LIE THEORY AND SPECIAL FUNCTIONS

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TO MY PARENTS
CERTIFICATE

Certified that Mr. Masroor Alam Khan, has carried out the research on 'Lie Theory and Special Functions' under my supervision and work is suitable for submission for the award of degree of Master of Philosophy in Mathematics.

( PROF. M.A. PATHAN )
Supervisor
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1.1 INTRODUCTION:

Special function has wide application in the field of Mathematics. Most of these applications lie in the field of Statistics, Physics, Engineering, Theory of Elasticity, Quantum Theory and Lie-Theory.

In this preliminaries chapter, we present some basic concept needed for the presentation of the subsequent chapters. This chapter has two parts. Part-I contains the theoretical background of special functions like hypergeometric, its confluent forms, their generalization including the Kampe de Feriet Function and some special cases of hypergeometric functions. A concept of generating function is also given. In Part-II, we discuss some definitions and examples of Lie groups and their Lie-algebras. Many of the standard notations, concepts and methods which are useful in the detailed study of group theoretic approach to special functions are also mentioned.
PART-I \hspace{1em} SPECIAL FUNCTIONS

1.2 GAUSSIAN HYPERGEOMETRIC SERIES:

The term hypergeometric was first used by Wallis in Oxford as early as 1655 in his work *Arithmatica Infinitorum* when referring to any series which could be regarded as a generalizing of the ordinary geometric series

$$\sum_{n=0}^{\infty} x^n \hspace{1em} \ldots \ldots (1.2.1)$$

but the main systematic development of what is now regarded as the hypergeometric series of one variable

$$\frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (n)!} \frac{z^n}{n} \hspace{1em} \ldots \ldots (1.2.2)$$

was undertaken by Gauss in 1812.

The Pochhammer symbol \((a)_n\) where 'a' denotes any number, real or complex and \(n\) is any integer positive, negative or zero, is defined by

$$\begin{cases} 1, & \text{if } n = 0 \\ a(a+1) \ldots (a+n-1), & \text{if } n \geq 1 \end{cases} \hspace{1em} \ldots \ldots (1.2.3)$$
and \((a)^{-n} = \frac{(-1)^n}{(1-a)^n}\) for \(n < 0\)  

\[ \ldots (1.2.4) \]

In particular \((1)_n = n!\)

hence the symbol \((a)_n\) is also referred to as the factorial function.

In terms of Gamma function, we have

\[(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad a \neq 0, -1, -2, \ldots \quad \ldots (1.2.5)\]

Furthermore, the binomial coefficient may be expressed as

\[
\binom{a}{n} = \frac{a(a-1) \cdots (a-n+1)}{n!} = \frac{(-1)^n (-a)_n}{n!} \quad \ldots (1.2.6)
\]

Equation \((1.2.5)\) also yields

\[(a)_{m+n} = (a)_m (a+m)_n \quad \ldots (1.2.7)\]

also we have

\[
(a)_{n-k} = \frac{(-1)^k (a)_n}{(1-a-n)_k}, \quad 0 \leq k \leq n \quad \ldots (1.2.8)
\]

For \(a = 1\)
\[(n-k)! = \frac{(-1)^{n-k} n!}{(-n)_k}, \quad 0 \leq k \leq n \quad \cdots \quad (1.2.9)\]

which may alternatively be written in the form

\[(-n)_k = \begin{cases} 
\frac{(-1)^k n!}{(n-k)!}, & 0 \leq k \leq n \\
0, & k > 0 
\end{cases} \quad \cdots \quad (1.2.10)\]

In (1.2.2), \(a, b, c\) are parameters and \(x\) is a variable (real or complex) of the series. All four of these quantities may be any numbers real or complex. However (1.2.2) is not defined if \(c\) is a negative integer i.e. \(c \neq 0, -1, -2, \ldots\) unless one of the \(a\) or \(b\) is also negative integer such that 
\(-c < -a\). In general, if either of the numerator parameters is a negative integer, the series terminates.

The series given by (1.2.2) is convergent when \(|z| < 1\), when \(z = 1\), provided that \(\text{Re}(c-a-b) > 0\) and when \(z = -1\), provided that \(\text{Re}(c-a-b) > -1\).

**Generalized Hypergeometric Functions**

A natural generalization of the hypergeometric series \(2F_1\) is the generalized hypergeometric function, which is defined as
\[ \int_{p}^{q} \left( \begin{array}{c} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q; \end{array} \right) z = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n z^n}{(b_1)_n (b_2)_n \cdots (b_q)_n n!} \]

\[ = \sum_{n=0}^{\infty} \left( \frac{p}{q} \right) \frac{(a_i)_n z^n}{(b_j)_n n!} \]

\[ \ldots \quad (1.2.11) \]

where \((a_i)_n\) is Pochhammer's symbol given by

\[ (a_i)_n = \frac{(a_i + n - 1)!}{(a_i)!} \]

Here \(p\) and \(q\) are positive integer or zero. The numerator parameters \(a_1, a_2, \ldots, a_p\) and the denominator parameters \(b_1, b_2, \ldots, b_q\) take on complex values, provided that \(b_j \neq 0, -1, -2, \ldots; j = 1, 2, \ldots q\).

**Convergence of \( \int_{p}^{q} \)**

(i) If \(p \leq q\), the series converges for all finite \(z\) i.e. \(|z| < \infty\) (real or complex) and diverges absolutely when \(|z| = 1\).

(ii) If \(P = q+1\), the series converges for \(|z| < 1\) and diverges \(|z| > 1\).
(iii) If $p > q+1$, the series converges only when $z = 0$ and diverges when $z \neq C$.

(iv) If $p = q+1$, the series is absolutely convergent on circle $|z| = 1$ i.e.

$$\text{Re}(\sum_{j=1}^{q}b_j - \sum_{j=1}^{p}a_j) > 0 \quad \text{for} \quad z = 1$$

and

$$\text{Re}(\sum_{j=1}^{q}b_j - \sum_{j=1}^{p}a_j) > -1 \quad \text{for} \quad z = -1$$

Special Case

I. When $p = q = 1$ in (1.2.11) reduces to confluent hypergeometric $\text{I}_1$ named as Kummer's series given by E. E. Kummer [25].

$$\lim_{|b| \to \infty} 2\text{F}_1(a, b; c; \frac{z}{b}) = 1\text{F}_1(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!} \quad \ldots \quad (1.2.12)$$

II. When $p = 2$, $q = 1$ in (1.2.11) reduces to the Gaussian hypergeometric series

$$2\text{F}_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!} \quad \ldots \quad (1.2.13)$$
putting $z = 1$, we get Gauss theorem

$$2F_1(a, b; c; 1) = \frac{|c|}{|c-a|} \frac{|c-b|}{|c-b|} \quad .... (1.2.14)$$

Note: - If the homogeneous linear differential equation of second order has atmost three singularities we may assume that these are at $0, \infty, 1$. If all these singularities are 'regular' (Cf. Poole [42]), then the equations can be reduced to the form

$$z(1-z) \frac{d^2u}{dz^2} - [c-(a+b+1)z] \frac{du}{dz} - abu = 0 \quad .... (1.2.15)$$

where $a, b, c$ (Independent of $z$) are parameters of the equation. Equation (1.2.15) is called the hypergeometric equation having solution (1.2.2).

**Confluent Hypergeometric Functions**

Since, the Gauss functions $2F_1(a, b; c; z)$ is a solution of the differential equation (1.2.15), replacing $z$ by $\frac{z}{b}$ in (1.2.15), we have

$$z(1 - \frac{z}{b}) \frac{d^2u}{dz^2} + [c-(1+ \frac{1+a}{b})z] \frac{du}{dz} - au = 0 \quad .... (1.2.16)$$
Obviously $\genfrac{[}{]}{0pt}{}{2}{1}(a,b;c;\frac{z}{b})$ is a solution of (1.2.16).

As $b \to \infty$

$$\lim_{b \to \infty} \genfrac{[}{]}{0pt}{}{2}{1}(a,b;c;\frac{z}{b}) = \genfrac{[}{]}{0pt}{}{1}{1}(a;c;z) \quad \ldots \quad (1.2.17)$$

is a solution of differential equation

$$z \frac{d^2u}{dz^2} - (c-z) \frac{du}{dz} - au = 0 \quad \ldots \quad (1.2.18)$$

The function

$$\genfrac{[}{]}{0pt}{}{1}{1}(a;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} \quad \ldots \quad (1.2.19)$$

is called the confluent hypergeometric function or Kummer function given by E. i. Kummer in 1836 [ ]. It is also denoted by Humbert symbol $\varphi(a;c;z)$.

The integral representation of $\genfrac{[}{]}{0pt}{}{1}{1}(a;c;z)$ is given by

$$\genfrac{[}{]}{0pt}{}{1}{1}(a;c;z) = \frac{c}{c-a} \int_{0}^{1} (1-u)^{c-a-1} u^{a-1} e^{zu} du \quad \ldots \quad (1.2.20)$$

for $\text{Re}(c) > \text{Re}(a) > 0$. 
Appell's functions

The theory of single hypergeometric function has led to a generalization involving double series. In the year 1883, Appell defined four complete series of second order $F_1$ to $F_4$ which are analogous to Gauss's $\sum F_1$ in the form

\[ F_1(a;b,c;d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_m (c)_n x^m y^n}{(d)_{m+n} m! n!} \quad \ldots (1.2.20) \]

\[ F_2(a;b,c;d,e;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_m (c)_n x^m y^n}{(d)_{m} (e)_{n} m! n!} \quad \ldots (1.2.21) \]

\[ F_3(a,b;c,d,e;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m} (b)_n (c)_m (d)_n x^m y^n}{(e)_{m+n} m! n!} \quad \ldots (1.2.22) \]

\[ F_4(a;b,c;d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n} x^m y^n}{(c)_{m} (d)_{n} m! n!} \quad \ldots (1.2.23) \]

The convergence conditions of Appell series are as mentioned below:

(i) The series $F_1$ and $F_3$ converges for $|x| < 1, |y| < 1$.

(ii) The series $F_2$ converges for $|x| + |y| < 1$.

(iii) The series $F_4$ converges when $|x^{1/2}| + |y^{1/2}| < 1$. 
The Kampe De Feriet Function

The four Appell series are unified and generalized by Kampe de Feriet (1921) who defined a general double hypergeometric series, See [Appell and Kampe de Feriet (1), p.150, (29)].

Kampe de Feriet's functions, denoted by $F_{l:m;n}^{P:q;k}$, is defined as follows [1, p.27]

\[
F_{l:m;n}^{P:q;k} \left[ \begin{array}{c}
(a_p; b_q; c_k; x, y) \\
(\alpha_l; \beta_m; \gamma_n)
\end{array} \right] = \sum_{i,j,s=0}^{\infty} \frac{[(a_p)]_i [b_q]_i [c_k]_s}{[\alpha_l]_i [\beta_m]_i [\gamma_n]_s} i! j! s! x^i y^j
\]

where, for convergence,

(i) $P+q < \ell+m+1$, $p+k < \ell+n+1$, $|x| < \infty$, $|y| < \infty$

(ii) Also when $p+q = \ell+m+1$, $p+k = \ell+n+1$ and

$$\left| x \frac{1}{(p-\ell)} + y \frac{1}{(p-\ell)} \right| < 1$$

$\max \left\{ |x|, |y| < 1, \text{ if } p \leq \ell \right\}.$
1.3 SOME SPECIAL CASES OF HYPERGEOMETRIC FUNCTIONS:

The most important special functions are the hypergeometric functions. Indeed all the other special functions and many elementary functions, are just special case of the hypergeometric functions.

In this section, we show the intimate relationships which exist between the hypergeometric functions and other special functions which we will define:

**Relation (1.3.1)**

\[ P_n(x) = _2F_1(-n,n+1;1; \frac{1-x}{2}) \]

where \( P_n(x) \) is the **Legendre Polynomials** and is defined by the relation

\[ (1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n \]

for \(|t| < 1\) and \(|x| \leq 1\).

Here \((1-2xt+t^2)^{-\frac{1}{2}}\) is the generating function of the generating relation (1.3.1).
The Legendre Polynomial $P_n(x)$ of order $n$ can also be defined by the equation

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^k k! (n-k)! (n-2k)!} x^{n-2k} \quad \ldots \quad (1.3.2)$$

where

$$\left[ \frac{n}{2} \right] = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Relation (1.3.2)

$$P_n(x) = \frac{(n+m)!(1-x)^2}{(n-m)! 2^m m!} \, _2F_1(m-n,m+n+1;m+1; \frac{1-x}{2})$$

where $P_n(x)$ is the associated Legendre Polynomials of the first kind, and is defined by

$$P_n(x) = (1-x)^{-\frac{1}{2}} \frac{d^m}{dx^m} (P_n(x)) \quad \ldots \quad (1.3.3)$$

Relation (1.3.3)

$$J_n(x) = \frac{\left(\frac{x}{2}\right)^n}{\Gamma(1+n)} \, _0F_1\left(\frac{1}{4} ; 1+n ; -\frac{x^2}{4}\right)$$
where \( J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k! (1+n+k)} \) \ldots (1.3.4)

\( n \) is a positive integer or zero, and if \( n \) is a negative integer, we put

\[ J_n(x) = (-1)^n J_{-n}(x) \] \ldots (1.3.5)

\( J_n(x) \) is called Bessel function \([10]\) of order \( n \) for all finite \( x \).

Bessel function \( J_n(x) \) may also be defined by means of a generating function, for integral \( n \) only, if \( t \neq 0 \) then for all finite \( x \)

\[ \exp \left[ \frac{x}{2} (t - \frac{1}{t}) \right] = \sum_{n=-\infty}^{\infty} J_n(x) t^n \] \ldots (1.3.6)

Relation (1.3.4)

\[ H_{2n}(x) = \frac{(-1)^n (2n)!}{n!} \quad _1F_1(-n; \frac{1}{2}; x^2) \]

and also

\[ H_{2n+1}(x) = \frac{(-1)^n 2(2n+1)!}{n!} \quad _1F_1(-n; \frac{3}{2}; x^2) \]
We define the Hermite polynomials by means of the generating relation

\[
\exp(2xt-t) = \sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!}
\]

\[\cdots \quad (1.3.7)\]

Hermite polynomials \(H_n(x)\) can also be defined by

\[
H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^{n-k}(2x)^{n-2k}}{k!(n-2k)!}
\]

\[\cdots \quad (1.3.8)\]

Relation (1.3.5)

\[
L_n(x) = \mathbf{1}_F_1(-n;1;x)
\]

where Laguerre Polynomials \(L_n(x)\) of order \(n\) are defined by means of generating relation

\[
(1-t)^{-1} \exp \left[ -\frac{xt}{1-t} \right] = \sum_{n=0}^{\infty} L_n(x) t^n
\]

\[\cdots \quad (1.3.9)\]

Also, \(L_n(x)\) in the series form is given as

\[
L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n! x^k}{(k!) (n-k)!}
\]

\[\cdots \quad (1.3.10)\]
Relation (1.3.6)

\[
L_n(x) = \sum_{r=0}^{k} \frac{(-1)^r (n+k)!}{r! (n-r)! (k+r)!} x^r
\]

where \( L_n(x) \) are the associated Laguerre polynomials defined by

\[
L_n(x) = \frac{1}{(n+k+1)!} \int_0^x (t-t^2)^{n+k} dt
\]

Relation (1.3.7)

\[
T_n(x) = 2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2})
\]

\[
U_n(x) = ((1-x)^n)^{\frac{1}{2}} 2F_1(-n+1, n+1; \frac{3}{2}; \frac{1-x}{2})
\]

We define the **Chebyshev** Polynomials of first kind, \( T_n(x) \) and second kind \( U_n(x) \), by

\[
T_n(x) = \text{Cos}(n \text{ Cos } x)
\]

\[
U_n(x) = \text{Sin}(n \text{ Cos } x)
\]
for a non-negative integer.

In series form, we have

$$T_n(x) = \sum_{r=0}^{[n/2]} \frac{(-1)^r n! (1-x)^r x^{n-2r}}{(2r)! (n-2r)!}$$

$$U_n(x) = \sum_{r=0}^{[(n-1)/2]} \frac{(-1)^r n! (1-x)^r x^{n-2r-1}}{(2r+1)! (n-2r-1)!}$$

The generating functions are

$$2^{-1} (1-xt) (1-2xt+t^2) = \sum_{n=0}^{\infty} T_n(x) t^n$$

$$2^{-1} (1-2xt+t^2) = \sum_{n=0}^{\infty} U_n(x) t^n$$

Relation (1.3.8)

$$P_n^{(\alpha, \beta)}(x) = \frac{(\alpha+1)_n}{n!} \, _2F_1(-n, n+\alpha+\beta+1; 1+\alpha; \frac{1-x}{2})$$

we define the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ by generating relation

$$\frac{\alpha+\beta}{2} \left( \frac{1}{1-2xt+t^2} \right)^{\frac{1}{2}} \left\{ 1-t+(1-2xt+t^2)^{\frac{1}{2}} \right\}^{\frac{1}{2}} \left\{ 1+t+(1-2xt+t^2)^{\frac{1}{2}} \right\}^{\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(x) t^n$$
$P_n^{(\alpha,\beta)}$ has the following power series expansions

(i) $P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (1+\alpha+\beta)_n+k}{k! (n-k)! (1+\alpha)_k (1+\alpha+\beta)_n} \left( \frac{x-1}{2} \right)^k$

(ii) $P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (1+\beta)_n}{k! (n-k)! (1+\alpha)_k (1+\beta)_{n-k}} \left( \frac{x-1}{2} \right)^k \left( \frac{x+1}{2} \right)^{n-k}$

(iii) $P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n-\alpha} \frac{(-1)^n (1+\beta)_n (1+\alpha+\beta)_n+k}{k! (n-k)! (1+\beta)_k (1+\alpha+\beta)_n} \left( \frac{x+1}{2} \right)^k$

Note: $P_n^{(\alpha,\beta)}(x) = P_n(x)$.

Relation (1.3.9)

$$H_n^{(\alpha,\beta)}(x) (k,p;x) = \binom{\alpha+n}{n} 3F_2 \left[ \begin{array}{c} -n, 1+\alpha+\beta+n; k; x \\ 1+\alpha, p; \end{array} \right]$$

where $H_n^{(\alpha,\beta)}(k,p;x)$ be the generalized Rice Polynomials.

Note: $H_n^{(\alpha,\beta)}(k,k;x) = P_n^{(\alpha,\beta)}(1-2x)$ \hspace{1cm} ...... (1.3.16)
### 1.4 Generating Functions:

The name 'generating function' was first introduced by Laplace in 1812. We define a generating function for the set of function $f_n(x)$ as follows [30].

**Definition:** Let $G(x,t)$ be a function that can be expanded in power of $t$ such that

$$G(x,t) = \sum_{n=0}^{\infty} c_n f_n(x) t^n$$

(1.4.1)

where $c_n$ is a function of $n$ and independent of $x$ and $t$. Then $G(x,t)$ is called a generating function of the set $\{f_n(x)\}$.

**Remark:** A set of function may have more than one generating function. However if,

$$G(x,t) = \sum_{n=0}^{\infty} h_n(x) t^n$$

then $G(x,t)$ is unique generator for the set $h_n(x)$ as the unefficient set.

Let us define a generating function of more than one variable.
Definition: Let $G(x_1, x_2, \ldots, x_p; t)$ be a function of $(p+1)$ variables. Suppose $G(x_1, x_2, \ldots, x_p; t)$ has a formal expansion in powers of $t$ such that

$$G(x_1, x_2, \ldots, x_p; t) = \sum_{n=-\infty}^{\infty} C_n f_n(x_1, x_2, \ldots, x_p) t^n \quad \ldots \quad (1.4.2)$$

where $C_n$ is independent of the variables $x_1, x_2, \ldots, x_p$ and $t$. Then we say that $G(x_1, x_2, \ldots, x_p; t)$ is a generating function for the $f_n(x_1, x_2, \ldots, x_p)$ corresponding to non-zero $C_n$. In particular, if

$$G(x, y, t) = \sum_{n=0}^{\infty} C_n f_n(x) g_n(y) t^n \quad \ldots \quad (1.4.3)$$

The expression determines the set of constant $\{C_n\}$ and two sets of function $\{f_n(x)\}$ and $\{g_n(y)\}$. Then $G(x, y, t)$ can be considered as a generator of any of these three sets and as unique generator of co-efficient set $\{C_n f_n(x), g_n(y)\}$.

Application of generating functions

A generating function may be used to defined a set of functions to determine a differential recurrence relation or a pure recurrence relation, to evaluate certain integrals etc.
We will use generating functions to define the following special functions: for example - Bessel's functions, and the polynomials of Legendre, Gegenbaur, Hermite and Laguerre.

We are given below certain generating relations [30, chapter 1]. The Hermite polynomial $H_n(x)$ is defined by the generating relation

$$\exp(2xt-t) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \quad \ldots \quad (1.4.4)$$

The Laguerre polynomial $L_n(x)$ satisfies the generating relations

$$\sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n = e^t \frac{\text{E}_1(-1;1;-xt)}{1-t} \quad \ldots \quad (1.4.5)$$

and

$$\sum_{n=0}^{\infty} L_n(x) t^n = (1-t)^{-1} \exp \frac{-xt}{1-t} \quad \ldots \quad (1.4.6)$$

The generating function of generalized Laguerre polynomial $(\alpha) L_n(x)$ satisfies the generating relation

$$\sum_{n=0}^{\infty} \frac{(\alpha) L_n(x)}{n!} t^n = (1-t)^{-1-\alpha} \exp \frac{-xt}{1-t} \quad \ldots \quad (1.4.7)$$
A bilinear generating functions

If a function $G(x,y,t)$ can be expanded in the form

$$G(x,y,t) = \sum_{n=0}^{\infty} g_n f_n(x) f_n(y) t^n$$

..... (1.4.8)

where $g_n$ is independent of $x$ and $y$ then $G(x,y,t)$ is called a bilinear generating function. For example, the Laguerre polynomials satisfies the following bilinear generating relation [30, p.17] or [10, p.212]

$$\sum_{n=0}^{\infty} \frac{\binom{\alpha}{n} \binom{\alpha}{n}}{(1+\alpha)_n} \frac{L_n(x) L_n(y)}{t^{n-1}} \frac{1}{a} \exp \frac{t(x+y)}{(1-t)}$$

$$= \left(\frac{1-t}{1-t}\right)^{\alpha} \frac{\exp \left(\frac{t(x+y)}{(1-t)}\right)}{(1-t)^{\alpha}}$$

$$= \binom{\alpha}{n} \binom{\alpha}{n} \frac{L_n(x) L_n(y)}{t^{n-1}} \frac{1}{a} \exp \frac{t(x+y)}{(1-t)}$$

..... (1.4.9)

Bilateral generating functions

If $H(x,y,t)$ can be expended in power of $t$ in the form

$$H(x,y,t) = \sum_{n=0}^{\infty} h_n f_n(x) g_n(y) t^n$$

..... (1.4.10)

where $h_n$ is independent of $x$ and $y$ and $f_n(x)$ and $g_n(y)$ are different functions, then $H(x,y,t)$ is called a bilateral generating function. For example [30, p.12].
PART II. LIE GROUP THEORY:

1.5 HISTORICAL BACKGROUND:

In the late 19th century Lie, \cite{26}, \cite{27} considered Lie groups (then called continuous groups) for the first time. His motivation was to treat the various geometries from a group theoretic point of view and to investigate the relationship between differential equations and the group of transformation preserving their solutions. It arose out of his work on differential equations, in which these groups would play the role of the Galois group of an algebraic equations. Lie groups were studied locally and the notion of Lie-algebras introduced.

In 1893, Killing, \cite{24} wrote a series of five papers in which he established, many of the basic structure theorems about complex Lie-algebra. Killing in fact, coined the term semi-simple and most remarkably, the classification simple Lie-algebras. Killing had been led to the classification problem by geometric considerations to a large extent independently of Lie's work.
Early in the 20th century, the theory of infinite dimensional Lie groups was studied by Cartan, E [4] Poincare's [40] in his first publication to mention the theory of continuous transformation groups.

From 1900-1930, Cartan [4] and Weyl, H [52] obtained a complete classification of semi-simple Lie-algebras and determined their representation and characters. They also devised useful method of investigating the structure of these algebras and did pioneering work of global structures of the underlying manifolds of Lie-groups. After him, (1930-1950), these results were systematized and redefined by Chevalley [6], Harish Chandra [17] and others. In the same period Iwasawa, K [22] classified Cartan's idea, showing that the only Lie-groups that are topologically important, are compact Lie groups. He also obtained the Iwasawa decomposition, which has become a basic tool in the study of semi-simple Lie-groups. At the same time, Iwasawa contributed to Hilbert's fifth problem which seeks to characterize Lie-groups among topological groups. These problems were solved by Gleason, A [15], Montgomery and Zippin [31] in 1952. Hopf, H [19] also used the properties of groups
extensively, were succeeded by systematic application of general theory of algebraic topology.

In an attempt in this direction, group theory has been found to have an important role. This approach is better known as Lie-theoretic method. The first significant advancement in this direction was made by Weisner, [50] who exhibits the group theoretic significance of generating functions. Miller, W. Jr. [35] and McBride [30] present Weisner's method in a systematic manner and thereby lay its firm foundation. Miller [35] also extends Weisner's theory further by relating it to factorization method of Infeld and Hull [21].

In the factorization method, a single second order differential equation is replaced, if possible, by a pair of first order differential equations for a whole set of special functions, that is a pair of equations of the form

\[ L_n^+ f_n = f_{n+1}, \quad L_n^- f_n = f_{n-1} \]

where \( L_n^+ \) and \( L_n^- \) are first order differential operators. The second order-differential equation can be written in the two alternate forms
\[ L_{n+1} L_n f_n = f_n, \quad L_{n-1} L_n f_n = f_n. \]

It is possible to identify the operators \( L_n \) and \( L_n \) (together with additional operators) with a Lie-algebra and the possible factorizations can be classified by the study of these Lie-algebras. The special functions constitute basis functions in representation spaces for Lie-algebras and many of their properties can be obtained in this way. This approach has been thoroughly investigated by Miller [34], [35], [37], Kalnin, Manocha and Miller [23], Chen and Feng [5], Gelfand and Sapir [14] and Manocha [29].

The qualitative and geometric background of Lie group theory applicable to special functions is found in standard treatises, Chevalley [6], Helgason [18], Miller ([33] to [37]), Cohen [7], Gleason [15], Hammermesh [16], Borel [2], Howe [2C]. We shall emphasize the theory of Lie groups given in Miller ([33] to [37]), Talman [46], Wawrzynczyk [48], Weisner [50] and Vilenkin since apparently this is the side of the theory that is best known to Physicists and is directly applicable to the theory of special functions to obtain the 'fine structure' of the concerned functions.
1.6 **LIE GROUP AND LIE-ALGEBRA**

**GLOBAL LIE GROUP:**

Definition [35] A **Lie group** is both an abstract group and an analytic manifold such that the operation of group multiplication and group inversion are analytic with respect to the manifold structure.

Definition [8] A **Lie group** is a set $G$ such that

1. $G$ is a group
2. $G$ is an analytic manifold
3. The mapping $(x,y) \rightarrow xy$ of the product manifold $G \times G \rightarrow G$ is analytic.

**Note:** (1.6.1) A topological group has two distinct kind of structure on it, one algebraic and other topological. Algebraically it is a group and topologically, it is a manifold.

**Note:** (1.6.2) [43,p.16] Every Lie-group is Hausdorff topological group.

**Note:** (1.6.3) [8,p.45] Any Lie-group is a topological group with respect to the topology induced by its analytic structure.
Example of Lie-groups

Example (1.6.1) Any abstract discrete topological group is a Lie group as a zero-dimensional smooth manifold.

Example (1.6.2) Any finite-dimensional vector space is a Lie group under addition.

Example (1.6.3) A unit circle $S^1: |z| = 1$ whose points are complex number $z = e^{i\theta}$ is a Lie group under multiplication.

A Lie group under multiplication is similarly a unit sphere $S^n$ in the space of quaternion whose points are quaternions $\xi$ for which $|\xi| = 1$.

We can show that if sphere $S^n$ is a Lie group, then it is necessary that $n = 1$ or $n = 3$, so that $S^1$ and $S^3$ are the only spheres admitting the structure of Lie-group.

Example (1.6.4) The group of symplectic matrices of even order $n = 2m$ denoted by $\text{sp}(m,\mathbb{R})$ form a Lie group and is called a real Linear symplectic group of dimension $m(2m+1)$.

Example (1.6.5) The intersection $\text{sp}(m,\mathbb{R})$ is called an orthogonal symplectic group. The Cayley image of non-exceptional matrices of this group are of the form
where \( D \) is a symmetric matrix and \( C \) is skew symmetric matrix. Since matrices of the form (1.5.1) also constitute a vector space, \( \text{sp}(m,\mathbb{R})_n \), \( O(2m) \) is a Lie-group, its dimension is \( m \).

**Example (1.6.6)** The rotation group \( O_3 \) in three-dimensional space is the group of real \( 3 \times 3 \) matrices \( A \) such that \( AA^t = I \) and \( \det A = 1 \) (see [16]). Here \( A^t \) is the transpose of \( A \) and \( I \) is the \( 3 \times 3 \) identity matrix. \( O_3 \) is a real three-parameter Lie-group.

**Example (1.6.7)** The matrix group \( T_3 \) is the set of all \( 4 \times 4 \) matrices

\[
g = \begin{bmatrix}
1 & 0 & 0 & \tau \\
0 & e^{-\tau} & 0 & c \\
0 & 0 & e^\tau & b \\
0 & 0 & 0 & 1
\end{bmatrix}, \tau, b, c, \epsilon \in \mathbb{T} \quad \text{..... (1.5.2)}
\]

The inverse of \( g \in T_3 \) is given by

\[
g^{-1} = \begin{bmatrix}
1 & 0 & 0 & \tau \\
c & e^\tau & 0 & -e^\tau c \\
c & 0 & e^{-\tau} & -e^\tau b \\
c & 0 & 0 & 1
\end{bmatrix} \quad \text{..... (1.5.3)}
\]
and the product \( g_1 g_2 \) by

\[
\begin{pmatrix}
1 & 0 & 0 & \tau_1 + \tau_2 \\
0 & e^{-\tau_1 - \tau_2} & 0 & c_1 + e^{-\tau_1} c_2 \\
0 & 0 & e^{\tau_1 + \tau_2} & b_1 + e^{\tau_1} b_2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

..... (1.5.4)

where \( g_1 \) and \( g_2 \) are matrices of the form (1.5.2). We can establish a co-ordinate system for \( T_3 \) by assigning to \( g \in T_3 \) the co-ordinates

\[
g \equiv (b, c, \tau)
\]

\( T_3 \) is clearly a three-dimensional complex local Lie group. Moreover, the above co-ordinates can be extended over all of \( \mathbb{C}^3 \). Thus, \( T_3 \) has the topology of \( \mathbb{C}^3 \) and is simply connected (see [41], chapter 8).

**Example (1.6.8)** The direct product \( G \times H \) of two smooth (or topological) groups \( G \) and \( H \) is a smooth (respectively, topological) group.

In particular, any torus \( T^n, n \geq 1 \), is a Lie group.
Note: The composition function \( \psi(x,y) = (x + y)^{1/3} \) defines the set \( \mathbb{R} \) of real number (with usual topology) as a topological group, but not as a Lie-group because \( \psi \) is not analytic.

**LOCAL LIE GROUPS**

For a particular case of Lie-groups these general consideration justify introducing a new mathematical concept of a local Lie-group which is a formalization of a neighbourhood of the identify in a Lie-group together with multiplication available in that neighbourhood. The definition of Local Lie-group given in [35] is as follows:

Let \( \mathbb{C}^n \) be the space of complex \( n \)-tuples \( g = (g_1, g_2, \ldots, g_n) \), where \( g_i \in \mathbb{C}, \ i = 1, 2, \ldots, n \) and define the origin \( e \) of \( \mathbb{C}^n \) by \( e = (0,0,\ldots,0) \). Suppose \( V \) is an open set in \( \mathbb{C}^n \) containing \( e \).

**Definition:** A complex \( n \)-dimensional local Lie-group \( \mathcal{G} \) in the neighbourhood \( V \subset \mathbb{C}^n \) is determined by a function \( \psi: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n \) such that

(i) \( \psi(g,h) \in \mathbb{C}^n, \ g,h \in V \).

(ii) \( \psi(g,h) \) is analytic in each of its \( 2n \)-argument.
(iii) If $\phi(g,h)\in V$, $\phi(h,k)\in V$ then

$$\phi(\phi(g,h),k) = \phi(g,\phi(h,k)).$$

(iv) $\phi(e,g) = g$, $\phi(g,e) = g$ for all $g \in V$.

Note: A local Lie-group is not necessarily a group in the usual sense. However, the group axioms are satisfied for the elements in a sufficiently small neighbourhood of $e$. 
INFINITESIMAL VECTOR OR TANGENT VECTOR

Let \( G \) be a local Lie-group in the neighbourhood \( \mathcal{V} \subseteq \mathbb{C}^n \) let \( t \mapsto g(t) = (g_1(t), \ldots, g_n(t)), t \in \mathbb{C} \) be an analytic mapping of a neighbourhood \( 0 \in \mathbb{C} \) into \( \mathcal{V} \) such that \( g(0) = e \).

We can consider such a mapping to be a complex analytic curve in \( G \) passing through \( e \).

**Definition:** An infinitesimal vector \( \alpha \) of an analytic curve \( g(t) \) in \( \mathcal{V} \) is the tangent vector to \( g(t) \) at \( e \) and define as \([35, p.2]\).

\[
\alpha = \frac{d}{dt} g(t) \bigg|_{t=0} = \frac{d}{dt} (g_1(t), \ldots, g_n(t)) \bigg|_{t=0} \in \mathbb{C}^n \hspace{1cm} (1.6.2)
\]

Every vector \( \alpha \in \mathbb{C}^n \) can be regarded as the tangent vector at \( e \) for some analytic curve.

**Note (1.6.2).** If \( g(t) \) and \( h(t) \) are analytic curve in \( G \) such that \( g(0) = h(0) = e \) with tangent vector \( \alpha \) and \( \beta \) respectively. Then the analytic curve \( g(t)h(t) \) has tangent vector \( \alpha + \beta \) at \( e \) and analytic curve \( g^{-1}(t) \) has the tangent vector \( -\alpha \) at \( e \).
**Definition:** For \( g(t), h(t) \) as given above, we define commutator \([\alpha,\beta]\) of \( \alpha \) and \( \beta \) to be tangent vector at \( e \) of the analytic curve

\[
k(t) = g(r)h(r)g(t)h(t), \quad t = r^2
\]

i.e.

\[
[\alpha, \beta] = \frac{d}{dt} \left[ k(t) \right]_{t=0}
\]

\[
= \frac{d}{dt} \left[ g(r)h(r)g(t)h(t) \right]_{t=0}
\]

\[
= \alpha \beta - \beta \alpha
\]

The commutator has following properties,

1. \([\alpha, \beta] = -[\beta, \alpha]\)
2. \([a_1 \alpha_1 + a_2 \alpha_2, \beta] = a_1[\alpha_1, \beta] + a_2[\alpha_2, \beta]\)
3. \([[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] = 0\)

where \( a_1, a_2 \in \mathfrak{g} \) and \( \alpha, \beta, \gamma \in \mathfrak{g} \).

**LIE-ALGEBRA**

**Definition:** The Lie-algebra \( L(G) \) of a Lie-group (local) \( G \) is the set of all tangent vectors at \( e \) equipped with the operations of vector addition and Lie-product.
Definition: A complex abstract Lie-algebra \( L(G) \) is a complex vector space together with a Lie-product \( [\alpha, \beta] \in L(G) \) defined for all \( \alpha, \beta \in \mathbb{C}^n \) such that the above conditions (1), (2), (3) are satisfied.

Clearly \( L(G) \) is a complex abstract Lie-algebra and any abstract Lie-algebra is in fact the Lie-algebra of some local Lie-group. We will always assume that \( L(G) \) is a finite dimensional Lie-algebra.

Further example of Lie group and their Lie-algebras

Example (1.6.9) General linear group \( GL(2, \mathbb{C}) \)

The complex general linear group \( GL(2, \mathbb{C}) \) is the set of all 2X2 matrices (non-singular)

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad a, b, c, d \in \mathbb{C} \quad \text{and} \quad ad - bc \neq 0
\]

where group operation is matrix-multiplication. Clearly the identity element \( e \in GL(2, \mathbb{C}) \) is the matrix

\[
e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
GL(2) is a 4-dimensional complex local Lie-group. These coordinate are valid only for \( g \) in a suitably small neighbourhood of \( e \).

Let \( g(t), g(0) = e \) be an analytic curve \( \text{GL}(2) \) with tangent vector \( \alpha \) at \( e \) then \( \alpha \) can be identified with the complex matrix

\[
\alpha = \begin{pmatrix}
\alpha_1 & \alpha_2 \\
\alpha_3 & \alpha_4
\end{pmatrix} = \left. \frac{d}{dt} \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \right|_{t=0}
\]

Lie-Algebra \( \text{L}[\text{LG}(2)] = \text{gl}(2) \).

\( \text{L}[\text{LG}(2)] = \text{gl}(2) \) is the space of all \( 2 \times 2 \) matrices \( \alpha \) with the relation \( [\alpha, \beta] = \alpha \beta - \beta \alpha; \alpha, \beta \in \text{gl}(2) \). The special elements

\[
\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

form a basis for \( \text{gl}(2) \) and satisfying the commutation relations

\[
[j^+, j^-] = j^-, [j^-, j^+] = 2 j^3, [j^+, j^-] = 2 j^3 \\
[\xi, j^+] = [\xi, j^-] = [\xi, j^+] = 0
\]

\( \cdots (1.6.3) \)

\( \cdots (1.6.4) \)

\( \cdots (1.6.5) \)
where $\Theta$ is the 2X2 matrix.

For any element $\alpha \in \mathfrak{gl}(2)$ can be written uniquely in the form

$$\alpha = a_1 \mathbf{1} + a_2 \mathbf{j} + a_3 \mathbf{j}^3 + a_4; \quad a_1, a_2, a_3, a_4 \in \mathbb{C}.$$ 

**Example (1.6.10): Special Linear Group $\text{SL}(2,\mathbb{C})$**

The complex special linear group $\text{SL}(2)$ is the abstract matrix group of all 2X2 non singular matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad a, b, c, d \in \mathbb{C}.$$ 

such that $ad - bc = 1$ or $d = \frac{1+bc}{a}$.

$\text{SL}(2)$ is a 3-dimensional local Lie group. Clearly, $\text{SL}(2)$ is a subgroup of $\text{GL}(2)$.

Let $g(t), g(0) = e$ is an analytic curve whose tangent vector $\alpha$ at $e$ then, $\alpha$ can be identified with the complex 2X2 matrix

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}_{t=0}$$
where \( \frac{d}{dt} d(t) \bigg|_{t=0} = \frac{d}{dt} (\frac{1+b(t)c(t)}{a(t)}) \bigg|_{t=0} = -a_1 \)

Lie-algebra \( L[SL(2)] = Sl(2) \) is the space of all \( 2 \times 2 \) complex matrices with trace zero and lie product is given by

\[
[\alpha, \beta] = \alpha \beta - \beta \alpha \quad \text{for } \alpha, \beta \in sl(2).
\]

The special elements

\[
\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}
\]

obey the commutation relations

\[
[3, +] = +, \quad [3, -] = -, \quad [+, -] = 2j \quad \ldots \quad (1.6.7)
\]

form a basis for \( sl(2) \). Every \( \alpha \in Sl(2) \) can be written uniquely in the form

\[
\alpha = a_1^+ j + a_2^- j + a_3^3 j; \quad a_1, a_2, a_3 \in \mathfrak{t}.
\]

1.7 LOCAL TRANSFORMATION GROUPS

Let \( G \) be a \( n \)-dimensional local Lie-group and \( U \) an open set in \( G \). Suppose there is given a mapping \( F : UXG \rightarrow \mathfrak{t}^m \). 

such that
\[ F(x,g) = xg \in \mathcal{F} \quad \text{for} \quad x \in U, \ g \in G. \]

Here \( e \) is the identity element of \( G \) and \( x \in U \) such that \( x = (x_1, x_2, \ldots, x_m) \).

**Definition:** \( G \) acts on manifold \( U \) as a local transformation group if the mapping \( F \) satisfies the conditions

1. \( Xg \) is analytic in the co-ordinates of \( x \) and \( g \)
2. \( xe = x \)
3. If \( xg \in U \) then \( (xg)g' = x(gg') \), \( g, g' \in G \).

**Definition:** The Lie derivative \( L_\alpha f \) of an analytic function \( f(x) \) is
\[
L_\alpha f(x) = \frac{d}{dt} \left[ \frac{(\exp \alpha t f)(x)}{t=0} \right] \quad \alpha \in L(G).
\]

The commutator \([L_\alpha, L_\beta]\) of the Lie-derivative \( L_\alpha, L_\beta \) is defined as \( [L_\alpha, L_\beta] = L_\alpha L_\beta - L_\beta L_\alpha \).

**Theorem (1.7.1)** [Lie's First Fundamental Theorem].

The unique solution of the equation
\[
\frac{dx}{dt} = L_\alpha x, \quad x(0) = x^0
\]
is the trajectory \( x(t) = x^0 \exp xt \).
Theorem (1.7.2) [Lie's second Fundamental Theorem]

The set of all Lie derivatives of a local Lie transformation group $G$ forms a Lie-algebra which is homomorphic image of $L(G)$.

In fact, $L_{a\alpha + b\beta} = aL_\alpha + bL_\beta$, $L_{[\alpha,\beta]} = [L_\alpha, L_\beta]$ for all $a, b \in \mathbb{C}$; $\alpha, \beta \in L(G)$.

Local Multiplier Representation

Let $G$ be a local Lie transformation group acting on an open neighbourhood $U \subseteq \mathbb{C}^m$, $0 \in U$ and $C_l$ be the set of all analytic functions in a neighbourhood of identity $C$.

Definition: A local multiplier representation $\mathcal{T}$ of $G$ on $C_l$ with multiplier $\gamma$, consist of a mapping $\mathcal{T}(g)$ of $C_l$ onto $C_l$ defined for $g \in G$, $f \in C_l$ by

$$[\mathcal{T}(g)f](x) = \gamma(x,g)f(x,g); \ x \in U.$$ 

Where $\gamma(x,g)$ is a complex-valued function analytic in $x$ and $g$ such that...
(1) \( \gamma(x,e) = 1 \)

(2) \( \gamma(\gamma, g_1 g_2) = \gamma(x, g_1) \gamma(x g_1, g_2); g_1, g_2, g_1 g_2 \in G. \)

Property (2) is equivalent to the relation

\[ \gamma(g_1 g_2)^T f(x) = [T(g_1)^\gamma T(g_2)^T f](x). \]

**Note (1.7.1).** The multiplier representation \( T, T' \) are isomorphic if there is an analytic isomorphism \( \mu \) of \( G_1 \), onto \( G_2 \) and analytic one to one co-ordinate transformation \( \phi \) of a neighbourhood of \( 0 \in U_1 \) onto a neighbourhood of \( 0 \in U_2 \) such that

(1) \( \phi(x_1 g_1) = \phi(x_1) \mu(g_1) \)

(2) \( \gamma_2(\phi(x_1), \mu(g_1)) = \gamma_1(x_1, g_1); \) for \( x_1 \in U_1, g_1 \in G_1. \)

**Definition:** The generalized Lie-derivative \( D_\alpha f \) on an analytic function \( f(x) \) under the 1-parameter subgroup \( \exp \alpha t \) is the analytic function

\[ D_\alpha f(x) = \frac{d}{dt} \left[ \gamma^T (\exp \alpha t) f \right]_{t=0} \]

For \( \gamma = 1 \) the generalized Lie-derivative becomes the ordinary Lie-Derivative.
THE COMPLEX 3-DIMENSIONAL LIE ALGEBRA

2.1 INTRODUCTION:

Bessel functions are related to the representation theory of $\mathfrak{g}(0,0)$ as demonstrated in Miller [35]. These functions appear in two distinct ways: as matrix elements of local irreducible representations of $G(0,0)$ and as basis functions for irreducible representations of $\mathfrak{g}(0,0)$. The first relationship will yield addition theorems and the second will yield generating functions and recursion relations for Bessel functions.

Since $\mathfrak{g}(0,0) \cong \mathfrak{t}_3^+(\mathbb{C})$, where $\mathfrak{g}(0,0)$ is special case of a 4-dimensional complex Lie algebra $\mathfrak{g}(a,b)$ for any pair of complex numbers $(a,b)$ and $\mathfrak{t}_3$ is the one dimensional Lie algebra generated by $\xi$, the representation theory of $\mathfrak{g}(0,0)$ is concerned with the subalgebra $\mathfrak{t}_3$. For a theory of Bessel functions, it is sufficient to study the representation theory of $\mathfrak{t}_3$ and the local Lie group $T_3$. 
The Euclidean group in the plane $E_3$ is a real parameter global Lie group where Lie algebra is a real form of $\mathfrak{J}_3$. The faithful irreducible representations of $E_3$ are well known with respect to a suitable bases, the matrix elements of these representations are proportional to Bessel function of integral order [Vilenkin [47], Wigner [53]]. The relationship between local representations of $T_3$ and unitary representations of $E_3$ are studied in Miller [35, chapter 3].

Some new algebras which have realizations by generalized Lie derivatives in one and two complex variables can be discovered which lead to new classes of special functions. Miller [35, Chapter 9] introduce a family of 3-dimensional Lie algebra $\mathfrak{g}_{p,q}$ which forms a natural generalization of $\mathfrak{J}_3 \cong \mathfrak{g}_{1,1}$. The special functions associated with this family form a natural generalization of Bessel functions, and the identities obeyed by these functions are analogous to those derived for Bessel functions [35, Chapter 3].

2.2 THE LIE ALGEBRA:

Corresponding to a pair of positive integers $(p,q)$, let
\[ \mathcal{G}_{p,q} \] be the complex 3-dimensional Lie algebra with basis \( j^3, j^+, j^- \) and commutation relations

\[
[j^3, j^+] = pj^+, [j^3, j^-] = -qj^-, [j^+, j^-] = 0 \quad \text{..... (2.2.1)}
\]

Clearly, \( \mathcal{G}_{1,1} \) is isomorphic to \( \mathcal{J}_3 \). In addition the following isomorphisms are easily established:

**Lemma:** \( \mathcal{G}_{p,q} \cong \mathcal{G}_{q,p} \); \( \mathcal{G}_{np, nq} \cong \mathcal{G}_{p,q} \)

for all positive integers \( p, q, n \).

According to this lemma every Lie algebra \( \mathcal{G}_{p', q'} \) is isomorphic to a Lie algebra \( \mathcal{G}_{p, q} \) such that

1. \( p \) and \( q \) are relatively prime positive integers
2. \( p \) is odd; and
3. if \( q \) is odd, then \( p > q \).

Consequently, from now on the pair \( (p, q) \) will be assumed to satisfy properties (1)-(3). It is obvious that two Lie algebras \( \mathcal{G}_{p, q}, \mathcal{G}_{p', q'} \) with subscripts satisfying these properties, are isomorphic if and only if \( p = p' \) and \( q = q' \).
Applying the technique develop in Miller [35, p.279] we can determine all of the transitive effective realizations of $\mathcal{G}_{p,q}$ by gd's in one or two complex variables. Thus, from [35, p.294(8.26)] there follows the realization

$$\gamma_1 : \frac{\partial}{\partial y}, e^{py}, e^{-qy}, r = 3, k = 2, s = 0 \ldots \quad (2.2.2)$$

Every transitive effective realization of $\mathcal{G}_{p,q}$ by gd's in complex variable in an element of $\mathcal{M}(\gamma_1)$.

Similarly, by making use of Lie's tables [Lie[26], P. 71-73], it can be shown that every transitive effective realization of $\mathcal{G}_{p,q}$ by gd's in two complex variables is an elements of $\mathcal{L}(\delta_j)$, $j = 1, 2, 3$, where

$$\delta_1 : \frac{\partial}{\partial z_1}, e^{pz_1} \frac{\partial}{\partial z_2}, e^{-qz_1} \frac{\partial}{\partial z_2} ; r = 3, k = 1, s = 0$$

$$\delta_2 : -pz_1 \frac{\partial}{\partial z_1} + qz_2 \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} ; r = 3, k = 1, s = 0 \quad (2.2.3)$$

$$\delta_3 : \frac{\partial}{\partial z_1}, e^{pz_1} \frac{\partial}{\partial z_2}, e^{-qz_1} ; r = 3, k = 1, s = 0$$
2.3 THE LIE GROUP $G_{p,q}$

Corresponding to a pair of relatively prime positive integers $(p,q)$ such that $p$ is odd, denote by $G_{p,q}$ the 3-dimensional complex Lie group with elements

$$g(b,c,\tau), \quad b,c, \tau \in \mathbb{C}$$

and group multiplication

$$g(b_1,c_1,\tau_1) g(b_2,c_2,\tau_2) = g(b_1 + e^{p\tau_1} b_2, c_1 + e^{-q\tau_1} c_2, \tau_1 + \tau_2)$$

It is easy to check that the multiplication is associative.

Furthermore, $e = g(0,0,0)$ is the identity element and

$$g(-e^{-p\tau} b, -e^{-q\tau} c, -\tau)$$

is the unique inverse of the group elements $g(b,c,\tau)$.

$G_{p,q}$ has a $4 \times 4$ matrix realization

$$g(b,c,\tau) = \begin{pmatrix}
1 & 0 & 0 & \tau \\
0 & e^{-q\tau} & 0 & c \\
0 & 0 & e^{p\tau} & b \\
0 & 0 & 0 & 1
\end{pmatrix}$$

$\ldots \ldots \ldots (2.3.2)$
where matrix multiplication corresponds to the group operation. $G_{1,1}$ is isomorphic to $T_3$.

The Lie algebra of $G_{p,q}$ can be computed from either of the expressions (2.3.1) or (2.3.2), and is easily recognized to be isomorphic to $\mathfrak{g}_{p,q}$. Indeed we can make the identification.

$$g(b,c,\tau) = \exp(bj^+) \exp(cj^-) \exp(\tau j^3) \quad \text{..... (2.3.3)}$$

where $j^+$, $j^-$, $j^3$ generate $\mathfrak{g}_{p,q}$ and satisfy the commutation relation (2.2.1).

### 2.4 REPRESENTATION OF $\mathfrak{g}_{p,q}$:

Let $\mathcal{P}$ be a representation of $\mathfrak{g}_{p,q}$ on an abstract vector space $V$ and let $\mathcal{P}(j^+) = J^+$, $\mathcal{P}(j^-) = J^-$, $\mathcal{P}(j^3) = J^3$, be operators on $V$. Clearly,

$$[J^3, J^+] = pJ^3, [J^3, J^-] = -qJ^-, [J^+, J^-] = 0 \quad \text{..... (2.4.1)}$$

Note that the operator $C_{p,q} = (J^+)^q (J^-)^p = (J^-)^p (J^+)^q$ commutes with all operators $\mathcal{P}(a)$, $a \in \mathfrak{g}_{p,q}$ on $V$. For $\mathcal{P}$ irreducible we would expect $C_{p,q}$ to be a multiple of $I$. 
We will classify all representations $\rho$ satisfying the properties

(i) $\rho$ is irreducible

(ii) Each eigenvalue of $J$ has multiplicity $\ldots (2.4.2)$ equal to one. There is a countable basis for $V$ consisting of eigen vectors of $J$.

We give without proof the results of the straightforward classification.

**Theorem:** Every representation $\rho$ of $\mathbb{C}^p,q$ satisfying (2.4.2) and for which $C^p,q \neq 0$, on $V$, is isomorphic to a representation of the form $Q_p,q(w,m_o)$ defined for $w,m_o \in \mathbb{C}$ such that $w \neq 0$ and $0 \leq \Re m_o < 1$. The spectrum of $J$ corresponding to $Q_p,q(w,m_o)$ is given by $S = \left\{ m_o + n \mid n \text{ is an integer} \right\}$ and there is a basis $\{ f_m \}$, $m \in S$, for $V$ such that

\[
\begin{align*}
J f_m &= m f_m, \\
J f_m &= w f_m + p,
\end{align*}
\]

\[
\begin{align*}
J f_m &= w f_m - q, \\
C^p,q f_m &= (J^+ q - p) f_m = p + q
\end{align*}
\]

There exist isomorphism $Q_p,q(w,m_o) \cong Q_p,q(w',m_o)$ if and only if
Consider the operators

\[ J^3 = z \frac{d}{dz} + C_1, \quad J^+ = C_2 z^p, \quad J^- = C_2 z^{-q} \] .... (2.4.4)

Let \( V_1 \) be the complex vector space of all finite linear combinations of the functions \( h_n(z) = z^n, \quad n = 0, \pm 1, \pm 2, \ldots \)
and define operators \( J^+, J^3 \) on \( V_1 \) by (2.4.4) where \( C_1 = m_0, \quad C_2 = w \). Define the basis vector \( f_m \) of \( V_1 \) by \( f_m(z) = h_n(z) \) where \( m = m_0/n \) and runs over the integers.

Then

\[ J^3 f_m = (z \frac{d}{dz} + m_0)z^n = mz^n = mf_m, \]
\[ J^+ f_m = (wz^p)z^n = wf_{m+p}, \quad J^- f_m = (wz^{-q})z^n = w_{m-q} \] .... (2.4.5)
\[ C^{p-q} f_m = w f_m \]

These equation agree with (2.4.3) and yield a realization of \( Q_p,q(w,m_0) \).

The differential operators (2.4.4) \( (C_1 = m_0, \quad C_2 = w) \) induced a multiplier representation \( A \) of \( G_{p,q} \) on the space
$V_2$ consisting of those functions $f(z)$ which are analytic and single valued for all $z \neq 0$. This multiplier representation is defined by operators $A(g)$, $g = (b, c, \tau) \in G_{p,q}$

$$[A(g)f](z) = \exp[w(bz + cz^*) + m_0 \tau]f(e^{\tau}z), f \in V_2 \quad \ldots \quad (2.4.6)$$

Clearly, $V_2$ is invariant under the operator $A(g)$ and the group property $A(g_1g_2) = A(g_1)A(g_2)$ is valid for all $g_1, g_2 \in G_{p,q}$.

The matrix elements $A_{\ell k}(g)$ of $A(g)$ with respect to the analytic basis $\{f_m = h_n\}$ of $V_2$ are defined by

$$[A(g)h_k](z) = \sum_{\ell = -\infty}^{\infty} A_{\ell k}(g)h(z), g \in G_{p,q}, k = 0, \pm 1, \pm 2, \ldots \quad (2.4.7)$$

or

$$\exp[w(bz + cz^*) + (m_0 + k)\tau]z^k = \sum_{\ell = -\infty}^{\infty} A_{\ell k}(g)z^\ell \quad \ldots \quad (2.4.8)$$

where $g = g(b, c, \tau)$. Explicitly,

$$A_{\ell k}(g) = e^{(m_0 + k)\tau} F_{\ell-k}^{p,q}(b, c) \quad \ldots \quad (2.4.9)$$

where

$$F_{\ell}^{p,q}(b, c) = w^{s_\ell + n_\ell} b^{n_\ell} c^{s_\ell} \sum_{j=0}^{\infty} \frac{(w b c)^j}{(n_\ell + jq)!(s_\ell + jp)!} \quad \ldots \quad (2.4.10)$$
The non-negative integer $s_{\ell}, n_{\ell}$ are uniquely determined by the properties:

(1) \[ \ell = n_{\ell} p - s_{\ell} q \]

(2) If \( \ell = n_{\ell} p - s_{\ell} q \) where \( n_{\ell}, s_{\ell} \) are non-negative integers, then \( n_{\ell} + s_{\ell} \geq n_{\ell} + s_{\ell} \).

Since \( p \) and \( q \) are relatively prime, the integers \( n_{\ell}, s_{\ell} \) can easily be shown to exist for all \( \ell \). For example, if \( p = q = 1 \) then \( s_{\ell} = 0, n_{\ell} = \ell \) for \( \ell \geq 0 \) and \( s_{\ell} = -\ell, n_{\ell} = 0 \) for \( \ell < 0 \).

If \( bc \neq 0 \) we can introduce new group parameters \( r, v \) defined by \( b = \frac{rv}{(p+q)}, c = \frac{-q}{(p+q)} \). In terms of these parameters the matrix elements are

\[
A_{\ell k}(g) = e^{(m_0+k)\ell \frac{1}{2}} \sum_{\ell-k}^{p, q} v I_{\ell-k}(w r) \tag{2.4.11}
\]

where

\[
I_{\ell}^{p, q}(r) = \left( \frac{r}{p+q} \right)^{\ell} \frac{1}{\Gamma(p+q)} \sum_{j=0}^{\infty} \frac{\ell!}{(n_{\ell} + jq)! (s_{\ell} + jp)!} \tag{2.4.12}
\]

It follows from the ratio test that \( I_{\ell}^{p, q}(r) \) is an entire function of \( r \) for all integer \( \ell \). Here \( I_{\ell}^{1, 1}(r) = (-i)^{\ell} J_\ell(ir) \), \( \ell \geq 0 \), is the ordinary "modified Bessel function". Thus we
can consider the functions $I_{\ell}^{p,q}(r)$ to be a group theoretic generalization of Bessel functions. Substitute (2.4.11) into (2.4.8) to obtain the simple generating function

$$
\exp\left[\frac{r}{p+q}(z+z')\right] = \sum_{=\infty}^{\infty} I_{\ell}^{p,q}(r) z^\ell \quad \cdots \quad (2.4.13)
$$

The group property of the operators $A(g)$ implies the addition theorem

$$
A_{\ell k}(g_1 g_2) = \sum_{j=-\infty}^{\infty} A_{\ell j}(g_1) A_{jk}(g_2) \quad \cdots \quad (2.4.14)
$$

valid for all $g_1, g_2 \in G_{p,q}$. This leads immediately to the identity

$$
F_{\ell}^{p,q}(b_1+b_2, c_1+c_2) = \sum_{j=-\infty}^{\infty} F_{\ell-j}^{p,q}(b_1, c_1) F_{j}^{p,q}(b_2, c_2) \quad \cdots \quad (2.4.15)
$$

We could also use the addition theorem to derive the identities involving the functions $I_{\ell}^{p,q}(r)$. 
3.1 INTRODUCTION:

Making use of group theoretic method see e.g. [35, chapters 2 and 3; 30, chapter 4 and 5] Weisner [13, p.145] obtained a mixed generating function

\[
(1 + \frac{z}{y}) e^{2wy-y} = \sum_{n=\infty}^{\infty} g_n y^n, \quad |y| > |z|
\]

\[\ldots \quad (3.1.1)\]

Where \( g_n = \sum_{k=0}^{\infty} \binom{\gamma}{k} H_{k+n}(w)z^k \), \( n = 0, \pm 1, \pm 2, \ldots \)

\( H_n(x) \) is Hermite polynomial \[10\] and \( \binom{\gamma}{k} = \frac{\gamma!}{k!(\gamma-k)!} \)

we also recall a number of results of Miller \[35, pp.83(4.14), 87(4.27),(4.124)\] involving Laguerre polynomials \( L_n(x) \) using Lie theoretic methods expansions of the types

\[
e^{xt} = \sum_{n=-\infty}^{\infty} g_n(x) t^n
\]

were studied by Halphen and Bird \[see[13, p.237]\] Meixner \[13, p.273\] determined all orthogonal polynomials \( g_n(x) \) which
possess a generating function of the type

\[ f(t) \exp[xu(t)] = \sum_{n=0}^{\infty} g_n(x) \frac{t^n}{n!} \quad \ldots \ldots (3.1.2) \]

The fact that a generating function of the type

\[ \frac{1}{(1-t)^{k-1}} = \sum_{n=0}^{\infty} \frac{1}{(k)_n} \frac{t^n}{n!}, \quad k = 0, 1, 2, \ldots \]

where \( g_n(x) = \sum_{n-m}^{n-1} \frac{(-1)^n}{n-m} (\frac{1}{k})_x \)

is the \( k \)-th Cesaro mean of the first \( n \)-partial sums of the series \( 1 + x + x^2 + \ldots \)

for the applications see Obrechkoff [13, p. 245]) and a well known mixed generating functions [see, 3, p. 176].

\[ \exp[x(y+t - \frac{1}{yt})] = \sum_{m,n=0}^{\infty} J_{m,n}(x) y^m t^n \quad \ldots \ldots (3.1.3) \]

where \( J_{m,n}(x) = \frac{x^{m+n}}{(m+1)(n+1)} \Gamma\left(\frac{3}{2}\right) \)

and for negative values of \( m, n \)

\[ J_{m,n}(x) = x^{n+m} \sum_{r=0}^{\infty} \frac{(-x)^r}{r!(r+1+m)} \]

raises the question whether results of these types may be
generalized in the form of a set of generating functions for the product of hypergeometric series \( ? \). The purpose of this note is to answer this question. The resulting formula (3.2.1) allow a considerable unification of the results given above and various special results which appear in the literature.

3.2 **THE MAIN GENERATING FUNCTION:**

If the function

\[
V = pF_q \left[ \begin{array}{c} (a_p); \\ xy \\ (b_q); \end{array} \right] F_s \left[ \begin{array}{c} (c_r); \\ xt \\ (d_s); \end{array} \right] uF_v \left[ \begin{array}{c} (e_u); \\ -\frac{x}{yt} \\ (f_v); \end{array} \right]
\]

is expanded as double series of powers of \( y \) and \( t \), we have

\[
V = \sum_{j=0}^{\infty} \frac{((a_p))_j (xy)^j}{((b_q))_j j!} \sum_{k=0}^{\infty} \frac{((c_r))_k (xt)^k}{((d_s))_k k!} \sum_{i=0}^{\infty} \frac{((e_u))_i (-\frac{x}{yt})^i}{((f_v))_i i!}
\]

\[
= \sum_{i=0}^{\infty} \frac{((e_u))_i (-x)^i}{((f_v))_i i!} \sum_{j=0}^{\infty} \frac{((a_p))_j (xy)^j}{((b_q))_j j!} \sum_{k=0}^{\infty} \frac{((c_r))_k x^k t^k}{((d_s))_k k!}
\]

\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{((e_u))_i ((a_p))_j ((c_r))_k (-1)^j (-x)^i y^j t^k}{((f_v))_i ((b_q))_j ((d_s))_k i! j! k!}
\]

\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{((e_u))_i ((a_p))_j ((c_r))_k (-1)^j (-x)^i y^j t^k}{((f_v))_i ((b_q))_j ((d_s))_k i! j! k!}
\]
Replacing \( j-i = m \) and \( k-i \), respectively, by \( m, n \),

\[ j+k = m+n+2i, \]

then after rearrangement, justified by the absolute convergence of the above series, it follows that

\[
\begin{align*}
V &= \sum_{i=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{((e_u)_i (a_p)_m i (c_r)_n i)(-1)}{((f_v)_i (b_q)_m i (d_s)_n i)(n+i)!} \\
&= \sum_{i=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{((-x)^i m n)(-x)^i y t}{(m+i)! (n+i)!} \\
&= \sum_{m,n=\infty}^{\infty} \frac{((-x)^i m n)(-x)^i y t}{(m+1)_i (n+1)_i i!} \\
&= \sum_{m,n=\infty}^{\infty} \frac{((-x)^i m n)(-x)^i y t}{(m+1)_i (n+1)_i i!}
\end{align*}
\]
\[ \sum_{m,n=\infty}^{\infty} \frac{((a_p)_m ((c_r)_n x^m y^n t)}{((b_q)_m ((d_s)_n m! n!)} \]

\[ p+u F_q+s+v+2 \begin{bmatrix} (a_p)_m, (c_r)_n, (e_u)_n; \\
(b_q)_m, (d_s)_n, (f_v)_m+1, n+1; \end{bmatrix} \]

\[ \Rightarrow p^F_q \begin{bmatrix} (a_p); \\
(b_q); \end{bmatrix} r^F_s \begin{bmatrix} (c_r); \\
(d_s); \end{bmatrix} u^F_v \begin{bmatrix} (e_u); \\
(f_v); \end{bmatrix} \]

\[ = \sum_{m,n=\infty}^{\infty} \frac{((a_p)_m ((c_r)_n x^m y^n t)}{((b_q)_m ((d_s)_n m! n!)} \]

\[ p+u F_q+s+v+2 \begin{bmatrix} (a_p)_m, (c_r)_n, (e_u)_n; \\
(b_q)_m, (d_s)_n, (f_v)_m+1, n+1; \end{bmatrix} \]

\[ \Rightarrow (3.2.1) \]

3.3 **SPECIAL CASES:**

In (3.2.1) setting \( p = q = r = s = u = v = l = 1, a_1 = -M \)
\( b_1 = 1+\alpha, c_1 = -N, d_1 = 1+\beta, e_1 = -R, f_1 = 1+\gamma, x = -x, y = -y \)
and \( t = -t, \) we get the following generating relation involving a product of three Laguerre Polynomials \( L_m^{(\alpha)}(x) \) [10, p.200]
\[ 1^F_1(-M; 1+\alpha; xy) \cdot 1^F_1(-N; 1+\beta; xt) \cdot 1^F_1(-R; 1+\gamma; \frac{x}{yt}) \]

\[ = \sum_{m,n=-\infty}^{\infty} \frac{(-M)_m (-N)_n x}{(1+\alpha)_m (1+\beta)_n} \frac{m+n}{m! n!} \frac{y t}{x} \]

\[ 3^F_5 \left[ \begin{array}{c}
-M+m, -N+n, -R; \\
1+\alpha+m, 1+\beta+n, 1+\gamma, m+1, n+1;
\end{array} \right] \]

\[ \frac{M!}{(1+\alpha)_M} L_M(\alpha)(xy) \cdot \frac{N!}{(1+\beta)_N} L_N(\beta)(xt) \cdot \frac{R!}{(1+\gamma)_R} L_R(\gamma)(\frac{x}{yt}) \]

\[ = \sum_{m,n=-\infty}^{\infty} \frac{(-M)_m (-N)_n x}{(1+\alpha)_m (1+\beta)_n} \frac{m+n}{m! n!} \frac{y t}{x} \]

\[ 3^F_5 \left[ \begin{array}{c}
-M+m, -N+n, -R; \\
1+\alpha+m, 1+\beta+n, 1+\gamma, m+1, n+1;
\end{array} \right] \]

\[ \frac{L_M(\alpha)(xy)}{L_N(\beta)(xt)} \cdot \frac{L_R(\gamma)(\frac{x}{yt})}{(1+\alpha)_M (1+\beta)_N (1+\gamma)_R} \]

\[ = \frac{(1+\alpha)_M (1+\beta)_N (1+\gamma)_R}{M! N! R!} \sum_{m,n=-\infty}^{\infty} \frac{(-M)_m (-N)_n x}{(1+\alpha)_m (1+\beta)_n} \frac{m+n}{m! n!} \frac{y t}{x} \]

\[ 3^F_2 \left[ \begin{array}{c}
-M+m, -N+n, -R; \\
1+\alpha+m, 1+\beta+n, 1+\gamma, m+1, n+1;
\end{array} \right] \]

\[ \ldots \ldots (3.3.1) \]
If in (3.2.1), we set \( p = r = u = 2, \ q = s = v = 1, \)
\( a_1 = -M, \ a_2 = 1+\alpha+\beta, \ b_1 = 1+\alpha, \ c_1 = -N, \ c_2 = 1+\gamma+\delta, \ d_1 = 1+\gamma, \)
\( e_1 = -R, \ e_2 = \xi+\eta+1, \ f_1 = 1+\xi, \ x = -x, \ y = -y \) and \( t = -t, \)
we get

\[
2F_1\begin{bmatrix}
-M, 1+\alpha+\beta; \\
1+\alpha;
\end{bmatrix}_{xy}
2F_1\begin{bmatrix}
-N, 1+\gamma+\delta; \\
1+\gamma;
\end{bmatrix}_{xt}
2F_1\begin{bmatrix}
-R, \xi+\eta+1; \\
1+\xi;
\end{bmatrix}_{yt}
\]

\[
= \sum_{m,n=\infty}^{\infty} \frac{(-M)^m (1+\alpha+\beta)_m (-N)^n (1+\gamma+\delta)_n}{(1+\alpha)_m (1+\gamma)_n m! n!} x^m y^n t^n
\]

\[
6F_5\begin{bmatrix}
-M+m, 1+\alpha+\beta+m, -N+n, 1+\gamma+\delta+n, -R, \xi+\eta+1; \\
1+\alpha+m, 1+\gamma+n, 1+\xi, m+1, n+1;
\end{bmatrix}
\]

\[
= \sum_{m,n=\infty}^{\infty} \frac{(\alpha,\beta-M)_m (\gamma,\delta-N)_n}{(1+\alpha)_m (1+\gamma)_n m! n!} x^m y^n t^n
\]

\[
6F_5\begin{bmatrix}
-M+m, 1+\alpha+\beta+m, -N+n, 1+\gamma+\delta+n, -R, \xi+\eta+1; \\
1+\alpha+m, 1+\gamma+n, 1+\xi, m+1, n+1;
\end{bmatrix}
\]

\[
= \sum_{m,n=\infty}^{\infty} \frac{(-M)^m (1+\alpha+\beta)_m (-N)^n (1+\gamma+\delta)_n}{(1+\alpha)_m (1+\gamma)_n m! n!} x^m y^n t^n
\]

\[
6F_5\begin{bmatrix}
-M+m, 1+\alpha+\beta+m, -N+n, 1+\gamma+\delta+n, -R, \xi+\eta+1; \\
1+\alpha+m, 1+\gamma+n, 1+\xi, m+1, n+1;
\end{bmatrix}
\]
\[
\begin{align*}
\pmb{p}_m^{(\alpha, \beta-\mu)}(1-2xy) \pmb{p}_n^{(\gamma, \delta-\nu)}(1-2xt) \pmb{p}_r^{(\xi, \eta-\rho)}(1 - \frac{x}{yt}) \\
= \sum_{m,n=-\infty}^{\infty} \frac{(-\mu)_m (1+\alpha+\beta)_m (-\nu)_n (1+\gamma+\delta)_n}{(1+\alpha)_m (1+\gamma)_n} x^m y^n t
\end{align*}
\]

\[
\delta F_5 \left[ \begin{array}{c}
-M+m, 1+\alpha+\beta+n, -N+n, 1+\gamma+\delta+n, -R, \xi+\eta+1; \\
1+\alpha+m, 1+\gamma+n, 1+\delta, m+1, n+1;
\end{array} \right] x^3 \] .... (3.3.2)

where \( p_n(x) \) is Jacobi polynomials [10, p.254(1)].

In (3.2.1) setting \( p = r = u = 3, q = s = v = 2 \),

\[
\begin{align*}
a_1 &= -M, a_2 = 1+\alpha+\beta, a_3 = a, b_1 = 1+\alpha, b_2 = b, c_1 = -N, \\
c_2 &= 1+\gamma+\delta, c_3 = c, d_1 = 1+\gamma, d_2 = d, e_1 = -R, e_2 = 1+\xi+\eta, \\
e_3 &= e, f_1 = 1+\delta, f_2 = f,
\end{align*}
\]

we get

\[
\begin{align*}
&3F_2 \left[ \begin{array}{c}
-M, 1+\alpha+\beta, a; \\
1+\alpha, b;
\end{array} \right] \quad 3F_2 \left[ \begin{array}{c}
-N, 1+\gamma+\delta, c; \\
1+\gamma, d;
\end{array} \right] \quad 3F_2 \left[ \begin{array}{c}
-R, 1+\xi+\eta, e; \\
1+\xi, f;
\end{array} \right] \\
= \sum_{m,n=-\infty}^{\infty} \frac{(-\mu)_m (1+\alpha+\beta)_m (a)_m (-\nu)_n (1+\gamma+\delta)_n (c)_n}{(1+\alpha)_m (1+\gamma)_n} x^m y^n t
\end{align*}
\]
where \( H_n^{(a,b,x)} \) is generalized Rice polynomials \([39, p.158 (2.3)]\).

For \( p = r = u = 1, \ q = s = v = 0, \ a_1 = a, \ c_1 = c, \ e_1 = e, \ x = -x, \ y = -y, \ t = -t \) \((3.2.1)\) gives

\[
\begin{align*}
1^F_0[a; xy] & 1^F_0[c; xt] 1^F_0[e; \frac{x}{yt}] \\
= & \sum_{m,n=-\infty}^{\infty} \frac{(a)_m (c)_n x^m y^n}{m! n!} 3^F_2 \left[ \begin{array}{c} a+m, c+n, e; \\ m+1, n+1; \end{array} \right] x^3
\end{align*}
\]

\((1-xy) -a (1-xt) -c (1 - \frac{x}{yt}) \)

\[
= \sum_{m,n=-\infty}^{\infty} \frac{(a)_m (c)_n x^m y^n}{m! n!} 3^F_2 \left[ \begin{array}{c} a+m, c+n, e; \\ m+1, n+1; \end{array} \right] x^3 \quad \ldots (3.3.4)
\]

On taking \( p = r = u = 0, \ q = s = v = 1, \ b_1 = 1+\alpha, \)

\(d_1 = 1+\beta\) and \( f_1 = 1+\gamma \) in \((3.2.1)\) and replacing \( y, x \) and \( t \)

by \( \frac{-x}{4}, \frac{-t}{4} \) and \( \frac{16}{y^2} \) respectively, we get

\[
\begin{align*}
0^F_1[1+\alpha; \frac{-x^2}{4}] & 0^F_1[1+\beta; \frac{-t^2}{4}] 0^F_1[1+\gamma; \frac{-y^2}{4}] \\
= & \sum_{m,n=-\infty}^{\infty} \frac{(-\frac{x}{4})^m (-\frac{t}{4})^n (\frac{16}{y^2})^n}{(1+\alpha)(1+\beta)n! m! n!} 0^F_5 \left[ \begin{array}{c} -; -; -; -; -; \\ 1+\alpha+m, 1+\beta+n, 1+\gamma, m+1, n+1; \end{array} \right] t^3
\end{align*}
\]
\[ \frac{\alpha!}{(\frac{x}{2})^\alpha} J_\alpha(x) \frac{\beta!}{(t/2)^\beta} J_\beta(t) \frac{\gamma!}{(y/2)^\gamma} J_\gamma(y) \]

\[ = \sum_{m,n=-\infty}^{\infty} \frac{(-1)^n 4^{-n} t^m x^y}{(1+\alpha)_m (1+\beta)_n m! n!} {}_0F_5 \left[ \begin{array}{c} -\alpha, -\beta, -\gamma, -m, -n \end{array} ; \frac{t}{64} \right] \]

\[ J_\alpha(x) J_\beta(t) J_\gamma(y) = \frac{x^\alpha y^\beta t^\gamma}{2^{\alpha+\beta+\gamma}} \sum_{m,n=-\infty}^{\infty} (-1)^n 4^{-n} t^m x^y \frac{1}{(1+\alpha)_m (1+\beta)_n m! n!} \]

\[ \times {}_0F_5 \left[ \begin{array}{c} -\alpha, -\beta, -\gamma, -m, -n \end{array} ; \frac{t}{64} \right] \]

\[ \ldots (3.3.5) \]

where \( J_n(x) \) is Bessel function \([10,p.108(1)]\).

On taking \( p = 1, q = r = s = u = v = 0 \) and \( a_1 = -\gamma \)
and replacing \( \frac{x}{yt}, xt \) and \( xy \) by \( \frac{2}{y} \), \( 2WY \) and \( \frac{2}{\gamma} \) respectively, we get (3.1.1).

For \( a = k+1, c = 1, e = 0, xy = t \Rightarrow y = \frac{t}{x} \) (3.3.4) gives

\[ -(k+1) \frac{1}{(1-t)} \frac{1}{(1-xt)} = \sum_{n=0}^{\infty} (k)_n n^{k-1} \]

\[ g_n(x) t^k \quad k = 0, 1, 2, \ldots \]
where \( g_n(x) = \sum_{m=0}^{n} (-1)^{n-m} \binom{k-1}{m} x^{n-m} \) \( \ldots \) (3.3.6)

In (3.2.1) setting \( p = u = 1, \quad q = r = s = u = 0, \quad a_1 = -2k, \quad e_1 = 0 \) we get the result [35, p.190 (5.101)]

\[
2k \cdot \frac{-(bz)}{1-b} \left( \frac{1}{1-b} \right)^{-2k-1} \left( \frac{1}{1-b} \right) = \sum_{\ell=0}^{\infty} b^\ell L_{\ell}^{(-2k-1)}(z), \quad |b| < 1
\]

\[
L_0^{(-2u-1)}(z) = 1 \quad \ldots \quad (3.3.7)
\]
CHAPTER-4

SOME TRANSFORMATION ON HYPERGEOMETRIC FUNCTIONS

4.1 INTRODUCTION:

In this chapter an attempt has been made to establish certain transformations of hypergeometric functions. We make use of certain known integral to establish our first and second transformation. To establish the rest of the transformations we use the known summation of $\mathbf{_{3}F_{2}}$ and a known integral.

4.2 EVALUATION OF CERTAIN INTEGRALS:

First, we shall prove the following integral

$$I_1 = \int_0^1 x^{\sigma-\lambda-\delta} (1-x)^{2 - \frac{1}{2} \mu} \frac{1}{\beta}(x) \frac{J_\delta(xt)}{J_\lambda(xt)} \ dx$$

$$= \frac{\left[ \frac{1}{2} (1+\sigma) \right] \left[ (1 + \frac{1}{2} \sigma) \right]^{\lambda+\delta}}{2^{\delta+\lambda+1-\mu} \left[ (1+\delta) \right] \left[ (1+\lambda) \right] \left[ 1+ \frac{1}{2} (\sigma-\gamma-\mu) \right] \left[ \frac{1}{2} (\sigma+\gamma-\mu+3) \right]}$$

$$= \mathbf{_{2}F_{2}}:\left[ \begin{array}{c} \frac{1}{2}(1+\sigma); 1 + \frac{1}{2} \sigma; -\frac{t}{4}; -\frac{t}{4} \\ 1 + \frac{1}{2}(\sigma-\gamma-\mu); \frac{1}{2}(\sigma+\gamma-\mu+3); 1+\lambda; 1+\delta/2 \end{array} \right]$$

Re($\mu$) < 1, Re($\sigma$) > -1 ...... (4.2.1)
Proof:

We have

\[ I_1 = \int_0^1 x^{\sigma-\delta} (1-x)^2 \frac{1}{2^\mu} P_\gamma(x) J_\delta(x) J_\lambda(x) \, dx \]

\[ = \int_0^1 x^{\sigma-\delta} (1-x)^2 \frac{1}{2^\mu} P_\gamma(x) \sum_{r=0}^{\infty} \frac{(-1)(\frac{xt}{2})^r}{\lambda+2s} \sum_{s=0}^{\infty} \frac{(-1)(\frac{xt}{2})^s}{1+\lambda+s} dx \]

\[ = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r \frac{xt}{2}^{\delta+2r}}{\delta+\lambda+2r+2s} \frac{1}{\lambda+2s} \frac{1}{1+\delta+r} \frac{1}{1+\lambda+s} \]

Now by using the integral [11,p. 172], we get

\[ I_1 = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r t^{\delta+2r+2s}}{\delta+\lambda+2r+2s} \frac{1}{\lambda+2s} \frac{1}{1+\delta+r} \frac{1}{1+\lambda+s} \]

\[ = \frac{r+s}{\delta+\lambda+2r+2s} \frac{1}{\lambda+2s} \frac{1}{1+\delta+r} \frac{1}{1+\lambda+s} \]

\[ \mu-1 \]

\[ \frac{\frac{1}{2} + \frac{1}{2}(\sigma+2r+2s)}{1+\frac{1}{2}(\sigma+2r+2s)} \frac{1+\frac{1}{2}(\sigma+2r+2s)}{1+\frac{1}{2}(\sigma+2r+2s)-\frac{1}{2} \gamma - \frac{1}{2} \mu} \frac{1}{\frac{1}{2}(\sigma+\gamma-\mu+3)+r+s} \]
\[ \text{Re}(\mu) < 1, \text{Re}(\sigma + 2r + 2s) > -1 \]

\[
\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^{r+s} \left(\frac{1}{2}\right)^{\delta + \lambda - \mu + 1} \left(\frac{1}{4}\right)^{r} \left(\frac{1}{4}\right)^{s} \left(\frac{1}{2}(1+\sigma)\right)^{r+s}}{\left(1+\delta\right)^{r+s}(1+\delta)(1+\lambda)^{r+s}(1+\lambda)\left(1+\frac{1}{2}(\sigma - \gamma - \mu)\right)^{r+s}} \cdot \frac{1}{2}(1+\sigma)
\]

\[
= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(1 + \frac{1}{2} \sigma)^{r+s}}{(1 + \frac{1}{2} \sigma)^{r+s}} \cdot \frac{\lambda + \delta + 2r + 2s}{\left(1 + \frac{1}{2}(\sigma - \gamma - \mu)\right)^{r+s} \left(\frac{1}{2}(\sigma + \gamma - \mu + 3)\right)^{r+s}}
\]

\[
= \frac{\left(\frac{1}{2}(1+\sigma)\right)^{r+s}}{(1+\delta)^{r+s}(1+\lambda)^{r+s}(1 + \frac{1}{2}(\sigma - \gamma - \mu))^2(\sigma + \gamma - \mu + 3)}
\]

\[
= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\frac{1}{2}(1+\sigma)^{r+s} \left(1 + \frac{1}{2} \sigma\right)^{r+s}}{(1+\frac{1}{2}(\sigma - \gamma - \mu))^{r+s} \left(\frac{1}{2}(\sigma + \gamma - \mu + 3)\right)^{r+s}} \cdot \frac{\lambda + \delta + 2r + 2s}{\left(1 + \frac{1}{2}(\sigma - \gamma - \mu)\right)^{r+s} \left(\frac{1}{2}(\sigma + \gamma - \mu + 3)\right)^{r+s}}
\]

\[
= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\frac{1}{2}(1+\sigma)^{r+s}}{(1+\delta)^{r+s}(1+\lambda)^{r+s}(1 + \frac{1}{2}(\sigma - \gamma - \mu))^2(\sigma + \gamma - \mu + 3)}
\]

which is the required result.
Next we prove an integral which is more generalized than the result (4.2.1).

\[ I_2 = \int_0^1 \frac{x^{\sigma-\lambda-\delta-2}}{(1-x)} P_\gamma(x) J_\delta(xt) J_\lambda(xt) J_k(xt) dx \]

\[
= \left[ \frac{1}{2(\sigma+k)} \right]^{\mu-(\delta + \lambda+k+1)} \delta + \lambda + k \\
\left[ 1+ \frac{1}{2}(\sigma+k) \right]^{\mu-(\delta + \lambda+k+1)} \delta + \lambda + k \\
\left[ (\delta+1) \right]^{\mu-(\delta + \lambda+k+1)} \delta + \lambda + k \\
\left[ (\lambda+1) \right]^{\mu-(\lambda+1)} \lambda + 1 \\
\left[ (k+1) \right]^{\mu-(k+1)} k + 1 \\
\left[ 1+ \frac{1}{2}(\sigma+k-\gamma-\mu) \right]^{\mu-(\delta + \lambda+k+1)} \delta + 1 \quad \lambda + 1 \quad k + 1 \]

\[
\left[ \frac{1}{2}(1+\sigma+k), 1+ \frac{1}{2}(\sigma+k) : -\frac{t}{4}, -\frac{t}{4}, -\frac{t}{4} \right] 
\]

\[
\text{Re } \mu < 1, \text{ Re } \sigma > -1 
\]

**Proof:** We have

\[ I_2 = \int_0^1 \frac{x^{\sigma-\lambda-\delta}}{(1-x^2)} P_\gamma(x) J_\delta(xt) J_\lambda(xt) J_k(xt) dx \]
\[
\int_0^1 x^{\omega - \lambda - \delta} (1-x^2)^{\frac{1}{2}\mu} p_\gamma(x) \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x^t}{2}\right)^{\delta+2m}}{m! \left(1+m+\delta\right)}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x^t}{2}\right)^{\lambda+2n}}{n! \left(1+n+\lambda\right)} \sum_{p=0}^{\infty} \frac{(-1)^p \left(\frac{x^t}{2}\right)^{k+2p}}{p! \left(1+p+k\right)} dx
\]

\[
= \sum_{m,n,p=0}^{\infty} \frac{(-1)^{m+n+p} \delta + \lambda + k + 2m + 2n + 2p}{\sqrt{2} \left[\delta + m + 1\right] \left[\lambda + n + 1\right] \left[k + p + 1\right] m! n! p!}
\]

\[
\int_0^1 x^{\omega + k + 2m + 2n + 2p} (1-x^2)^{\frac{1}{2}\mu} p_\gamma(x) dx
\]

which on using the integral [11, p. 172], gives

\[
I_2 = \sum_{m,n,p=0}^{\infty} \frac{(-1)^{m+n+p} \delta + \lambda + k + 2m + 2n + 2p}{\sqrt{2} \left[\delta + m + 1\right] \left[\lambda + n + 1\right] \left[k + p + 1\right] m! n! p!}
\]

\[
2^{\mu-1} \frac{1}{\sqrt{\frac{1}{2} \left(1+\sigma+k+2m+2n+2p\right)}} \frac{1}{\sqrt{\frac{1}{2} \left(\sigma+k+2m+2n+2p\right)}}
\]

\[
\left| \frac{1}{\sqrt{\frac{1}{2} \left(\sigma+k+2m+2n+2p+\gamma-\mu\right)}} \right| \frac{1}{\sqrt{\frac{1}{2} \left(\sigma+k+2m+2n+p+\gamma-\mu+3\right)}}
\]

\(\text{Re}(\mu) < 1, \text{Re}(\sigma) > -1\)
\[
\begin{align*}
\delta + \lambda + k & \quad 2^{\mu-(\delta + \lambda + k+1)} \left[ \frac{1}{2}(1+\sigma+k) \right] \\
& \quad 1 + \frac{1}{2}(\sigma+k)
\end{align*}
\]

\[
\frac{\delta + \lambda + k}{2^{\mu-(\delta + \lambda + k+1)}} \left[ \frac{1}{2}(1+\sigma+k) \right] \\
1 + \frac{1}{2}(\sigma+k)
\]

\[
\sum_{m,n,p=0}^{\infty} \left( \frac{1}{2}(1+\sigma+k) \right)_{m+n+p} \left( 1 + \frac{1}{2}(\sigma+k) \right)_{m+n+p} \left( \frac{1}{2}(\sigma+k+y-\mu+3) \right)_{m+n+p}
\]

\[
\begin{align*}
\delta + \lambda + k & \quad 2^{\mu-(\delta + \lambda + k+1)} \left[ \frac{1}{2}(1+\sigma+k) \right] \\
& \quad 1 + \frac{1}{2}(\sigma+k)
\end{align*}
\]

\[
\frac{\delta + \lambda + k}{2^{\mu-(\delta + \lambda + k+1)}} \left[ \frac{1}{2}(1+\sigma+k) \right] \\
1 + \frac{1}{2}(\sigma+k)
\]

\[
\begin{array}{c}
F \quad 2:0;0;0 \left[ \frac{1}{2}(1+\sigma+k), 1 + \frac{1}{2}(\sigma+k) : \quad \_ \_ \_ \_ ; \quad \_ \_ \_ \_ ; \quad \_ \_ \_ \_ ; \quad \_ \_ \_ \_ ; \quad -\frac{t}{4}, -\frac{t}{4}, -\frac{t}{4} \right] \\
F \quad 2:1;1;1 \left[ 1 + \frac{1}{2}(\sigma+k-y-\mu), \frac{1}{2}(\sigma+k+y-\mu+3) : \delta+1; \lambda+1; k+1; \right]
\end{array}
\]

which is the required result.

Now evaluating the integral \[ I_2 \] by using [28, p.7.(3.4)] we get,

\[
J_\delta(xt)J_\lambda(xt)J_k(xt) = \frac{x^{\delta + \lambda + k}}{\delta + \lambda + k} \frac{t^{\delta + \lambda + k}}{2^{\delta + \lambda + k}} \delta! \lambda! k!
\]
\[
\sum_{m=-\infty}^{\infty} \sum_{n=m}^{\infty} \frac{(-1)^m}{m+n} \frac{2m+2n}{(1+\delta)_m (1+\lambda)_n} \frac{2m+2n}{m! n!} \left[ {}_2F_3 \left( \begin{array}{c} -(\lambda+n), -n; \\ 1+\delta+m, 1+k, 1+m; \\ \frac{x^2 t^2}{4} \end{array} \right) \right]
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{n=m}^{\infty} \sum_{p=0}^{\infty} \frac{\delta+\lambda+k+2m+2n+2p}{2(1+\delta)_m (1+\lambda)_n} \frac{\delta+\lambda+k}{(1+\delta)_m (1+\lambda)_n} \frac{\delta! \lambda! k!}{\Omega m! n! p!}
\]

\[
\frac{(-\lambda-n)_p}{(1+\delta+m)_p (1+k)_p (1+m)_p}
\]

Hence \( J_\delta(xt) J_\lambda(\lambda+xt) J_k(\lambda+xt) \)

\[
= \sum_{m=-\infty}^{\infty} \sum_{n=m}^{\infty} \sum_{p=0}^{\infty} \frac{\delta+\lambda+k+2m+2n+2p}{2(1+\delta)_m (1+\lambda)_n} \frac{\delta+\lambda+k}{(1+\delta)_m (1+\lambda)_n} \frac{\delta! \lambda! k!}{\Omega m! n! p!}
\]

\[
\frac{(-\lambda-n)_p}{(1+\delta+m)_p (1+k)_p (1+m)_p}
\]

\[\ldots (4.2.3)\]

Substituting the value of \( J_\delta(\lambda+xt) J_\lambda(\lambda+xt) J_k(\lambda+xt) \) from (4.2.3) in left hand side of (4.2.2), we get
\[ I_2 = \int_0^1 x^{-\lambda-\delta} (1-x)^{\frac{1}{2}\mu} p_\gamma(x) J_\delta(xt) J_\lambda(xt) J_k(xt) \, dx \]

\[ = \sum_{m=-\infty}^{\infty} \sum_{n=m}^{\infty} \sum_{p=0}^{\infty} \frac{\delta+\lambda+k}{t} \frac{(-\lambda-n)_p}{(n)_p} \frac{(-t_4)^m}{(t_4)^m} \frac{(-t_4)^n}{(t_4)^n} \frac{1}{2^{\frac{t_4}{2}}} p_\gamma(x) \]

\[ \frac{1}{(1+\delta+m)_p (1+k)_p (1+m)_p} \int_0^1 x^{-\sigma+k+2m+2n+2p} (1-x)^{\frac{1}{2}\mu} p_\gamma(x) \, dx \]

which on using the integral [11, P. 172], gives

\[ I_2 = \frac{\delta+\lambda+k}{t} \frac{\frac{1}{2}(1+\sigma+k)}{(1+\frac{1}{2}(\sigma+k))^2} \frac{\mu-(\delta + \lambda + k+1)}{\sigma+k+\gamma-\mu} \frac{1}{(1+\frac{1}{2})(\sigma+k+\gamma-\mu)} \frac{1}{(1+\frac{1}{2})(\sigma+k+\gamma-\mu+3)} \]

\[ = \sum_{m=-\infty}^{\infty} \sum_{n=m}^{\infty} \sum_{p=0}^{\infty} \frac{(-\lambda-n)_p}{(n)_p} \frac{(-t_4)^m}{(t_4)^m} \frac{(-t_4)^n}{(t_4)^n} \frac{1}{2^{\frac{t_4}{2}}} p_\gamma(x) \]

\[ \frac{(\frac{1}{2}(1+\sigma+k))_{m+n} (\frac{1}{2}(1+\sigma+k)+m+n)_p (1+\frac{1}{2}(\sigma+k))_{m+n} (1+\frac{1}{2}(\sigma+k)+m+n)_p}{1+\frac{1}{2}(\sigma+k-\gamma-\mu)_{m+n} (1+\frac{1}{2}(\sigma+k-\gamma-\mu)+m+n)_p (\frac{1}{2}(\sigma+k+\gamma-\mu+3)_{m+n} (\frac{1}{2}(\sigma+k+\gamma-\mu+3)+m+n)_p} \]
\[
\begin{align*}
\frac{\delta + \lambda + k}{t} & \quad \frac{1}{2(1+\sigma+k)} \quad \frac{1}{2(\sigma+k)} \quad \frac{2}{\mu-(\delta + \lambda + k+1)} \\
\delta! \quad \lambda! \quad k! & \quad \frac{1}{1+\frac{1}{2}(\sigma+k+\gamma-\mu)} \quad \frac{1}{\frac{1}{2}(\sigma+k+\gamma-\mu+3)} \\
\end{align*}
\]

\[
\sum_{m=-\infty}^{\infty} \sum_{n=m}^{\infty} \frac{(\frac{1}{2}(1+\sigma+k))_{m+n} (1+\frac{1}{2}(\sigma+k))_{m+n} \left(\frac{-t^2}{4}\right)^{m+n}}{(1+\delta)_{m} (1+\lambda)_{n} (1+\frac{1}{2}(\sigma+k-\gamma-\mu))_{m+n} \left(\frac{1}{2}(\sigma+k+\gamma-\mu+3)\right)_{m+n}}^{m+n!} \\
\]

\[
\begin{bmatrix}
-\lambda-n, -n, \frac{1}{2}(1+\sigma+k)+m+n, 1+\frac{1}{2}(\sigma+k)+m+n; \\
4^{F_5} \quad \left(\frac{-t^2}{4}\right)^{m+n}
\end{bmatrix}
\]

Hence

\[
\begin{align*}
\int_0^1 x^{\sigma-\lambda-\delta} \left(1-x^2\right)^{\frac{-1}{2}\mu} P_\gamma(x) J_\delta(xt) J_\lambda(xt) J_k(xt) dx \\
\delta + \lambda + k & \quad \frac{1}{2(1+\sigma+k)} \quad \frac{1}{2(\sigma+k)} \quad \frac{2}{\mu-(\delta + \lambda + k+1)} \\
\delta! \quad \lambda! \quad k! & \quad \frac{1}{1+\frac{1}{2}(\sigma+k+\gamma-\mu)} \quad \frac{1}{\frac{1}{2}(\sigma+k+\gamma-\mu+3)} \\
\sum_{m=-\infty}^{\infty} \sum_{n=m}^{\infty} \frac{(\frac{1}{2}(1+\sigma+k))_{m+n} (1+\frac{1}{2}(\sigma+k))_{m+n} \left(\frac{-t^2}{4}\right)^{m+n}}{(1+\delta)_{m} (1+\lambda)_{n} (1+\frac{1}{2}(\sigma+k-\gamma-\mu))_{m+n} \left(\frac{1}{2}(\sigma+k+\gamma-\mu+3)\right)_{m+n}}^{m+n!} \\
\end{align*}
\]
4.3 TRANSFORMATIONS AND REDUCTION FORMULAS

To obtain the certain transformations and reduction formulas, we make use of the results established in section 4.2.

First Transformation

From (4.2.2) and (4.2.3), we have

\[
\begin{align*}
\delta + \lambda + k & \quad \mu - (\delta + \lambda + k + 1) \\
\frac{t}{2} & \quad \frac{1}{2} (1+\sigma+k) \\
\frac{1}{2} (1+\sigma+k) & \quad \frac{1}{2} (\sigma+k) \\
\frac{1}{2} (\sigma+k-\gamma-\mu) & \quad \frac{1}{2} (\sigma+k+\gamma-\mu+3) \\
\end{align*}
\]

\[
\begin{cases}
\frac{1}{2} (1+\sigma+k), \frac{1}{2} (\sigma+k) : & -t^2/4, -t^2/4, -t^2/4 \\
1+\frac{1}{2} (\sigma+k-\gamma-\mu), \frac{1}{2} (\sigma+k+\gamma-\mu+3) : & \delta+1, \lambda+1, k+1;
\end{cases}
\]
\[
\sum_{m=\infty}^{\infty} \sum_{n=m}^{\infty} \frac{\left(\frac{1}{2}(1+\sigma+k)\right)^{m+n} \left(1+\frac{1}{2}(\sigma+k)\right)^{m+n}}{\left(1+\delta\right)^{m}(1+\lambda)^{n}\left(1+\frac{1}{2}(\sigma+k-\gamma-\mu)\right)^{m+n}\left(\frac{1}{2}(\sigma+k+\gamma-\mu+3)\right)^{m+n}} \frac{(\frac{t^2}{4})^{m+n}}{m!n!}
\]

\[
\left[ -\lambda - n, -n, \frac{1}{2}(1+\sigma+k) + m + n, \frac{1}{2}(\sigma+k) + m + n; \right.
\]

\[
\begin{array}{c}
{4F5} \left[ \frac{t^2}{4} \right] \\
\left[ 1 + \delta + m, 1 + k, \frac{1}{2}(\sigma+k-\gamma-\mu) + m + n, \frac{1}{2}(\sigma+k+\gamma-\mu+3) + m + n; \right. \\
\end{array}
\]

which on setting \( \frac{1}{2}(1+\sigma) = a, \frac{1}{2}(\gamma-1) = b, \frac{1}{2} = c, \lambda + 1 = d, \delta + 1 = e, k + 1 = f \) and \( \frac{t^2}{4} = x \)
Second Transformation:

On using the result [45, p(71-72)(1.1) and (4.2.1) we have

\[
\frac{\frac{1}{2} \Gamma(1+\sigma) \Gamma\left(1+\frac{\sigma}{2}\right)}{2^{\sigma+\lambda+1-\mu} \Gamma\left(1+\frac{\sigma}{2}\right) \Gamma\left(1+\lambda\right) \Gamma\left(1+\frac{1}{2} (\sigma+\gamma-\mu)\right) \Gamma\left(\frac{1}{2} (\sigma+\gamma-\mu+3)\right)}
\]

\[
\begin{align*}
\begin{bmatrix}
\binom{1}{2} (1+\sigma); & 1+\frac{1}{2} \sigma; & \cdots; & \cdots; \\
\binom{1}{2} (\sigma-\gamma-\mu); & \frac{1}{2} (\sigma+\gamma-\mu+3); & 1+\lambda; & 1+\delta;
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
F_{2:1}^{2:0;0;0}
\end{align*}
\]

\[
\begin{align*}
F_{2:1}^{2:1;1;1}
\end{align*}
\]

\[
\begin{align*}
\left\{ a+\frac{1}{2} (f-1), a+\frac{f}{2}; \quad \cdots; \cdots; \end{align*}
\]

\[
\begin{align*}
a-b-c+\frac{1}{2} (f-1), a+b-c+\frac{f}{2} + l; d; e; f;
\end{align*}
\]

\[
\begin{align*}
= \sum_{m=-\infty}^{\infty} \sum_{n=m}^{\infty} \frac{(a+\frac{1}{2} (f-1))_{m+n} (a+\frac{f}{2})_{m+n}}{(a)_{m} (d)_{n} (a-b-c+\frac{1}{2} (f-1))_{m+n} (a+b-c+\frac{f}{2} + l)_{m+n}}
\]

\[
\begin{align*}
\left\{ l-d-n, -n, a+\frac{1}{2} (f-1)+m+n, a+\frac{f}{2}+m+n; \\
e+m, f, l+m, a-b-c+\frac{1}{2} (f-1)+m+n, a+b-c+\frac{f}{2} + m+n+1;
\end{align*}
\]

\[
\begin{align*}
\text{Re}(a) > 0, \text{Re}(c) < \frac{1}{2}
\end{align*}
\]

\[\ldots \quad (4.3.1)\]
\[
\begin{align*}
\text{Re}(\mu) < 1, \text{ Re}(\mu) > -1
\end{align*}
\]

which on setting \( \frac{1}{2}(1+\sigma) = a, \frac{1}{2}(\gamma-1) = b, \frac{1}{2} = c, 1+\lambda = d \)

\( 1+\delta = e \), and \( \frac{-t^2}{4} = x \), yields

\[
\begin{align*}
\text{Re}(a) > 0, \text{ Re}(c) < \frac{1}{2}
\end{align*}
\]
4.4 TRANSFORMATION OF HYPERGEOMETRIC FUNCTIONS OF UNIT ARGUMENT

In this section, the following results are to be established here:

First Result:

\[
\frac{\binom{p+1}{2f+1}}{(2f+1)^3} 3F_2 \left[ \begin{array}{c} \gamma, \delta, f; \\ \frac{\gamma+\delta+2}{2}, 2f+1, 1 \end{array} \right] \\
2^{\gamma+\delta-2f-2} \frac{\binom{f-\frac{1}{2}(\gamma+\delta)}{\frac{1}{2}(\gamma+\delta+2)}}{(\gamma-\delta) \frac{1}{\gamma} \frac{1}{\delta}} \\
\left[ \frac{(2f-\gamma+\delta)}{(2f+1)} \right] \frac{\binom{1/2(\gamma+1)}{\delta/2}}{\frac{1}{2}(\delta+1)} - \frac{(2f+\gamma-\delta)}{\frac{1}{2}(\gamma-1)} \frac{\binom{1/2(\delta+1)}{\gamma/2}}{\frac{1}{2}(\delta+2+1)} \right] \ldots (4.4.1)
\]

Re\(f\) > 0, Re\(2f-\gamma-\delta\) > 0.

Proof: On using \([38, P. 48(3.2.1)]\), we have

\[
\int_0^{1} t^{f-1} (1-t)^{1-f} \left[ \beta + (\alpha - \beta) t \right]^{2f-1} 2F_1 \left( \frac{1}{2}(\gamma+\delta+2); \frac{\alpha t}{\beta + (\alpha - \beta) t} \right) dt
\]

\[
= \frac{2^{\gamma+\delta-2f-2} \binom{1/2(\gamma+\delta)}{\gamma+\delta+2}}{(\gamma-\delta) \frac{1}{\gamma} \frac{1}{\delta}}
\]

\[
\frac{1}{\alpha} \frac{1}{\beta+1} \frac{1}{\delta}
\]
\[
\frac{(2\gamma - \gamma + \delta)^{\frac{1}{2}(\gamma + 1)}}{\Gamma\left(1 + \frac{\delta}{2}\right)} - \frac{(2\gamma + \gamma - \delta)^{\frac{1}{2}(\gamma + 1)}}{\Gamma\left(1 + \frac{\delta}{2}\right)} \tag{4.4.2}
\]

\[\text{Re}(\gamma) > 0, \text{Re}(2\gamma - \gamma - \delta) > 0, \alpha, \beta, \beta + (\alpha - \beta) t \neq 0, 0 \leq t \leq 1\]

also, we have

\[
\int_0^1 t^{\gamma - 1} (1-t)^{-2\gamma - 1} \left[\beta + (\alpha - \beta) t\right] \frac{\alpha t}{\beta + (\alpha - \beta) t} \, dt
\]

\[
= \sum_{n=0}^{\infty} \frac{(\gamma)^n (\delta)^n (\alpha)^n}{(\gamma + \delta + 2)^n n!} \int_0^1 t^{\gamma + n - 1} (1-t)^{\beta + (\alpha - \beta) t - (2\gamma + n + 1)} \, dt
\]

\[
= \sum_{n=0}^{\infty} \frac{(\gamma)^n (\delta)^n (\alpha)^n}{(\gamma + \delta + 2)^n n! \beta^{2\gamma + n + 1}} \int_0^1 t^{\gamma + n - 1} (1-t)^{\frac{\alpha - \beta}{\beta}} \, dt
\]

\[
= \sum_{n=0}^{\infty} \frac{(\gamma)^n (\delta)^n (\alpha/\beta)^n \Gamma(\gamma + n) \Gamma(\gamma + 1)}{(\gamma + \delta + 2)^n n! \beta^{2\gamma + 1} \Gamma(2\gamma + n + 1)} \frac{2^\gamma + n + 1, \gamma + n; \frac{\beta - \alpha}{\beta}}{2^\gamma + n + 1, \gamma + n; \frac{\beta - \alpha}{\beta}} \Gamma\left(1 - \frac{\beta - \alpha}{\beta}\right) - \gamma - n
\]

\[
= \frac{\Gamma(\gamma + 1)}{(2\gamma + 1)^{\alpha + \beta}} \alpha^{\gamma + \delta + 2} \beta^{2\gamma + 1} \left(\begin{array}{c}
\gamma, \delta, \beta; \\
\frac{\gamma + \delta + 2}{2}, 2\gamma + 1;
\end{array}\right) \tag{4.4.3}
\]
On comparing (4.4.2) and (4.4.3), we get the required result:

**Second Result:**

\[
\frac{j}{2^j-1} \quad {}_3F_2\left( \frac{\gamma+\delta}{2}, \frac{\gamma}{2}, j; \frac{j}{2} \right) = \frac{2^{-2\delta+j+\gamma+\delta-1} \frac{1}{2}(\gamma+\delta) \frac{j}{2}(\gamma+\delta+2)}{\Gamma \Gamma}
\]

\[
\frac{(2^j-\gamma+\delta-2)\frac{1}{2}(\gamma+1) \frac{\delta}{2}}{\Gamma \frac{1}{2}(\delta+1)} + \frac{(2^j+\gamma-\delta-2) \frac{1}{2}(\gamma+1) \frac{1}{2}(\delta+1)}{\Gamma \frac{1}{2}(-\delta/2)}
\]

\[
..... (4.4.4)
\]

\[
\text{Re}(j) > 1, \text{ Re}(2^j-\gamma-\delta) > 2,
\]

**Proof:** On using \[38, p. (48-49) (3.2.2)], we have

\[
\int_0^1 t^{j-1} (1-t)^{j-2} \left[ \beta + (\alpha-\beta) t \right]^{-2\delta+j+1} 2^j_{-1} [\gamma, \delta; \frac{1}{2}(\gamma+\delta); \frac{dt}{\beta+(\alpha-\beta) t}]
\]

\[
= \frac{2^{-2\delta+j+\gamma+\delta-1} \frac{1}{2}(\gamma+\delta) \frac{j}{2}(\gamma+\delta+2)}{\Gamma \Gamma}
\]

\[
\frac{(2^j-\gamma+\delta-2)\frac{1}{2}(\gamma+1) \frac{\delta}{2}}{\Gamma \frac{1}{2}(\delta+1)} + \frac{(2^j+\gamma-\delta-2) \frac{1}{2}(\gamma+1) \frac{1}{2}(\delta+1)}{\Gamma \frac{1}{2}(-\delta/2)}
\]

\[
..... (4.4.5)
\]

\[
\text{Re}(j) > 1, \text{ Re}(2^j-\gamma-\delta) > 2
\]
also

\[
\int_0^1 t^{j-1} (1-t)^{j-2} \left[ \beta + (\alpha - \beta) t \right]^{-2j+1} _2F_1 \left[ \gamma, \delta; \frac{1}{2} (\gamma + \delta); \frac{ct}{\beta + (\alpha - \beta) t} \right] dt
\]

\[
= \sum_{n=0}^{\infty} \frac{(\gamma)_n (\delta)_n (\alpha)^n}{\left( \frac{1}{2} (\gamma + \delta) \right)_n} \frac{1}{n!} \int_0^1 t^{j+n} (1-t)^{j-2} \left[ \beta + (\alpha - \beta) t \right]^{-2j-n+1} dt
\]

\[
= \sum_{n=0}^{\infty} \frac{(\gamma)_n (\delta)_n (\alpha)^n}{\left( \frac{1}{2} (\gamma + \delta) \right)_n} \frac{1}{n!} \frac{2j+n-1}{\beta} \int_0^1 t^{j+n-1} (1-t)^{j-2} \left[ 1+ (\frac{\alpha-\beta}{\beta}) t \right]^{-(2j+n-1)} dt
\]

\[
= \sum_{n=0}^{\infty} \frac{(\gamma)_n (\delta)_n (\alpha)^n}{\left( \frac{1}{2} (\gamma + \delta) \right)_n} \frac{1}{n!} \frac{2j+n-1}{\beta} \frac{2j+n-1}{(2j+n-1)} \frac{2j+n-1}{\beta} \left( \frac{\alpha-\beta}{\beta} \right)_{2j+n-1} \left( \frac{\beta-\alpha}{\beta} \right)_{2j+n-1} < 1
\]

\[
= \sum_{n=0}^{\infty} \frac{(\gamma)_n (\delta)_n (\alpha/\beta)^n (\beta)^n}{\left( \frac{1}{2} (\gamma + \delta) \right)_n} \frac{1}{\beta^{2j-1}} \frac{1}{(2j-1)_n} \frac{1}{(2j-1)_n} \left( \frac{\beta-\alpha}{\beta} \right)^{-n-j}
\]

\[
= \frac{1}{\left( (2j-1)_n \right)} \frac{1}{\beta^{2j-1}} \frac{1}{(2j-1)_n} \frac{1}{(2j-1)_n} \left( \frac{\beta-\alpha}{\beta} \right)^{-n-j}
\]

\[
= \frac{1}{\left( (2j-1)_n \right)} \frac{1}{\beta^{2j-1}} \frac{1}{(2j-1)_n} \frac{1}{(2j-1)_n} \left( \frac{\beta-\alpha}{\beta} \right)^{-n-j}
\]

\[
= \frac{1}{\left( (2j-1)_n \right)} \frac{1}{\beta^{2j-1}} \frac{1}{(2j-1)_n} \frac{1}{(2j-1)_n} \left( \frac{\beta-\alpha}{\beta} \right)^{-n-j}
\]

\[
= \frac{1}{\left( (2j-1)_n \right)} \frac{1}{\beta^{2j-1}} \frac{1}{(2j-1)_n} \frac{1}{(2j-1)_n} \left( \frac{\beta-\alpha}{\beta} \right)^{-n-j}
\]

\[
= \frac{1}{\left( (2j-1)_n \right)} \frac{1}{\beta^{2j-1}} \frac{1}{(2j-1)_n} \frac{1}{(2j-1)_n} \left( \frac{\beta-\alpha}{\beta} \right)^{-n-j}
\]

\[
= \frac{1}{\left( (2j-1)_n \right)} \frac{1}{\beta^{2j-1}} \frac{1}{(2j-1)_n} \frac{1}{(2j-1)_n} \left( \frac{\beta-\alpha}{\beta} \right)^{-n-j}
\]

\[
= \frac{1}{\left( (2j-1)_n \right)} \frac{1}{\beta^{2j-1}} \frac{1}{(2j-1)_n} \frac{1}{(2j-1)_n} \left( \frac{\beta-\alpha}{\beta} \right)^{-n-j}
\]

\[
= \frac{1}{\left( (2j-1)_n \right)} \frac{1}{\beta^{2j-1}} \frac{1}{(2j-1)_n} \frac{1}{(2j-1)_n} \left( \frac{\beta-\alpha}{\beta} \right)^{-n-j}
\]

\[
= \frac{1}{\left( (2j-1)_n \right)} \frac{1}{\beta^{2j-1}} \frac{1}{(2j-1)_n} \frac{1}{(2j-1)_n} \left( \frac{\beta-\alpha}{\beta} \right)^{-n-j}
\]

\[
= \frac{1}{\left( (2j-1)_n \right)} \frac{1}{\beta^{2j-1}} \frac{1}{(2j-1)_n} \frac{1}{(2j-1)_n} \left( \frac{\beta-\alpha}{\beta} \right)^{-n-j}
\]

\[
= \frac{1}{\left( (2j-1)_n \right)} \frac{1}{\beta^{2j-1}} \frac{1}{(2j-1)_n} \frac{1}{(2j-1)_n} \left( \frac{\beta-\alpha}{\beta} \right)^{-n-j}
\]

\[
= \frac{1}{\left( (2j-1)_n \right)} \frac{1}{\beta^{2j-1}} \frac{1}{(2j-1)_n} \frac{1}{(2j-1)_n} \left( \frac{\beta-\alpha}{\beta} \right)^{-n-j}
\]

On comparing (4.4.5) and (4.4.6), we get the required result.
Third Result:

\[
\frac{1}{(2\sigma-\mu)} \binom{\gamma}{\mu, 2\sigma-\mu} = \frac{2^{-2\gamma-1}}{\Gamma(\mu) \Gamma(2\sigma-\mu-\gamma)}
\]

\[
\left[ \frac{1/2(\mu-\gamma+1)}{\sigma-1/2(\mu-\gamma)} \frac{1/2(\mu+\gamma)}{1/2(\mu+\gamma+1)} - \frac{1/2(\mu-\gamma)}{\sigma-1/2(\mu+\gamma+1/2)} \frac{1/2(\mu+\gamma)}{\sigma-1/2(\mu-\gamma-1)} \right] \quad \ldots \quad (4.4.7)
\]

Re(\sigma) > 0, Re(\mu) > 0, Re(\sigma-\mu) > 0.

Proof: On using [38, p. 51-52 (3.2.7)], we have

\[
\int_0^1 t^{\sigma-1} (1-t)^{\sigma-\mu-1} 2F_1(\gamma, -\gamma, \mu; t) dt
\]

\[
= \frac{2^{-2\gamma-1}}{\Gamma(\mu) \Gamma(2\sigma-\mu-\gamma)} \left[ \frac{1/2(\mu-\gamma+1)}{\sigma-1/2(\mu-\gamma)} \frac{1/2(\mu+\gamma)}{1/2(\mu+\gamma+1)} - \frac{1/2(\mu-\gamma)}{\sigma-1/2(\mu+\gamma+1/2)} \frac{1/2(\mu+\gamma)}{\sigma+1/2(\gamma+\mu+1)} \right]
\]

\[
\ldots \ldots \quad (4.4.8)
\]

Re(\sigma) > 0, Re(\mu) > 0, Re(\sigma-\mu) > 0.

also

\[
\int_0^1 t^{\sigma-1} (1-t)^{\sigma-\mu-1} 2F_1(\gamma, -\gamma, \mu; t) dt
\]
\[ \sum_{n=0}^{\infty} \frac{(\gamma)_n (-\gamma)_n}{(\mu)_n n!} \int_{0}^{1} t^{\sigma+n-1} (1-t)^{\sigma-\mu-1} \, dt \]

\[ = \sum_{n=0}^{\infty} \frac{(\gamma)_n (-\gamma)_n}{(\mu)_n n!} \beta(\sigma+n, \sigma-\mu) \]

\[ = \sum_{n=0}^{\infty} \frac{(\gamma)_n (-\gamma)_n}{(\mu)_n n!} \frac{\Gamma(\sigma+n) \Gamma(\sigma-\mu)}{\Gamma(2\sigma+n-\mu)} \]

\[ = \sum_{n=0}^{\infty} \frac{(\gamma)_n (-\gamma)_n (\sigma)_n}{(\mu)_n n! (2\sigma-\mu)_n} \frac{\Gamma(\sigma) \Gamma(\sigma-\mu)}{(2\sigma-\mu)} \]

\[ = \frac{\Gamma(\sigma) \Gamma(\sigma-\mu)}{(2\sigma-\mu)} \left[ \begin{array}{c} \gamma, -\gamma, \sigma; \\ \mu, 2\sigma-\mu; 1 \end{array} \right] \quad \text{... (4.4.9)} \]

On comparing (4.4.8) and (4.4.9), we get the required result.

**Fourth Result:**

\[ \frac{1}{(2\sigma-\mu)} \begin{Bmatrix} \gamma, -1-\gamma, \sigma; \\ \mu, 2\sigma-\mu; 1 \end{Bmatrix} = \frac{2^{-2(\gamma+1)}}{(\mu-\gamma)(2\sigma-\mu-\gamma)} \left[ \begin{array}{c} (2\sigma-\mu) \frac{1}{2}(\mu-\gamma+1) \frac{1}{2}(\gamma+\mu) \\ \sigma+\frac{1}{2}(\gamma+1) \frac{1}{2}(\mu+\gamma+1) \end{array} \right] - \frac{\mu \frac{1}{2}(\mu+\gamma) \sigma-\frac{1}{2}(1+\gamma-1)}{(\mu+\gamma+2) \sigma+\frac{1}{2}(\gamma+1)} \quad \text{... (4.4.1C) } \]
Proof: On using \([38, p. 52 (3.2.8)]\), we have

\[
\begin{align*}
\int_0^1 t^{\sigma-1} (1-t)^{\sigma-\mu-1} & \quad _2F_1(\gamma, -1-\gamma, \mu; t) \, dt \\
= & \quad 2^{-2(\gamma+1)} \frac{\Gamma(\sigma+\mu) \Gamma(\sigma-\mu)}{\Gamma(\mu-\gamma) \Gamma(2\sigma-\mu-\gamma)} \left[ \frac{(2\sigma-\mu) \frac{1}{2}(\mu-\gamma+1) \Gamma(\frac{1}{2}(\mu+\gamma))}{\Gamma(\sigma+\frac{1}{2}(\gamma-\mu+1)) \Gamma(\frac{1}{2}(\mu+\gamma+1))} \right. \\
& \quad \left. - \frac{\mu \frac{1}{2}(\mu+\gamma) \Gamma(\sigma-\frac{1}{2}(\mu+\gamma+1))}{\Gamma(\frac{1}{2}(\mu+\gamma+2)) \Gamma(\sigma+\frac{1}{2}(\gamma-\mu+1))} \right] \quad \text{...... (4.4.11)}
\end{align*}
\]

also, \[
\int_0^1 t^{\sigma-1} (1-t)^{\sigma-\mu-1} \quad _2F_1(\gamma, -1-\gamma; \mu; t) \, dt
\]

\[
= \sum_{n=0}^{\infty} \frac{(\gamma)_n (-1-\gamma)_n}{(\mu)_n n!} \int_0^1 t^{\sigma+n-1} (1-t)^{\sigma-\mu-1} \, dt
\]

\[
= \sum_{n=0}^{\infty} \frac{(\gamma)_n (-1-\gamma)_n}{(\mu)_n (2\sigma-\mu)_n n!} \quad B(\sigma+n, \sigma-\mu)
\]

\[
= \frac{\Gamma(\sigma-\mu) \Gamma(\sigma)}{\Gamma(2\sigma-\mu)} \sum_{n=0}^{\infty} \frac{(\gamma)_n (-1-\gamma)_n}{(\mu)_n (2\sigma-\mu)_n n!}
\]

\[
= \frac{\Gamma(\sigma-\mu) \Gamma(\sigma)}{\Gamma(2\sigma-\mu)} \quad _3F_2(\gamma, -1-\gamma, \sigma; \mu, 2\sigma; 1) \quad \text{...... (4.4.12)}
\]

On comparing (4.4.11) and (4.4.12) we get the required result.


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