# CATEGORY THEORY AND ITS APPLICATIONS

DISSERTATION

SUBMITTED FOR THE AWARD OF THE DEGREE OF

**Master of Philosophy**

IN

**MATHEMATICS**

*By*


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*Dedicated*

*to*

*My Parents*

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# Certificate

This is to certify that the dissertation entitled "Category Theory and its Applications" has been completed by **Mr. Gulam Muhiuddin** under my supervision and the work is suitable for the submission for the award of degree of Master of Philosophy in Mathematics.

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# PREFACE

The present dissertation entitled "*Category Theory and its Applications*" contains the work done by various researchers on category theory and some of its applications in computer science.

This exposition comprises six chapters. Each chapter contains a brief introduction and is divided into various sections. The definitions, examples and results in the text have been specified with double decimal numbering. The first figure indicates the chapter, the second denotes the section and the third mentions the number of definition or example or proposition or theorem as the case may be in a particular chapter. For example Theorem 4.3.2 refers to the second theorem appearing in the section 3 of chapter 4.

Chapter 0 is devoted to the historical development of category theory which is introduced by Eilenberg and MacLane [17-18] in 1945. Chapter 1 contains basic concepts, definitions and some basic results which are useful to develop the theory in the subsequent chapters.

In Chapter 2 the properties of special objects such as initial, terminal and zero objects and special morphisms such as monomorphism, epimorphism and isomorphism together with retraction and coretraction are discussed. It is obtained that a morphism which is both monomorphism and epimorphism need not be an isomorphism. Further, some constructions in category theory such as product, co-product, equalizers and kernels are discussed.

Chapter 3 deals with the study of some structural categories such as semi-additive, additive, normal, exact, abelian etcetera. Further, it is shown that every normal (or an abelian) category is balanced. Chapter 4 has been devoted to the study of some special types of functors. In fact the preservation properties of functors have been studied and the notion of additive functors and exact functors are discussed.

Finally, in Chapter 5 some applications of category theory in computer science have been given. Specially, the relation between category theory & computer science, categories with products-circuits and categories with sums-flow charts are discussed.

In the end of the dissertation, a bibliography has been given which by no means is comprehensive but mentions only the papers and books referred to in the main body of the dissertation.

# CONTENTS

# CHAPTER 0

# HISTORICAL DEVELOPMENT OF CATEGORY THEORY

## §0.1 Introduction

Categories, functors and natural transformations were introduced by Samual Eilenberg and Saunders MacLane in 1945. Initially, the notion were applied in topology, especially algebraic topology, as an important part of the transition from homology (an intutive and geometric concept) to homology theory, an axiomatic approach. It has been claimed on behalf of Ulam, that comparable ideas were current in the later 1930s in the Polish school. Eilenberg/MacLane have said that their goal was to understand natural transformations; in order to do that, functors had to be defined; and to define functors one needed categories.

### §0.2 Category Theory

Category theory now occupies a central position not only in contemporary mathematics, but also in theoretical computer science and even in mathematical physics. It can roughly be described as a general mathematical theory of structures and systems of structures. It is at the very least a very powerful language or conceptual framework which allows us to see, among other things, how structures of different kinds are related to one another as well as the universal components of a family of structures of a given kind. Beside its intrinsic mathematical interest and its role in the development of contemporary mathematics, thus as an object of study for the epistemology of mathematics itself, the theory is philosophically relevant in many other ways. As a general formal tool, it can be used to study and clarify fundamental concepts such as the concept of space, the concept of system or even the concept of truth. It can also be applied for the study of logical systems, which in this context are called "categorical doctrines", both at the syntactic level, more generally the proof-theoretical level and at the semantic level. As a framework, it is considered by many as constituting an alternative to set theory as a foundation for mathematics.

As such, it raises many issues with respect to the nature of mathematical entities and mathematical knowledge.

## §0.3 Brief Historical Sketch

It is difficult to do justice to the short but intricate history of the field. in particular it is not possible to mention all those who have contributed to its rapid development. Here are some of the main threads that have to be mentioned. Categories, functors, natural transformations, limits and colimits appeared almost out of nowhere in 1945 in Eilenberg & MacLane's paper entitled "General Theory of Natural Equivalences". We said "almost", because when one looks at their 1942 paper "Group Extensions and Homology", one discovers specific functors and natural transformations at work, limited to groups. In fact, it was basically the need to clarify and abstract from their 1942 results that Eilenberg & MacLane came up with the notions of category theory. The central notion for them was the notion of natural transformation. In order to give a general definition of the latter, they defined the notion of functor, borrowing the terminology from Carnap, and in order to give a general definition of functor, they defined the notion of category, borrowing this time from Kant and Aristotle. After their 1945 paper, it was not clear that the concepts of category theory would be more than a convenient language and so it remained for approximately fifteen years. It was used as such by Eilenberg and Steenrod in their influential book on the foundations of algebraic topology, published in 1952 and by Cartan and Eilenberg in their ground breaking book on homological algebra, published in 1956. (It is interesting to note, however, that although categories are defined in Eilenberg & Steenrod's book, they are not in Cartan & Eilenberg's work! They are simply assumed in that latter). These books allowed new generations of mathematicians to learn algebraic topology and homological algebra directly in the categorical language and to master the method of diagrams. Indeed, many results published in these two books seems to be inconceivable, or at the very least considerably more intricate, without the method of diagram chasing. Then, in 1957 and in 1958, the situation radically changed. In 1957, Grothendieck published his landmark "Sur quelques points d'algebre homologique" in which categories are used intrinsically to define and construct more general theories which are then applied to specific fields, in particular, in the following years, algebraic geometry, and in 1958 Kan published "Adjoint functors" and showed that the latter concept subsumes the important concepts of limits and colimits and could be used to capture fundamental conceptual situations (which in his case were in homotopy theory). From then on, category theory became more than a convenient language and this, for two reasons. First, using the axiomatic method and the categorical language, Grothendieck defined abstractly types of categories, e.g., additive and abelian categories, showed how to perform various constructions in these categories and proved various results for them. In a nutshell, Grothendieck showed how a part of homological algebra could be developed in such an abstract setting. From then on, a specific category of structures, e.g., a category of sheaves over a topological space X, could be seen as being a token of an abstract category of a certain type, e.g., an abelian category, and one could therefore immediately see how the methods of homological algebra for instance could be applied in this case, e.g., in algebraic geometry. Furthermore, it made sense to look for other types of abstract categories, types of abstract categories which would encapsulate the fundamental and formal aspects of various mathematical fields in the same way that abelian categories encapsulated fundamental aspects of homological algebra. Second, mostly under the influence of Freyd and Lawvere, category theorists progressively saw how pervasive the concept of adjoint functors is. Not only can the existence of adjoints to given functors be used to define abstract categories, and presumably those which are defined by such means have a privileged status, but as we have mentioned, many important theorems and even theories in various fields can be seen as being equivalent to the existence of specific functors between particular categories. By the early seventies, the concept of adjoint functors was considered to be the central concept of category theory.

With these developments, category theory became an autonomous part of mathematics, and pure category theory could be developed. And indeed, it did grow rapidly not only as a discipline but also in its applications, mainly in its original context, namely algebraic topology and homological algebra, but also in algebraic geometry and, after the appearance of Lawvere's thesis in 1963, in universal algebra. The latter work also constitutes a landmark in the history of this field. For it is in his thesis that Lawvere proposed the idea of developing the category of categories as a foundation for category theory, set theory and, thus, the whole of mathematics, as well as using categories for the study of theories, that is the logical aspects of mathematics. In the sixties, Lawvere outlined that basic framework for the development of an entirely original approach to logic and the foundations of mathematics: he proposed an axiomatization of the category of categories (Lawvere 1966), an axiomatization of the category of sets (Lawvere 1964), characterized Cartesian closed categories and showed their connections to logical systems and various logical paradoxes (Lawvere 1969), showed that the quantifiers and the comprehension schemes could be captured as adjoint functors to given elementary operations (Lawvere 1969, 1970, 1971) and finally argued for the role of adjoint functors in foundations in general, through the notion of "categorical doctrines" (Lawvere 1969). At the same time, Lambek described categories in terms of deductive systems and used categorical methods for proof theoretical purposes [30]. The 1970s saw the development and application of the concept in many different directions. (For more on the history of topos theory, see [40] ). The very first applications outside algebraic geometry were in set theory where various independent results were given a topos theoretical analysis.

Finally, from the 1980s to this day, category theory found new applications. On the one hand, it now has many applications to theoretical computer science where it has firm roots and contributes, among other things, to the development of the semantics of programming and the development of new logical systems ( [45], [46], [48] ). On the other hand, its applications to mathematics are becoming more diversified and it even touches upon theoretical physics where higher-dimensional category theory, which is to category theory what higher-dimensional geometry is to plane geometry, is used in the study of the so-called "quantum groups", or in quantum field theory [5].

## §0.4 Philosophical Significance

Category theory challenges philosophers in two non-exclusive ways. On the one hand, it is certainly the task of philosophy to clarify the general epistemological status of category theory and, in particular, its foundational status. On the other hand, category theory can be used by philosophers in their exploration of philosophical and logical problems. These two aspects can be illustrated briefly in turn.

Category theory is now a common tool in the toolbox of mathematicians. It unifies and provides a fruitful organization of mathematics. Arguments in favour of category theory and arguments against category theory as a foundational framework have been advanced (See [7] for a survey of the relationships between category theory and set theory, [20], [6] for arguments against category theory and [38] for a quick overview and a proposal). This is in itself a complicated issue which is rendered even more difficult by the fact that the foundations of category theory itself still have to be clarified. Given that most of philosophy of mathematics of the last 50 years or so has been done under the assumption that mathematics is more or less set theory in disguise, the retreat of set theory in favour of category theory would necessarily have an important impact on philosophical thinking.

The use of category theory for logical and philosophical studies is already well underway. Indeed, categorical logic, the study of logic with the help of categorical means, has been around for about 30 years now and is still vigorous. Category theory also provides relevant information to more general philosophical questions. For instance, Ellerman 1987 has tried to show that category theory constitutes a theory of universals which has properties radically different from set theory considered as a theory of universals [39]. If we move from universals to concepts in general, we can see how category theory could be useful even in cognitive science. Indeed, Macnamara and Reyes have already tried to use categorical logic to provide a different logic of reference [37]. Awodey, Landry, Makkai, Marquis and McLarty have tried to show how it sheds an interesting light on structuralists approach to mathematical knowledge ([2], [31], [32], [41]).

Thus, category theory is philosophically relevant in many ways which will undoubtedly have to be taken into account in the years to come.

# CHAPTER 1

# BASIC CONCEPTS

## §1.1 Introduction

This chapter deals with the study of category theory, functors and natural transformations which form the pillar of the category theory. This chapter is based on the work of Blyth[8], Eilenberg[18], Freyd[22], MacLane[35], Mitchell[42] and Schubert[47] etc.

Section 1.2 deals with the basic definition of categories and relative examples due to Blyth[8], MacLane[35] and Schubert[47] etc. Section 1.3 deals with the definition of functors and some examples of functors which states that functors are structure preserving maps between categories. In the last Section 1.4 the notion of natural transformation is introduced which describes that a natural transformation is a relation between two functors.

## §1.2 Categories

First of all we shall give a brief idea about the concept of category theory. The notion of function is one of the most fundamental concepts in mathematics and science. Functions are used to model variation– for example, the motion of a particle in space; the variation of a quantity like temperature over a space; the symmetries of a geometric object, or of physical laws; the variation of the state of a system over time.

A category is a abstract structure: a collection of objects, together with a collection of morphisms between them. For example, the object could be geometric figures and the morphisms could be ways of transforming one into another; or the objects might be data types and the morphisms programs.

Category theory is the algebra of functions; the principal operation on functions is taken to be composition. Whenever we calculate by composing functions (for example, in iteration a function) there is a category behind our calculations.

Now we define category as follows:

**Definition 1.2.1** A category $\mathcal{C}$, consists of the following data:

(*i*) a class $|\mathcal{C}|$ of objects $A$,$B$,$C$,.... called the class of objects of $\mathcal{C}$

(*ii*) for each ordered pair $(A, B)$ of $\mathcal{C}$, a set (possibly empty) $Mor_{\mathcal{C}}(A, B)$ called the set of morphisms from $A$ to $B$ (sometimes we denote $Mor_{\mathcal{C}}(A, B)$ by $Mor(A, B)$ )

(*iii*) for each ordered triplet ($A$,$B$,$C$) of objects of $\mathcal{C}$ we can define a map $Mor(B, C)\times Mor(A, B) \to Mor(A, C)$ called composition of morphisms. If $\beta \in Mor(B, C)$, $\alpha \in Mor(A, B)$ then the image of the pair ($\beta$,$\alpha$) is designated by $\beta\alpha$ (read as $\beta$ following $\alpha$), we can also write $\beta$ o $\alpha$.

The data are subjected to the following axioms:

$\mathbf{C_1}$ : The set $Mor(A, B)$ is pairwise disjoint.

$\mathbf{C_2}$ : *Associativity of composition:* If $\gamma\beta$ and $\beta\alpha$ both are defined then $(\gamma\beta)\alpha$ and $\gamma(\beta\alpha)$ are defined and $(\gamma\beta)\alpha = \gamma(\beta\alpha)$ holds.

$\mathbf{C_3}$ : *Existence of identity:* For each object $A$ there is an identity $I_A \in Mor(A, A)$ for which $I_A\alpha = \alpha$ and $\beta I_A = \beta$ hold whenever the left side is defined.

**Notation:** $\alpha \in Mor(A, B)$ is usually denoted by $\alpha : A \to B$ or $A \xrightarrow{\alpha} B$. $A$ is called the domain (source) and $B$ is called the codomain (target) of $\alpha$.

**Remark 1.2.1** The class of all morphisms of $\mathcal{C}$ is denoted by

$$Mor\mathcal{C} = \bigcup_{(A,B)\in|\mathcal{C}|\times|\mathcal{C}|} Mor(A, B).$$

**Remark 1.2.2** The identity morphism $I_A$ is uniquely determined by the object $A$. For this, let $I_A$ and $I'_A$ be two identity morphisms for $A$. Then by axioms $\mathbf{C_3}$(Definition 1.2.1), $I'_A I_A = I_A$ therefore, $I_A = I'_A$. Conversely, $A$ is determined by $I_A$ because the set of morphisms are pairwise disjoint as let $A \neq A'$, then we have $I_A \in Mor(A', A')$, but $I_A \in Mor(A, A)$ which is contradiction of the fact that $Mor(A, A)$ and $Mor(A', A')$ are pairwise disjoint. Hence $A$ is uniquely determined by $I_A$.

**Remark 1.2.3** By Remark 1.2.2 we obtain that there is one-one correspondence between the objects and subclass of morphisms consisting identities.

This shows that objects play secondary role in the definition of category. We can define a category without objects [21].

## Some standard categories with their notations

**Ens** the category of sets, whose class of objects is the class of all *sets* and the class of morphisms is the class of all *functions* on sets.

**Grp** the category of groups, whose class of objects is the class of all *groups* and the class of morphisms is the class of all *group homomorphisms* between them.

**Sgp** the category of subgroups, whose class of objects is the class of all *subgroups* and the class of morphisms is the class of all *group homomorphisms* between them.

**Ab** the category of abelian groups, whose class of objects is the class of all *abelian groups* and the class of morphisms is the class of all *group homomorphisms* which preserve the abelian structure of group.

**DivAb** the category of divisible abelian groups, whose class of objects is the class of all *divisible abelian groups* and the class of morphisms is the class of all *group homomorphisms* which preserve the abelian structure of group.

**Ord** the category of ordered sets, whose class of objects is the class of all *sets* on which there is defined an ordering $\leq$ (i.e. a relation that is reflexive, anti-symmetric and transitive) and the class of morphisms is the class of all *morphisms* $f : A \to B$ that are isotone (or order-preserving) in the sense that if $x \leq y$ in $A$ then $f(x) \leq f(y)$ in $B$.

**Top** the category of topological spaces, whose class of objects is the class of all *topological spaces* and the class of morphisms is the class of all *continuous maps* between them.

**Top**$_H$ the category of Hausdorff topological spaces, whose class of objects is the class of all *Hausdorff topological spaces* and the class of morphisms is the class of all *continuous maps* between them.

$_R$**Mod** the category of modules over R, whose class of objects is the class of all *R-modules* and the class of morphisms is the class of all *module homomorphisms* between them.

**Mon** the category of monoids, whose class of objects is the class of all *monoids* which are groups and the class of morphisms is the class of all *group homomorphisms* between them.

**MetSp** the category of metric spaces, whose class of objects is the class of all *metric spaces* and the class of morphisms is the class of all *continuous maps* between them.

**Vect**$_F$ the category of finite dimensional vector spaces, whose class of objects is the class of all *finite dimensional vector spaces* and the class of morphisms is the class of all *linear transformations* between them.

**Ring** the category of rings, whose class of objects is the class of all *rings* and the class of morphisms is the class of all *ring homomorphisms* between them.

**Remark 1.2.4** Sometimes the morphisms in the category need not be actual functions or mappings in the usual sense.

**Definition 1.2.2 (Small and large categories)** If the class of objects of a category $\mathcal{C}$ is a set, the category is called small category otherwise it is called large category.

**Example 1.2.1** If we take all sets $A$, $B$, $C$, .... to be objects and all functions $f : A \to B$, $g : B \to C$, .... to be morphisms, we get a category **Ens** called the category of sets. Here we take the composition as composition of functions, defined by the rule $gof(a) = g(f(a))$. This is a large category.

All the standard categories as defined above are the examples of large categories.

**Examples of small categories:**

**Example 1.2.2** We can construct a category with one object $A$ and one morphism, which must therefore, be the identity morphism i.e. $I_A : A \to A$.

**Example 1.2.3** Category with one object $A$ and two morphisms $I_A : A \to A$ and $\alpha : A \to A$. To specify the category we have to observe all composites $I_A o I_A$, $I_A o\alpha$, $\alpha o I_A$, $\alpha o\alpha$ and to check the identity and associative law.
Trivially, compositions $I_A o I_A$, $I_A o\alpha$ and $\alpha o I_A$ are defined. Further, we shall check the only composition $\alpha o\alpha$. For this, there are two possible choices, either $\alpha o\alpha = I_A$ or $\alpha o\alpha = \alpha$.

'$ase(i)$ suppose $\alpha o\alpha = I_A$ i.e. the composition table is:

| | $1_A$ | $\alpha$ |
|---|---|---|
| $1_A$ | $1_A$ | $\alpha$ |
| $\alpha$ | $\alpha$ | $1_A$ |

In fact, this does give a category. All that needs to be checked is the associative law. Here composition is the fully defined operation.
we may recognize the composition table as
⋆ addition modulo 2 or

⋆ the cyclic group of order 2,

and there are the well known to be associative.

*Case(ii)* Suppose the $\alpha o \alpha = \alpha$

This is also yields a category with the composition table as.

| | $1_A$ | $\alpha$ |
|---|---|---|
| $1_A$ | $1_A$ | $\alpha$ |
| $\alpha$ | $\alpha$ | $\alpha$ |

From this table it is clear that associative law holds and $I_A$ is the identity of $A$.

For this, take A to be the set $\{0, 1\}$. Let $I_A$ be the identity function $A \to A$ and $\alpha$ is the function given by

$$\alpha : 0 \mapsto 0$$
$$1 \mapsto 0$$

Clearly, $\alpha o \alpha = \alpha^2 : 0 \mapsto 0 \mapsto 0,$

$1 \mapsto 0 \mapsto 0$

Hence $\alpha^2 = \alpha$ as required.

Here is another way of representing this example in terms of sets and functions. Consider $A$ be the Cartesian plane and $I_A$ the identity function. Now take $\alpha$ to be projection onto the $x$-*axis* i.e.

$$\alpha : (x, y) \mapsto (x, 0)$$

Then clearly $\alpha^2 = \alpha(x, 0) = (x, 0) = \alpha(x, y)$

i.e. $\alpha^2 = \alpha$ as required.

**Example 1.2.4** Given any group $G = \{1, f, g, ....\}$ we get a category with one object as the set $G$ and morphims as the elements of $G$. Composition is the product of elements in the group which is, of course, associative.

**Definition 1.2.3 (Subcategory)** Let $\mathcal{C}$ be a category then a category $\mathcal{C}'$ is said to be a *subcategory* of $\mathcal{C}$ if

(i) Each object of $\mathcal{C}'$ is also an object of $\mathcal{C}$.

(ii) For all objects $A$ and $B$ in $\mathcal{C}'$ we have that $Mor_{\mathcal{C}'}(A, B)$ is a subset of $Mor_{\mathcal{C}}(A, B)$.

(iii) The composition of any two morphisms in $\mathcal{C}'$ is same as their composition in $\mathcal{C}$.

(iv) For each object $A$ in $\mathcal{C}'$ the subset $Mor_{\mathcal{C}'}(A, A)$ of $Mor_{\mathcal{C}}(A, A)$ contains the element $I_A$ of $Mor_{\mathcal{C}}(A, A)$.

**Definition 1.2.4** For every category $\mathcal{C}$ we can form a subcategory containing all the objects of $\mathcal{C}$ and the morphisms as the only identities morphisms, we call this category as discrete subcategory.

**Definition 1.2.5 (Full subcategory)** A subcategory $\mathcal{C}'$ is called a full subcategory of $\mathcal{C}$ if

$$Mor_{\mathcal{C}'}(A, B) = Mor_{\mathcal{C}}(A, B)$$

**Example 1.2.5** The category of finite sets is the full subcategory of **Ens** whose objects are the finite sets in **Ens**. Therefore, the category of finite sets has all finite sets as objects, the set $Mor(X, Y)$ of all morphisms from the finite set $X$ to the finite $Y$ is just the set of all maps from $X$ to $Y$. While the composition $Mor(X, Y) \times Mor(Y, Z) \to Mor(X, Z)$ for all triplet of finite sets $X$,$Y$,$Z$ is given by $(f, g) \to gf$ where $gf$ is the usual composition of the map $f : X \to Y$ and $g : Y \to Z$.

**Example 1.2.6** The category **Grp** is defined to be the full subcategory of the category **Mon** whose class of objects are the monoids which are groups. Therefore, the objects of **Grp** are all groups, $Mor(X, Y)$ is the set of all morphisms of groups from the group $X$ to the group $Y$ for all objects $X$ and $Y$ in **Grp** and the composition $Mor(X, Y) \times Mor(Y, Z) \to Mor(X, Z)$ for all triplet of groups $X$,$Y$,$Z$ is given by $(f, g) \to gf$ where $gf$ is the usual composition of the morphisms of groups

$f : X \to Y$ and $g : Y \to Z$.

**Example 1.2.7** The category $\mathbf{Ring_1}$ is a subcategory of the category **Ring**. This subcategory is not full subcategory since for any pair of objects $A,B \in \mathbf{Ring_1}$, the $Mor(A, B)$ has zero morphism when it is considered in the category **Ring** but it has no zero morphism when we consider it in $\mathbf{Ring_1}$.

**Observations:**

In a category $\mathcal{C}$, the following statements hold:

(*i*) $\mathcal{C}$ is a full subcategory of $\mathcal{C}$.

(*ii*) Two categories $\mathcal{C}$ and $\mathcal{C}'$ are the same if and only if $\mathcal{C}$ is a subcategory of $\mathcal{C}'$ and $\mathcal{C}'$ is a subcategory of $\mathcal{C}$.

(*iii*) If $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ and $\mathcal{C}''$ is a subcategory of $\mathcal{C}'$, then $\mathcal{C}''$ is a subcategory of $\mathcal{C}$.

(*iv*) If $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$ and $\mathcal{C}''$ is a full subcategory of $\mathcal{C}'$, then $\mathcal{C}''$ is a full subcategory of $\mathcal{C}$.

(*v*) If $\mathcal{C}'$ and $\mathcal{C}''$ are full subcategories of $\mathcal{C}$, then $\mathcal{C}'$=$\mathcal{C}''$ if and only if $|\mathcal{C}'|$=$|\mathcal{C}''|$.

**Definition 1.2.6 (Dual category)** For any category $\mathcal{C}$, we can form a category $\mathcal{C}^*$, known as the dual category of $\mathcal{C}$ if the following conditions hold:

(*i*) The class of objects of $\mathcal{C}^*$ is similar that of the class of objects of $\mathcal{C}$.

(*ii*) For every pair of objects $A,B \in \mathcal{C}^*$ we have

$$Mor_{\mathcal{C}^*}(A, B) = Mor_{\mathcal{C}}(B, A).$$

(*iii*) If $\alpha^*$, $\beta^* \in \mathcal{C}^*$ and $\alpha^* o \beta^*$ is defined in $\mathcal{C}^*$ then

$$\alpha^* o \beta^* = (\beta o \alpha)^*.$$

**Remark 1.2.5** If $\alpha : A \to B$ be a morphism in $\mathcal{C}$ then $\alpha^* \in \mathcal{C}^*$ be a morphism $\alpha^* : B^* \to A^*$.

**Example 1.2.8** The dual of the category with one object $A$ and three morphisms $I_A$, $e_1$, $e_2$ satisfying

$$e_i e_j = e_j \qquad (i, j = 1, 2)$$

is the category with one object $A^*$ and three morphisms $I_A{}^*$, $e_1{}^*$, $e_2{}^*$ satisfying

$$e_j{}^* e_i{}^* = e_i{}^* \qquad (i, j = 1, 2)$$

Compare the composition tables of these two categories:

| | $1_A$ | $e_1$ | $e_2$ |
|---|---|---|---|
| $1_A$ | $1_A$ | $e_1$ | $e_2$ |
| $e_1$ | $e_1$ | $e_1$ | $e_2$ |
| $e_2$ | $e_2$ | $e_1$ | $e_2$ |

| | $1_A^*$ | $e_1^*$ | $e_2^*$ |
|---|---|---|---|
| $1_A^*$ | $1_A^*$ | $e_1^*$ | $e_2^*$ |
| $e_1^*$ | $e_1^*$ | $e_1^*$ | $e_1^*$ |
| $e_2^*$ | $e_2^*$ | $e_2^*$ | $e_2^*$ |

Now it is an amazing but not obvious fact that the dual of many well-known categories are also well-known categories.

## §1.3 Functors

Within a category $\mathcal{C}$ we have the morphism sets Mor(X,Y) which serve to establish connection between different objects of the category. Now the language of categories has been developed to delineate the various areas of mathematical theory: thus it is natural that we should wish to be able to describe connections between different categories. We now formulate the notion of a transformation from one category to another. Such a transformation is called a functor, which can be defined as :

**Definition 1.3.1** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A pair of functions $F = (F_{ob}, F_{mor})$, where

$$F_{ob} : |\mathcal{C}| \to |\mathcal{D}|$$

and

$$F_{mor} : Mor_{\mathcal{C}} \rightarrow Mor_{\mathbf{D}}$$

is called a functor which assigns to each object of $\mathcal{C}$ an object of $\mathcal{D}$ and to morphisms of $\mathcal{C}$ a morphism of $\mathcal{D}$, satisfying the following conditions:

$F_1$ : If $\alpha : X \rightarrow Y$ is a morphism in $\mathcal{C}$ then $F(\alpha) : F(X) \rightarrow F(Y)$is a morphism in $\mathcal{D}$.

$F_2$ : $F(I_X) = I_{F(X)}$ for every object $X \in \mathcal{C}$.

$F_3$ : If $\alpha\beta$ is defined in $\mathcal{C}$ then $F(\alpha)F(\beta)$ is defined in $\mathcal{D}$ and $F(\alpha\beta)=F(\alpha)F(\beta)$ or $F(\alpha\beta)=F(\beta)F(\alpha)$

In axiom $F_3$ if $F(\alpha\beta)=F(\alpha)F(\beta)$ holds then $F$ is called covariant functor and if $F(\alpha\beta)=F(\beta)F(\alpha)$ holds then $F$ is called contravariant functor.

We shall make a connection that whenever we speak simply of a functor we shall mean a covariant functor. A functor $F$ from $\mathcal{C}$ to $\mathcal{D}$ is denoted by $F : \mathcal{C} \longrightarrow \mathcal{D}$.

**Remark 1.3.1** Every functor $F : \mathcal{C} \rightarrow \mathcal{D}$ induces a function

$$F_{A,B} : Mor_{\mathcal{C}}(A, B) \rightarrow Mor_{\mathbf{D}}(F(A), F(B))$$

For every pair of objects $(A, B)$ in $\mathcal{C}$.

**Definition 1.3.2 (Composition of two functors)** Let $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$ be two functor then their composition $GoF : \mathcal{A} \rightarrow \mathcal{C}$ be defined as,

$$GoF(X) = G(F(X)) \ for \ all \ objects \ X \in \mathcal{A}$$

and

$$GoF(\alpha) = G(F(\alpha)) \ for \ all \ morphisms \ \alpha \in \mathcal{A}.$$

**Remark 1.3.2** The composition functor $GoF$ is a covariant functor if $F$ and $G$ are of the same variance, $GoF$ is a contravariant functor if $F$ and $G$ are of the opposite variance.

**Remark 1.3.3** Every homomorphism monoid to monoid (ring to ring or group to group) can be regarded as a functor.

Now we consider some examples of functors.

**Example 1.3.1** For any category $\mathcal{C}$, assigning every object $A$ to $A$ and every morphisms $\alpha$ to the same morphism $\alpha$ in $\mathcal{C}$, we can define a functor

$$I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$$

such that

$$I_{\mathcal{C}}(A) = A \ \ and \ \ I_{\mathcal{C}}(\alpha) = \alpha \ \ for \ all \ \alpha \in A.$$

This functor is known as **identity functor**.

**Example 1.3.2** Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$ then a covariant functor $\mathbf{I} : \mathcal{C}' \to \mathcal{C}$ can be defined as

$$\mathbf{I}(A) = A \ for \ all \ objects \ A \in \mathcal{C}'$$

and

$$\mathbf{I}(\alpha) = \alpha \ for \ all \ morphisms \ \alpha \in \mathcal{C}'.$$

This functor is known as **inclusion functor**.

**Example 1.3.3** Since every group is a set and every group homomorphism is a function. A covariant functor $F$ can be defined from the category **Grp** of groups to the category **Ens** of sets by assigning

(*i*) To every group $G$ in **Grp**, the underlying set $F(G)$ in **Ens**.

(*ii*) To every group homomorphism $\alpha : G \to G'$ in **Grp**, the underlying function $F(\alpha) : F(G) \to F(G')$ in **Ens**.

This function is known as **forgetful functor** because it forgets the group structure in **Ens**.

In connection of functors we have the following definitions:

**Definition 1.3.3 (Cat)** Since the composition of functors is also a functor and this operation (composition) is also associative whenever it is defined. With the help of these facts, we can construct a new category **Cat** by taking objects of category as all small categories and morphisms as the functors between them.

**Definition 1.3.4 (Full functor)** A functor $F : \mathcal{C} \to \mathcal{D}$ is called full if the function $Mor_{\mathcal{C}}(A, B) \to Mor_{\mathcal{D}}(F(A), F(B))$ induced by $F$ is surjective (onto).

**Definition 1.3.5 (Faithful functor)** A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be faithful if the function $Mor_{\mathcal{C}}(A, B) \to Mor_{\mathcal{D}}(F(A), F(B))$ induced by $F$ is injective (one-one).

**Definition 1.3.6 (Representive functor)** A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be representive if for every object $B \in \mathcal{D}$ there exist an object $A \in \mathcal{C}$ such that

$$F(A) \cong B.$$

**Definition 1.3.7 (Imbedding)** A faithful functor $F : \mathcal{C} \to \mathcal{D}$ which takes distinct objects to distinct objects is said to be imbedding.

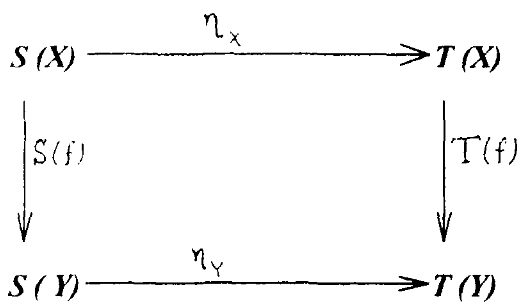
## §1.4 Natural Transformations

In this section we introduce the concept of natural transformation which plays a key role in the development of the language of categories and functors. Natural transformation can be defined as follows:

**Definition 1.4.1** Let **S** and **T** be two covariant functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. A family of morphisms,

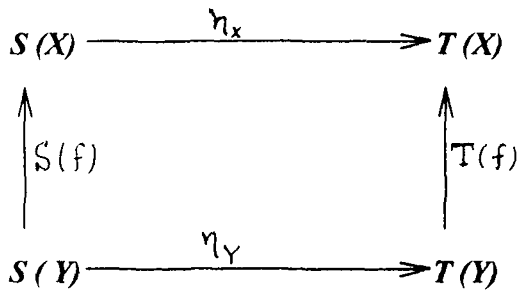
$$\eta = \{\eta_X | X \in Obj\mathcal{C} \ and \ \eta_X : S(X) \to T(X)\} \subseteq \mathcal{D}$$

is called the natural transformation from $S$ to $T$ if for every morphism $f : X \to Y$ in $\mathcal{C}$ such that the diagram




commutes.
The natural transformation $\eta$ from $S$ to $T$ is denoted by $\eta : S \dashv T$.

If $S$ and $T$ are contravariant then the above diagram is represented by the following commutative diagram




If each $\eta_X$ is an isomorphism then we say that $\eta$ is a natural isomorphism.

**Definition 1.4.2** The functors $S$,$T$ : $\mathcal{C} \to \mathcal{D}$ are said to be naturally equivalent, denoted by $S \cong T$, if there is a natural isomorphism $\eta : S \to T$.

**Theorem 1.4.1** A natural transformation $\eta : S \to T$ is a natural isomorphism if and only if there is a natural transformation $\mu : T \to S$ such that

$$\mu o \eta = I_S \text{ and } \eta o \mu = I_T.$$

**Proof** If $\eta : S \to T$ is a natural isomorphism where $S$,$T$ : $\mathcal{C} \to \mathcal{D}$, assign to every object $A$ of $\mathcal{C}$ the morphism $\mu_A = {\eta_A}^{-1} : T(A) \to S(A)$. This clearly define a natural transformation $\mu : T \to S$. Since

$$(\mu o \eta)_A = \mu_A o \eta_A = \eta_A{}^{-1} o \eta_A = I_{S(A)},$$
$$(\eta o \mu)_A = \eta_A o \mu_A = \eta_A o \eta_A{}^{-1} = I_{T(A)}.$$

We deduce that $\mu o \eta = I_S$ and $\eta o \mu = I_T$.

Conversely, if there exists a natural transformation $\mu : T \to S$ such that $\mu o \eta = I_S$ and $\eta o \mu = I_T$ then for every object $A$ of $\mathcal{C}$ we have

$$\mu_A o \eta_A = (\mu o \eta)_A = I_{S(A)},$$
$$\eta_A o \mu_A = (\eta o \mu)_A = I_{T(A)}$$

whence each $\eta_A$ is an isomorphism.

**Proposition 1.4.1** Composition of two natural transformations is also a natural transformation.

**Proof** Let $\eta : S \dashv T$ and $\psi : T \dashv U$ be two natural transformations, where $S,T,U$:$\mathcal{C} \to \mathcal{D}$ be covariant functors from category $\mathcal{C}$ to category $\mathcal{D}$.

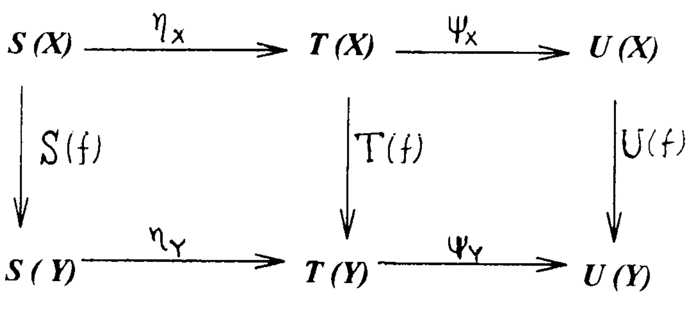
Now we take,

$$\phi_X = \psi_X \eta_X$$

i.e.

$$S(X) \xrightarrow{\phi_X} U(X) \quad S(X) \xrightarrow{\eta_X} T(X) \xrightarrow{\psi_X} U(X) \ for\ all\ objects\ X \in \mathcal{C}$$

then for any morphism $f : X \to Y$ in $\mathcal{C}$ such that in the diagram




the left and right squares commute since $\eta$ and $\psi$ are natural. Hence the outside rectangle commutes i.e. $\psi o \eta$ is natural.

**Example 1.4.1** Consider the functor

$$I : \mathbf{Grp} \to \mathbf{Grp}$$

which carries each group $G \in \mathbf{Grp}$ to its inner automorphisms group $I(G)$ in **Grp** and morphism $\alpha : G \to G'$ to the corresponding homomorphism

$$I(\alpha) : I(G) \to I(G') \textit{ since for each } G \in \mathbf{Grp}, G \cong \mathbf{Grp}.$$

Then we define a natural transformation

$$\eta : I \to I$$

from identity functor to $I : \mathbf{Grp} \to \mathbf{Grp}$, by taking

$$\eta_G : G \simeq I(G) \textit{ for all } G \in \mathbf{Grp}.$$

# CHAPTER 2

# SPECIAL OBJECTS, MORPHISMS AND SOME CONSTRUCTIONS IN CATEGORIES

## §2.1 Introduction

This chapter has been devoted to the study of certain special types of morphisms and objects. Most of the results of this chapter are based on the work of Blyth [8], MacLane [36] and Mitchell [42] etc.

Section 2.2 deals with the notion of monomorphism, epimorphism and isomorphism which states that a morphism which is both monomorphism and epimorphism need not be an isomorphism. In the Section 2.3 the notion of initial object, terminal object and zero object is introduced which states that an object in a category is called zero object which is initial and terminal both but converse need not true. Section 2.4 deals with the categorical porduct and co-product. In the last Section 2.5 the notion of equalizers and kernels are discussed.

## §2.2 Special Morphisms

We have already discussed the examples of categories in which the objects are sets endowed with some additional structure and the morphisms are structure-preserving mappings. Such categories are called **concrete**. One of the main objectives of the theory of categories is to obtain general theorems with applications in concrete categories. To see how this can be achieved, we first show how notions that arise in concrete categories can be generalized to arbitrary categories. For this we have to find properties that are independent of 'element-wise' arguments. By way of illustration, we observe that in the category **Ens** the following statements concerning a mapping $f : X \to Y$ are equivalent:

$(i)$ $f$ is injective (in the sense that $x \neq y \implies f(x) \neq f(y)$)

$(ii)$ $f$ is left cancellable (in the sense that $fog = foh \implies g = h$)

Thus the notion of injectivity, which is usually defined as in $(i)$ using elements, can be expressed as in (ii) in terms only of mappings.

Now the property of being left cancellable can clearly be considered in an arbitrary category, and leads to the following notion.

**Definition 2.2.1 (Monomorphism)** A morphism $f : X \to Y$ in a category $\mathcal{C}$ is called a monomorphism if for every pair of morphisms $g,h : A \to X$ such that

$$A \xrightarrow{g} X \xrightarrow{f} Y = A \xrightarrow{h} X \xrightarrow{f} Y$$

i.e.

$$fog = foh$$

$\Longrightarrow$

$$g = h \qquad \text{(i.e. } f \text{ is left cancellable)}$$

We have just seen that in **Ens** a morphism is monic if and only if it is injective. This is also true, for example in **Sgp**, **Grp**, ${}_R$**Mod**. However in a concrete category every injective morphism is monic. But the converse is not true, as the following example illustrates.

**Example 2.2.1** Consider the category **DivAb** of divisible abelian groups (it is subcategory of **Ab**).
Now take two objects $Q$ and $Q/Z$ in **DivAb**, both are the abelian groups, this follows from the observations:

$$p/q = n(p/nq)$$

and

$$p/q + Z = n(p/nq + Z).$$

Consider the natural morphism

$$\eta : Q \to Q/Z$$

defined by

$$\eta(p/q) = p/q + Z \ \ \forall \ \ p/q \in Q$$

then it is always onto (epimorphism). Trivially, this morphism $\eta$ is not injective as Ker $\eta = Z \neq 0$ in $Q$, but it is left cancellable (monomorphism).
Now we will check that it is left cancellable.
For this, let $f, g : A \to Q$ *are morphism in this category and* $f \neq g$, there exists $a \in A$ such that

$$f(a) \neq g(a)$$

i.e.

$$f(a) - g(a) \neq 0 \;\; in \;\; Q$$
$$= r/s(s \neq \pm 1).$$

Since $A$ is divisible we can find $b \in A$ such that

$$a = r.b$$

Now,

$$\begin{aligned} r[f(b) - g(b)] &= rf(b) - rg(b) \\ &= f(rb) - g(rb) \\ &= f(a) - g(a) \\ &= r/s \end{aligned}$$

i.e.

$$r[f(b) - g(b)] = r/s$$

$\Longrightarrow$
$$f(b) - g(b) = 1/s$$

Since,

$$\eta(1/s) = 1/s + Z \neq 0$$

$\Longrightarrow$
$$\eta(f(b) - g(b)) \neq 0$$

$\Longrightarrow$
$$\eta(f(b)) - \eta(g(b)) \neq 0$$

$$\Longrightarrow \qquad (\eta o f)(b) - (\eta o g)(b) \neq 0$$

$$\Longrightarrow \qquad (\eta o f - \eta o g)(b) \neq 0$$

$$\Longrightarrow \qquad \eta o f - \eta o g \neq 0$$

$$\Longrightarrow \qquad \eta o f \neq \eta o g$$

$\Longrightarrow$ $\eta$ is left cancellable

$\Longrightarrow$ $\eta$ is monomorphism in **DivAb** but not injective.

Which completes our claim.

**Definition 2.2.2 (Epimorphism)** A morphism $f : X \to Y$ in a category $\mathcal{C}$ is called a epimorphism if for every pair of morphisms $g$,$h : Y \to B$ such that

$$X \xrightarrow{f} Y \xrightarrow{g} B = X \xrightarrow{f} Y \xrightarrow{h} B$$

i.e.

$$gof = hof$$

$$\Longrightarrow \qquad g = h \qquad \text{(i.e. } f \text{ is right cancellable).}$$

**Remark 2.2.1** A morphism $f : X \to Y$ which is epimorphism in a category $\mathcal{C}$ may not be surjective, as the following example illustrates.

**Example 2.2.2** Consider the category **Ring** (*or* **Sgp**) of rings (w.r.t. multiplication). Now since $Z$, $Q \in$ **Ring**, we define an inclusion morphism $i : Z \to Q$ which is injective but not surjective, but it is epimorphism (right cancellable) in the category.

Now we will show that it is epimorphism (right cancellable) but not surjective.

For this, let $g$, $h : Q \to A$ be a morphism in **Ring** (*or* **Sgp**) with,

$$Z \xrightarrow{i} Q \xrightarrow{g} A = Z \xrightarrow{i} Q \xrightarrow{h} A$$

such that

$$goi = hoi$$

$\Longrightarrow$ $$(goi)(n) = (hoi)(n) \;\; \forall \;\; n \in Z$$

$\Longrightarrow$ $$g(i(n)) = h(i(n))$$

$\Longrightarrow$ $$g(n) = h(n) \;\; \forall \;\; n \in Z$$

Now for any $m/n \in Q$ we have

$$\begin{aligned} g(m/n) &= g(m.n^{-1}.1) \\ &= g(m)g(n^{-1})g(1) \\ &= h(m)g(n^{-1})h(1) \qquad [\; since \;\; g(1) = h(1) \;] \\ &= h(m)g(n^{-1})h(nn^{-1}) \\ &= h(m)g(n^{-1})h(n)h(n^{-1}) \\ &= h(m)g(n^{-1})g(n)h(n^{-1}) \qquad [\; since \;\; g(n) = h(n) \;] \\ &= h(m)g(n^{-1}n)h(n^{-1}) \\ &= h(m)g(1)h(n^{-1}) \\ &= h(m)h(1)h(n^{-1}) \\ &= h(m.1.n^{-1}) \\ &= h(mn^{-1}) \\ &= h(m/n) \end{aligned}$$

$\Longrightarrow$ $$g(m/n) = h(m/n) \;\; \forall \;\; m/n \in Q$$

$\Longrightarrow$ $$g = h$$

$\Longrightarrow$ $i$ is right cancellable, but not surjective.

Therefore, $i$ is epimorphism, but not surjective in the category **Ring** of rings. Which completes our claim.

**Definition 2.2.3 (Bimorphism)** A morphism which is monomorphism as well as epimorphism both is called bimorphism.

**Definition 2.2.4** A morphism $f : X \to Y$ in a category $\mathcal{C}$ is called *retraction* if and only if there exists a morphism $g : Y \to X$ in $\mathcal{C}$ such that

$$fog = I_Y$$

Dually,

**Definition 2.2.5** A morphism $f : X \to Y$ in a category $\mathcal{C}$ is called *coretraction* if and only if there exists a morphism $g : Y \to X$ in a category $\mathcal{C}$ such that

$$gof = I_X.$$

**Remark 2.2.2** Every retraction is a epimorphism but converse need not be true.

Let us a consider a morphism $f : A \to B$ be a retraction in a category $\mathcal{C}$ then there exists $g : B \to A$ in $\mathcal{C}$ such that

$$fog = I_B.$$

Now consider two morphisms $h_1, h_2 : B \to X$ such that

$$A \xrightarrow{f} B \xrightarrow{h_1} X = A \xrightarrow{f} B \xrightarrow{h_2} X$$

i.e.

$$h_1of = h_2of$$

$$\begin{aligned} (h_1of)og &= (h_2of)og \\ h_1o(fog) &= h_2o(fog) \\ h_1oI_B &= h_2oI_B \end{aligned}$$

$$h_1 = h_2$$

$\Longrightarrow$ $f$ is right cancellable

$\Longrightarrow$ $f$ is epimorphism.

Conversely, it is not true, for this we construct a counter example as follows:
In ${}_Z\mathbf{Mod}$ consider the object $Q_p$ defined by

$$Q_p = \{x \in Q | x = kp^{-n}, k \in Z \;\&\; n \in N\}$$

where $p$ is a prime. This collection forms a subgroup of $Q$ and $Z$ is a subgroup of $Q_p$ i.e. $Z \subset Q_p$.
Now we define a morphism

$$f : Q_p/Z \to Q_p/Z$$

such that

$$f(x + Z) = px + Z.$$

Then it is readily seen that $f$ is a $Z$-*morphism*.
Since

$$\begin{aligned} kp^{-n} + Z &= p(kp^{-n-1} + Z) \\ &= p(kp^{-n-1}) + Z. \end{aligned}$$

We see that $f$ is surjective. Now we shall only to show that $f$ has no right inverse.
On contrary, assume that $f$ has a right inverse in ${}_Z\mathbf{Mod}$ then there exists

$$h : Q_p/Z \to Q_p/Z \;\; in \;\; Z - module \;\; such \;\; that$$

$$foh = I \;(identity \;\; on \;\; Q_p/Z)$$

$$\implies \qquad (foh)(n) = n \;\; \forall \;\; n \in Q_p/Z.$$

Now in particular as $k = 1$, $n = 1$ we have,

$$\begin{aligned} p^{-1} + Z &= f(h(p^{-1} + Z)) \qquad [\, since \; foh = I \,] \\ &= p(h(p^{-1} + Z)) \\ &= h(p(p^{-1} + Z)) \\ &= h(I + Z) \\ &= h(0 + Z) \\ &= 0 + Z. \end{aligned}$$

Thus we have $\quad p^{-1} + Z = 0 + Z.$

Which is never possible since $x + Z = 0 + Z$ if and only if $x \in Z$ but here $p$ is prime.

$\Longrightarrow$ $\quad h$ does not exist

$\Longrightarrow$ $\quad f$ has no right inverse

$\Longrightarrow$ $\quad f$ is not retraction.

Which completes our claim.

**Remark 2.2.3** Every coretraction (section) is a monomorphism but converse need not be true.

Let us a consider a morphism $f : A \to B$ be a coretraction (section) in a category $\mathcal{C}$ then there exists $g : B \to A$ in $\mathcal{C}$ such that

$$fog = I_B.$$

Now consider two morphisms $h_1, h_2 : X \to A$ in $\mathcal{C}$ such that

$$X \xrightarrow{h_1} A \xrightarrow{f} B = X \xrightarrow{h_2} A \xrightarrow{f} B$$

i.e.

$$foh_1 = foh_2$$

$$\begin{aligned} go(foh_1) &= go(foh_2) \\ (gof)oh_1 &= (gof)oh_2 \\ I_A oh_1 &= I_A oh_2 \end{aligned}$$

$$h_1 = h_2$$

$\Longrightarrow$ $\quad f$ is left cancellable

$\Longrightarrow$ $\quad f$ is monomorphism.

Conversely, it is not true i.e. every monomorphism need not be a coretraction (section). For this we construct a counter example as follows:

Consider a category ${}_Z\mathbf{Mod}$ and a morphism

$$f : Z \to Z$$

such that

$$f(n) = 2n \ \ \forall \ \ n \in Z.$$

Now we check for the left inverse.
Suppose there exist a $Z$-morphism $g : Z \to Z$ such that

$$gof = I_Z$$

for any integer $n \in Z$ we have,

$$\begin{aligned} 2.g(n) &= g(2n) \qquad (\ module \ \ homomorphism\ ) \\ &= g(f(n)) \qquad (\ since \ \ f(n) = 2n\ ) \\ &= gof(n) \\ &= I_Z(n) \\ &= n \end{aligned}$$

i.e. $\qquad 2.g(n) = n \ \ \forall \ \ n \in Z.$

In particular, if we take $n = 1$ then

$$2.g(1) = 1.$$

For convenience we can consider this relation as an equation of the form

$$2x = 1 \ \ in \ \ Z.$$

But there is no value of $x$ in $Z$ to solve the equation $2x = 1$
$\Longrightarrow$ $\qquad g$ can not be defined
$\Longrightarrow$ $\qquad$ it is not left cancellable
$\Longrightarrow$ $\qquad$ it is not a coretraction (section).
Which completes our claim.

**Definition 2.2.6 (Isomorphism)** A morphism $f : X \to Y$ in a category $C$ is called *isomorphism* if there exist a morphism $g : Y \to X$ in $C$ such that

$$X \xrightarrow{f} Y \xrightarrow{g} X = I_X$$

i.e.

$$gof = I_X$$

and

$$Y \xrightarrow{g} X \xrightarrow{f} Y = I_Y$$

i.e.

$$fog = I_Y.$$

This uniquely determined morphism $g$ is also an isomorphism which is called the inverse of $f$ and is often denoted by $f^{-1}$.

**Observations:**

Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms in $C$ then the following statements hold:

(*a*) If $f$ and $g$ are both monomorphism (epimorphism), then the composition $gf : X \to Z$ is an monomorphism (epimorphism).

(*b*) If $gf : X \to Z$ is an epimorphism, then so is $g$.

(*c*) If $gf : X \to Z$ is an monomorphism, then so is $f$.

(*d*) If $f : X \to Y$ is an isomorphism, then $f$ is both a monomorphism and an epimorphism.

In connection with the last of the above observations, it is worth observing that a morphism in a category which is both a monomorphism and epimorphism (i.e. bimorphism) need not be an isomorphism. It is clear from the following example.

**Example 2.2.3** Consider the category **DivAb** of divisible abelian groups (it is subcategory of **Ab**).
Now take two objects $Q$ and $Q/Z$ in **DivAb**, both are the abelian groups, this follows from the observations:

$$p/q = n(p/nq)$$

and

$$p/q + Z = n(p/nq + Z).$$

Consider the natural morphism

$$\eta : Q \to Q/Z$$

defined by

$$\eta(p/q) = p/q + Z \ \ \forall \ \ p/q \in Q$$

then it is always onto (epimorphism). Trivially this morphism $\eta$ is not injective as $Ker\eta = Z \neq 0$ in $Q$, but it is left cancellable (monomorphism) as we discussed in Example 2.2.1.
Therefore, $\eta$ is bimorphism (monomorphism and epimorphism both) but not isomorphism (as it is not injective).
Which completes our claim.

**Remarks 2.2.4** Two objects in a category are called *isomorphic* if there is an isomorphism between them.

**Remarks 2.2.5** An isomorphism in a category $\mathcal{C}$ is a coretraction (section) and retraction both.

By the Example 2.2.3 it is clear that every bimorphism in a category need not be isomorphism in general, but there are certain categories in which these two concepts are equivalent. To characterize this behavior of some categories we shall define a special category, known as balanced category.

**Definition 2.2.7 (Balanced category)** A category $\mathcal{C}$ is called balanced if every bimorphism in $\mathcal{C}$ is an isomorphism.

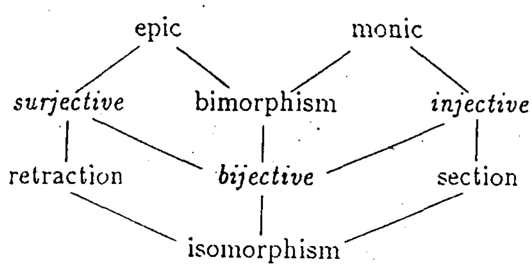
**Example 2.2.4** The category **Ens** is balanced category, since every function which is injection and surjection both is also a bijection.

**Example 2.2.5** The category **Grp** is balanced category, since every group homomorphism which is monomorphism and epimorphism both is also an isomorphism.

**Example 2.2.6** The category ${}_R$**Mod** is balanced category.

The category **DivAb**, **Ring**, **Sgp** and $\mathbf{Top}_H$ are not balanced. As we have discussed in Remark 2.2.9 that in **DivAb**, bimorphism is not to be an isomorphism, hence it is not balanced.

For concrete categories the following diagram (in which an increasing line segment is taken to mean $\Longrightarrow$) summarises the above discussion:




## §2.3 Special Objects

In **Ens** the empty set $\phi$ and the one point set $\{\star\}$ both have properties which characterize them, and which can be formulated purely in terms of functions.

1. **A characteristic property of the one point set** $\{\star\}$. Given any set $X$ there is exactly one function from $X$ to $\{\star\}$.

The function takes any $x \in X$ to $\star$. Clearly this is a function and is the only possible function from $X$ to $\{\star\}$.

2. **A characteristic property of the empty set** $\phi$. Given any set $X$ there is exactly one function from $\phi$ to $X$.

For this, since a function from $X$ to $Y$ is a subset $U$ of $X \times Y$ satisfying the property that to each $y \in Y$ there is exactly one pair in $U$ with first coordinate $y$. Hence a function from $\phi$ to $X$ is a subset of $\phi \times X = \phi$, and there is only one such subset , namely $\phi$, and it satisfies the property vacuously.

**Definition 2.3.1 (Initial object)** An object $I \in \mathcal{C}$ is called an initial object of the category $\mathcal{C}$ if for every object $X \in \mathcal{C}$ the set $Mor_{\mathcal{C}}(I, X)$ is singleton.

**Definition 2.3.2 (Terminal object)** An object $T \in \mathcal{C}$ is called an terminal(co-initial) object of the category $\mathcal{C}$ if for every object $X \in \mathcal{C}$ the set $Mor_{\mathcal{C}}(X, T)$ is singleton.

**Remark 2.3.1** Any two terminal (initial) objects in a category are isomorphic i.e. terminal and initial objects in a category are unique upto isomorphism.

**Remark 2.3.2** If a category $\mathcal{C}$ has terminal (initial) object then the corresponding dual category $\mathcal{C}^*$ has initial (terminal) object.

**Definition 2.3.3 (Zero object)** An object $Z \in \mathcal{C}$ is called a zero object of the category $\mathcal{C}$ if it is initial and terminal both.

**Remark 2.3.3** [8] A category that has initial and terminal objects both need not has a zero object.

For example, in the category **Ens** every singleton set is terminal object and empty set is the only initial object i.e. $Mor_{\mathcal{C}}(\phi, X)$=singleton (there is only one function with no assignment) but it has no zero object.

**Example 2.3.1** In the category **Grp** of groups the trivial group $\{e\}$ is initial and terminal both hence zero object.

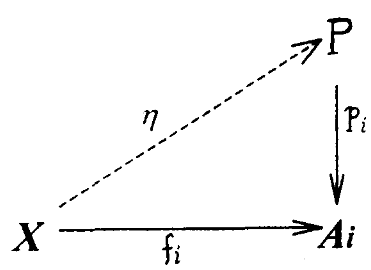
**Example 2.3.2** For an ordered set considered as a category, terminal object is the greatest element (if exists) and initial object is the least element.

**Example 2.3.3** In the category **Top**, any space of one point is terminal object and empty space is initial.

**Definition 2.3.4 (Zero morphism)** Let $\mathcal{C}$ be a category with zero object. A morphism $A \to B$ in $\mathcal{C}$ is called a zero morphism if and only if it factored through a zero object i.e. $A \to B = A \to O \to B$, where $O$ denotes the zero object.

## §2.4 Product and Co-product

**Definition 2.4.1 (Product)** Let $\{A_i\}_{i\in I}$ be a family of objects in a category $\mathcal{C}$. An object $P$ in $\mathcal{C}$ together with a family of morphisms $\{p_i : P \to A_i\}_{i\in I}$ is called the product of the family $\{A_i\}_{i\in I}$ if for any object $X \in \mathcal{C}$ and family of morphisms $\{f_i : X \to A_i\}_{i\in I}$, there exists a unique morphism $\eta : X \to P$ such that the diagram

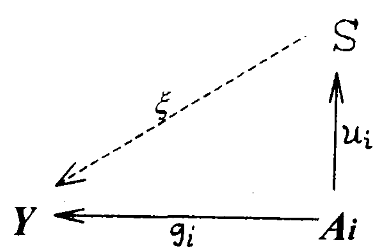



commutes.

**Notation:** The product of the family of objects $\{A_i\}_{i\in I}$ is denoted by $\prod_{i\in I} A_i$.

Dually,

**Definition 2.4.2 Co-product (Sum)** Let $\{A_i\}_{i\in I}$ be a family of objects in a category $\mathcal{C}$. An object $S$ in $\mathcal{C}$ together with a family of morphisms $\{u_i : A_i \to S\}_{i\in I}$ is called the co-product (sum) of the family $\{A_i\}_{i\in I}$ if for any object $Y \in \mathcal{C}$ and family of morphisms $\{g_i : A_i \to Y\}_{i\in I}$, there exists a unique morphism $\xi : S \to Y$ such that the diagram




commutes.

**Notation:** The co-product (sum) of the family of objects $\{A_i\}_{i\in I}$ is denoted by $\sum\limits_{i\in I} A_i$.

**Remark 2.4.1** A category has product if it has product for every family of objects.

**Remark 2.4.2** A category has finite product (co-product) if it has product (co-product) for every finite family of objects.

**Remark 2.4.3** The product of the empty family of objects is the terminal object of the category and dually the co-product of the empty family of objects is the initial object of the category.

**Proposition 2.4.1** If $P$ and $P'$ are products of the family of objects $\{A_i\}_{i\in I}$ in a category then there exists an isomorphism between $P$ and $P'$.

**Example 2.4.1** In the category **Ens**, cartesian product is the categorical product and disjoint union is the co-product of the objects of the category.

**Example 2.4.2** In the category **Grp**, the external direct product of the groups is the categorical product and the free product of groups is the co-product of the objects of the category.

**Example 2.4.3** In the category **Top**, cartesian product is the categorical product and disjoint union is the co-product of the objects of the category.

## §2.5 Equalizers and Kernels

**Definition 2.5.1 (Equalizer)** Let $\alpha_1, \alpha_2 : A \to B$ be two morphisms in a category $\mathcal{C}$. An object $K$ together with morphism $u : K \to A$ is called equalizer of the pair of morphisms $\alpha_1, \alpha_2$ if

$E_1 : \alpha_1 u = \alpha_2 u$

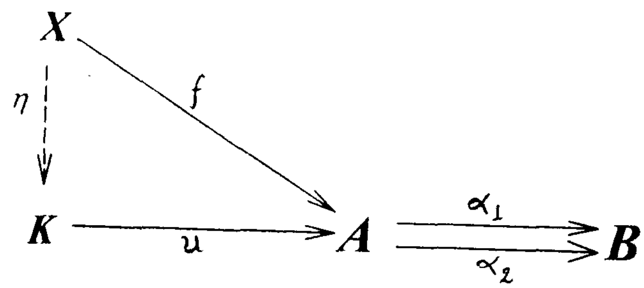
$E_2$ : For every morphism $f : X \to A$ such that

$$\alpha_1 f = \alpha_2 f$$

there exists a unique morphism $\eta : X \to K$ such that

$$u\eta = f$$

i.e. the following diagram




commutes.

Dually,

**Definition 2.5.2 (Co-equalizer)** Let $\alpha_1$, $\alpha_2 : A \to B$ be two morphisms in a category $\mathcal{C}$. An object $F$ together with morphism $v : B \to F$ is called co-equalizer of the pair of morphisms $\alpha_1$, $\alpha_2$ if

$E_1^*$ : $v\alpha_1 = v\alpha_2$

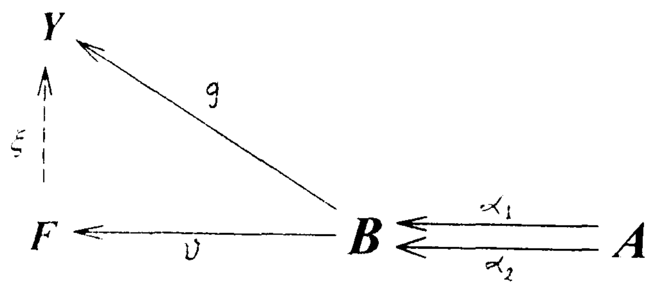
$E_2^*$ : For any morphism $g : B \to Y$ such that

$$g\alpha_1 = g\alpha_2$$

there exists a unique morphism $\xi : F \to Y$ such that

$$\xi v = g$$

i.e. the following diagram




commutes.

**Remark 2.5.1** Any two equalizers of a pair of morphisms $\alpha_1, \alpha_2 : A \to B$ in a category are isomorphic.

**Proposition 2.5.1** Equalizer morphism is a monomorphism and dually co-equalizer morphism is an epimorphism.

**Example 2.5.1** The category **Ens** of sets has equalizers and co-equalizers.

**Example 2.5.2** The category **Grp** of groups has equalizers and co-equalizers.

**Example 2.5.3** [8] Let $f, g : A \longrightarrow B$ be two morphisms in **Ens**, **Grp** or ${}_R$**Mod**. Let

$$T = \{x \in A : f(x) = g(x)\}$$

be considered appopriately as a subset, a subgroup or a submodule. Then the canonical inclusion $i : T \longrightarrow A$ is a equalizer of $f, g$ .

**Definition 2.5.3 (Kernel)** Let $C$ be a category with a zero object, then the equalizer of the morphism $\alpha : A \longrightarrow B$ and $O : A \longrightarrow B$ is called the kernel of the morphism $\alpha$.

**Notation:** The kernel of the morphism $\alpha$ is denoted by $Ker(\alpha)$.

Dually,

**Definition 2.5.4 (Co-kernel)** Let $C$ be a category with a zero object, then the co-equalizer of the morphism $\alpha : A \longrightarrow B$ and $O : A \longrightarrow B$ is called the co-kernel of the morphism $\alpha$.

**Notation:** The co-kernel of the morphism $\alpha$ is denoted by $Coker(\alpha)$.

**Remark 2.5.2** If $f : A \to B$ is a zero morphism then

$$Ker(f) = I_A \ \ and \ \ Coker(f) = I_B.$$

**Remark 2.5.3** If $f : A \to B$ be a monomorphism then $Ker(f) = 0$.

Dually,

**Remark 2.5.4** If $f : A \to B$ be an epimorphism then $Coker(f) = 0$.

**Proposition 2.5.2** The composition $\alpha\ \beta$ defined in $\mathcal{C}$ then the following statements hold:

(*i*) If $\beta$ is mono then $Ker(\alpha\beta) = Ker\alpha$.

(*ii*) If $\alpha$ is epi then $Coker(\alpha\beta) = Coker\beta$.

**Example 2.5.4** The category **Grp** of groups and the category $_R$**Mod** of modules has kernel and co-kernel.

**Example 2.5.5** In **Grp** and $_R$**Mod** the usual notion of kernel yields the categorical equivalent. Indeed, let $f : A \longrightarrow B$ be a group homomorphism (or module hmomorphism). Then the (algebraic) kernel of $f$ is the subgroup (or submodule)

$$Kerf = \{x \in A : f(x) = O_B\}.$$

The canonical embedding $i : Kerf \longrightarrow A$ is then a kernel of $f$ in the categorical sense (see Example 2.5.3). Thus **Grp** and $_R$**Mod** have kernels.

# CHAPTER 3

# STRUCTURES ON CATEGORIES

## §3.1 Introduction

This chapter has been devoted to the study of structures on categories due to Blyth[8], MacLane[35], Mitchell[42] and Freyd[22] etc.

Section 3.2 deals with the notion of semi-additive and additive category due to Blyth[8] and MacLane[35]. In Section 3.3 normal and co-normal categories are discussed and it is obtained that every normal category is balanced. Section 3.4 deals with the notion of exact sequence and exact categories. Finally, in the last Section 3.5 the notion of abelian categories is introduced. Further, it is shown that every abelian category is balanced.

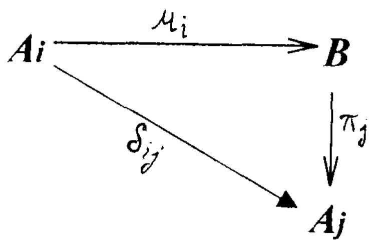
To develope this chapter we need the following notion:

**Definition 3.1.1** Let $\mathcal{C}$ be a category with zero object and let $\{A_i\}_{i\in I}$ be a family of objects of $\mathcal{C}$. An object $B$ in $\mathcal{C}$ together with morphism $A_i \xrightarrow{\mu_i} B \xrightarrow{\pi_i} A_i$ for all $i \in I$ is called biproduct if

(*i*) $(B, \pi_i)_{i\in I}$ is the product of $\{A_i\}_{i\in I}$.

(*ii*) $(B, \mu_i)_{i\in I}$ is the co-product of the family $\{A_i\}_{i\in I}$ in $\mathcal{C}$.

(*iii*) The diagram




is commutative.

where

$$\delta_{ij} = \begin{cases} id & if\ i = j \\ 0 & if\ i \neq j. \end{cases}$$

**Example 3.1.1** If $\{A_i\}_{i \in I}$ is a family of R-modules and $I = \{1, 2, \ldots, n\}$ then we have that

$$\prod_{i=1}^{n} A_i = \bigoplus_{i=1}^{n} A_i,$$

together with the canonical injections and surjections, is a biproduct of $A_1, A_2, \ldots, A_i$.

## §3.2 Semi-additive and Additive Categories

**Definition 3.2.1 (Semi-additive category)** A category $\mathcal{C}$ together with a zero object is said to be semi-additive if for every pair of objects $A, B$ of $\mathcal{C}$ there is a law of composition $\star$ on $Mor_{\mathcal{C}}(A, B)$ such that

(*i*) $(Mor_{\mathcal{C}}(A, B), \star)$ is a commutative semi group with identity 0.

(*ii*) $o$ is bilinear, in the sense that the following identities hold:

$$ho(f \star g) = (hof) \star (hog)$$

$$(f \star g)ok = (fok) \star (gok).$$

**Example 3.2.1** ${}_R$**Mod** is a semi-additive category.

**Example 3.2.2** Let $G$ be an additive abelian group regarded as a category **Grp**. Then $G$ is semi-additive (the addition on the morphism sets being the group operation).

**Example 3.2.3** The additive monoid $N$ considered as a category is semi-additive.

**Example 3.2.4** Let $\mathbf{Lat_0}$ be the subcategory of **Lat** consisting of those lattices that have a smallest element 0, the morphisms being the 0-preserving lattice morphisms. Then $\mathbf{Lat_0}$ is semi-additive.

In fact $\{0\}$ is a zero object; and if $f, g \in Mor(L, M)$ then with $f \star g$ defined by

$$(f \star g)(x) = f(x) \star g(x) \quad for\ all\ x \in L.$$

It is clear that properties $(i)$ and $(ii)$ of Definition 3.2.1 are readily seen to hold.

**Theorem 3.2.1** [8] In a category $\mathcal{C}$, the following statements are equivalent:

$(i)$ $\mathcal{C}$ has finite biproducts

$(ii)$ $\mathcal{C}$ is semi-additive and has finite products

$(iii)$ $\mathcal{C}$ is semi-additive and has finite co-products.

**Definition 3.2.2 (Additive category)** A category $\mathcal{C}$ is said to be additive if it is semi-additive and $(Mor(A, B), +)$ forms an additive abelian group for all objects $A, B$ of $\mathcal{C}$.

**Example 3.2.5** The categories **Ab** and ${}_R$**Mod** are additive.

**Example 3.2.6** A ring (always with 1) is to be regarded as an additive category with only one object [47].

Now we define the notion of additive category due to MacLane[36] as follows:

**Definiton 3.2.3** An additive category is a category $\mathcal{C}$ in which each set $Mor(A, B)$ of morphisms has the structure of an abelian group, subject to the following three axioms:

$A_1$ : There is a zero object.

$A_2$ : **(Biproduct)** To each pair of objects $A_1$ and $A_2$ there exists an object $B$ with four morphisms $A_1 \xrightarrow{p_1} B \xrightarrow{p_2} A_2$ and $A_2 \xrightarrow{u_2} B \xrightarrow{u_1} A_1$ which satisfy the identities $p_1u_1 = I_{A_1}$, $p_2u_2 = I_{A_2}$ and $p_1u_1 + p_2u_2 = I_B$.

**Remark 3.2.1** In additive category $p_1u_2 = 0$ and $p_2u_1 = 0$.

**Proposition 3.2.1** A morphism $\alpha$ in an additive category is monomorphism if and only if $\alpha f = 0$ always implies $f = 0$. Similarly, $\beta$ is epimorphism if and only if $g\beta = 0$ always implies $g = 0$.

**Proof** Suppose $\alpha : A \longrightarrow B$ be a monomorphism then for any equality

$$K \xrightarrow{f_1} A \xrightarrow{\alpha} B = K \xrightarrow{f_2} A \xrightarrow{\alpha} B$$

we have

$$f_1 = f_2.$$

Let $\alpha f = 0$ for some $f : K \to A$ then

$$K \xrightarrow{f} A \xrightarrow{\alpha} B = K \xrightarrow{0} B = K \xrightarrow{0} A \xrightarrow{\alpha} B$$

$\Longrightarrow$ $\quad f = 0 \quad$ since $\alpha$ is mono.

Again, let

$$K \xrightarrow{f_1} A \xrightarrow{\alpha} B = K \xrightarrow{f_2} A \xrightarrow{\alpha} B$$

i.e.

$$\alpha f_1 = \alpha f_2$$

$\Longrightarrow$ $\quad \alpha f_1 - \alpha f_2 = 0$

$\Longrightarrow$ $\quad \alpha(f_1 - f_2) = 0$ (by distributive law)

$\Longrightarrow$ $\quad f_1 - f_2 = 0$

$\Longrightarrow$ $\quad f_1 = f_2$

$\Longrightarrow$ $\quad \alpha$ is mono.

Hence $\alpha$ is mono if and only if $\alpha f = 0$ always implies $f = 0$.
Similarly, we can prove that $\beta$ is epimorphism if and only if $g\beta = 0$ always implies $g = 0$.

**Proposition 3.2.2** The object $B$ in the axiom $A_3$ of Definition 3.2.2 together with morphism $B \xrightarrow{p_1} A_1$ , $B \xrightarrow{p_2} A_2$ is the product of the objects $A_1$ and $A_2$.

**Proof** Let $f_1 : C \to A_1$ and $f_2 : C \to A_2$ be any pair of morphisms with common domain $C$.

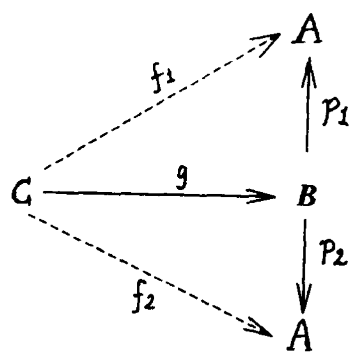
Construct a morphism $g : C \rightarrow B$ such that

$$g = u_1 f_1 + u_2 f_2$$

then we obtain,

$$\begin{aligned} p_1 g &= p_1(u_1 f_1 + u_2 f_2) \\ &= p_1(u_1 f_1) + p_1(u_2 f_2) \quad [By\ distributive\ law] \\ &= (p_1 u_1) f_1) + (p_1 u_2) f_2 \quad [By\ associative\ law] \\ &= I_{A_1} f_1 + 0.f_2 \quad [By\ Remark\ 3.2.1] \\ &= f_1 + 0 \\ &= f_1. \end{aligned}$$

Similarly, $p_2 g = f_2$ i.e. the following diagram




commutes.

The morphism $g : C \rightarrow B$ is unique by construction. Hence object $B$ together with morphisms $p_1 : B \rightarrow A_1$ and $p_2 : B \rightarrow A_2$ is a product of $A_1$ and $A_2$.

Similarly, we can prove the following proposition.

**Proposition 3.2.3** The object $B$ in the axiom $A_3$ in the Definition 3.2.2 together with morphisms $u_1 : A_1 \rightarrow B$ and $u_2 : A_2 \rightarrow B$ is a co-product (sum) of the objects $A_1$ and $A_2$.

## §3.3 Normal and Co-normal Categories

In an arbitrary category with zero object, morphism need not have kernel and co-kernel.

For example if $R$ is a commutative ring with a 1 then *R-algebra* is a *R-module* that is also a ring with a 1. For example, the set $Mat_{n\times n}(Z)$ of $n \times n$ matrices over $Z$ is a *Z-algebra*. Now in category ${}_RAlg$ of *R-algebras* the only condidiate for the (algebraic) kernel of morphism is a submodule that is also an ideal. But this is in general fails to have an identity element and therefore fails to have an object in the category. Hence not every morphism in ${}_RAlg$ has a kernel in the categorical sense.

Likewise, in the certain types of topological spaces the quotient spaces (the candidiates for co-kernels) do not in general inherit the properties of the parent spaces, so fail to be objects of the category under consideration. Consequently, not every morphism has a co-kernel.

Although kernels and co-kernels need not exist in general. Now we define a special type of category in which every morphism has kernel and co-kernel.

**Definition 3.3.1 (Normal category)** A category $\mathcal{C}$ with zero object is called normal category if the following conditions hold:

$(i)$ $\mathcal{C}$ has zero object

$(ii)$ every morphism in $\mathcal{C}$ has a kernel and co-kernel

$(iii)$ every monic morphism in $\mathcal{C}$ is a kernel.

**Example 3.3.1** The category ${}_R$**Mod** is normal. In fact axioms $(i)$ and $(ii)$ are clearly satisfied. As for axioms $(iii)$, $f : A \to B$ is monic then $f$ is a kernel of $\eta : B \longrightarrow B/Im(f)$. To see this , observe by Example 2.5.4 that the embedding $i : Imf \to B$ is a kernel of $\eta$. Since $f$ is monic, it is injective, so $\bar{f} : A \to Imf$ is given by $\bar{f}(a) = f(a)$ is an isomorphism. Then, using the fact that kernels are

unique to with in composition by an isomorphism, we deduce from $f = io\bar{f}$ that $f$ is a kernel of $\eta$.
Which completes our result.

In a normal category every monic morphism $f : A \longrightarrow B$ is a kernel, say a kernel of $h : B \longrightarrow C$. Then from $Kerh \sim Kercokerkerh$, we obtain $f \sim Kercokerf$. Thus we see that $f$ is monic if and only if $f \sim Kercokerf$.
Similarly, $f$ is epic if and only if $f \sim Cokerkerf$. We shall make use frequently of these observations in establishing the following results concerning normal categories.

**Theorem 3.3.1** [8] A normal category is balanced.

**Corollary 3.3.1** [8] A normal category has finite products and equalizers.

Dually, we define co-normal category.

**Definition 3.3.2 (Co-normal category)** A category $\mathcal{C}$ is said to be co-normal if the following conditions hold:

$(i)$ $\mathcal{C}$ has zero object

$(ii)$ every morphism in $\mathcal{C}$ has a kernel and co-kernel

$(iii)$ every epic morphism in $\mathcal{C}$ is a co-kernel.

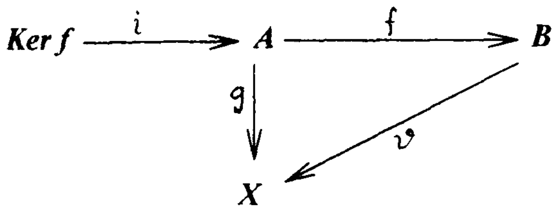
**Example 3.3.2** The category $_R$**Mod** is co-normal.

To see this, it is sufficient to prove that every epic morphism in $_R$**Mod** is a co-kernel of any of its kernels. Now a kernel of $f : A \longrightarrow B$ is the inclusion $i : kerf \longrightarrow A$. If now $g : A \longrightarrow X$ is such that $goi = 0$ then clearly $Kerf \subseteq Kerg$ such that $f(a) = f(b)$ implies $g(a) = g(b)$. Since $f$ is epic, hence surjective. We can therefore, define a morphism

$$v : B \longrightarrow X$$

such that

$$v(f(a)) = g(a) \ for \ every \ a \in A$$




clearly, $vof = g$ and so $f$ is a co-kernel of $i$.

**Example 3.3.3** The category **Grp** is co-normal but not normal. In fact the same argument used in Example 2.4.2 shows that **Grp** is co-normal. However, it is not normal. For this, we know that a co-kernel of $f : A \to B$ is $\eta : B \longrightarrow B/N$ where $N$ is the smallest normal subgroup containing $Imf$, and a kernel of $\eta$ is the imbedding $i : N \longrightarrow B$. If $f$ is monic then $f$ need not be a kernel of its co-kernel $\eta$, for the imbedding $j : Imf \longrightarrow N$ need not be an isomorphism.

**Definition 3.3.3 (Binormal category)** A category $\mathcal{C}$ is said to be binormal if it is both normal and co-normal.

## §3.4 Exact Sequence and Exact Categories

In this section the exact categories are defined in the terminology of Buchsbaum[9] and Mitchell[12] and their equivalence is proved. Most of the results in this section are taken from Buchsbaum[9]. All the results as we discussed here are also true in ordinary theory of groups and modules etc.

We shall often write $Kercokerf$ as $Imf$ and call this an image of $f$. Like wise, we often write $Cokerkerf$ as $Coimf$ and call this a coimage of $f$. The notion of image and coimage give rise to the following important concept.

**Definition 3.4.1 (Exact sequence)** If $\mathcal{C}$ is a binormal category then a sequence

$$\cdots\cdots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \longrightarrow \cdots\cdots$$

of objects and morphism in $\mathcal{C}$ is said to be exact at $A_i$ if every image of $f_{i-1}$ is a kernel of $f_i$; and coexact at $A_i$ if every coimage of $f_i$ is a co-kernel of $f_{i-1}$. The sequence is said to be exact (coexact) if it is exact (coexact) at every $A_i$.

Now we define exact category due to Mitchell[42] as follows:

**Definition 3.4.2 (Exact category)** A category $\mathcal{C}$ is said to be exact if satisfying the following axioms:

$E_1$ : $\mathcal{C}$ has zero object.

$E_2$ : Every morphism in category $\mathcal{C}$ has kernel and co-kernel.

$E_3$ : Every monomorphism (epimorphism) in category $\mathcal{C}$ be a kernel (co-kernel) of some morphism.

$E_4$ : Every morphism $\alpha : A \to B$ in category $\mathcal{C}$ can be decomposed as

$$A \xrightarrow{\alpha} B = A \xrightarrow{q} I \xrightarrow{\gamma} B$$

where $q$ is an epimorphism and $\gamma$ is a monomorphism.

**Example 3.4.1** The category **Grp** of groups is an exact category.

**Example 3.4.2** The category ${}_R$**Mod** of modules is an exact category.

**Proposition 3.4.1** [42] In an exact category $\mathcal{C}$, the following statements hold:

(*i*) $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact in $\mathcal{C}$ if and only if $A^* \xleftarrow{\alpha^*} B^* \xleftarrow{\beta^*} C^*$ is exact in $\mathcal{C}^*$.

(*ii*) $O \longrightarrow A \xrightarrow{\alpha} B$ is exact in $\mathcal{C}$ if and only if $\alpha$ is a monomorphism.

(*iii*) $A \xrightarrow{\alpha} B \longrightarrow O$ is exact in $\mathcal{C}$ if and only if $\alpha$ is a epimorphism.

(*iv*) $O \longrightarrow A \xrightarrow{\alpha} B \longrightarrow O$ is exact in $\mathcal{C}$ if and only if $\alpha$ is a isomorphism.

**Proof (i)** Consider

$$A \xrightarrow{q} I \xrightarrow{v} B \xrightarrow{r} J \xrightarrow{w} C$$

where $v$ is the image of $\alpha$ and $w$ is the image of $\beta$. Then $r$ is the coimage of $\beta$. If $A \longrightarrow B \longrightarrow C$ is exact, then $v$ is the kernel of $\beta$ and hence also the kernel of $r$. Therefore, $r$ is the co-kernel of $v$ and hence also the co-kernel of $\alpha$. In the dual category, $r$ then becomes the kernel of $\alpha$ as well as the image of $\beta$ and so $A^* \longleftarrow B^* \longleftarrow C^*$ is exact.

$(ii)$ If $\alpha$ is a monomorphism then its kernel is $O$, and so clearly $O \longrightarrow A \longrightarrow B$ is exact. Consequently, if $O \longrightarrow A \longrightarrow B$ is exact then $\alpha$ has kernel $O$. Let $A \xrightarrow{q} I \xrightarrow{v} B$ be a factorization of $\alpha$ as an epimorphism followed by a monomorphism. Then $q$ is the co-kernel of the kernel of $\alpha$. Since the latter is $O$, $q$ must be an isomorphism. But $\alpha = vq$ must be monomorphism.

$(iii)$ Follows from $(i)$ and $(ii)$.

$(iv)$ Since a normal category is balanced, $(iv)$ follows from $(ii)$ and $(iii)$.

From the above Proposition 3.4.1 we obtained the following remarks.

**Remark 3.4.1** A morphism $\alpha$ in an exact category is monomorphism if and only if $Ker(\alpha) = O$.
Dually,

**Remark 3.4.2** A morphism $\beta$ in an exact category is epimorphism if and only if $Coker(\alpha) = O$.

**Lemma 3.4.1** [22] For sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in an exact category, the following conditions are equivalent:

$(i)$ $Im(\alpha) = Ker(\beta)$

$(ii)$ $Coim(\beta) = Coker(\alpha)$

$(ii)$ $Coim(\beta) = Coker(\alpha)$

(*iii*) $A \xrightarrow{\alpha} B \xrightarrow{\beta} C = 0$ and $K \xrightarrow{u} B \xrightarrow{v} F = 0$

where $u : K \longrightarrow B$ is a kernel of $\beta$ and $v : B \longrightarrow F$ is a co-kernel of $\alpha$.

**Lemma 3.4.2** [22] A sequence $O \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow O$ in an exact category is exact if and only if one of the following condition holds:

(*i*) $\alpha$ is monomorphism and $\beta$ is a co-kernel of $\alpha$

(*ii*) $\beta$ is epimorphism and $\alpha$ is a kernel of $\beta$.

**Lemma 3.4.3** [9] let $\alpha : A \longrightarrow B$ be a monomorphism in an exact category $\mathcal{C}$. Then there is a morphism $\beta$ and an object $C$ such that
$O \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow O$ is exact.
Dually,

**Lemma 3.4.4** [9] let $\gamma : C \longrightarrow D$ be an epimorphism in an exact category $\mathcal{C}$. Then there is a morphism $\delta$ and an object $E$ such that

$$O \longrightarrow E \xrightarrow{\delta} C \xrightarrow{\gamma} D \longrightarrow O \textit{ is exact}.$$

**Remark 3.4.3** From the Lemma 3.5.2 and Lemma 3.5.3 we obtained that every monomorphism(epimorphism) in an exact category be a kernel(co-kernel) of some morphism in the category.

## §3.5 Abelian Categories

Certain of the categories we introduced in previous sections possess significant additional structure. Thus in category **Ab**, ${}_R$**Mod** the morphism sets all have abelian group structure and we have the notion of exact sequences. We proceed in this section to extract certain essential features of such categories and define the important notion of an abelian category. Also this section consists of a study of the formal properties of abelian categories. It is a very important fact about such categories that the axioms which characterize them are self dual, so that any theorem proved about abelian categories yields two dual theorems when applied to a particular abelian category such as ${}_R\mathbf{Mod}$.

We define abelian category due to Freyd [22]as follows:

**Definition 3.5.1 (Abelian category)** A category $\mathcal{C}$ is said to be abelian if satisfying the following axioms:

$A_1$ : $\mathcal{C}$ has a zero object.

$A_2$ : For every pair of object there is a product and

$A_2{}^*$ : a co-product (sum).

$A_3$ : Every morphism has a kernel and

$A_3{}^*$ : a co-kernel.

$A_4$ : Every monomorphism is a kernel of a morphism.

$A_4{}^*$ : Every epimorphism is a co-kernel of a morphism.

**Observations:**

(*i*) The axioms $A_4$, and $A_4{}^*$ imply that abelian category is both normal and co-normal.

(*ii*) The axioms $A_1$, $A_3$ and $A_3{}^*$ imply that every morphism $\alpha : A \to B$ in an abelian category can be factored as

$$A \xrightarrow{\alpha} B = A \xrightarrow{f} I \xrightarrow{g} B.$$

Where $f$ is an epimorphism and $g$ is monomorphism.

**Remark 3.5.1** From above observations (*i*) and (*ii*), we obtain that every abelian category is an exact category.

**Example 3.5.1** The category **Ab** of abelian groups is an abelian category.

**Example 3.5.2** The category ${}_R\mathbf{Mod}$ over a ring $R$ is an abelian category.

**Example 3.5.3** The category $\mathbf{Vect}_F$ of vector spaces over a field $F$ is an abelian category.

**Proposition 3.5.1** Every abelian category is balanced.

**Proof** Let $\alpha : A \longrightarrow B$ be a morphism in an abelian category $\mathcal{C}$, which is monomorphism and epimorphism both, then we have

$$Coker(\alpha) = B \longrightarrow O, \quad since\ \alpha\ is\ epimorphism$$

$$Ker(B \longrightarrow O) = B \xrightarrow{I_B} B \ \ (By\ Remark\ 2.5.2).$$

Since every monomorphism is the kernel of its own co-kernel.

$\Rightarrow$ $\qquad Ker(B \longrightarrow O) = A \xrightarrow{\alpha} B$

$\Rightarrow$ $\qquad$ there exist a morphism $\eta : B \longrightarrow A$ such that

$$B \xrightarrow{\eta} A \xrightarrow{\alpha} B = B \xrightarrow{I_B} B. \qquad (3.5.1)$$

Dually, we note that $O \longrightarrow A$ is the kernel of $A \xrightarrow{\alpha} B$ and both $A \xrightarrow{\alpha} B$ and $A \xrightarrow{I_A} A$ are co-kernels of $O \longrightarrow A$.

Hence there is a morphism $\xi : B \longrightarrow A$ such that

$$A \xrightarrow{\alpha} B \xrightarrow{\xi} A = A \xrightarrow{I_A} A. \qquad (3.5.2)$$

Therefore, from equations (3.5.1) and (3.5.2) we have, $\alpha$ is an isomorphism.
Hence every bimorphism is an isomorphism in $\mathcal{C}$ i.e. $\mathcal{C}$ is balanced.

Now we define the notion of ablian category due to Blyth as follows:

**Definition 3.5.2 (Abelian category)** A binormal category $\mathcal{C}$ is said to be abelian if it is additive.

**Proposition 3.5.2** [22] In an abelian category kernel and co-kernel are inverse morphisms.

**Proposition 3.5.3** [8] A category $\mathcal{C}$ is abelian if and only if it is binormal and has finite biproduct.

# CHAPTER 4

# SOME SPECIAL FUNCTORS

## §4.1 Introduction

This chapter has been devoted to the study of certain types of functors. Most of the results of this chapter are based on the work of Blyth[8], Mitchell[42], MacLane[35], Freyd[22] etc.

Section 4.2 deals with the preservation properties of functors which states that if $T : A \longrightarrow B$ and $S : B \longrightarrow C$ are covariant functors both having a certain preservation property, then $ST$ also has that property. In Section 4.3 the notion of additive functors are discussed. In the last Section 4.4 some results have been presented on exact functors and obtained that an exact covariant functor 'preserves' short exact sequence and that an exact contravariant functor 'reverses' short exact sequences.

## §4.2 Preservation Properties of Functors

**Definition 4.2.1 (Monofunctor)** A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is called monofunctor if $F(\alpha)$ is monomorphism in $\mathcal{D}$ whenever $\alpha$ is monomorphism in $\mathcal{C}$.

**Definition 4.2.2 (Epifunctor)** A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is called epifunctor if $F(\alpha)$ is epimorphism in $\mathcal{D}$ whenever $\alpha$ is epimorphism in $\mathcal{C}$.

**Definition 4.2.3 (Zero preserving functor)** If $\mathcal{C}$ and $\mathcal{D}$ be two categories with zero objects then a covariant functor $F : \mathcal{C} \to \mathcal{D}$ is called a zero preserving functor if $F(0)$ is a zero object in $\mathcal{D}$ for $0$ a zero object in $\mathcal{C}$.

In this case $F$ necessarily takes zero morphism into zero morphism. Conversely, if $F$ takes zero morphism into zero morphism, then using the fact that a zero object is characterized by its identity morphism being zero we see that $F$ must be zero preserving.

**Definition 4.2.4 (Kernel preserving functor)** A covariant functor $F$ is called kernel preserving if $F(u)$ is the kernel of $F(\alpha)$ when $u : K \to A$ is the kernel of $\alpha : A \to B$.
Taking $K = A = B = 0$, we see that a kernel preserving functor is necessarily zero preserving.

The properties of functors defined in this section are called preservation properties of functors i.e if $T : A \to B$ and $S : B \to C$ are covariant functors both having a certain preservation property, then $ST$ also has that property [42].

## §4.3 Additive Functors

**Definition 4.3.1 (Additive functor)** Let $\mathcal{C}$ and $\mathcal{D}$ be additive category then a functor $F : \mathcal{C} \to \mathcal{D}$ is called additive if for all morphisms $\alpha, \beta : A \to B$ in $\mathcal{C}$ we have

$$F(\alpha + \beta) = F(\alpha) + F(\beta).$$

**Example 4.3.1** Let ${}_R\mathbf{Mod}$ be a category of modules over a ring $R$ then

$$Mor(A, -) : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$$

is an additive functor for all $A \in {}_R\mathbf{Mod}$.

For this, let us consider

$$Mor(A, -) : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$$

defined by

$$Mor(A, -)X = Mor(A, X) \quad for\ all \quad X \in {}_R\mathbf{Mod}.$$

Now for every morphism

$$f : X \to Y \ in \ {}_R\mathbf{Mod}$$

i.e.

$$Mor(A, -)f = Mor(A, f) : Mor(A, X) \to Mor(A, Y)$$

defined by

$$Mor(A,f)(\alpha) = f\circ\alpha \in Mor(A,Y) \quad for\ any \quad \alpha \in Mor(A,X).$$

Now if we take $f, g \in {}_R\mathbf{Mod}$ then

$$f + g : X \to Y$$

i.e.

$$Mor(A,-)(f+g) = Mor(A,f+g) : Mor(A,X) \to Mor(A,Y)$$

such that

$$\begin{aligned} Mor(A,f+g)(\alpha) &= (f+g)\alpha \\ &= f\alpha + g\alpha \quad for\ all \quad \alpha \in Mor(A,X) \\ &= Mor(A,f)(\alpha) + Mor(A,g)(\alpha) \\ &= (Mor(A,f) + Mor(A,g))(\alpha) \end{aligned}$$

i.e.

$$Mor(A,f+g) = Mor(A,f) + Mor(A,g)$$

i.e.

$$Mor(A,-)(f+g) = Mor(A,-)f + Mor(A,-)g$$

$\Longrightarrow$ $Mor(A,-)$ is a covariant additive functor.

**Example 4.3.2** If $\mathcal{C}$ is an abelian category and $M$ is an object of $\mathcal{C}$ we have the associated 'set valued' functors $Mor_{\mathcal{C}}(M,-)$ and $Mor_{\mathcal{C}}(-,M)$. We can also consider the associated 'group valued' functors defined by

$$h_M : \mathcal{C} \to \mathbf{Ab}$$

such that

$$h_M X = Mor_{\mathcal{C}}(M,X);$$

$$h_M f = Mor_{\mathcal{C}}(I_M, f) : v \longmapsto f\circ v$$

and

$$h^M : \mathcal{C} \to \mathbf{Ab}$$

such that

$$h^M X = Mor_{\mathcal{C}}(M, X);$$

$$h^M f = Mor_{\mathcal{C}}(f, I_M) : v \longmapsto vof.$$

If $U$ : **Ab** $\rightarrow$ **Ens** is the forgetful functor then clearly we have

$$Uoh_M = Mor_{\mathcal{C}}(M, -)$$

and

$$Uoh^M = Mor_{\mathcal{C}}(-, M).$$

The group valued functors $h_M$ and $h^M$ are additive.
For example, $h_M(\alpha + \beta)$ sends

$$v \longmapsto (\alpha + \beta)ov = \alpha ov + \beta ov$$

and so is the same as $h_M\alpha + h_M\beta$ i.e. $h_M(\alpha + \beta) = h_M\alpha + h_M\beta$.

**Theorem 4.3.1** [8] Additive functors preserve zero objects.

## §4.4 Exact Functors

**Definition 4.4.1 (Left exact functor)** Let $\mathcal{C}$ and $\mathcal{D}$ be binormal categories. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called left exact functor if for every left exact sequence, $O \rightarrow A_1 \rightarrow A_2 \rightarrow A_3$ in $\mathcal{C}$ its image under $F$ i.e. $O \rightarrow F(A_1) \rightarrow F(A_2) \rightarrow F(A_3)$ is also left exact in $\mathcal{D}$.

**Definition 4.4.2 (Right exact functor)** Let $\mathcal{C}$ and $\mathcal{D}$ be binormal categories. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called right exact functor if for every right exact sequence, $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow O$ in $\mathcal{C}$ its image under $F$ i.e. $F(A_1) \rightarrow F(A_2) \rightarrow F(A_3) \rightarrow O$ is also right exact in $\mathcal{D}$.

**Definition 4.4.3 (Exact functor)** A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called exact functor if it is left as well as right exact functor.

**Theorem 4.4.1** Let $\mathcal{C}$ be an abelian category. For every object $A$ of $\mathcal{C}$ the covariant functor $h_A : \mathcal{C} \rightarrow \mathbf{Ab}$ is left exact and the contravariant functor $h^A : \mathcal{C} \rightarrow \mathbf{Ab}$ is also left exact.

**Proof** Suppose that we have an exact $\mathcal{C}$-*sequence*

$$O \longrightarrow B \longrightarrow C \longrightarrow D$$

and consider the associated **Ab**-*sequence*

$$O \longrightarrow h_A B \longrightarrow h_A C \longrightarrow h_A D$$

in which we recall, $h_A\alpha$ is simply composition on the left by $\alpha$.
If $f \in Kerh_A\alpha$ then $\alpha of = 0$ and so, $\alpha$ being monic, $f = 0$.
Thus $h_A(\alpha)$ is injective, hence monic, in **Ab**.
If now $g \in Kerh_A\beta$ then $\beta og = 0$ and so, $\alpha$ being a kernel of $\beta$, there exist $k$ such that $\alpha ok = g$, i.e. $h_A(k) = g$.
Thus we have

$$Kerh_A\beta \subseteq Imh_A\alpha.$$

But $\beta o\alpha = 0$ so, by Theorem 4.3.1, $h_A\beta oh_A\alpha = 0$
and consequently,

$$Imh_A\alpha \subseteq Kerh_A\beta.$$

Hence

$$Kerh_A\alpha = Kerh_A\beta.$$

Which shows that the associated **Ab**-*sequence* is exact.
Therefore, $h_A$ is left exact.

A similar proof shows that the contravariant functor $h^A$ is also left exact.

In general, the functors $h_A$ and $h^A$ fail to be exact, as following example shows

**Example 4.4.1** In $_Z$**Mod** consider the short exact sequence

$$O \longrightarrow Z \xrightarrow{i} Q \xrightarrow{\eta} Q/Z \longrightarrow O$$

let A=Z/2Z and consider the functor

$$h_A : {}_Z\mathbf{Mod} \longrightarrow \mathbf{Ab}.$$

This is left exact (by Theorem 4.4.1), but not right exact since the induced morphism

$$Mor_Z(Z/2Z, Q) \longrightarrow Mor_Z(Z/2Z, Q/Z)$$

can not be epic (surjective). In fact, the group on the left collapses to $\{0\}$ whereas that on the right does not.
To see this,

let

$$v : Z/2Z \longrightarrow Q \textit{ be a group morphism}$$

and let

$$x = v(1 + 2Z).$$

We have

$$\begin{aligned} 2x &= 2v(1+2Z) \\ &= v(2+2Z) \\ &= v(0+2Z) \\ &= 0. \end{aligned}$$

Whence $x = 0$ and consequently, $v = 0$.

On the other hand,

$$0 + 2Z \longrightarrow 0 + Z,$$
$$1 + 2Z \longrightarrow \frac{1}{2} + Z,$$

describes a nonzero element of $Mor_Z(Z/2Z, Q/Z)$

$$\Longrightarrow \qquad Mor_Z(Z/2Z, Q/Z) \neq \{0\}.$$

So no epimorphism is possible.

Which completes our claim.

**Example 4.4.2** In ${}_Z\mathbf{Mod}$ consider the short exact sequence

$$O \longrightarrow Z \stackrel{i}{\longrightarrow} Q \stackrel{\eta}{\longrightarrow} Q/Z \longrightarrow O$$

and the left exact contravariant functor

$$h^A : {}_Z\mathbf{Mod} \longrightarrow \mathbf{Ab}.$$

This functor is not right exact since the induced morphism

$$Mor_Z(Q, Z) \longrightarrow Mor_Z(Z, Z)$$

can not be epic (surjective). In fact, the group on the left collapses to $\{0\}$ whereas that on the right does not.
To see this,
Let

$$v : Q \longrightarrow Z \;\; \textit{be a group morphism}$$

and suppose that

$$v(1) \neq 0.$$

Then for every non-zero integer $r$ we have $v(1) = rv(1/r)$, whence r divides $v(1)$. But, by the fundamental theorem of arithmetic, $v(1)$ has only finitely many divisors. Hence we must have $v(1) = 0$.
For all non-zero integers $p$, $q$ we then have

$$\begin{aligned} 0 &= pv(1) \\ &= pv(q/q) \\ &= pqv(1/q) \\ &= qv(p/q). \end{aligned}$$

Whence $$v(p/q) = 0$$

$\Longrightarrow$ $$v = 0 \quad (\text{ as } p\ ,\ q \text{ are non zero}).$$

On the other hand, $I_Z$ is clearly a non-zero element of $Mor_Z(Z, Z)$.

Therefore, no epimorphism is possible.

Which completes our claim.

In view of these observations, it is natural to investigate those objects $A$ of an abelian category $\mathcal{C}$ for which the functors $h_A$ and $h^A$ are exact. We can in fact do this at a more general level [8].

## CHAPTER 5

# APPLICATIONS IN COMPUTER SCIENCE

## §5.1 Introduction

This chapter has been devoted to the study of the applications of category theory in the computer science which is based on the work of Barr M. [4] and Walters ([49]-[51]) etc.

Section 5.2 deals with the study of the relation between category theory and computer science. Section 5.3 deals with the study of categories with product-circuits and categories with sums-flow charts which states that a circuit diagram is just a representation of the decomponent of a function using products of categories and a flow chart is a representation of the decomponent of a function using sums of categories. Further, we observe that products and sums are key notions in analyzing computation.

In this chapter throughout the discussion for the convenience we call function in place of morphism.

## §5.2 Relation between Category Theory and Computer Science

It is always exciting and fruitful when two disparate scientific fields are found to have much in common. Each field is enriched by the different perspective and insights of the other. This has happened recently with category theory and theoretical computer science.

The relation between category theory and computer science constitute an extremely active area of the research at the moment. Among the many places where the research is being done are: Aarhus, Pennsylvania, Pisa, Stanford and Sydney. Topics of current interest include the connection between category theory and functional programming, abstract data structures, object-oriented programming and hardware design.

This dissertation is an introduction to category theory in which several of the connection with computer science are discussed in sufficient detail and a feeling for the rich possibilities arising from the happy connection between these two subjects.

In brief we see that how is category theory related to the computer science as follows:

⋆ An important aspect of computer science is the construction of function out of a given set of simple functions, using various operations on functions like composition, and repeated composition. Category theory is exactly the appropriate algebra for such constructions.

⋆ Computing is concerned with machines-that is, dynamical systems, which have sets of states which vary over time. They are built up out of functions or elementary machines by an essentially algebraic process. Again underlying this is the theory of functions and composition.

⋆ Since category theory is an algebra of functions we can consider categories which are purely formal, and which don't really consist of functions. This is the syntactical side of computer science. Programs and languages are formal things which are intended to describe or specify actual functions. Category theory is well adapted to deal with the relation syntax and semantics.

⋆ A category is a mixture of graphical information and algebraic operations. Computer science is similarly a mixture of graphs and algebra.

Some computer science topics we will deal with in this chapter are *boolean algebra, circuit theory and flow charts.*

## §5.3 Categories with Products-Circuits and Categories with Sums-Flow Charts

Now recall the definition of product and sum as we discussed in Chapter 2. Continuing with the definition of product we have the following notions:

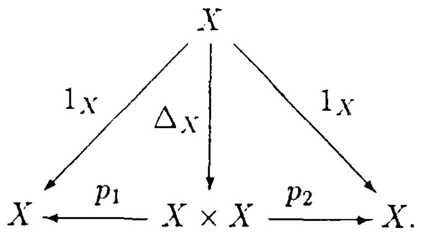
**Definition 5.3.1** Let $X$ be a set then a function defined by

$$\triangle : X \to X \times X$$

such that

$$\triangle_x = (x, x)$$

is called the *diagonal function*. It is sometimes called the *copy function* since it produces two copies of $x$. In an arbitrary category with products the diagonal function is defined as the function with component $1_X$, $1_X$. That is, $\triangle_x$ is the unique function making the following diagram




commutes.

**Definition 5.3.2** In **Ens**, given two functions $f : X_1 \to Y_1$, $g : X_2 \to Y_2$ there is a function denoted by

$$f \times g : X_1 \times X_2 \to Y_1 \times Y_2$$

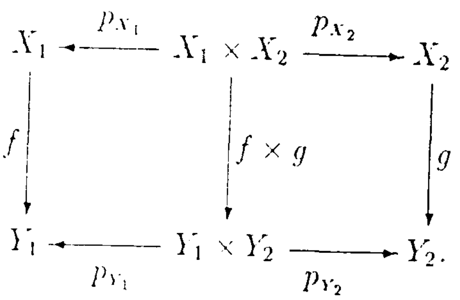
$$(x_1, y_2) \longmapsto (f(x_1), g(x_2)).$$

This function $f \times g$ may be thought of as the two functions $f$ and $g$ in *parallel*.
In an arbitrary category with products we define this operation as follows:
Given functions $f : X_1 \to Y_1$, $g : X_2 \to Y_2$ in a category with products, the function $f \times g$ is defined to be the unique function from $X_1 \times X_2$ to $Y_1 \times Y_2$ such that

$$p_{Y_1}(f \times g) = f p_{X_1},$$

$$p_{Y_2}(f \times g) = g p_{X_2},$$

where the $p$'s are projections. That is, $f \times g$ is the unique function such that the following diagram




commutes.

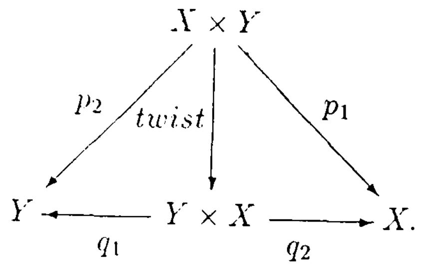
**Definition 5.3.3** In **Ens**, given two sets $X$, $Y$ there is a function

$$twist_{X,Y} : X \times Y \to Y \times X$$

$$(x, y) \longmapsto (y, x).$$

In an arbitrary category with products, $twist_{X,Y} : X \times Y \to Y \times X$ is defined as follows:

Let $p_1$, $p_2$ be the projections of $X \times Y$ and $q_1$, $q_2$ the projections of $Y \times X$. Then $twist_{X,Y}$ is the unique function such that the following diagram




commutes.

## Categories with products-circuits

Let us consider **B**={0,1}. The following is a category, which we shall call **Circ**:

- *Objects*: $\mathbf{B}^0$, $\mathbf{B}^1$, $\mathbf{B}^2$, $\mathbf{B}^3$,....
  where $\mathbf{B}^0=\{\star\}$, $\mathbf{B}^1=\mathbf{B}$, and $\mathbf{B}^m=\{(x_1, x_2, \ldots, x_m): x_i \in \mathbf{B}\}$ for $m > 1$.

- *Morphisms*: all functions between these sets.

There are 2 functions from $\mathbf{B}^0$ to $\mathbf{B}^1$, namely

$$\begin{aligned} true : \mathbf{B^0} &\to \mathbf{B^1} \\ \star &\longmapsto 1 \end{aligned}$$

and

$$\begin{aligned} false : \mathbf{B^0} &\to \mathbf{B^1} \\ \star &\longmapsto 0. \end{aligned}$$

Now we define some interesting functions in this category as follows:

(*a*) A function define by

$$\begin{aligned} \neg : \mathbf{B^1} &\to \mathbf{B^1} \\ 0 &\longmapsto 1 \\ 1 &\longmapsto 0 \end{aligned}$$

is knows as **not**.

(*b*) A function define by

$$\begin{aligned} \& : \mathbf{B^2} &\to \mathbf{B^1} \\ (0,0) &\longmapsto 0 \\ (0,1) &\longmapsto 0 \\ (1,0) &\longmapsto 0 \\ (1,1) &\longmapsto 1 \end{aligned}$$

is knows as **and**.

(c) A function define by

$$
\begin{aligned}
or : \mathbf{B^2} &\to \mathbf{B^1} \\
(0,0) &\longmapsto 0 \\
(0,1) &\longmapsto 1 \\
(1,0) &\longmapsto 1 \\
(1,1) &\longmapsto 1
\end{aligned}
$$

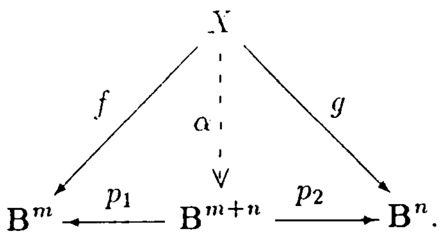
is knows as **or**.

Now we claim that this category has product. In fact, the product of $\mathbf{B^m}$ and $\mathbf{B^n}$ is $\mathbf{B^{m+n}}$ with the following projections:

$$
\mathbf{B}^m \xleftarrow{\quad p_1 \quad} \mathbf{B}^{m+n} \xrightarrow{\quad p_2 \quad} \mathbf{B}^n
$$

$$
(x_1, \ldots, x_m) \longleftarrow\!\shortmid (x_1, x_2, \ldots, x_m, \ldots, x_{m+n}) \longmapsto (x_{m+1}, \ldots, x_{m+n}).
$$

Now we check the property of products. Consider the following diagram




Suppose

$$f(x) = (f_1(x), f_2(x), f_3(x), \ldots\ldots, f_m(x))$$

and

$$g(x) = (g_1(x), g_2(x), g_3(x), \ldots\ldots, g_n(x)).$$

Then

$$\alpha(x) = (f_1(x), f_2(x), f_3(x), \ldots\ldots, f_m(x), g_1(x), g_2(x), g_3(x), \ldots\ldots, g_n(x)),$$

and clearly $p_1\alpha = f$, $p_2\alpha = g$. Further, $p_1\alpha$ is the only function with this property.

Further we will show that what kind of functions can be constructed using products. Let us consider the following remark.

**Remark 5.3.1** All functions can be constructive in **Circ**, starting with *true*, *false*, $\neg$, &, *or*, *identityfunctions* and *projections* using only composition and the property of products.

We will not give a formal proof of this result, but instead we will give an example which makes the general case clear.
Consider the following function

$$
\begin{aligned}
f : \mathbf{B^3} &\longrightarrow \mathbf{B} \\
(0,0,0) &\longmapsto 1 \\
(0,0,1) &\longmapsto 0 \\
(0,1,0) &\longmapsto 0 \\
(0,1,1) &\longmapsto 0 \\
(1,0,0) &\longmapsto 0 \\
(1,0,1) &\longmapsto 1 \\
(1,1,0) &\longmapsto 0 \\
(1,1,1) &\longmapsto 0.
\end{aligned}
$$

We claim that

$$f(x, y, z) = (\neg x \& \neg y \& \neg z) \quad or \quad (x \& \neg y \& z).$$

To see this, notice that $f(x, y, z)$ is 1 if either of the two parts $(\neg x \& \neg y \& \neg z)$ $or$ $(x \& \neg y \& z)$ is 1.

The first part is 1 precisely when $x = 0$ and $y = 0$ and $z = 0$; the second part is 1 precisely when $x = 1$ and $y = 0$ and $z = 1$. Hence the result.

Using this expression for $f$ we can decompose $f$ into $\neg$, &, *or*, using products and composition as follows:

$$\begin{array}{ccc} \mathbf{B}^3 & \xrightarrow{\Delta_{\mathbf{B}^3}} & \mathbf{B}^3 \times \mathbf{B}^3 = \mathbf{B}^6 \\ (x, y, z) & \longmapsto & (x, y, z, x, y, z) \end{array}$$

$$\begin{array}{ccc} \mathbf{B}^6 & \xrightarrow{\neg \times \neg \times \neg \times 1_{\mathbf{B}} \times \neg \times 1_{\mathbf{B}}} & \mathbf{B}^6 \\ (x, y, z, x, y, z) & \longmapsto & (\neg x, \neg y, \neg z, x, \neg y, z) \end{array}$$

$$\begin{array}{ccc} \mathbf{B}^6 & \xrightarrow{\& \times 1_{\mathbf{B}} \times \& \times 1_{\mathbf{B}}} & \mathbf{B}^4 \\ (\neg x, \neg y, \neg z, x, \neg y, z) & \longmapsto & (\neg x \& \neg y, \neg z, x \& \neg y, z) \end{array}$$

$$\begin{array}{ccc} \mathbf{B}^4 & \xrightarrow{\& \times \&} & \mathbf{B}^2 \\ (\neg x \& \neg y, \neg z, x \& \neg y, z) & \longmapsto & (\neg x \& \neg y \& \neg z, x \& \neg y \& z) \end{array}$$

$$\begin{array}{ccc} \mathbf{B}^2 & \xrightarrow{or} & \mathbf{B} \\ (\neg x \& \neg y \& \neg z, x \& \neg y \& z) & \longmapsto & f(x, y, z). \end{array}$$

**Note:** We have used the following easily checked facts about this category

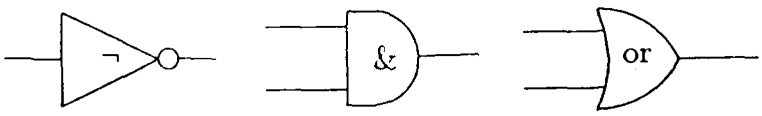
$$(\mathbf{B}^{\mathbf{m}} \times \mathbf{B}^{\mathbf{n}}) \times \mathbf{B}^{\mathbf{p}} = \mathbf{B}^{\mathbf{m}+\mathbf{n}+\mathbf{p}} = \mathbf{B}^{\mathbf{m}} \times (\mathbf{B}^{\mathbf{n}} \times \mathbf{B}^{\mathbf{p}})$$

and the same is true for the functions, namely,

$$(f \times g) \times h = f \times (g \times h).$$

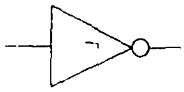
It is because of these facts that we have omitted brackets above.

This decomposition corresponds to the way such a function might be implemented using *boolean gates*; that is, using a circuits (without feedback) consisting of wires and component:

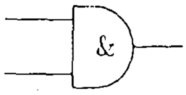


The set B is the set of possible states (the state space) of each wire- each wire can be at zero volt and one volt, say.

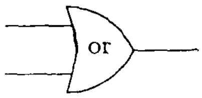
The function $\neg : \mathbf{B} \longrightarrow \mathbf{B}$ is implemented by the component




The function $\& : \mathbf{B}^2 \longrightarrow \mathbf{B}$ is implemented by the component




The function $or : \mathbf{B}^2 \longrightarrow \mathbf{B}$ is implemented by the component




**Observations:**

- We can spilt up wires. This corresponds to the diagonal function,

$$\triangle : \mathbf{B} \longrightarrow \mathbf{B}^2.$$

- We can put two components side by side. This corresponds to the function,

$$f \times g : \mathbf{B} \times \mathbf{B} \longrightarrow \mathbf{B} \times \mathbf{B}.$$

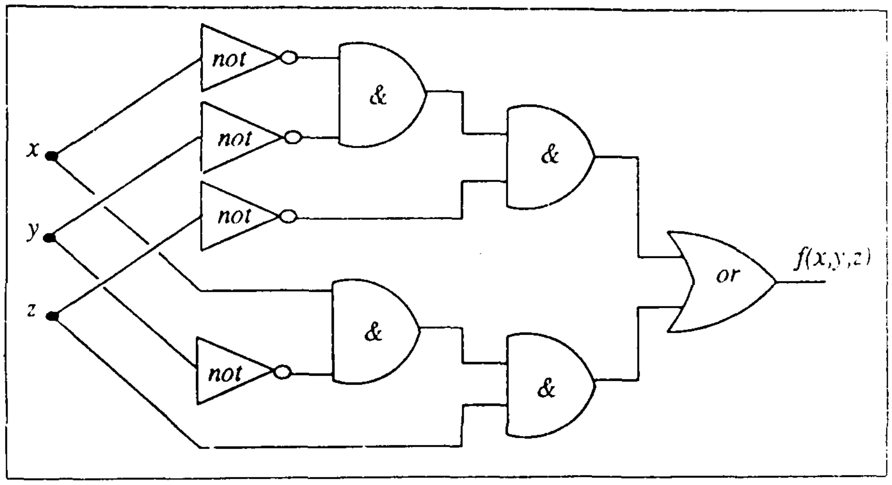
- We can put two components in series. This corresponds to the composition,

$$gof : \mathbf{B} \longrightarrow \mathbf{B}.$$

- We can twist two of the wires. This corresponds to the function,

$$twist : \mathbf{B} \times \mathbf{B} \longrightarrow \mathbf{B} \times \mathbf{B}.$$

Now we will draw a circuit which implements the function $f : \mathbf{B}^3 \longrightarrow \mathbf{B}$ in the above example and notice that how the circuit corresponds exactly to the decomposition given above.




Going from left to right in this circuit corresponds exactly to the successive functions in the composite:

$$\mathbf{B}^3 \xrightarrow{\Delta} \mathbf{B}^6 \xrightarrow{\neg \times \neg \times \neg \times 1 \times \neg \times 1} \mathbf{B}^6 \xrightarrow{\& \times 1 \times \& \times 1} \mathbf{B}^4 \xrightarrow{\& \times \&} \mathbf{B}^2 \xrightarrow{\text{or}} \mathbf{B}.$$

**Observations:**

(*i*) Using wires, we can implement products.

(*ii*) Every function $\mathbf{B^m} \longrightarrow \mathbf{B^n}$ can be implemented using $\neg$, &, *or*, *true*, *false*, using products and composition.

(*iii*) A decomposition of function $f$ into $\neg$, &, *or*, using products and composition to an implementation of $f$ by a circuit.

Now we construct a new category as follows:

- *Objects*: $\mathbf{R}^0$, $\mathbf{R}^1$, $\mathbf{R}^2$, $\mathbf{R}^3$,....
- *Morphisms*: all functions between these sets.

Here are some particular functions in this category:

(*i*) To each real number $r$ there is a function

$$\lceil r \rceil : \mathbf{R^0} \longrightarrow \mathbf{R} \text{ ( \textit{called the name of} } r\text{)}$$

which takes the single point of $\mathbf{R}^0$ to $r \in \mathbf{R}$.

(*ii*) *add* : $\mathbf{R^2} \longrightarrow \mathbf{R}$ which takes $(x, y)$ to $x + y$.

(*iii*) *multiply* : $\mathbf{R^2} \longrightarrow \mathbf{R}$ which takes $(x, y)$ to $xy$.

The polynomial functions can be constructed from these particular functions using only composition and the properties of product.

We will not give a formal proof of this result but instead we will give an example as an illustration.

**Example 5.3.1** $f(x,y) = 3x^2 + 2xy + 1 : \mathbf{R^2} \longrightarrow \mathbf{R}$ can be constructed as the composite of the following functions:

$$\begin{array}{rcl} \mathbb{R}^2 & \xrightarrow{\Delta \times 1_{\mathbb{R}}} & \mathbb{R}^3 \\ (x,y) & \longmapsto & (x,x,y), \end{array}$$

$$\begin{array}{rcl} \mathbb{R}^3 & \xrightarrow{\Delta \times 1_{\mathbb{R}^2}} & \mathbb{R}^4 \\ (x,x,y) & \longmapsto & (x,x,x,y), \end{array}$$

$$\begin{array}{rcl} \mathbb{R}^0 \times \mathbb{R}^2 \times \mathbb{R}^0 \times \mathbb{R}^2 \times \mathbb{R}^0 & \xrightarrow{\lceil 3 \rceil \times 1_{\mathbb{R}^2} \times \lceil 2 \rceil \times 1_{\mathbb{R}^2} \times \lceil 1 \rceil} & \mathbb{R}^7 \\ (x,x,x,y) & \longmapsto & (3,x,x,2,x,y,1), \end{array}$$

$$\begin{array}{rcl} \mathbb{R}^7 & \xrightarrow{mult \times 1_{\mathbb{R}} \times mult \times 1_{\mathbb{R}^2}} & \mathbb{R}^5 \\ (3,x,x,2,x,y,1) & \longmapsto & (3x,x,2x,y,1), \end{array}$$

$$\begin{array}{rcl} \mathbb{R}^5 & \xrightarrow{mult \times mult \times 1_{\mathbb{R}}} & \mathbb{R}^3 \\ (3x,x,2x,y,1) & \longmapsto & (3x^2,2xy,1), \end{array}$$

$$\begin{array}{rcl} \mathbb{R}^3 & \xrightarrow{add \times 1_{\mathbb{R}}} & \mathbb{R}^2 \\ (3x^2,2xy,1) & \longmapsto & (3x^2+2xy,1), \end{array}$$

$$\begin{array}{rcl} \mathbb{R}^2 & \xrightarrow{add} & \mathbb{R} \\ (3x^2+2xy,1) & \longmapsto & (3x^2+2xy+1). \end{array}$$

**Note:**

(i) We have used the following easily checked facts about this category

$$(\mathbf{R}^m \times \mathbf{R}^n) \times \mathbf{R}^p = \mathbf{R}^{m+n+p} = \mathbf{R}^m \times (\mathbf{R}^n \times \mathbf{R}^p)$$

and the same is true for the functions, namely,

$$(f \times g) \times h = f \times (g \times h).$$

It is because of these facts that we have omitted brackets above.

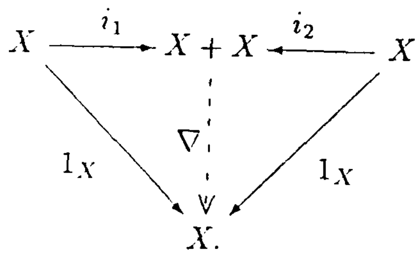
(*ii*) $|r| \times f : \mathbf{R}^0 \times \mathbf{R}^1 \longrightarrow \mathbf{R}^1 \times \mathbf{R}^1$ takes $x$ to $(r, f(x))$.

Now further we will discuss the some special functions using the property of co-products(sums)as continuing with the definition of sums we have the following notions:

**Definition 5.3.4** Let $X$ be a set then a function defined by

$$\nabla : X + X \longrightarrow X$$
$$(x, 0) \longmapsto x$$
$$(x, 1) \longmapsto x$$

called the *codiagonal function*.
i.e. the diagram

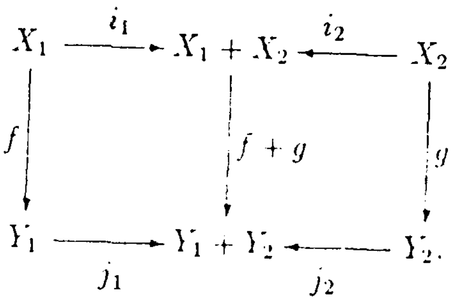



commutes.

**Definition 5.3.5** Given $f : X_1 \longrightarrow Y_1$, $g : X_2 \longrightarrow Y_2$, there is a function

$$f + g : X_1 + X_2 \longrightarrow Y_1 + Y_2,$$

defined to be the unique function the following diagram

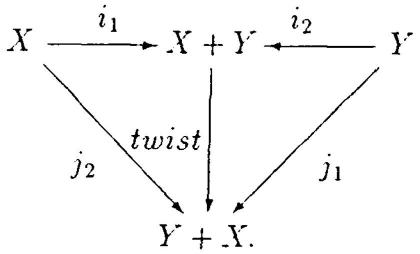



commutes.

**Definition 5.3.6** In **Ens**, given two sets $X$, $Y$ there is a function

$$twist_{X,Y} : X + Y \to Y + X$$
$$(x, 0) \longmapsto (x, 1)$$
$$(y, 1) \longmapsto (y, 0).$$

In an arbitrary category with sums, $twist_{X,Y} : X + Y \to Y + X$ is defined as follows: Let $i_1$, $i_2$ be the injections of $X + Y$ and $j_1$, $j_2$ the injections of $Y + X$. Then $twist_{X,Y}$ is the unique function such that the following diagram




commutes.

## Categories with sums-flow charts

The following is a category, which we shall call **Flow**:

- *Objects*: $0.\mathbf{R} = \phi$, $1.\mathbf{R} = \mathbf{R}$, $2.\mathbf{R}$, $3.\mathbf{R}$ ..... where
  $m.\mathbf{R} = \{(x,0) : x \in \mathbf{R}\} \cup \{(x,1) : x \in \mathbf{R}\} \cup .... \cup \{(x, m-1) : x \in \mathbf{R}\}$
  $(m > 1)$

- *Morphisms*: all functions between these sets.

The category **Flow** has sums which are strictly associative. In fact,

$$m.\mathbf{R} + n.\mathbf{R} = (m+n)\mathbf{R}.$$

The injections are:

$$m{\bullet}\mathrm{IR} \xrightarrow{i_1} (m+n){\bullet}\mathrm{IR} \xleftarrow{i_2} n{\bullet}\mathrm{IR}$$

$$(x,k) \longmapsto (x,k)$$

$$(x, l+m) \longleftarrow\!\!\!\mid (x,l).$$

It is easy to see that the property of a sum holds. Given $f : m.\mathbf{R} \longrightarrow Z$ and $g : n.\mathbf{R} \longrightarrow Z$ then

$$\binom{f}{g}(x,i) = \begin{cases} f(x,i) & \text{if } 0 \le i \le m-1; \\ g(x, i-m) & \text{if } m \le i \le m+n-1. \end{cases}$$

As before, here also we will take a special class of functions and see what function can be generated out of them using sums and composition.

Here we will take the following as the special functions:

- all continuous functions from $\mathbf{R}$ to $\mathbf{R}$

- the function which tests whether $x$ is positive or not:

$$test_{x>0} : \mathbf{R} \to \mathbf{2.R} = \mathbf{R} + \mathbf{R}$$

$$x \longmapsto (x,0) \quad (if\ x \le 0)$$

$$x \longmapsto (x,1) \quad (if\ x > 0).$$

**Remark 5.3.2** Out of these special functions using composition and the property of sums we can construct a number of functions as clear from the following examples.

Example 5.3.2 The discontinuous function.

$$f : \mathbf{R} \longrightarrow \mathbf{R}$$
$$x \longmapsto \begin{cases} sin x & \text{if } x \leq 0 \\ e^x & \text{if } x > 0 \end{cases}$$

can be constructed as the composite of the following functions:

$$\mathbb{R} \xrightarrow{\quad test_{x>0} \quad} \mathbb{R} + \mathbb{R}$$
$$x \longmapsto \begin{cases} (x,0) & \text{if } x \leq 0 \\ (x,1) & \text{if } x > 0, \end{cases}$$

$$\mathbb{R} + \mathbb{R} \xrightarrow{\quad \lceil \sin x \rceil + \lceil e^x \rceil \quad} \mathbb{R} + \mathbb{R}$$
$$(x,0) \text{ if } x \leq 0 \longmapsto (\sin x, 0) \text{ if } x \leq 0$$
$$(x,1) \text{ if } x > 0 \longmapsto (e^x, 1) \text{ if } x > 0,$$

$$\mathbb{R} + \mathbb{R} \xrightarrow{\quad \nabla \quad} \mathbb{R}$$
$$(\sin x, 0) \text{ if } x \leq 0 \longmapsto \sin x \text{ if } x \leq 0$$
$$(e^x, 1) \text{ if } x > 0 \longmapsto e^x \text{ if } x > 0.$$

**Example 5.3.3** The test function.

$$test_{x>1} : \mathbf{R} \longrightarrow \mathbf{R} + \mathbf{R}$$
$$x \longmapsto \begin{cases} (x,0) & \text{if } x < 1 \\ (x,1) & \text{if } x \geq 1 \end{cases}$$

can be constructed as the composite of the following functions:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\lceil 1-x \rceil} & \mathbb{R} \\ x & \longmapsto & (1-x), \end{array}$$

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{test_{x>0}} & \mathbb{R}+\mathbb{R} \\ (1-x) & \longmapsto & \begin{cases} (1-x,0) & \text{if } 1-x \leq 0 \\ (1-x,1) & \text{if } 1-x > 0, \end{cases} \end{array}$$

$$\begin{array}{ccc} \mathbb{R}+\mathbb{R} & \xrightarrow{twist} & \mathbb{R}+\mathbb{R} \\ (1-x,0) \text{ if } x \geq 1 & \longmapsto & (1-x,1) \text{ if } x \geq 1 \\ (1-x,1) \text{ if } x < 1 & \longmapsto & (1-x,0) \text{ if } x < 1, \end{array}$$

$$\begin{array}{ccc} \mathbb{R}+\mathbb{R} & \xrightarrow{\lceil 1-x \rceil + \lceil 1-x \rceil} & \mathbb{R}+\mathbb{R} \\ (1-x,0) \text{ if } x < 1 & \longmapsto & (x,0) \text{ if } x < 1 \\ (1-x,1) \text{ if } x \geq 1 & \longmapsto & (x,1) \text{ if } x \geq 1. \end{array}$$

**Example 5.3.4** The piecewise-continuous function,

$$\begin{array}{l} f : \mathbf{R} \longrightarrow \mathbf{R} \\ x \longmapsto \begin{cases} sinx & \text{if } x \leq 0 \\ e^{x} & \text{if } 0 < x < 1 \\ cosx & \text{if } 1 \leq x \end{cases} \end{array}$$

can be constructed as the composite of the following functions:

$$
\begin{array}{rcl}
\mathbb{R} & \xrightarrow{\quad test_{x>0} \quad} & \mathbb{R}+\mathbb{R} \\
x & \longmapsto & \begin{cases} (x,0) & \text{if } x \leq 0 \\ (x,1) & \text{if } x > 0, \end{cases}
\end{array}
$$

$$
\begin{array}{rcl}
\mathbb{R}+\mathbb{R} & \xrightarrow{\quad 1_{\mathbb{R}} + test_{x \geq 1} \quad} & \mathbb{R}+\mathbb{R}+\mathbb{R} \\
(x,0) \text{ if } x \leq 0 & \longmapsto & (x,0) \text{ if } x \leq 0 \\
(x,1) \text{ if } x > 0 & \longmapsto & \begin{cases} (x,1) & \text{if } 0 < x < 1 \\ (x,2) & \text{if } 1 \leq x, \end{cases}
\end{array}
$$

$$
\begin{array}{rcl}
\mathbb{R}+\mathbb{R}+\mathbb{R} & \xrightarrow{\quad \lceil \sin x \rceil + \lceil e^x \rceil + \lceil \cos x \rceil \quad} & \mathbb{R}+\mathbb{R}+\mathbb{R} \\
(x,0) \text{ if } x \leq 0 & \longmapsto & (\sin x, 0) \text{ if } x \leq 0 \\
(x,1) \text{ if } 0 < x < 1 & \longmapsto & (e^x, 1) \text{ if } 0 < x < 1 \\
(x,2) \text{ if } 1 \leq x & \longmapsto & (\cos x, 2) \text{ if } 1 \leq x,
\end{array}
$$

$$
\begin{array}{rcl}
\mathbb{R}+\mathbb{R}+\mathbb{R} & \xrightarrow{\quad 1_{\mathbb{R}} + \nabla \quad} & \mathbb{R}+\mathbb{R} \\
(\sin x,0) \text{ if } x \leq 0 & \longmapsto & (\sin x, 0) \text{ if } x \leq 0 \\
(e^x,1) \text{ if } 0 < x < 1 & \longmapsto & (e^x, 1) \text{ if } 0 < x < 1 \\
(\cos x,2) \text{ if } 1 \leq x & \longmapsto & (\cos x, 1) \text{ if } 1 \leq x,
\end{array}
$$

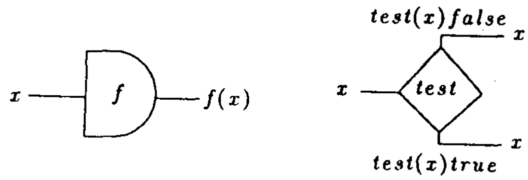
$$
\begin{array}{rcl}
\mathbb{R}+\mathbb{R} & \xrightarrow{\quad \nabla \quad} & \mathbb{R} \\
(\sin x,0) \text{ if } x \leq 0 & \longmapsto & f(x) \text{ if } x \leq 0 \\
(e^x,1) \text{ if } 0 < x < 1 & \longmapsto & f(x) \text{ if } 0 < x < 1 \\
(\cos x,1) \text{ if } 1 \leq x & \longmapsto & f(x) \text{ if } 1 \leq x.
\end{array}
$$

It is more or less clear that any piecewise continuous function with a finite number of discontinuities can be constructed in this way. Some functions with an infinite number of discontinuities like,

$$\mathbf{R} \longrightarrow \mathbf{R}$$
$$x \longmapsto \begin{cases} 0 & \text{if } cos^2x < 0.5 \\ 1 & \text{if } cos^2x \geq 0.5 \end{cases}$$

can also be constructed.

Notice that corresponding to such decompositions there is a flow chart (without feedback) which implements the function. A flow chart may be built up out of components like functions and tests:

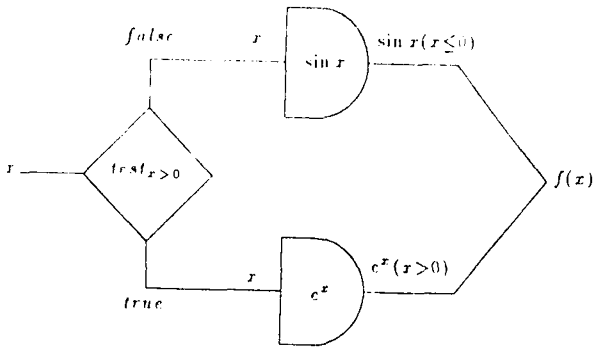



The way a flow chart can be built up is analogous, but dual, to the way circuits are built up. Components may be joined in series (which corresponds to composition) or side by side (which corresponds to the sum of functions).Two edges may be joined, which corresponds to the codiagonal function. Each edge of the flow chart has state space $\mathbf{R}$; that is, when following through a flow chart we carry with us one real number. Below we give the flow charts corresponding to the last Examples 5.3.2, 5.3.3 and 5.3.4. A study of these flow charts will show that a flow chart is just a graphical representation of the decomposition of a function using sums.

Example 5.3.2*

$$f : \mathrm{R} \longrightarrow \mathrm{R}$$

$$x \longmapsto \begin{cases} \sin x & \text{if } x \leq 0 \\ e^{x} & \text{if } x > 0 \end{cases}$$



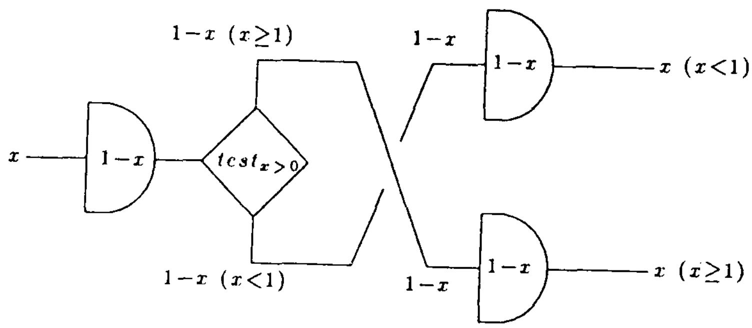

Going from left to right in this flow chart corresponds exactly to the successive functions in the following composite:

$$\mathrm{IR} \xrightarrow{test_{x>0}} \mathrm{IR} + \mathrm{IR} \xrightarrow{\lceil \sin x \rceil + \lceil e^{x} \rceil} \mathrm{IR} + \mathrm{IR} \xrightarrow{\nabla} \mathrm{IR}.$$

Example 5.3.3*

$$test_{x>1} : \mathrm{R} \longrightarrow \mathrm{R} + \mathrm{R}$$

$$x \longmapsto \begin{cases} (x, 0) & \text{if } x < 1 \\ (x, 1) & \text{if } x \geq 1 \end{cases}$$



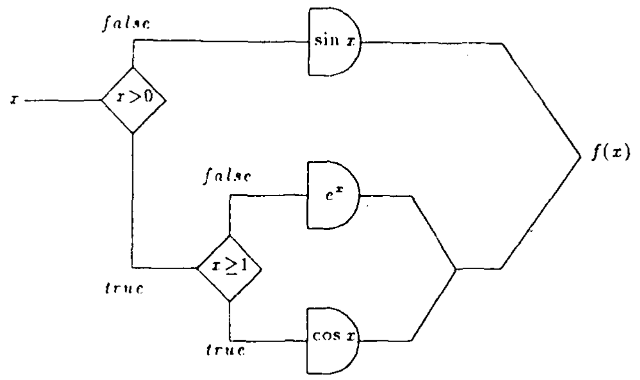

Going from left to right in this flow chart corresponds exactly to the successive functions in the following composite:

$$\mathrm{IR} \xrightarrow{\lceil 1 - x \rceil} \mathrm{IR} \xrightarrow{test_{x>0}} \mathrm{IR} + \mathrm{IR} \xrightarrow{twist} \mathrm{IR} + \mathrm{IR} \xrightarrow{\lceil 1 - x \rceil + \lceil 1 - x \rceil} \mathrm{IR} + \mathrm{IR}.$$

**Example 5.3.4***

$$f : \mathbf{R} \longrightarrow \mathbf{R}$$

$$x \longmapsto \begin{cases} sin x & \text{if } x \leq 0 \\ e^x & \text{if } 0 < x < 1 \\ cos x & \text{if } 1 \leq x \end{cases}$$




Going from left to right in this flow chart corresponds exactly to the successive functions in the following composite:

$$\mathsf{IR} \xrightarrow{test_{x>0}} 2{\bullet}\mathsf{IR} \xrightarrow{1_{\mathsf{IR}} + test_{x\geq 1}} 3{\bullet}\mathsf{IR} \xrightarrow{\lceil \sin x \rceil + \lceil e^x \rceil + \lceil \cos x \rceil} 3{\bullet}\mathsf{IR}$$

$$\xrightarrow{1_{\mathsf{IR}} + \nabla} 2{\bullet}\mathsf{IR} \xrightarrow{\nabla} \mathsf{IR}.$$

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