $\theta_n$ defined by (5.3.5) does not satisfy Voronovskaja type property in the usual sense.

5.5 Rate of the $\lambda$-equi-statistical convergence

In this section, we study the rate of the $\lambda$-equi-statistical convergence of a sequence of positive linear operators defined on $\mathcal{C}(X)$. We begin by presenting the following definition.

Definition 5.5.1. Let $(a_n)$ be a positive non-increasing sequence. A sequence $(f_n)$ is equi-statistically convergent to a function $f$ with the rate $o(a_n)$ if, for every $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{\Lambda_n(x, \epsilon)}{a_n} = 0$$

uniformly with respect to $x \in X$ or, equivalently, if (for every $\epsilon > 0$),

$$\lim_{n \to \infty} \frac{\|\Lambda_n(\cdot, \epsilon)\|_{C(X)}}{a_n} = 0,$$

where

$$\Lambda_n(x, \epsilon) = \frac{1}{\lambda_n} |\{k : k \in I_n \text{ and } |f_k(x) - f(x)| \geq \epsilon\}|.$$

In this case, it is denoted as follows:

$$f_n - f = o(a_n) \text{ (}$\lambda$\text{-equi-stat}) \text{ on } X.$$ 

We now prove the following basic lemma.
\[ u_0 = \frac{u_0}{(x)C^1 \omega (x)\mu \phi}\]

Hypotheses of Lemma 5.2.1, we conclude that

Now, by taking the limit as \( u \to \infty \) in the second member of (6.2) and using the

\[ \frac{u_q}{(x)C^1 \omega (x)\mu \phi} + \frac{u_p}{(x)C^1 \omega (x)\mu \phi} \leq \frac{u_0}{(x)C^1 \omega (x)\mu \phi} \]

By the first member of (6.2), we find

\[ \{u_q, u_p\} \max = u_0 \]

Moreover, since

\[ (\varepsilon)_{\omega (x)\mu \phi} \eta (x) = (\varepsilon)_{\omega (x)\mu \phi} \eta (x) \]

Then, clearly, we have

\[ \{ \varepsilon \geq |(x)\theta - (x)\phi| \quad \text{and} \quad u \leq \eta : \gamma \} = (e \omega (x)\mu \phi) \]

and

\[ \{ \varepsilon \geq |(x)f - (x)\eta f| \quad \text{and} \quad u \leq \eta : \gamma \} = (e \omega (x)\mu \phi) \]

\[ \{ \varepsilon \geq |(x)(\theta + f) - (x)(\theta + f)| \quad \text{and} \quad u \leq \eta : \gamma \} = (e \omega (x)\mu \phi) \]

Following sets:

**Proof.** In order to prove the statement (1), for \( e \in E \) and \( x \in X \) we define the

\[ X \quad u_0 \quad (\varepsilon)_{\omega (x)\mu \phi} (\eta) = |f - \eta|, \quad (\eta) \]

if we let

\[ X \quad u_0 \quad (\varepsilon)_{\omega (x)\mu \phi} (\eta) = \phi - \eta \]

and

\[ X \quad u_0 \quad (\varepsilon)_{\omega (x)\mu \phi} (\eta) = f - \eta \]

pose also that

**Lemma 5.2.1.** Let \( (\varepsilon) \) be a sequence of functions belonging to \( C^1 \) and \( u_0 \) and \( u_1 \) be sequences of functions belonging to \( C^1 \).
which evidently completes the proof of the statement (i) of Lemma 5.5.1. The proofs of the statements (ii), (iii) and (iv) of Lemma 5.5.1 can also be given along the same or similar lines.

We now state and prove the following result.

**Theorem 5.5.1.** Let $X$ be a compact subset of the real numbers. Also let $L_n$ be a sequence of positive linear operators acting from $C(X)$ into itself. Assume that each of the following conditions holds true:

(a) $L_n(e_0) - (e_0) = o(a_n)$ (λ-equi-stat) on $X$;
(b) $w(f, \delta_n) = o(b_n)$ (λ-equi-stat) on $X$, where

$$\delta_n(x) = \sqrt{L_n(\phi^2; x)} \text{ and } \phi(y) = y - x.$$

Then, for all $f \in C(X)$, the following statement holds true:

$$L_n(f) - f = o(c_n) \text{ (λ-equi-stat) on } X,$$

where

$$c_n = \max\{a_n, b_n\}.$$

**Proof.** Let $f \in C(X)$ and $x \in X$. Then it is well known that

$$|L_n(f; x) - f(x)| \leq M|L_n(e_0; x) - e_0(x)| + (L_n(e_0; x) + \sqrt{L_n(e_0; x)})w(f, \delta_n),$$

where

$$M = \|f\|_{C(X)}.$$

This shows that

$$|L_n(f; x) - f(x)| \leq M|(L_n(e_0; x) - e_0(x)) + 2w(f, \delta_n)$$

$$+ w(f, \delta_n)[L_n(e_0; x) - e_0(x)] + w(f, \delta_n)\sqrt{|L_n(e_0; x) - e_0(x)|}.$$

Now, by using the conditions (a) and (b) of Theorem 5.5.1, in conjunction with Lemma 5.5.1, in the above inequality, we arrive at the statement (5.5.3) of Theorem 5.5.1.

This completes the proof of Theorem.
5.6 Concluding remarks and observations

In this concluding section of our investigation, we present several remarks and observations concerning the results which we have proved here.

Remark 5.6.1. Let \((f_r)_{r \in \mathbb{N}}\) be the sequence of functions given by (5.2.1). Then, since
\[
f_r \rightsquigarrow 0 \text{ (\(\lambda\)-equi-stat)},
\]
we conclude that
\[
\theta_n(e_i) \rightsquigarrow e_i \text{ (\(\lambda\)-equi-stat)} \quad (i = 0, 1, 2).
\]
Therefore, by Theorem 5.3.1, we can see that
\[
\theta_n(f; x) \rightsquigarrow f \text{ (\(\lambda\)-equi-stat)}
\]
on \([0, 1]\) for all \(f \in [0, 1]\). However, since \(f_r\) is not uniformly \(\lambda\)-equi-statistically convergent and not uniformly convergent (in the ordinary sense) to the function \(f = 0\), the classical Korovkin theorem does not work for the operators defined by (5.3.5). Consequently, Theorem 5.3.1 is a non-trivial extension of the classical Korovkin Theorem.

Remark 5.6.2. Let \((f_r)_{r \in \mathbb{N}}\) be the sequence of functions given by (5.2.2) with \(X = [0, 1]\) and let \(\lambda_n = [\sqrt{n}]\). Since
\[
f_r \rightsquigarrow 0 \text{ (\(\lambda\)-equi-stat)},
\]
we have (5.6.1). By applying (5.6.1) and Theorem 5.3.1, we have (5.6.2). But, since \(f_r\) does not converge to \(f = 0\) (equi-stat), Theorem 2.1 of Karakuş et al. [31] does not work. Therefore, Theorem 5.3.1 is also a non-trivial extension of Theorem 2.1 of Karakuş et al.

Remark 5.6.3. Suppose that we replace the conditions (a) and (b) in Theorem 5.5.1 by the following condition:
\[
L_n(e_i) - e_i = o(a_n) \text{ (\(\lambda\)-equi-stat)} \quad \text{on} \quad X \quad (i = 0, 1, 2).
\]
Then, since
\[
L_n(\phi^2; x) = L_n(e_2; x) - 2xL_n(e_1; x) + x^2L_n(e_0; x),
\]
we may write
\[
L_n(\phi^2; x) \leq K \sum_{i=0}^{2} |(L_n(e_i; x) - e_i(x))|,
\]
where
\[
K = 1 + 2\|e_1\|_{C(X)} + \|e_2\|_{C(X)}.
\]
Now it follows from (5.6.3), (5.6.4) and Lemma 5.5.1 that

\[ \delta_n = \sqrt{L_n(\phi^2)} = o(d_n) \text{ (\lambda-equi-stat) on } X, \quad (5.6.5) \]

where

\[ d_n = \max \{a_{n_0}, a_{n_1}, a_{n_2}\}. \]

Hence

\[ w(f, \delta_n) = o(d_n) \text{ (\lambda-equi-stat) on } X. \]

Using (5.6.5) in Theorem 5.5.1, we can thus see that, for all

\[ f \in C(X), L_n(f) - f = o(d_n) \text{ (\lambda-equi-stat) on } X. \quad (5.6.6) \]

Therefore, if we use the condition (5.6.3) in Theorem 5.5.1 instead of the conditions (a) and (b), then we obtain the rates of \( \lambda \)-equi-statistical convergence of the sequence of positive linear operators in Theorem 5.3.1.
Chapter 6

Approximation for Periodic Functions via Weighted Statistical Convergence

6.1 Introduction

In this chapter, we use the notion of weighted statistical convergence to prove the Korovkin approximation theorem for the functions 1, cos and sin in the space of all continuous 2π-periodic functions on the real line and show that our result is stronger. We also study the rate of weighted statistical convergence.

Recently, Karakaya and Chishti [30] has defined the concept of weighted statistical convergence which was further modified/ corrected in [54].

Let \( p = (p_k)_{k=0}^{\infty} \) be a sequence of non-negative numbers such that \( p_0 > 0 \) and \( P_n = \sum_{k=0}^{n} p_k \to \infty \) as \( n \to \infty \). Set \( t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k x_k, n = 0, 1, 2, \ldots \).

We define the weighted density of \( K \) by \( \delta_N(K) = \lim \frac{1}{P_n} |K_n| \) if the limit exists. We say that the sequence \( x = (x_k) \) is weighted statistically convergent (or \( S-N \)-convergent) to \( L \) if for every \( \varepsilon > 0 \), the set \( \{k \in \mathbb{N} : p_k |x_k - L| \geq \varepsilon \} \) has weighted density zero, i.e.

\[
\lim \frac{1}{P_n} |\{k \leq P_n : p_k |x_k - L| \geq \varepsilon \}| = 0.
\]

In this case, we write \( L = S-N \)-lim \( x \).

Let \( x = (x_k) \) be a sequence defined by

\[
x_k = \begin{cases} \sqrt{k} & \text{if } k = n^2, n \in \mathbb{N}, \\ 0 & \text{if } k \neq n^2.
\end{cases}
\]

That is \( (x_k) = (1, 0, 0, 2, 0, 0, 0, 0, 3, 0, \ldots, 0, 4, 0, 0, \ldots) \). Let \( p_k = k \). Then \( p_k x_k = \)

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\( (1, 0, 0, 8, 0, 0, 0, 0, 27, 0, \ldots, 0, 64, 0, 0, \ldots) \). Since
\[
\lim_{n \to \infty} \frac{1}{P_n} | \{ k \leq P_n : p_k [x_k - 0] \geq \epsilon \} | = 0 \leq \lim_{n \to \infty} \frac{1}{\sqrt{P_n}} \to 0,
\]
\((x_k)\) is weighted statistical convergent to 0 but not convergent.

In this chapter, we prove Korovkin second theorem by applying the notion of weighted statistical convergence. We also give an example to justify that our result is stronger than Theorem 1.5.2.

### 6.2 Main result

We write \( L_n(f; x) \) for \( L_n(f(s); x) \); and we say that \( L \) is a positive operator if
\[
L(f; x) \geq 0 \text{ for all } f(x) \geq 0.
\]

**Theorem 6.2.1.** Let \((T_k)\) be a sequence of positive linear operators from \( C_{2\pi}(\mathbb{R}) \) into \( C_{2\pi}(\mathbb{R}) \). Then for all \( f \in C_{2\pi}(\mathbb{R}) \)
\[
\lim_{k \to \infty} \| T_k(f; x) - f(x) \|_{2\pi} = 0 \quad (6.2.1)
\]
if and only if
\[
\lim_{k \to \infty} \| T_k(1; x) - 1 \|_{2\pi} = 0, \quad (6.2.2)
\]
\[
\lim_{k \to \infty} \| T_k(\cos t; x) - \cos x \|_{2\pi} = 0, \quad (6.2.3)
\]
\[
\lim_{k \to \infty} \| T_k(\sin t; x) - \sin x \|_{2\pi} = 0. \quad (6.2.4)
\]

**Proof.** Since each \( f_1, f_2, f_3 \) belongs to \( C_{2\pi}(\mathbb{R}) \), conditions (6.2.2)-(6.2.4) follow immediately from (6.2.1). Let the conditions (6.2.2)-(6.2.4) hold and \( f \in C_{2\pi}(\mathbb{R}) \).

Let \( I \) be a closed subinterval of length \( 2\pi \) of \( \mathbb{R} \). Fix \( x \in I \). By the continuity of \( f \) at \( x \), it follows that for given \( \epsilon > 0 \) there is a number \( \delta > 0 \) such that for all \( t \)
\[
|f(t) - f(x)| < \epsilon, \quad (6.2.5)
\]
whenever \( |t - x| < \delta \). Since \( f \) is bounded, it follows that
\[
|f(t) - f(x)| \leq \| f \|_{2\pi}, \quad (6.2.6)
\]

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for all $t \in \mathbb{R}$. For all $t \in (x - \delta, 2\pi + x - \delta]$, it is well-known that

$$|f(t) - f(x)| < \varepsilon + \frac{2 \|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}} \psi(t), \quad (6.2.7)$$

where $\psi(t) = \sin^2 \left(\frac{t - x}{2}\right)$. Since the function $f \in C_{2\pi}(\mathbb{R})$ is $2\pi$-periodic, the inequality (6.2.7) holds for $t \in \mathbb{R}$.

Now, operating $T_k(1; x)$ to this inequality, we obtain

$$|T_k(f; x) - f(x)| \leq (\varepsilon + |f(x)|) |T_k(1; x) - 1| + \varepsilon + \frac{\|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}} (|T_k(1; x) - 1| + |T_k(\cos t; x) - \cos x| + |T_k(\sin t; x) - \sin x|)$$

$$\leq \varepsilon + |f(x)| + \frac{\|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}} (|T_k(1; x) - 1| + |T_k(\cos t; x) - \cos x| + |T_k(\sin t; x) - \sin x|)$$

$$\leq \varepsilon + (\varepsilon + |f(x)| + \frac{\|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}}) (|T_k(1; x) - 1| + |T_k(\cos t; x) - \cos x| + |T_k(\sin t; x) - \sin x|)$$

$$\leq \varepsilon + (\varepsilon + \|f\|_{2\pi}) (|T_k(1; x) - 1| + |T_k(\cos t; x) - \cos x| + |T_k(\sin t; x) - \sin x|)$$

$$\leq \varepsilon + K \left( \left\|T_k(1; x) - 1\right\|_{2\pi} + \left\|T_k(\cos t; x) - \cos x\right\|_{2\pi} + \left\|T_k(\sin t; x) - \sin x\right\|_{2\pi} \right),$$

where $K = \sup_{\pi \in I} \{\varepsilon + |f|_{2\pi} + \frac{\|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}}\}$. Hence

$$\left\|T_k(f; x)p_k - f(x)\right\|_{\infty} \leq \varepsilon + K \left( \left\|T_k(1; x)p_k - 1\right\|_{2\pi} + \left\|T_k(\cos t; x)p_k - \cos x\right\|_{2\pi} + \left\|T_k(\sin t; x)p_k - \sin x\right\|_{2\pi} \right).$$

(6.2.9)

For a given $r > 0$ choose $\varepsilon' > 0$ such that $\varepsilon' < r$. Define the following sets

$$D = \{k \leq n : \|T_k(f; x)p_k - f(x)\|_{2\pi} \geq r\},$$

$$D_1 = \{k \leq n : \|T_k(1; x)p_k - 1\|_{2\pi} \geq \frac{r - \varepsilon'}{4K}\},$$

$$D_2 = \{k \leq n : \|T_k(\cos t; x)p_k - \cos x\|_{2\pi} \geq \frac{r - \varepsilon'}{4K}\},$$

$$D_3 = \{k \leq n : \|T_k(\sin t; x)p_k - \sin x\|_{2\pi} \geq \frac{r - \varepsilon'}{4K}\}.$$
Then

$$D \subset D_1 \cup D_2 \cup D_3,$$

and so

$$\delta_{R}(D) \leq \delta_{R}(D_1) + \delta_{R}(D_2) + \delta_{R}(D_3).$$

Therefore, using conditions (6.2.2), (6.2.3) and (6.2.4), we get

$$S_{R} - \lim_{n} \|T_n(f, x) - f(x)\|_{2\pi} = 0.$$ 

This completes the proof of the theorem.

### 6.3 Rate of weighted statistical convergence

In this section, we study the rate of weighted statistical convergence of a sequence of positive linear operators defined from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$.

**Definition 6.3.1.** Let $(a_n)$ be a positive non-increasing sequence. We say that the sequence $x = (x_k)$ is weighted statistically convergent to the number $L$ with the rate $o(a_n)$ if for every $\varepsilon > 0$,

$$\lim_{n} \frac{1}{a_n} \sum_{k \leq P_n : |x_k - L| \geq \varepsilon} = 0.$$ 

In this case, we write $x_k - L = S_{R} - o(a_n)$.

As usual we have the following auxiliary result whose proof is standard.

**Lemma 6.3.1.** Let $(a_n)$ and $(b_n)$ be two positive non-increasing sequences. Let $x = (x_k)$ and $y = (y_k)$ be two sequences such that $x_k - L_1 = S_{R} - o(a_n)$ and $y_k - L_2 = S_{R} - o(b_n)$. Then

(i) $\alpha(x_k - L_1) = S_{R} - o(a_n)$, for any scalar $\alpha$,

(ii) $(x_k - L_1) \pm (y_k - L_2) = S_{R} - o(c_n),$

(iii) $(x_k - L_1)(y_k - L_2) = S_{R} - o(c_n b_n),$

where $c_n = \max\{a_n, b_n\}$. 

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Now, we recall the notion of modulus of continuity. The modulus of continuity of \( f \in C_{2\pi}(\mathbb{R}) \), denoted by \( \omega(f, \delta) \) is defined by

\[
\omega(f, \delta) = \sup_{|x-y|<\delta} |f(x) - f(y)|.
\]

It is well known that

\[
|f(x) - f(y)| \leq \omega(f, \delta)(\frac{|x-y|}{\delta} + 1).
\] (6.3.1)

Then we have the following result.

**Theorem 6.3.1.** Let \((T_k)\) be a sequence of positive linear operators from \( C_{2\pi}(\mathbb{R}) \) into \( C_{2\pi}(\mathbb{R}) \). Suppose that

(i) \( \|T_k(1; x) - 1\|_{2\pi} = S_{N} - o(a_n), \)

(ii) \( \omega(f, \lambda_k) = S_{N} - o(b_n), \) where \( \lambda_k = \sqrt{T_k(\varphi_x; x)} \) and \( \varphi_x(y) = \sin^2(\frac{y-x}{2}). \)

Then for all \( f \in C_{2\pi}(\mathbb{R}) \), we have

\[
\|T_k(f; x) - f(x)\|_{2\pi} = S_{N} - o(c_n),
\]

where \( c_n = \max\{a_n, b_n\}. \)

**Proof.** Let \( f \in C_{2\pi}(\mathbb{R}) \) and \( x \in [-\pi, \pi] \). Using (6.3.1), we have

\[
|T_k(f; x) - f(x)| \leq T_k(|f(y) - f(x)|; x) + |f(x)| |T_k(1; x) - 1|
\]

\[
\leq T_k\left(\frac{|x-y|}{\delta} + 1; x\right)\omega(f, \delta) + |f(x)| |T_k(1; x) - 1|
\]

\[
\leq T_k(1 + \frac{\pi^2}{\delta^2} \sin^2(\frac{y-x}{2}); x)\omega(f, \delta) + |f(x)| |T_k(1; x) - 1|
\]

\[
\leq (T_k(1; x) + \frac{\pi^2}{\delta^2} T_k(\varphi_x; x))\omega(f, \delta) + |f(x)| |T_k(1; x) - 1|.
\]

Put \( \delta = \lambda_k = \sqrt{T_k(\varphi_x; x)}. \) Hence we get

\[
\|T_k(f; x) - f(x)\|_{2\pi} \leq \|f\|_{2\pi} \|T_k(1; x) - 1\|_{2\pi} + \omega(f, \lambda_k) + \omega(f, \lambda_k)\|T_k(1; x) - 1\|_{2\pi}
\]

\[
\leq K\{\|T_k(1; x) - 1\|_{2\pi} + \omega(f, \lambda_k) + \omega(f, \lambda_k)\|T_k(1; x) - 1\|_{2\pi}\},
\]

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where $K = \max\{\|f\|_{2\pi}, 1 + \pi^2\}$. Hence

$$\|T_k(f; x)p_k - f(x)\|_{2\pi} \leq K\{\|T_k(1; x)p_k - 1\|_{2\pi} + \omega(f, \lambda_k)p_k + \omega(f, \lambda_k)p_k\|T_k(1; x)p_k - 1\|_{2\pi}\}.$$ 

Now, using Definition 6.3.1 and Conditions (i) and (ii), we get the desired result.

This completes the proof of the theorem.

6.4 Example and the concluding remark

Finally, we construct an example of a sequence of positive linear operators satisfying the conditions of Theorem 6.2.1 but does not satisfy the conditions of Theorem 1.5.2.

For any $n \in \mathbb{N}$, denote by $S_n(f)$ the $n$-th partial sum of the Fourier series of $f$, i.e.

$$S_n(f)(x) = \frac{1}{2}a_0(f) + \sum_{k=1}^{n}a_k(f)\cos{kx} + b_k(f)\sin{kx}.$$ 

For any $n \in \mathbb{N}$, write

$$F_n(f) = \frac{1}{n + 1} \sum_{k=0}^{n}S_k(f).$$ 

A standard calculation gives that for every $t \in \mathbb{R}$

$$F_n(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1}{n + 1} \sum_{k=0}^{n} \frac{\sin((2k + 1)(x - t)/2)}{\sin((x - t)/2)} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1}{n + 1} \sum_{k=0}^{n} \frac{\sin^2((n + 1)(x - t)/2)}{\sin^2((x - t)/2)} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \varphi_n(x - t) dt,$$

where

$$\varphi_n(x) = \begin{cases} 
\frac{\sin^2((n+1)(x-t)/2)}{(n+1)\sin^2((x-t)/2)} & \text{if } x \text{ is not a multiple of } 2\pi, \\
n + 1 & \text{if } x \text{ is a multiple of } 2\pi.
\end{cases}$$

The sequence $(\varphi_n)_{n \in \mathbb{N}}$ is a positive kernel which is called the Fejér kernel, and the corresponding operators $F_n$, $n \geq 1$, are called the Fejér convolution operators.
Note that the Theorem 1.5.2 is satisfied for the sequence $(F_n)$. In fact, we have, for every $f \in C_{2\pi}(\mathbb{R})$,

$$\lim_{n \to \infty} F_n(f) = f.$$ 

Let $L_k : C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$ be defined by

$$L_k(f; x) = (1 + x_k)F_k(f; x),$$

where the sequence $x = (x_k)$ is defined by (6.1.1). Now

$$L_n(1; x) = 1,$$

$$L_n(\cos t; x) = \frac{n}{n + 1} \cos x,$$

$$L_n(\sin t; x) = \frac{n}{n + 1} \sin x.$$ 

So that we have

$$S_{\infty} \lim_{n \to \infty} \|L_n(1; x) - 1\|_{2\pi} = 0,$$

$$S_{\infty} \lim_{n \to \infty} \|L_n(\cos t; x) - \cos x\|_{2\pi} = 0,$$

$$S_{\infty} \lim_{n \to \infty} \|L_n(\sin t; x) - \sin x\|_{2\pi} = 0,$$

that is, the sequence $(L_n)$ satisfies the conditions (6.2.2), (6.2.3) and (6.2.4). Hence by Theorem 6.2.1, we have

$$S_{\infty} \lim_{n \to \infty} \|L_n(f) - f\|_{2\pi} = 0,$$

i.e. our theorem holds. But on the other hand, Theorem 1.5.2 does not hold for our operator defined by (6.4.1), since the sequence $(L_n)$ is not convergent.

Hence our Theorem 6.2.1 is stronger than that of 1.5.2.
Bibliography


List of Publications


(v) Generalized q-Bernstein-Schurer operators and some approximation theorems, accepted for publication in *Journal of Function Spaces and Applications*.
Statistical Approximation Properties of Modified $q$-Stancu-Beta Operators

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Abstract. In this paper we define the modified $q$-Stancu-Beta operators and study the weighted statistical approximation by these operators with the help of the Korovkin type approximation theorem. We also establish the rates of statistical convergence by means of the modulus of continuity and the Lipschitz type maximal function. Our results show that rates of convergence of our operators are at least as fast as classical Stancu-Beta operators.

2010 Mathematics Subject Classification: Primary: 41A10, 41A25, 41A36; Secondary: 40A30

Keywords and phrases: Statistical convergence, $q$-Stancu-Beta operators, rate of statistical convergence, modulus of continuity, positive linear operator, Korovkin type approximation theorem.

1. Introduction and preliminaries

After the paper of Phillips [18] who generalized the classical Bernstein polynomials based on $q$-integers, many generalizations of well-known positive linear operators, based on $q$-integers were introduced and studied by several authors. Recently the statistical approximation properties have also been investigated for $q$-analogue polynomials. For instance, in [19] $q$-analogue of Bernstein–Kantorovich operators; in [10] $q$-Baskakov–Kantorovich operators; in [17] $q$-Szász–Mirakjan operators; in [4] and [7] $q$-Bleimann, Butzer and Hahn operators; in [1] and [14] $q$-analogue of MKZ operators and in [4] $q$-analogue of Stancu-Beta operators were defined and their statistical approximation properties were investigated.

In this paper, we first introduce a new modification of the operators defined by Aral and Gupta [5] and study the weighted statistical approximation properties of the modified $q$-Stancu-Beta operators with the help of the Korovkin type approximation theorem. We also estimate the rate of statistical convergence of the sequence of the operators to the function $f$.

First, we recall certain notations of $q$-calculus as follows. Details on $q$-integers can be found in [3]. For each nonnegative integer $k$, the $q$-integer $[k]_q$ is defined by

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Approximation for periodic functions via weighted statistical convergence

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ABSTRACT

Korovkin type approximation theorems are useful tools to check whether a given sequence \((L_n)_{n=1}^{\infty}\) of positive linear operators on \([0, 1]\) of all continuous functions on the real interval \([0, 1]\) is an approximation process. That is, these theorems exhibit a variety of test functions which assure that the approximation property holds on the whole space if it holds for them. Such a property was discovered by Korovkin in 1953 for the functions 1, \(x\) and \(x^2\) in the space \(C[0, 1]\) as well as for the functions 1, \(\cos\) and \(\sin\) in the space of all continuous \(2\pi\)-periodic functions on the real line. In this paper, we use the notion of weighted statistical convergence to prove the Korovkin approximation theorem for the functions 1, \(\cos\) and \(\sin\) in the space of all continuous \(2\pi\)-periodic functions on the real line and show that our result is stronger. We also study the rate of weighted statistical convergence.

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1. Introduction and preliminaries

Let \(\mathbb{N}\) be the set of all natural numbers and \(K \subseteq \mathbb{N}\) and \(K_n = \{k \leq n : k \in K\}\). Then the natural density of \(K\) is defined by

\[
\delta(K) = \lim_{n \to \infty} \frac{|K_n|}{n} \text{ if the limit exists,}
\]

where the vertical bars indicate the number of elements in the enclosed set. The sequence \(x = (x_k)\) is said to be statistically convergent to \(L\) if for every \(\varepsilon > 0\), the set \(K_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\) has natural density zero (cf. Fast [6]), i.e. for each \(\varepsilon > 0\),

\[
\lim_{n \to \infty} \frac{1}{n} |\{j \leq n : |x_j - L| \geq \varepsilon\}| = 0.
\]

In this case, we write \(L = \text{st-lim} x\). Note that every convergent sequence is statistically convergent but not conversely.

Recently, Karakaya and Chishti [7] has defined the concept of weighted statistical convergence which was further modified/corrected in [15].

Let \(p = (p_k)_{k=0}^{\infty}\) be a sequence of nonnegative numbers such that \(p_0 > 0\) and \(P_n = \sum_{k=0}^{n} p_k \to \infty\) as \(n \to \infty\). Set \(\tau_n = \sum_{k=0}^{n} p_k, n = 0, 1, 2, \ldots\).

We define the weighted density of \(K\) by \(\delta_p(K) = \lim_{n \to \infty} \frac{|K_n|}{\tau_n}\) if the limit exists. We say that the sequence \(x = (x_k)\) is weighted statistically convergent (or \(S_p\)-convergent) to \(L\) if for every \(\varepsilon > 0\), the set \(\{k \in \mathbb{N} : p_k |x_k - L| \geq \varepsilon\}\) has weighted density zero, i.e.

\[
\lim_{n \to \infty} \frac{1}{\tau_n} |\{k \leq n : p_k |x_k - L| \geq \varepsilon\}| = 0.
\]
Operators constructed by means of q-Lagrange polynomials and A-statistical approximation

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ABSTRACT

In this paper we construct some positive linear operators by means of q-Lagrange polynomials and prove some approximation results via A-statistical convergence. We also define and study the rate of A-statistical approximation of these operators by using the notion of modulus of continuity and Lipschitz class.

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1. Introduction and preliminaries

An application of statistical summability gave rise to the theory of statistical approximation (e.g. [4,6,10,14-19,21-23], etc.) which have been an active area of research for the last one decade. Recently the statistical approximation properties have also been investigated for q-analogue of several operators. For instance, in [2] q-Butzer and Hahn operators; in [3,18] q-analogue of Stancu-Beta operators; in [8] q-Bleimann, Butzer and Hahn operators; in [11] q-Baskakov–Kantorovich operators; in [13] q-analogue of the Meyer-König and Zeller Operators; in [20,21] q-Szász–Mirakjan operators; and in [22] q-analogues of Bernstein–Kantorovich operators were defined and their statistical approximation properties were investigated. In this paper, we construct a new family of operators with the help of q-analogue of Chan–Chyan–Srivastava polynomials, and study the statistical approximation properties via A-statistical convergence. We also study some approximation properties for the rate of A-statistical convergence with the help of modulus of continuity and Lipschitz class.

First we recall the following definitions:

Let N denote the set of all natural numbers. Let K ⊆ N and K n = {k ∈ N : k ≤ K}. Then the natural density of K is defined by δ(K) = lim n→∞ n−1|Kn| if the limit exists, where |Kn| denotes the cardinality of the set Kn. A sequence x = (x k) of real numbers is said to be statistically convergent to L (cf. [9]) provided that, for every ε > 0, the set {k ∈ N : |x k - L| > ε} has natural density zero, i.e., for each ε > 0.

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k \leq n} |x_k - L| = 0. \]

In this case, we write st - lim x k = L. Note that every convergent, sequence is statistically convergent, but not conversely.

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Generalized equi-statistical convergence of positive linear operators and associated approximation theorems

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ABSTRACT

The concepts of equi-statistical convergence, statistical pointwise convergence and statistical uniform convergence for sequences of functions were introduced recently by Balcerzak et al. [M. Balcerzak, K. Dems, A. Kornisarski, Statistical convergence and ideal convergence for sequences of functions, J. Math. Anal. Appl. 328 (2007) 715–729]. In this paper, we use the notion of λ-statistical convergence in order to generalize these concepts. We establish some inclusion relations between them. We apply our new notion of λ-equistatistical convergence to prove a Korovkin type approximation theorem and we show that our theorem is a non-trivial extension of some well-known Korovkin type approximation theorems. Finally, we prove a Voronovskaja type approximation theorem via the concept of λ-equistatistical convergence. Some interesting examples are also displayed here in support of our definitions and results.

1. Introduction and preliminaries

The following concept of statistical convergence for sequences of real numbers was introduced by Fast [1] and further studied by Fridy [2] and many other authors.

In terms of the set \( \mathbb{N} \) of positive integers, let

\[
K \subseteq \mathbb{N} \quad \text{and} \quad K \ell = \{j: j \leq n \text{ and } j \in K\}.
\]

Then the natural density of \( K \) is defined by

\[
\delta(K) := \lim_{n \to \infty} \frac{|K \ell|}{n},
\]

if the limit exists, where \(|K \ell|\) denotes the cardinality of the set \( K \ell \).

A sequence \( x = (x_j) \) of real numbers is said to be statistically convergent to the number \( L \) if, for every \( \varepsilon > 0 \), the set

\[
\{j: j \in \mathbb{N} \text{ and } |x_j - L| \geq \varepsilon\}
\]

has natural density zero, that is, if, for each \( \varepsilon > 0 \), we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} 1\{j: j \leq n \text{ and } |x_j - L| \geq \varepsilon\} = 0.
\]

The concept of λ-statistical convergence was introduced recently in [3] as follows.

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Dear Dr. Mursaleen,

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Best regards,

Simone Secchi
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Generalized $q$-Bernstein-Schurer operators and some approximation theorems

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Abstract

In this chapter, we study statistical approximation properties of $q$-Bernstein-Schurer operators and also establish some direct theorems. We compute error estimation by using modulus of continuity with the help of Matlab and give its algorithm. Furthermore, we show graphically the convergence of the $q$-Bernstein-Schurer operators to various functions.

Keywords and phrases: Statistical convergence; $q$-Bernstein-Schurer operator; modulus of continuity; error estimate; positive linear operator; Korovkin type approximation theorem.

AMS Subject Classifications (2010): 41A10, 41A25, 41A36, 40A30

1. Introduction and preliminaries

In 1987, Lupas [13] introduced the first $q$-analogue of Bernstein operator and investigated its approximating and shape-preserving properties. Another $q$-generalization of the classical Bernstein polynomials is due to Phillips [25]. After that many generalizations of well-known positive linear operators, based on $q$-integers were introduced and studied by several authors. Recently the statistical approximation properties have also been investigated for $q$-analogue polynomials. For instance, in [13] $q$-analogues of Bernstein- Kantorovich operators; in [10] $q$-Baskakov-Kantorovich operators; in [23] $q$-Szász-Mirakjan operators; in [4] and [7] $q$-Bleimann, Butzer and Hahn operators; in [14] $q$-analogue of Baskakov and Baskakov-Kantorovich operators; in [15] $q$-analogue of Szász Kantorovich operators; in [5] and [21] $q$-analogue of Stancu-Beta operators and in [20] $q$-Lagrange polynomials were defined and their classical approximation or statistical approximation properties were investigated.

Schurer [26] introduced the following operators $L_{m,p} : C[0,p+1] \rightarrow C[0,1]$ defined for any $m \in \mathbb{N}$ and any function $f \in C[0,p+1]$

$$L_{m,p}(f;x) = \sum_{k=0}^{m+p} \binom{m+p}{k} x^k(1-x)^{m+p-k}f\left(\frac{k}{m}\right), \quad x \in [0,1]. \quad (1.1)$$

Recently, Muraru [17] introduced the $q$-analogue of these operators and investigated their approximation properties and rate of convergence using modulus of continuity.