"FIXED POINT THEOREMS IN CONE METRIC SPACES"

DISSERTATION

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF

MASTER OF PHILOSOPHY

IN

MATHEMATICS

By

VISHAL KUMAR YADAV

Under The Supervision Of

PROF. MOHD. IMDAD

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
Aligarh (India)
2011

Dedicated

To

My

Loving parents

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH-202002 (U.P.), INDIA
Ph: (O): 0571-2701019; (R): 09897260980
Fax: 0571-2704229
E-mail: mhimdad@yahoo.co.in

Dr. Mohd. Imdad
Professor

Dated: 15th July, 2011

CERTIFICATE

This is to certify that the dissertation entitled ***"Fixed Point Theorems in Cone Metric Spaces"*** has been written by **Mr. Vishal Kumar Yadav** under my guidance in the Department of Mathematics, Aligarh Muslim University, Aligarh as a partial fulfillment for the award of **Master of Philosophy in Mathematics**. To the best of my knowledge, the exposition has not been submitted to any other university/ institution for the award of the degree.

It is further certified that **Mr. Vishal Kumar Yadav** has fulfilled the prescribed conditions and nature given in the statutes and ordinances of the Aligarh Muslim University, Aligarh.

CHAIRMAN
DEPARTMENT OF MATHEMATICS
A.M.U., ALIGARH

Dr. Mohammad Imdad
(Supervisor)

CONTENTS

ACKNOWLEDGMENT

I am extremely grateful to Almighty **GOD**, the most Gracious and Compassionate for granting me a blessed life. Indeed, without his help and will nothing can be accomplished.

The first person, I would like to thank is my supervisor **Prof. Mohd. Imdad**, Department of Mathematics, A.M.U., Aligarh for sparing me his valuable time whenever I approached him. I owe him lots of gratitude for his advises and tips in the course of developing scientific temper besides highlighting the roles of moral and ethical values in education and research.

I express my special thanks to Prof. Afzal Beg, Chairman, Department of Mathematics, A.M.U., Aligarh for providing me necessary facilities in the department. I would like to thank all the faculty members of department who have always been a source of inspiration for me.

Further, I wish to express my sincere thanks to my seniors and co-workers namely: Dr. Javid Ali, Mr. Mohd. Tanveer, Mr. Abu Zaid Ansari, Mr. Izharuddin, Mr. Faisal Khan, Mr. Kamran Khan, Mr. Shuaib Khan, Miss. Anupam Sharma, Miss. Nazia Parveen, Miss. Shikha Varshney, Miss. Tarannum Bano and Miss. Sana Khan. I also avail this opportunity to thank my personal friends namely: Mr. Ejaz Mustafa, Mr. Pradeep Sharma, Mr. Nirpendra Yadav and Mr. Mohan Tiwari for their encouragement, affection and support in the entire course of my association with them. I also express my heartful thanks to my senior colleague Mr. M. Hasan for his help in several ways throughout this work. Also, I wish to express sincere thanks to my teacher Mr. P. S. Yadav, who helped me all the time.

I have no words to express my gratitude and thanks to my father Mr. Narendra Yadav, mother Mrs. Kusum Yadav and my brother Mr. Vikash Yadav for their limitless sacrifices to enrich my future. Their selfless care, love, support, prayers and advises helped me to face the challenge especially in the period of doubt and despair.

I express my indebtedness to the seminar library staff for providing me books, journals and other facilities. Finally, I express my special thanks to University Grants Commission, India, for providing me fellowship during this work.

Vishal Yadav

(Vishal Kumar Yadav)

PREFACE

The presence or absence of a fixed point in respect of a mapping is an intrinsic property of the mapping. However, many necessary or sufficient conditions for the existence of fixed points involve a mixture of algebraic, order theoretic or topological properties of the mapping or its domain. The fixed point theory has so many applications in various fields within as well as outside the Mathematics namely: Approximation theory, Successive approximation, Integral equations, Game theory, Optimal control, Optimization, Economics and several others.

The origin of the fixed point theory which dates to the later part of nineteenth century heavily rests on use of successive approximations to establish the existence and uniqueness of solutions, particularly to differential equations. This method is associated with many big names which include Cauchy, Liouville, Lipschitz, Peano, Fredholm and above all Picard. In Fact the precursors of a fixed point theoretic approach are explicite in the work of Picard. However, it is the Polish mathematician Stefann Banach who is credited for placing the underlying ideas into an abstract frame of work suitable for broad applications well beyond the scope of elementary differential and integral equations. The fixed point theory of nonexpansive mappings gained new impetus largely as a result of the pioneering work of Felix Browder in the mid nineteen sixties and the evolution of nonlinear functional analysis as an active and vital branch of mathematics. Pivotal in this development were the 1965 existence theorems of Browder, Gohde and Kirk and the early metric result of Edelstein.

In 2007, Huang and Zhang [25] proposed a generalization of metric notion by replacing the set of real numbers by orderied Banach space and define a new space termed as cone metric space and gave the fixed point theorems on cone metric space which is the generalization of some fixed point theorems on metric space. After this Rezapour and Hamlbarani [44], Vetro [38] and Jungck [12] also gave fixed point theorems on cone metric spaces.

As usual, Chapter 1 of the dissertation is devoted to the background material wherein we collect the definitions; examples and basic results on cone metric space. Section 1.4 deals with definitions and basic fixed point theorems.

Chapter 2 deals with fixed point theorems on contractive mappings in cone metric space for single valuued maps. In Section 2.3 we present results on common fixed point theorems for a pair of maps whereas Section 2.4 contains some basic definitions and periodic point theorems employing property P and Q in respect of cone metric spaces. In the last section, we present fixed point theorems for a sequence of mappings.

Chapter 3 is broadly devoted to some core fixed point theorems for multivalued mappings. In Section 3.2, we present the basic definitions and examples besides defining lower semicontinuity and upper semicontinuity with examples. The final section of thiis chapter is devoted to fixed point theorems for multivalued mappings.

In Chapter 4, we present the topological properties of cone metric space including it;s equivalence with metric space. In Section 4.2, we deal with topological properties of cone metric space. In the remaining part of this chapter, we discuss metrizability and completion of cone metric space.

In the end, a bibliography is given which by no means is an exhaustive one but lists only those books and papers which have been referred to in this exposition.

CHAPTER 1

CONCEPTS AND CORE RESULTS ON CONE METRIC SPACE

1.1 Introduction

The concept of a metric space is essentially due to a French mathematician Maurice Frechet $(1878-1973)$.Through the definition presently in use is the one formulated by German mathematician Flix Hausdorff $(1868-1942)$ in 1914. Frechet introduced this notion in his doctoral thesis presented to the University of Paris in 1906 and for many years pioneered the study of such spaces and their application to other areas of mathematics. There are many generalizaions of metric space which include Topological spaces, Fuzzy metric spaces, Probablistic metric spaces, Intutionstiic metric spaces, b-metric spaces, Generalized metric spaces etc.

In 2007 Huang and Zhang gave generelization of metric space by replacing the real numbers with an ordered Banach space and define a new space termed as cone metric space and gave fixed point theorems on cone metric space which is the generalization of some fixed point theorems on metric space . After Huang and Zhang [25], Sh. Rezapour [44] , P. Vetro [38], G. Jungck [12] also gave fixed point theorems on cone metric space . In this chapter we discuss some basic properties, examples, and resuts of cone metric space .

1.2 Some Basic Definitions

In this section we give some basic definitions and examples of cone in metric setting .

Definition 1.2.1 Let E be a real Banach space and P a subset of E. Then P is called a cone in E if it satisfies
i) P is closed, non empty and $P \neq \{0\}$,
ii) $a.b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P \Rightarrow ax + by \in P$,
iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$ while $x \ll y$ will stand for $y - x \in intP$.

We always suppose E is a Banach space, P is a cone in E with $intP \neq \phi$ and $\leq$ is partial ordering with respect to P.

Definition 1.2.2 The cone P is called normal if there exists a constant $M > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \quad implies \quad \| x \| \leq M \| y \|$$

or equivalently, a cone $P \subset E$ is a normal cone if

$$inf\{\| x + y \|: \ x, y \in P, \ \ \| x \| = \| y \| = 1\} > 0. \tag{1.1}$$

The least positive integer M satisfying above is called normal constant. Sh. Rezapour and R. Hamlbarani [44] proved that $M \geq 1$.

It follows from (1.1) that P is non normal if and only if there exist sequences $x_n, y_n \in P$ such that

$$0 \leq x_n \leq x_n + y_n, \ x_n + y_n \to 0 \ but \ x_n \not\to 0$$

Example 1.2.3 Let $E = C_{\mathbb{R}}([0, 1])$ with supremum norm and $P = \{f \in E : f \geq 0\}$. Then P is a cone with normal constant $M = 1$.

Example 1.2.4 Let E be a real vector space

$$E = \left\{ax + b | a, b \in \mathbb{R}; x \in [\frac{1}{2}, 1]\right\}$$

with supremum norm and

$$P = \{ax + b \in E | a \leq 0, b \geq 0\}$$

then P is a normal cone in E with normal constant $M > 1$.

Example 1.2.5 let $E = l^1$, $P = \{\{x_n\}_{n\leq 1} \in E : x_n \geq 0, for\ all\ n\}$, (X, ρ) a metric space and $d : X \times X \to E$ defined by $d(x, y) = \{\frac{\rho(x,y)}{2^n}\}_{n\geq 1}$. Then (X, d) is a cone metric space and the normal constant of P is equal to $M = 1$.

Example 1.2.6 Let $E = C^1_{\mathbb{R}}([0, 1])$, with $\| x \|=\| x \|_\infty + \| x' \|_\infty, P = \{x \in E : x(t) \geq 0\}$. This cone is non normal . Consider, $x_n(t) = t^n/n$ and $y_n(t) = 1/n$. Then $0 \leq x_n \leq y_n$ and $\lim\limits_{n\to\infty} y_n = 0$, but $\| x_n \|= max_{t\in[0,1]}|t^n/n| + max_{t\in[0,1]}|t^{n-1}| = 1/n + 1 > 1$, hence x_n does not converge to zero.

Example 1.2.7 Let $E = C^2_{\mathbb{R}}([0, 1])$ with the norm $\| f \|=\| f \|_\infty + \| f' \|_\infty$ and $P = \{f \in E : f \geq 0\}$. Put $x_n = \frac{1-sinnt}{n+2}$ and $y_n = \frac{1+sinnt}{n+2}$. Then $0 \leq x_n \leq x_n + y_n$, $\| x_n \|=\| y_n \|= 1$ and $\| x_n + y_n \|= \frac{2}{n+2} \to 0$. Therefore the cone P is non normal.

Lemma 1.2.8 There is not normal cone with normal constant $M < 1$.

Proof. Let (X, d) be a cone metric space and P a normal cone with normal constant $M < 1$. Choose a non-zero element $x \in P$ and $0 < \varepsilon < 1$ such that $M < 1-\varepsilon$ .Then, $(1 - \varepsilon)x \leq x$, but $(1 - \varepsilon)\|x\| > M\|x\|$. This is a contradiction.

Proposition 1.2.9 For each $k > 1$, there is a normal cone with normal constant $M > k$.

Proof. Let $k > 1$ be given. Consider the real vector space

$$E = \{ax + b|a, b \in \mathbb{R}; x \in [1 - \frac{1}{k}, 1]\}$$

with supremum norm and the cone

$$P = \{ax + b \in E|a \leq 0, b \geq 0\}$$

in E. First, we show that P is regular (and so normal). Let $\{a_nx + b_n\}_{n\geq 1}$ be an increasing sequence which is bounded from above, that is, there is an element $cx+d \in E$ such that

$$a_1x + b_1 \leq a_2x + b_2 \leq ..... \leq a_nx + b_n \leq .... \leq cx + d,$$

for all $x \in [1 - \frac{1}{k}, 1]$. Then, $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are two sequences in $\mathbb{R}$ such that

$$b_1 \leq b_2 \leq ... \leq d, \quad a_1 \geq a_2 \geq ... \geq c.$$

Thus, $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are convergent. Let $a_n \to a$ and $b_n \to b$. Then, $ax + b \in P$ and $a_nx + b_n \to ax + b$. Therefore P is regular. Hence by Lemma 1.2.8, there is $M \geq 1$ such that $0 \leq g \leq f$ implies $\| g \| \leq M \| f \|$, for all $g, f \in E$. Now we show that $M > k$. First note that $f(x) = -kx + k \in P$, $g(x) = k \in P$ and $f - g \in P$. So, $0 \leq g \leq f$. Therefore, $k = \| g \| \leq M \| f \| = M$. On the other hand, if we consider $f(x) = -(k + \frac{1}{k})x + k$ and $g(x) = k$, then $f \in P, g \in P$ and $f - g \in P$. Also , $\| g \| = k$ and $\| f \| = 1 - \frac{1}{k} + \frac{1}{k^2}$. Thus, $k = \| g \| > k \| f \| = k + \frac{1}{k} - 1$. This shows that $M > k$.

Definition 1.2.10 The cone $P \subset E$ is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq x_2 \leq x_3 \leq ... \leq x_n \leq ... \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\| x_n - x \| \to 0$ $(n \to \infty)$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent.

Lemma 1.2.11 Every regular cone is normal.

Proof. Let P a regular cone which is not normal. For each $n \geq 1$, choose $t_n, s_n \in P$ such that $t_n - s_n \in P$ and $n^2 \| t_n \| < \| s_n \|$. For each $n \geq 1$, put $y_n = \frac{t_n}{\|t_n\|}$ and $x_n = \frac{s_n}{\|t_n\|}$. Then x_n, y_n, $y_n - x_n \in P$, $\| y_n \| = 1$ and $n^2 < \| x_n \|$, for all $n \geq 1$. Since the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} \| y_n \|$ is convergent and P is closed, there is an element $y \in P$ such that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} y_n = y$. Now, note that

$$0 \leq x_1 \leq x_1 + \frac{1}{2^2}x_2 \leq x_1 + \frac{1}{2^2}x_2 + \frac{1}{3^2}x_3 \leq \ldots \leq y.$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^2}x_n$ is convergent because P is regular. Hence,

$$\lim_{n\to\infty} \frac{\| x_n \|}{n^2} = 0,$$

which is a contradiction.

The converse of Lemma 1.2.11 is not true.

Example 1.2.12 Let $E = C_{\mathbb{R}}([0,1])$ with the supremum norm and $P = \{f \in E : f \geq 0\}$. Then, P is a cone with normal constant $M = 1$. Now, consider the following sequence of elements of E which is decreasing and bounded from below but not convergent in E,

$$x \geq x^2 \geq x^3 \geq \ldots \geq 0.$$

Therefore, the converse of above Lemma is not true.

Definition 1.2.13 The cone P is called

i) minihedral if $sup\{x, y\}$ exists for all $x, y \in E$.

ii) strongly minihedral if every subset of E which is bounded from above has a supremum.

iii) solid if $intP \neq \phi$

Example 1.2.14 Let $E = \mathbb{R}^n$ with $P = \{(x_1, x_2, \ldots x_n) : x_i \geq 0 \; for \; all \; i = 1, 2, \ldots n\}$. Then cone P is normal, minihedral, strongly minihedral and solid.

Example 1.2.15 Let $E = \mathbb{R}^2$ and $P = \{(x_1, 0) : x_1 \geq 0\}$. Then P is strongly minihedral but not minihedral.

Example 1.2.16 Let $D \subseteq \mathbb{R}^n$ be a compact set, $E = C(D)$ and $P = \{f \in E : f(x) \geq 0 \; for \; all \; x \in D\}$. The cone P is normal, solid and minihedral but is not strongly minihedral, and regular.

1.3 Cone Metric Space

Definition 1.3.1 Let X be a non empty set. Suppose the mapping $d : X \times X \to E$ satisfies

i) $d(x,y) > 0$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$,

ii) $d(x,y) = d(y,x)$ for all $x, y \in X$,

iii) $d(x,y) \leq d(x,z) + d(y,z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space. It is obvious that cone metric spaces generalize metric spaces.

Example 1.3.2 Let $E = \mathbb{R}^2, P = \{(x,y) \in E \,|x,y \geq 0\} \subset \mathbb{R}^2, X = \mathbb{R}$ and $d : X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space.

Example 1.3.3 Let $E = \mathbb{R}^n$, $P = \{(x_1, x_2, ...x_n) \in E | x_i \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that

$$d(x,y) = (|x-y|, \alpha_1|x-y|, \alpha_2|x-y|, ..., \alpha_{n-1}|x-y|)$$

where $\alpha_i \geq 0, for\ all\ 1 \leq i \leq n-1$. Then (X,d) is a cone metric space.

Example 1.3.4 Let $E = C_{\mathbb{R}}([0,1])$ with the supremum norm and $P = \{f \in E : f \geq 0\}$. Then P is a cone with normal constant $M = 1$ and (X,ρ) a metric space. Define $d : X \times X \to E$ by $d(x,y) = \rho(x,y)\varphi$ where $\varphi : [0,1] \to \mathbb{R}$ such that $\varphi(t) = e^t$. Then (X,d) is a cone metric space.

Example 1.3.5 Let $E = (C_{\mathbb{R}}([0,\infty)), \|\cdot\|_\infty)$, $P = \{f \in E : f(x) \geq 0\}, (X,\rho)$ a metric space and $d : X \times X \to E$ defined by $d(x,y) = \rho(x,y)\varphi$ where $\varphi : [0,1] \to \mathbb{R}^+$ is continuous. Then (X,d) is a normal cone metric space and the normal constant of P is equal to $M = 1$.

Example 1.3.6 Let $q > 0$, $E = l^q$, $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0,\ for\ all\ n\}$, (X, ρ) a metric space and defined by $d(x, y) = \{(\frac{\rho(x,y)}{2^n})^{\frac{1}{q}}\}_{n\geq 1}$. Then, (X, d) is a cone metric space and the normal constant of P is equal to $M = 1$.

Definition 1.3.7 Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $o \ll c$ there is N such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by

$$\lim_{n\to\infty} x_n = x \quad or \quad x_n \to x \quad (n \to \infty)$$

Lemma 1.3.8 Let (X, d) be a cone metric space, P be a normal cone with normal constant M. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ $(n \to \infty)$.

Proof. Suppose that $\{x_n\}$ converges to x. For every real $\varepsilon > 0$, choose $c \in E$ with $0 \ll c$ and $M \parallel c \parallel < \varepsilon$. Then there is N, for all $n > N$, $d(x_n, x) \ll c$. So that when $n > N$, $\parallel d(x_n, x) \parallel \leq M \parallel c \parallel < \varepsilon$. This means $d(x_n, x) \to 0$ $(n \to \infty)$.

Conversely, suppose that $d(x_n, x) \to 0$ $(n \to \infty)$. For $c \in E$ with $0 \ll c$, there is $\delta > 0$, such that $\parallel x \parallel < \delta$ implies $c - x \in intP$. For this δ there is N, such that for all $n > N$, $\parallel d(x_n, x) \parallel < \delta$. So $c - d(x_n, x) \in intP$. This means $d(x_n, x) \ll c$. Therefore $\{x_n\}$ converges to x.

Remark 1.3.9 Converse part of above Lemma 1.3.8 is always true but if cone is non normal then direct is not true. we demonstrate this by an example.

Let $X = E = C^2_{\mathbb{R}}([0, 1])$ with the norm $\parallel f \parallel = \parallel f \parallel_\infty + \parallel f' \parallel_\infty$ and $P = \{f \in E : f \geq 0\}$ that is not normal cone. Consider $x_n = \frac{1-sinnt}{n+2}$ and $y_n = \frac{1+sinnt}{n+2}$ so $0 \leq x_n \leq x_n + y_n \to 0$, and $\parallel x_n \parallel = \parallel y_n \parallel = 1$. Define cone metric $d : X \times X \to E$ with $d(f, g) = f + g$, for $f \neq g, d(f, f) = 0$. Since $0 \leq x_n \ll c$, namely, $d(x_n, 0) \ll c$ but $d(x_n, 0) \not\to 0$. Indeed $x_n \to 0$ in (X, d) but $x_n \not\to 0$ in E. Even for

$n > m, d(x_n, x_m) = x_n + x_m \ll c$ and $\| d(x_n, x_m) \| = \| x_n + x_m \| = 2$. In particular $d(x_n, x_{n+1}) \ll c$ but $d(x_n, x_{n+1}) \not\to 0$.

Lemma 1.3.10 Let (X, d) be a cone metric space, P be a normal cone with normal constant M. Let $\{x_n\}$ be a squence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$. That is, the limit of $\{x_n\}$ is unique.

Proof. For any $c \in E$ with $0 \ll c$, there is N such that for all $n > N$, $d(x_n, x) \ll c$ and $d(x_n, y) \ll c$. We have

$$d(x, y) \leq d(x_n, x) + d(x_n, y) \leq 2c.$$

Hence $\| d(x, y) \| \leq 2M \| c \|$. Since c is arbitrary, hence $d(x, y) = 0$, therefore $x = y$.

Definition 1.3.11 Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 \ll c$, there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 1.3.12 Let (X, d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Lemma 1.3.13 Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence in X.

Proof. For any $c \in E$ with $0 \ll c$, there is N such that for all $n, m > N$, $d(x_n, x) \ll c/2$ and $d(x_m, x) \ll c/2$. Hence $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) \ll c$. Therefore $\{x_n\}$ is a Cauchy sequence.

Lemma 1.3.14 Let (X, d) be a cone metric space, P be a normal cone with normal constant M. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.

Proof. Suppose that $\{x_n\}$ is a Cauchy sequence. For every $\varepsilon > 0$, choose $c \in E$ with $0 \ll c$ and $M \| c \| < \varepsilon$. Then there is N, for all $n, m > N, d(x_n, x_m) \ll c$.

So that when $n, m > N$, $\| d(x_n, x_m) \| \leq M \| c \| < \varepsilon$. This means $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.

Conversely, suppose that $d(x_n, x_m) \to 0$ $(n, m \to \infty)$. For $c \in E$ with $0 \ll c$, there is $\delta > 0$, such that $\| x \| < \delta$ implies $c - x \in intP$. For this δ there is N, such that for all $n, m > N$, $\| d(x_n, x_m) \| < \delta$. So $c - d(x_n, x_m) \in intP$. This means $d(x_n, x_m) \ll c$. Therefore $\{x_n\}$ is a Cauchy sequence.

Lemma 1.3.15 Let (X, d) be a cone metric space, P be a normal cone with nomal constant M. Let $\{x_n\}$ and $\{y_n\}$ be two sequence in X and $x_n \to x$, $y_n \to y$ $(n \to \infty)$. Then $d(x_n, x_m) \to d(x, y)$ $(n \to \infty)$.

Proof. For every $\varepsilon > 0$, choose $c \in E$ with $0 \ll c$ and $\| c \| < \frac{\varepsilon}{4M+2}$. From $x_n \to x$ and $y_n \to y$, there is N such that for all $n > N$, $d(x_n, x) \ll c$ and $d(y_n, y) \ll c$. We have

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y_n, y) \leq d(x, y) + 2c,$$

$$d(x, y) \leq d(x_n, x) + d(x_n, y_n) + d(y_n, y) \leq d(x_n, y_n) + 2c.$$

Hence

$$0 \leq d(x, y) + 2c - d(x_n, y_n) \leq 4c$$

and

$$\| d(x_n, y_n) - d(x, y) \| \leq \| d(x, y) + 2c - d(x_n, y_n) \| + \| 2c \| \leq (4M + 2) \| c \| < \varepsilon.$$

Therefore $d(x_n, y_n) \to d(x, y)$ $(n \to \infty)$.

Definition 1.3.16 Let (X, d) be a cone metric space. If for any sequence $\{x_n\}$ in X, there is a subsequence $\{x_{n_i}\}$ such that $\{x_{n_i}\}$ is convergent in X. Then X is called a sequentially compact cone metric space.

1.4 Some Basic Fixed Point Results

In what follows, we collect some basic definitions needed throughout this exposition.

Definition 1.4.1 Let X be a nonempty set and $T : X \to X$ be a self map. We say that $x \in X$ is a fixed point of T if $Tx = x$ and denote by $F(T)$ or $Fix(T)$, the set of all fixed points of T. In other words, a point which remains invariant under the transformation T is called a fixed point of T.

Example 1.4.2
(i) If $X = \mathbb{R}$ and $T(x) = x^2 + 5x + 4$, then $F(T) = \{-2\}$;
(ii) If $X = \mathbb{R}$ and $T(x) = x^2 - x$, then $F(T) = \{0, 2\}$;
(iii) If $X = \mathbb{R}$ and $T(x) = x + 2$, then $F(T) = \varphi$;
(iv) If $X = \mathbb{R}$ and $T(x) = x$, then $F(T) = \mathbb{R}$.

Remark 1.4.3 For a given self map the following properties obviously hold:
(1) $F(T) \subset F(T^n)$, for each $n \in \mathbb{N}$;
(2) $F(T^n) = \{x\}$, for some $n \in \mathbb{N} \Rightarrow F(T) = \{x\}$.
The converse of (1) is not true, in general, as demonstrated by the following example.

Example 1.4.4 Let $T : \{1,2,3\} \to \{1,2,3\}$, with $T(1) = 3$, $T(2) = 2$ and $T(3) = 1$. Then $F(T^2) = \{1,2,3\}$ but $F(T) = \{2\}$.

In 1912, Brouwer proved the earliest fixed point theorem which runs as follows:

Theorem 1.4.5. If
(i) K is a compact convex subset of a Euclidean space R^n and
(ii) $T : K \to K$ is a continuous function,
then T has a fixed point in K.

An immediate corollary of this theorem on the real line can be stated in the following way:

Corollary 1.4.6 Every continuous self mapping of a closed interval has a fixed point.

Most of the problems in Functional Analysis arise in function as well as sequence spaces and therefore, it is natural to ask if Brouwer theorem can be extended to these spaces. Kakutani produced an example to show that Theorem 1.2.1 cannot be extended to infinite dimensional spaces.

Example 1.4.7 Let $C = \{x \in l^2 : \|x\| \le 1\}$ be the unit ball in Hilbert space l^2. For each $x = \{x_1, x_2, x_3, ...\}$ in C, define a map $T : C \to C$ by $Tx =$

$\{\sqrt{1-\|x\|^2}, x_1, x_2, ..., x_n, ...\}$. Since $\|Tx\| = 1$, T is continuous, but T does not admit any fixed point.

Definition 1.4.8 Let (X, d) be a metric space. A mapping $T : X \to X$ is called
(1) *Lipschitzian (or L-Lipschitzian)* if there exists $L > 0$ such that

$$d(Tx, Ty) \leq L\ d(x, y), \quad \text{for all } x, y \in X;$$

(2) *strict contraction (or a-contraction)* if T is a-Lipschitzian, with $a \in [0, 1)$;
(3) *nonexpansive* if T is 1-Lipschitzian;
(4) *contractive* if $d(Tx, Ty) < d(x, y)$, for all $x, y \in X$, $x \neq y$;
(5) *isometry* if $d(Tx, Ty) = d(x, y)$.

Example 1.4.9
(i) $T : \mathbb{R} \to \mathbb{R}$, with $T(x) = \frac{x}{2} + 3$, $x \in \mathbb{R}$, is a strict contraction and $F(T) = \{6\}$.
(ii) If $x = \{x_n\} \in l^2$, then the mapping $T : l^2 \to l^2$ defined by $T(x) = \{\frac{x_n}{2}\}$ is a contraction mapping on l^2.
(iii) The mapping $T : [\frac{1}{2}, 2] \to [\frac{1}{2}, 2]$, defined by $T(x) = \frac{1}{x}$, is a 4-Lipschitzian with $F(T) = \{1\}$, while the functions T in Example 1.2.1 part (iii) and (iv) are all isometries.
(iv) $T : [1, \infty) \to [1, \infty)$, with $T(x) = x + \frac{1}{x}$, is contractive and $F(T) = \phi$.

The following theorem is of fundamental importance in the metrical fixed point theory which is popularly referred as *Banach contraction principle or contraction mapping principle.*

Theorem 1.4.10 If
(i) (X, d) is a complete metric space and
(ii) $T : X \to X$ is a contraction map,
then T has a unique fixed point p and $T^n(x) \to p$ (as $n \to \infty$), *for each* $x \in X$.

Remark 1.4.11 While considering Lipschitzian mappings, a natural question arises whether it is possible to weaken contraction assumption a little bit in Banach contraction principle and still obtain the existence of a fixed point. In general, the answer to this question is no. In this regard, the following interesting example is available in Khamsi and Kirk.

Example 1.4.12 Let $C[0,1]$ be the complete metric space of real valued continuous functions defined on $[0,1]$ with respect to supremum metric and consider the closed subspace Z of $C[0,1]$ consisting of those functions $f \in C[0,1]$ satisfying $f(1) = 1$. Since Z is a closed subspace of $C[0,1]$, therefore Z is also complete. Now, define $T : Z \to Z$ by $Tf(t) = tf(t)$ for all $t \in [0,1]$. Then we can say $d(Tf, Tg) < d(f, g)$ whenever $f \neq g$ but T has no fixed point as $Tf = f \Rightarrow tf = f \Rightarrow f(t) = 0$ for all $t \in [0,1)$. On the other hand, $f(1) = 1$ which contradicts the continuity of T and so T cannot have a fixed point in Z.

In 1930, Schauder extended Brouwer's result to infinite dimensional spaces, which runs as follows:

Theorem 1.4.13 If
(i) K is a compact convex subset of a Banach space E and
(ii) $T : K \to K$ is a continuous function,
then T has at least one fixed point.

Schauder also proved a theorem for a compact map which is known as second form of Schauder fixed point theorem. For this we need the following definition.

Definition 1.4.14 A self mapping T of a Banach space E is called completely continuous compact map if T is continuous and T maps bounded set to precompact set.

Remark 1.4.15 A compact map is always continuous but converse need not be true. For example, an identity function defined on an infinite dimensional normed space is continuous but not compact.

Here we represent the another form of Schauder fixed point theorem.

Theorem 1.4.16 If
(i) K is a bounded closed convex subset of a Banach space E and
(ii) $T : K \to K$ is a compact map,
then T has at least one fixed point.

CHAPTER 2
FIXED POINT THEOREMS OF CONTRACTIVE MAPPINGS

2.1 Introduction

Fixed point theory is a rich, interesting and highly applied branch of mathematics. The classical fixed point theorems are utilized very effectively in the existence theories of differential equations, integral equations, functional equations, partial differential equations, random differential equations and other related areas. By a fixed point theorem, we shall understand a statement which asserts that under what conditions a mapping T of a set X admits one or more point x of X such that $Tx = x$. Indeed, the most significant result of fixed point theory was given by the Polish mathematician Stefan Banach in 1922 which is popularly referred as *classical Banach Contraction Principle.*

The fixed point theory has got its origin in Brouwer wherein he proved his pioneer theorem (to be stated later) which laid the foundation of Topological Fixed Point Theory. The fixed point theory for nonexpansive mappings defined on Banach spaces was initiated by Browder and Kirk.

The study of fixed points of functions satisfying certain contractive conditions has been at the center of vigorous research activity and it has a wide range of applications in different areas such as nonlinear and adaptive control systems, fractal image decoding, and convergence of recurrent networks.

In 2007 Huang and Zhang [25] generalized the classical notion of metric space by replacing the real numbers with an ordering Banach space and term the new notion as cone metric space. They proved some fixed point theorems of contractive mappings on cone metric spaces which form the subject material for this dissertation

2.2 Fixed Point Theorems

In this section we shall give some fixed point theorems of contractive mappings.

Theorem 2.2.1 Let (X, d) be a complete cone metric space, P be a normal cone with normal constant M. Suppose the mapping $T : X \to X$ satisfies the contractive condition

$$d(Tx, Ty) \leq kd(x, y), \ \textit{for all} \ x, y \in X$$

where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$ iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Choose $x_0 \in X$. Set

$$x_1 = Tx_0, x_2 = Tx_1 = T^2 x_0, ..., x_{n+1} = Tx_n = T^{n+1} x_0......$$

We have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \leq kd(x_n, x_{n-1}) \\ &\leq k^2 d(x_{n-1}, x_{n-2}) \leq ... \leq k^n d(x_1, x_0) \end{aligned}$$

So for $n > m$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m) \\ &\leq (k^{n-1} + k^{n-2} + ... + k^m) d(x_1, x_0) \leq \tfrac{k^m}{1-k} d(x_1, x_0) \end{aligned}$$

We get $\| d(x_n, x_m) \| \leq \frac{k^m}{1-k} M \| d(x_1, x_0) \|$. This implies $d(x_n, x_m) \to 0 (n, m \to \infty)$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $x^* \in X$ such that $x_n \to x^* (n \to \infty)$. Since

$$d(Tx^*, x^*) \leq d(Tx_n, Tx_*) + d(Tx_n, x^*) \leq kd(x_n, x^*) + d(x_{n+1}, x^*),$$

$$\| d(Tx^*, x^*) \| \leq M(k \| d(x_n, x^*) \| + \| d(x_{n+1}, x^*) \|) \to 0.$$

Hence $\| d(Tx^*, x^*) \| = 0$. This implies $Tx^* = x^*$. So x^* is a fixed point of T. Now if y^* is another fixed point of T, then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq kd(x^*, y^*).$$

Hence $\| d(x^*, y^*) \| = 0$ and $x^* = y^*$. Therefore the fixed point of T is unique.

Corollary 2.2.2 Let (X, d) be a complete cone metric space, P a normal cone with normal constant M. For $c \in E$ with $0 \ll c$ and $x_0 \in X$, set $B(x_0, c) = \{x \in X \mid d(x_0, x) \leq c\}$. Suppose the mapping $T : X \to X$ satisfies the contractive condition

$$d(Tx, Ty) \leq kd(x, y), \ \textit{for all} \ x, y \in B(x_0, c),$$

where $k \in [0,1)$ is a constant and $d(Tx_0, x_0) \leq (1-k)c$. Then T has a unique fixed point in $B(x_0, c)$.

Corollary 2.2.3 Let (X,d) be a complete cone metric space, P a normal cone with normal constant M. Suppose a mapping $T : X \to X$ satisfies for some positive integer n,

$$d(T^n x, T^n y) \leq k d(x,y), \ for \ all \ x, y \in X$$

where $k \in [0,1)$ is a constant. Then T has a unique fixed point in X.

Theorem 2.2.4 Let (X,d) be a sequentially compact cone metric space, P be a regular cone. Suppose the mapping $T : X \to X$ satisfies the contractive condition

$$d(Tx, Ty) < d(x,y), \ for \ all \ x, y \in X, x \neq y.$$

Then T has a unique fixed point in X.

Theorem 2.2.5 Let (X,d) be a complete cone metric space, P a normal cone with normal constant M. Suppose the mapping $T : X \to X$ satisfies the contractive condition

$$d(Tx, Ty) \leq k(d(Tx, x) + d(Ty, y)), \ for \ all \ x, y \in X$$

where $k \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X and for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Theorem 2.2.6 Let (X,d) be a complete metric space, P be a normal cone with normal constant M. Suppose the mapping $T : X \to X$ satisfies the contractive condition

$$d(Tx, Ty) \leq k(d(Tx, y) + d(Ty, x)), \ for \ all \ x, y \in X,$$

where $k \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Remark 2.2.7 Theorems 2.2.1-2.2.6 generalize the fixed point theorems of contractive mappings in metric space to cone metric space.

We conclude with an example.

Let $E = R^2$, the Euclidean plane, and $P = \{(x, y) \in R^2 \mid x.y \geq 0\}$ be a normal cone in E. Let $X = \{(x, 0) \in R^2 \mid 0 \leq x \leq 1\} \cup \{(0, x) \in R^2 \mid 0 \leq x \leq 1\}$.The mapping $d : X \times X \to E$ is defined by

$$d((x,0),(y,0)) = (\frac{4}{3} \mid x - y \mid, \mid x - y \mid)$$

$$d((0,x),(0,y)) = (\mid x - y \mid, \frac{2}{3} \mid x - y \mid)$$

$$d((x,0),(0,y)) = d((0,y),(x,0)) = (\frac{4}{3}x + y, x + \frac{2}{3}y).$$

Then (X, d) is a complete cone metric space. Let the mapping $T : X \to X$ with

$$T((x,0)) = (0,x) \text{ and } T((0,x)) = (\frac{1}{2}x, 0).$$

Then T satisfies the contractive condition

$$d(T((x_1,x_2)), T((y_1,y_2))) \leq kd((x_1,x_2),(y_1,y_2)), \ for\ all\ (x_1,x_2),(y_1,y_2) \in X,$$

with constant $k = \frac{3}{4} \in [0, 1)$. It is obvious that T has a unique fixed point $(0, 0) \in X$. On the other hand, we see that T is not a contractive mapping in the Euclidean metric on X.

In 2008 Sh. Rezapour and R. Hamlbarani [44] proved that there are no normal cones with normal constant $M < 1$ and for each $k > 1$ there are cones with normal constant $M > k$. Also by providing non-normal cones and omitting the assumption of normality in some results of [25], we obtain generalizations of the results.

Lemma 2.2.8 There is not normal cone with normal constant $M < 1$.

Proof. Let (X, d) be a cone metric space and P be a normal cone with normal constant $M < 1$. Choose a non-zero element $x \in P$ and $0 < \varepsilon < 1$ such that $M < 1 - \varepsilon$. Then, $(1 - \varepsilon)x \leq x$, but $(1 - \varepsilon)\|x\| > M\|x\|$. This is a contradiction.

Proposition 2.2.9 For each $k > 1$, there is a normal cone with normal constant $M > k$.

Proof. Let $k > 1$ be given. Consider the real vector space

$$E = \{ax + b | a, b \in \mathbb{R}; x \in [1 - \frac{1}{k}, 1]\}$$

with supremum norm and the cone

$$P = \{ax + b \in E | a \leq 0, b \geq 0\}$$

in E. First, we show that P is regular (and so normal). Let $\{a_n x + b_n\}_{n\geq 1}$ be an increasing sequence which is bounded from above, that is, there is an element $cx + d \in E$ such that

$$a_1 x + b_1 \leq a_2 x + b_2 \leq ..... \leq a_n x + b_n \leq .... \leq cx + d,$$

for all $x \in [1 - \frac{1}{k}, 1]$. Then, $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are two sequences in $\mathbb{R}$ such that

$$b_1 \leq b_2 \leq ... \leq d, \quad a_1 \geq a_2 \geq ... \geq c.$$

Thus, $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are convergent. Let $a_n \to a$ and $b_n \to b$. Then, $ax + b \in P$ and $a_n x + b_n \to ax + b$. Therefore P is regular. Hence by Lemma 2.2.8, there is $M \geq 1$ such that $0 \leq g \leq f$ implies $\| g \| \leq M \| f \|$, for all $g, f \in E$. Now we show that $M > k$. First note that $f(x) = -kx + k \in P$, $g(x) = k \in P$ and $f - g \in P$. So, $0 \leq g \leq f$. Therefore, $k = \| g \| \leq M \| f \| = M$. On the other hand, if we consider $f(x) = -(k + \frac{1}{k})x + k$ and $g(x) = k$, then $f \in P, g \in P$ and $f - g \in P$. Also, $\| g \| = k$ and $\| f \| = 1 - \frac{1}{k} + \frac{1}{k^2}$. Thus, $k = \| g \| > k \| f \| = k + \frac{1}{k} - 1$. This shows that $M > k$.

Theorem 2.2.10 Let (X, d) be a complete cone metric space and the mapping $T : X \to X$ satisfy the contractive condition

$$d(Tx, Ty) \leq kd(x, y) \; for \; all \; x, y \in X$$

where $k \in [0, 1)$ is a constant. Then, T has a unique fixed point in X. For each $x \in X$, the iterative sequence$\{T^n x\}_{n\geq 1}$ converges to the fixed point.

Corollary 2.2.11 Let (X, d) be a complete cone metric space. Suppose a mapping $T : X \to X$ satisfies for some positive integer n,

$$d(T^n x, T^n y) \leq kd(x, y),$$

for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X.

Theorem 2.2.12 Let (X, d) be a complete cone metric space and the mapping $T : X \to X$ satisfy the contractive condition

$$d(Tx, Ty) \leq k(d(Tx, x) + d(Ty, y)),$$

for all $x, y \in X$; where $k \in [0, \frac{1}{2})$ is a constant. Then, T has a unique fixed point in X. For each $x \in X$, the iterative sequence $\{T^n x\}_{n \geq 1}$ converges to the fixed point.

Theorem 2.2.13 Let (X, d) be a complete cone metric space and the mapping $T : X \to X$ satisfy the contractive condition

$$d(Tx, Ty) \leq k(d(Tx, y) + d(x, Ty)),$$

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$ is a constant. Then, T has a unique fixed point in X, the iterative sequence $\{T^n x\}_{n \geq 1}$ converges to the fixed point.

Theorem 2.2.14 Let (X, d) be a complete cone metric space and the mapping $T : X \to X$ satisfy the contractive condition

$$d(Tx, Ty) \leq kd(x, y) + ld(y, Tx),$$

for all $x, y \in X$, where $k, l \in [0, 1)$ are constants. Then, T has a unique fixed point in X. Also the fixed point of T is unique whenever $k + l < 1$.

P. Vetro [38] gave the fixed point results by using the notion of g-weak contraction mapping in the setting of cone metric space. These results generalize some common fixed points results in metric space and some of results of Huang and Zhang [25] are in cone metric space.

Definition 2.2.15 Let (X, d) be a cone metric space, P be a normal cone with normal constant M. Let $f, g : X \to X$ be mappings, f is a g-weak contraction if

$$d(f(x), f(y)) \leq \alpha\ d(f(x), g(x)) + \beta\ d(f(y), g(y)) + \gamma\ d(g(x), g(y)),$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \in [0, 1)$ and $\alpha + \beta + \gamma < 1$.
Suppose $f(X) \subset g(X)$ and for every $x_0 \in X$, we consider the sequence $\{x_n\} \subset X$

defined by $g(x_n) = f(x_{n-1})$ for all $n \in N$, we say that $(f(x_n))$ is a f-g-sequence of initial point x_0.

Theorem 2.2.16 Let (X, d) be a cone metric space, P be a normal cone with normal constant M and let $f, g : X \to X$ be such that $f(X) \subset g(X)$. Suppose that f is a weak g-contraction such that

$$f(g(x)) = g(g(x)) \; if \; f(x) = g(x).$$

If $f(X)$ *or* $g(X)$ is a complete subspace of X, then the mappings f and g have a unique common fixed point in X. Moreover for any $x_0 \in X$, the f-g-sequence $(f(x_n))$ of initial point x_0 converges to the fixed point.

From above theorem if we choose $g = I_X$ the identity mapping on X, we obtain the following corollary:

Corollary 2.2.17 Let (X, d) be a complete cone metric space, P be a normal cone with normal constant M and let $f : X \to X$. Suppose that f satisfies the contractive condition

$$d(f(x), f(y)) \leq \alpha \; d(f(x), x) + \beta \; d(f(y), y) + \gamma \; d(x, y), \; for \; all \; x, y \in X.$$

Then the mapping f has a unique fixed point in X. Moreover for any $x_0 \in X$, the sequence $(f^n(x_0))$ converges to the fixed point.

Remark 2.2.18 We obtain Theorem 2.2.1 respectively Theorem 2.2.4 of [24] from Corollary if we choose $\alpha = \beta = 0$, respectively $\alpha = \beta$ and $\gamma = 0$.

Corollary 2.2.19 Let (X, d) be a cone metric space, P be a normal cone with normal constant M and let $f, g : X \to X$ be such that $f(X) \subset g(X)$. Suppose that f is a g-weak contraction and that the condition

$$d(f(g(x)), g(g(x))) \leq Kd(f(x), g(x))$$

hold for each $x \in X$, where K is a positive constant. If $f(X)$ *or* $g(X)$ is a complete subspace of X, then the mappings f and g have a unique common fixed point in X.

Moreover for $x_0 \in X$, the f-g-sequence $(f(x_n))$ of initial point x_0 converges to the fixed point.

2.3 Common Fixed Point Theorems for Two Maps

Definition 2.3.1 Let f and g be self mappings on a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Definition 2.3.2 A pair of self-mappings (f, g) on a cone metric space (X, d) is said to be compatible if for arbitrary sequence x_n in X such that $lim_{n\to\infty} f(x_n) = lim_{n\to\infty} g(x_n) = t \in X$, and for arbitrary $c \in P$ with $c \in$int$P$, there exists $n_0 \in \mathbb{N}$ such that $d(fgx_n, gfx_n) \ll c$ whenever $n > n_0$.

Definition 2.3.3 Two self mappings f and g of a set X are said to be weakly compatible if they commute at their coincidence points; that is, if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

Proposition 2.3.4 Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

Proof. Since $w = fx = gx$ and f and g are weakly compatible, we have $fw = fgx = gfx = gw$, i.e., $fw = gw$ is a point of coincidence of f and g. But w is the only point of coincidence f and g, so $w = fw = gw$. Moreover if $z = fz = gz$, then z is a point of coincidence of f and g, and therefore $z = w$ by uniqueness. Thus w is a unique common fixed point of f and g.

Theorem 2.3.5 Let (X, d) be a cone metric space, and P a normal cone with normal constant M. Suppose mappings $f, g : X \to X$ satisfy

$$d(fx, fy) \leq kd(gx, gy), \quad for\ all\ x, y \in X,$$

where $k \in [0, 1)$ is a constant. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Example 2.3.6 Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset R^2$, $d : R \times R \to E$ such that

$$d(x, y) = (|x - y|, \alpha|x - y|),$$

where $\alpha > 0$ is a constant. Define

$$fx = \begin{cases} \frac{\alpha}{\beta+1}x, & x \neq 0 \\ \gamma, & x = 0 \end{cases}$$

and

$$gx = \begin{cases} \alpha x, & x \neq 0 \\ \gamma, & x = 0 \end{cases}$$

where $\beta \geq 1$, and $\gamma \neq 0$. It may be verified that

$$d(fx, fy) \leq kd(gx, gy), \ for \ all \ x, y \in X,$$

where $k = \frac{1}{\beta} \in (0, 1]$. Moreover f and g have a coincidence point X.

In above example f and g do not commute at the coincidence point 0, and therefore are not weakly compatible. And f and g do not have common fixed point. Thus, this example demonstrates the crucial role of weak compatibility in our results.

Theorem 2.3.7 Let (X, d) be a cone metric space and P a normal cone with normal constant M. Suppose that the mappings $f, g : X \to X$ satisfy the contractive condition

$$d(fx, fy) \leq k(d(fx, gx) + d(fy, gy)), \ for \ all \ x, y \in X,$$

where $k \in [0, \frac{1}{2})$ is a constant. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Theorem 2.3.8 Let (X, d) be a cone metric space, and P a normal cone with normal constant M. Suppose that the mappings $f, g : X \to X$ satisfy the contractive condition

$$d(fx, fy) < k(d(fx, gy) + d(fy, gx)), \; for\ all\ x, y \in X,$$

where $k \in [0, \frac{1}{2})$ is a constant. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible , f and g have a unique common fixed point.

G. Jungck, M. Abbas [26] establish the existence of coincidence points and common fixed point for pair of mappings satisfying certain contractive conditions in cone metric space. After this M. Abbas and B.E. Rhoades obtained [13] fixed point theorems for mappings without appealing to commutativity conditions, defined in a cone metric space.

Theorem 2.3.9 Let (X, d) be a complete cone metric space, and P a normal cone with normal constant M. Suppose that the mappings f and g are two self maps of X satisfying

$$d(fx, gx) \leq \alpha\ d(x, y) + \beta\ [d(x, fx) + d(y, gy)] + \gamma\ [d(x, gy) + d(y, fx)] \qquad (2.1)$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then f and g have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g, and conversely.

Corollary 2.3.10 Let (X, d) be a complete cone metric space, and P a normal cone with normal constant M. Suppose that a self map f of X satisfies

$$d(f^p x, f^q y) \leq \alpha\ d(x, y) + \beta\ [d(x, f^p x) + d(y, f^q y)] + \gamma\ [d(x, f^q y) + d(y, f^p x)] \quad (2.2)$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$, $\alpha + 2\beta + 2\gamma < 1$, and p and q are fixed positive integers. Then f has a unique fixed point in X.

Corollary 2.3.11 Let (X, d) be a complete cone metric space, and P a normal cone with normal constant M. Suppose the mapping $f : X \to X$ satisfies

$$d(fx, fy) \leq \alpha\ d(x, y) + \beta\ [d(x, fx) + d(y, fy)] + \gamma\ [d(x, fy) + (y, fx)]$$

for all $x, y \in X$ $\alpha, \beta, \gamma \geq 0$ and $\alpha+2\beta+2\gamma < 1$. Then f has a unique fixed point in X.

Corollary 2.3.12 Let (X, d) be a complete cone metric space, and P be a normal cone with normal constant M. Suppose that mapping $f : X \to X$ satisfies

$$d(fx, fy) \leq a_1 d(x, y) + a_2 d(x, fx) + a_3 d(y, fy) + a_4 d(x, fy) + a_5 d(y, fx)$$

for all $x, y \in X$, where $a_i \geq 0$ for each $i \in \{1, 2...5\}$ and $\sum\limits_{i=1}^{5} a_i < 1$. Then f has a unique fixed point in X.

Corollary 2.3.13: Let (X, d) be a complete cone metric space, and P be a normal cone with normal constant M. Suppose the mapping $f : X \to X$ satisfies

$$d(fx, fy) \leq \alpha\ d(x, y) + \beta\ [d(x, fx) + d(y, fy)]$$

for all $x, y \in X$, where $\alpha, \beta, \geq 0$ and $\alpha + 2\beta < 1$. Then f has a unique fixed point in X.

2.4 On Periodic Point Theorems

It is obvious that if f is a map which has a fixed point p, then p is also a fixed point of f^n for every natural number n. However the converse is false. For example, consider $X = [0, 1]$, and f defined by $fx = 1 - x$. Then f has a unique fixed point at $\frac{1}{2}$, but every even iterate of f is the identity map, which has every point of $[0, 1]$ as a fixed point. On the other hand, if $X = [0, \pi], fx = cos\ x$, then every iterate of f has the same fixed point as f. If a map satisfies $F(f) = F(f^n)$ for each $n \in \mathbb{N}$, where $F(f)$ denotes a set of all fixed points of f, then it is said to have property P [17]. We shall say that f and g have property Q if $F(f) \cap F(g) = F(f^n) \cap F(g^n)$.

Theorem 2.4.1 Let f be a self-map of a cone metric space (X, d), and P a normal cone with normal constant M, satisfying

i)$d(fx, f^2x) \leq \lambda d(x, fx)$, for all $x \in X$, where $0 \leq \lambda < 1$
or (ii) with strict inequality, $\lambda = 1$ and for all $x \in X, x \neq fx$. If $F(f) \neq \phi$, then f has property P.

Proof. We shall always assume that $n > 1$, since the statement for $n = 1$ is trivial. Let $u \in F(f^n)$. Suppose that f satisfies (i). Then

$$\begin{aligned} d(u, fu) &= d(f(f^{n-1}u), f^2(f^{n-1}u)) \leq \lambda d(f^{n-1}u, f^n u) \\ &\leq \lambda^2 d(f^{n-2}u, f^{n-1}u) \leq .... \leq \lambda^n d(u, fu) \end{aligned}$$

Then by normality of cone metric space

$$\| d(u, fu) \| \leq \lambda^n M \| d(u, fu) \| .$$

Now the right hand side of the above inequality approaches zero as $n \to \infty$. Hence $\| d(u, fu) \| = 0$, and $u = fu$. Suppose that f satisfies (ii). If $fu = u$, then there is nothing to prove. Suppose, $fu \neq u$. Then a repetition of the argument for case (i) leads to

$$d(u, fu) < d(u, fu)$$

a contradiction. Therefore, in all cases, $u = fu$ and $F(f^n) = F(f)$.

Theorem 2.4.2 Let (X, d) be a complete cone metric space, and P be a normal cone with normal constant M. Suppose the mappings $f, g : X \to X$ satisfy (2.1). Then f and g have property Q.

Proof. From Theorem 2.3.9 f and g have a common fixed point in X. Let $u \in F(f^n) \cap F(g^n)$. Now,

$$\begin{aligned} d(u, gu) &= d(f(f^{n-1}u), g(g^n u)) \\ &\leq \alpha d(f^{n-1}u, f^n u) + \beta[d(f^{n-1}u, f^n u) + d(f^n u, g^{n+1}u)] \\ &\quad + \gamma[d(f^{n-1}u, g^{n+1}u) + d(g^n u, f^n u)] \\ &\leq \alpha d(f^{n-1}u, u) + \beta[d(f^{n-1}u, u) + d(u, gu)] + \gamma[d(f^{n-1}u, u) + d(u, gu)], \end{aligned}$$

which further implies that

$$d(u, gu) \leq \delta d(f^{n-1}u, u)$$

where $\delta = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} < 1$, and we have

$$\begin{aligned} d(u, gu) &= d(f^n u, g^{n+1}u) \leq \delta d(f^{n-1}u, u) \\ &\leq \ ... \leq \delta^n d(u, fu). \end{aligned}$$

Then by normality of cone metric space

$$\| d(u, gu) \| \leq \delta^n M \| d(u, fu) \| .$$

Now the right hand side of the above inequality approaches zero as $n \to \infty$. Hence $\| d(u, gu) \| = 0$ and $u = gu$, which, from Theorem 2.3.9 implies that $u = fu$.

Theorem 2.4.3 Let (X, d) be a complete cone metric space, and P a normal cone with normal constant M. Suppose that the mapping $f : X \to X$ satisfies (2.2). Then f has property P.

Proof. From Corollary 2.3.11, f has a unique fixed point. Let $u \in F(f^n)$. Now,

$$\begin{aligned} d(u, fu) &= d(f(f^{n-1}u), f(f^n u)) \\ &\leq \alpha d(f^{n-1}u, f^n u) + \beta[d(f^{n-1}u, f^n u) + d(f^n u, f^{n+1}u)] \\ &\quad + \gamma[d(f^{n-1}u, f^{n+1}u) + d(f^{n-1}u, f^n u)] \\ &\leq \alpha d(f^{n-1}u, u) + \beta[d(f^{n-1}u, u) + d(u, fu)] + \gamma[d(f^{n-1}u, u) + d(u, fu)], \end{aligned}$$

which further implies that

$$d(u, fu) \leq \delta d(f^{n-1}u, u),$$

where $\delta = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} < 1$, and we have

$$\begin{aligned} d(u, fu) &= d(f^n u, f^{n+1}u) \leq \delta d(f^{n-1}u, f^n u) \\ &\leq \; ..... \leq \delta^n d(u, fu). \end{aligned}$$

From (1.1),

$$\| d(u, fu) \| \leq \delta^n M \| d(u, fu) \| .$$

Now the right hand side of the above inequality approaches to zero as $n \to \infty$. Hence $\| d(u, fu) \| = 0$ and $u = fu$.

In 2009 K. Jha [24] proved a common fixed point theorem for a pair of weakly compatible mappings in a cone metric space, without exploiting the notion of the continuity which can be stated as follows.

Theorem 2.4.4 Let (X, d) be a cone metric space, and P be a normal cone with normal constant M. Suppose the mappings $f, g : X \to X$ satisfy the contractive condition

$$d(fx, fy) \leq r[d(fx, gy) + d(fy, gx) + d(fy, gy)]$$

where $r \in [0, \frac{1}{4})$ is a constant. If the range of g contains the range of f and $g(X)$ is complete subspace of X, then f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Now we give an example to illustrate the above Theorem.

Example 2.4.5 Let $E = I^2$, for $I = [0, 1]$, $P = \{(x, y) \in E : x, y \geq 0\} \subset I^2$, $d : I \times I \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha > 0$ is a constant. Define $fx = \frac{\alpha x}{1+\alpha x}$, for all $x \in I$ and $gx = \alpha x$ for all $x \in I$. Then, for $\alpha = 1$ both the mappings f and g are weakly compatible and satisfy all the conditions of the above theorem with $x = 0$ as a unique common fixed point.

Remark 2.4.6 The above theorem extends the results of Abbas and Jungck [37]. Also it improves the results of Huang and Zhang [24].

Theorem 2.4.7 Let (X, d) be a cone metric space and let $a_i \geq 0$ $(i = 1, 2, 3, 4, 5)$ be constants with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. Suppose that the mappings $f, g : X \to X$ satisfy the condition

$$d(fx, fy) \leq a_1 d(gx, gy) + a_2 d(fx, gx) + a_3 d(fy, gy) + a_4 d(gx, fy) + a_5 d(fx, gy)$$

for all $x, y \in X$. If the range of g contains the range of f and $g(X)$ is a complete subspace, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible then f and g have a unique fixed point.

Remark 2.4.8 Obviously, Theorem 2.3.5 in [37] is a special case of Theorem 2.4.7 with $a_2 = a_3 = a_4 = a_5 = 0, a_1 = k$, and P is a normal cone.
Theorem 2.3.7 in [37] is a special case of Theorem 2.4.7 with $a_1 = a_4 = a_5 = 0, a_2 = a_3 = k$, and P a normal cone.

In Theorem 2.4.7, if $g = I_X$ is the identity map on X, and X is a complete cone metric space, then, as an immediate consequence of Theorem 2.4.7 we obtain the following result.

Corollary 2.4.9 Let (X, d) be a complete cone metric space and let $a_i \geq 0$ $(i = 1, 2, 3, 4, 5)$ be constants with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. Suppose that the mapping $f : X \to X$ satisfies the condition

$d(fx, fy) \leq a_1 d(x, y) + a_2 d(x, fx) + a_3 d(y, fy) + a_4 d(x, fy) + a_5 d(y, fx),$ *for all* $x, y \in X$

Then f has a unique fixed point x^* in X, and for any $x_0 \in X$, the successive iterates

$$x_n = fx_{n-1}, \quad (n = 1, 2, 3....)$$

converges to x^*.

Remark 2.4.10 Obviously, Theorem 2.2.1 in [24] is a special case of Corollary 2.4.9 with $a_2 = a_3 = a_4 = a_5 = 0, a_1 = k$, and P is a normal cone.
Theorem 2.2.4 in [24] is a special case of Corollary 2.4.9 with $a_1 = a_4 = a_5 = 0, a_2 = a_3 = k$, and P is a normal cone.
Theorem 2.2.5 in [24] is a special case of Corollary 2.4.9 with $a_1 = a_2 = a_3 = 0, a_4 = a_5 = k$, and P is a normal cone.
Therefore, our Corollary 2.4.9 has generalized and unified the mains results of Huang and Zhang in [24].

2.5 Fixed Point Theorems for Sequence of Mappings

In 2010 Xianjiu Huang *et. al.* [56] proved common fixed point theorems for a sequence of mappings in cone metric spaces. These theorems generalize the results of Huang and Zhang [25].

Theorem 2.5.1 Let (X, d) be a complete cone metric space. P be a normal cone with normal constant M. Suppose the sequence of mappings $\{T_n\} : X \to X$ satisfy for some positive integer m

$$d(T_i^m x, T_j^m y) \leq a_{i,j} d(x, y) \text{ } for \text{ } all \text{ } i, j = 1, 2, ..., x, y \in X,$$

where $a_{i,j}$ and k are constants with $0 < a_{i,j} < k < 1$. Then the sequence $\{T_n\}_n$ has a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X and $x_1 = T_1^m x_0,\ x_2 = T_2^m x_1, \ldots.$ Then for all $p > 0$

$$d(x_1, x_2) = d(T_1^m x_0, T_2^m x_1) \leq a_{1,2} d(x_0, x_1)$$

$$d(x_2, x_3) = d(T_2^m x_1, T_3^m x_2) \leq a_{2,3} d(x_1, x_2) \leq a_{1,2} a_{2,3} d(x_0, x_1)$$

and so on. By induction we have

$$d(x_n, x_{n+1}) \leq \prod_{i=1}^{n} a_{i,i+1} d(x_0, x_1).$$

So for $n > m$ we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \ldots. + d(x_{m+1}, x_m) \\ &\leq \textstyle\prod_{i=1}^{n-1} a_{i,i+1} d(x_0, x_1) + \ldots.. + \prod_{i=1}^{m} a_{i,i+1} d(x_0, x_1) \\ &\leq (k^{n-1} + \ldots + k^m) d(x_0, x_1) \leq \tfrac{k^m}{1-k} d(x_0, x_1). \end{aligned}$$

We get $\| d(x_n, x_m) \| \leq \frac{k^m}{1-k} M \| d(x_0, x_1) \|$. This implies that $d(x_n, x_m) \to 0$ $(n, m \to \infty)$. Hence x_n is a Cauchy sequence by Lemma 1.1. By the completeness of X, there is $x^* \in X$ such that $x_n \to x^*(n \to \infty)$. Now, we prove that x^* is a periodic point of T_i.

Thus for some $m \in \mathbb{N}$, we have

$$\begin{aligned} d(x^*, T_i^m x^*) &\leq d(x^*, x_n) + d(x_n, T_i^m x^*) \\ &= d(x^*, x_n) + d(T_n^m x_{n-1}, T_i^m x^*) \\ &\leq d(x^*, x_n) + a_{n,i} d(x_{n-1}) \\ &\leq d(x^*, x_n) + k d(x_{n-1}, x^*). \end{aligned}$$

Thus, $\| d(x^*, T_i^m x^*) \| \leq M(\| d(x^*, x_n) \| + k \| d(x_{n-1}, x^*) \|) \to 0$.
Hence $\| d(x^*, T_i^m x^*) \| = 0$. This implies $x^* = T_i^m x^*$. So, x^* is a periodic point of T_i.
Now, if y^* is another periodic point of T_i, then

$$d(x^*, y^*) = d(T_i^m(x^*), T_j^m(y^*)) \leq a_{i,j} d(x^*, y^*) \leq k d(x^*, y^*).$$

Hence $\| d(x^*, y^*) \| = 0$ and $x^* = y^*$, that is, x^* is a unique periodic point of T_i.
Also,

$$T_i x^* = T_i(T_i^m x^*) = T_i^m(T_i x^*),$$

that is, T_ix^* is also a periodic point of T_i. Therefore, $x^* = T_ix^*$, that is, x^* is a unique common fixed point of the sequence $\{T_n\}_n$.

Example 2.5.2 Let $E = \mathbb{R}^2$, the Euclidean plane, and $P = \{(x, y) \in \mathbb{R}^2 | x, y \geq 0\}$ be a normal cone in E. Let $X = \{(x, 0) \in R^2 | x \geq 0\} \bigcup \{(0, x) \in R^2 | x \geq 0\}$. The mapping $d : X \times X \to E$ is defined by

$$d((x,0),(y,0)) = \left(\frac{4}{3}|x-y|, |x-y|\right),$$

$$d((0,x),(0,y)) = \left(|x-y|, \frac{2}{3}|x-y|\right),$$

$$d((x,0),(0,y)) = d((0,y),(x,0)) = \left(\frac{4}{3}x + y, x + \frac{2}{3}y\right)$$

Then (X, d) is a complete cone metric space.
Let sequence of mappings $T_n : X \to X$ with

$$T_n((x,0)) = (0, 2^n x) \ and \ T_n((0,x)) = \left(\frac{1}{2^{n+1}}x, 0\right).$$

For $m = 2, a_{i,j} = \frac{3}{4}$, then T satisfies the contractive condition

$$d(T_i^2((x_1,x_2)),\ T_j^2((y_1,y_2))) \leq a_{i,j} d((x_1,x_2),(y_1,y_2))$$

for all $(x_1, x_2), (y_1, y_2) \in X,\ i, j = 1, 2, ...$ with constant $a_{i,j} = \frac{3}{4} \in (0, 1)$.
Thus all conditions of the theorem are satisfied and $(0, 0)$ is a unique common fixed point of the sequence $\{T_n\}_n$.

Corollary 2.5.3 Let (X, d) be a complete cone metric space. P be a normal cone with normal constant M. Suppose the mappings $T : X \to X$ satisfies the contractive condition

$$d(Tx, Ty) \leq kd(x, y) \ for \ all \ x, y \in X,$$

where $k \in (0, 1)$ is a constant. Then T has a unique fixed point in X.

Proof. In the above theorem take $T_n = T$ for all $n = 1, 2, ...; m = 1$ and $a_{i,j} = k$ then the proof follows.

Corollary 2.5.4 Let (X, d) be a complete cone metric space. P be a normal cone with normal constant M. Suppose a mappings $T : X \to X$ satisfies for some positive integer m

$$d(T^m x, T^m y) \leq kd(x, y) \ for \ all \ x, y \in X,$$

where $k \in (0, 1)$ is a constant. Then T has a unique fixed point in X.

CHAPTER 3
FIXED POINTS OF MULTIVALUED MAPPINGS

3.1 Introduction

The first natural instance when set-valued maps occur is the inverse of a single-valued map. Kuratowski realized the importance of set-valued maps (also referred as multivalued maps or point-to-set maps or multifunctions) and devoted considerable space in his famous book on topology. Fixed point theory has a basic role in applications of many branches of mathematics, and finding the fixed point of multifunctions is a generalization of fixed point theory in a sense for usual mappings. There are many works about fixed point of contractive maps. In some works about non-convex analysis, specially in ordered normed spaces, the authors define an ordered by using a cone in a vector space (for example [29]). In this chapter we give some definitions, examples and some fixed point theorems which are taken mainly by Sh. Rezapour, R.H. Haghi [43], M. Asadi [30] and D. Wardoski [9].

3.2 Basic Definitions

Definition 3.2.1 Let X and Y be two nonempty sets. A set-valued map or multivalued map or point-to-set map or multifunction $T : X \to Y$ from X to Y is a map that associates every $x \in X$ to a subset $T(x)$ of Y, the set $T(x)$ is called the image of x under T. T is called proper if there exists at least an element $x \in X$ such that $T(x) \neq \emptyset$. In this case the set $Dom(T) = \{x \in X : T(x) \neq \emptyset\}$ is called the domain of T. Actually, a set-valued map T is characterized by its graph, the subset of $X \times Y$ defined by

$$Graph(T) = \{(x,y) : y \in T(x)\}.$$

Indeed, if A is a nonempty subset of the product space $X \times Y$, then the graph of a set valued map T is defined by

$$y \in T(x) \quad \text{if and only if } (x,y) \in A.$$

Example 3.2.2 Let $X = [0,1]$ and $Y = [0,1]$. Define a map $T : X \to Y$ by

$$Tx = \begin{cases} \left\{\frac{1}{2}x\right\}, & \text{for } x \in [0,1) \\ \left[\frac{1}{2}, 1\right], & \text{for } x = 1. \end{cases}$$

There are two distinct ways to extend the concept of continuity to the set-valued map. The concept of two kinds of semicontinuity of a set-valued map was introduced by G. Bouligand and K. Kuratowski.

Definition 3.2.3 Let X and Y be two topological spaces. Then a set-valued function $T : X \to Y$ is said to be upper(lower) semicontinuous if the inverse image of a closed(open) set is closed(open). A multivalued function is continuous if it is both upper and lower semicontinuous.
Or,
T is said to be lower semicontinuous (l.s.c) at $x \in X$ if

$$x_n \to x \Rightarrow T(x) \leq \liminf_{n\to\infty} T(x_n).$$

And upper semicontinuous (u.s.c) at $x \in X$ if

$$x_n \to x \Rightarrow T(x) \geq \limsup_{n\to\infty} T(x_n).$$

Example 3.2.4 Let $X = R$ be a metric space with usual metric. Then the map $T : X \to 2^X$ defined by

$$T(x) = \begin{cases} [-1,1] & \text{if } x = 0 \\ \{0\} & \text{if } x \neq 0, \end{cases}$$

is upper semicontinuous at zero but not lower semicontinuous at zero.

Example 3.2.5 Let $X = R$ be a metric space with usual metric. Then the map $T : X \to 2^X$ defined by

$$T(x) = \begin{cases} \{0\} & \text{if } x = 0 \\ [-1,1] & \text{if } x \neq 0, \end{cases}$$

is lower semicontinuous at zero but not upper semicontinuous at zero.

Definition 3.2.6 An element $x \in X$ is said to be an end point of a set valued map $T : X \to N(X)$, if $Tx = \{x\}$. We denote the set of all end points of T by $End(T)$.

Definition 3.2.7 An element $x \in X$ is said to be a fixed point of a set valued map $T : X \to N(X)$, if $x \in Tx$. Denote $Fix(T) = \{x \in X | x \in Tx\}$.

3.3 Results on Set-valued Contraction in Cone Metric Space

In this section we give some fixed point theorems on multifunctions for this we need the following definitions and lemmas.

Definition 3.3.1 Let (X, d) be a cone metric space and $B \subseteq X$.

(i) $b \in B$ is called an interior point of B whenever there is $0 \ll p$ such that

$$N(b,p) \subseteq B$$

where $N(b,p) = \{y \in X : d(y,b) \ll p\}$.

(ii) A subset $A \subseteq X$ is called open if each element of A is an interior point of A.

The family $\beta = \{N(x, e) : x \in X, 0 \ll e\}$ is a sub-basis for a topology on X. We denote this cone topology by τ_c.

Lemma 3.3.2 Let (X, d) be a cone metric space, P be a normal cone with normal constant $M = 1$ and A be a compact set in (X, τ_c). Then for every $x \in X$ there exists $a_0 \in A$ such that

$$\| d(x, a_0) \| = \inf_{a \in A} \| d(x, a) \| .$$

Proof. Let $x \in X$ be given. Define $f_x : X \to [0, \infty)$ by $f_x(y) = \| d(x, y) \|$. Let $y \in X$ and $\varepsilon > 0$. Choose $0 \ll c$ such that $\| c \| < \varepsilon$ and suppose that $z \in N(y, c)$. Note that, by using the normality and the relations

$$d(y, x) \leq d(y, z) + d(z, x), \quad d(z, x) \leq d(z, y) + d(y, x),$$

we have

$$\| d(y,x) \| - \| d(z,x) \| \leq \| d(y,z) \| < \varepsilon,$$

$$\| d(z,x) \| - \| d(y,x) \| \leq \| d(z,y) \| < \varepsilon.$$

Thus, $|f_x(y) - f_x(z)| < \varepsilon$. Hence, f_x is continuous on X. Because A is compact, $f_x(A)$ is a compact subset of $[0,\infty)$ and so there exists $a_0 \in A$ such that

$$\| d(x,a_0) \| = \inf_{a\in A} \| d(x,a) \| .$$

Lemma 3.3.3 Let (X,d) be a cone metric space, P be a normal cone with normal constants $M = 1$, and A, B be two compact sets in (X,τ_c). Then,

$$\sup_{x\in B} d'(x,A) < \infty,$$

where $d'(x,A) = \inf_{a\in A} \| d(x,a) \|$.

Proof. Define $g_A : X \to [0,\infty)$ by $g_A(x) = inf_{a\in A} \| d(x,a) \|$. Let $x \in X$ and $\varepsilon > 0$. Choose $0 \ll c$ such that $\| c \| < \varepsilon$ and suppose that $y \in N(x,c)$. By using the normality of cone metric space and the relation $d(x,a) \leq d(x,y) + d(y,a)$ we have

$$g_A(x) - g_A(y) \leq \| d(x,y) \| .$$

Thus, $|g_A(x) - g_A(y)| < \varepsilon$. Hence, g_A is continuous on X. Because B is compact, $g_A(B)$ is compact subset of $[0,\infty)$. This completes the proof.

Definition 3.3.4 Let (X,d) be a cone metric space, P be a normal cone with normal constants $M = 1$, $\mathcal{H}_c(X)$ be the set of all compact subsets of (X,τ_c) and $A \in \mathcal{H}_c(X)$. By using Lemma 3.3.2, we can define

$$h_A : \mathcal{H}_c(X) \to [0,\infty) \text{ and } d_H : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \to [0,\infty)$$

by

$$h_A(B) = \sup_{x\in A} d'(x,B) \text{ and } d_H(A,B) = \max\{h_A(B), h_B(A)\}$$

respectively.

Remark 3.3.5 Let (X,d) be a cone metric space with normal constant $M = 1$. Define $\rho : X \times X \to [0,\infty)$ by $\rho(x,y) = \| d(x,y) \|$. Then, (X,ρ) is a metric space.

This implies that for each $A, B \in \mathcal{H}_c(X)$ and $x, y \in X$, we have the following relations:

(i) $d'(x, A) \leq \| d(x, y) \| + d'(y, A)$,

(ii) $d'(x, A) \leq d'(x, B) + h_B(A)$,

(iii) $d'(x, A) \leq \| d(x, y) \| + d'(y, B) + h_B(A)$.

Theorem 3.3.6 Let (X, d) be a complete cone metric space with normal constant $M = 1$, and the multifunction $T : X \to \mathcal{H}_c(X)$ satisfies the relation

$$d_H(Tx, Ty) \leq c(d'(Tx, x) + d'(Ty, y))$$

for all $x, y \in X$, where $c \in (0, \frac{1}{2})$ is a constant. Then T has a fixed point.

Proof. Let $x_0 \in X$ be given. By Lemma 3.3.2, there is $x_1 \in Tx_0$ such that

$$d'(x_0, Tx_0) = \| d(x_0, x_1) \| .$$

If x_n has been given, then choose $x_{n+1} \in Tx_n$ such that $d'(x_n, Tx_n) = \| d(x_n, x_{n+1}) \|$. Thus, we have

$$\begin{aligned} \| d(x_n, x_{n+1}) \| &= d'(x_n, Tx_n) \leq h_{Tx_{n-1}}(Tx_n) \leq d_H(Tx_{n-1}, Tx_n) \\ &\leq c(d'(Tx_{n-1}, x_{n-1}) + d'(Tx_n, x_n)) \\ &= c(\| d(x_n, x_{n-1}) \| + \| d(x_n, x_{n+1}) \|), \end{aligned}$$

for all $n \geq 1$. Hence

$$\| d(x_n, x_{n+1}) \| \leq \frac{c}{1-c} \| d(x_n, x_{n-1}) \|$$

for all $n \geq 1$. Put $s = \frac{c}{1-c}$. Then, for $n > m$ we have

$$\begin{aligned} \| d(x_n, x_m) \| &\leq \sum_{i=m+1}^{n} \| d(x_i, x_{i-1}) \| \\ &\leq (s^{n-1} + \ldots + s^m) \| d(x_0, x_1) \| \\ &\leq \frac{s^m}{1-s} \| d(x_0, x_1) \| . \end{aligned}$$

This implies that

$$\lim_{m,n\to\infty} \| d(x_n, x_m) \| = 0.$$

Then $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in X. Thus, there exists $x^* \in X$ such that $x_n \to x^*$. Now by using Remark 3.3.5, we have

$$d'(x^*, Tx^*) \leq d'(x^*, Tx_n) + h_{Tx_n}(Tx^*) \leq d'(x^*, Tx_n) + d_H(Tx_n, Tx^*)$$

$$\leq \| d(x^*, x_{n+1}) \| + c(d'(Tx_n, x_n) + d'(Tx^*, x^*))$$

for all $n \geq 1$. Hence

$$d'(x^*, Tx^*) \leq \frac{c}{1-c} d'(Tx_n, x_n) + \frac{1}{1-c} \| d(x^*, x_{n+1}) \|$$

$$= \frac{c}{1-c} \| d(x_{n+1}, x_n) \| + \frac{1}{1-c} \| d(x^*, x_{n+1}) \|,$$

for all $n \geq 1$. Therefore, $d'(x^*, Tx^*) = 0$. By Lemma 3.3.2, $x^* \in Tx^*$.

Example 3.3.7 Let $X = \{a_1, a_2, a_3, \ldots\}$ be a countable set, $E = (l^2, \| \ . \ \|_2)$ and $P = \{\{x_n\}_{n\geq 1} \in l^2 : x_n \geq 0\ (\forall n \geq 1)\}$. Put $x_i = \{\frac{3^i}{n}\}_{n\geq 1}$ for all $i \geq 1$ and note that $x_i \in l^2$ $(i \geq 1)$. Define the map $d : X \times X \to P$ by

$$d(a_i, a_j) = |x_i - x_j| = \left\{\frac{|3^i - 3^j|}{n}\right\}_{n\geq 1}.$$

It is easy to see that (X, d) is a cone metric space, the normal constant of P is $M = 1$, and there is no Cauchy sequence in (X, d). Hence, (X, d) is a complete cone metric space. Now, define the multifunction

$$T : X \to \mathcal{H}_c(X)$$

by $Ta_1 = \{a_1\}$ and $Ta_i = \{a_1, a_2, \ldots a_{i-1}\}$ for all $i \geq 1$. Then, we have $Ta_1 = Ta_2$ and for each $i \geq 3$

$$d_H(Ta_1, Ta_i) = max\{d'(a_1, Ta_i)\}, \sup_{b\in Ta_i}\{d'(Ta_1, b)\} = \| d(a_1, a_{i-1}) \|_2$$

$$= \| x_1 - x_{i-1} \|_2 = \left\{\sum_{n=1}^{\infty}\left|\frac{3}{n} - \frac{3^{i-1}}{n}\right|^2\right\}^{\frac{1}{2}} = (3^{i-1} - 3)\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Also, we have $d'(a_i, Ta_i) = 0$ and

$$d'(a_i, Ta_1) = \| d(a_i, a_1) \|_2 \quad = \| x_i - x_1 \|_2 \quad = (3^i - 3) \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Thus $d_H(Ta_1, Ta_i) \leq \frac{1}{3}(d'(a_1, Ta_i) + d'(a_i, Ta_1))$. Now, suppose that $j > i > 1$. Then, $Ta_i \subset Ta_j$ and so $\sup_{b \in Ta_i} d'(b, Ta_j) = 0$. Hence,

$$d_H(Ta_i, Ta_j) = \sup_{b \in Ta_j} d'(b, Ta_i) = \| d(a_{j-1}, a_{i-1}) \|_2$$

$$= \| x_{i-1} - x_{j-1} \|_2 \quad = (3^{j-1} 3^{i-1}) \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Also, we have $d'(a_i, Ta_j) = 0$ and

$$d'(a_j, Ta_i) = \| d(a_j, a_{i-1}) \|_2 \quad = (3^j - 3^{i-1}) \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Thus $d_H(Ta_i, Ta_j) \leq \frac{1}{3}(d'(a_i, Ta_j) + d'(a_j, Ta_i))$. Therefore, T satisfies assumptions of Theorem 3.3.6 and a_1 is the unique fixed point of T.

In 2009 D. Wardowski [9] gave endpoint and fixed point theorems for contractive set-valued maps in cone metric spaces inspired by the idea of contraction for set-valued maps in metric spaces which was initiated by Feng and Liu.

Let X be a non empty set. Denote $N(X)$ a collection of all non empty subsets of X, $C(X)$ a collection of all nonempty closed subset of X and $K(X)$ a collection of all nonempty sequentially compact subsets of X.

Let (X, d) be a cone metric space. Let $T : X \to C(X)$. For $x \in X$, we denote

$$D(x, Tx) = \{d(x, z) : z \in Tx\},$$

$$S(x, Tx) = \{u \in D(x, Tx) : \| u \| = \inf\{\| v \| : v \in D(x, Tx)\}\}.$$

Theorem 3.3.8 Let (X, d) be a complete cone metric space, P be a normal cone with normal constant M, and let $T : X \to C(X)$. Assume that a function $I : X \to \mathbb{R}$

defined by $I(x) = \inf_{y\in Tx} \| d(x,y) \|, x \in X$ is lower semicontinuous. If there exist $\lambda \in [0,1)$, $b \in (\lambda, 1]$ such that $\forall x \in X \; \exists \; y \in Tx$,

$$v \in D(y, Ty) \; \forall u \in D(x, Tx), \;\; bd(x,y) \leq u \text{ and } v \leq \lambda d(x,y) \tag{3.1}$$

then $Fix(T) \neq \phi$.

Proof. Let $x_0 \in X$ be arbitrary and fixed. Take any $u_0 \in D(x_0, Tx_0)$. From (3.1) there exist $x_1 \in Tx_0$, $u_1 \in D(x_1, Tx_1)$ such that

$$bd(x_0, x_1) \leq u_0 \text{ and } u_1 \leq \lambda d(x_0, x_1).$$

Further, for x_1 there exist $x_2 \in Tx_1$, $u_2 \in D(x_2, Tx_2)$ satisfying

$$bd(x_1, x_2) \leq u_1 \text{ and } u_2 \leq \lambda d(x_1, x_2).$$

Inductively, for x_n there exist $x_{n+1} \in Tx_n, u_{n+1} \in D(x_{n+1}, Tx_{n+1})$ such that

$$bd(x_n, x_{n+1} \leq u_n) \tag{3.2}$$

and

$$u_{n+1} \leq \lambda d(x_n, x_{n+1}). \tag{3.3}$$

By (3.2) and (3.3) we get, for any $n \in \mathbb{N}$ the following inequalities:

$$u_{n+1} \leq \lambda d(x_n, x_{n+1}) \leq \frac{\lambda}{b} u_n \leq \frac{\lambda^2}{b} d(x_{n-1}, x_n) \leq \ldots \leq \frac{\lambda^{n+1}}{b^n} d(x_0, x_1) \leq \left(\frac{\lambda}{b}\right)^{n+1} u_0.$$

From the above and from the fact the cone P is normal we obtain,

$$\| u_{n+1} \| \leq M\left(\frac{\lambda}{b}\right)^{n+1} \| u_0 \|, \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

From (3.4), we get $\| u_n \| \to 0$, where $n \to \infty$. This gives the result that $\{u_n\}$ is convergent to 0. Furthermore, from (3.2) and (3.3), for any $n \in \mathbb{N}$ we have

$$(b - \lambda) d(x_n, x_{n+1}) = bd(x_n, x_{n+1}) - \lambda d(x_n, x_{n+1}) \leq u_n - u_{n+1}. \tag{3.5}$$

Now let $m, n \in \mathbb{N}$ be such that $n < m$. By (3.5) we get the inequalities

$$d(x_m, x_n) \leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \leq \frac{1}{b-\lambda} \sum_{j=n}^{m-1} (u_j - u_{j+1}) = \frac{1}{b-\lambda}(u_n - u_m),$$

which gives

$$\| d(x_m, x_n) \| \leq \frac{M}{b-\lambda} \| u_n - u_m \| .$$

From the above and from a convergence of the sequence $\{u_n\}$ to 0, we get that $\{x_n\}$ is a Cauchy sequence. From the completeness of X there exists $x^* \in X$, such that $x_n \to x^*$, $n \to \infty$. We observe from the fact that $u_n \in D(x_n, Tx_n)$ there exists a sequence $\{z_n\}$ such that for any $n \in \mathbb{N}$, $z_n \in Tx_n$ and $u_n = d(x_n, z_n)$. From the convergence of the sequence $\{u_n\}$ and from lower semicontinuity of the function I we obtain

$$\inf_{y \in Tx^*} \| d(x^*, y) \| \leq \liminf_{n\to\infty} \inf_{y \in Tx_n} \| d(x_n, y) \| \leq \liminf_{n\to\infty} \| d(x_n, z_n) \| = 0.$$

Thus

$$\inf_{y \in Tx^*} \| d(x^*, y) \| = 0. \tag{3.6}$$

Suppose that $x^* \notin Tx^*$. From (3.6) there exists a sequence $\{y_n\} \subset Tx^*$ such that $lim_{n\to\infty} \| d(x^*, y_n) \| = 0$. For any $m, n \geq 0$ we have

$$d(y_m, y_n) \leq d(y_m, x^*) + d(x^*, y_n)$$

so,

$$\| d(y_m, y_n) \| \leq M \| d(y_m, x^*) \| + M \| d(x^*, y_n) \| .$$

Thus we get the result that $\{y_n\}$ is a Cauchy sequence. From the completeness of (X, d) there exists $y^* \in X$ such that $\{y_n\}$ is convergent to y^*. Since Tx^* is closed, we get $y^* \in Tx^*$. Now for any $n \in \mathbb{N}$ we obtain

$$d(x^*, y^*) \leq d(x^*, y_n) + d(y_n, y^*),$$

and consequently

$$\| d(x^*, y^*) \| \leq M \| d(x^*, y_n) \| + M \| d(y_n, x^*) \| .$$

That gives $x^* = y^*$, which is a contradiction. Therefore $x^* \in Tx^*$.

Theorem 3.3.9 Let (X, d) be a complete cone metric space , P be a normal cone with normal constant M, and let $T : X \to K(X)$. Assume that a function $I : X \to \mathbb{R}$ of the form $I(x) = inf_{y \in Tx} \| d(x, y) \|, x \in X$ is lower semicontinuous. The following

hold:
(i) If there exist $\lambda \in [0,1), b \in (\lambda, 1]$ such that

$$\forall x \in X \ \exists \ y \in Tx, \ \ v \in S(y,Ty) \ \forall u \in S(x,Tx) \ \ bd(x,y) \leq u \ \text{ and } \ v \leq \lambda d(x,y)$$

then $Fix(T) \neq \phi$.
(ii) If there exists $\lambda \in [0,1), b \in (\lambda, 1]$ such that

$$\forall x \in X, \ \ y \in Tx \ \exists \ v \in S(y,Ty) \ \forall u \in S(x,Tx) \ \ bd(x,y) \leq u \ \text{ and } \ v \leq \lambda d(x,y), \tag{3.7}$$

then $Fix(T) = End(T) \neq \phi$.

Proof. (i) From Theorem 3.3.8 and due to the fact $S(x,Tx) \subset D(x,Tx)$ for all $x \in X$, we only need to prove that $S(x,Tx) \neq \phi$ for all $x \in X$.
Let $x \in X$ be arbitrary and fixed. Denote $c = inf_{z\in Tx} \parallel d(x,z) \parallel$ and suppose that there is not any $z \in Tx$ such that $\parallel d(x,z) \parallel = c$. Let $\{z_n\} \subset Tx$ be a sequence such that a sequence $c_n = \parallel d(x,z_n) \parallel$ is convergent to c. Since Tx is sequentially compact, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to z_0 \in Tx, k \to \infty$. Furthermore, we obtain for all $k \in \mathbb{N}$ the following inequalities:

$$c < \parallel d(x,z_0) \parallel \leq \parallel d(x,z_0) - d(x,z_{n_k}) \parallel + \parallel d(x,z_{n_k}) \parallel .$$

From the above, we obtain $c < \parallel d(x,z_0) \parallel \leq c$, which is a contradiction. So, $S(x,Tx) \neq \phi$.
(ii) In order to show that $End(T) \neq \phi$, let us first observe that from Theorem 3.3.9 (i) we get the existence of x^* such that $x^* \in Tx^*$. Taking any $y \in Tx^*$, we get that for all $u \in S(x^*,Tx^*), bd(x^*,y) \leq u$. Since $x^* \in Tx^*$ we get $0 \in S(x^*,Tx^*)$ and hence $bd(x^*,y) \leq 0$, which gives $x^* = y$. Thus we get $Tx^* = \{x^*\}$.

Theorem 3.3.8 for single-valued maps reduces to the following:

Theorem 3.3.10 Let (X,d) be a complete cone metric space, P be a normal cone with normal constant M, and let $T : X \to X$. Assume that a function $I : X \to \mathbb{R}$ such that $I(x) = \parallel d(x,Tx) \parallel, x \in X$ is lower semicontinuous. If there exists $\lambda \in [0,1)$ such that

$$\forall x \in X \ \ d(Tx,T^2x) \leq \lambda d(x,Tx),$$

then $Fix(T) \neq \phi$.
Firstly, we illustrate Theorem 3.3.8

Example 3.3.11 Let $X = [0,1]$, $E = \mathbb{R}^2$ be a Banach space with the maximum norm, $P = \{(x,y) \in E : x, y \geq 0\}$ be a normal cone and let $d : X \times X \to E$ be of the form $d(x,y) = (|x-y|, \beta|x-y|), \beta \in (0,1)$. Then the pair (X,d) is a complete cone metric space.
Define the map $T : X \to C(X)$ by

$$Tx = \begin{cases} \{\frac{1}{2}x\}, & \text{for } x \in [0,1) \\ [\frac{1}{2}, 1], & \text{for } x = 1. \end{cases}$$

Since

$$I(x) = \inf_{y \in Tx} \| d(x,y) \| = \begin{cases} \frac{1}{2}x, & \text{for } x \in [0,1) \\ 0, & \text{for } x = 1, \end{cases}$$

the map I is lower semicontinuous. Moreover, for any $x \in [0,1)$ and $y = \frac{1}{2x}$, we have

$$D(x,Tx) = \{d(x, 1/2x)\} = \{(1/2x, \beta 1/2x)\}$$

and

$$D(y,Ty) = \{d(1/2x, 1/4x)\} = \{1/4x, \beta 1/4x\}.$$

Now, taking $\lambda = 1/2$ and $b = 1$ we get

$$bd(x,y) \leq u \quad \text{for each } u \in D(x,Tx)$$

and

$$v \leq \lambda d(x,y), \qquad \text{for } v = (1/4x, \beta 1/4x).$$

In the case $x = 1$, condition (3.1) is satisfied as well. Indeed, putting $y = 1$ we get

$$bd(x,y) = 0 \leq u \quad \text{for any } u \in D(x,Tx) \textit{ and}$$

$$v \leq \lambda d(x,y), \quad \text{for } v = 0 \in D(y,Ty).$$

Therefore all the assumptions of Theorem 3.3.8 are satisfied and also

$$Fix(T) = \{0,1\} \neq \phi.$$

In 2009 M. Asadi *et. al.* [30] introduced a new order on the subsets of cone metric spaces and gave some fixed point theorems for contractive set-valued maps, omit the assumption of normality.

Definition 3.3.12 Let A and B are subsets of E, we write $A \leq B$ if and only if there exist $x \in A$ such that for all $y \in B$, $x \leq y$. Also for $x \in E$, we write $x \leq B$ if and only if $\{x\} \leq B$ and similarly $A \leq x$ if and only if $A \leq \{x\}$.

Firstly we prove the closedness of $Fix(T)$ without the assumption of normality.

Lemma 3.3.13 Let (X, d) be a complete cone metric space and $T : X \to C(X)$. If the function $f(x) = inf_{y \in Tx} \parallel d(x,y) \parallel$ for $x \in X$ is lower semicontinuous, then $Fix(T)$ is closed.

Proof. Let $x_n \in Tx_n$ and $x_n \to x$. We show that $x \in Tx$. Since

$$f(x) \leq \liminf_{n \to \infty} f(x_n) = \liminf_{n \to \infty} \inf_{y \in Tx_n} \parallel d(x_n, y) \parallel.$$

$$\leq \liminf_{n \to \infty} \parallel d(x_n, x_n) \parallel = 0.$$

So $f(x) = 0$ which implies $d(y_n, x) \to 0$ for some $y_n \in Tx$. Let $c \in E$ with $c \gg 0$. Then there exists N such that for $n \geq N$, $d(y_n, x) \ll (1/2)c$. Now, for $n > m$, we have,

$$d(y_n, y_m) \leq d(y_n, x) + d(x, y_m) \ll \frac{1}{2}c + \frac{1}{2}c = c.$$

So $\{y_n\}$ is a Cauchy sequence in complete metric space, hence there exists $y^* \in X$ such that $y_n \to y^*$. Since Tx is closed, thus $y^* \in Tx$. Now by uniqueness of limit we conclude that $x = y^* \in Tx$.

Theorem 3.3.14 Let (X, d) be a complete cone metric space, $T : X \to C(X)$, a set valued map and the function $f : X \to P$ defined by $f(x) = d(x, Tx), x \in X$ with lsc property. If there exist real numbers $a, b, c, e \geq 0$ and $q > 1$ with $k = aq + b + ceq < 1$ such that for all $x \in X$ there exists $y \in Tx$,

$$d(x, y) \leq qD(x, Tx),$$

$$D(y,Tx) \preceq ed(x,y),$$

$$D(y,Ty) \preceq ad(x,y) + bD(x,Tx) + cD(y,Tx),$$

then $Fix(T) \neq \phi$.

Example 3.3.15 Let $X = E = C^2([0,1],\mathbb{R})$ with norm $\| f \| = \| f \|_\infty + \| f' \|_\infty$ and $P = \{f \in E : f \geq 0\}$ that is not normal cone. Define cone metric $d : X \times X \to E$ with $d(f,g) = f^2 + g^2$, for $f \neq g, d(f,f) = 0$ and the set valued mapping $T : X \to C(X)$ by $Tf = \{-f, 0, f\}$. In this space every Cauchy sequence converges to zero. The function $F(f) = d(f,Tf) = inf_{g \in Tf} d(f,g) = inf\{0, f^2, 2f^2\} = 0$ have lsc property. Also we have $D(f,Tf) = \{0, f^2, 2f^2\}$ and $D(f,Tg) = \{f^2, f^2 + g^2\}$. Now for $q > 1, e \geq 1, a, b, c \geq 0, k = aq + b + ceq < 1$ and for all $f \in X$ take $g = 0 \in Tf$. Therefore, it satisfies all of the hypothesis of Theorem 3.3.14. So T has a fixed point $f \in Tf$. For sample take $a = b = c = 1/6, e = 1$, and $q = 2$.

Theorem 3.3.16 Let (X,d) be a complete cone metric space, $T : X \to K(X)$, a set-valued map, and a function $f : X \to P$ defined by $f(x) = d(x,Tx), x \in X$ with lsc property. The following conditions hold:
(i) If there exist real numbers $a, b, c, e \geq 0$ and $q > 1$ with $k = aq + b + ceq < 1$ such that for all $x \in X$, there exists $y \in Tx$,

$$d(x,y) \preceq qS(x,Tx),$$

$$S(y,Tx) \preceq ed(x,y),$$

$$S(y,Ty) \preceq ad(x,y) + bS(x,Tx) + cS(y,Tx),$$

then $Fix(T) \neq \phi$.
(ii) If there exist real numbers $a, b, c, e \geq 0$ and $q > 1$ with $k = aq + b + ceq < 1$ such that for all $x \in X$ and $y \in Tx$,

$$d(x,y) \preceq qS(x,Tx),$$

$$S(y,Tx) \preceq ed(x,y),$$

$$S(y,Ty) \preceq ad(x,y) + bS(x,Tx) + cS(y,Tx),$$

then $Fix(T) = End(T) \neq \phi$.

Lemma 3.3.17 Let (X, d) be a cone metric space, P be a normal cone with constant one and $T : X \to C(X)$, a set valued map, then

$$\| d(x, Tx) \| = \| \inf_{y \in Tx} d(x, y) \| = \inf_{y \in Tx} \| d(x, y) \| .$$

Proof. Put $\alpha = inf_{y \in Tx} \| d(x,y) \|$ and $\beta = inf_{y \in Tx} d(x, y)$. We show that $\alpha = \| \beta \|$. Let $y \in Tx$ then $\beta \leq d(x, y)$ and so $\| \beta \| \leq \| d(x, y) \|$, which implies $\| \beta \| \leq \alpha$. For the inverse, let for all $0 \leq r \leq \alpha$. Then $r \leq \| d(x, y) \|$ for all $y \in Tx$. Since $\beta = inf_{y \in Tx} d(x, y)$, for every c that $c \gg 0$ there exists $y \in Tx$ such that $d(x, y) < \beta + c$, so $r \leq \| d(x, y) \| < \| \beta + c \| < \| \beta \| + \| c \|$, for all $c \gg 0$. Thus $r \leq \| \beta \|$.

Remark 3.3.18 Let (X, d) be a cone metric space, P be a normal cone with constant one and $T : X \to C(X)$ be a set-valued map. Then the function $f : X \to P$ defined by $f(x) = d(x, Tx), x \in X$ with lsc property, and $g : E \to \mathbb{R}^+$ with $g(x) = \| x \|$. Then $gof(x) = inf_{y \in Tx} \| d(x, y) \|$, is lower semi-continuous.

Now the Theorems 3.3.8 and 3.3.9 are stated as the following corollaries without the assumption of normality, and by Lemma 3.3.17 and Remark 3.3.18 we have the same theorems.

Corollary 3.3.19 Let (X, d) be a complete cone metric space, $T : X \to C(X)$ be a set-valued map and the function $f : X \to P$ defined by $f(x) = d(x, Tx), x \in X$ with lsc property. If there exists real numbers $0 \leq \lambda < 1$, $\lambda < b \leq 1$ such that for all $x \in X$ there exists $y \in Tx$ one has $D(y, Ty) \leq \lambda d(x, y)$ and $bd(x, y) \leq D(x, Tx)$ then $Fix(T) \neq \phi$.

Corollary 3.3.20 Let (X, d) be a complete metric space, $T : X \to K(X)$ be a set valued map and the function $f : X \to P$ defined by $f(x) = d(x, Tx), x \in X$ with lsc property. The following hold:

(i) If there exist real numbers $0 \leq \lambda < 1, \lambda < b \leq 1$ such that for all $x \in X$ there exists $y \in Tx$ one has $S(y, Ty) \leq \lambda d(x, y)$ and $bd(x, y) \leq S(x, Tx)$, then $Fix(T) \neq \phi$.
(ii) If there exist real numbers $0 \leq \lambda < 1, \lambda < b \leq 1$ such that for all $x \in X$ and every $y \in Tx$ one has $S(y, Ty) \leq \lambda d(x, y)$ and $bd(x, y) \leq S(x, Tx)$, then $Fix(T) = End(T) \neq \phi$.

CHAPTER 4

TOPOLOGICAL PROPERTIES OF CONE METRIC SPACE

4.1 Introduction

In this chapter we discuss some topological concepts and definitions are generalized to cone metric spaces given by D. Turkoglu and M. Abuloha [11] in 2010. It is proved that every cone metric space is first countable topological space and that sequentially compact subsets are compact. Also, we define diametrically contractive mappings and asymptotically diametrically contractive mappings on cone metric spaces to obtain some fixed point theorems by assuming that our cone is strongly minihedral.

The main question is *"Are cone metric spaces a real generalization of metric spaces"*. Firstly M. A. Khamsi [32] gave remark on cone metric spaces and obtained equivalent metric from cone metric. In 2011 M. Asadi, S. M. Vaezpour and H. Soleimani [31] proved that every cone metric space is metrizable and the equivalent metric satisfies the same contractive conditions as the cone metric. So most of the fixed point theorems which have been proved are straightforward results from the metric case. But if cone is non normal same are not true.

4.2 Topological Cone Metric Spaces

In this section we introduce some basic topological concepts and definitions in cone metric space and prove that every cone metric space is a topological space. The following lemmas will be used extensively throughout this chapter.

Lemma 4.2.1 Let (X, d) be a cone metric space. Then for each $c \gg 0, c \in E$, there exists $\delta > 0$ such that $(c - x) \in intP$ $(i.e.\ x \ll c)$ whenever $\| x \| < \delta,\ x \in E$.

Proof. Since $c \gg 0$, then $c \in intP$. Hence, find $\delta > 0$ such that $\{x \in E : \| x - c \| < \delta\} \subset intP$. Now if $\| x \| < \delta$ then $\| (c - x) - c \| = \| -x \| = \| x \| < \delta$, and hence

$(c-x) \in intP$.

Lemma 4.2.2 Let (X,d) be a cone metric space. Then, for each $c_1 \gg 0$ and $c_2 \gg 0$, $c_1, c_2 \in E$, there exists $c \gg 0$, $c \in E$ such that $c \ll c_1$ and $c \ll c_2$.

Proof. Since $c_2 \gg 0$ then by Lemma 4.2.1, find $\delta > 0$ such that $\| x \| < \delta$ implies $x \ll c_2$. Choose n_0 such that $\frac{1}{n_0} < \frac{\delta}{\|c_1\|}$. Let $c = \frac{c_1}{n_0}$ then $\| c \| = \| \frac{c_1}{n_0} \| = \frac{\|c_1\|}{n_0} < \delta$ and hence, $c \ll c_2$. But also it is clear that $c \gg 0$ and $c \ll c_1$.

Proposition 4.2.3 Every cone metric space (X,d) is a topological space.

Proof. For $c \gg 0, c \in E$, let $B(x,c) = \{y \in X : d(x,y) \ll c\}$ and $\beta = \{B(x,c) : x \in X, c \gg 0\}$. Then, $\tau_c = \{U \subset X : \forall\, x \in U,\ \exists\ B \in \beta,\ x \in B \subset U\}$ is a topology on X. Indeed,

$\tau_1)$ $\phi, X \in \tau_c$.

$\tau_2)$ Let $U, V \in \tau_c$ and let $x \in U \cap V$. Then $x \in U$ and $x \in V$, find $c_1 \gg 0, c_2 \gg 0$ such that $x \in B(x,c_1) \subset U$ and $x \in B(x,c_2) \subset V$. Then by Lemma 4.2.2 find $c \gg 0$ such that $c \ll c_1$ and $c \ll c_2$. Then, clearly $x \in B(x,c) \subset B(x,c_1) \cap B(x,c_2) \subset U \cap V$. Hence, $U \cap V \in \tau_c$.

$\tau_3)$ Let $U_\alpha \in \tau_c$ for each $\alpha \in \triangle$, and let $x \in \bigcup_{\alpha\in\triangle} U_\alpha$. Then $\exists\ \alpha_0 \in \triangle$ such that $x \in U_{\alpha_0}$. Hence, find $c \gg 0$ such that $x \in B(x,c) \subset U_\alpha \subset \bigcup_{\alpha\in\triangle} U_\alpha$. That is, $\bigcup_{\alpha\in\triangle} U_\alpha \in \tau_c$.

It is important to note that any cone metric space (X,d) is a Hausdorff space. Indeed, if $x \neq y$ are two point in X, then $d(x,y) = c > 0$ so that $\left[B(x,\frac{c}{3}) \cap B(y,\frac{c}{3})\right] \in \tau_c$ and $B(x,\frac{c}{3}) \cap B(y,\frac{c}{3}) = \phi$. Hence, we gaurantee that the limits are unique.

Definition 4.2.4 Let (X,d) be a cone metric space. A subset $A \subset (X,d)$ is called sequentially closed if whenever $x_n \in A$ with $x_n \to x$ then $x \in A$.

Proposition 4.2.5 Let (X,d) be a cone metric space. The ball $\overline{B(x,c)} = \{y \in X : d(x,y) \leq c\}$, $c \gg 0$, $c \in E$ is sequentially closed.

Proof. Let $y_n \in \overline{B(x,c)}$ be a sequence such that $y_n \to y$. Then $d(y_n, x) \leq c$ and $d(y_n, x) \to 0$ as $n \to \infty$. Then $y \in \overline{B(x,c)}$ if and only if $d(x,y) \leq c$ if and only if $c - d(x,y) \in P$. But then by continuity of cone metric space $d(y_n, x) \to d(x,y)$. Since P is closed, then $lim_{n\to\infty}(c - d(y_n, x)) = c - d(x,y) \in P$.

Definition 4.2.6 Let (X,d) be a cone metric space. Then $A \subset X$ is called bounded above if there exists $c \in E, c \gg 0$ such that $d(x,y) \leq c$, for all $x, y \in A$, and is called bounded if $\delta(A) = sup\{d(x,y) : x, y \in A\}$ exists in E. If the supremum does not exist, we say that A is unbounded.

Proposition 4.2.7 Let P be a strongly minihedral cone with normal constant M and $A \subset E$. Then A is bounded if and only if $\delta'(A) = \sup_{x,y\in A} \parallel d(x,y) \parallel < \infty$.

Proof. Assume A is bounded. Then there exists $c \in E,\ c \gg 0$ such that $d(x,y) \leq c,\ \forall\ x, y \in A$, then $\parallel d(x,y) \parallel \leq M \parallel c \parallel < \infty,\ \forall\ x, y \in A$. Hence,

$$\sup_{x,y\in A} \parallel d(x,y) \parallel < \infty.$$

Conversely, assume that $\delta'(A) = \sup_{x,y\in A} \parallel d(x,y) \parallel = K < \infty$, and fix some $c_1 \gg 0$. Then by Lemma 4.2.1, find $\delta > 0$ such that $\parallel z \parallel < \delta$ implies $c_1 \gg z$. For each $x, y \in A$ let $c_{x,y} = \frac{\delta d(x,y)}{2\|d(x,y)\|}$. Then $\parallel c_{x,y} \parallel = \frac{\delta}{2} < \delta$. Hence, $c_1 \gg \frac{\delta d(x,y)}{2\|d(x,y)\|}$ and so $c_1 - \frac{\delta d(x,y)}{2\|d(x,y)\|} \in intP$. Therefore, $\frac{2\|d(x,y)\|}{\delta}c_1 - \frac{\delta d(x,y)}{2\|d(x,y)\|}\frac{2\|d(x,y)\|}{\delta} \in intP$, then $\frac{2\|d(x,y)\|}{\delta}c_1 - d(x,y) \in intP$, that is, $d(x,y) \ll \frac{2\|d(x,y)\|}{\delta}c_1 \leq \frac{2M}{\delta}c_1 = c \in intP$, from which it follows that $d(x,y) \leq c$. Since P is strongly minihedral then A is bounded.

Definition 4.2.8 Let (X,d) be a cone metric space and $c \gg 0,\ c \in E$. A finite subset $N = \{e_1, e_2, e_3, ..., e_n\}$ of X is called a c-net for A if for each $p \in A$, there exist $e_{i_0} \in N$ such that $d(p, e_{i_0}) \ll c$.

Definition 4.2.9 Let (X,d) be a cone metric space. A subset A of (X,d) is called totally bounded if for each $c \gg 0, c \in E$, A can be composed into union of sets $N_i, i = 1, 2, 3..., n$ $(A \subset \bigcup_{i=1}^{n} N_i)$ where $\delta(N_i) \ll c$.

Proposition 4.2.10 Let (X,d) be a cone metric space, and $A \subset X$. Then A is totally bounded if and only if for each $c \gg 0, c \in E$, A possesses a c-net.

Proof. Assume A is totally bounded and let $c \gg 0,\ c \in E$. Then find $N_1, N_2, ..., N_n$ such that $A \subset \bigcup_{i=1}^{n} N_i,\ \delta(N_i) \ll c$. From each N_i choose an element $e_i, i = 1, 2, 3..., n$. Let $N = \{e_1, e_2, ..., e_n\}$. We show that N is a c-net for A. Let $p \in A$. Then there exists $e_{i_0},\ i_0 = \{1, 2, .., n\}$ such that $p \in N_{i_0}$. Using that both P and e_{i_0} are in N_{i_0} and that $\delta(N_{i_0}) \ll c$, we conclude that $d(p, e_{i_0}) \ll c$.

Conversely, let $c \gg 0, c \in E$. Then find a finite set $N = \{e_1, e_2, ..., e_n\}$ such that for each $p \in A$ there exists $e_{i_o} \in N$ with $d(p, e_{i_0}) \ll c$. Let $N_i = B(e_i, c) = \{x \in X : d(x, e_i) \ll c\}; i = 1, 2, .., n$. Then clearly $A \subset \bigcup_{i=1}^{n} N_i$ and $\delta(N_i) \ll c$.

Definition 4.2.11 Let (X, d) be a cone metric space. A subset A of (X, d) is called compact if each cover of A by subsets from τ_c can be reduced to finite subcover.

Proposition 4.2.12 Let (X, d) be a cone metric space and $A \subset X$. If A is sequentially compact, then it is totally bounded.

Proof. Assume that there exists $c \gg 0, c \in E$ such that A can not have a c-net. Hence, for fixed $x_1 \in A$ there exists $x_2 \in A$ such that $c - d(x_1, x_2) \notin intP$, then also $\{x_1, x_2\}$ can not be a c-net for A, hence there exists $x_3 \in A$ such that $c - d(x_1, x_3) \notin intP$, and $c - d(x_3, x_2) \notin intP$. Like this we construct a sequence $x_n \in A$ such that $c - d(x_n, x_m) \notin intP,\ \forall\ n, m \in N$. That is, any subsequence of $\{x_n\}$ cannot be Cauchy and $\{x_n\}$ cannot have convergent subsequence. Therefore A is not sequentially compact.

Proposition 4.2.13 Every cone metric space (X, d) is first countable.

Proof. Let $p \in X$. Fix $c \gg 0, c \in E$. We show that $\beta_p = \{B(p, \frac{c}{n}) : n \in N\}$ is a local base at P. Let u be open with $p \in u$. Find $c_1 \gg 0$ such that $p \in B(p, c_1) \subset u$. Also by Lemma 4.2.1, find n_0 such that $\frac{c}{n_0} \ll c_1$. Hence, $B(p, \frac{c}{n_0}) \subset B(p, c_1) \subset u$.

Lemma 4.2.14 Let P be a cone in E and $\{x_n\}, \{y_n\}$ be two sequence in E. If

$y_n \to y, x_n \to x$ as $n \to \infty$ in $(E, \|\ .\ \|)$ and $x_n \leq y_n$ for all n, then $x \leq y$.

Proof. $x_n \leq y_n$ implies $y_n - x_n \in P$. Since P is closed and $(y_n - x_n) \to (y - x)$, then $(y - x) \in P$. Hence, $x \leq y$.

Definition 4.2.15 A map $T : (X, d) \to (X, d)$ is called continuous at $x \in X$, if for each $V \in \tau_c$ containing Tx, there exists $U \in \tau_c$ containing x such that $T(U) \subset V$. If T is continuous at each $x \in X$ then it is called continuous.

Definition 4.2.16 Let (X, d) be a cone metric space. A map $T : (X, d) \to (X, d)$ is called sequentially continuous if $x_n \in X$, $x_n \to x$ implies $Tx_n \to Tx$.

Proposition 4.2.17 Let (X, d) be a cone metric space, and $T : (X, d) \to (X, d)$ be any map. Then, T is continuous if and only if T is sequentially continuous.

Proof. Assume $x_n \to x$ and let $c \gg 0$. Since T is continuous at $x \in X$, then find $c_1 \gg 0$ such that $T(B(x, c_1)) \subset B(Tx, c)$. By convergence of x_n, find n_0 such that $d(x_n, x) \ll c_1$, $\forall\, n \geq n_0$. But then $d(Tx_n, Tx) \ll c$, $\forall\, n \geq n_0$. Since (X, d) is a first countable topological space, then the converse holds.

Lemma 4.2.18 Let (X, d) be a cone metric space and A be a sequentially compact subset of (X, d). Then, there exist $x_0, y_0 \in A$ such that $\delta(A) = sup\{d(x, y) : x, y \in A\} = d(x_0, y_0)$.

Proof. For fix $c \gg 0, c \in E$, we have $\delta(A) - \frac{c}{n} < \delta(A)$. By the definition of supremum, for each $n \in N$ find $y_n, x_n \in A$ such that $\delta(A) - \frac{c}{n} < d(x_n, y_n) \leq \delta(A)$. By sequential compactness of A, we may assume, for the sake of simplicity, that $x_n \to x_0 \in A$ and $y_n \to y_0 \in A$. By Lemma 4.2.14, $lim_{n\to\infty}(\delta(A) - \frac{c}{n}) < lim_{n\to\infty} d(x_n, y_n) \leq \delta(A)$ and hence, $\delta(A) < lim_{n\to\infty} d(x_n, y_n) \leq \delta(A)$. That is, $(\delta(A) - lim_{n\to\infty} d(x_n, y_n)) \in P$ and $lim_{n\to\infty}(d(x_n, y_n) - \delta(A)) \in P$. By Proposition 4.2.7, $\delta(A) = lim_{n\to\infty} d(x_n, y_n) = d(x_0, y_0)$.

Definition 4.2.19 A mapping $T : X \to X$ on a complete cone metric space (X, d)

is said to be diametrically contractive if $\delta(TA) < \delta(A)$ for all closed bounded subset $A \subset X$, such that $\delta(A)$ exists and $\delta(A) > 0$.

It is clear that each diametrically contractive map is contractive.

Theorem 4.2.20 Let (X, d) be sequentially compact cone metric space with strongly minihedral cone and $T : X \to X$ be diametrically contractive mapping. Then T has a fixed point.

Definition 4.2.21 A mapping $T : X \to X$ on a complete cone metric space (X, d) is said to be asymptotically diametrically contractive if

$$\delta_a(\{TA_n\}) < \delta_a(\{A_n\})$$

for all nonincreasing sequence $\{A_n\}$ of closed bounded subset of X with $\delta_a(\{A_n\}) > 0$, where

$$\delta_a(\{A_n\}) = \lim_{n\to\infty} \delta(A_n)$$

is called the asymptotic diameter of the sequence $\{A_n\}$.

It is clear that every asymptotically diametrically contractive is diametrically contractive. However, if (X, d) is sequentially compact then the converse is also true. Indeed, we have

Proposition 4.2.22 Let (X, d) be a sequentially compact cone metric space and $T : X \to X$ be a contracting mapping. Then, T is asymptotically diametrically contractive mapping.

Lemma 4.2.23 Let (X, d) be a cone metric space, P be strongly minihedral and let $A \subset X$ be bounded. Then $\delta(A) = \delta(\overline{A})$.

Theorem 4.2.24 Let (X, d) be a complete cone metric space, P strongly minihedral and $T : X \to X$ be an asymptotically diametrically contractive mapping.

Assume T has a bounded orbit $\{T^n x_0\}_{n=0}^{\infty}$ for some $x_0 \in X$. Then T has a unique fixed point $z \in X$, and for each $x \in X$, $\{T^n x\}_{n=0}^{\infty}$ converges to z.

4.3 Equivalence of Cone Metric Spaces And Metric Spaces

In this section we have to show that cone metric spaces have a metric type structure which is proved by M. A. Khamsi [31].

Definition 4.3.1 Let (X, d) be a cone metric space. A mapping $T : X \to X$ is called Lipschitzian if there exists $k \in \mathbb{R}$ such that

$$d(Tx, Ty) \leq kd(x, y),$$

for all $x, y \in X$. The smallest constant k which satisfies the above inequality is called the Lipschitzian constant of T, denoted $Lip(T)$. In particular T is a contraction if $Lip(T) \in [0, 1)$. Indeed we have the following result.

Theorem 4.3.2 Let (X, d) be a metric cone over the Banach space E with the cone P which is normal with the normal constant M. The mapping $D : X \times X \longrightarrow [0, \infty)$ defined by $D(x, y) = \| d(x, y) \|$ satisfies the following properties:

(*i*) $D(x, y) = 0$ if and only if $x = y$;

(*ii*) $D(x, y) = D(y, x)$, for any $x, y \in X$;

(*iii*) $D(x, y \leq M(D(x, z_1) + D(z_1, z_2) + ... + D(z_n, y))$, for any points $x, y, z_i \in X. i = 1, 2, ..., n$.

Proof. The proofs of (i) and (ii) are trivial. In order to prove (iii), let $x, y, z_1, z_2, ..., z_n$ be any points in X. Using triangle inequality satisfied by d, we get

$$d(x, y) \leq d(x, z_1) + d(x, z_2) + ... + d(x, z_n).$$

Since P is normal with constant M we get

$$\| d(x, y) \| \leq M \| d(x, z_1) + d(x, z_2) + ... + d(x, z_n) \|,$$

which implies

$$\| d(x,y) \| \leq M(\| d(x,z_1) \| + \| d(x,z_2) \| + ... + \| d(x,z_n) \|).$$

This completes the proof of the theorem.

Note that the property (iii) is discouraging since it does not give the classical triangle inequality satisfied by a distance. But there are many examples where triangle inequality fails.

The above result suggests the following definition.

Definition 4.3.3 Let X be a set. Let $D : X \times X \longrightarrow [0,\infty)$ be a function which satisfies

(i) $D(x,y) = 0$ if and only if $x = y$;

(ii) $D(x,y) = D(y,x)$, for any $x, y \in X$;

(iii) $D(x,y) \leq K(D(x,z_1) + D(z_1,z_2) + ... + D(z_n,y))$, for any points $x, y, z_i \in X, i = 1, 2, ..., n$, for some constant $K > 0$;

$(iii)'$ $D(x,y) \leq K(D(x,z) + D(z,y))$.

The pair (X, D) is called a metric type space.

Similarly we define convergence and completeness in metric type space.

Definition 4.3.4 Let (X, D) be a metric type space.

(i) The sequence $\{x_n\}$ converges to $x \in X$ if and only if $\lim_{n\to\infty} D(x_n, x) = 0$.

(ii) The sequence $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\to\infty} D(X_n, x_m) = 0$.

(X, D) is complete if and only if any Cauchy sequence in X is convergent.

Definition 4.3.5 Let $T : X \longrightarrow X$ be a map. T is called Lipschitzian if there exists a constant $\lambda \geq 0$ such that

$$D(Tx, Ty) \leq \lambda D(x,y)$$

for any $x, y \in X$. The smallest constant λ will be denoted Lip(T).

Theorem 4.3.6 Let (X, D) be a complete metric type space. Let $T : X \longrightarrow X$ be a map such that T^n is Lipschitzian for all $n \geq 0$ and $\sum_{n=0}^{\infty} Lip(T^n) < \infty$. Then T has a unique fixed point $\omega \in X$. Moreover for any $x \in X$, the orbit $\{T^n x\}$ converges to ω.

Example 4.3.7 Let X be the set Lebesgue measurable functions on $[0, 1]$ such that

$$\int_0^1 |f(x)|^2 dx < \infty.$$

Define $D : X \times X \to [0, \infty)$ by

$$D(f, g) = \int_0^1 |f(x) - g(x)|^2 dx$$

Then D satisfies the following properties:

(i) $D(f, g) = 0$ if and only if $f = g$;

(ii) $D(f, g) = D(g, f)$, for any $f, g \in X$;

(iii') $D(f, g) \leq 2(D(f, h) + D(h, g))$, for any points $f, g, h \in X$.

In the next result we consider the case of metric type spaces (X, D) when D satisfies (iii'). Recall that a subset Y of X is said to be bounded whenever $\sup\{D(x, y); x, y \in Y\} < \infty$.

Theorem 4.3.8 Let (X, D) be a complete metric type space, where D satisfies (iii') instead of (iii). Let $T : X \longrightarrow X$ be a map such that T^n is Lipschitzian for all $n \geq 0$ and $\lim_{n\to\infty} Lip(T^n) = 0$. Then T has a unique fixed point if and only if there exists a bounded orbit. Moreover if T has a fixed point ω, then for any $x \in X$, the orbit $\{T^n x\}$ converges to ω.

Corollary 4.3.9 Let (X, D) be a complete metric cone over the Banach space E with the cone P which is normal with the normal constant M. Consider

$D : X \times X \longrightarrow [0, \infty)$ defined by $D(x,y) = \| d(x,y) \|$. Let $T : X \longrightarrow X$ be a contraction with constant $k < 1$. Then

$$D(T^n x, T^n y) \leq Mk^n D(x,y)$$

for any $x, y \in X$ and $n \geq 0$. Hence $Lip(T^n) \leq Mk^n$, for any $n \geq 0$. Therefore $\sum_{n>0} Lip(T^n)$ is convergent, which implies T has a unique fixed point ω, and any orbit converges to ω.

4.4 Metrizability of Cone Metric Space

M. Asadi,*et.al* [30] proved that the cone metric spaces are metrizable and defined the equivalent metric in different approaches. However there is main question " *Will the equivalent metric satisfy the same contractive conditions which the cone one does?.*" M. Asadi,*et. al* answered affirmatively for a few contractive conditions but it is impossible to answer the question in general.

By renorming the Banach spaces which have been partially ordered by a cone, we can obtain a new norm which converts it to normal cone, so every cone metric space is metrizable.

Theorem 4.4.1 For every cone metric $D : X \times X \rightarrow E$ there exists metric $d : X \times X \rightarrow \mathbb{R}^+$ which is equivalent to D on X.

Proof. Define $d(x,y) = \inf\{\| u \| : D(x,y) \leq u\}$. We shall prove that d is an equivalent metric to D. If $d(x,y) = 0$ then there exists u_n such that $\| u_n \| \rightarrow 0$ and $D(x,y) \leq u_n$. So $u_n \rightarrow 0$ and consequently for all $c \gg 0$ there exists $N \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq N$. Thus for all $c \gg 0$, $0 \leq D(x,y) \ll c$. Namely $x = y$.
If $x = y$ then $D(x,y) = 0$ which implies that $d(x,y) \leq \| u \|$ for all $0 \leq u$. Put $u = 0$ it implies $d(x,y) \leq \| 0 \| = 0$, on other hand $0 \leq d(x,y)$. Therefore $d(x,y) = 0$. It is clear that $d(x,y) = d(y,x)$. To prove triangle inequality, for $x, y, z \in X$ we have,

$$\forall\, \varepsilon > 0\ \exists\, u_1,\ \ \| u_1 \| < d(x,z) + \varepsilon,\ D(x,z) \leq u_1,$$

$$\forall\, \varepsilon > 0\ \exists\, u_2,\ \ \| u_2 \| < d(z,y) + \varepsilon,\ D(z,y) \leq u_2.$$

But $D(x,y) \le D(x,z) + D(z,y) \le u_1 + u_2$, therefore

$$d(x,y) \le \| u_1 + u_2 \| \le \| u_1 \| + \| u_2 \| \le d(x,z) + d(z,y) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary so $d(x,y) \le d(x,z) + d(z,y)$.
Now we shall prove that, for all $\{x_n\} \subseteq X$ and $x \in X, x_n \to x$ in (X,d) if and only if $x_n \to x$ in (X,D). We have

$$\forall\ n,m \in \mathbb{N}\ \exists\ u_{nm}\ \textit{such that}\ \| u_{nm} \| < d(x_n,x) + \frac{1}{m},\ D(x_n,x) \le u_{nm}.$$

Put $v_n = u_{nm}$ then $\| v_n \| < d(x_n,x) + \frac{1}{n}$ and $D(x_n,x) \le v_n$. Now if $x_n \to x$ in (X,d) then $d(x_n,x) \to 0$ and so $v_n \to 0$ too. Therefore for all $c \gg 0$ there exists $N \in \mathbb{N}$ such that $v_n \ll c$ for all $n \ge N$. This implies that $D(x_n,x) \ll c$ for all $n \ge N$. Namely $x_n \to x$ in (X,D).
Conversely, for every real $\varepsilon > 0$, choose $c \in E$ with $c \gg 0$ and $\| c \| < \varepsilon$. Then there exists $N \in \mathbb{N}$ such that $D(x_n,x) \ll c$ for all $n \ge N$. This means that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n,x) \le \| c \| < \varepsilon$ for all $n \ge N$. Therefore $d(x_n,x) \to 0$ as $n \to \infty$ so $x_n \to x$ in (X,d).

Example 4.4.2 Let $0 \ne a \in P \subseteq \mathbb{R}^n$ with $\| a \| = 1$ and for every $x,y \in \mathbb{R}^n$ define

$$D(x,y) = \begin{cases} a, & x \ne y \\ 0, & x = y. \end{cases}$$

Then D is a cone metric on $\mathbb{R}^n$ and its equivalent metric d is

$$d(x,y) = \begin{cases} 1, & x \ne y \\ 0, & x = y, \end{cases}$$

which is discrete metric.

Example 4.4.3 Let $a,b \ge 0$ and consider the cone metric $D : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ with $D(x,y) = (ad_1(x,y), bd_2(x,y))$ where d_1, d_2 are metrices on $\mathbb{R}$. Then its equivalent metric is $d(x,y) = \sqrt{a^2+b^2}\ \| (d_1(x,y), d_2(x,y)) \|$. In particular if $d_1(x,y) = |x-y|$ and $d_2(x,y) = \alpha|x-y|$, where $\alpha \ge 0$ then D is the same famous cone metric which has been introduced in [24] and its equivalent metric is $d(x,y) = \sqrt{1+\alpha^2}|x-y|$.

Example 4.4.4 For $q > 0, b > 1, E = l^q, P = \{\{x_n\}_{n\geq 1} : x_n \geq 0, for\ all\ n\}$ and (X, ρ) a metric space, define $D : X \times X \to E$ which is the same cone metric as [41,Example 3]

$$D(x,y) = \Big\{\Big(\frac{\rho(x,y)}{b^n}\Big)^{\frac{1}{q}}\Big\}_{n\geq 1}.$$

Then its equivalent metric on X is,

$$d(x,y) = \Big\|\Big\{\Big(\frac{\rho(x,y)}{b^n}\Big)^{\frac{1}{q}}\Big\}_{n\geq 1}\Big\|_{l^q} = \Big(\sum_{n=1}^{\infty}\frac{\rho(x,y)}{b^n}\Big)^{\frac{1}{q}} = \Big(\frac{\rho(x,y)}{b-1}\Big)^{\frac{1}{q}}.$$

Lemma 4.4.5 Let $D, D^* : X \times X \to E$ be cone metrices, $d, d^* : X \times X \to \mathbb{R}^+$ their equivalent metrices respectively and $T : X \to X$ a self map. If $D(Tx, Ty) \leq D^*(x, y)$, then $d(Tx, Ty) \leq d^*(x, y)$.

Proof. By the definition of d^*,

$$\forall\ \varepsilon > 0\ \exists\ v\ such\ that\ \parallel v \parallel < d^*(x,y) + \varepsilon,\ D^*(x,y) \leq v.$$

Therefore if $D(Tx, Ty) \leq D^*(x, y) \leq v$, then we have

$$d(Tx,Ty) \leq \parallel v \parallel \leq d^*(x,y) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary so $d(Tx, Ty) \leq d^*(x, y)$.

Example 4.4.6 Let $E = \mathbb{R}, P = \mathbb{R}^+$ and $D : X \times X \to E$ be a cone metric, $d : X \times X \to \mathbb{R}^+$ its equivalent metric, $T : X \to X$ be a self map and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\varphi(x) = \frac{x}{1+x}$. If $D^* = \varphi(D)$, then its equivalent metric is $d^* = \varphi(d)$, and if $D(Tx, Ty) \leq \varphi(D(x, y)) = \frac{D(x,y)}{1+D(x,y)}$, then by Lemma 4.4.5, $d(Tx, Ty) \leq \varphi(d(x, y)) = \frac{d(x,y)}{1+d(x,y)}$.

Definition 4.4.7 A self map φ on normed space X is bounded if

$$\parallel \varphi \parallel = \sup_{0\neq x\in X}\frac{\parallel \varphi(x) \parallel}{\parallel x \parallel} < \infty.$$

Theorem 4.4.8 Let $D : X \times X \to E$ be a cone metric, $d : X \times X \to \mathbb{R}^+$ its equivalent metric, $T : X \to X$ a self map and $\varphi : P \to P$ be a bounded map. Then there exists $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $D(Tx, Ty) \leq \varphi(D(x, y))$ for every $x, y \in X$ implies $d(Tx, Ty) \leq \psi(\| D(x, y) \|)$ for all $x, y \in X$. Moreover if ψ is decreasing map or φ is linear and increasing map then, $d(x, y) \leq \psi(d(x, y))$ for all $x, y \in X$,

Corollary 4.4.9 Let D, D^* be cone metrices, d, d^* their equivalent metrices, $T : X \to X$ be a map, $\lambda \in [0, \frac{1}{2})$ and $\alpha, \beta \in [0, 1)$. For $x, y \in X$,

i) $D(Tx, Ty) \leq \alpha D(x, y) \Rightarrow d(Tx, Ty) \leq \alpha d(x, y)$.

ii) $D(Tx, Ty) \leq \lambda(D(Tx, x) + D(Ty, y)) \Rightarrow d(Tx, Ty) \leq \lambda(d(Tx, x) + d(Ty, y))$.

iii) $D(Tx, Ty) \leq \lambda(D(Tx, y) + D(Ty, x)) \Rightarrow d(Tx, Ty) \leq \lambda(d(Tx, y) + d(Ty, x))$.

iv) $D(Tx, Ty) \leq \lambda D(x, y) + \beta D(Tx, y) \Rightarrow d(Tx, Ty) \leq \alpha d(x, y) + \beta d(Tx, y)$.

v)There exists $u \in \{D(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}[D(x, Ty)+D(y, Tx)]\}$ such that $D(Tx, Ty) \leq \alpha u$ where $\alpha \in (0, 1)$ then

$$d(Tx, Ty) \leq \alpha \; max\{d(x, y); d(x, Tx); d(y, Ty); \tfrac{1}{2}[d(x, Ty) + d(y, Tx)]\}.$$

Theorem 4.4.10 Let $(E, \| . \|)$ a real Banach space with a positive cone P. There exists an equivalent norm on E such that P is a normal cone with constant $M = 1$, with respect to this norm.

Corollary 4.4.11 Every cone metric space (X, D) is metrizable.

4.5 Completion of Cone Metric Space

Definition 4.5.1 A cone normed space is an ordered pair $(X, \| \cdot \|_c)$, where X is a vector space over $\mathbb{R}$ and $\| \cdot \|_c : X \longrightarrow (E, \| \cdot \|)$ is a function satisfying:

(*i*) $0 < \| x \|_c$, for all $x \in X$.

(*ii*) $\| x \|_c = 0$ if and only if $x = 0$.

(*iii*) $\| \alpha x \|_c = | \alpha | \| x \|_c$, for each $x \in X$ and $\alpha \in \mathbb{R}$.

(iv) $\| x + y \|_c \leq \| x \|_c + \| y \|_c$, for all $x, y \in X$.

It is clear that each cone normed space is a cone metric space. In fact, the cone metric is given by $d(x, y) = \| x - y \|_c$. Complete cone normed spaces are called *cone Banach spaces.*

According to the definition of convergence in cone metric spaces, we see that $x_n \to x$ in $(X, \| \cdot \|_c)$ if and only if for all $c \gg 0$ in E there exists n_0 such that $\| x_n - x \|_c \ll c$ for all $n \geq n_0$ and, if the cone is normal, if and only if $\lim_{n\to\infty} \| \| x_n - x \|_c \| = 0$. Also, $x_n \in (X, \| \cdot \|_c)$ will be Cauchy if and only if for all $c \gg 0$ in E there exists n_0 such that $\| x_n - x_m \|_c \ll c$ for all $m, n \geq n_0$ and the cone is normal, if and only if $\lim_{m,n\to\infty} \| \| x_n - x_m \|_c \| = 0$.

Before proceeding to prove a scalar norm completion theorem, we first give the meaning of isometries of cone metric spaces.

Definition 4.5.2 Let (X, d) and (Y, ρ) be cone metric spaces. A mapping T of X into Y is said to be an isometry if it preserves cone distances, that is for all $x_1, x_2 \in X$,

$$\rho(Tx_1, Tx_2) = d(x_1, x_2).$$

It is clear that if T is bijective and an isometry, then it is together with its inverse, (sequentially) continuous and hence (X, d) and (Y, ρ) become topological isomorphic. Throughout, we shall say that cone metric space X is isometric with the cone metric space Y if there exists a bijective isometry of X onto Y. In the sequel, one has to note that every cone isometry is one to one.

Proposition 4.5.3 Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences in a cone metric space (X, d) over a normal cone with constant M. Then $\lim_{n\to\infty} d(x_n, y_n)$ exists in $(E, \| \cdot \|)$.

Theorem 4.5.4 For a cone metric space (X, d) over a normal cone there exists a complete cone metric space (X^s, d_s) which has a subspace W that is isometric with X and dense in X^s. The space (X^s, d_s) is unique upto isometry, that is, if Z is any complete cone metric space having a dense subspace U isometric with X, then Z and X^s are isometric.

As every cone normed space is a cone metric space, we define the meaning of isometry of cone normed spaces.

Definition 4.5.5 Two cone normed spaces $(X, \|\cdot\|_{c_1})$ and $(Y, \|\cdot\|_{c_2})$ are said to be isometric if there exists a bijective linear operator $T : X \longrightarrow Y$ such that

$$\| Tx \|_{c_2} = \| x \|_{c_1}, \quad \text{for all} \quad x \in X.$$

Theorem 4.5.6 Let $(X, \|\cdot\|_c)$ be a cone normed space over a normal cone. Then there is a cone Banach space $(X^s, \|\cdot\|_s)$ and an isometry T from X onto a subspace W of X^s, which is dense in X^s. The space X^s is unique upto isometry.

References

[1] A. Azam, M. Arshad and I. Beg, *Common fixed points of two maps in cone metric spaces*, Rend. Circ. Mat. Palermo, 57 , 433-441, (2008).

[2] A. J. B. Potter, *A fixed point theorem for positive k-set contractions*, Proceedings of the Edinburgh Mathematical Society II, vol. 19, pp. 93-102, 1974.

[3] A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. 28, 326-329, (1969).

[4] A. Latif and F. Y. Shaddad, *Fixed point results for multivalued maps in cone metric spaces* Fixed Point Theory and Applications, (2010).

[5] B.E. Rhoades, *A comparison of various definition of contractive mappings*, Trans. Amer. Math. Soc. 266, 257-290, (1977).

[6] B.S. Choudhury and N. Metiya, *Fixed point of weak contraction in cone metric spaces*, Nonlinear Analysis 72, 1589-1593, (2010).

[7] D. Ilic and V. Rakocevic, *Common fixed points for maps on cone metric space*, Journal of Mathematical Analysis and Applications, vol. 341, no. 2, pp. 876-882, 2008.

[8] D. Klim and D.Wardowski, *Dynamic processes and fixed points of set-valued nonlinear contractions in cone metric spaces,* Nonlinear Analysis Theory, Methods and Applications, vol. 71, no. 11, pp. 5170-5175, 2009.

[9] D. Wardowski, *End points and fixed point of set valued contractions in cone metric spaces*, Nonlinear Analysis: Theory, Methods and Applications 71, 512-516, (2009).

[10] D. Ili and V. Rakoevi, *Quasi-contraction on cone metric space*, Appl. Math. Lett., doi:10.1016/j.aml.2008.

[11] D. Turkoglu and M. Abuloha, *Cone metric spaces and fixed point theorems in diametrically contractive mappings*, Acta Math. Sin. (Engl. Ser.).

[12] G. Jungck, S. Radenovic, S. Radojevic and V. Rakocevic, *Common fixed theorems of weakly compatile pairs on cone metric spaces*, Fixed Point Theory and Applications, vol. 59, Article ID 643-840,13 pages, 2009.

[13] G. Jungck and B. E. Rhoades, *Fixed points for set valued functions without continuity*, Indian Journal of Pure and Applied Mathematics, vol. 29, no. 3, pp. 227-238, 1998.

[14] G. Jungck, *Commuting maps and fixed points*, Amer. Math. Monthly 83, 261-263, (1976).

[15] G. Jungck ,*Compatible mappings and common fixed points*, Internat. J. Math. Sci., 9(4), 771-779, (1986).

[16] H. Covitz and S.B. Nadler , *Multi-valued contraction mappings in generalized metric spaces*, Israel J. Math. 8:5, (1970).

[17] H. Demirer, M. Ozturk and M. Basarr, *On some maps with P and Q Properties in non-normal cone metric spaces*, International Mathematical Forum, Vol. 6, no. 8, 381-387, 2011.

[18] H.E. Kunze, D. La Torre, and E.R. Vrscay , *Contraction multifunctions, fixed point inclusions and iterated multifunction systems*, J. Math. Anal. Appl. 330:159-173, (2007).

[19] I. Altun and G. Durmaz, *Some fixed point theorems on ordered cone metric spaces*, Rend. Circ. Mat. Palermo 58, 319-325, (2009).

[20] I. Altum and D. Djoric, *Fixed point and common fixed point theorems on ordered cone metric spaces*, Applied Math. Letters, 310-316, (2010) .

[21] I. Sahin, and M. Telci, *Fixed points of contractive mappings on complete cone metric spaces*, Hacettepe J. Math. Stat. 38 (1), 59-67, (2009).

[22] J. R. Morales and E. Rojas, *Cone metric spaces and fixed point theorems of T-contractive mappings* , Revista Notes de Mathematica vol 4(2), 66-78,(2008).

[23] J. R. Morales and E. Rojas, *Cone metric spaces and fixed point theorems of T-Kannan contractive mapping*, Int. Jour. of Math. Anal. vol 4, 175-184, (2010).

[24] K. Jha, *A common fixed point theorem in a cone metric space*, vol 5 (2009).

[25] L.G. Huang , X. Zhang ,*Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., 332 , 1467-1475, (2007).

[26] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, Journal of Mathematical Analysis and Applications, vol. 341, no. 1, pp. 416-420, 2008.

[27] M. A. Krasnosel *Fixed points of cone-compressing or cone-extending operators*, Soviet Mathematics. Doklady, vol. 1, pp. 1285-1288, 1960.

[28] M. Abbas and B. E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Applied Mathematics Letters, vol. 22, no. 4, pp. 511-515, 2009.

[29] M. Abbas and B. E. Rhoades and T. Nazir, *Common fixed points for four maps in cone metric spaces* Applied Mathematics and Computation (2010).

[30] M. Asadi, H. Soleimani and S. M. Vaezpour, *An order on subsets of cone metric spaces and fixed points of set-valued contractions*, Fixed Point Theory and Applications, vol. 2009, Article ID 723203, 8 pages, 2009.

[31] M. Arshad, A. Azam and P. Vetro, *Some common fixed point results on cone metric spaces*, Fixed Point Theory Appl. 11, Article ID 493965, (2009).

[32] M. A. Khamsi, *Remarks on cone metric space and fixed point theorems of contractive mappings* , Fixed Point Theory Appl. (2010.)

[33] M. Asadi, S. M. Vaezpour, H. Soleimani, *Metrizability of cone metric spaces* ar Xiv: 1102.2353v1 [math. FA](2011).

[34] M. Asadi, S. M. Vaezpour, H. Soleimani, *Metrizability of cone metric spaces via renorming the Banach space.* ar Xiv: 1102.2040v1 [math. FA](2011).

[35] M. Asadi H. Soleimani, *Examples in cone metric spaces* (2011).

[36] M. Khani, M. Pourmahdian, *On the metrizability of cone metric spaces* , Topology and its applications, 190-193, (2011).

[37] M. Edelstein, *An extension of Banachs contraction principle*, Proceedings of the American Mathematical Society, vol. 12, no. 1, pp. 7-10, 1961.

[38] P. Vetro, *Common fixed points in cone metric spaces,* Rendiconti del Circolo Matematico di Palermo, vol. 56, no. 3, pp. 464-468, 2007.

[39] P. Raja and S. M. Vaezpour, *Some extensions of Banachs contraction principle in complete cone metric spaces*, Fixed Point Theory and Applications, Article ID 768294, 11 pages, 2008.

[40] P. Raja and S. M. Vaezpour, *Some fixed point theorems in complete cone metric spaces*, submitted.

[41] P. Raja and S. M. Vaezpour, *Fixed point theorems for certain contractive mappings in complete cone metric spaces*, submitted.

[42] R. Pant, R. Mohan P.K. Mishra, *Some common fixed point theorems in cone metric space*, Int. Jour. of Science Tech. and Management (2011).

[43] Sh. Rezapour, and R. Hamlbarani Haghi, *Fixed points of multifunctions on regular cone metric spaces*, Expositiones Mathematicae 28 (1), 71-77, (2010).

[44] Sh. Rezapour and R. Hamlbarani, *Some notes on the paper: Cone metric spaces and fixed point theorems of contractive mappings*, Journal of Mathematical Analysis and Applications, vol. 345, no. 2, pp. 719-724, 2008.

[45] Sh. Rezapour, *A review on topological properties of cone metric spaces*, in Analysis, Topology and Applications (ATA 08), Vrnjacka Banja, Serbia, May-June 2008.

[46] Sh. Rezapour. R. H. Haghi, *Two results about fixed point of multifunctions*, Bull.Iranian Mathematical Society, to appear.

[47] Sh. Rezapour. R. H. Haghi, *Fixed point of multifunctions on cone metric spaces*, Numer. Funct. Anal. and Opt. 30, 825-832,(2009).

[48] S. Radenovic, *Common fixed points under contractive conditions in cone metric spaces*,Computers and Mathematics with Applications, vol. 58, no. 6, pp. 1273-1278, 2009.

[49] S. Radenovic and B.E. Rhoades, *Fixed point theorem for two non-self mappings in cone metric spaces*, Computers and Mathematics with Applications, 57:, 1701-1707,(2009).

[50] S. Radenovic, *Common fixed points under contractive conditions in cone metric spaces* , Computers and Math. with Applications, 1273-127, (2009) 8.

[51] S. M. Veaspour and P. Raja, *Some extensions of Banach contraction principle in complete cone metric spaces,* Fixed Point Theory and Applications, vol. 2008, Article ID 768294, 11 pages, 2008.

[52] T. Abdeljawad, *Completion of cone metric spaces*, Hacettepe Jour. of Math. and Stat. vol 39, 67-74, (2010).

[53] W.A. Kirk, *Fixed points of asymptotic contractions* , J. Math. Anal. Appl. 277, 645-650, (2003).

[54] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.

[55] W. Shih Du, *A note on cone metric fixed point theory and its equivalence,* Nonlinear Analy., 2259-2261, (2010).

[56] X. Huang, C. Zhu and Xi. Wen, *A common fixed point theorem in cone metric spaces*, Int. J. of math. anal., 721-726, (2010).

[57] Y. Feng, S. Liu, *Fixed point theorems for multi-valued contractive maps and multi-valued Caristi type maps*, J. Math. Anal. Appl. 317, 108-112, (2006).

[58] Z. Kadelburg, S. Raenovic and V. Rakocevic, *A note on the equivalence of some metric and cone metric fixed point results*, App. Matematics Letters, 370-374, (2011).

[59] Z. Kadelburg, S. Raenovic and V. Rakocevic, *Remarks on Quasi-contraction on a cone metric space*, Applied Math. Letters, 1674-1679, (2009).

[60] Z. Kadelburg, S. Raenovic, and B. Rosic, *Strict contractive conditions and common fixed point theorems in cone metric spaces* Fixed Point Theory and Applications, (2009).

[61] Z. Kadelburg, P.P Murthy and S. Radenovic, *Common fixed points for expansive mappings in cone metric spaces*, Int. Jour. of Math. Analy, Vol. 5, 1309-1319, (2011).