SOME PROBLEMS CONCERNING GENERALIZED VARIATIONAL INEQUALITIES

THESIS

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Dedicated

To

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Certificate

This is to certify that the contents of this thesis entitled ***"Some Problems Concerning Generalized Variational Inequalities"*** is the original research work of ***Ms. Farhat Usman*** carried out under my supervision. To the best of my knowledge and belief the work incorporated in the thesis, either partially or fully has not been submitted to any other university or institution for the award of any other degree or diploma.

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Preface

Mathematics is a central element of our current technology but few people realize that this celebrated high technology is so strongly based on Mathematics. The theory of variational inequalities is a powerful and elegant tool of the current mathematical technology and have become a rich source of inspiration for scientist and engineers. There are numerous standard textbooks and monographs dealing with various aspects of this domain. In the last four decades, this theory has been extended and generalized in various directions because of the applications to a wide class of problems arising in various branches of mathematical, physical and engineering sciences and optimization. There are three different aspects to study variational inequalities (i)*Mathematical Modelling:* To convert the problems of real life or the problems from science, engineering and social sciences into a variational inequality problem is called mathematical modelling. (ii) *Existence Theory:* To study the existence of solutions of variational inequalities. (iii) *Numerical Methods:* To find the algorithms for computing the approximate solutions of variational inequalities, which converge to the exact solution.

It is mentioned by *Aubin* [15] in his book that the Nash equilibrium problem for differentiable functions can be formulated in the form of a variational inequality problem defined over the product of sets. Further, *Pang* [97] showed that not only Nash equilibrium problem but also various equilibrium-type problems, like, traffic equilibrium, spatial equilibrium, and general equilibrium programming problems from operations research, economics, game theory, mathematical physics and other areas, can also be uniformly modelled as a variational inequality problem defined over the product of sets which is equivalent to the problem of system of variational inequalities.

This thesis deals with the existence theory and numerical methods of different kinds of variational inclusions, variational-like inclusions, system of variational inequalities, system of generalized variational inclusions, system of variational-like inclusions, variational inclusions with fuzzy mappings.

Chapter 1 deals with the brief introduction of variational inequalities, variational-like inequalities, variational inclusions and system of variational inequalities besides some basic definitions and results from functional analysis.

In Chapter 2, we consider a system of mixed variational inequalities and a system of mixed variational-like inclusions in the setting of Banach spaces. By applying the notion of J-proximal mapping and J^{η}-proximal mapping and their Lipschitz continuity, we suggest the iterative methods for computing the approximate solutions of these systems. The existence and convergence of solutions obtained by defined algorithms are also studied.

In Chapter 3, we consider a system of set-valued variational inclusions and a system of generalized variational inclusions with H-accretive operators in uniformly smooth Banach spaces. An iterative algorithm for computing the approximate solutions of these systems is defined. Some existence and convergence results are also derived. In the last section, we consider a system of generalized H-resolvent equations in uniformly smooth Banach spaces. An equivalence relation is established between system of generalized H-resolvent equations and system of generalized variational inclusions.

In Chapter 4, we consider the generalized variational-like inclusions for fuzzy mappings. We develop an Ishikawa type perturbed iterative algorithm and a Mann type perturbed iterative algorithm for computing the approximate solutions. The existence and convergence analysis is also studied. Further, we consider a class of mixed variational inclusions for fuzzy mappings. The existence and convergence analysis for this class of mixed variational inclusions for fuzzy mappings discussed by using the definition of relaxed strongly accretive operators. In the last section, we introduce generalized T-resolvent equations with fuzzy mappings. An equivalence relation is established between the mixed variational inclusions for fuzzy mappings and the generalized T-resolvent equations with fuzzy mappings.

In Chapter 5, we introduce and study a system of generalized variational inclusions with (A, η)-accretive mappings in Banach spaces. By using the resolvent

List Of Accepted Papers Based On This Thesis

[1] R. Ahmad and F. Usman, *System of generalized variational inclusions with H-accretive operators in uniformly smooth Banach spaces,* **J. Comput. Appl. Math.,** (available online) doi:10.1016/j.cam.2008.12.008.

[2] R. Ahmad and F. Usman, *Ishikawa type perturbed iterative algorithm for generalized variational-like inclusions for fuzzy mappings,* accepted in **J. Fuzzy Math.**

[3] R. Ahmad, F. Usman and S. S. Irfan, *Mixed variational inclusions and T-resolvent equations with fuzzy mappings,* **J. Fuzzy Math.,** 17(2) 2009.

[4] R. Ahmad and F. Usman, *Approximate solutions for generalized resolvent equations with (A,η)-accretive and relaxed cocoercive mappings in Banach spaces,* under minor revision in **Internat. J. Comput. Math.**

[5] R. Ahmad and F. Usman, *System of mixed variational inequalities in reflexive Banach spaces,* To appear in **JAMI.**

List Of Submitted Papers Based On This Thesis

[1] R. Ahmad and F. Usman, *System of mixed variational-like inclusions,* submitted in **SouthEast Asian Bull. Math.**

[2] R. Ahmad and F. Usman, *System of set-valued variational inclusions,* submitted in **J. Convex Anal.**

[3] R. Ahmad and F. Usman, *System of generalized H-resolvent equations with corresponding system of genaralized variational inclusions,* submitted in **Internat. J. Comput. Math.**

[4] R. Ahmad and F. Usman, *Approximation-solvability of a system of generalized variational inclusions,* submitted in **European J. Pure and Appl. Math.**

[5] R. Ahmad and F. Usman, *An iterative algorithm for Nonlinear relaxed cocoercive generalized variational inclusions in Banach spaces without Hausdorff metric,* submitted in **Math. Commun.**

Contents

Chapter 1

Preliminaries

1.1. Introduction

Variational inequalities introduced by *Stampacchia* [112] have enjoyed vigorous growth for the last forty years. Variational inequality theory describes a broad spectrum of interesting and important developments involving a link among various fields of mathematics, physics, economics and engineering sciences [2,4,14,16,17,21,26,29, 33,35,36,46,48,58,59,61,70,74,98,105,119,122-124,126]. It turned out that the odd-order and nonsymmetric free, moving, unilateral obstacle and equilibrium can be studied via the general variational inequality approach.

In recent past, considerable interest has been shown in developing various extensions and generalizations of variational inequalities related to multivalued operators, nonconvex optimization and structural analysis. This theory was developed simultaneously not only to study the fundamental facts about the qualitative behaviour of solutions of nonlinear problems, but also to solve them more efficiently numerically.

In different sections of this chapter, we discuss various notions which are essential for presentation of results in the subsequent chapters.

1.2. Some Basic Concepts And Results

In this section, we present some basic notations, definitions and known results of functional analysis which will be used in the subsequent chapters.

Throughout this thesis, unless otherwise specified, we assume that E is a real Banach space endowed with a norm $||.||$, $E^\star$ is the topological dual of E, $\langle \cdot, \cdot \rangle$ is the duality pairing between E and $E^\star$, d is the metric induced by the norm $\|.\|$, $CB(E)$ is the family of all nonempty closed and bounded subsets of E, 2^E is the family of all nonempty subsets of E, $D(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$D(A,B)=\max\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\Big\}.$$

where $d(x,B)=\inf_{y\in B} d(x,y)$ and $d(A,y)=\inf_{x\in A} d(x,y)$.

We denote by H a real Hilbert space and by H^* its dual.

Theorem 1.2.1.[111]. Let K be a nonempty, closed and convex subset of Hilbert space H. Then for all $z\in H$, there exists unique $u\in K$ such that

$$\|z-u\|=\inf_{v\in K}\|z-v\|. \tag{1.2.1}$$

Definition 1.2.1. The point u satisfying (1.2.1) is called the *projection of z onto K* and we write

$$u=P_K(z). \tag{1.2.2}$$

Lemma 1.2.1.[83]. If K is a nonempty, closed and convex subset of H and z is a given point in H, then $u\in K$ satisfies the inequality

$$\langle u-z, v-u\rangle \geq 0, \qquad \text{for all } \ v\in K,$$

if and only if

$$u=P_K(z),$$

where P_K is the projection of H onto K.

Lemma 1.2.2.[83]. The projection P_K defined by (1.2.2) is nonexpansive, i.e.,

$$\|P_K(u)-P_K(v)\|\leq\|u-v\|, \qquad \text{for all } \ u,v\in H.$$

Theorem 1.2.2.(*Riesz representation theorem*)**[111].** If f is a bounded linear functional on a Hilbert space H, there exists a unique vector $v\in H$ such that

$$f(u)=\langle u,v\rangle, \qquad \text{for all } \ u\in H \ \text{ and } \ \|f\|=\|v\|.$$

Definition 1.2.2. Let X and Y be a topological vector space. A multivalued mapping $P : X \to 2^Y$ is said to be:

(i) *upper semicontinuous* at $x_0 \in X$, if for every open set V in Y containing $P(x_0)$, there exists an open neighbourhood U of x_0 in X such that $P(x) \subseteq V$, for all $x \in U$;

(ii) *closed*, if for every net $\{x_\lambda\}$ converges to x_* and $\{y_\lambda\}$ converges to y_* such that for all λ, $y_\lambda \in P(x_\lambda)$ implies that $y_* \in P(x_*)$.

Definition 1.2.3. Let E be a Banach space and $f : E \to R \cup \{+\infty\}$. Then f is said to be *convex*, if

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v),$$

holds for all $t \in (0,1)$ and u, $v \in E$.

Definition 1.2.4.[7]. Let E be a real Banach space. Then

(i) a mapping $\mathcal{J} : E \to 2^{E^\star}$ is called *normalized duality mapping* defined by

$$\mathcal{J}(x) = \{f \in E^\star : \langle x, f\rangle = \|x\|\|f\|, \text{ for all } x \in E\}$$

(ii) a mapping $\mathcal{J}_q : E \to 2^{E^\star}$ is called *generalized duality mapping* defined by

$$\mathcal{J}_q(x) = \{f \in E^\star : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}, \text{ for all } x \in E\}$$

For $q = 2$, the generalized duality mapping coincides with the usual normalized duality mapping.

Definition 1.2.5.[34]. Let $T : E \to CB(E)$ be multivalued mapping and let $D(.,.)$ be the Häusdorff metric on $CB(E)$, T is said to be ξ-*Lipschitz continuous*, if for any $x, y \in E$ such that

$$D(Tx, Ty) \leq \xi\|x - y\|,$$

where $\xi > 0$ is a constant.

Theorem 1.2.3.(*Nadler*)[87]. Let (X, d) be a complete metric space. If $F : X \to CB(X)$ is a multivalued contraction mapping, then F has a fixed point.

Definition 1.2.6.[3]. A Banach space E is said to be *uniformly convex* if for any given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$, $||x|| \leq 1, ||y|| \leq 1$ and $||x - y|| = \epsilon$ we have

$$||x + y|| \leq 2(1 - \delta).$$

The function

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{||x + y||}{2} : ||x|| = 1,\ ||y|| = 1,\ ||x - y|| = \epsilon\right\}$$

is called the *modulus of the convexity* of the space E.

Definition 1.2.7.[3]. A Banach space E is said to be *uniformly smooth* if for any given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{||x + y|| + ||x - y||}{2} - 1 \leq \epsilon ||y||$$

holds.

The function

$$\rho_E(t) = \sup\left\{\frac{||x + y|| - ||x - y||}{2} - 1 : ||x|| = 1,\ ||y|| = t\right\}$$

is called the *modulus of the smoothness* of the space E.

Remark 1.2.1. The space E is *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon > 0$, and it is *uniformly smooth* if and only if $\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0$.

Definition 1.2.8.[3]. The Banach space E is called *q-uniformly smooth*, if there exists a constant $C > 0$ such that

$$\rho_E(t) \leq Ct^q, \quad q > 1.$$

Definition 1.2.9. Let $T, g : E \to E$ be two single valued mappings. Then T is said to be:

(i) *accretive*, if for all $x, y \in E$ there exists $j(x - y) \in \mathcal{J}(x - y)$ such that

$$\langle T(x) - T(y), j(x - y) \rangle \geq 0;$$

(ii) *strictly accretive*, if for all $x, y \in E$ there exists $j(x-y) \in \mathcal{J}(x-y)$ such that

$$\langle T(x) - T(y), j(x-y)\rangle \geq 0;$$

and the equality holds if and only if $x = y$;

(iii) *strongly accretive*, if for all $x, y \in E$ there exists $j(x-y) \in \mathcal{J}(x-y)$ and a constant $\delta_T > 0$ such that

$$\langle T(x) - T(y), j(x-y)\rangle \geq \delta_T \|x-y\|^2;$$

(iv) *strongly accretive* with respect to g if for all $x, y \in E$ there exist a constant δ_T such that

$$\langle T(x) - T(y), j(g(x) - g(y))\rangle \geq \delta_T \|g(x) - g(y)\|^q;$$

(v) *Lipschitz continuous* if for all $x, y \in E$ there exists a constant λ_T such that

$$\|T(x) - T(y)\| \leq \lambda_T \|x-y\|.$$

Definition 1.2.10. Let $H : E \to E^*$ and $\eta : E \times E \to E$ be single-valued mappings. Then

(i) η is said to be *monotone*, if for all $x, y \in E$

$$\langle x - y, \eta(x,y)\rangle \geq 0;$$

(ii) η is said to be *strictly monotone*, if for all $x, y \in E$

$$\langle x - y, \eta(x,y)\rangle \geq 0;$$

and equality holds if and only if $x = y$;

(iii) η is said to be *strongly monotone*, if for all $x, y \in E$ there exists a constant $\delta > 0$ such that

$$\langle x - y, \eta(x,y)\rangle \geq \delta \|x-y\|^2;$$

(iv) η is said to be H-*strongly accretive* with constant $\alpha > 0$, if for all $x, y \in E$

$$\langle \eta(x,y), H(x) - H(y)\rangle \geq \alpha \|x-y\|^2;$$

(v) η is said to be *Lipschitz continuous*, if for all $x, y \in E$, there exists a constant $\lambda > 0$ such that

$$\|\eta(x, y)\| \leq \lambda\|x - y\|;$$

(vi) H is said to be *η-strongly accretive*, if for all $x, y \in E$ there exists a constant $\alpha > 0$ such that

$$\langle \eta(x, y), H(x) - H(y) \rangle \ \geq \ \alpha\|x - y\|^2.$$

Definition 1.2.11. A multi-valued mapping $A : E \to 2^E$ is said to be:

(i) *accretive*, if for any $x, y \in E$, there exists $j(x - y) \in \mathcal{J}(x - y)$ such that for all $u \in A(x)$ and $v \in A(y)$,

$$\langle u - v, j(x - y) \rangle \geq 0;$$

(ii) *k-strongly accretive* $k \in (0, 1)$, if for any $x, y \in E$, there exists $j(x - y) \in \mathcal{J}(x - y)$ such that for any $u \in A(x)$, $v \in A(y)$

$$\langle u - v, j(x - y) \rangle \geq k\|x - y\|^2;$$

(iii) *m-accretive*, if A is accretive and $(I + \rho A)(E) = E$, for every (equivalently, for some) $\rho > 0$, where I is the identity mapping (equivalently, if A is accretive and $(I + A)(E) = E$);

(iv) *Lipschitz continuous*, if for all $x_1 \in A(u_1)$, $x_2 \in A(u_2)$, there exists $\lambda_A > 0$ such that for any $u_1, u_2 \in E$

$$\|x_1 - x_2\| \leq \lambda_A\|u_1 - u_2\|.$$

Remark 1.2.2. If $E = H$ is a Hilbert space, then $A : D(A) \subset E \to 2^E$ is an m-accretive mapping if and only if it is a maximal monotone mapping.

Lemma 1.2.3.[63]. Let $g : E \to E$ be a continuous and k-strongly accretive mapping. Then g maps E onto E.

Definition 1.2.12. Let $T, F : E \to 2^E$ be set-valued mappings. The mapping $g : E \times E \to E$ is said to be:

(i) *Lipschitz continuous* in first argument with respect to T if there exists a constant $\lambda_{g_1} > 0$ such that

$$\|g(u_1, .) - g(u_2, .)\| \leq \lambda_{g_1} \|u_1 - u_2\|$$

for all $u_1 \in T(x_1),\ u_2 \in T(x_2)$ and $x_1, x_2 \in E$;

(ii) *Lipschitz continuous* in second argument with respect to F if there exists a constant λ_{g_2} such that

$$\|g(., v_1) - g(., v_2)\| \leq \lambda_{g_2} \|v_1 - v_2\|$$

for all $v_1 \in F(x_1),\ v_2 \in F(x_2)$ and $x_1, x_2 \in E$.

Definition 1.2.13. Let $\varphi : E \to R \cup \{+\infty\}$ be a proper functional, φ is said to be *subdifferential* at a point $x \in E$, if there exists a point $f^\star \in E^\star$ such that

$$\varphi(y) - \varphi(x) \geq \langle f^\star, y - x \rangle, \quad \text{for all} \quad y \in E;$$

where $f^\star$ is called a subgradient of φ at x. The set of all subgradient of φ at x is denoted by $\partial\varphi(x)$.

The mapping $\partial\varphi : E \to 2^{E^\star}$ defined by

$$\partial\varphi(x) = \{f^\star \in E^\star : \varphi(y) - \varphi(x) \geq \langle f^\star, y - x \rangle, \quad \text{for all} \quad y \in E\}$$

is said to be *subdifferential* of φ at x.

Lee et al. [81] introduced the following concept of η-subdifferential.

Definition 1.2.14. Let $\eta : E \times E \to E$ and $\varphi : E \to R \cup \{+\infty\}$. A vector $w^\star \in E^\star$ is called an *η-subgradient* of φ at $x \in \text{dom } \varphi$, if

$$\langle w^\star, \eta(y, x) \rangle \leq \varphi(y) - \varphi(x), \qquad \text{for all} \quad y \in E.$$

Each φ can be associated with the following *η-subdifferential map* $\partial_\eta\varphi$ defined by

$$\partial_\eta \varphi(x) = \begin{cases} \{w^\star \in E^* : \langle w^\star, \eta(y,x)\rangle \leq \varphi(y) - \varphi(x), \quad \text{for all } y \in E\}, & x \in \text{dom } \varphi, \\ \emptyset & x \notin \text{dom } \varphi. \end{cases}$$

Definition 1.2.15. A Banach space E is reflexive if the mapping $J : x \to Fx$ from E into $E^{\star\star}$, where $F_x(f) = f(x)$, $f \in E^\star$, is an onto mapping.

Definition 1.2.16.[18]. Let $A : D(A) \subset E \to 2^E$ be an m-accretive mapping. For any $\rho > 0$, the mapping $J_\rho^A : E \to D(A)$ associated with A defined by

$$J_\rho^A(u) = (I + \rho A)^{-1}(u), \quad \text{for all } \ u \in E,$$

is called the *resolvent operator.*

Remark 1.2.3.[18]. The resolvent operator J_ρ^A is single-valued and nonexpansive, that is,

$$\|J_\rho^A(x) - J_\rho^A(y)\| \leq \|x - y\|, \quad \text{for all } \ x, y \in E.$$

Definition 1.2.17. Let $A : D(A) \subset E \to 2^E$ be an m-accretive mapping. For any $\rho > 0$, the resolvent operator $J_\rho^A : E \to D(A)$ associated with A, is said to be:

(i) *retraction*, if

$$(I + \rho A)^{-1} \circ (I + \rho A)^{-1}(u) = (I + \rho A)^{-1}(u), \quad \text{for all } \ u \in E,$$

where I is the identity operator;

(ii) *nonexpansive retraction*, if

$$\|J_\rho^A(z_1) - J_\rho^A(z_2)\| \leq \|z_1 - z_2\| \quad \text{for all } \ z_1, z_2 \in E.$$

Proposition 1.2.1.[7,101]. Let E be a real Banach space and $\mathcal{J} : E \to 2^{E^\star}$ be a normalized duality mapping. Then for any $x, y \in E$, the following holds:

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle$, for all $j(x + y) \in \mathcal{J}(x + y)$;

(ii) $\langle x-y, j(x)-j(y)\rangle \leq 2d^2\tau_E(4\|x-y\|/c)$,

where $c = \sqrt{(\|x\|^2+\|y\|^2)/2}$.

Proposition 1.2.2.[120,121]. Let E be a real uniformly smooth Banach space. Then E is q-uniformly smooth if and only if there exists a constant $C_q > 0$ such that for all $x, y \in E$

$$\|x+y\|^q \leq \|x\|^q + q\langle y, \mathcal{J}_q(x)\rangle + C_q\|y\|^q.$$

Definition 1.2.18.[50]. Let $H : E \to E$ be an operator. A multivalued mapping $M : E \to 2^E$ is said to be *H-accretive* if M is accretive and $(H+\rho M)(E) = E$ for all $\rho > 0$.

Remark 1.2.4. If $H = I$, then Definition 1.2.18 reduces to the usual definition of m-accretive operator.

Definition 1.2.19. Let $H : E \to E$ be a strictly accretive operator and $M : E \to 2^E$ be an H-accretive multivalued mapping. The *H-resolvent operator* $J^M_{H,\rho} : E \to E$ associated with H and M is defined by

$$J^M_{H,\rho}(x) = (H+\rho M)^{-1}(x), \qquad \text{for all } \ x \in E.$$

Theorem 1.2.4.[50]. Let $H : E \to E$ be a strongly accretive operator with constant r and $M : E \to 2^E$ be an H-accretive multivalued mapping. Then the H-resolvent operator $J^M_{H,\rho} : E \to E$ associated with H and M is Lipschitz continuous with constant $\frac{1}{r}$, that is,

$$\|J^M_{H,\rho}(x) - J^M_{H,\rho}(y)\| \leq \frac{1}{r}\|x-y\|, \qquad \text{for all } \ x, y \in E.$$

Definition 1.2.20. Let $f : H \to H$ be a mapping. A multivalued mapping $S : H \to 2^H$ is said to be:

(i) *relaxed Lipschitz continuous* with respect to f, if there exists a constant $k \geq 0$ such that

$$\langle f(u) - f(v), x-y\rangle \leq -k\|x-y\|^2,$$

for all $x, y \in H$, $u \in S(x)$, $v \in S(y)$;

(ii) *relaxed monotone* with respect to f, if there exists a constant $c > 0$ such that

$$\langle f(u) - f(v), x - y\rangle \geq -c\|x - y\|^2,$$

for all $x, y \in H$, $u \in S(x)$, $v \in S(y)$.

For the justification of Definition 1.2.20, we construct the following two examples.

Example 1.2.1. Let $H = \mathbb{R}$ and $S = I$, the identity mapping. Suppose $f(x) = -3x$, $0 \leq \epsilon \leq 3$, $k = (3 - \epsilon)$. Then it is easy to see that f is relaxed Lipschitz continuous mapping.

Example 1.2.2. Let $H = \mathbb{R}$ and $S = I$, the identity mapping. Suppose $f(x) = 3x$, $\epsilon = 4, 5, \ldots$, $c = (\epsilon - 3)$. Then it is easy to see that f is relaxed monotone mapping.

Definition 1.2.21. Let $\eta : H \times H \to H$ be a given map. A multivalued mapping $Q : H \to 2^H$ is called *η-monotone* if for all $x, y \in H$

$$\langle u - v, \eta(x, y)\rangle \ \geq \ 0,$$

for all $u \in Q(x)$, $v \in Q(y)$.

Remark 1.2.5. Q is called *maximal η-monotone* if and only if it is η-monotone and there is no other η-monotone multivalued mapping whose graph strictly contains the graph of Q.

Definition 1.2.22. Let $\eta : E \times E \to E$ be a single-valued mapping. Then the set-valued mapping $M : E \to 2^E$ is said to be:

(i) *η-accretive*, if

$$\langle u - v, j_q(\eta(x, y))\rangle \geq 0,$$

for all $x, y \in E$, $u \in M(x)$, $v \in M(y)$;

(ii) *strictly η-accretive*, if M is η-accretive and equality holds if and only if $x = y$;

(iii) *r-strongly η-accretive*, if there exists a constant $r > 0$ such that

$$\langle u - v, j_q(\eta(x, y))\rangle \geq r\|x - y\|^q,$$

for all $x, y \in E$, $u \in M(x)$, $v \in M(y)$;

(iv) *m-relaxed η-accretive*, if there exists a constant $m > 0$ such that

$$\langle u - v, j_q(\eta(x, y))\rangle \geq -m\|x - y\|^q,$$

for all $x, y \in E$, $u \in M(x)$, $v \in M(y)$.

Remark 1.2.6.

(i) If $r = 0$ and equality holds if and only if $x = y$, then (iii) of Definition 1.2.22 reduces to the definition of strictly η-accretive mappings.

(ii) If $\eta(x, y) = x - y$, then (iii) of Definition 1.2.22 reduces to the definition of r-strongly accretive mappings.

Example 1.2.3. Let $E =\mathbb{R}$, $M(x) = x$, $\eta(x, y) = (-2x) - (-2y)$, then it is easy to see that M is a 2-relaxed η-accretive function.

Definition 1.2.23. Let $A : E \to E$, $\eta : E \times E \to E$ be two single-valued mappings. Then a set-valued mapping $M : E \to 2^E$ is called *(A, η)-accretive*, if M is m-relaxed η-accretive and $(A + \rho M)(E) = E$, for every $\rho > 0$.

Remark 1.2.7.

(i) If $m = 0$, then Definition 1.2.23 reduces to the definition of (H, η)-accretive operators [54] which includes generalized m-accretive operators [69], H-accretive operators [50] and classical m-accretive operators.

(ii) When $m = 0$ and $E = H$ is a Hilbert space, then Definition 1.2.23 reduces to the definition of (H, η)-monotone operators [53,55] which includes maximal η-monotone operators [68] and classical maximal monotone operators [129].

Definition 1.2.24. Let $A : E \to E$ be a strictly η-accretive mapping and $M : E \to 2^E$ be an (A, η)-accretive mapping. Then *resolvent operator* $J_{\eta,M}^{\rho,A} : E \to E$ is defined by

$$J_{\eta,M}^{\rho,A}(x) = (A + \rho M)^{-1}(x), \quad \text{for all} \;\; x \in E.$$

Lemma 1.2.4.[77]. Let $\eta : E \times E \to E$ be τ-Lipschitz continuous, $A : E \to E$ be r-strongly η-accretive mapping and $M : E \to 2^E$ be an (A, η)-accretive mapping. Then the resolvent operator $J_{\eta,M}^{\rho,A} : E \to E$ is $\frac{\tau^{q-1}}{r-\rho m}$-Lipschitz continuous, i.e.,

$$\|J_{\eta,M}^{\rho,A}(x) - J_{\eta,M}^{\rho,A}(y)\| \leq \frac{\tau^{q-1}}{r - \rho m}\|x - y\|, \quad \text{for all} \;\; x, y \in E,$$

where $\rho \in (0, \frac{r}{m})$ is a constant.

Definition 1.2.25. A mapping $g : E \to E$ is said to be (b, ξ)-*relaxed cocoercive*, if there exists constants b, $\xi > 0$ such that

$$\langle g(x) - g(y), j_q(x - y)\rangle \geq -b\|g(x) - g(y)\|^q + \xi\|x - y\|^q,$$

for all $x, y \in E$.

1.3. Variational Inequalities

Many problems of elasticity and fluid mechanics can be expressed in terms of an unknown u, representing the displacement of a mechanical system, satisfying

$$a(u, v - u) \geq F(v - u), \qquad \text{for all} \;\; v \in K, \tag{1.3.1}$$

where K is a nonempty, closed, convex subset of a Hilbert space H, $a(.,.)$ is a bilinear form and F is a bounded linear functional on H. The relations of the type (1.3.1) are called *variational inequalities.*

If the bilinear form $a(.,.)$ is continuous, then by Riesz representation theorem 1.2.2, we have

$$a(u, v) = \langle A(u), v\rangle, \quad \text{for all} \quad u, v \in H, \tag{1.3.2}$$

where A is a continuous linear operator on H.

Then inequality (1.3.1) is equivalent to find $u \in K$ such that

$$\langle A(u), v-u\rangle \geq \langle F, v-u\rangle, \quad \text{for all} \quad v \in K. \tag{1.3.3}$$

If the operators A and F are nonlinear, then variational inequality (1.3.3) is known as *strongly nonlinear variational inequality*, introduced and studied by *Noor* [92].

If $F \equiv 0$, then (1.3.3) is equivalent to find $u \in K$ such that

$$\langle A(u), v-u\rangle \geq 0, \quad \text{for all} \quad v \in K. \tag{1.3.4}$$

The variational inequality of the type (1.3.4) was introduced and studied by *Fichera* [57] in 1964. *Lions and Stampacchia* [82] proved the existence of unique solution of (1.3.4) using essentially the projection techniques.

It is worth mentioning that the unilateral contact problems involving contact laws of monotone nature do not lead to the formulation of variational inequalities directly. However, it has been shown by *Panagiotopoulus* [95], using the notions of Clarke's generalized gradient and Rockafeller's upper subderivative, that the non-convex unilateral contact problems can only be characterized by a class of strongly nonlinear variational inequalities (1.3.3).

Till now, variational inequalities have been generalized and extended in various directions. Variational-like inequality is one of its generalized form, which was introduced and studied by *Parida et al.* [96].

Let K be a closed convex set in $\mathbb{R}^n$. Given two continuous maps $F : K \to \mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$, then the *variational-like inequality problem* is to find $u \in K$ such that

$$\langle F(u), \eta(u,v)\rangle \geq 0, \qquad \text{for all} \quad v \in K. \tag{1.3.5}$$

If $\eta(u,v) = v-u$, then variational-like inequality (1.3.5) is equivalent to the variational inequality (1.3.4).

A useful and important generalization of variational inequalities is a mixed type variational inequality containing nonlinear term. Due to the presence of the nonlinear term, the projection method can not be used to study the existence of a solution for the mixed type variational inequalities. In 1994, *Hassouni and Moudafi* [60] used the resolvent operator technique for maximal monotone mappings to study mixed type variational inequalities with single-valued mappings, which are called *variational inclusions* and developed a perturbed algorithm for finding approximate solutions of mixed variational inequalities.

Let H be a real Hilbert space endowed with a norm $\|.\|$ and inner product $\langle .,. \rangle$ and given continuous mappings $T, g : H \to H$, with Im(g) $\cap$ dom $(\partial\varphi) \neq \emptyset$.

Then the following problem of finding $u \in H$ such that $g(u) \cap \text{dom } (\partial\varphi) \neq \emptyset$ and

$$\langle T(u) - A(u), v - g(u) \rangle \geq \varphi(g(u)) - \varphi(v), \qquad \text{for all } \ v \in H, \tag{1.3.6}$$

where A is a nonlinear continuous mapping on H, $\partial\varphi$ denotes the subdifferential of a proper, convex and lower semicontinuous function $\varphi : H \to R \cup \{+\infty\}$, dom $(\partial\varphi)$ denotes the domain of $\partial\varphi$.

Problem (1.3.6) is called *variational inclusion problem*, introduced and studied in [60].

1.4. System Of Variational Inequalities

In the recent past, systems of variational inequalities are used as tools to solve various equilibrium-type problems like, Nash equilibrium, traffic equilibrium, spatial equilibrium and general equilibrium programming problems, problems from operations research, economics, game theory, mathematical physics and other areas; see for example [12,15,32,56,73,75,76,85,86,97] and references therein. *Pang* [97] uniformly modeled these equilibrium-type problems in the form of a variational inequality defined on a product of sets. He decomposed the original variational inequality into a system of variational inequalities, which are easy to solve, to establish some solution methods for finding the approximate solutions of above mentioned equilibrium-type problems. Later, it is found that these two problems, variational inequality defined on a product of sets and system of variational inequalities, are equivalent.

Further, *Ansari and Yao* [12] introduced and studied the following *system of variational inequalities* in Häusdorff topological vector space.

Let I be an index set and for each $i \in I$, let E_i be a Häusdorff topological vector space with its topological dual E_i^*. Let $\{K_i\}_{i\in I}$ be a family of nonempty, convex subsets with each K_i in E_i. Let $K = \prod_{i\in I} K_i$, $K_i = \prod_{j\neq i,\ j\in I} K_j$ and $E = \prod_{i\in I} E_i$. For each $i \in I$, let $A_i : K \to E_i^*$ be a given function.

Find $\bar{x} = (\bar{x}_i, \bar{x}^i) \in K$ such that for each $i \in I$,

$$\langle A_i(\bar{x}), y_i - \bar{x}_i\rangle \geq 0, \qquad \text{for all} \quad y_i \in K_i,$$

Kassay and Kolumbán [72] introduced the following *system of variational inequalities* and proved the existence of solutions using Ky-Fan's lemma.

Let H_1 and H_2 are two Hilbert spaces, $A \subset H_1$ and $B \subset H_2$ are two nonempty, closed and convex sets. Let $F : H_1 \times H_2 \to H_1$, $G : H_1 \times H_2 \to H_2$ be the single-valued mappings.

Find $(a, b) \in A \times B$ such that

$$\langle F(a,b), x - a\rangle \geq 0, \qquad \text{for all } x \in A,$$

$$\langle G(a,b), y - b\rangle \geq 0, \qquad \text{for all } y \in B.$$

Verma [114] introduced and studied the following *system of nonlinear variational inequalities.*

Let H be a real Hilbert space endowed with the inner product $\langle .,. \rangle$ and norm $||.||$. Let $A \subset H$ be a closed, convex subset of H. $T : A \to H$ is a nonlinear mapping and $\rho, \gamma > 0$ are constants.

Find $(a, b) \in A \times A$ such that

$$\langle \rho T(b) + a - b, x - a\rangle \geq 0, \qquad \text{for all } x \in A,$$

$$\langle \gamma T(a) + b - a, x - b\rangle \geq 0, \qquad \text{for all } y \in A.$$

Since then many authors have generalized and extended the system of variational inequalities (inclusions) in different directions using different techniques; see for examples [30,49,51,52,54,67,102,104,116,117] and references therein.

Chapter 2

Systems Of Mixed Variational Inequalities And Mixed Variational-like Inclusions

2.1. Introduction

Pang [97], *Cohen and Chaplais* [32], *Bianchi* [20], *Ansari and Yao* [12] considered a system of scalar variational inequalities and *Ansari et al.* [13] introduced and studied a system of vector equilibrium problems and a system of vector variational inequalities by using fixed point theorem. *Allevi et al.* [11] considered a system of generalized vector variational inequalities and established some existence results with relative pseudo monotonicity. *Kassay and Kolumbán* [72] introduced a system of variational inequalities and proved an existence theorem by using Ky-Fan's lemma. *Peng and Yang* [104] introduced a system of quasi-variational inequality problems and proved an existence theorem by maximal element theorems.

Recently, *Peng* [102] introduced a system of generalized mixed quasi-variational-like inclusions with (H, η)-accretive operators, i.e., a family of generalized mixed quasi-variational-like inclusions with (H, η)-accretive operators defined on a product of sets in Banach spaces.

In 2002, *Ding and Xia* [45] introduced the concept of J-proximal mapping for a lower semicontinuous subdifferentiable proper (may not be convex) functional, which is an extension of the resolvent operator technique, to propose an iterative algorithm for computing the approximate solutions of variational-like inequality problems. The generalization of J-proximal mapping is introduced and studied by *Ahmad et al.* [10], called J^{η}-proximal mapping.

In Section 2.2, we consider a system of mixed variational inequalities in Banach spaces. By using J-proximal mapping and its Lipschitz continuity introduced by *Ding and Xia* [45], an iterative algorithm for finding the approximate solutions of system of mixed variational inequalities is suggested.

In Section 2.3, we consider a system of mixed variational-like inclusions in Banach spaces which is a generalization of problem considered in section 3.2. By applying the notion of J^{η}-proximal mapping and its Lipschitz continuity introduced by *Ahmad et al.* [10], the existence of solutions for system of mixed variational-like inclusions is proved. The convergence analysis is also studied.

The following definitions and results will be used to prove the results of Section 2.2 and Section 2.3.

Definition 2.1.1.[45]. Let E be a Banach space with the dual space $E^{\star}$, $\varphi : E \to R\cup\{+\infty\}$ be a proper subdifferentiable (may not convex) functional and $J : E \to E^{\star}$ be a mapping. If for any given point $x^{\star} \in E^{\star}$ and $\rho > 0$, there is unique point $x \in E$ satisfying

$$\langle Jx - x^{\star}, y - x\rangle + \rho\varphi(y) - \rho\varphi(x) \geq 0, \quad \text{for all} \;\; y \in E.$$

The mapping $x^{\star} \to x$, denoted by $J_{\rho}^{\partial\varphi}(x^{\star})$, is said to be *J-proximal mapping* of φ. We have $x^{\star} - Jx \in \rho\partial\phi(x)$, it follows that $J_{\rho}^{\partial\phi}(x^{\star}) = (J + \rho\partial\phi)^{-1}(x^{\star})$.

Remark 2.1.1. If E is Hilbert space, φ is a convex lower semicontinuous proper functional on E and J is the identity mapping on E, then the J-proximal mapping of φ reduces to the resolvent operator of φ on Hilbert space.

Lemma 2.1.1.(Ding and Tan [44]). Let D be a nonempty convex subset of a topological vector space and $f : D \times D \to R \cup \{+\infty\}$ such that

(i) for any $x \in D$, $y \to f(x, y)$ is lower semicontinuous on each compact subset of D;

(ii) for each finite set $\{x_1, \cdots, x_n\} \in D$ and for each $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and

$\sum_{i=1}^{n} \lambda_i \equiv 1$,

$$\min_{1 \leq i \leq n} f(x_i, y) \leq 0;$$

(iii) there exists a nonempty compact convex subset D_0 of D and a nonempty compact subset K of D such that for each $y \in D \backslash K$, there is an $x \in \text{co}(D_0 \cup \{y\})$ satisfying $f(x, y) > 0$, then there exists $\hat{y} \in D$ such that $f(x, \hat{y}) \leq 0$, for all $x \in D$.

Definition 2.1.2.[10]. Let E be a Banach space with the dual space E^*, $\varphi : E \to R \cup \{+\infty\}$ be a proper η-subdifferentiable (may not be convex) functional, $\eta : E \times E \to E$ and $J : E \to E^*$ be the mappings. If for any given point $x^* \in E^*$ and $\rho > 0$, there is a unique point $x \in E$ satisfying

$$\langle Jx - x^*, \eta(y, x) \rangle + \rho\varphi(y) - \rho\varphi(x) \geq 0, \quad \text{for all } y \in E,$$

then the mapping $x^* \to x$, denoted by $J_\rho^{\partial_\eta \varphi}(x^*)$ is said to be *J^η-proximal mapping* of φ. We have $x^* - Jx \in \rho\partial_\eta\varphi(x)$, it follows that

$$J_\rho^{\partial_\eta \varphi}(x^*) = (J + \rho\partial_\eta\varphi)^{-1}(x^*).$$

Remark 2.1.2. If $\varphi : E \to R \cup \{+\infty\}$ is proper subdifferentiable and $\eta(y, x) = y - x$ for all $x, y \in E$, then Definition 2.1.2 of J^η-proximal mapping coincides with the definition of J-proximal mapping.

Definition 2.1.3. A functional $f : E \times E \to R \cup \{+\infty\}$ is said to be *0-diagonally quasi-concave* (in short 0-DQCV) in y, if for any finite subset $\{x_1, \cdots, x_n\} \subset E$ and for any $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$,

$$\min_{1 \leq i \leq n} f(x_i, y) \leq 0.$$

Some sufficient conditions are given here which guarantee the existence and Lipschitz continuity of the J-proximal mapping and J^η-proximal mapping of a proper functional on reflexive Banach spaces.

Theorem 2.1.1.[45]. Let E be a reflexive Banach space with the dual space E^* and $\varphi : E \to R \cup \{+\infty\}$ be a lower semicontinuous, subdifferentiable, proper functional which may not be convex. Let $J : E \to E^*$ be an α-strongly accretive continuous mapping. Then for any $\rho > 0$ and any $x^* \in E^*$, there exists a unique $x \in E$ such that

$$\langle Jx - x^*, y - x\rangle + \rho\varphi(y) - \rho\varphi(x) \geq 0, \quad \text{for all } \; y \in E.$$

That is $x = J_\rho^{\partial\varphi}(x^*)$ and so the J-proximal mapping $J_\rho^{\partial\varphi}$ of φ is well defined and is $1/\alpha$-Lipschitz continuous.

Theorem 2.1.2.[10]. Let E be a reflexive Banach space with the dual space E^* and $\varphi : E \to R \cup \{+\infty\}$ be lower semicontinuous, η-subdifferentiable, proper functional which may not be convex. Let $J : E \to E^*$ be a mapping and let $\eta : E \times E \to E$ be Lipschitz continuous with constant $\tau > 0$, J-strongly accretive with constant $\alpha > 0$ such that $\eta(x,y) = -\eta(y,x)$ for all $x, y \in E$ and for any $x \in E$, the function $h(y,x) = \langle x^* - Jx, \eta(y,x)\rangle$ is 0-DQCV in y. Then for any $\rho > 0$, and any $x^* \in E^*$, there exists a unique $x \in E$ such that

$$\langle Jx - x^*, \eta(y,x)\rangle + \rho\varphi(y) - \rho\varphi(x) \geq 0, \quad \text{for all } \; y \in E.$$

That is, $x = J_\rho^{\partial_\eta\varphi}(x^*)$ and so the J^η-proximal mapping of φ is well defined and τ/α-Lipschitz continuous.

The following MatLab programming shows that $\eta : E \times E \to E$ satisfies conditions (1)-(3) in Theorem 2.1.2 and condition (4) is shown seperately.

Example 2.1.1. Let $E = R$ and $J = I$

```
function value=eta(x,y)
if abs(x*y)<1/4
value=2*x-2*y;
elseif abs(x*y)>=1/4 & abs(x*y)<1/2
value=8*abs(x*y)*(x-y);
elseif abs(x*y)>=1/2
value=4*(x-y);
end
```

Then it is easy to see that:

(1) $\langle \eta(x,y), x-y\rangle \geq 2|x-y|^2$ for all $x, y \in R$, i.e., η is 2-strongly accretive;

(2) $\eta(x,y) = -\eta(y,x)$ for all $x, y \in R$;

(3) $|\eta(x,y)| \leq 4|x-y|$ for all $x, y \in R$, i.e., η is 4-Lipschitz continuous;

(4) for any $x \in R$, the function $h(y,u) = \langle x-u, \eta(y,u)\rangle = (x-u)\eta(y,u)$ is 0-DQCV in y.

If it is false, then there exists a finite set $\{y_1, \cdots, y_n\}$ and $u_0 = \sum_{i=1}^{n} \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$ such that for each $i = 1, 2, \ldots\ldots, n$

$$0 < h(y_i, u_0) = \begin{cases} (x-u_0)(2y_i - 2u_0) & \text{if } |y_i u_0| < 1/4, \\ (x-u_0)8|y_i u_0|(y_i - u_0) & \text{if } 1/4 \leq |y_i u_0| < 1/2, \\ 4(x-u_0)(y_i - u_0) & \text{if } 1/2 \leq |y_i u_0|. \end{cases}$$

It follows that $(x-u_0)(2y_i - 2u_0) > 0$, for each $i = 1, 2, \ldots\ldots, n$ and hence we have

$$0 < \sum_{i=1}^{n} \lambda_i (x-u_0)(2y_i - 2u_0) = (x-u_0)(2u_0 - 2u_0) = 0,$$

which is not possible. Hence $h(y,u)$ is 0-DQCV in y. Therefore, η satisfies all assumptions in Theorem 2.1.2.

2.2. System Of Mixed Variational Inequalities

This section is devoted to the study of a system of mixed variational inequalities in Banach spaces. By using J-proximal mapping and its Lipschitz continuity for a nonconvex, lower semicontinuous, subdifferentiable, proper functional, an iterative algorithm for computing the approximate solutions of system of mixed variational inequalities is suggested. The existence and convergence of solutions of our system are proved.

Let E_1 and E_2 be any two real Banach spaces. Let $S : E_1 \times E_2 \to E_1^*$, $T : E_1 \times E_2 \to E_2^*$, $f_1 : E_1 \to E_1$ and $f_2 : E_2 \to E_2$ be the single-valued mappings, H :

$E_1 \to CB(E_1^*)$ and $F : E_2 \to CB(E_2^*)$ be set-valued mappings. Let $\varphi_1 : E_1 \times E_1 \to R \cup \{+\infty\}$ be lower semicontinuous, subdifferentiable (may not be convex), proper functional on E_1 satisfying $f_1(x) \in \text{dom } (\partial\varphi_1(\cdot, x))$ and $\varphi_2 : E_2 \times E_2 \to R \cup \{+\infty\}$ be lower semicontinuous, subdifferentiable (may not be convex), proper functional on E_2 satisfying $f_2(y) \in \text{dom } (\partial\varphi_2(\cdot, y))$, where $\partial\varphi_1(\cdot, x)$ is subdifferential of $\varphi_1(\cdot, x)$ and $\partial\varphi_2(\cdot, y)$ is subdifferential of $\varphi_2(\cdot, y)$. We consider the following *system of mixed variational inequalities*:

Find $(x, y) \in E_1 \times E_2$, $u \in H(x)$ and $v \in F(y)$ such that

$$\langle S(x, v), a - f_1(x)\rangle \geq \varphi_1(f_1(x), x) - \varphi_1(a, x), \qquad \text{for all } a \in E_1,$$

$$\langle T(u, y), b - f_2(y)\rangle \geq \varphi_2(f_2(y), y) - \varphi_2(b, y), \qquad \text{for all } b \in E_2. \quad (2.2.1)$$

If $E_1{=}H_1$, $E_2{=}H_2$, where H_1 and H_2 are Hilbert spaces, $f_1{=}f_2{=}I$, where I is the identity mapping, H and F are single-valued mapping, $\varphi_1(x, \cdot){=}\varphi_1(x)$ and $\varphi_2(y, \cdot){=}\varphi_2(y)$, then the Problem (2.2.1) reduces to the following problem:

Find $(x, y) \in H_1 \times H_2$ such that

$$\langle S(x, F(y)), a - x\rangle + \varphi_1(a) - \varphi_1(x) \geq 0, \qquad \text{for all } a \in H_1,$$

$$\langle T(H(x), y), b - y\rangle + \varphi_2(b) - \varphi_2(y) \geq 0, \qquad \text{for all } b \in H_2, \quad (2.2.2)$$

which is called a system of nonlinear mixed variational inequalities. Some special cases of the Problem (2.2.2) can be found in [114]. Further, if $F{=}H{=}I$, then the Problem (2.2.2) reduces to the system of nonlinear variational inequalities problem considered by *Cho et al.* [30].

We mention the following theorem which transfer our problem *system of mixed variational inequalities* (2.2.1) into a fixed point problem.

Theorem 2.2.1. (x, y, u, v), where $(x, y) \in E_1 \times E_2$, $u \in H(x)$, $v \in F(y)$ is a solution of the *system of mixed variational inequalities* (2.2.1) if and only if it satisfies

$$f_1(x) = J_\rho^{\partial\varphi_1(\cdot,x)}[J_1(f_1(x)) - \rho S(x, v)],$$

$$f_2(y) = J_\gamma^{\partial\varphi_2(\cdot,y)}[J_2(f_2(y)) - \gamma T(u,y)],$$

where $J_1 : E_1 \to E_1^*$, $J_2 : E_2 \to E_2^*$, $J_\rho^{\partial\varphi_1(\cdot,x)} = (J_1 + \rho\partial\varphi_1(\cdot,x))^{-1}$, $J_\gamma^{\partial\varphi_2(\cdot,y)} = (J_2 + \gamma\partial\varphi_2(\cdot,y))^{-1}$ and $\rho > 0$, $\gamma > 0$ are constants.

Proof. The fact is directly follows from Definition 2.1.1.

We propose the following proximal point algorithm to compute the approximate solutions of our problem *system of mixed variational inequalities* (2.2.1).

Algorithm 2.2.1. For any given $(x_0, y_0) \in E_1 \times E_2$, we choose $u_0 \in H(x_0)$, $v_0 \in F(y_0)$ and compute $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by iterative schemes as follows:

$$f_1(x_{n+1}) = J_\rho^{\partial\varphi_1(\cdot,x_n)}[J_1(f_1(x_n)) - \rho S(x_n, v_n)], \tag{2.2.3}$$

$$f_2(y_{n+1}) = J_\gamma^{\partial\varphi_2(\cdot,y_n)}[J_2(f_2(y_n)) - \gamma T(u_n, y_n)], \tag{2.2.4}$$

and choose $u_{n+1} \in H(x_{n+1})$ and $v_{n+1} \in F(y_{n+1})$ such that

$$\|u_{n+1} - u_n\| \le \left(1 + \frac{1}{n+1}\right) D(H(x_{n+1}), H(x_n)), \tag{2.2.5}$$

$$\|v_{n+1} - v_n\| \le \left(1 + \frac{1}{n+1}\right) D(F(y_{n+1}), F(y_n)), \tag{2.2.6}$$

where $\rho > 0$ and $\gamma > 0$ are constants and $n = 0, 1, 2, \ldots\ldots$

We use Algorithm 2.2.1 to compute the approximate solutions of *system of mixed variational inequalities* (2.2.1). The convergence analysis is also studied.

Theorem 2.2.2. Let E_1 and E_2 be two reflexive Banach spaces with their duals E_1^* and E_2^*, respectively. Let $S : E_1 \times E_2 \to E_1^*$ and $T : E_1 \times E_2 \to E_2^*$ are Lipschitz continuous in both the arguments with constants λ_{S_1}, λ_{S_2} and λ_{T_1}, λ_{T_2}, respectively. For $i = 1, 2$, let $f_i : E_i \to E_i$ is Lipschitz continuous with constants λ_{f_i} and strongly accretive with constants δ_{f_i} such that $f(E_i) = E_i$, $J_i : E_i \to E_i^*$ be Lipschitz continuous with constants λ_{J_i} and strongly accretive with constants α_i. Let $\varphi_1 : E_1 \times E_1 \to R \cup \{+\infty\}$ be lower semicontinuous, subdifferential (may not be convex), proper functional on E_1 satisfying $f_1(x) \in \text{dom } (\partial\varphi_1(\cdot, x))$ and $\varphi_2 : E_2 \times E_2 \to R \cup \{+\infty\}$ be lower semicontinuous, subdifferential (may not be convex), proper functional on E_2 satisfying $f_2(y) \in \text{dom}\ (\partial\varphi_2(\cdot, y))$, for all $x \in E_1$ and $y \in E_2$. Let $H : E_1 \to CB(E_1^*)$ and $F : E_2 \to CB(E_2^*)$ be D-Lipschitz continuous mappings with constants λ_{D_H} and λ_{D_F}, respectively.

If there exists constants $\rho > 0$ and $\gamma > 0$ such that

$$||J_\rho^{\partial\varphi_1(\cdot,x_n)}(x^*) - J_\rho^{\partial\varphi_1(\cdot,x_{n-1})}(x^*)|| \le \mu^*||x_n - x_{n-1}||, \tag{2.2.7}$$

$$\text{for any}\ \ x_n, x_{n-1} \in E_1,\ x^* \in E_1^*,$$

$$||J_\gamma^{\partial\varphi_2(\cdot,y_n)}(y^*) - J_\gamma^{\partial\varphi_2(\cdot,y_{n-1})}(y^*)|| \le \mu^{**}||y_n - y_{n-1}||, \tag{2.2.8}$$

$$\text{for any}\ \ y_n, y_{n-1} \in E_2,\ y^* \in E_2^*$$

and the following condition is satisfied:

$$\begin{cases} 0 < \sqrt{\frac{4(\lambda_{J_1}\lambda_{f_1})^2 + 8\rho^2\lambda_{S_1}^2 + 2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}} + \sqrt{\frac{8\gamma^2(\lambda_{T_1}\lambda_{D_H})^2}{(2\delta_{f_2}+3)\alpha_2^2}} < 1 \\ 0 < \sqrt{\frac{8\rho^2(\lambda_{S_2}\lambda_{D_F})^2}{(2\delta_{f_1}+3)\alpha_1^2}} + \sqrt{\frac{4(\lambda_{J_2}\lambda_{f_2})^2 + 8\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}} < 1. \end{cases} \tag{2.2.9}$$

Then the *system of mixed variational inequalities* (2.2.1) admits a solution (x, y, u, v) and the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ converge to x, y, u, and v, respectively, where $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ are the sequences generated by Algorithm 2.2.1.

Proof. We can write

$$||x_{n+1} - x_n||^2 = ||f_1(x_{n+1}) - f_1(x_n) - f_1(x_{n+1}) + f_1(x_n) - x_{n+1} + x_n||^2.$$

By Proposition 1.2.1, we have

$$\begin{aligned} ||x_{n+1} - x_n||^2 \le\ & ||f_1(x_{n+1}) - f_1(x_n)||^2 \\ & -2\langle f_1(x_{n+1}) - f_1(x_n) + x_{n+1} - x_n, j(x_{n+1} - x_n)\rangle. \end{aligned} \tag{2.2.10}$$

By (2.2.3), we have

$$f_1(x_{n+1}) = J_\rho^{\partial\varphi_1(\cdot,x_n)}[J_1(f_1(x_n)) - \rho S(x_n, v_n)].$$

Thus

$$\|f_1(x_{n+1}) - f_1(x_n)\|^2 = \|J_\rho^{\partial\varphi_1(\cdot,x_n)}[J_1(f_1(x_n)) - \rho S(x_n, v_n)]$$
$$-J_\rho^{\partial\varphi_1(\cdot,x_{n-1})}[J_1(f_1(x_{n-1})) - \rho S(x_{n-1}, v_{n-1})]\|^2.$$

Since $\|x+y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$, therefore by the assumption (2.2.7) and Theorem 2.1.1, we have

$$\begin{aligned}
\frac{1}{2}\|f_1(x_{n+1}) - f_1(x_n)\|^2 &\leq \|J_\rho^{\partial\varphi_1(\cdot,x_n)}[J_1(f_1(x_n)) - \rho S(x_n, v_n)] \\
&\quad -J_\rho^{\partial\varphi_1(\cdot,x_n)}[J_1(f_1(x_{n-1})) - \rho S(x_{n-1}, v_{n-1})]\|^2 \\
&\quad +\|J_\rho^{\partial\varphi_1(\cdot,x_n)}[J_1(f_1(x_{n-1})) - \rho S(x_{n-1}, v_{n-1})] \\
&\quad -J_\rho^{\partial\varphi_1(\cdot,x_{n-1})}[J_1(f_1(x_{n-1})) - \rho S(x_{n-1}, v_{n-1})]\|^2 \\
&\leq \frac{1}{\alpha_1^2}\|[J_1(f_1(x_n)) - \rho S(x_n, v_n)] - [J_1(f_1(x_{n-1})) \\
&\quad -\rho S(x_{n-1}, v_{n-1})]\|^2 + \mu^{*2}\|x_n - x_{n-1}\|^2 \\
&\leq \frac{2}{\alpha_1^2}\|J_1(f_1(x_n)) - J_1(f_1(x_{n-1}))\|^2 + \frac{2\rho^2}{\alpha_1^2}\|S(x_n, v_n) \\
&\quad -S(x_{n-1}, v_{n-1})\|^2 + \mu^{*2}\|x_n - x_{n-1}\|^2. \qquad (2.2.11)
\end{aligned}$$

By the Lipschitz continuity of J_1 and f_1, we have

$$\|J_1(f_1(x_n)) - J_1(f_1(x_{n-1}))\| \leq \lambda_{J_1}\lambda_{f_1}\|x_n - x_{n-1}\|. \qquad (2.2.12)$$

By the Lipschitz continuity of $S(\cdot,\cdot)$ in both the arguments, (2.2.6) and D-Lipschitz continuity of F, we have

$$\begin{aligned}
\|S(x_n, v_n) - S(x_n, v_{n-1})\| &\leq \lambda_{S_2}\|v_n - v_{n-1}\| \\
&\leq \lambda_{S_2}\left(1 + \frac{1}{n}\right) D(F(y_n), F(y_{n-1})) \\
&\leq \lambda_{S_2}\left(1 + \frac{1}{n}\right) \lambda_{D_F}\|y_n - y_{n-1}\|. \qquad (2.2.13)
\end{aligned}$$

$$\|S(x_n, v_{n-1}) - S(x_{n-1}, v_{n-1})\| \leq \lambda_{S_1}\|x_n - x_{n-1}\|. \qquad (2.2.14)$$

Using (2.2.13) and (2.2.14), it follows that

$$\|S(x_n, v_n) - S(x_{n-1}, v_{n-1})\|^2 \leq 2\|S(x_n, v_n) - S(x_n, v_{n-1})\|^2$$
$$+2\|S(x_n, v_{n-1}) - S(x_{n-1}, v_{n-1})\|^2$$

$$\leq 2(\lambda_{S_2}\lambda_{D_F})^2\left(1+\frac{1}{n}\right)^2\|y_n-y_{n-1}\|^2$$
$$+2\lambda_{S_1}^2\|x_n-x_{n-1}\|^2. \tag{2.2.15}$$

By (2.2.12) and (2.2.15), (2.2.11) becomes

$$\|f_1(x_{n+1})-f_1(x_n)\|^2 \leq \left[\frac{4}{\alpha_1^2}(\lambda_{J_1}\lambda_{f_1})^2+\frac{8}{\alpha_1^2}\rho^2\lambda_{S_1}^2+2\mu^{*2}\right]\|x_n-x_{n-1}\|^2$$
$$+\frac{8}{\alpha_1^2}\rho^2(\lambda_{S_2}\lambda_{D_F})^2\left(1+\frac{1}{n+1}\right)^2\|y_n-y_{n-1}\|^2. \tag{2.2.16}$$

By using the strong accretiveness of f_1 with constant δ_{f_1} and (2.2.16), (2.2.10) becomes

$$\|x_{n+1}-x_n\|^2 \leq \|f_1(x_{n+1})-f_1(x_n)\|^2-2\langle f_1(x_{n+1})-f_1(x_n)+x_{n+1}-x_n, j(x_{n+1}-x_n)\rangle$$
$$\leq \left[\frac{4}{\alpha_1^2}(\lambda_{J_1}\lambda_{f_1})^2+\frac{8}{\alpha_1^2}\rho^2\lambda_{S_1}^2+2\mu^{*2}\right]\|x_n-x_{n-1}\|^2$$
$$+\frac{8}{\alpha_1^2}\rho^2(\lambda_{S_2}\lambda_{D_F})^2\left(1+\frac{1}{n}\right)^2\|y_n-y_{n-1}\|^2$$
$$-(2\delta_{f_1}+2)\|x_{n+1}-x_n\|^2. \tag{2.2.17}$$

It follows that

$$\|x_{n+1}-x_n\|^2 \leq \left[\frac{4(\lambda_{J_1}\lambda_{f_1})^2}{(2\delta_{f_1}+3)\alpha_1^2}+\frac{8\rho^2\lambda_{S_1}^2}{(2\delta_{f_1}+3)\alpha_1^2}+\frac{2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}\right]\|x_n-x_{n-1}\|^2$$
$$+\frac{8\rho^2(\lambda_{S_2}\lambda_{D_F})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_1}+3)\alpha_1^2}\|y_n-y_{n-1}\|^2$$
$$=\theta_1\|x_n-x_{n-1}\|^2+\theta_2\|y_n-y_{n-1}\|^2$$
$$\leq \theta_1\|x_n-x_{n-1}\|^2+\theta_2\|y_n-y_{n-1}\|^2$$
$$+2\sqrt{\theta_1}\sqrt{\theta_2}\|x_n-x_{n-1}\|\|y_n-y_{n-1}\|$$
$$=(\sqrt{\theta_1}\|x_n-x_{n-1}\|+\sqrt{\theta_2}\|y_n-y_{n-1}\|)^2.$$

Thus, we have

$$\|x_{n+1}-x_n\| \leq \sqrt{\theta_1}\|x_n-x_{n-1}\|+\sqrt{\theta_2}\|y_n-y_{n-1}\|, \tag{2.2.18}$$

where

$$\theta_1 = \frac{4(\lambda_{J_1}\lambda_{f_1})^2}{(2\delta_{f_1}+3)\alpha_1^2} + \frac{8\rho^2\lambda_{S_1}^2}{(2\delta_{f_1}+3)\alpha_1^2} + \frac{2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}$$

and

$$\theta_2 = \frac{8\rho^2(\lambda_{S_2}\lambda_{D_F})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_1}+3)\alpha_1^2}.$$

We can also write

$$\|y_{n+1}-y_n\|^2 = \|f_2(y_{n+1})-f_2(y_n)-f_2(y_{n+1})+f_2(y_n)-y_{n+1}+y_n\|^2.$$

By Proposition 1.2.1, we have

$$\|y_{n+1}-y_n\|^2 \le \|f_2(y_{n+1})-f_2(y_n)\|^2 - 2\langle f_2(y_{n+1})-f_2(y_n)$$
$$+y_{n+1}-y_n, j(y_{n+1}-y_n)\rangle. \qquad (2.2.19)$$

By (2.2.4), we have

$$f_2(y_{n+1}) = J_\gamma^{\partial\varphi_2(\cdot,y_n)}[J_2(f_2(y_n)) - \gamma T(u_n,y_n)].$$

Thus

$$\|f_2(y_{n+1})-f_2(y_n)\|^2 = \|J_\gamma^{\partial\varphi_2(\cdot,y_n)}[J_2(f_2(y_n)) - \gamma T(u_n,y_n)]$$
$$-J_\gamma^{\partial\varphi_2(\cdot,y_{n-1})}[J_2(f_2(y_{n-1})) - \gamma T(u_{n-1},y_{n-1})]\|^2.$$

Using the same argument as for (2.2.11), we have

$$\frac{1}{2}\|f_2(y_{n+1})-f_2(y_n)\|^2 \le \frac{2}{\alpha_2^2}\|J_2(f_2(y_n)) - J_2(f_2(y_{n-1}))\|^2 + \frac{2\gamma^2}{\alpha_2^2}\|T(u_n,y_n)$$
$$-T(u_{n-1},y_{n-1})\|^2 + \mu^{**2}\|y_n-y_{n-1}\|^2. \qquad (2.2.20)$$

By the Lipschitz continuity of J_2 and f_2, we have

$$\|J_2(f_2(y_n)) - J_2(f_2(y_{n-1}))\| \le \lambda_{J_2}\lambda_{f_2}\|y_n-y_{n-1}\|. \qquad (2.2.21)$$

By the Lipschitz continuity of $T(\cdot,\cdot)$ in both the arguments, (2.2.5) and D-Lipschitz continuity of H, we have

$$\|T(u_n,y_n) - T(u_n,y_{n-1})\| \le \lambda_{T_2}\|y_n-y_{n-1}\|. \qquad (2.2.22)$$

$$\begin{aligned}\|T(u_n, y_{n-1}) - T(u_{n-1}, y_{n-1})\| &\leq \lambda_{T_1}\|u_n - u_{n-1}\| \\ &\leq \lambda_{T_1}\left(1+\frac{1}{n}\right) D(H(x_n), H(x_{n-1})) \\ &\leq \lambda_{T_1}\lambda_{D_F}\left(1+\frac{1}{n}\right)\|x_n - x_{n-1}\|. \qquad (2.2.23)\end{aligned}$$

Using (2.2.22) and (2.2.23), it follows that

$$\begin{aligned}&\|T(u_n, y_n) - T(u_{n-1}, y_{n-1})\|^2 \\ &\quad\leq 2\|T(u_n, y_n) - T(u_n, y_{n-1})\|^2 + 2\|T(u_n, y_{n-1}) - T(u_{n-1}, y_{n-1})\|^2 \\ &\quad\leq 2\lambda_{T_2}^2\|y_n - y_{n-1}\|^2 + 2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2\|x_n - x_{n-1}\|^2. \qquad (2.2.24)\end{aligned}$$

By (2.2.21) and (2.2.24), (2.2.20) becomes

$$\begin{aligned}\|f_2(y_{n+1}) - f_2(y_n)\|^2 &\leq \left[\frac{4}{\alpha_2^2}(\lambda_{J_2}\lambda_{f_2})^2 + \frac{8}{\alpha_2^2}\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\right]\|y_n - y_{n-1}\|^2 \\ &\quad+\frac{8}{\alpha_2^2}\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2\|x_n - x_{n-1}\|^2. \qquad (2.2.25)\end{aligned}$$

By using the strong accretiveness of f_2 with constant δ_{f_2} and (2.2.25), (2.2.19) becomes

$$\begin{aligned}\|y_{n+1} - y_n\|^2 &\leq \|f_2(y_{n+1}) - f_2(y_n)\|^2 - 2\langle f_2(y_{n+1}) - f_2(y_n) + y_{n+1} - y_n, j(y_{n+1} - y_n)\rangle \\ &\leq \left[\frac{4}{\alpha_2^2}(\lambda_{J_2}\lambda_{f_2})^2 + \frac{8}{\alpha_2^2}\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\right]\|y_n - y_{n-1}\|^2 \\ &\quad+\frac{8}{\alpha_2^2}\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2\|x_n - x_{n-1}\|^2 \\ &\quad-(2\delta_{f_2}+2)\|y_{n+1} - y_n\|^2. \qquad (2.2.26)\end{aligned}$$

It follows that

$$\begin{aligned}\|y_{n+1} - y_n\|^2 &\leq \left[\frac{4(\lambda_{J_2}\lambda_{f_2})^2 + 8\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}\right]\|y_n - y_{n-1}\|^2 \\ &\quad+\frac{8\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_2}+3)\alpha_2^2}\|x_n - x_{n-1}\|^2\end{aligned}$$

$$
\begin{aligned}
&= \theta_3\|y_n - y_{n-1}\|^2 + \theta_4\|x_n - x_{n-1}\|^2 \\
&\leq \theta_3\|y_n - y_{n-1}\|^2 + \theta_4\|x_n - x_{n-1}\|^2 \\
&\qquad +2\sqrt{\theta_3}\sqrt{\theta_4}\|y_n - y_{n-1}\|\|x_n - x_{n-1}\| \\
&= (\sqrt{\theta_3}\|y_n - y_{n-1}\| + \sqrt{\theta_4}\|x_n - x_{n-1}\|)^2.
\end{aligned}
$$

Thus, we have

$$
\|y_{n+1} - y_n\| \leq \sqrt{\theta_3}\|y_n - y_{n-1}\| + \sqrt{\theta_4}\|x_n - x_{n-1}\|, \tag{2.2.27}
$$

where

$$
\theta_3 = \frac{4(\lambda_{J_2}\lambda_{f_2})^2 + 8\tau_2^2\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}
$$

and

$$
\theta_4 = \frac{8\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_2}+3)\alpha_2^2}.
$$

By (2.2.18) and (2.2.27), we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| &\leq (\sqrt{\theta_1} + \sqrt{\theta_4})\|x_n - x_{n-1}\| \\
&\qquad +(\sqrt{\theta_2} + \sqrt{\theta_3})\|y_n - y_{n-1}\| \\
&= \theta_n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|),
\end{aligned} \tag{2.2.28}
$$

where

$$
\begin{aligned}
\theta_n = \max\Bigg\{ &\sqrt{\frac{4(\lambda_{J_1}\lambda_{f_1})^2 + 8\rho^2\lambda_{S_1}^2 + 2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}} \\
&+\sqrt{\frac{8\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_2}+3)\alpha_2^2}}, \sqrt{\frac{8\rho^2(\lambda_{S_2}\lambda_{D_F})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_1}+3)\alpha_1^2}} \\
&+\sqrt{\frac{4(\lambda_{J_2}\lambda_{f_2})^2 + 8\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}}\Bigg\}.
\end{aligned}
$$

Let

$$\theta = \max\left\{ \sqrt{\frac{4(\lambda_{J_1}\lambda_{f_1})^2 + 8\rho^2\lambda_{S_1}^2 + 2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}} + \sqrt{\frac{8\gamma^2(\lambda_{T_1}\lambda_{D_H})^2}{(2\delta_{f_2}+3)\alpha_2^2}}, \sqrt{\frac{8\rho^2(\lambda_{S_2}\lambda_{D_F})^2}{(2\delta_{f_1}+3)\alpha_1^2}} + \sqrt{\frac{4(\lambda_{J_2}\lambda_{f_2})^2 + 8\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}} \right\}.$$

Then $\theta_n \to \theta$ as $n \to \infty$. By (2.2.9), we know that $0 < \theta < 1$ and so (2.2.28) implies that $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences. Thus, there exists $x \in E_1$ and $y \in E_2$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Now we prove that $u_n \to u \in H(x)$ and $v_n \to v \in F(y)$. In fact, it follows from the D-Lipschitz continuity of H, F,(2.2.5) and (2.2.6) that

$$\|u_n - u_{n-1}\| \le \left(1 + \frac{1}{n}\right) \lambda_{D_H} \|x_n - x_{n-1}\|, \tag{2.2.29}$$

$$\|v_n - v_{n-1}\| \le \left(1 + \frac{1}{n}\right) \lambda_{D_F} \|y_n - y_{n-1}\|. \tag{2.2.30}$$

From (2.2.29) and (2.2.30), we know that $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences. We can assume that $u_n \to u$ and $v_n \to v$ as $n \to \infty$.

Further,

$$\begin{aligned} d(u, H(x)) &\le \|u - u_n\| + d(u_n, H(x)) \\ &\le \|u - u_n\| + D\left(H(x_n), H(x)\right) \\ &\le \|u - u_n\| + \lambda_{D_H}\|x_n - x\| \to 0, \text{ as } n \to +\infty. \end{aligned}$$

Hence $d(u, H(x)) = 0$ and therefore $u \in H(x)$. Similarly, we can show that $v \in F(y)$. By continuity of f_1, f_2, J_1, J_2, S, T, $J_\rho^{\partial\varphi_1}$, $J_\gamma^{\partial\varphi_2}$, φ_1, φ_2, H, F, and Algorithm 2.2.1, we know that x, y, u and v satisfy the following relations

$$f_1(x) = J_\rho^{\partial\varphi_1(\cdot,x)}[J_1(f_1(x)) - \rho S(x, v)],$$

$$f_2(y) = J_\gamma^{\partial\varphi_2(\cdot,y)}[J_2(f_2(y)) - \gamma T(u, y)].$$

By Theorem 2.2.1, $(x, y) \in E_1 \times E_2$, $u \in H(x)$ and $v \in F(y)$ is a solution of Problem (2.2.1). This completes the proof.

2.3. System Of Mixed Variational-Like Inclusions

In this section, we consider a system of mixed variational-like inclusions in Banach spaces. By applying the notion of J^{η}-proximal mapping and its Lipschitz continuity for a nonconvex, lower semicontinuous, η-subdifferentiable proper functional, we define an iterative method for system of mixed variational-like inclusions.

Let E_1 and E_2 be any two real Banach spaces, $S : E_1 \times E_2 \to E_1^*$, $T : E_1 \times E_2 \to E_2^*$, $f_1 : E_1 \to E_1$, $f_2 : E_2 \to E_2$, $\eta_1 : E_1 \times E_1 \to E_1$ and $\eta_2 : E_2 \times E_2 \to E_2$ be the single-valued mappings, $H : E_1 \to 2^{E_1}$ and $F : E_2 \to 2^{E_2}$ be any two multivalued mappings. Let $\varphi_1 : E_1 \times E_1 \to R \cup \{+\infty\}$ be lower semicontinuous (may not be convex), η_1-subdifferentiable, proper functional on E_1 satisfying $f_1(E_1) \cap \text{dom}\, \partial_{\eta_1}\varphi_1 \neq \emptyset$ and $\varphi_2 : E_2 \times E_2 \to R \cup \{+\infty\}$ be lower semicontinuous (may not be convex), η_2-subdifferentiable, proper functional and E_2 satisfying $f_2(E_2) \cap \text{dom}\, \partial_{\eta_2}\varphi_2 \neq \emptyset$, where $\partial_{\eta_1}\varphi_1$ is η_1-subdifferential of φ_1 and $\partial_{\eta_2}\varphi_2$ is η_2-subdifferential of φ_2. Then we consider the following *system of mixed variational-like inclusions*:

Find $(x, y) \in E_1 \times E_2$, $u \in H(x)$ and $v \in F(y)$ such that

$$\langle S(x, v), \eta_1(a, f_1(x))\rangle \geq \varphi_1(f_1(x), x) - \varphi_1(a, x), \quad \text{for all} \quad a \in E_1,$$

$$\langle T(u, y), \eta_2(b, f_2(y))\rangle \geq \varphi_2(f_2(y), y) - \varphi_2(b, y), \quad \text{for all} \quad b \in E_2. \quad (2.3.1)$$

It is clear that for a suitable choices of the mappings involved in the formulation of the *system of mixed variational-like inclusions* (2.3.1), we can derive many systems of variational inequalities (inclusions) considered and studied in the literature.

We suggest a fixed point formulation which shows the equivalence between our problem *system of mixed variational-like inclusions* (2.3.1) and a fixed point problem.

Theorem 2.3.1. (x, y, u, v), where $(x, y) \in E_1 \times E_2$, $u \in H(x)$ and $v \in F(y)$ is a solution of *system of mixed variational-like inclusions* (2.3.1) if and only if (x, y, u, v) satisfies

$$f_1(x) = J_{\rho}^{\partial_{\eta_1}\varphi_1(\cdot, x)}[J_1(f_1(x)) - \rho S(x, v)],$$

$$f_2(y) = J_{\gamma}^{\partial_{\eta_2}\varphi_2(\cdot,y)}[J_2(f_2(y)) - \gamma T(u,y)],$$

where $J_1 : E_1 \to E_1^*$, $J_2 : E_2 \to E_2^*$, $J_{\rho}^{\partial_{\eta_1}\varphi_1(\cdot,x)} = (J_1 + \rho\partial_{\eta_1}\varphi_1(\cdot,x))^{-1}$, $J_{\gamma}^{\partial_{\eta_2}\varphi_2(\cdot,y)} = (J_2 + \gamma\partial_{\eta_2}\varphi_2(\cdot,y))^{-1}$, $\rho > 0$ and $\gamma > 0$ are constants.

Proof. The fact directly follows from Definition 2.1.2.

The above fixed point formulation enables us to suggest the following proximal point algorithm.

Algorithm 2.3.1. For any given $(x_0, y_0) \in E_1 \times E_2$, we choose $u_0 \in H(x_0)$, $v_0 \in F(y_0)$ and compute $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by iterative schemes as follows:

$$f_1(x_{n+1}) = J_{\rho}^{\partial_{\eta_1}\varphi_1(\cdot,x_n)}[J_1(f_1(x_n)) - \rho S(x_n, v_n)], \tag{2.3.2}$$

$$f_2(y_{n+1}) = J_{\gamma}^{\partial_{\eta_2}\varphi_2(\cdot,y_n)}[J_2(f_2(y_n)) - \gamma T(u_n, y_n)] \tag{2.3.3}$$

and choose $u_{n+1} \in H(x_{n+1})$ and $v_{n+1} \in F(y_{n+1})$ such that

$$u_n \in H(x_n), \quad \|u_{n+1} - u_n\| \le \left(1 + \frac{1}{n+1}\right) D(H(x_{n+1}), H(x_n)), \tag{2.3.4}$$

$$v_n \in F(y_n), \quad \|v_{n+1} - v_n\| \le \left(1 + \frac{1}{n+1}\right) D(F(y_{n+1}), F(y_n)), \tag{2.3.5}$$

where $\rho > 0$ and $\gamma > 0$ are constants and $n = 0, 1, 2, \ldots\ldots$

Now the existence of solutions for *system of mixed variational-like inclusions* (2.3.1) is proved and the convergence of iterative sequences generated by the Algorithm 2.3.1 is also studied.

Theorem 2.3.2. Let E_1 and E_2 be two reflexive Banach spaces with their duals E_1^* and E_2^*, respectively. For $i = 1, 2$, let $\eta_i : E_i \times E_i \to E_i$ be Lipschitz continuous with constants τ_i such that $\eta_i(x_1, x_2) = -\eta_i(x_2, x_1)$ for all $x_1, x_2 \in E_i$, J_i-strongly accretive with constants α_i and for any $x_1 \in E_i$, the function $h_i(x_2, x_1) = \langle x_1^* - J_i x_1, \eta_i(x_2, x_1)\rangle$ is 0-DQCV in x_2. Let $J_i : E_i \to E_i^*$ be Lipschitz continuous with constant λ_{J_i}, $f_i : E_i \to E_i$ is Lipschitz continuous with constant λ_{f_i} and strongly accretive with constant δ_{f_i} such that $f_i(E_i) = E_i$, $\varphi_i : E_i \times E_i \to R \cup \{+\infty\}$ be lower semicontinuous, η_i-subdifferentiable, proper functional satisfying $f_i(E_i) \cap \mathrm{dom}\, \partial_{\eta_i}\varphi_i \neq \emptyset$. Let $S : E_1 \times E_2 \to E_1^*$ is Lipschitz continuous in both the

$$f_2(y) = J_\gamma^{\partial_{\eta_2}\varphi_2(\cdot,y)}[J_2(f_2(y)) - \gamma T(u,y)],$$

where $J_1 : E_1 \to E_1^*$, $J_2 : E_2 \to E_2^*$, $J_\rho^{\partial_{\eta_1}\varphi_1(\cdot,x)} = (J_1 + \rho\partial_{\eta_1}\varphi_1(\cdot,x))^{-1}$, $J_\gamma^{\partial_{\eta_2}\varphi_2(\cdot,y)} = (J_2 + \gamma\partial_{\eta_2}\varphi_2(\cdot,y))^{-1}$, $\rho > 0$ and $\gamma > 0$ are constants.

Proof. The fact directly follows from Definition 2.1.2.

The above fixed point formulation enables us to suggest the following proximal point algorithm.

Algorithm 2.3.1. For any given $(x_0, y_0) \in E_1 \times E_2$, we choose $u_0 \in H(x_0)$, $v_0 \in F(y_0)$ and compute $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by iterative schemes as follows:

$$f_1(x_{n+1}) = J_\rho^{\partial_{\eta_1}\varphi_1(\cdot,x_n)}[J_1(f_1(x_n)) - \rho S(x_n, v_n)], \tag{2.3.2}$$

$$f_2(y_{n+1}) = J_\gamma^{\partial_{\eta_2}\varphi_2(\cdot,y_n)}[J_2(f_2(y_n)) - \gamma T(u_n, y_n)] \tag{2.3.3}$$

and choose $u_{n+1} \in H(x_{n+1})$ and $v_{n+1} \in F(y_{n+1})$ such that

$$u_n \in H(x_n), \quad \|u_{n+1} - u_n\| \leq \left(1 + \frac{1}{n+1}\right) D(H(x_{n+1}), H(x_n)), \tag{2.3.4}$$

$$v_n \in F(y_n), \quad \|v_{n+1} - v_n\| \leq \left(1 + \frac{1}{n+1}\right) D(F(y_{n+1}), F(y_n)), \tag{2.3.5}$$

where $\rho > 0$ and $\gamma > 0$ are constants and $n = 0, 1, 2, \ldots\ldots.$

Now the existence of solutions for *system of mixed variational-like inclusions* (2.3.1) is proved and the convergence of iterative sequences generated by the Algorithm 2.3.1 is also studied.

Theorem 2.3.2. Let E_1 and E_2 be two reflexive Banach spaces with their duals E_1^* and E_2^*, respectively. For $i = 1, 2$, let $\eta_i : E_i \times E_i \to E_i$ be Lipschitz continuous with constants τ_i such that $\eta_i(x_1, x_2) = -\eta_i(x_2, x_1)$ for all $x_1, x_2 \in E_i$, J_i-strongly accretive with constants α_i and for any $x_1 \in E_i$, the function $h_i(x_2, x_1) = \langle x_1^* - J_i x_1, \eta_i(x_2, x_1)\rangle$ is 0-DQCV in x_2. Let $J_i : E_i \to E_i^*$ be Lipschitz continuous with constant λ_{J_i}, $f_i : E_i \to E_i$ is Lipschitz continuous with constant λ_{f_i} and strongly accretive with constant δ_{f_i} such that $f_i(E_i) = E_i$, $\varphi_i : E_i \times E_i \to R \cup \{+\infty\}$ be lower semicontinuous, η_i-subdifferentiable, proper functional satisfying $f_i(E_i) \cap \text{dom}\, \partial_{\eta_i}\varphi_i \neq \emptyset$. Let $S : E_1 \times E_2 \to E_1^*$ is Lipschitz continuous in both the arguments with constants λ_{S_1} and λ_{S_2}, respectively and $T : E_1 \times E_2 \to E_2^*$ is Lipschitz continuous in both the arguments with constants λ_{T_1} and λ_{T_2}, respectively. Let $H : E_1 \to CB(E_1^*)$ and $F : E_2 \to CB(E_2^*)$ be D-Lipschitz continuous with constants λ_{D_H} and λ_{D_F}, respectively.

If there exists constants $\rho > 0$ and $\gamma > 0$ such that

$$\|J_\rho^{\partial_{\eta_1}\varphi_1(\cdot,x_n)}(x^*) - J_\rho^{\partial_{\eta_1}\varphi_1(\cdot,x_{n-1})}(x^*)\| \le \mu^*\|x_n - x_{n-1}\| \tag{2.3.6}$$

$$\text{for any } \; x_n, x_{n-1} \in E_1, \; x^* \in E_1^*$$

and

$$\|J_\gamma^{\partial_{\eta_2}\varphi_2(\cdot,y_n)}(y^*) - J_\gamma^{\partial_{\eta_2}\varphi_2(\cdot,y_{n-1})}(y^*)\| \le \mu^{**}\|y_n - y_{n-1}\| \tag{2.3.7}$$

$$\text{for any } \; y_n, y_{n-1} \in E_2, \; y^* \in E_2^*$$

and the following condition is satisfied:

$$\begin{cases} 0 < \sqrt{\frac{4\tau_1^2(\lambda_{J_1}\lambda_{f_1})^2+8\tau_1^2\rho^2\lambda_{S_1}^2+2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}} + \sqrt{\frac{8\tau_2^2\gamma^2(\lambda_{T_1}\lambda_{D_H})^2}{(2\delta_{f_2}+3)\alpha_2^2}} < 1 \\ 0 < \sqrt{\frac{8\tau_1^2\rho^2(\lambda_{S_2}\lambda_{D_F})^2}{(2\delta_{f_1}+3)\alpha_1^2}} + \sqrt{\frac{4\tau_2^2(\lambda_{J_2}\lambda_{f_2})^2+8\tau_2^2\gamma^2\lambda_{T_2}^2+2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}} < 1. \end{cases} \tag{2.3.8}$$

Then the *system of mixed variational-like inclusions* (2.3.1) admits a solution (x, y, u, v) and the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ converge to x, y, u, and v, respectively, where $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ are the sequences generated by Algorithm 2.3.1.
Proof. We can write

$$\|x_{n+1} - x_n\|^2 = \|f_1(x_{n+1}) - f_1(x_n) - f_1(x_{n+1}) + f_1(x_n) - x_{n+1} + x_n\|^2.$$

By Proposition 1.2.1, we have

$$\|x_{n+1} - x_n\|^2 \le \|f_1(x_{n+1}) - f_1(x_n)\|^2 - 2\langle f_1(x_{n+1}) - f_1(x_n) + x_{n+1} - x_n, \\ j(x_{n+1} - x_n)\rangle. \tag{2.3.9}$$

By (2.3.2), we have

$$f_1(x_{n+1}) = J_\rho^{\partial_{\eta_1}\varphi_1(\cdot,x_n)}[J_1(f_1(x_n)) - \rho S(x_n, v_n)].$$

Hence, we have

$$\|f_1(x_{n+1}) - f_1(x_n)\|^2 = \|J_\rho^{\partial_{\eta_1}\varphi_1(\cdot,x_n)}[J_1(f_1(x_n)) - \rho S(x_n, v_n)] - J_\rho^{\partial_{\eta_1}\varphi_1(\cdot,x_{n-1})}[J_1(f_1(x_{n-1})) - \rho S(x_{n-1}, v_{n-1})]\|^2.$$

Since $\|x+y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$, therefore by the assumption (2.3.6) and Theorem 2.1.2, we have

$$\begin{aligned}
\frac{1}{2}\|f_1(x_{n+1}) - f_1(x_n)\|^2 \leq\ & \|J_\rho^{\partial_{\eta_1}\varphi_1(\cdot,x_n)}[J_1(f_1(x_n)) - \rho S(x_n, v_n)] \\
& - J_\rho^{\partial_{\eta_1}\varphi_1(\cdot,x_n)}[J_1(f_1(x_{n-1})) - \rho S(x_{n-1}, v_{n-1})]\|^2 \\
& + \|J_\rho^{\partial_{\eta_1}\varphi_1(\cdot,x_n)}[J_1(f_1(x_{n-1})) - \rho S(x_{n-1}, v_{n-1})] \\
& - J_\rho^{\partial_{\eta_1}\varphi_1(\cdot,x_{n-1})}[J_1(f_1(x_{n-1})) - \rho S(x_{n-1}, v_{n-1})]\|^2 \\
\leq\ & \frac{\tau_1^2}{\alpha_1^2}\|[J_1(f_1(x_n)) - \rho S(x_n, v_n)] - [J_1(f_1(x_{n-1})) \\
& - \rho S(x_{n-1}, v_{n-1})]\|^2 + \mu^{*2}\|x_n - x_{n-1}\|^2 \\
\leq\ & \frac{2\tau_1^2}{\alpha_1^2}\|J_1(f_1(x_n)) - J_1(f_1(x_{n-1}))\|^2 + \frac{2\tau_1^2\rho^2}{\alpha_1^2}\|S(x_n, v_n) \\
& - S(x_{n-1}, v_{n-1})\|^2 + \mu^{*2}\|x_n - x_{n-1}\|^2. \qquad (2.3.10)
\end{aligned}$$

By the Lipschitz continuity of J_1 and f_1, we have

$$\begin{aligned}
\|J_1(f_1(x_n)) - J_1(f_1(x_{n-1}))\| &\leq \lambda_{J_1}\|f_1(x_n) - f_1(x_{n-1})\| \\
&\leq \lambda_{J_1}\lambda_{f_1}\|x_n - x_{n-1}\|. \qquad (2.3.11)
\end{aligned}$$

By the Lipschitz continuity of $S(\cdot,\cdot)$ in both the arguments, (2.3.5) and D-Lipschitz continuity of F, we have

$$\begin{aligned}
\|S(x_n, v_n) - S(x_n, v_{n-1})\| &\leq \lambda_{S_2}\|v_n - v_{n-1}\| \\
&\leq \lambda_{S_2}\left(1 + \frac{1}{n}\right) D(F(y_n), F(y_{n-1})) \\
&\leq \lambda_{S_2}\left(1 + \frac{1}{n}\right)\lambda_{D_F}\|y_n - y_{n-1}\|. \qquad (2.3.12)
\end{aligned}$$

$$\|S(x_n, v_{n-1}) - S(x_{n-1}, v_{n-1})\| \leq \lambda_{S_1}\|x_n - x_{n-1}\|. \qquad (2.3.13)$$

Using (2.3.12) and (2.3.13), it follows that

$$\begin{aligned}\|S(x_n, v_n) - S(x_{n-1}, v_{n-1})\|^2 &\le 2\|S(x_n, v_n) - S(x_n, v_{n-1})\|^2 \\ &\quad +2\|S(x_n, v_{n-1}) - S(x_{n-1}, v_{n-1})\|^2 \\ &\le 2(\lambda_{S_2}\lambda_{D_F})^2 \left(1+\frac{1}{n}\right)^2 \|y_n - y_{n-1}\|^2 \\ &\quad +2\lambda_{S_1}^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{2.3.14}$$

By (2.3.11) and (2.3.14), (2.3.10) becomes

$$\begin{aligned}\|f_1(x_{n+1}) - f_1(x_n)\|^2 &\le \left[\frac{4\tau_1^2}{\alpha_1^2}(\lambda_{J_1}\lambda_{f_1})^2 + \frac{8\tau_1^2}{\alpha_1^2}\rho^2\lambda_{S_1}^2 + 2\mu^{*2}\right]\|x_n - x_{n-1}\|^2 \\ &\quad +\frac{8\tau_1^2}{\alpha_1^2}\rho^2(\lambda_{S_2}\lambda_{D_F})^2\left(1+\frac{1}{n}\right)^2\|y_n - y_{n-1}\|^2. \end{aligned} \tag{2.3.15}$$

Since f_1 is strongly accretive with constant δ_{f_1}, by (2.3.9), we have

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &\le \|f_1(x_{n+1})-f_1(x_n)\|^2 - 2\langle f_1(x_{n+1})-f_1(x_n)+x_{n+1}-x_n, j(x_{n+1}-x_n)\rangle \\ &\le \left[\frac{4\tau_1^2}{\alpha_1^2}(\lambda_{J_1}\lambda_{f_1})^2 + \frac{8\tau_1^2}{\alpha_1^2}\rho^2\lambda_{S_1}^2 + 2\mu^{*2}\right]\|x_n - x_{n-1}\|^2 \\ &\quad +\frac{8\tau_1^2}{\alpha_1^2}\rho^2(\lambda_{S_2}\lambda_{D_F})^2\left(1+\frac{1}{n}\right)^2\|y_n - y_{n-1}\|^2 \\ &\quad -(2\delta_{f_1}+2)\|x_{n+1}-x_n\|^2. \end{aligned} \tag{2.3.16}$$

It follows that

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &\le \left[\frac{4\tau_1^2(\lambda_{J_1}\lambda_{f_1})^2}{(2\delta_{f_1}+3)\alpha_1^2} + \frac{8\tau_1^2\rho^2\lambda_{S_1}^2}{(2\delta_{f_1}+3)\alpha_1^2} + \frac{2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}\right]\|x_n - x_{n-1}\|^2 \\ &\quad +\frac{8\tau_1^2\rho^2(\lambda_{S_2}\lambda_{D_F})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_1}+3)\alpha_1^2}\|y_n - y_{n-1}\|^2 \\ &= \theta_1\|x_n - x_{n-1}\|^2 + \theta_2\|y_n - y_{n-1}\|^2 \\ &\le \theta_1\|x_n - x_{n-1}\|^2 + \theta_2\|y_n - y_{n-1}\|^2 \\ &\quad +2\sqrt{\theta_1}\sqrt{\theta_2}\|x_n - x_{n-1}\|\|y_n - y_{n-1}\| \\ &= (\sqrt{\theta_1}\|x_n - x_{n-1}\| + \sqrt{\theta_2}\|y_n - y_{n-1}\|)^2. \end{aligned}$$

Thus, we have

$$\|x_{n+1} - x_n\| \le \sqrt{\theta_1}\|x_n - x_{n-1}\| + \sqrt{\theta_2}\|y_n - y_{n-1}\|, \tag{2.3.17}$$

where

$$\theta_1 = \frac{4\tau_1^2(\lambda_{J_1}\lambda_{f_1})^2}{(2\delta_{f_1}+3)\alpha_1^2} + \frac{8\tau_1^2\rho^2\lambda_{S_1}^2}{(2\delta_{f_1}+3)\alpha_1^2} + \frac{2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}$$

and

$$\theta_2 = \frac{8\tau_1^2\rho^2(\lambda_{S_2}\lambda_{D_F})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_1}+3)\alpha_1^2}.$$

We can also write

$$\|y_{n+1} - y_n\|^2 = \|f_2(y_{n+1}) - f_2(y_n) - f_2(y_{n+1}) + f_2(y_n) - y_{n+1} + y_n\|^2.$$

By Proposition 1.2.1, we have

$$\|y_{n+1} - y_n\|^2 \le \|f_2(y_{n+1}) - f_2(y_n)\|^2 - 2\langle f_2(y_{n+1}) - f_2(y_n) + y_{n+1} - y_n, \\ j(y_{n+1} - y_n)\rangle. \tag{2.3.18}$$

By (2.3.3), we have

$$f_2(y_{n+1}) = J_\gamma^{\partial_{\eta_2}\varphi_2(\cdot,y_n)}[J_2(f_2(y_n)) - \gamma T(u_n, y_n)].$$

Hence, we have

$$\|f_2(y_{n+1}) - f_2(y_n)\|^2 = \|J_\gamma^{\partial_{\eta_2}\varphi_2(\cdot,y_n)}[J_2(f_2(y_n)) - \gamma T(u_n, y_n)] \\ - J_\gamma^{\partial_{\eta_2}\varphi_2(\cdot,y_{n-1})}[J_2(f_2(y_{n-1})) - \gamma T(u_{n-1}, y_{n-1})]\|^2.$$

Using the same argument as for (2.3.10), we have

$$\frac{1}{2}\|f_2(y_{n+1}) - f_2(y_n)\|^2 \le \frac{2\tau_2^2}{\alpha_2^2}\|J_2(f_2(y_n)) - J_2(f_2(y_{n-1}))\|^2 + \frac{2\tau_2^2\gamma^2}{\alpha_2^2}\|T(u_n, y_n) \\ - T(u_{n-1}, y_{n-1})\|^2 + \mu^{**2}\|y_n - y_{n-1}\|^2. \tag{2.3.19}$$

By the Lipschitz continuity of J_2 and f_2, we have

$$\|J_2(f_2(y_n)) - J_2(f_2(y_{n-1}))\| \le \lambda_{J_2}\|f_2(y_n) - f_2(y_{n-1})\| \\ \le \lambda_{J_2}\lambda_{f_2}\|y_n - y_{n-1}\|. \tag{2.3.20}$$

By the Lipschitz continuity of $T(\cdot,\cdot)$ in both the arguments, (2.3.4) and D-Lipschitz *continuity of* H, we have

$$\|T(u_n, y_n) - T(u_n, y_{n-1})\| \leq \lambda_{T_2}\|y_n - y_{n-1}\|. \tag{2.3.21}$$

$$\begin{aligned}\|T(u_n, y_{n-1}) - T(u_{n-1}, y_{n-1})\| &\leq \lambda_{T_1}\|u_n - u_{n-1}\| \\ &\leq \lambda_{T_1}\left(1+\frac{1}{n}\right) D(H(x_n), H(x_{n-1})) \\ &\leq \lambda_{T_1}\lambda_{D_F}\left(1+\frac{1}{n}\right)\|x_n - x_{n-1}\|. \end{aligned} \tag{2.3.22}$$

Using (2.3.21) and (2.3.22), it follows that

$$\begin{aligned}\|T(u_n, y_n) - T(u_{n-1}, y_{n-1})\|^2 &\leq 2\|T(u_n, y_n) - T(u_n, y_{n-1})\|^2 \\ &\quad +2\|T(u_n, y_{n-1}) - T(u_{n-1}, y_{n-1})\|^2 \\ &\leq 2\lambda_{T_2}^2\|y_n - y_{n-1}\|^2 + 2(\lambda_{T_1}\lambda_{D_H})^2 \\ &\quad +2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{2.3.23}$$

By (2.3.20) and (2.3.23), (2.3.19) becomes

$$\begin{aligned}\|f_2(y_{n+1}) - f_2(y_n)\|^2 &\leq \left[\frac{4\tau_2^2}{\alpha_2^2}(\lambda_{J_2}\lambda_{f_2})^2 + \frac{8\tau_2^2}{\alpha_2^2}\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\right]\|y_n - y_{n-1}\|^2 \\ &\quad +\frac{8\tau_2^2}{\alpha_2^2}\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{2.3.24}$$

Since f_2 is strongly accretive with constant δ_{f_2}, by (2.3.18), we have

$$\begin{aligned}\|y_{n+1} - y_n\|^2 &\leq \|f_2(y_{n+1}) - f_2(y_n)\|^2 - 2\langle f_2(y_{n+1}) - f_2(y_n) + y_{n+1} - y_n, j(y_{n+1} - y_n)\rangle \\ &\leq \left[\frac{4\tau_2^2}{\alpha_2^2}(\lambda_{J_2}\lambda_{f_2})^2 + \frac{8\tau_2^2}{\alpha_2^2}\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\right]\|y_n - y_{n-1}\|^2 \\ &\quad +\frac{8\tau_2^2}{\alpha_2^2}\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2\|x_n - x_{n-1}\|^2 \\ &\quad -(2\delta_{f_2} + 2)\|y_{n+1} - y_n\|^2. \end{aligned} \tag{2.3.25}$$

It follows that

$$\begin{aligned}
\|y_{n+1}-y_n\|^2 &\le \left[\frac{4\tau_2^2(\lambda_{J_2}\lambda_{f_2})^2}{(2\delta_{f_2}+3)\alpha_2^2}+\frac{8\tau_2^2\gamma^2\lambda_{T_2}^2}{(2\delta_{f_2}+3)\alpha_2^2}+\frac{2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}\right]\|y_n-y_{n-1}\|^2\\
&\quad+\frac{8\tau_2^2\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_2}+3)\alpha_2^2}\|x_n-x_{n-1}\|^2\\
&=\theta_3\|y_n-y_{n-1}\|^2+\theta_4\|x_n-x_{n-1}\|^2\\
&\le \theta_3\|y_n-y_{n-1}\|^2+\theta_4\|x_n-x_{n-1}\|^2\\
&\quad+2\sqrt{\theta_3}\sqrt{\theta_4}\|y_n-y_{n-1}\|\|x_n-x_{n-1}\|\\
&=(\sqrt{\theta_3}\|y_n-y_{n-1}\|+\sqrt{\theta_4}\|x_n-x_{n-1}\|)^2.
\end{aligned}$$

Thus, we have

$$\|y_{n+1}-y_n\|\le\sqrt{\theta_3}\|y_n-y_{n-1}\|+\sqrt{\theta_4}\|x_n-x_{n-1}\|, \tag{2.3.26}$$

where

$$\theta_3=\frac{4\tau_2^2(\lambda_{J_2}\lambda_{f_2})^2}{(2\delta_{f_2}+3)\alpha_2^2}+\frac{8\tau_2^2\gamma^2\lambda_{T_2}^2}{(2\delta_{f_2}+3)\alpha_2^2}+\frac{2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}$$

and

$$\theta_4=\frac{8\tau_2^2\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_2}+3)\alpha_2^2}.$$

By (2.3.17) and (2.3.26), we have

$$\begin{aligned}
&\|x_{n+1}-x_n\|+\|y_{n+1}-y_n\|\\
&\le(\sqrt{\theta_1}+\sqrt{\theta_4})\|x_n-x_{n-1}\|+(\sqrt{\theta_2}+\sqrt{\theta_3})\|y_n-y_{n-1}\|\\
&=\left[\sqrt{\frac{4\tau_1^2(\lambda_{J_1}\lambda_{f_1})^2+8\tau_1^2\rho^2\lambda_{S_1}^2+2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}}\right.\\
&\quad\left.+\sqrt{\frac{8\tau_2^2\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_2}+3)\alpha_2^2}}\right]\|x_n-x_{n-1}\|\\
&\quad+\left[\sqrt{\frac{8\tau_1^2\rho^2(\lambda_{S_2}\lambda_{D_F})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_1}+3)\alpha_1^2}}\right.\\
&\quad\left.+\sqrt{\frac{4\tau_2^2(\lambda_{J_2}\lambda_{f_2})^2+8\tau_2^2\gamma^2\lambda_{T_2}^2+2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}}\right]\|y_n-y_{n-1}\|
\end{aligned}$$

$$= \theta_n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|), \tag{2.3.27}$$

where

$$\theta_n = \max\Bigg\{\sqrt{\frac{4\tau_1^2(\lambda_{J_1}\lambda_{f_1})^2 + 8\tau_1^2\rho^2\lambda_{S_1}^2 + 2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}}$$
$$+\sqrt{\frac{8\tau_2^2\gamma^2(\lambda_{T_1}\lambda_{D_H})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_2}+3)\alpha_2^2}}, \sqrt{\frac{8\tau_1^2\rho^2(\lambda_{S_2}\lambda_{D_F})^2\left(1+\frac{1}{n}\right)^2}{(2\delta_{f_1}+3)\alpha_1^2}}$$
$$+\sqrt{\frac{4\tau_2^2(\lambda_{J_2}\lambda_{f_2})^2 + 8\tau_2^2\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}}\Bigg\}.$$

Let

$$\theta = \max\Bigg\{\sqrt{\frac{4\tau_1^2(\lambda_{J_1}\lambda_{f_1})^2 + 8\tau_1^2\rho^2\lambda_{S_1}^2 + 2\mu^{*2}\alpha_1^2}{(2\delta_{f_1}+3)\alpha_1^2}}$$
$$+\sqrt{\frac{8\tau_2^2\gamma^2(\lambda_{T_1}\lambda_{D_H})^2}{(2\delta_{f_2}+3)\alpha_2^2}}, \sqrt{\frac{8\tau_1^2\rho^2(\lambda_{S_2}\lambda_{D_F})^2}{(2\delta_{f_1}+3)\alpha_1^2}}$$
$$+\sqrt{\frac{4\tau_2^2(\lambda_{J_2}\lambda_{f_2})^2 + 8\tau_2^2\gamma^2\lambda_{T_2}^2 + 2\mu^{**2}\alpha_2^2}{(2\delta_{f_2}+3)\alpha_2^2}}\Bigg\}.$$

Then $\theta_n \to \theta$ as $n \to \infty$. By (2.3.8), we know that $0 < \theta < 1$ and so (2.3.27) implies that $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences. Thus, there exists $x \in E_1$ and $y \in E_2$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Now we prove that $u_n \to u \in H(x)$ and $v_n \to v \in F(y)$. In fact, it follows from the D-Lipschitz continuity of H, F, (2.3.4) and (2.3.5) that

$$\|u_n - u_{n-1}\| \le \left(1+\frac{1}{n}\right)\lambda_{D_H}\|x_n - x_{n-1}\|, \tag{2.3.28}$$

$$\|v_n - v_{n-1}\| \le \left(1+\frac{1}{n}\right)\lambda_{D_F}\|y_n - y_{n-1}\|. \tag{2.3.29}$$

From (2.3.28) and (2.3.29), we know that $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences. We can assume that $u_n \to u$ and $v_n \to v$ as $n \to \infty$.

Further,

$$
\begin{aligned}
d(u, H(x)) &\leq \|u - u_n\| + d(u_n, H(x)) \\
&\leq \|u - u_n\| + D\left(H(x_n), H(x)\right) \\
&\leq \|u - u_n\| + \lambda_{D_H}\|x_n - x\| \to 0, \text{ as } n \to +\infty.
\end{aligned}
$$

Hence $d(u, H(x)) = 0$ and therefore $u \in H(x)$. Similarly, we can show that $v \in F(y)$. By continuity of f_1, f_2, J_1, J_2, S, T, $J_\rho^{\partial_{\eta_1}\varphi_1}$, $J_\gamma^{\partial_{\eta_2}\varphi_2}$, η_1, η_2, φ_1, φ_2, H, F and Algorithm 2.3.1, we know that x, y, u and v satisfy the following relations

$$f_1(x) = J_\rho^{\partial_{\eta_1}\varphi_1(\cdot,x)}[J_1(f_1(x)) - \rho S(x, v)],$$

$$f_2(y) = J_\gamma^{\partial_{\eta_2}\varphi_2(\cdot,y)}[J_2(f_2(y)) - \gamma T(u, y)].$$

By Theorem 2.3.1, $(x, y) \in E_1 \times E_2$, $u \in H(x)$ and $v \in F(y)$ is a solution of Problem (2.3.1). This completes the proof.

Chapter 3

Systems Of Variational Inclusions In Uniformly Smooth Banach Spaces

3.1. Introduction

In the last decade, variational inclusions, generalized forms of variational inequalities, have been extensively studied and generalized in various directions to study a wide class of problems arising in mechanics, optimization, nonlinear programming, economics, finance and applied sciences, etc; see for example [1,5,8,22,24, 25,41,115]. Several authors used resolvent operator technique to propose and analyze the iterative algorithms for computing the approximate solutions of different kinds of variational inclusions. *Fang and Huang* [49] studied variational inclusions by introducing a class of generalized monotone operators, H-monotone operators and defined an associated resolvent operator. *Fang and Huang* [50] further extended the notion of H-monotone operators to the Banach spaces, called H-accretive operators. They also gave some properties of the resolvent operator associated with the H-accretive operator.

Yan et.al. [125] introduce and study a new system of set-valued variational inclusions with H-monotone operators in Hilbert spaces. By using the resolvent operator associated with H-monotone operator due to *Fang and Huang*, the authors constructed a new iterative algorithm for solving the system of set-valued variational inclusions and proved the existence of solutions for the system of set-valued variational inclusions and the convergence of iterative sequences generated by the algorithm. As generalization of system of variational inequalities, *Agarwal et.al.* [6]

introduced a system of generalized nonlinear mixed quasi-variational inclusions and investigated the sensitivity analysis of solutions for their system.

The concept of resolvent equations is equally important and is initially used by *Noor* [90]. This technique has been used to develop some numerical methods for solving the mixed variational inequalities and variational inclusions; see for examples [9,93,94] and references therein. The resolvent equations include the Wiener-Hopf (normal maps) equations as a special case. The Wiener-Hopf equation were introduced by *Shi* [108] and *Robinson* [107] in the connection with variational inequalities. The Wiener-Hopf equations technique was use to develop various numerical methods for solving the variational inequalities and complementarity problems.

In Section 3.2, we introduce and study a system of set-valued variational inclusions. An iterative algorithm for computing the approximate solutions of system of set-valued variational inclusions is defined and convergence criteria is also discussed.

In Section 3.3, we introduce and study a system of generalized variational inclusions with H-accretive operators in uniformly smooth Banach spaces. We prove the convergence of iterative algorithm for this system of generalized variational inclusions.

In Section 3.4, we introduce and study a system of generalized H-resolvent equations in uniformly smooth Banach spaces and also mention the corresponding system of generalized variational inclusions. An equivalence relation is established between system of generalized H-resolvent equations and system of generalized variational inclusions. Further, we prove the existence of solutions for the system of generalized H-resolvent equations and the convergence of iterative sequences generated by the algorithm.

We introduce the following definition which is used to prove Theorem 3.3.1 and Theorem 3.4.1 and is supported by an example and numerical example.

Definition 3.1.1. The H-resolvent operator $J^M_{H,\rho} : E \to E$ is said to be *retraction* if

$$[J^M_{H,\rho}(x)]^2 = J^M_{H,\rho}(x), \quad \text{for all} \quad x \in E.$$

Example 3.1.1. For $\rho = 1$, let

$$H = \begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{pmatrix} \quad \text{an } n \times n \text{ matrix}$$

and

$$M = \begin{pmatrix} b_{11} & b_{12} & \cdots\cdots & b_{1n} \\ b_{21} & b_{22} & \cdots\cdots & b_{2n} \\ . & & & \\ . & & & \\ . & & & \\ b_{n1} & b_{n2} & \cdots\cdots & b_{nn} \end{pmatrix} \quad \text{an } n \times n \text{ matrix}$$

We define the following operations for matrices H and M:

(i) $a_{ij} + b_{ij} = 1$ if $i = j$;

(ii) $a_{ij} + b_{ij} = 0$ if $i \neq j$,

then we have

$$[J^M_{H,\rho}(x)]^2 = J^M_{H,\rho}(x), \quad \text{for all } x \in E.$$

Numerical Example 3.1.2. Here we present the following MatLab programming for the justification of Definition 3.1.1. The program is valid for any value of $\rho > 0$.

```
H=input('Enter the matrix H: ');
k=input('Enter the value of k: ');
n=size(H,1); lambda=eye(n) - H;
more on;
for hrow=1 : k
disp(['Hrow= ' int2str(hrow)]);
M=(1/hrow)*lambda;
disp('The matrix M= ');
disp(M);
end
```

For illustration, we take H to be a real non-singular 3×3 matrix then our program will generate M to be a 3×3 matrix for the values of ρ in between 1 to $k = 5$ such that

$$[J^M_{H,\rho}(x)]^2 = J^M_{H,\rho}(x).$$

Enter the matrix H: [1 3 7;4 6 9;-1 6 8]

Enter the value of k: 5

Hrow= 1

The matrix M=

$$\begin{matrix} 0 & -3 & -7 \\ -4 & -5 & -9 \\ 1 & -6 & -7 \end{matrix}$$

Hrow= 2

The matrix M=

$$\begin{matrix} 0 & -1.5000 & -3.5000 \\ -2.0000 & -2.5000 & -4.5000 \\ 0.5000 & -3.0000 & -3.5000 \end{matrix}$$

Hrow= 3

The matrix M=

$$\begin{matrix} 0 & -1.0000 & -2.3333 \\ -1.3333 & -1.6667 & -3.0000 \\ 0.3333 & -2.0000 & -2.3333 \end{matrix}$$

Hrow= 4

The matrix M=

$$\begin{matrix} 0 & -0.7500 & -1.7500 \\ -1.0000 & -1.2500 & -2.2500 \\ 0.2500 & -1.5000 & -1.7500 \end{matrix}$$

Hrow= 5

The matrix M=

$$\begin{matrix} 0 & -0.6000 & -1.4000 \\ -0.8000 & -1.0000 & -1.8000 \\ 0.2000 & -1.2000 & -1.4000 \end{matrix}$$

3.2. System Of Set-valued Variational Inclusions

In this section, we introduce and study a system of set-valued variational inclusions in the setting of uniformly smooth Banach spaces. An iterative algorithm for computing the approximate solutions of this system is suggested. By using the definition of nonexpansive retraction, we prove convergence result for the approximate solutions obtained by the Algorithm 3.2.1.

Let E_1 and E_2 be any two real Banach spaces. Let $S : E_1 \times E_2 \to E_1$, $T : E_1 \times E_2 \to E_2$, $p : E_1 \to E_1$ and $q : E_2 \to E_2$ be single-valued mappings, $G : E_1 \to CB(E_1)$, $F : E_2 \to CB(E_2)$, $M : E_1 \times E_1 \to 2^{E_1}$ and $N : E_2 \times E_2 \to 2^{E_2}$ be set-valued mappings, $f : E_1 \to E_1$ and $g : E_2 \to E_2$ be nonlinear mappings with $f(E_1) \cap D(M) \neq \emptyset$ and $g(E_2) \cap D(N) \neq \emptyset$. We consider the following *system of set-valued variational inclusions:*

Find $(x, y) \in E_1 \times E_2$, $u \in G(x)$ and $v \in F(y)$ such that

$$\begin{aligned} &0 \in S(x - p(x), v) + M(f(x), x), \\ &0 \in T(u, y - q(y)) + N(g(y), y). \end{aligned} \tag{3.2.1}$$

Some special cases:

(i) If $x = 2p(x)$, $y = 2q(y)$, $M(f(x), x) = M(f(x))$ and $N(g(y), y) = N(g(y))$, then Problem (3.2.1) reduces to the problem of finding $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$ such that

$$\begin{aligned} &0 \in S(p(x), v) + M(f(x)), \\ &0 \in T(u, q(y)) + N(g(y)). \end{aligned} \tag{3.2.2}$$

Problem (3.2.2) is considered by *Lan et.al.* [78] in Hilbert spaces with A-monotone operators.

(ii) If $p(x) = 0 = q(y)$, $M(f(x), x) = M(x)$ and $N(g(y), y) = N(y)$, then Problem (3.2.1) reduces to the problem of finding $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$ such that

$$0 \in S(x,v) + M(x),$$
$$0 \in T(u,y) + N(y). \tag{3.2.3}$$

Problem (3.2.3) is considered by *Huang and Fang* [67] in Hilbert spaces.

We mention the following lemma which ensures that the *system of set-valued variational inclusions* (3.2.1) is equivalent to a fixed point problem.

Lemma 3.2.1. (x,y,u,v), where $(x,y) \in E_1 \times E_2$, $u \in G(x)$ and $v \in F(y)$ is a solution of the *system of set-valued variational inclusions* (3.2.1) if and only if (x,y,u,v) satisfies

$$f(x) = J_\rho^{M(\cdot,x)}(f(x) - \rho S(x - p(x), v)),$$
$$g(y) = J_\gamma^{N(\cdot,y)}(g(y) - \gamma T(u, y - q(y))), \tag{3.2.4}$$

where $\rho > 0$ and $\gamma > 0$ are constants.

Proof. The fact is directly follow from the Definition 1.2.16 of resolvent operator.

Based on Lemma 3.2.1 and Nadler's Theorem 1.2.3 [87], we suggest the following Algorithm for solving the *system of set-valued variational inclusions* (3.2.1).

Algorithm 3.2.1. For any given $(x_0, y_0) \in E_1 \times E_2$, we choose $u_0 \in G(x_0)$, $v_0 \in F(x_0)$ and compute the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by iterative schemes as follows:

$$x_{n+1} = x_n - f(x_n) + J_\rho^{M(\cdot,x_n)}(f(x_n) - \rho S(x_n - p(x_n), v_n)), \tag{3.2.5}$$

$$y_{n+1} = y_n - g(y_n) + J_\gamma^{N(\cdot,y_n)}(g(y_n) - \gamma T(u_n, y_n - q(y_n))), \tag{3.2.6}$$

and choose $u_{n+1} \in G(x_{n+1})$ and $v_{n+1} \in F(y_{n+1})$ such that

$$u_n \in G(x_n), \qquad \|u_n - u_{n+1}\| \leq (1 + (n+1)^{-1}) D(G(x_n), G(x_{n+1})), \tag{3.2.7}$$

$$v_n \in F(y_n), \qquad \|v_n - v_{n+1}\| \leq (1 + (n+1)^{-1}) D(F(y_n), F(y_{n+1})). \tag{3.2.8}$$

$n = 1, 2, \ldots\ldots\ldots$ and $\rho, \gamma > 0$ are constants.

Now we study the convergence of iterative sequences generated by Algorithm 3.2.1 and prove the existence of solutions of the *system of set-valued variational inclusions* (3.2.1).

Theorem 3.2.1. Let E_1 and E_2 be any two real uniformly smooth Banach spaces with module of smoothness $\tau_{E_1}(t) \leq C_1 t^2$ and $\tau_{E_2}(t) \leq C_2 t^2$ for some $C_1, C_2 > 0$. Let $M : E_1 \times E_1 \to 2^{E_1}$ and $N : E_2 \times E_2 \to 2^{E_2}$ be m-accretive mappings, $S : E_1 \times E_2 \to E_1$ and $T : E_1 \times E_2 \to E_2$ are single-valued mappings such that S and T are Lipschitz continuous in first argument with constants λ_{S_1} and λ_{T_1}, respectively; and Lipschitz continuous in second argument with constants λ_{S_2} and λ_{T_2}, respectively. Let $f : E_1 \to E_1$, $g : E_2 \to E_2$, $p : E_1 \to E_1$ and $q : E_2 \to E_2$ be strongly accretive mappings with constants δ_f, δ_g, δ_p and δ_q, respectively; and Lipschitz continuous with constants λ_f, λ_g, λ_p and λ_q, respectively such that $f(E_1) \cap D(M) \neq \phi$ and $g(E_2) \cap D(N) \neq \phi$. Let $G : E_1 \to CB(E_1)$ and $F : E_2 \to CB(E_2)$ be D-Lipschitz continuous mappings with constants λ_{D_G} and λ_{D_F}, respectively. Suppose that there exists constants $\psi, \varphi > 0$ and $\rho, \gamma > 0$ such that for each $x \in E_1$, $y \in E_2$, $x^* \in E^*$

$$\begin{aligned} \|J_\rho^{M(\cdot,x_n)}(x^*) - J_\rho^{M(\cdot,x_{n-1})}(x^*)\| &\leq \psi\|x_n - x_{n-1}\|, \\ \|J_\gamma^{N(\cdot,y_n)}(x^*) - J_\gamma^{N(\cdot,y_{n-1})}(x^*)\| &\leq \varphi\|y_n - y_{n-1}\| \end{aligned}$$

and the following conditions are satisfied:

$$\sqrt{1 - 2\delta_f + 64C_1\lambda_f^2} + \lambda_f + \psi + \rho\lambda_{S_1}\sqrt{1 - 2\delta_p + 64C_1\lambda_p^2} + \gamma\lambda_{T_1}\lambda_{D_G} < 1,$$

$$\sqrt{1 - 2\delta_g + 64C_1\lambda_g^2} + \lambda_g + \varphi + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_1\lambda_q^2} + \rho\lambda_{S_2}\lambda_{D_F} < 1. \quad (3.2.9)$$

Then the *system of set-valued variational inclusions* (3.2.1) admits a solutions (x, y, u, v) and the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ generated by Algorithm 3.2.1 converge strongly to x, y, u and v respectively.

Proof. From Algorithm 3.2.1, nonexpansiveness of the operator J_ρ^M and by assumption, we have

$$\|x_{n+1} - x_n\| = \|x_n - f(x_n) + J_\rho^{M(\cdot,x_n)}(f(x_n) - \rho S(x_n - p(x_n), v_n))$$

$$
\begin{aligned}
&-[x_{n-1} - f(x_{n-1}) + J_\rho^{M(\cdot,x_{n-1})}(f(x_{n-1}) \\
&\qquad -\rho S(x_{n-1} - p(x_{n-1}), v_{n-1}))]\| \\
&\leq \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| \\
&\quad + \|J_\rho^{M(\cdot,x_n)}(f(x_n) - \rho S(x_n - p(x_n), v_n)) \\
&\quad - J_\rho^{M(\cdot,x_{n-1})}(f(x_{n-1}) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1}))\| \\
&\leq \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| \\
&\quad + \|J_\rho^{M(\cdot,x_n)}(f(x_n) - \rho S(x_n - p(x_n), v_n)) \\
&\quad - J_\rho^{M(\cdot,x_n)}(f(x_{n-1}) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1}))\| \\
&\quad + \|J_\rho^{M(\cdot,x_n)}(f(x_{n-1}) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1})) \\
&\quad - J_\rho^{M(\cdot,x_{n-1})}(f(x_{n-1}) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1}))\| \\
&\leq \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| + \|(f(x_n) \\
&\quad - \rho S(x_n - p(x_n), v_n) - (f(x_{n-1}) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1}))\| \\
&\qquad + \psi\|x_n - x_{n-1}\| \\
&\leq \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| + \|f(x_n) - f(x_{n-1})\| \\
&\quad + \rho\|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n)\| \\
&\quad + \rho\|S(x_{n-1} - p(x_{n-1}), v_n) - S(x_{n-1} - p(x_{n-1}), v_{n-1})\| \\
&\qquad + \psi\|x_n - x_{n-1}\|.
\end{aligned} \tag{3.2.10}
$$

By Proposition 1.2.1, we have

$$\|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|^2 \leq (1 - 2\delta_f + 64C_1\lambda_f^2)\|x_n - x_{n-1}\|^2. \tag{3.2.11}$$

As f is Lipschitz continuous with constant λ_f, we have

$$\|f(x_n) - f(x_{n-1})\| \leq \lambda_f\|x_n - x_{n-1}\|. \tag{3.2.12}$$

By using the Lipschitz continuity of S in second argument and F is D-Lipschitz continuous, we have

$$\begin{aligned}
&\|S(x_{n-1}-p(x_{n-1}),v_n)-S(x_{n-1}-p(x_{n-1}),v_{n-1})\| \\
&\qquad \le \lambda_{S_2}\|v_n-v_{n-1}\| \\
&\qquad \le \lambda_{S_2}(1+n^{-1})D(F(y_n),F(y_{n-1})) \\
&\qquad \le \lambda_{S_2}\lambda_{D_F}(1+n^{-1})\|y_n-y_{n-1}\|. \qquad (3.2.13)
\end{aligned}$$

By using the Lipschitz continuity of S in first argument, we have

$$\begin{aligned}
&\|S(x_n-p(x_n),v_n)-S(x_{n-1}-p(x_{n-1}),v_n)\| \\
&\qquad \le \lambda_{S_1}\|x_n-x_{n-1}-(p(x_n)-p(x_{n-1}))\|. \qquad (3.2.14)
\end{aligned}$$

Using the same arguments as for (3.2.11), we have

$$\|x_n-x_{n-1}-(p(x_n)-p(x_{n-1}))\|^2 \le (1-2\delta_p+64C_1\lambda_p^2)\|x_n-x_{n-1}\|^2. \qquad (3.2.15)$$

By (3.2.14) and (3.2.15), we have

$$\begin{aligned}
&\|S(x_n-p(x_n),v_n)-S(x_{n-1}-p(x_{n-1}),v_n)\| \\
&\qquad \le \lambda_{S_1}\sqrt{1-2\delta_p+64C_1\lambda_p^2}\|x_n-x_{n-1}\|. \qquad (3.2.16)
\end{aligned}$$

By using (3.2.11)-(3.2.13) and (3.2.16), (3.2.10) becomes

$$\begin{aligned}
\|x_{n+1}-x_n\| &\le \sqrt{1-2\delta_f+64C_1\lambda_f^2}\|x_n-x_{n-1}\|+\lambda_f\|x_n-x_{n-1}\| \\
&\quad +\rho\lambda_{S_1}\sqrt{1-2\delta_p+64C_1\lambda_p^2}\|x_n-x_{n-1}\|+\rho\lambda_{S_2}\lambda_{D_F} \\
&\quad \times(1+n^{-1})\|y_n-y_{n-1}\|+\psi\|x_n-x_{n-1}\| \\
&\le \left[\sqrt{1-2\delta_f+64C_1\lambda_f^2}+\lambda_f+\psi+\rho\lambda_{S_1}\sqrt{1-2\delta_p+64C_1\lambda_p^2}\,\right] \\
&\quad \times\|x_n-x_{n-1}\|+\rho\lambda_{S_2}\lambda_{D_F}(1+n^{-1})\|y_n-y_{n-1}\|. \qquad (3.2.17)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|y_n-g(y_n)+J_\gamma^{N(\cdot,y_n)}(g(y_n)-\gamma T(u_n,y_n-q(x_n))) \\
&\quad -[y_{n-1}-g(x_{n-1})+J_\gamma^{N(\cdot,y_{n-1})}(g(y_{n-1})-\gamma T(u_{n-1},y_{n-1}-q(y_{n-1})))]\| \\
&\le \|y_n-y_{n-1}-(g(y_n)-g(y_{n-1}))\| \\
&\quad +\|J_\gamma^{N(\cdot,y_n)}(g(y_n)-\gamma T(u_n,y_n-q(y_n)))
\end{aligned}$$

$$
\begin{aligned}
&- J_\gamma^{N(\cdot,y_n)}(g(y_{n-1}) - \gamma T(u_{n-1}, y_{n-1} - q(y_{n-1})))\| \\
&+ \|J_\gamma^{N(\cdot,y_n)}(g(y_{n-1}) - \gamma T(u_{n-1}, y_{n-1} - q(y_{n-1}))) \\
&- J_\gamma^{N(\cdot,y_{n-1})}(g(y_{n-1}) - \gamma T(u_{n-1}, y_{n-1} - q(y_{n-1})))\| \\
\leq\ & \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\| + \|g(y_n) - g(y_{n-1})\| \\
&+ \gamma\|T(u_n, y_n - q(y_n)) - T(u_{n-1}, y_n - q(y_n))\| \\
&+ \gamma\|T(u_{n-1}, y_n - q(y_n)) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
&+ \varphi\|y_n - y_{n-1}\|.
\end{aligned} \tag{3.2.18}
$$

Using the same argument as for (3.2.11), we have

$$
\|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\|^2 \leq (1 - 2\delta_g + 64C_2\lambda_g^2)\|y_n - y_{n-1}\|^2. \tag{3.2.19}
$$

As g is Lipschitz continuous with constants λ_g, we have

$$
\|g(y_n) - g(y_{n-1})\| \leq \lambda_g\|y_n - y_{n-1}\|. \tag{3.2.20}
$$

By using the Lipschitz continuity of T in first argument and G is D-Lipschitz continuous, we have

$$
\begin{aligned}
&\|T(u_n, y_n - q(x_n)) - T(u_{n-1}, y_n - q(y_n))\| \\
&\leq \lambda_{T_1}\|u_n - u_{n-1}\| \\
&\leq \lambda_{T_1}(1 + n^{-1})D(G(x_n), G(x_{n-1})) \\
&\leq \lambda_{T_1}\lambda_{D_G}(1 + n^{-1})\|x_n - x_{n-1}\|.
\end{aligned} \tag{3.2.21}
$$

By using the Lipschitz continuity of T in second argument, we have

$$
\begin{aligned}
&\|T(u_{n-1}, y_n - q(y_n)) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
&\leq \lambda_{T_2}\|y_n - y_{n-1} - (q(y_n) - q(y_{n-1}))\|.
\end{aligned} \tag{3.2.22}
$$

Using the same argument as for (3.2.11), we have

$$
\|y_n - y_{n-1} - (q(y_n) - q(y_{n-1}))\|^2 \leq (1 - 2\delta_q + 64C_2\lambda_q^2)\|y_n - y_{n-1}\|. \tag{3.2.23}
$$

By (3.2.22) and (3.2.23), we have

$$\|T(u_{n-1}, y_n - q(y_n)) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\|$$
$$\leq \lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2}\|y_n - y_{n-1}\|. \tag{3.2.24}$$

Using (3.2.19)-(3.2.21) and (3.2.24), (3.2.18) becomes

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq \sqrt{1 - 2\delta_g + 64C_2\lambda_g^2}\|y_n - y_{n-1}\| + \lambda_g\|y_n - y_{n-1}\| \\
&\quad + \gamma\lambda_{T_1}\lambda_{D_G}(1 + n^{-1})\|x_n - x_{n-1}\| + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2} \\
&\quad \times\|y_n - y_{n-1}\| + \varphi\|y_n - y_{n-1}\| \\
&\leq \left[\sqrt{1 - 2\delta_g + 64C_2\lambda_g^2} + \lambda_g + \varphi + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2}\,\right] \\
&\quad \times\|y_n - y_{n-1}\| + \gamma\lambda_{T_1}\lambda_{D_G}(1 + n^{-1})\|x_n - x_{n-1}\|.
\end{aligned} \tag{3.2.25}$$

Equation (3.2.17) and (3.2.25) implies that

$$\begin{aligned}
&\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \\
&\leq \Big[\sqrt{1 - 2\delta_f + 64C_1\lambda_f^2} + \lambda_f + \psi + \rho\lambda_{S_1}\sqrt{1 - 2\delta_p + 64C_1\lambda_p^2} \\
&\quad + \gamma\lambda_{T_1}\lambda_{D_G}(1 + n^{-1})\Big]\|x_n - x_{n-1}\| + \Big[\sqrt{1 - 2\delta_g + 64C_2\lambda_g^2} \\
&\quad + \lambda_g + \varphi + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2} + \rho\lambda_{S_2}\lambda_{D_F}(1 + n^{-1})\Big] \\
&\quad \times\|y_n - y_{n-1}\| \\
&\leq \theta_n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|),
\end{aligned} \tag{3.2.26}$$

where

$$\theta_n = \max\Big\{\sqrt{1 - 2\delta_f + 64C_1\lambda_f^2} + \lambda_f + \psi + \rho\lambda_{S_1}\sqrt{1 - 2\delta_p + 64C_1\lambda_p^2} + \gamma\lambda_{T_1}\lambda_{D_G}(1 + n^{-1}),$$
$$\sqrt{1 - 2\delta_g + 64C_2\lambda_g^2} + \lambda_g + \varphi + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2} + \rho\lambda_{S_2}\lambda_{D_F}(1 + n^{-1})\Big\}.$$

Let

$$\theta = \max\left\{ \sqrt{1-2\delta_f+64C_1\lambda_f^2}+\lambda_f+\psi+\rho\lambda_{S_1}\sqrt{1-2\delta_p+64C_1\lambda_p^2}+\gamma\lambda_{T_1}\lambda_{D_G},\right.$$

$$\left.\sqrt{1-2\delta_g+64C_2\lambda_g^2}+\lambda_g+\varphi+\gamma\lambda_{T_2}\sqrt{1-2\delta_q+64C_2\lambda_q^2}+\rho\lambda_{S_2}\lambda_{D_F}\right\}.$$

Then $\theta_n \to \theta$ as $n \to \infty$. By condition (3.2.9) we know that $0 < \theta < 1$ and so (3.2.26) implies that $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences. Thus, there exists $x \in E_1$ and $y \in E_2$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Now we prove that $u_n \to u \in G(x)$ and $v_n \to v \in F(y)$. In fact, it follows from (3.2.13) and (3.2.21) that $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences. Let $u_n \to u$ and $v_n \to v$, respectively. We will show that $u \in G(x)$ and $v \in F(y)$.

Since $u_n \in G(x_n)$ and

$$\begin{aligned} d(u_n, G(x)) \le\ & \max\left\{ d(u_n, G(x)),\ \sup_{v\in G(x)} d(G(x_n), v)\right\} \\ & \le \max\left\{ \sup_{y\in G(x_n)} d(y, G(x)),\ \sup_{v\in G(x)} d(G(x_n), v)\right\} \\ & = D\left(G(x_n), G(x)\right), \end{aligned}$$

we have

$$\begin{aligned} d(u, G(x)) &\le \|u-u_n\| + d(u_n, G(x)) \\ &\le \|u-u_n\| + D\left(G(x_n), G(x)\right) \\ &\le \|u-u_n\| + \lambda_{D_G}\|x_n - x\| \to 0, \text{ as } n \to +\infty, \end{aligned}$$

since $G(x)$ is closed, we have $u \in G(x)$. Similarly $v \in F(x)$. By continuity and Algorithm 3.2.1, we know that $x, y, u,$ and v satisfy the following relation

$$f(x) = J_\rho^{M(\cdot,x)}(f(x) - \rho S(x-p(x), v)),$$

$$g(y) = J_\gamma^{N(\cdot,y)}(g(y) - \gamma T(u, y-q(y)).$$

By Lemma 3.2.1, (x, y, u, v) is a solution of Problem (3.2.1). This completes the proof.

3.3. System Of Generalized Variational Inclusions With H-accretive Operators

In this section, we study a system of generalized variational inclusions with H-accretive operators in uniformly smooth Banach spaces. An iterative algorithm is defined for computing approximate solutions of this system of generalized variational inclusions with H-accretive operators. The convergence criteria is also discussed.

Let E be a real Banach space. Let $G, F : E \to CB(E)$ be multivalued mappings and $f, g, p, q : E \to E$, $S, T : E \times E \to E$, $H_1, H_2 : E \to E$ are all single-valued mappings. Let $M : E \times E \to 2^E$ be a multivalued mapping such that for each $x \in E$, $M(\cdot, x)$ is H_1-accretive and $N : E \times E \to 2^E$ be a multivalued mapping such that for each $y \in E$, $N(\cdot, y)$ is H_2-accretive. We consider the following *system of generalized variational inclusions with H-accretive operators*:

Find $x, y \in E$, $u \in G(x)$, $v \in F(y)$ such that

$$0 \in S(x - p(x), v) + M(f(x), x),$$

$$0 \in T(u, y - q(y)) + N(g(y), y). \tag{3.3.1}$$

The following fixed point formulation convert *system of generalized variational inclusions with H-accretive operators* (3.3.1) into a fixed point problem.

Lemma 3.3.1. $x, y \in E$, $u \in G(x)$, $v \in F(y)$ is the solution of *system of generalized variational inclusions with H-accretive operators* (3.3.1) if and only if it satisfies

$$f(x) = J_{H_1,\rho}^{M(\cdot,x)}[H_1(f(x)) - \rho S(x - p(x), v)],$$

$$g(y) = J_{H_2,\gamma}^{N(\cdot,y)}[H_2(g(y)) - \gamma T(u, y - q(y))],$$

where $\rho > 0$ and $\gamma > 0$ are constants.

Proof. The proof of the above lemma is a direct consequence of the Definition 1.2.19 of H-resolvent operator and hence is omitted.

We invoke Lemma 3.3.1 and Nadler's theorem 1.2.3 [87] to propose the following iterative algorithm.

Algorithm 3.3.1. For any given $x_0, y_0 \in E$, we choose $u_0 \in G(x_0)$, $v_0 \in F(y_0)$ and compute $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by iterative schemes as follows:

$$x_{n+1} = x_n - f(x_n) + J_{H_1,\rho}^{M(\cdot,x_n)}[H_1(f(x_n)) - \rho S(x_n - p(x_n), v_n)],$$

$$y_{n+1} = y_n - g(y_n) + J_{H_2,\gamma}^{N(\cdot,y_n)}[H_2(g(y_n)) - \gamma T(u_n, y_n - q(y_n))]$$

and choose $u_{n+1} \in G(x_{n+1})$ and $v_{n+1} \in F(y_{n+1})$ such that

$$\|u_n - u_{n+1}\| \leq D(G(x_n), G(x_{n+1})),$$

$$\|v_n - v_{n+1}\| \leq D(F(y_n), F(y_{n+1})),$$

where $\rho > 0$ and $\gamma > 0$ are constants and $n = 0, 1, 2, \ldots\ldots$

Now we study the existence of solutions of *system of set-valued variational inclusions with H-accretive operators* (3.3.1) and the convergence of approximate solutions obtained by the Algorithm 3.3.1.

Theorem 3.3.1. Let E be a real uniformly smooth Banach space with module of smoothness $\tau_E(t) \leq Ct^2$ for some $C > 0$. Let $H_1, H_2 : E \to E$ be strongly accretive and Lipschitz continuous operators with constants r_1, r_2 and λ_{H_1} λ_{H_2}, respectively. Let $f, g, p, q : E \to E$ be strongly accretive mappings with constants δ_f, δ_g, δ_p and δ_q, respectively; and Lipschitz continuous with constants λ_f, λ_g, λ_p and λ_q, respectively. Suppose that $S, T : E \times E \to E$ be both Lipschitz continuous mappings in the first argument with constants λ_{S_1}, λ_{T_1}, respectively; and in the second argument with constants λ_{S_2}, λ_{T_2}, respectively. Let $G, F : E \to CB(E)$ be D-Lipschitz continuous mappings with constants λ_{D_G} and λ_{D_F}, respectively. Let $M : E \times E \to 2^E$ be a multivalued mapping such that for each $x \in E$, $M(\cdot, x)$ is H_1-accretive and $N : E \times E \to 2^E$ be a multivalued mapping such that for each $y \in E$, $N(\cdot, y)$ is H_2-accretive and the H_1-resolvent operator associated with M and H_2-resolvent operator associated with N are retractions.

If there exists constants $\rho > 0$ and $\gamma > 0$ such that

$$0 < B(f) + \frac{1}{r_1}\sqrt{\frac{\lambda_{H_1}\lambda_f + \rho\lambda_{S_1}B(p)}{1-\rho\lambda_{S_1}B(p)}} + \mu^* + \frac{1}{r_2}\gamma\lambda_{T_1}\lambda_{D_G} < 1$$

$$0 < B(g) + \frac{1}{r_2}\sqrt{\frac{\lambda_{H_2}\lambda_g + \gamma\lambda_{T_2}B(q)}{1-\gamma\lambda_{T_2}B(q)}} + \mu^{**} + \frac{1}{r_1}\rho\lambda_{S_2}\lambda_{D_F} < 1, \tag{3.3.2}$$

where

$$B(f) = \sqrt{1 - 2\delta_f + 64C\lambda_f^2}; \qquad B(p) = \sqrt{1 - 2\delta_p + 64C\lambda_p^2};$$

$$B(g) = \sqrt{1 - 2\delta_g + 64C\lambda_g^2}; \qquad B(q) = \sqrt{1 - 2\delta_q + 64C\lambda_q^2}\,.$$

Suppose that

$$\|J_{H_1,\rho}^{M(\cdot,x_n)}(x) - J_{H_1,\rho}^{M(\cdot,x_{n-1})}(x)\| \leq \mu^*\|x_n - x_{n-1}\|, \tag{3.3.3}$$

$$\text{for all } x,\ x_n,\ x_{n-1} \in E$$

and

$$\|J_{H_2,\gamma}^{N(\cdot,y_n)}(y) - J_{H_2,\gamma}^{N(\cdot,y_{n-1})}(y)\| \leq \mu^{**}\|y_n - y_{n-1}\|. \tag{3.3.4}$$

$$\text{for all } y,\ y_n,\ y_{n-1} \in E$$

Then the *system of set-valued variational inclusions with H-accretive operators* (3.3.1) admits a solution (x, y, u, v) and the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ converge to x, y, u and v, respectively, where $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ are the sequences generated by Algorithm 3.3.1.

Proof. From Algorithm 3.3.1 and Theorem 1.2.4, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|x_n - f(x_n) + J_{H_1,\rho}^{M(\cdot,x_n)}[H_1(f(x_n)) - \rho S(x_n - p(x_n), v_n)] \\
&\quad - [x_{n-1} - f(x_{n-1}) + J_{H_1,\rho}^{M(\cdot,x_{n-1})}[H_1(f(x_{n-1})) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1})]]\| \\
&\leq \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| \\
&\quad + \|J_{H_1,\rho}^{M(\cdot,x_n)}[H_1(f(x_n)) - \rho S(x_n - p(x_n), v_n)] \\
&\qquad - J_{H_1,\rho}^{M(\cdot,x_n)}[H_1(f(x_{n-1})) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1})]\| \\
&\quad + \|J_{H_1,\rho}^{M(\cdot,x_n)}[H_1(f(x_{n-1})) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1})]
\end{aligned}$$

$$- J_{H_1,\rho}^{M(\cdot,x_{n-1})}[H_1(f(x_{n-1})) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1})]\|$$

$$\leq \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| + \frac{1}{r_1}\|H_1(f(x_n)) - H_1(f(x_{n-1}))$$

$$- \rho[S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n)]\|$$

$$+ \frac{1}{r_1}\rho\|S(x_{n-1} - p(x_{n-1}), v_n) - S(x_{n-1} - p(x_{n-1}), v_{n-1})\|$$

$$+ \|J_{H_1,\rho}^{M(\cdot,x_n)}[H_1(f(x_{n-1})) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1})]$$

$$- J_{H_1,\rho}^{M(\cdot,x_{n-1})}[H_1(f(x_{n-1})) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1})]\|. \quad (3.3.5)$$

Since f is strongly accretive with constant δ_f and Lipschitz continuous with constant λ_f, by Proposition 1.2.1, we have

$$\|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|^2 \leq (1 - 2\delta_f + 64C\lambda_f^2)\|x_n - x_{n-1}\|^2$$

$$= B^2(f)\|x_n - x_{n-1}\|^2, \quad (3.3.6)$$

where $B^2(f) = (1 - 2\delta_f + 64C\lambda_f^2)$.

Since H_1 is Lipschitz continuous with constant λ_{H_1}, f is Lipschitz continuous with constant λ_f, p is strongly accretive with constant δ_p and Lipschitz continuous with constant λ_p, S is Lipschitz continuous in the first argument with constant λ_{S_1} and using Proposition 1.2.1, we have

$$\|H_1(f(x_n)) - H_1(f(x_{n-1})) - \rho[S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n)]\|^2$$

$$\leq \|H_1(f(x_n)) - H_1(f(x_{n-1}))\|^2 - 2\rho\langle S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n),$$

$$j(H_1(f(x_n)) - H_1(f(x_{n-1})) - \rho[S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n)])\rangle$$

$$\leq \lambda_{H_1}\lambda_f\|x_n - x_{n-1}\|^2 + 2\rho\|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n)\|$$

$$\times\|H_1(f(x_n)) - H_1(f(x_{n-1})) - \rho[S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n)]\|. \quad (3.3.7)$$

Now as S is Lipschitz continuous in the first argument with constant λ_{S_1} and using the same argument as for (3.3.6), we have

$$\|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n)\|$$

$$\leq \lambda_{S_1}\|x_n - x_{n-1} - (p(x_n) - p(x_{n-1}))\|$$

$$\leq \lambda_{S_1}\sqrt{1-2\delta_p+64C\lambda_p^2}\|x_n-x_{n-1}\|. \tag{3.3.8}$$

Thus,

$$\begin{aligned}
&\|H_1(f(x_n))-H_1(f(x_{n-1}))-\rho[S(x_n-p(x_n),v_n)-S(x_{n-1}-p(x_{n-1}),v_n)]\|^2\\
&\quad\leq \lambda_{H_1}\lambda_f\|x_n-x_{n-1}\|^2+2\rho\lambda_{S_1}\sqrt{1-2\delta_p+64C\lambda_p^2}\|x_n-x_{n-1}\|\\
&\qquad\times\|H_1(f(x_n))-H_1(f(x_{n-1}))-\rho[S(x_n-p(x_n),v_n)-S(x_{n-1}-p(x_{n-1}),v_n)]\|\\
&\quad\leq \lambda_{H_1}\lambda_f\|x_n-x_{n-1}\|^2+\rho\lambda_{S_1}\sqrt{1-2\delta_p+64C\lambda_p^2}\Big\{\|x_n-x_{n-1}\|^2\\
&\qquad+\|H_1(f(x_n))-H_1(f(x_{n-1}))-\rho[S(x_n-p(x_n),v_n)-S(x_{n-1}-p(x_{n-1}),v_n)]\|^2\Big\},
\end{aligned} \tag{3.3.9}$$

which implies that

$$\begin{aligned}
&\|H_1(f(x_n))-H_1(f(x_{n-1}))-\rho[S(x_n-p(x_n),v_n)-S(x_{n-1}-p(x_{n-1}),v_n)]\|^2\\
&\quad\leq \frac{\lambda_{H_1}\lambda_f+\rho\lambda_{S_1}\sqrt{1-2\delta_p+64C\lambda_p^2}}{1-\rho\lambda_{S_1}\sqrt{1-2\delta_p+64C\lambda_p^2}}\|x_n-x_{n-1}\|^2\\
&\quad\leq \frac{\lambda_{H_1}\lambda_f+\rho\lambda_{S_1}B(p)}{1-\rho\lambda_{S_1}B(p)}\|x_n-x_{n-1}\|^2,
\end{aligned} \tag{3.3.10}$$

where $B(p)=\sqrt{1-2\delta_p+64C\lambda_p^2}$.

It follows from the Lipschitz continuity of S in the second argument with constant λ_{S_2} and D-Lipschitz continuity of F with constant λ_{D_F}, that

$$\begin{aligned}
&\|S(x_{n-1}-p(x_{n-1}),v_n)-S(x_{n-1}-p(x_{n-1}),v_{n-1})\|\\
&\qquad\leq \lambda_{S_2}\|v_n-v_{n-1}\|\\
&\qquad\leq \lambda_{S_2}D(F(y_n),F(y_{n-1}))\\
&\qquad\leq \lambda_{S_2}\lambda_{D_F}\|y_n-y_{n-1}\|.
\end{aligned} \tag{3.3.11}$$

Using (3.3.6)-(3.3.11) and condition (3.3.3), (3.3.5) becomes

$$\begin{aligned}
\|x_{n+1}-x_n\|\leq B(f)\|x_n-x_{n-1}\|&+\frac{1}{r_1}\sqrt{\frac{\lambda_{H_1}\lambda_f+\rho\lambda_{S_1}B(p)}{1-\rho\lambda_{S_1}B(p)}}\|x_n-x_{n-1}\|\\
&+\frac{1}{r_1}\rho\lambda_{S_2}\lambda_{D_F}\|y_n-y_{n-1}\|+\mu^*\|x_n-x_{n-1}\|
\end{aligned}$$

$$= \left[B(f) + \frac{1}{r_1}\sqrt{\frac{\lambda_{H_1}\lambda_f + \rho\lambda_{S_1}B(p)}{1-\rho\lambda_{S_1}B(p)}} + \mu^*\right]\|x_n - x_{n-1}\|$$
$$+\frac{1}{r_1}\rho\lambda_{S_2}\lambda_{D_F}\|y_n - y_{n-1}\|. \quad (3.3.12)$$

Again by Algorithm 3.3.1 and Theorem 1.2.4, we have

$$\|y_{n+1}-y_n\| = \|y_n-g(y_n)+J_{H_2,\gamma}^{N(\cdot,y_n)}[H_2(g(y_n))-\gamma T(u_n, y_n-q(y_n))]$$
$$-[y_{n-1} - g(y_{n-1}) + J_{H_2,\gamma}^{N(\cdot,y_{n-1})}[H_2(g(y_{n-1})) - \gamma T(u_{n-1}, y_{n-1} - q(y_{n-1}))]\|$$
$$\leq \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\|$$
$$+ \|J_{H_2,\gamma}^{N(\cdot,y_n)}[H_2(g(y_n)) - \gamma T(u_n, y_n - q(x_n))]$$
$$- J_{H_2,\gamma}^{N(\cdot,y_n)}[H_2(g(y_{n-1})) - \gamma T(u_{n-1}, y_{n-1} - q(y_{n-1}))]\|$$
$$+ \|J_{H_2,\gamma}^{N(\cdot,y_n)}[H_2(g(y_{n-1})) - \gamma T(u_{n-1}, y_{n-1} - q(y_{n-1}))]$$
$$- J_{H_2,\gamma}^{N(\cdot,y_{n-1})}[H_2(g(y_{n-1})) - \gamma T(u_{n-1}, y_{n-1} - q(y_{n-1}))]\|$$
$$\leq \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\| + \frac{1}{r_2}\|H_2(g(y_n)) - H_2(g(y_{n-1}))$$
$$- \gamma[T(u_n, y_n - q(y_n)) - T(u_n, y_{n-1} - q(y_{n-1}))]\|$$
$$+ \frac{1}{r_2}\gamma\|T(u_n, y_{n-1} - q(y_{n-1})) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\|$$
$$+ \|J_{H_2,\gamma}^{N(\cdot,y_n)}[H_2(g(y_{n-1})) - \gamma T(u_{n-1}, y_{n-1} - q(y_{n-1}))]$$
$$- J_{H_2,\gamma}^{N(\cdot,y_{n-1})}[H_2(g(y_{n-1})) - \gamma T(u_{n-1}, y_{n-1} - q(y_{n-1}))]\|. \quad (3.3.13)$$

Since g is strongly accretive with constant δ_g and Lipschitz continuous with constant λ_g and using the same argument as for (3.3.6), we have

$$\|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\|^2 \leq (1 - 2\delta_g + 64C\lambda_g^2)\|y_n - y_{n-1}\|^2$$
$$= B^2(g)\|y_n-y_{n-1}\|^2, \quad (3.3.14)$$

where $B^2(g) = (1 - 2\delta_g + 64C\lambda_g^2)$.

Since H_2 is Lipschitz continuous with constant λ_{H_2}, g is Lipschitz continuous with constant λ_g, q is strongly accretive with constant δ_q and Lipschitz continuous with constant λ_q, T is Lipschitz continuous in the second argument with constant λ_{T_2} and using Proposition 1.2.1, we have

$$\begin{aligned}
&\|H_2(g(y_n)) - H_2(g(y_{n-1})) - \gamma[T(u_n, y_n - q(y_n)) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))]\|^2 \\
&\quad \leq \|H_2(g(y_n)) - H_2(g(y_{n-1}))\|^2 - 2\gamma\langle T(u_n, y_n - q(y_n)) - T(u_n, y_{n-1} - q(y_{n-1})), \\
&\qquad\qquad j(H_2(g(y_n)) - H_2(g(y_{n-1})) - \gamma[T(u_n, y_n - q(y_n)) - T(u_n, y_{n-1} - q(y_{n-1}))])\rangle \\
&\quad \leq \lambda_{H_2}\lambda_g\|y_n - y_{n-1}\|^2 + 2\gamma\|T(u_n, y_n - q(y_n)) - T(u_n, y_{n-1} - q(y_{n-1}))\| \\
&\qquad\qquad \times\|H_2(g(y_n)) - H_2(g(y_{n-1})) - \gamma[T(u_n, y_n - q(y_n)) - T(u_n, y_{n-1} - q(y_{n-1}))]\|.
\end{aligned} \tag{3.3.15}$$

Now as T is Lipschitz continuous in the second argument with constant λ_{T_2} and using the same argument as for (3.3.6), we have

$$\begin{aligned}
&\|T(u_n, y_n - q(y_n)) - T(u_n, y_{n-1} - q(y_{n-1}))\| \\
&\qquad \leq \lambda_{T_2}\|y_n - y_{n-1} - (q(y_n) - q(y_{n-1}))\| \\
&\qquad \leq \lambda_{T_2}\sqrt{1 - 2\delta_q + 64C\lambda_q^2}\|y_n - y_{n-1}\|.
\end{aligned} \tag{3.3.16}$$

Thus,

$$\begin{aligned}
&\|H_2(g(y_n)) - H_2(g(y_{n-1})) - \gamma[T(u_n, y_n - q(y_n)) - T(u_n, y_{n-1} - q(y_{n-1}))]\|^2 \\
&\quad \leq \lambda_{H_2}\lambda_g\|y_n - y_{n-1}\|^2 + 2\gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C\lambda_q^2}\|y_n - y_{n-1}\| \\
&\qquad \times\|H_2(g(y_n)) - H_2(g(y_{n-1})) - \gamma[T(u_n, y_n - q(y_n)) - T(u_n, y_{n-1} - q(y_{n-1}))]\| \\
&\quad \leq \lambda_{H_2}\lambda_g\|y_n - y_{n-1}\|^2 + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C\lambda_q^2}\Big\{\|y_n - y_{n-1}\|^2 \\
&\qquad + \|H_2(g(y_n)) - H_2(g(y_{n-1})) - \gamma[T(u_n, y_n - q(y_n)) - T(u_n, y_{n-1} - q(y_{n-1}))]\|^2\Big\},
\end{aligned} \tag{3.3.17}$$

which implies that

$$\begin{aligned}
&\|H_2(g(y_n)) - H_2(g(y_{n-1})) - \gamma[T(u_n, y_n - q(y_n)) - T(u_n, y_{n-1} - q(y_{n-1}))]\|^2 \\
&\qquad \leq \frac{\lambda_{H_2}\lambda_g + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C\lambda_q^2}}{1 - \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C\lambda_q^2}}\|y_n - y_{n-1}\|^2 \\
&\qquad \leq \frac{\lambda_{H_2}\lambda_g + \gamma\lambda_{T_2}B(q)}{1 - \gamma\lambda_{T_2}B(q)}\|y_n - y_{n-1}\|^2,
\end{aligned} \tag{3.3.18}$$

where $B(q) = \sqrt{1 - 2\delta_q + 64C\lambda_q^2}$.

It follows from the Lipschitz continuity of T in the first argument with constant λ_{T_1} and D-Lipschitz continuity of G with constant λ_{D_G}, that

$$\begin{aligned}
&\|T(u_n, y_{n-1} - q(y_{n-1})) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
&\qquad \leq \lambda_{T_1}\|u_n - u_{n-1}\| \\
&\qquad \leq \lambda_{T_1} D(G(x_n, G(x_{n-1})) \\
&\qquad \leq \lambda_{T_1}\lambda_{D_G}\|x_n - x_{n-1}\|. \qquad (3.3.19)
\end{aligned}$$

Using (3.3.14)-(3.3.19) and condition (3.3.4), (3.3.13) becomes

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq B(g)\|y_n - y_{n-1}\| + \frac{1}{r_2}\sqrt{\frac{\lambda_{H_2}\lambda_g + \gamma\lambda_{T_2}B(q)}{1 - \gamma\lambda_{T_2}B(q)}}\|y_n - y_{n-1}\| \\
&\quad + \frac{1}{r_2}\gamma\lambda_{T_1}\lambda_{D_G}\|x_n - x_{n-1}\| + \mu^{**}\|y_n - y_{n-1}\| \\
&= \left[B(g) + \frac{1}{r_2}\sqrt{\frac{\lambda_{H_2}\lambda_g + \gamma\lambda_{T_2}B(q)}{1 - \gamma\lambda_{T_2}B(q)}} + \mu^{**}\right]\|y_n - y_{n-1}\| \\
&\quad + \frac{1}{r_2}\gamma\lambda_{T_1}\lambda_{D_G}\|x_n - x_{n-1}\|. \qquad (3.3.20)
\end{aligned}$$

Combining (3.3.12) and (3.3.20), we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \\
&\quad \leq \left[B(f) + \frac{1}{r_1}\sqrt{\frac{\lambda_{H_1}\lambda_f + \rho\lambda_{S_1}B(p)}{1 - \rho\lambda_{S_1}B(p)}} + \mu^* + \frac{1}{r_2}\gamma\lambda_{T_1}\lambda_{D_G}\right]\|x_n - x_{n-1}\| \\
&\qquad + \left[B(g) + \frac{1}{r_2}\sqrt{\frac{\lambda_{H_2}\lambda_g + \gamma\lambda_{T_2}B(q)}{1 - \gamma\lambda_{T_2}B(q)}} + \mu^{**} + \frac{1}{r_1}\rho\lambda_{S_2}\lambda_{D_F}\right]\|y_n - y_{n-1}\| \\
&\quad \leq \theta[\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|], \qquad (3.3.21)
\end{aligned}$$

where

$$\begin{aligned}
\theta = \max\Big\{&B(f) + \frac{1}{r_1}\sqrt{\frac{\lambda_{H_1}\lambda_f + \rho\lambda_{S_1}B(p)}{1 - \rho\lambda_{S_1}B(p)}} + \mu^* + \frac{1}{r_2}\gamma\lambda_{T_1}\lambda_{D_G}, \\
&B(g) + \frac{1}{r_2}\sqrt{\frac{\lambda_{H_2}\lambda_g + \gamma\lambda_{T_2}B(q)}{1 - \gamma\lambda_{T_2}B(q)}} + \mu^{**} + \frac{1}{r_1}\rho\lambda_{S_2}\lambda_{D_F}\Big\}.
\end{aligned}$$

H- resolvent equations and the convergence of iterative sequences generated by the algorithms is also dicussed.

Let E_1 and E_2 be any two real Banach spaces, $S : E_1 \times E_2 \to E_1$, $T : E_1 \times E_2 \to E_2$, $p : E_1 \to E_1$, $q : E_2 \to E_2$, $H_1 : E_1 \to E_1$ and $H_2 : E_2 \to E_2$ be single-valued mappings, $G : E_1 \to CB(E_1)$, $F : E_2 \to CB(E_2)$ be multi-valued mappings. Let $M : E_1 \times E_1 \to 2^{E_1}$ be H_1-accretive and $N : E_2 \times E_2 \to 2^{E_2}$ be H_2-accretive mappings. Let $f : E_1 \to E_1$ and $g : E_2 \to E_2$ be nonlinear mappings with $f(E_1) \cap D(M(\cdot, x)) \neq \emptyset$ and $g(E_2) \cap D(N(\cdot, y)) \neq \emptyset$, respectively. Then we consider the following *system of generalized H-resolvent equations*:

Find $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$, $z' \in E_1$, $z'' \in E_2$ such that

$$\begin{aligned} S(x - p(x), v) + \rho^{-1} R_{H_1,\rho}^{M(\cdot,x)}(z') = 0, \quad \rho > 0, \\ T(u, y - q(y)) + \gamma^{-1} R_{H_2,\gamma}^{N(\cdot,y)}(z'') = 0, \quad \gamma > 0, \end{aligned} \tag{3.4.1}$$

where $R_{H_1\rho}^{M(\cdot,x)} = I - H_1(J_{H_1,\rho}^{M(\cdot,x)})$, $R_{H_2,\gamma}^{N(\cdot,y)} = I - H_2(J_{H_2\gamma}^{N(\cdot,y)})$ and $J_{H_1,\rho}^{M(\cdot,x)}$, $J_{H_2,\gamma}^{N(\cdot,y)}$ are the resolvent operators associated with M and N, respectively.

Now we mention the corresponding *system of generalized variational inclusions* of the *system of generalized H-resolvent equations* (3.4.1).

Find $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$ such that

$$\begin{aligned} 0 \in S(x - p(x), v) + M(f(x), x), \\ 0 \in T(u, y - q(y)) + N(g(y), y). \end{aligned} \tag{3.4.2}$$

Lemma 3.4.1. $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$ is a solution of *system of generalized variational inclusions* (3.4.2) if and only if (x, y, u, v) satisfies

$$f(x) = J_{H_1,\rho}^{M(\cdot,x)}[H_1(f(x)) - \rho S(x - p(x), v)],$$

$$g(y) = J_{H_2,\gamma}^{N(\cdot,y)}[H_2(g(y)) - \gamma T(u, y - q(y))],$$

where $\rho > 0$ and $\gamma > 0$ are constants.

Proof. The proof of Lemma 3.4.1 is a direct consequence of the Definition 1.2.19 of H-resolvent operator, and hence is omitted.

The following proposition established an equivalence relation between the *system of generalized H-resolvent equations* (3.4.1) and corresponding *system of generalized variational inclusions* (3.4.2).

Proposition 3.4.1. The *system of generalized variational inclusions* (3.4.2) has a solution (x, y, u, v) with $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$ if and only if *system of generalized H-resolvent equations* (3.4.1) has a solution (z', z'', x, y, u, v) with $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$, $z' \in E_1$, $z'' \in E_2$ such that

$$f(x) = J_{H_1,\rho}^{M(\cdot,x)}(z'), \tag{3.4.3}$$

$$g(y) = J_{H_2,\gamma}^{N(\cdot,y)}(z''), \tag{3.4.4}$$

where $z' = H_1(f(x)) - \rho S(x - p(x), v)$ and $z'' = H_2(g(y)) - \gamma T(u, y - q(y))$.

Proof. Let (x, y, u, v) be a solution of *system of generalized variational inclusions* (3.4.2). Then by Lemma 3.4.1, it satisfies the following equations

$$f(x) = J_{H_1,\rho}^{M(\cdot,x)}[H_1(f(x)) - \rho S(x - p(x), v)],$$

$$g(y) = J_{H_2,\gamma}^{N(\cdot,y)}[H_2(g(y)) - \gamma T(u, y - q(y))].$$

Let $z' = H_1(f(x)) - \rho S(x - p(x), v)$ and $z'' = H_2(g(y)) - \gamma T(u, y - q(y))$, then we have

$$f(x) = J_{H_1,\rho}^{M(\cdot,x)}(z'),$$

$$g(y) = J_{H_2,\gamma}^{N(\cdot,y)}(z''),$$

$$z' = H_1(J_{H_1,\rho}^{M(\cdot,x)}(z')) - \rho S(x - p(x), v) \text{ and } z'' = H_2(J_{H_2,\gamma}^{N(\cdot,y)}(z'')) - \gamma T(u, y - q(y)),$$

it follows that

$$\begin{aligned} (I - H_1(J_{H_1,\rho}^{M(\cdot,x)}))(z') &= z' - H_1(J_{H_1,\rho}^{M(\cdot,x)}(z')) \\ &= H_1(J_{H_1,\rho}^{M(\cdot,x)}(z')) - \rho S(x - p(x), v) \\ &\quad - H_1(J_{H_1,\rho}^{M(\cdot,x)}(z')) \\ &= -\rho S(x - p(x), v), \end{aligned}$$

and similarly

$$(I - H_2(J_{H_2,\gamma}^{N(\cdot,y)}))(z'') = -\gamma T(u, y - q(y)),$$

i.e.,

$$S(x-p(x),v)+\rho^{-1}R_{H_1,\rho}^{M(\cdot,x)}(z')=0,$$
$$T(u,y-q(y))+\gamma^{-1}R_{H_2,\gamma}^{N(\cdot,y)}(z'')=0.$$

Thus, (z',z'',x,y,u,v) is a solution of *system of generalized H-resolvent equations* (3.4.1).

Conversely, let (z',z'',x,y,u,v) be a solution of *system of generalized H-resolvent equations* (3.4.1), then

$$\rho S(x-p(x),v)=-R_{H_1,\rho}^{M(\cdot,x)}(z'), \tag{3.4.5}$$

$$\gamma T(u,y-q(y))=-R_{H_2,\gamma}^{N(\cdot,y)}(z''). \tag{3.4.6}$$

Now

$$\begin{aligned}\rho S(x-p(x),v)&=-R_{H_1,\rho}^{M(\cdot,x)}(z')\\&=-(I-H_1(J_{H_1,\rho}^{M(\cdot,x)}))(z')\\&=(H_1(J_{H_1,\rho}^{M(\cdot,x)}))(z')-z'\\&=(H_1(J_{H_1,\rho}^{M(\cdot,x)}))[H_1(f(x))-\rho S(x-p(x),v)]\\&\qquad-[H_1(f(x))-\rho S(x-p(x),v)],\end{aligned}$$

which implies that

$$f(x)=J_{H_1,\rho}^{M(\cdot,x)}[H_1(f(x))-\rho S(x-p(x),v)],$$

and

$$\begin{aligned}\gamma T(u,y-q(y))&=-R_{H_2,\gamma}^{N(\cdot,y)}(z'')\\&=-(I-H_2(J_{H_2,\gamma}^{N(\cdot,y)}))(z'')\\&=(H_2(J_{H_2,\gamma}^{N(\cdot,y)}))(z'')-z''\\&=(H_2(J_{H_2,\gamma}^{N(\cdot,y)}))[H_2(g(y))-\gamma T(u,y-q(y))]\\&\qquad-[H_2(g(y))-\gamma T(u,y-q(y))],\end{aligned}$$

which implies that

$$g(y)=J_{H_2,\gamma}^{N(\cdot,y)}[H_2(g(y))-\gamma T(u,y-q(y))].$$

Thus, we have

$$f(x) = J_{H_1,\rho}^{M(\cdot,x)}[H_1(f(x)) - \rho S(x - p(x), v)],$$
$$g(y) = J_{H_2,\gamma}^{N(\cdot,y)}[H_2(g(y)) - \gamma T(u, y - q(y))].$$

Thus, by Lemma 3.4.1, (x, y, u, v) is a solution of *system of generalized variational inclusions* (3.4.2).

Alternative Proof. Let

$$z' = H_1(f(x)) - \rho S(x - p(x), v) \text{ and } z'' = H_2(g(y)) - \gamma T(u, y - q(y)),$$

using (3.4.3) and (3.4.4), we can write

$$z' = (H_1(J_{H_1,\rho}^{M(\cdot,x)}))(z') - \rho S(x - p(x), v) \text{ and } z'' = (H_2(J_{H_2,\gamma}^{N(\cdot,y)}))(z'') - \gamma T(u, y - q(y))$$

which implies that

$$S(x - p(x), v) + \rho^{-1} R_{H_1,\rho}^{M(\cdot,x)}(z') = 0, \quad \rho > 0,$$

$$T(u, y - q(y)) + \gamma^{-1} R_{H_2,\gamma}^{N(\cdot,y)}(z'') = 0, \quad \gamma > 0,$$

the required system of *generalized H-resolvent equations* (3.4.1).

We suggest a number of iterative methods for computing the approximate solutions of system of *generalized H-resolvent equations* (3.4.1).

Algorithm 3.4.1. For given $(x_0, y_0) \in E_1 \times E_2$, $u_0 \in G(x_0)$, $v_0 \in F(y_0)$, $z'_0 \in E_1$, $z''_0 \in E_2$, compute $\{z'_n\}$, $\{z''_n\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by the iterative schemes as follows:

$$f(x_n) = J_{H_1,\rho}^{M(\cdot,x_n)}(z'_n), \tag{3.4.7}$$

$$g(y_n) = J_{H_2,\gamma}^{N(\cdot,y_n)}(z''_n), \tag{3.4.8}$$

$$u_n \in G(x_n) : \ \|u_{n+1} - u_n\| \ \leq \ D(G(x_{n+1}), G(x_n)), \tag{3.4.9}$$

$$v_n \in F(y_n) : \ \|v_{n+1} - v_n\| \ \leq \ D(F(y_{n+1}), F(y_n)), \tag{3.4.10}$$

$$z'_{n+1} = [H_1(f(x_n)) - \rho S(x_n - p(x_n), v_n)], \tag{3.4.11}$$

$$z''_{n+1} = [H_2(g(y_n)) - \gamma T(u_n, y_n - q(y_n))]. \tag{3.4.12}$$

$n = 0, 1, 2, \ldots\ldots\ldots$

The *system of generalized H-resolvent equations* (3.4.1) can also be written as

$$z' = H_1(f(x)) - S(x-p(x),v) + (I-\rho^{-1})R_{H_1,\rho}^{M(\cdot,x)}(z'),$$

$$z'' = H_2(g(y)) - T(u,y-q(y)) + (I-\gamma^{-1})R_{H_2,\gamma}^{N(\cdot,y)}(z'').$$

We use this fixed-point formulation to suggest the following iterative method.

Algorithm 3.4.2. For given $(x_0, y_0) \in E_1 \times E_2$, $u_0 \in G(x_0)$, $v_0 \in F(y_0)$, $z_0' \in E_1$, $z_0'' \in E_2$,, compute $\{z_n'\}$, $\{z_n''\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by the iterative schemes as follows:

$$f(x_n) = J_{H_1,\rho}^{M(\cdot,x_n)}(z_n'),$$

$$g(y_n) = J_{H_2,\gamma}^{N(\cdot,y_n)}(z_n''),$$

$$u_n \in G(x_n): \ \|u_{n+1} - u_n\| \ \leq \ D(G(x_{n+1}), G(x_n)),$$

$$v_n \in F(y_n): \ \|v_{n+1} - v_n\| \ \leq \ D(F(y_{n+1}), F(y_n)),$$

$$z_{n+1}' = H_1(f(x_n)) - S(x_n - p(x_n), v_n) + (I-\rho^{-1})R_{H_1,\rho}^{M(\cdot,x_n)}(z_n'),$$

$$z_{n+1}'' = H_2(g(y_n)) - T(u_n, y_n - q(y_n)) + (I-\gamma^{-1})R_{H_2,\gamma}^{N(\cdot,y_n)}(z_n'').$$

$n = 0, 1, 2, \ldots\ldots\ldots$

Now we study the existence of solutions of *system of generalized H-resolvent equations* (3.4.1) and the convergence of the iterative sequences generated by the Algorithm 3.4.1.

Theorem 3.4.1. Let E_1 and E_2 be any two real uniformly smooth Banach spaces with module of smoothness $\tau_{E_1}(t) \leq C_1 t^2$ and $\tau_{E_2}(t) \leq C_2 t^2$ for $C_2, C_2 > 0$, respectively. Let $G : E_1 \to CB(E_1)$ and $F : E_2 \to CB(E_2)$ be D-Lipschitz continuous mappings with constants λ_{D_G} and λ_{D_F}, respectively. Let $H_1 : E_1 \to E_1$ and $H_2 : E_2 \to E_2$ be strongly accretive and Lipschitz continuous mappings with constants r_1, r_2 and λ_{H_1}, λ_{H_2}, respectively. Let $M : E_1 \times E_1 \to 2^{E_1}$ be H_1-accretive operator and $N : E_2 \times E_2 \to 2^{E_2}$ be H_2-accretive operator such that the H_1-resolvent operator associated with M and H_2-resolvent operator associated with N are retractions. Let $f, p : E_1 \to E_1$, $g, q : E_2 \to E_2$ be strongly accretive mappings with constants δ_f, δ_p, δ_g and δ_q, respectively and Lipschitz continuous with constants λ_f,

λ_p, λ_g and λ_q, respectively. Let $S : E_1 \times E_2 \to E_1$ and $T : E_1 \times E_2 \to E_2$ be Lipschitz continuous in first and second arguments with constants λ_{S_1}, λ_{S_2} and λ_{T_1}, λ_{T_2}, respectively.

If there exists constants $\rho > 0$ and $\gamma > 0$, such that

$$0 < \frac{B_1'/2 + 1 + \rho\lambda_{S_1}\sqrt{1 - 2\delta_p + 64C_1\lambda_p^2} + \gamma\lambda_{T_1}\lambda_{D_G}}{r_1\left(1 - \frac{B_2'}{2}\right)} < 1,$$

$$0 < \frac{B_1''/2 + 1 + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2} + \rho\lambda_{S_2}\lambda_{D_F}}{r_2\left(1 - \frac{B_2''}{2}\right)} < 1, \quad (3.4.13)$$

where

$$B_1' = 2\sqrt{1 - 2r_1\lambda_f^2 + 64C_1\lambda_{H_1}^2\lambda_f^2}, \qquad B_2' = 2\sqrt{1 - 2\delta_f + 64C_1\lambda_f^2},$$

$$B_1'' = 2\sqrt{1 - 2r_2\lambda_g^2 + 64C_2\lambda_{H_2}^2\lambda_f^2}, \qquad B_2'' = 2\sqrt{1 - 2\delta_g + 64C_2\lambda_g^2}\,.$$

Then there exists $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$, $z' \in E_1$, $z'' \in E_2$, satisfying the *system of generalized H-resolvent equations* (3.4.1) and the iterative sequences $\{z_n'\}$, $\{z_n''\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ generated by Algorithm 3.4.1 converge strongly to z', z'', x, y, u and v, respectively.

Proof. From Algorithm 3.4.1, we have

$$\begin{aligned}
\|z_{n+1}' - z_n'\| &= \|H_1(f(x_n)) - \rho S(x_n - p(x_n), v_n) - [H_1(f(x_{n-1})) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1})]\| \\
&\le \|x_n - x_{n-1} - (H_1(f(x_n)) - H_1(f(x_{n-1})))\| + \|x_n - x_{n-1}\| \\
&\quad + \rho\|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_{n-1})\|. \qquad (3.4.14)
\end{aligned}$$

Since H_1 is strongly accretive with constant r_1 and Lipschitz continuous with constant λ_{H_1}, f is Lipschitz continuous with constant λ_f and by Proposition 1.3..12, we have

$$\begin{aligned}
&\|x_n - x_{n-1} - (H_1(f(x_n)) - H_1(f(x_{n-1})))\|^2 \\
&\quad \le \|x_n - x_{n-1}\|^2 + 2\langle -(H_1(f(x_n)) - H_1(f(x_{n-1}))), \\
&\qquad\qquad j(x_n - x_{n-1} - (H_1(f(x_n)) - H_1(f(x_{n-1}))))\rangle \\
&\quad = \|x_n - x_{n-1}\|^2 + 2\langle -(H_1(f(x_n)) - H_1(f(x_{n-1}))), j(x_n - x_{n-1})\rangle
\end{aligned}$$

$$+2\langle -(H_1(f(x_n)) - H_1(f(x_{n-1}))),$$
$$j(x_n - x_{n-1} - (H_1(f(x_n)) - H_1(f(x_{n-1}))) - j(x_n - x_{n-1}))\rangle$$
$$\leq \|x_n - x_{n-1}\|^2 - 2r_1\|f(x_n) - f(x_{n-1})\|^2$$
$$+4d^2\tau_E\left(\frac{4\|H_1(f(x_n)) - H_1(f(x_{n-1}))\|}{d}\right)$$
$$\leq \|x_n - x_{n-1}\|^2 - 2r_1\lambda_f^2\|x_n - x_{n-1}\|^2 + 64C_1\lambda_{H_1}^2\lambda_f^2\|x_n - x_{n-1}\|^2$$
$$\leq \left(1 - 2r_1\lambda_f^2 + 64C_1\lambda_{H_1}^2\lambda_f^2\right)\|x_n - x_{n-1}\|^2. \quad (3.4.15)$$

Since S is Lipschitz continuous in both arguments, F is D-Lipschitz continuous, we have

$$\|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_{n-1})\|$$
$$= \|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n)$$
$$+S(x_{n-1} - p(x_{n-1}), v_n) - S(x_{n-1} - p(x_{n-1}), v_{n-1})\|$$
$$\leq \|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n)\|$$
$$+\|S(x_{n-1} - p(x_{n-1}), v_n) - S(x_{n-1} - p(x_{n-1}), v_{n-1})\|$$
$$\leq \lambda_{S_1}\|x_n - x_{n-1} - (p(x_n) - p(x_{n-1}))\| + \lambda_{S_2}\|v_n - v_{n-1}\|$$
$$\leq \lambda_{S_1}\|x_n - x_{n-1} - (p(x_n) - p(x_{n-1}))\| + \lambda_{S_2}D(F(y_n), F(y_{n-1}))$$
$$\leq \lambda_{S_1}\|x_n - x_{n-1} - (p(x_n) - p(x_{n-1}))\| + \lambda_{S_2}\lambda_{D_F}\|y_n - y_{n-1}\|. \quad (3.4.16)$$

By Proposition 1.2.1, we have

$$\|x_n - x_{n-1} - (p(x_n) - p(x_{n-1}))\|^2 \leq \left(1 - 2\delta_p + 64C_1\lambda_p^2\right)\|x_n - x_{n-1}\|^2. \quad (3.4.17)$$

Using (3.4.17), (3.4.16) becomes

$$\|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_{n-1})\|$$
$$\leq \lambda_{S_1}\sqrt{1 - 2\delta_p + 64C_1\lambda_p^2}\|x_n - x_{n-1}\| + \lambda_{S_2}\lambda_{D_F}\|y_n - y_{n-1}\|. \quad (3.4.18)$$

Using (3.4.15), (3.4.18), (3.4.14) becomes

$$\|z'_{n+1} - z'_n\| \leq \sqrt{1 - 2r_1\lambda_f^2 + 64C_1\lambda_{H_1}^2\lambda_f^2}\|x_n - x_{n-1}\| + \|x_n - x_{n-1}\|$$
$$+\rho\left(\lambda_{S_1}\sqrt{1 - 2\delta_p + 64C_1\lambda_p^2}\|x_n - x_{n-1}\| + \lambda_{S_2}\lambda_{D_F}\|y_n - y_{n-1}\|\right)$$

$$= \left(\sqrt{1-2r_1\lambda_f^2+64C_1\lambda_{H_1}^2\lambda_f^2}+1+\rho\lambda_{S_1}\sqrt{1-2\delta_p+64C_1\lambda_p^2}\right)\|x_n-x_{n-1}\|$$
$$+\rho\lambda_{S_2}\lambda_{D_F}\|y_n-y_{n-1}\|$$
$$= \left(B_1'/2+1+\rho\lambda_{S_1}\sqrt{1-2\delta_p+64C_1\lambda_p^2}\right)\|x_n-x_{n-1}\|$$
$$+\rho\lambda_{S_2}\lambda_{D_F}\|y_n-y_{n-1}\|, \tag{3.4.19}$$

where $B_1' = 2\sqrt{1-2r_1\lambda_f^2+64C_1\lambda_{H_1}^2\lambda_f^2}$.

Again, from Algorithm 3.4.1, we have

$$\|z''_{n+1}-z''_n\| = \|H_2(g(y_n))-\gamma T(u_n,y_n-q(y_n))-[H_2(g(y_{n-1}))-\gamma T(u_{n-1},y_{n-1}-q(y_{n-1}))]\|$$
$$\leq \|y_n-y_{n-1}-(H_2(g(y_n))-H_2(g(y_{n-1})))\|+\|y_n-y_{n-1}\|$$
$$+\gamma\|T(u_n,y_n-q(y_n))-T(u_{n-1},y_{n-1}-q(y_{n-1}))\|. \tag{3.4.20}$$

Since H_2 is strongly accretive with constant r_2 and Lipschitz continuous with constant λ_{H_2}, g is Lipschitz continuous with constant λ_g and by proposition 1.2.1, we have

$$\|y_n-y_{n-1}-(H_2(g(y_n))-H_2(g(y_{n-1})))\|^2$$
$$\leq \left(1-2r_2\lambda_g^2+64C_2\lambda_{H_2}^2\lambda_g^2\right)\|y_n-y_{n-1}\|^2. \tag{3.4.21}$$

Since T is Lipschitz continuous in both arguments, G is D-Lipschitz continuous, we have

$$\|T(u_n,y_n-q(y_n))-T(u_{n-1},y_{n-1}-q(y_{n-1}))\|$$
$$\leq \|T(u_n,y_n-q(y_n))-T(u_{n-1},y_n-q(y_n))\|$$
$$+\|T(u_{n-1},y_n-q(y_n))-T(u_{n-1},y_{n-1}-q(y_{n-1}))\|$$
$$\leq \lambda_{T_1}\|u_n-u_{n-1}\|+\lambda_{T_2}\|y_n-q(y_n)-(y_{n-1}-q(y_{n-1}))\|$$
$$\leq \lambda_{T_1}D(G(x_n),G(x_{n-1}))+\lambda_{T_2}\|y_n-y_{n-1}-(q(y_n)-q(y_{n-1}))\|$$
$$\leq \lambda_{T_1}\lambda_{D_G}\|x_n-x_{n-1}\|+\lambda_{T_2}\|y_n-y_{n-1}-(q(y_n)-q(y_{n-1}))\|. \tag{3.4.22}$$

Using same argument as for (3.4.17), we have

$$\|y_n-y_{n-1}-(q(y_n)-q(y_{n-1}))\|^2 \leq \left(1-2\delta_q+64C_2\lambda_q^2\right)\|y_n-y_{n-1}\|^2. \tag{3.4.23}$$

Using (3.4.23), (3.4.22) becomes

$$\|T(u_n, y_n - q(y_n)) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\|$$

$$\leq \lambda_{T_1}\lambda_{D_G}\|x_n - x_{n-1}\| + \lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2}\|y_n - y_{n-1}\|. \quad (3.4.24)$$

Using (3.4.21), (3.4.24), (3.4.20) becomes

$$\begin{aligned}\|z''_{n+1} - z''_n\| &\leq \sqrt{1 - 2r_2\lambda_g^2 + 64C_2\lambda_{H_2}^2\lambda_g^2}\|y_n - y_{n-1}\| + \|y_n - y_{n-1}\| \\ &\quad +\gamma\left(\lambda_{T_1}\lambda_{D_G}\|x_n - x_{n-1}\| + \lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2}\|y_n - y_{n-1}\|\right) \\ &= \left(\sqrt{1 - 2r_2\lambda_g^2 + 64C_2\lambda_{H_2}^2\lambda_g^2} + 1 + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2}\right)\|y_n - y_{n-1}\| \\ &\quad +\gamma\lambda_{T_1}\lambda_{D_G}\|x_n - x_{n-1}\| \\ &= \left(B''_1/2 + 1 + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2}\right)\|y_n - y_{n-1}\| \\ &\quad +\gamma\lambda_{T_1}\lambda_{D_G}\|x_n - x_{n-1}\|, \qquad (3.4.25)\end{aligned}$$

where $B''_1 = 2\sqrt{1 - 2r_2\lambda_g^2 + 64C_2\lambda_{H_2}^2\lambda_g^2}$.

By (3.4.19) and (3.4.25), we have

$$\begin{aligned}&\|z'_{n+1} - z'_n\| + \|z''_{n+1} - z''_n\| \\ &\leq \left(B'_1/2 + 1 + \rho\lambda_{S_1}\sqrt{1 - 2\delta_p + 64C_1\lambda_p^2} + \gamma\lambda_{T_1}\lambda_{D_G}\right)\|x_n - x_{n-1}\| \\ &\quad + \left(B''_1/2 + 1 + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2} + \rho\lambda_{S_2}\lambda_{D_F}\right)\|y_n - y_{n-1}\|. \\ &\qquad (3.4.26)\end{aligned}$$

Also from (3.4.7) and (3.4.8), we have

$$\begin{aligned}\|x_n - x_{n-1}\| &= \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1})) + J_{H_1,\rho}^M(z'_n) - J_{H_1,\rho}^M(z'_{n-1})\| \\ &\leq \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| + \|J_{H_1,\rho}^M(z'_n) - J_{H_1,\rho}^M(z'_{n-1})\| \\ &\leq \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| + \frac{1}{r_1}\|z'_n - z'_{n-1}\|. \qquad (3.4.27)\end{aligned}$$

Using same argument as for (3.4.17), we have

$$\|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|^2 \leq \left(1 - 2\delta_f + 64C_1\lambda_f^2\right)\|x_n - x_{n-1}\|^2. \quad (3.4.28)$$

Using (3.4.28), (3.4.27) becomes

$$\|x_n - x_{n-1}\| \leq \sqrt{1 - 2\delta_f + 64C_1\lambda_f^2}\|x_n - x_{n-1}\| + \frac{1}{r_1}\|z'_n - z'_{n-1}\|$$
$$\leq \frac{B'_2}{2}\|x_n - x_{n-1}\| + \frac{1}{r_1}\|z'_n - z'_{n-1}\|,$$
where $B'_2 = 2\sqrt{1 - 2\delta_f + 64C_1\lambda_f^2}$.

Which implies that

$$\|x_n - x_{n-1}\| \leq \frac{1}{r_1\left(1 - \frac{B'_2}{2}\right)}\|z'_n - z'_{n-1}\|, \tag{3.4.29}$$

and

$$\|y_n - y_{n-1}\| = \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1})) + J^N_{H2,\gamma}(z''_n) - J^N_{H2,\gamma}(z''_{n-1})\|$$
$$\leq \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\| + \|J^N_{H2,\gamma}(z''_n) - J^N_{H2,\gamma}(z''_{n-1})\|$$
$$\leq \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\| + \frac{1}{r_2}\|z''_n - z''_{n-1}\|. \tag{3.4.30}$$

Using same argument as for (3.4.17), we have

$$\|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\|^2 \leq \left(1 - 2\delta_g + 64C_2\lambda_g^2\right)\|y_n - y_{n-1}\|^2. \tag{3.4.31}$$

Using (3.4.31), (3.4.30) becomes

$$\|y_n - y_{n-1}\| \leq \sqrt{1 - 2\delta_g + 64C_2\lambda_g^2}\|y_n - y_{n-1}\| + \frac{1}{r_2}\|z''_n - z''_{n-1}\|$$
$$\leq \frac{B''_2}{2}\|y_n - y_{n-1}\| + \frac{1}{r_2}\|z''_n - z''_{n-1}\|,$$
where $B''_2 = 2\sqrt{1 - 2\delta_g + 64C_2\lambda_g^2}$.

Which implies that

$$\|y_n - y_{n-1}\| \leq \frac{1}{r_2\left(1 - \frac{B''_2}{2}\right)}\|z''_n - z''_{n-1}\|. \tag{3.4.32}$$

Using (3.4.29) and (3.4.32), (3.4.26) becomes

$$\|z'_{n+1} - z'_n\| + \|z''_{n+1} - z''_n\|$$
$$\leq \left[\frac{B'_1/2 + 1 + \rho\lambda_{S_1}\sqrt{1 - 2\delta_p + 64C_1\lambda_p^2} + \gamma\lambda_{T_1}\lambda_{D_G}}{r_1\left(1 - \frac{B'_2}{2}\right)}\right]\|z'_n - z'_{n-1}\|$$

$$+\left[\frac{B_1''/2+1+\gamma\lambda_{T_2}\sqrt{1-2\delta_q+64C_2\lambda_q^2}+\rho\lambda_{S_2}\lambda_{D_F}}{r_2\left(1-\frac{B_2''}{2}\right)}\right]\|z_n''-z_{n-1}''\|$$

$$\leq \theta(\|z_n'-z_{n-1}'\|+\|z_n''-z_{n-1}''\|), \qquad (3.4.33)$$

where

$$\theta=\max\left\{\frac{B_1'/2+1+\rho\lambda_{S_1}\sqrt{1-2\delta_p+64C_1\lambda_p^2}+\gamma\lambda_{T_1}\lambda_{D_G}}{r_1\left(1-\frac{B_2'}{2}\right)},\right.$$
$$\left.\frac{B_1''/2+1+\gamma\lambda_{T_2}\sqrt{1-2\delta_q+64C_2\lambda_q^2}+\rho\lambda_{S_2}\lambda_{D_F}}{r_2\left(1-\frac{B_2''}{2}\right)}\right\}.$$

By (3.4.13), we know that $0<\theta<1$ and so (3.4.33) implies that $\{z_n'\}$ and $\{z_n''\}$ are both Cauchy sequences. Thus, there exists $z'\in E_1$ and $z''\in E_2$ such that $z_n'\to z'$ and $z_n''\to z''$ as $n\to\infty$.

From (3.4.29) and (3.4.32), it follows that $\{x_n\}$ and $\{y_n\}$ are also Cauchy sequences, that is, there exists $x\in E_1$ and $y\in E_2$ such that $x_n\to x$ and $y_n\to y$ as $n\to\infty$.

Also from (3.4.9) and (3.4.10), we have

$$\|u_{n+1}-u_n\|\leq D(G(x_{n+1}),G(x_n))\leq \lambda_{D_G}\|x_{n+1}-x_n\|,$$

$$\|v_{n+1}-v_n\|\leq D(F(y_{n+1}),F(y_n))\leq \lambda_{D_F}\|y_{n+1}-y_n\|,$$

and hence, $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences, so there exist $u\in E_1$ and $v\in E_2$ such that $u_n\to u$ and $v_n\to v$, respectively.

Now, we will show that $u\in G(x)$ and $v\in F(y)$. In fact, since $u_n\in G(x_n)$ and

$$\begin{aligned} d(u_n,G(x)) &\leq \max\left\{d(u_n,G(x)),\ \sup_{w_1\in A(x)} d(G(x_n),w_1)\right\}\\ &\leq \max\left\{\sup_{w_2\in G(x_n)} d(w_2,G(x)),\ \sup_{w_1\in G(x)} d(G(x_n),w_1)\right\}\\ &= D(G(x_n),\,G(x)), \end{aligned}$$

we have

$$\begin{aligned} d(u,\,G(x)) &\leq \|u-u_n\|+d(u_n,\,G(x))\\ &\leq \|u-u_n\|+D(G(x_n),\,G(x))\\ &\leq \|u-u_n\|+\lambda_{D_G}\|x_n-x\|\to 0 \ \text{ as } \ n\to\infty. \end{aligned}$$

Which implies that $d(u, G(x)) = 0$. Since $G(x) \in CB(E)$, it follows that $u \in G(x)$. Similarly, we can show that $v \in F(y)$. By continuity of f, g, p, q, H_1, H_2, G, F, M, N, S, T, $J^M_{H_1,\rho}(\cdot, x)$, $J^N_{H_2,\gamma}(\cdot, y)$ and Algorithm 3.4.1, we have

$$z' = H_1(f(x)) - \rho S(x - p(x), v) = H_1(J^M_{H_1,\rho}(\cdot, x)(z')) - \rho S(x - p(x), v) \in E_1,$$

and

$$z'' = H_2(g(y)) - \gamma T(u, y - q(y)) = H_2(J^N_{H_2,\gamma}(\cdot, y)(z'')) - \gamma T(u, y - q(y)) \in E_2.$$

By Proposition 3.4.1, (z', z'', x, y, u, v) is a solution of Problem (3.4.1). This completes the proof.

Chapter 4

Generalized Variational Inclusions For Fuzzy Mappings

4.1. Introduction

It is well known that the fuzzy set theory, which was introduced by *Zadeh* [128] in 1965, has gained importance in analysis from both theoritical and practical point of view. Applications of the fuzzy set theory can be found in many branches of mathematical and engineering sciences; see [23,27,43,47,66,91,99,100,130]. Variational inequality theory provides us a unified frame work for dealing with a wide class of problems arising in elasticity, structural analysis, economics, physical and engineering sciences, etc; see [28,31,37,40,42,84,89,118] and references therein.

In 1989, *Chang and Zhu* [27] first introduced the concept of variational inequalities for fuzzy mappings and extended some of the results of *Lassonade* [80], *Shih and Tan* [109], *Takahashi* [113] and *Yen* [127] in the fuzzy setting. They investigated existence theorems for some kinds of variational inequalities for fuzzy mappings, which were the fuzzy extensions of some theorems in [109,113].

Several classes of variational inequalities and complementarity problems for fuzzy mappings were considered and studied by *Chang and Haung* [23], *Noor* [91], *Haung* [66], *Park and Jeoug* [99,100], *Ding and Park* [43] and *Ding* [38,39] in Hilbert spaces.

In Section 4.2, we study generalized variational-like inclusions for fuzzy mappings. We develop an Ishikawa type perturbed iterative algorithm and a Mann type perturbed iterative algorithm for computing the approximate solutions of generalized variational inclusions for fuzzy mappings.

In Section 4.3, we consider a class of mixed variational inclusions for fuzzy mappings. The existence and convergence analysis is discussed by using the definition of relaxed strongly accretive operators.

In Section 4.4, we introduce the generalized T-resolvent equations with fuzzy mappings in connection with the mixed variational inclusions for fuzzy mappings discussed in Section 4.3. An equivalence relation is established between the mixed variational inclusions for fuzzy mappings and the generalized T-resolvent equations with fuzzy mappings. Further, we prove the existence of solutions and the convergence of iterative sequences generated by the algorithm.

We assume that H is a Hilbert space with norm $\|.\|$ and inner product $\langle .,. \rangle$. $\mathcal{F}(H)$ denotes the collection of all *fuzzy set* over H. A mapping $F : H \to \mathcal{F}(H)$ is said to be *fuzzy mapping.* For each $x \in H$, $F(x)$ (denote it by F_x, in the sequel) is a fuzzy set on H and $F_x(y)$ is the membership function of y in F_x.

A fuzzy mapping $F : E \to \mathcal{F}(E)$ is said to be *closed* if for each $x \in E$, the function $y \to F_x(y)$ is upper semicontinuous i.e., for any given net $\{y_\alpha\} \subset H$ satisfying $y_\alpha \to y_0 \in E$, $\lim \sup_\alpha F_x(y_\alpha) \leq F_x(y_0)$.

For $A \in \mathcal{F}(E)$ and $\lambda \in [0,1]$, the set $(A)_\lambda = \{x \in E : Ax \geq \lambda\}$ is called a *λ-cut set* of A.

Let $A : E \to \mathcal{F}(E)$ is a closed fuzzy mapping satisfying the following condition (I):

Condition (I). There exists a function $a : E \to [0,1]$ such that for each $x \in E, (A_x)_{a(x)}$ is a nonempty bounded subset of E.

It is clear that if A is closed fuzzy mapping satisfying the condition (I), then for each $x \in E$, the set $(A_x)_{a(x)} \in CB(E)$.

Definition 4.1.1. A mapping $g : H \to H$ is said to be:

(i) *monotone*, if for all $x, y \in H$

$$\langle g(x) - g(y), x - y \rangle \geq 0;$$

(ii) *strictly monotone*, if for all $x, y \in H$

$$\langle g(x) - g(y), x - y \rangle \geq 0;$$

and equality holds if and only if $x = y$;

(iii) *strongly monotone*, if for all $x, y \in H$ there exists a constant $\delta > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geq \delta \|x - y\|^2.$$

Definition 4.1.2. Let E be a q-uniformly smooth real Banach space let $B : E \to CB(E)$ be a multivalued mapping. The mapping $t : E \to E$ is said to be *relaxed strongly accretive* with respect to B, if there exists a constant $k \geq 0$ such that

$$\langle t(u) - t(v), j_q(x - y) \rangle \leq -k\|x - y\|^q, \text{ for all} x, y \in E, \ u \in B(x), \ v \in B(y).$$

Example 4.1.1. Let $E = R$, $B = I$, the identity mapping. Let $t(x) = -2x$, $\epsilon \geq 0$, $k = (2 - \epsilon)$. Then it is easy to see that t is relaxed strongly accretive mapping.

4.2. Generalized Variational-like Inclusions For Fuzzy Mappings

In this section, we consider a class of generalized variational-like inclusions for fuzzy mappings. First, we established an equivalence of generalized variational-like inclusions for fuzzy mappings with some fixed point problems and develop Ishikawa type perturbed iterative algorithm and a Mann type perturbed iterative algorithm for this class of generalized variational-like inclusions for fuzzy mappings. The existence and convergence for our problem is also discussed.

Let $M, S, T : H \to \mathcal{F}(H)$ be three fuzzy mappings, $m, f, g, P : H \to H$ and $\eta : H \times H \to H$ be the single-valued mappings. We consider the following *generalized variational-like inclusion problem for fuzzy mappings:*

Find $x \in H$, $u \in (M(x))_r$, $v \in (S(x))_r$, $w \in (T(x))_r$, $r \in (0, 1]$ such that
$x \in \text{dom } \varphi$ and

$$\langle P(u) - (f(v) - m(w)), \eta(y, g(x)) \rangle \ \geq \ \varphi(g(x)) - \varphi(y), \quad \text{for all } y \in H \quad (4.2.1)$$

where $\varphi : H \to R \cup \{+\infty\}$ and dom $\varphi = \{x \in H : \varphi(x) < \infty\}$.

Some special cases:

(i) If $m \equiv 0$, $\eta(y, g(x)) = y - g(x)$ and P, f and T are identity mappings, then Problem (4.2.1) is equivalent to finding $x \in H$, $u \in (M(x))_r$, $v \in (S(x))_r$, $r \in (0,1]$ such that $g(x) \cap \text{dom } \partial\varphi \neq \phi$ and

$$\langle u - v, y - g(x) \rangle \geq \varphi(g(x)) - \varphi(y), \quad \text{for all } y \in H. \tag{4.2.2}$$

Problem (4.2.2) is called variational inclusion problem for fuzzy mappings which is considered and studied by *Park and Jeong* [99].

(ii) If $\varphi = \delta_K$, the indicator function of the nonempty closed convex set K in H, then Problem (4.2.2) is equivalent to finding $x \in H$, $u \in (M(x))_r$, $v \in (S(x))_r$, $r \in (0,1]$ such that $g(x) \in K$ and

$$\langle u - v, y - g(x) \rangle \geq 0, \quad \text{for all } y \in H, \tag{4.2.3}$$

which is called completely generalized strongly variational inequality problem for fuzzy mappings.

Assumption (U). The mapping $\eta : H \times H \to H$ satisfies the condition

$$\eta(y, x) + \eta(x, y) = 0, \qquad \text{for all } x, y \in H.$$

Remark 4.2.1. If $\eta : H \times H \to H$ satisfies Assumption (U) and $\varphi : H \to R \cup \{+\infty\}$, then it is easy to see that the η-subdifferential mapping $\partial_\eta \varphi : H \to 2^H$ is η-monotone.

We need the following result due to *Lee et al.* [81] to transform our problem *generalized variational-like inclusion problem for fuzzy mappings* (4.2.1) into a fixed point problem.

Proposition 4.2.1. Let $\eta : H \times H \to H$ be a strictly monotone mapping and $Q : H \to 2^H$ an η-monotone multivalued mapping. If the range $(I + \lambda Q)$, $R(I + \lambda Q) = H$, for $\lambda > 0$ and I is the identity mapping, then Q is maximal η-monotone. Further, the inverse mapping $(I + \lambda Q)^{-1} : H \to H$ is single-valued.

We assume that $\eta : H \times H \to H$ is strictly monotone and satisfies Assumption (U) and $\varphi : H \to R \cup \{+\infty\}$ is a mapping such that $R(I + \lambda \partial_\eta \varphi) = H$, for $\lambda > 0$.

From Proposition 4.2.1, we note that the mapping

$$J_\lambda^\varphi(x) \;=\; (I+\lambda\partial_\eta\varphi)^{-1}(x), \qquad \text{for all } \; x\in H,$$

is single-valued.

Lemma 4.2.1. (x,u,v,w), where $x\in H$, $u\in (M(x))_r$, $v\in (S(x))_r$, $w\in (T(x))_r$ is a solution of *generalized variational-like inclusion problem for fuzzy mappings* (4.2.1) if and only if it satisfies

$$g(x) \;=\; J_\lambda^\varphi[g(x)-\lambda(P(u)-(f(v)-m(w)))], \tag{4.2.4}$$

where $\lambda>0$ is a constant, $J_\lambda^\varphi=(I+\lambda\partial_\eta\varphi)^{-1}$ is so-called proximal mapping and I stands for the identity operator on H.

Proof. From the definition of J_λ^φ, we have

$$g(x)-\lambda(P(u)-(f(v)-m(w)))\in g(x)+\lambda\partial_\eta\varphi(g(x))$$

and hence

$$(f(v)-m(w))-P(u)\in\partial_\eta\varphi(g(x)).$$

By using the definition of η-subdifferential, we have

$$\langle (f(v)-m(w))-P(u),\eta(y,g(x))\rangle \;\le\; \varphi(y)-\varphi(g(x)), \qquad \text{for all } \; y\in H.$$

Thus, (x,u,v,w) is a solution of *generalized variational-like inclusion problem for fuzzy mappings* (4.2.1).

From the above Lemma 4.2.1, we see that *generalized variational-like inclusion problem for fuzzy mappings* (4.2.1) is equivalent to the fixed point problem of type

$$x\in N(x),$$

where

$$N(x)=x-g(x)+J_\lambda^\varphi[g(x)-\lambda(P(u)-(f(v)-m(w)))]. \tag{4.2.5}$$

Using this fixed point formulation, we suggest the following perturbed Ishikawa type and Mann type iterative algorithms.

Ishikawa Type Perturbed Iterative Algorithm:

Let $M, S, T : H \to \mathcal{F}(H)$ and $g, P, f, m : H \to H$. For any $x_0 \in H$, the iterative scheme is defined by $u_n \in (M(x_n))_r$, $v_n \in (S(x_n))_r$, $w_n \in (T(x_n))_r$, $\bar{u}_n \in (M(y_n))_r$, $\bar{v}_n \in (S(y_n))_r$, $\bar{w}_n \in (T(y_n))_r$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n[y_n - g(y_n) + J_\lambda^{\varphi_n}(g(y_n) - \lambda(P(\bar{u}_n) - (f(\bar{v}_n) - (m(\bar{w}_n)))))] + e_n,$$

$$y_n = (1-\beta_n)x_n + \beta_n[x_n - g(x_n) + J_\lambda^{\varphi_n}(g(x_n) - \lambda(P(u_n) - (f(v_n) - (m(w_n)))))] + \beta_n r_n,$$

for $n \geq 0$, where e_n and r_n in H, for all $n \geq 0$ are errors, $\{\varphi_n\}$ is the sequence approximating φ, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying $\alpha_0 = 1$, $0 \leq \alpha_n, \beta_n \leq 1$, for $n \geq 0$, $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$ and $\lambda > 0$ is a constant.

If $\beta_n = 0$, for all $n \geq 0$ in Ishikawa type perturbed iterative algorithm, then we have the following Mann type perturbed iterative algorithm.

Mann Type Perturbed Iterative Algorithm:

Let $M, S, T : H \to \mathcal{F}(H)$ and $g, P, f, m : H \to H$. For any $x_0 \in H$, the iterative scheme is defined by $u_n \in (M(x_n))_r$, $v_n \in (S(x_n))_r$, $w_n \in (T(x_n))_r$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n[x_n - g(x_n) + J_\lambda^{\varphi_n}(g(x_n) - \lambda(P(u_n) - (f(v_n) - (m(w_n)))))] + e_n,$$

for $n \geq 0$, where $\{\alpha_n\}$ is a sequence satisfying $\alpha_0 = 1$, $0 \leq \alpha_n \leq 1$, for $n > 0$ and $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, $e_n \in H$, for all n, is an error which is taken into account for a possible inexact computation of the proximal point, $\{\varphi_n\}$ is the sequence approximating φ and $\lambda > 0$ is a constant.

We need the following lemma due to *Lee et al.* [81], to prove the main result.

Lemma 4.2.2. Let $\eta : H \times H \to H$ be strongly monotone and Lipschitz continuous with constants $\sigma > 0$ and $r > 0$, respectively, and satisfy the Assumption (U). Then

$$\|J_\lambda^\varphi(x) - J_\lambda^\varphi(y)\| \leq \tau\|x - y\|, \qquad \text{for all } x, y \in H$$

where $\tau = \dfrac{r}{\sigma}$.

We prove the existence of solutions of *generalized variational-like inclusion problem for fuzzy mappings* (4.2.1) and the convergence of the iterative sequences generated by the Ishikawa type perturbed iterative algorithm is discussed.

Theorem 4.2.1. Let $\eta : H \times H \to H$ be strongly monotone and Lipschitz continuous with constants $\sigma > 0$ and $r > 0$, respectively and satisfy assumption (U). Let $M, S, T : H \to \mathcal{F}(H)$ be Lipschitz continuous with corresponding constants γ, h and d, respectively, S be relaxed Lipschitz with respect to f with constant k and T be relaxed monotone with respect to m with constant c. Let $g, f, m, P : H \to H$ be Lipschitz continuous with corresponding constants l_g, l_f, l_m and l_P, respectively and g is strongly monotone with constant $\delta > 0$. For each n, let $\varphi_n : H \to R \cup \{+\infty\}$ and $\varphi : H \to R \cup \{+\infty\}$ be mappings such that $R(I + \lambda\partial_\eta\varphi_n) = R(I + \lambda\partial_\eta\varphi) = H$, for $\lambda > 0$.

If

$$\left|\lambda - \frac{\tau^2(k-c) - \tau l_P\gamma(1-L)}{\tau^2(l_f h + l_m d)^2 - \tau^2 l_P^2\gamma^2}\right|$$
$$< \frac{\sqrt{[\tau^2(c-k) + \tau l_P\gamma(1-L)]^2 - [(\tau^2(l_f h + l_m d)^2 - \tau^2 l_P^2\gamma^2)(\tau^2 - 1 - L^2 + 2L)]}}{\tau^2(l_f h + l_m d)^2 - \tau^2 l_P^2\gamma^2}, \tag{4.2.6}$$

$$\tau^2(c-k) > \tau l_P\gamma(L-1) + \sqrt{(\tau^2(l_f h + l_m d)^2 - \tau^2 l_P^2\gamma^2)(\tau^2 - 1 - L^2 + 2L)},$$

$$l_f h + l_m d \;>\; l_P\gamma,$$

for $L = (1+\tau)\sqrt{1 - 2\delta + l_g^2} < 1$, then $(x^\star, u^\star, v^\star, w^\star)$ is a solution of *generalized variational-like inclusion problem for fuzzy mappings* (4.2.1).

Moreover, if

$$\lim_{n\to\infty} \|J_\lambda^{\varphi_n}(z) - J_\lambda^{\varphi}(z)\| = 0, \qquad \text{for all } z \in H,$$

and $\{x_n\}$, $\{\bar{u}_n\}$, $\{\bar{v}_n\}$ and $\{\bar{w}_n\}$ are defined by Ishikawa type perturbed iterative algorithm with conditions:

(i) $\lim_{n\to\infty} \|e_n\| = 0 = \lim_{n\to\infty} \|r_n\| = 0$ and

(ii) $\sum_{i=0}^{n} \prod_{j=i+1}^{n} (1 - \alpha_j(1-c))$ converges, $0 \le c < 1$.

Then $\{x_n\}$, $\{\bar{u}_n\}$, $\{\bar{v}_n\}$ and $\{\bar{w}_n\}$ strongly converge to x^*, u^*, v^* and w^*, respectively.

Proof. Define a multivalued mapping $F : H \to 2^H$ by

$$F(x) = \bigcup_{u\in(M(x))_r, v\in(S(x))_r, w\in(T(x))_r} [x - g(x) + J_\lambda^\varphi(g(x) - \lambda(P(u) - (f(v) - m(w))],$$

for each $x \in H$. For any $x, y \in H$, $a \in F(x)$, $b \in F(y)$, there exist $u_1 \in (M(x))_r$, $v_1 \in (S(x))_r$, $w_1 \in (T(x))_r$, $u_2 \in (M(y))_r$, $v_2 \in (S(y))_r$, $w_2 \in (T(y))_r$ such that

$$a = x - g(x) + J_\lambda^\varphi[g(x) - \lambda(P(u_1) - (f(v_1) - m(w_1)))],$$

$$b = y - g(y) + J_\lambda^\varphi[g(y) - \lambda(P(u_2) - (f(v_2) - m(w_2)))].$$

By Lemma 4.2.2, it follows that

$$\begin{aligned} \|a-b\| &= \|x - g(x) + J_\lambda^\varphi[g(x) - \lambda(P(u_1) - (f(v_1) - m(w_1)))] \\ &\quad - \{y - g(y) + J_\lambda^\varphi[g(y) - \lambda(P(u_2) - (f(v_2) - m(w_2)))]\}\| \\ &\leq \|x - y - (g(x) - g(y))\| + \tau\|g(x) - \lambda(P(u_1) - (f(v_1) - m(w_1))) \\ &\quad - [g(y) - \lambda(P(u_2) - (f(v_2) - m(w_2)))]\| \\ &\leq (1+\tau)\|x - y - (g(x) - g(y))\| + \tau\|x - y + \lambda(f(v_1) - f(v_2)) \\ &\quad - \lambda(m(w_1) - m(w_2))\| + \tau\lambda\|P(u_1) - P(u_2)\|. \end{aligned} \tag{4.2.7}$$

By using the Lipschitz continuity and the strong monotonicity of g, we have

$$\begin{aligned} \|x - y - (g(x) - g(y))\|^2 &= \|x-y\|^2 - 2\langle x - y, g(x) - g(y)\rangle + \|g(x) - g(y)\|^2 \\ &\leq (1 - 2\delta + l_g^2)\|x-y\|^2. \end{aligned} \tag{4.2.8}$$

Since M, S and T are Lipschitz continuous and f, m and P are Lipschitz continuous, we have

$$\|P(u_1) - P(u_2)\| \leq l_P\|u_1 - u_2\| \leq l_P\gamma\|x-y\|, \tag{4.2.9}$$

$$\|f(v_1) - f(v_2)\| \leq l_f\|v_1 - v_2\| \leq l_f h\|x-y\|, \tag{4.2.10}$$

$$\|m(w_1) - m(w_2)\| \leq l_m\|w_1 - w_2\| \leq l_m d\|x-y\|. \tag{4.2.11}$$

Further, since S is relaxed Lipschitz and T is relaxed monotone, we have

$$\|x - y + \lambda(f(v_1) - f(v_2)) - \lambda(m(w_1) - m(w_2))\|^2$$
$$= \|x-y\|^2 + 2\lambda\langle f(v_1) - f(v_2), x - y\rangle - 2\lambda\langle mw_1 - mw_2, x - y\rangle$$
$$+\lambda^2\|f(v_1) - f(v_2) - (m(w_1) - m(w_2))\|^2$$
$$\leq [1 - 2\lambda(k - c) + \lambda^2(l_f h + l_m d)^2]\|x - y\|^2. \tag{4.2.12}$$

By (4.2.7)-(4.2.12), we obtain

$$\|a - b\| \leq \left[(1+\tau)\sqrt{1 - 2\delta + l_g^2} + \tau\sqrt{1 - 2\lambda(k-c) + \lambda^2(l_f h + l_m d)^2} \right.$$
$$\left. + \tau\lambda l_P \gamma\right] \|x - y\|$$
$$= \theta\|x - y\|,$$

where $\theta = (1 + \tau\sqrt{1 - 2\delta + l_g^2} + \tau\sqrt{1 - 2\lambda(k-c) + \lambda^2(l_f h + l_m d)^2} + \tau\lambda l_P\gamma$.

It follows from condition (4.2.6) that $0 < \theta < 1$. Since $a \in F(x)$, $b \in F(y)$ are arbitrary, we obtain

$$H(F(x), F(y)) \leq \theta\|x - y\|, \qquad \text{for all } x, y \in H. \tag{4.2.13}$$

It follows from (4.2.13) and by Theorem 3.1 of *Siddiqi and Ansari* [110] that F has a fixed point $x^* \in H$ i.e., $x^* \in F(x^*)$. By the definition of F, there exist $u^* \in (M(x^*))_r$, $v^* \in (S(x^*))_r$, $w^* \in (T(x^*))_r$ such that (x^*, u^*, v^*, w^*) is a solution of *generalized variational-like inclusion problem for fuzzy mappings* (4.2.1).

Next we prove that the iterative sequences $\{x_n\}$, $\{\bar{u}_n\}$, $\{\bar{v}_n\}$ and $\{\bar{w}_n\}$ defined by Ishikawa type perturbed iterative algorithm strongly converges to x^*, u^*, v^* and w^*, respectively.

Since *generalized variational-like inclusion problem for fuzzy mappings* (4.2.1) has a solution (x^*, u^*, v^*, w^*) then, by Lemma 4.2.1, we have

$$x^* = x^* - g(x^*) + J_\lambda^\varphi[g(x^*) - \lambda(P(u^*) - (f(v^*) - m(w^*)))].$$

Further, since S is relaxed Lipschitz and T is relaxed monotone, we have

$$\begin{aligned}
&\|x-y+\lambda(f(v_1)-f(v_2))-\lambda(m(w_1)-m(w_2))\|^2\\
&\qquad = \|x-y\|^2+2\lambda\langle f(v_1)-f(v_2),x-y\rangle-2\lambda\langle mw_1-mw_2,x-y\rangle\\
&\qquad\qquad +\lambda^2\|f(v_1)-f(v_2)-(m(w_1)-m(w_2))\|^2\\
&\qquad \leq [1-2\lambda(k-c)+\lambda^2(l_fh+l_md)^2]\|x-y\|^2. \qquad (4.2.12)
\end{aligned}$$

By (4.2.7)-(4.2.12), we obtain

$$\begin{aligned}
\|a-b\| &\leq \Bigg[(1+\tau)\sqrt{1-2\delta+l_g^2}+\tau\sqrt{1-2\lambda(k-c)+\lambda^2(l_fh+l_md)^2}\\
&\qquad\qquad +\tau\lambda l_P\gamma\Bigg]\|x-y\|\\
&= \theta\|x-y\|,
\end{aligned}$$

where $\theta=(1+\tau\sqrt{1-2\delta+l_g^2}+\tau\sqrt{1-2\lambda(k-c)+\lambda^2(l_fh+l_md)^2}+\tau\lambda l_P\gamma$.

It follows from condition (4.2.6) that $0<\theta<1$. Since $a\in F(x)$, $b\in F(y)$ are arbitrary, we obtain

$$H(F(x),F(y)) \leq \theta\|x-y\|, \qquad \text{for all } x,y\in H. \qquad (4.2.13)$$

It follows from (4.2.13) and by Theorem 3.1 of *Siddiqi and Ansari* [110] that F has a fixed point $x^*\in H$ i.e., $x^*\in F(x^*)$. By the definition of F, there exist $u^*\in(M(x^*))_r$, $v^*\in(S(x^*))_r$, $w^*\in(T(x^*))_r$ such that (x^*,u^*,v^*,w^*) is a solution of *generalized variational-like inclusion problem for fuzzy mappings* (4.2.1).

Next we prove that the iterative sequences $\{x_n\}$, $\{\bar{u}_n\}$, $\{\bar{v}_n\}$ and $\{\bar{w}_n\}$ defined by Ishikawa type perturbed iterative algorithm strongly converges to x^*, u^*, v^* and w^*, respectively.

Since *generalized variational-like inclusion problem for fuzzy mappings* (4.2.1) has a solution (x^*,u^*,v^*,w^*) then, by Lemma 4.2.1, we have

$$x^*=x^*-g(x^*)+J_\lambda^\varphi[g(x^*)-\lambda(P(u^*)-(f(v^*)-m(w^*)))].$$

By making use of the same arguments used for obtaining (4.2.8), (4.2.9) and (4.2.12), we get

$$\|y_n - x^* - (g(y_n) - g(x^*))\| \leq \sqrt{1 - 2\delta + l_g^2}\|y_n - x^*\|,$$

$$\|P(\bar{u}_n) - P(u^*)\| \leq l_P\gamma\|y_n - x^*\|,$$

$$\begin{aligned}&\|y_n - x^* + \lambda(f(\bar{v}_n) - f(v^*)) - \lambda(m(\bar{w}_n) - m(w^*))\| \\ &\qquad\leq \sqrt{1 - 2\lambda(k - c) + \lambda^2(l_f h + l_m d)^2}\ \|y_n - x^*\|.\end{aligned}$$

By setting

$$h(u^*) = g(x^*) - \lambda(P(u^*) - (f(v^*) - m(w^*)))$$

and

$$h(y_n) = g(y_n) - \lambda(P(\bar{u}_n) - (f(\bar{v}_n) - m(\bar{w}_n))).$$

We have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|(1 - \alpha_n)x_n + \alpha_n[y_n - g(y_n) + J_\lambda^{\varphi_n}(h(y_n))] + e_n \\ &\qquad -(1 - \alpha_n)x^* - \alpha_n[x^* - g(x^*) + J_\lambda^{\varphi}(h(x^*))]\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|y_n - x^* - (g(y_n) - g(x^*))\| \\ &\qquad +\alpha_n\|J_\lambda^{\varphi_n}(h(y_n)) - J_\lambda^{\varphi}(h(x^*))\| + \|e_n\|. \qquad (4.2.14)\end{aligned}$$

By Lemma 4.2.2, we have

$$\begin{aligned}&\|J_\lambda^{\varphi_n}(h(y_n)) - J_\lambda^{\varphi}(h(x^*))\| \\ &\quad= \|J_\lambda^{\varphi_n}(h(y_n)) - J_\lambda^{\varphi_n}(h(x^*)) + J_\lambda^{\varphi_n}(h(x^*)) - J_\lambda^{\varphi}(h(x^*))\| \\ &\quad\leq \tau\|h(y_n) - h(x^*)\| + \|J_\lambda^{\varphi_n}(h(x^*)) - J_\lambda^{\varphi}(h(x^*))\| \\ &\quad\leq \tau[\|y_n - x^* - (g(y_n) - g(x^*))\| + \|y_n - x^* + \lambda(f(\bar{v}_n) \\ &\qquad\quad -f(v^*)) - \lambda(m(\bar{w}_n) - m(w^*))\| + \lambda\|P(\bar{u}_n) - P(u^*)\|] \\ &\qquad\quad +\|J_\lambda^{\varphi_n}(h(x^*)) - J_\lambda^{\varphi}(h(x^*))\| \\ &\quad\leq \tau\left[\sqrt{1 - 2\delta + l_g^2} + \sqrt{1 - 2\lambda(k - c) + \lambda^2(l_f h + l_m d)^2} + \lambda l_P\gamma\right] \\ &\qquad\quad \times\|y_n - x^*\| + \|J_\lambda^{\varphi_n}(h(x^*)) - J_\lambda^{\varphi}(h(x^*))\|. \qquad (4.2.15)\end{aligned}$$

On combining (4.2.14) and (4.2.15), we get

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\sqrt{1-2\delta+l_g^2}\,\|y_n - x^*\| \\
&\quad + \alpha_n \tau \Bigg[\sqrt{1-2\delta+l_g^2} + \sqrt{1-2\lambda(k-c)+\lambda^2(l_f h + l_m d)^2} \\
&\quad + \lambda l_P \gamma \Bigg] \|y_n - x^*\| + \alpha_n \|J_\lambda^{\varphi_n}(h(x^*)) - J_\lambda^{\varphi}(h(x^*))\| + \|e_n\| \\
&= (1-\alpha_n)\|x_n - x^*\| + \alpha_n \theta \|y_n - x^*\| + \alpha_n \epsilon_n + \|e_n\|, \qquad (4.2.16)
\end{aligned}
$$

where $\theta = (1 + \tau\sqrt{1-2\delta+l_g^2} + \tau\sqrt{1-2\lambda(k-c)+\lambda^2(l_f h + l_m d)^2} + \tau\lambda l_P \gamma$ and

$$\epsilon_n = \|J_\lambda^{\varphi_n}(h(x^*)) - J_\lambda^{\varphi}(h(x^*))\|.$$

Next

$$
\begin{aligned}
\|y_n - x^*\| &= \|(1-\beta_n)x_n + \beta_n[x_n - g(x_n) + J_\lambda^{\varphi_n}(h(x_n))] + \beta_n r_n \\
&\quad - (1-\beta_n)x_n - \beta_n[x^* - g(x^*) + J_\lambda^{\varphi}(h(x^*))]\| \\
&\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \beta_n\|J_\lambda^{\varphi_n}(h(x_n)) - J_\lambda^{\varphi}(h(x^*))\| + \beta_n\|r_n\|. \qquad (4.2.17)
\end{aligned}
$$

By making use of the same arguments used for obtaining (4.2.15), we get

$$
\begin{aligned}
&\|J_\lambda^{\varphi_n}(h(x_n)) - J_\lambda^{\varphi}(h(x^*))\| \\
&\quad \leq \tau \Bigg[\sqrt{1-2\delta+l_g^2} + \sqrt{1-2\lambda(k-c)+\lambda^2(l_f h + l_m d)^2} \\
&\quad + \lambda l_P \gamma \Bigg] \|x_n - x^*\| + \epsilon_n. \qquad (4.2.18)
\end{aligned}
$$

On combining (4.2.17) and (4.2.18), we get

$$
\begin{aligned}
\|y_n - x^*\| &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n \theta \|x_n - x^*\| + \beta_n \epsilon_n + \beta_n \|r_n\| \\
&\leq (1 - \beta_n(1-\theta))\|x_n - x^*\| + \beta_n(\epsilon_n + \|r_n\|) \\
&\leq \|x_n - x^*\| + \beta_n(\epsilon_n + \|r_n\|), \qquad (4.2.19)
\end{aligned}
$$

since $(1 - \beta_n(1-\theta)) \leq 1$.

On combining (4.2.16) and (4.2.19), we get

$$\|x_{n+1} - x^*\| \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta\|x_n - x^*\|$$

$$+\alpha_n\theta\beta_n(\epsilon_n + \|r_n\|) + \alpha_n\epsilon_n + \|e_n\|$$

$$= (1-\alpha_n(1-\theta))\|x_n - x^*\| + \alpha_n\epsilon_n + \theta\alpha_n\beta_n(\epsilon_n + \|r_n\|) + \|e_n\|$$

$$\leq \prod_{i=0}^{n}(1+\alpha_i(1-\theta))\|x_0 - x^*\| + \sum_{j=0}^{n}\alpha_j \prod_{i=j+1}^{n}(1-\alpha_i(1-\theta))\epsilon_j$$

$$+\theta\sum_{j=0}^{n}\alpha_j\beta_j \prod_{i=j+1}^{n}(1-\alpha_j(1-\theta))(\epsilon_j + \|r_j\|)$$

$$+\sum_{j=0}^{n}\prod_{i=j+1}^{n}(1-\alpha_i(1-\theta))\|e_j\|, \tag{4.2.20}$$

where $\prod\limits_{i=j+1}^{n}(1-\alpha_i(1-\theta)) = 1$, when $j = n$.

Now, let B denote the lower triangular matrix with entries

$$b_{nj} = \alpha_j \prod_{i=j+1}^{n}(1-\alpha_i(1-\theta)).$$

Then B is multiplicative; see *Rhoades* [106], so that

$$\lim_{n\to\infty}\sum_{j=0}^{n}\alpha_j \prod_{i=j+1}^{n}(1-\alpha_i(1-\theta))\epsilon_j = 0,$$

$$\lim_{n\to\infty}\theta\sum_{j=0}^{n}\alpha_j\beta_j \prod_{i=j+1}^{n}(1-\alpha_i(1-\theta))(\epsilon_j + \|r_j\|) = 0.$$

Since $\lim\limits_{n\to\infty}\|r_n\| = 0$ and $\lim\limits_{n\to\infty}\epsilon_n = \|J_\lambda^{\varphi_n}(h(x^*)) - J_\lambda^{\varphi}(h(x^*))\| = 0$.

Let D be the lower triangular matrix with entries

$$d_{nj} = \prod_{i=j+1}^{n}(1-\alpha_i(1-\theta)).$$

Condition (ii) implies that D is multiplicative, and hence

$$\lim_{n\to\infty}\sum_{j=0}^{\infty}\prod_{i=j+1}^{n}(1-\alpha_i(1-\theta))\|e_j\| = 0,$$

Since $\lim_{n\to\infty} \|e_n\| = 0$. Also

$$\lim_{n\to\infty} \prod_{i=0}^{n} (1 - \alpha_i(1-\theta)) = 0,$$

since $\sum_{i=0}^{n} \alpha_i = \infty$. Hence, it follows from inequality (4.2.20) that

$$\lim_{n\to\infty} \|x_{n+1} - x^*\| = 0,$$

i.e., the sequence $\{x_n\}$ strongly converges to $x^* \in H$. Since $\bar{u}_n \in (M(y_n))_r$, $u^* \in (M(x^*))_r$ and M is Lipschitz continuous, we have

$$\|\bar{u}_n - u^*\| \ \leq \ H(M(y_n), M(x^*)) \ \leq \ \gamma\|y_n - x^*\| \to 0 \text{ as } n \to \infty.$$

i.e., $\{\bar{u}_n\}$ strongly converges to u^*. Similarly, we can prove that $\{\bar{v}_n\}$ and $\{\bar{w}_n\}$ strongly converge to v^* and w^*, respectively.

We remark that, if $\beta_n = 0$, for all $n \geq 0$, Theorem 4.2.1 gives the conditions under which the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ defined by Mann type perturbed iterative algorithm strongly converge to x^*, u^*, v^* and w^*, respectively.

4.3. Mixed Variational Inclusions With Fuzzy Mappings

In this section, we consider a class of mixed variational inclusions with fuzzy mappings in Banach spaces. The existence of solutions of mixed variational inclusions problem with fuzzy mappings is discussed and the convergence of iterative sequences generated by the proposed algorithms is also studied by using the definition of relaxed strongly accretive operators.

Let $A, B, C : E \to \mathcal{F}(E)$ are closed fuzzy mappings satisfying condition (I). Then there exists three functions $a, b, c : E \to [0,1]$ such that for each $x \in E$, we have $(Ax)_{a(x)}, (Bx)_{b(x)}, (Cx)_{c(x)} \in CB(E)$. Therefore we define multivalued mappings $\tilde{A}, \tilde{B}, \tilde{C} : E \to CB(E)$ by $\tilde{A}(x) = (Ax)_{a(x)}$, $\tilde{B}(x) = (Bx)_{b(x)}$, $\tilde{C}(x) = (Cx)_{c(x)}$, for each $x \in E$. In the sequel, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are called the multivalued mappings induced by the fuzzy mappings A, B and C, respectively.

Let $p, t, h, g, T : E \to E$ are single-valued mappings and $M : E \to 2^E$ be a T-accretive multivalued mapping. Let $A, B, C : E \to \mathcal{F}(E)$ are fuzzy mappings. Let $a, b, c : E \to [0, 1]$ be given functions. For any given $f \in E$, $\lambda > 0$, we consider the following *mixed variational inclusion problem with fuzzy mappings:*

Find x, u, v, $w \in E$ such that $A_x(u) \geq a(x)$, $B_x(v) \geq b(x)$, $C_x(w) \geq c(x)$ and

$$f \in (p(u) - (t(v) - h(w))) + \lambda M(g(x)). \tag{4.3.1}$$

We remark that for suitable choices of A, B, C, p, t, h, g and M, the *mixed variational inclusion problem with fuzzy mappings* (4.3.1) reduces to various new as well as known classes of variational inclusions and variational inequalities; see for example [50,64,65,88] and references therein.

We first transform our *mixed variational inclusions problem with fuzzy mappings* (4.3.1) into fixed point problem and then establish an iterative algorithm for finding the approximate solutions of *mixed variational inclusion problem with fuzzy mappings* (4.3.1).

Lemma 4.3.1. (x, u, v, w), where $x \in E$, $u \in \tilde{A}(x)$, $v \in \tilde{B}(x)$ and $w \in \tilde{C}(x)$ is a solution of *mixed variational inclusion problem with fuzzy mappings* (4.3.1) if and only if it satisfies

$$g(x) = J_T^{M,\rho\lambda}[\rho f + T(g(x)) - \rho(p(u) - (t(v) - h(w)))],$$

where $J_T^{M,\rho\lambda} = (T + \rho\lambda M)^{-1}$ and $\rho > 0$ is a constant.

Proof. $x \in E$, $u \in \tilde{A}(x)$, $v \in \tilde{B}(x)$, $w \in \tilde{C}(x)$ is a solution of *mixed variational inclusions problem with fuzzy mappings* (4.3.1).

$$\Leftrightarrow f \in (p(u) - (t(v) - h(w))) + \lambda M(g(x))$$

$$\Leftrightarrow \rho f \in \rho(p(u) - (t(v) - h(w))) + \rho\lambda M(g(x))$$

$$\Leftrightarrow \rho f \in -[T(g(x)) - \rho(p(u) - (t(v) - h(w)))] + (T + \rho\lambda M)(g(x))$$

$$\Leftrightarrow g(x) = J_T^{M,\rho\lambda}[\rho f + T(g(x)) - \rho(p(u) - (t(v) - h(w)))].$$

Using Lemma 4.3.1 and Nadler's theorem 1.2.3 [87] , we suggest an iterative algorithm for finding the approximate solutions of *mixed variational inclusion problem with fuzzy mappings* (4.3.1) as follows.

Algorithm 4.3.1. For any given $x_0 \in E$, we choose $u_0 \in \tilde{A}(x_0)$, $v_0 \in \tilde{B}(x_0)$ and $w_0 \in \tilde{C}(x_0)$ and compute the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ by iterative schemes as follows:

$$g(x_{n+1}) = J_T^{M,\rho\lambda}[\rho f + T(g(x_n)) - \rho(p(u_n) - (t(v_n) - h(w_n)))],$$

$$u_n \in \tilde{A}(x_n), \quad \|u_{n+1} - u_n\| \leq \left(1 + \frac{1}{1+n}\right) D(\tilde{A}(x_{n+1}), \tilde{A}(x_n)),$$

$$v_n \in \tilde{B}(x_n), \quad \|v_{n+1} - v_n\| \leq \left(1 + \frac{1}{1+n}\right) D(\tilde{B}(x_{n+1}), \tilde{B}(x_n)),$$

$$w_n \in \tilde{C}(x_n), \quad \|w_{n+1} - w_n\| \leq \left(1 + \frac{1}{1+n}\right) D(\tilde{C}(x_{n+1}), \tilde{C}(x_n)),$$

where $\rho > 0$ is a constant and $n = 0, 1, 2, \ldots\ldots..$

Theorem 4.3.1. Let E be a q-uniformly smooth Banach space and $T : E \to E$ be strongly accretive and Lipschitz continuous operator with constants γ and λ_T, respectively. Let $g, p, h : E \to E$ be both strongly accretive and Lipschitz continuous mappings with constants r, α, β and λ_g, λ_p and λ_h, respectively. Let $A, B, C : E \to \mathcal{F}(E)$ be closed fuzzy mappings satisfying Condition (I) and let $\tilde{A}, \tilde{B}, \tilde{C} : E \to CB(E)$ be the multivalued mappings induced by the fuzzy mappings A, B and C, respectively. Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are D-Lipschitz continuous mappings with constants λ_A, λ_B and λ_C, respectively. Let $t : E \to E$ be relaxed strongly accretive with respect to $\tilde{B}$ with constant k and Lipschitz continuous with constant λ_T. Suppose that $M : E \to 2^E$ be a T-accretive multivalued mapping and there exists a constant $\rho > 0$ such that

$$[1 - (\alpha - k + \beta)\rho q]\lambda_T^q \lambda_g^q + \rho^q c_q[\lambda_p \lambda_A + \lambda_t \lambda_B + \lambda_h \lambda_C]^q \leq \gamma^q r^q, \tag{4.3.2}$$

where c_q is the constant as in Proposition 1.2.2. Then the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ generated by the Algorithm 4.3.1 converge strongly to x, u, v and w, respectively and (x, u, v, w) is a solution of *mixed variational inclusion problem with fuzzy mappings* (4.3.1).

Proof. By the strong accretivity of g with constant r, we have

$$\begin{aligned}
\|g(x_{n+1})-g(x_n)\|\|x_{n+1}-x_n\|^{q-1} &= \|g(x_{n+1})-g(x_n)\|\|J_q(x_{n+1}-x_n)\| \\
&\geq \langle g(x_{n+1})-g(x_n), J_q(x_{n+1}-x_n)\rangle \\
&\geq r\|x_{n+1}-x_n\|^q,
\end{aligned}$$

which implies that

$$\|x_{n+1}-x_n\| \leq \frac{1}{r}\|g(x_{n+1})-g(x_n)\|. \tag{4.3.3}$$

It follows from Theorem 1.2.4 and Algorithm 4.3.1, that

$$\begin{aligned}
\|g(x_{n+1})-g(x_n)\| &= \|J_T^{M,\rho\lambda}[\rho f+T(g(x_n))-\rho(p(u_n)-(t(v_n)-h(w_n)))] \\
&\quad -J_T^{M,\rho\lambda}[\rho f+T(g(x_{n-1}))-\rho(p(u_{n-1})-(t(v_{n-1})-h(w_{n-1})))]\| \\
&\leq \frac{1}{\gamma}\|T(g(x_n))-\rho(p(u_n)-(t(v_n)-h(w_n)))-(T(g(x_{n-1})) \\
&\quad -\rho(p(u_{n-1})-(t(v_{n-1})-h(w_{n-1}))))\|.
\end{aligned} \tag{4.3.4}$$

Since P is Lipschitz continuous with constant λ_p and $\tilde{A}$ is D-Lipschitz continuous with constant λ_A, we have

$$\begin{aligned}
\|p(u_n)-p(u_{n-1})\| &\leq \lambda_p\|u_n-u_{n-1}\| \\
&\leq \lambda_p\left(1+\frac{1}{n}\right)D(\tilde{A}(x_n),\tilde{A}(x_{n-1})) \\
&\leq \lambda_p\lambda_A\left(1+\frac{1}{n}\right)\|x_n-x_{n-1}\|.
\end{aligned} \tag{4.3.5}$$

Since t is Lipschitz continuous with constant λ_t and $\tilde{B}$ is D-Lipschitz continuous with constant λ_B, we have

$$\begin{aligned}
\|t(v_n)-t(v_{n-1})\| &\leq \lambda_t\|v_n-v_{n-1}\| \\
&\leq \lambda_t\left(1+\frac{1}{n}\right)D(\tilde{B}(x_n),\tilde{B}(x_{n-1})) \\
&\leq \lambda_t\lambda_B\left(1+\frac{1}{n}\right)\|x_n-x_{n-1}\|.
\end{aligned} \tag{4.3.6}$$

Also, since h is Lipschitz continuous with constant λ_h and $\tilde{C}$ is D-Lipschitz continuous with constant λ_C, we have

$$\begin{aligned}
\|h(w_n) - h(w_{n-1})\| &\leq \lambda_h \|w_n - w_{n-1}\| \\
&\leq \lambda_h \left(1 + \frac{1}{n}\right) D(\tilde{C}(x_n), \tilde{C}(x_{n-1})) \\
&\leq \lambda_h \lambda_C \left(1 + \frac{1}{n}\right) \|x_n - x_{n-1}\|. \qquad (4.3.7)
\end{aligned}$$

Using (4.3.5)-(4.3.7), Proposition 1.2.2 and Lipschitz continuity of T and g with constants λ_T and λ_g, respectively, strong accretivity of p and h with constants α and β, respectively and relaxed strong accretivity of t with respect to $\tilde{B}$ with constant k, we estimate

$$\begin{aligned}
&\|T(g(x_n)) - \rho(p(u_n) - (t(v_n) - h(w_n))) - [T(g(x_{n-1})) - \rho(p(u_{n-1}) - (t(v_{n-1}) - h(w_{n-1})))]\|^q \\
&\leq \|T(g(x_n)) - T(g(x_{n-1}))\|^q \\
&\quad - \rho q \langle p(u_n) - p(u_{n-1}), J_q(T(g(x_n)) - T(g(x_{n-1}))) \rangle \\
&\quad + \rho q \langle t(v_n) - t(v_{n-1}), J_q(T(g(x_n)) - T(g(x_{n-1}))) \rangle \\
&\quad - \rho q \langle h(w_n) - h(w_{n-1}), J_q(T(g(x_n)) - T(g(x_{n-1}))) \rangle \\
&\quad + \rho^q c_q (\|p(u_n) - (t(v_n) - h(w_n))) - (p(u_{n-1}) \\
&\qquad - (t(v_{n-1}) - h(w_{n-1})))\|)^q \\
&\leq \lambda_T^q \lambda_g^q \|x_n - x_{n-1}\|^q - \rho q \alpha \lambda_T^q \lambda_g^q \|x_n - x_{n-1}\|^q \\
&\quad + \rho q k \lambda_T^q \lambda_g^q \|x_n - x_{n-1}\|^q - \rho q \beta \lambda_T^q \lambda_g^q \|x_n - x_{n-1}\|^q \\
&\quad + \rho^q c_q \left[\lambda_p \lambda_A \left(1 + \frac{1}{n}\right) + \lambda_t \lambda_B \left(1 + \frac{1}{n}\right) \right. \\
&\qquad \left. + \lambda_h \lambda_C \left(1 + \frac{1}{n}\right) \right]^q \|x_n - x_{n-1}\|^q \\
&\leq (1 - (\alpha - k + \beta) \rho q) \lambda_T^q \lambda_g^q + \rho^q c_q \left[(\lambda_p \lambda_A + \lambda_t \lambda_B + \lambda_h \lambda_C) \right. \\
&\qquad \left. \times \left(1 + \frac{1}{n}\right)^q \right] \|x_n - x_{n-1}\|^q. \qquad (4.3.8)
\end{aligned}$$

Using (4.3.4) and (4.3.8), (4.3.3) becomes

$$\|x_{n+1} - x_n\| \leq \theta_n \|x_n - x_{n-1}\|, \qquad (4.3.9)$$

where

$$\theta_n = \frac{1}{\gamma r}\sqrt[q]{[1-(\alpha-k+\beta)\rho q]\lambda_T^q\lambda_g^q + \rho^q c_q\left[(\lambda_p\lambda_A + \lambda_t\lambda_B + \lambda_h\lambda_C)\left(1+\frac{1}{n}\right)\right]^q}.$$

Letting $n \to \infty$, we see that $\theta_n \to \theta$, where

$$\theta = \frac{1}{\gamma r}\sqrt[q]{[1-(\alpha-k+\beta)\rho q]\lambda_T^q\lambda_g^q + \rho^q c_q[(\lambda_p\lambda_A + \lambda_t\lambda_B + \lambda_h\lambda_C)]^q}.$$

Since $\theta < 1$ by condition (4.3.2), $\theta_n < 1$ for n sufficiently large. Therefore (4.3.9) implies that $\{x_n\}$ is a Cauchy sequence in E and hence there exists $x \in E$ such that $x_n \to x$. By D-Lipschitz continuity of $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$, we have

$$\|u_n - u_{n-1}\| \le \left(1+\frac{1}{n}\right)\lambda_A\|x_n - x_{n-1}\|,$$

$$\|v_n - v_{n-1}\| \le \left(1+\frac{1}{n}\right)\lambda_B\|x_n - x_{n-1}\|,$$

$$\|w_n - w_{n-1}\| \le \left(1+\frac{1}{n}\right)\lambda_C\|x_n - x_{n-1}\|.$$

It follows that $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are Cauchy sequences in E. Hence there exist $u, v, w \in E$ such that $u_n \to u$, $v_n \to v$, and $w_n \to w$. Further, since $u_n \in \tilde{A}(x_n)$, we have

$$\begin{aligned} d(u, \tilde{A}(x)) &\le \|u - u_n\| + d(u_n, \tilde{A}(x)) \\ &\le \|u - u_n\| + D(\tilde{A}(x_n), \tilde{A}(x)) \\ &\le \|u - u_n\| + \lambda_A\|x_n - x\| \to 0, \end{aligned}$$

and hence $u \in \tilde{A}(x)$. Similarly $v \in \tilde{B}(x)$ and $w \in \tilde{C}(x)$. It follows from Algorithm 4.3.1 that

$$g(x) = J_T^{M,\rho\lambda}[T(g(x)) - \rho(p(u) - (t(v) - h(w)))].$$

By Lemma 4.3.1, it follows that (x, u, v, w) is a solution of Problem (4.3.1). This completes the proof.

4.4. T-resolvent Equations With Fuzzy Mappings

In this section, in connection with *mixed variational inclusion problem with fuzzy mappings* (4.3.1), we introduce the following *T-resolvent equation problem with fuzzy mappings:*

Find $z, x \in E$, $u \in \tilde{A}(x)$, $v \in \tilde{B}(x)$, $w \in \tilde{C}(x)$ such that

$$(p(u) - (t(v) - h(w))) - f + \rho^{-1} R_T^{M,\rho\lambda}(z) = 0, \tag{4.4.1}$$

where $R_T^{M,\rho\lambda} = I - T(J_T^{M,\rho\lambda})$, I is the identity operator, $J_T^{M,\rho\lambda}$ is the T-resolvent operator and $\rho > 0$ is a constant.

We establish an equivalence relation between the *mixed variational inclusion problem with fuzzy mappings* (4.3.1) and *T-resolvent equation problem with fuzzy mappings* (4.4.1). Further using this equivalence an iterative algorithm is suggested to compute the approximate solutions of *T-resolvent equation problem with fuzzy mappings* (4.4.1).

Proposition 4.4.1. The *mixed variational inclusion problem with fuzzy mappings* (4.3.1) has a solution (x, u, v, w), where $x \in E$, $u \in \tilde{A}(x)$, $v \in \tilde{B}(x)$ and $w \in \tilde{C}(x)$ if and only if *T-resolvent equation problem with fuzzy mappings* (4.4.1) has a solution (z, x, u, v, w), where $z, x \in E$, $u \in \tilde{A}(x)$, $v \in \tilde{B}(x)$ and $w \in \tilde{C}(x)$,

$$g(x) = J_T^{M,\rho\lambda}(z) \tag{4.4.2}$$

and

$$z = \rho f + T(g(x)) - \rho(p(u) - (t(v) - h(w))), \tag{4.4.3}$$

$\rho > 0$ is a constant.

Proof. Let (x, u, v, w) be a solution of *mixed variational inclusion problem with fuzzy mappings* (4.3.1). Then by Lemma 4.3.1, it is a solution of the following equation

$$g(x) = J_T^{M,\rho\lambda}[\rho f + T(g(x)) - \rho(p(u) - (t(v) - h(w)))]. \tag{4.4.4}$$

Let $z = \rho f + T(g(x)) - \rho(p(u) - (t(v) - h(w)))$, then from (4.4.4), we have

$$g(x) = J_T^{M,\rho\lambda}(z).$$

By using the fact that $R_T^{M,\rho\lambda} = I - T(J_T^{M,\rho\lambda})$, we obtain

$$\begin{aligned} z &= \rho f + T(J_T^{M,\rho\lambda}(z)) - \rho(p(u) - (t(v) - h(w))) \\ &\Leftrightarrow z - T(J_T^{M,\rho\lambda}(z)) = \rho f - \rho(p(u) - (t(v) - h(w))) \\ &\Leftrightarrow [I - T(J_T^{M,\rho\lambda})](z) = \rho f - \rho(p(u) - (t(v) - h(w))) \\ &\Leftrightarrow R_T^{M,\rho\lambda}(z) = \rho f - \rho(p(u) - (t(v) - h(w))). \end{aligned}$$

Hence

$$(p(u) - (t(v) - h(w))) - f + \rho^{-1} R_T^{M,\rho\lambda}(z) = 0.$$

Based on Proposition 4.4.1, we suggest the following iterative algorithm to compute the approximate solutions of *T-resolvent equation problem with fuzzy mappings* (4.4.1).

Algorithm 4.4.1. For any given $z_0, x_0 \in E$ we choose $u_0 \in \tilde{A}(x_0)$, $v_0 \in \tilde{B}(x_0)$ and $w_0 \in \tilde{C}(x_0)$ and compute the sequences $\{z_n\}$, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ by iterative schemes as follows:

$$g(x_{n+1}) = J_T^{M,\rho\lambda}(z_{n+1}), \tag{4.4.5}$$

$$u_n \in \tilde{A}(x_n), \quad \|u_{n+1} - u_n\| \leq \left(1 + \frac{1}{1+n}\right) D(\tilde{A}(x_{n+1}), \tilde{A}(x_n)),$$

$$v_n \in \tilde{B}(x_n), \quad \|v_{n+1} - v_n\| \leq \left(1 + \frac{1}{1+n}\right) D(\tilde{B}(x_{n+1}), \tilde{B}(x_n)),$$

$$w_n \in \tilde{C}(x_n), \quad \|w_{n+1} - w_n\| \leq \left(1 + \frac{1}{1+n}\right) D(\tilde{C}(x_{n+1}), \tilde{C}(x_n)),$$

$$z_{n+1} = \rho f + T(g(x_n)) - \rho(p(u_n) - (t(v_n) - h(w_n))),$$

where $\rho > 0$ is a constant and $n = 0, 1, 2, \ldots\ldots\ldots$

Theorem 4.4.1. Let E be a q-uniformly smooth Banach space and $T : E \to E$ be strongly accretive and Lipschitz continuous operator with constants γ and λ_T, respectively. Let $g, p, h : E \to E$ be both strongly accretive and Lipschitz continuous mappings with constants r, α, β and λ_g, λ_p and λ_h, respectively. Let $A, B, C : E \to \mathcal{F}(E)$ be closed fuzzy mappings satisfying Condition (I) and let $\tilde{A}, \tilde{B}, \tilde{C} : E \to CB(E)$ be the multivalued mappings induced by the fuzzy mappings A, B and C respectively. Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are D-Lipschitz continuous mappings with constants λ_A, λ_B and λ_C, respectively. Let $t : E \to E$ be relaxed strongly accretive with respect to $\tilde{B}$ with constant k and Lipschitz continuous with constant λ_T. Suppose that $M : E \to 2^E$ be a T-accretive multivalued mapping and there exists a constant $\rho > 0$ such that

$$[1-(\alpha-k+\beta)\rho q]\lambda_T^q\lambda_g^q + \rho^q c_q[\lambda_p\lambda_A + \lambda_t\lambda_B + \lambda_h\lambda_C]^q \leq (2\gamma\lambda_g - \gamma r)^q. \qquad (4.4.6)$$

where c_q is the constant as in Proposition 1.2.2. Then the iterative sequences $\{z_n\}, \{x_n\}, \{u_n\}, \{v_n\}$ and $\{w_n\}$ generated by the Algorithm 4.4.1 converge strongly to z, x, u, v and w, respectively and (z, x, u, v, w) is a solution of *T-resolvent equation problem with fuzzy mappings* (4.4.1).

Proof. From Algorithm 4.4.1 and (4.3.8), we have

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|\rho f + T(g(x_n)) - \rho(p(u_n) - (t(v_n) - h(w_n))) \\
&\quad -[\rho f + T(g(x_{n-1})) - \rho(p(u_{n-1}) - (t(v_{n-1}) - h(w_{n-1})))]\| \\
&= \|T(g(x_n)) - \rho(p(u_n) - (t(v_n) - h(w_n))) \\
&\quad -[(T(g(x_{n-1})) - \rho(p(u_{n-1}) - (t(v_{n-1}) - h(w_{n-1})))]\| \\
&\leq \sqrt[q]{[1-(\alpha-k+\beta)\rho q]\lambda_T^q\lambda_g^q + \rho^q c_q\left[(\lambda_p\lambda_A + \lambda_t\lambda_B + \lambda_h\lambda_C)\left(1+\frac{1}{n}\right)\right]^q} \\
&\qquad \times \|x_n - x_{n-1}\|^q.
\end{aligned}$$

From (4.3.3) and (4.4.2), we obtain

$$\begin{aligned}
\|x_n - x_{n-1}\| &\leq \frac{1}{r}\|g(x_n) - g(x_{n-1})\| \\
&= \frac{1}{r}\|g(x_n) - g(x_{n-1}) + g(x_n) - g(x_{n-1}) - J_T^{M,\rho\lambda}(z_n) + J_T^{M,\rho\lambda}(z_{n-1})\| \\
&\leq \frac{1}{r}[2\|g(x_n) - g(x_{n-1})\| - \|J_T^{M,\rho\lambda}(z_n) - J_T^{M,\rho\lambda}(z_{n-1})\|] \\
&\leq \frac{2}{r}\lambda_g\|x_n - x_{n-1}\| - \frac{1}{r\gamma}\|z_n - z_{n-1}\|,
\end{aligned}$$

since g is Lipschitz continuous with constant λ_g and $J_T^{M,\rho\lambda}$ is $\frac{1}{\gamma}$-Lipschitz continuous. Therefore, we have

$$\|x_n - x_{n-1}\| \leq \frac{1}{(2\gamma\lambda_g - \gamma r)}\|z_n - z_{n-1}\|, \tag{4.4.8}$$

combining (4.4.7) and (4.4.8), we get

$$\|z_{n+1} - z_n\| \leq k_n\|z_n - z_{n-1}\|, \tag{4.4.9}$$

where

$$k_n = \frac{1}{(2\gamma\lambda_g - \gamma r)}\sqrt[q]{[1 - (\alpha - k + \beta)\rho q]\lambda_T^q\lambda_g^q + \rho^q c_q\left[(\lambda_p\lambda_A + \lambda_t\lambda_B + \lambda_h\lambda_C)\left(1 + \frac{1}{n}\right)\right]^q}.$$

Letting $n \to \infty$, we see that $k_n \to k$, where

$$k = \frac{1}{(2\gamma\lambda_g - \gamma r)}\sqrt[q]{[1 - (\alpha - k + \beta)\rho q]\lambda_T^q\lambda_g^q + \rho^q c_q[(\lambda_p\lambda_A + \lambda_t\lambda_B + \lambda_h\lambda_C)]^q}.$$

Since $k < 1$ by condition (4.4.6), $k_n < 1$ for n sufficiently large. Therefore (4.4.9) implies that the sequence $\{z_n\}$ is a Cauchy sequence in E. So there exists $z \in E$ such that $z_n \to z$ as $n \to \infty$. From (4.4.8), we know that the sequence $\{x_n\}$ is a Cauchy sequence in E, so there exists $x \in E$ such that $x_n \to x$. Also from Algorithm 4.4.1 and D-Lipschitz continuity of $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$, we have

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq \left(1 + \frac{1}{1+n}\right) D(\tilde{A}(x_{n+1}), \tilde{A}(x_n)) \\
&\leq \left(1 + \frac{1}{1+n}\right) \lambda_A\|x_{n+1} - x_n\|, \\
\|v_{n+1} - v_n\| &\leq \left(1 + \frac{1}{1+n}\right) D(\tilde{B}(x_{n+1}), \tilde{B}(x_n)) \\
&\leq \left(1 + \frac{1}{1+n}\right) \lambda_B\|x_{n+1} - x_n\|, \\
\|w_{n+1} - w_n\| &\leq \left(1 + \frac{1}{1+n}\right) D(\tilde{C}(x_{n+1}), \tilde{C}(x_n)) \\
&\leq \left(1 + \frac{1}{1+n}\right) \lambda_C\|x_{n+1} - x_n\|,
\end{aligned}$$

and hence $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are also Cauchy sequences in E, so there exist u, v, $w \in E$ such that $u_n \to u$, $v_n \to v$ and $w_n \to w$. By using the same arguments as in the proof of Theorem 4.3.1, it is easy to see that $u \in \tilde{A}(x)$, $v \in \tilde{B}(x)$ and $w \in \tilde{C}(x)$.

Now by using the continuity of the operators T, g, p, t, h, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $J_T^{M,\rho\lambda}$ and Algorithm 4.4.1, we have

$$z = \rho f + T(g(x)) - \rho(p(u) - (t(v) - h(w))).$$

By Proposition 4.4.1, it follows that (z, x, u, v, w) is a solution of Problem (4.4.1). This completes the proof.

Chapter 5

Generalized Variational Inclusions With (A, η)-accretive Mappings

5.1. Introduction

Lan et al. [79] introduced the concept of (A, η)-accretive mappings, which generalizes the existing η-subdifferential operators, maximal η-monotone operators, generalized monotone operators (H-monotone operators), A-monotone operators, (H, η)-monotone operators, (A, η)-monotone operators in Hilbert spaces, H-accretive mappings, generalized m-accretive mappings and (H, η)-accretive mappings in Banach spaces. He also studied some properties of (A, η)-accretive mappings and defined resolvent operator associated with (A, η)-accretive mappings.

In Section 5.2, we introduce and study a system of generalized variational inclusions with (A, η)-accretive mappings which is more general than a system of variational inclusions recently considered by *Jin* [71]. By using the resolvent operator technique associated with (A, η)-accretive mappings, we define a new iterative algorithm for computing the approximate solutions of system of generalized variational inclusions.

5.2. System Of Generalized Variational Inclusions

Let $g : E \to E$, $N, S : E \times E \to E$; $A_i : E \to E$, $\eta_i : E \times E \to E$ $(i = 1, 2)$ be nonlinear mappings and $T, F : E \to CB(E)$ be set-valued mappings. Suppose $M_i : E \to 2^E$ be (A_i, η_i)-accretive mappings $(i = 1, 2)$. We consider the following *system of generalized variational inclusions with (A, η)-accretive mappings*:

Find $x, y \in E$, $u \in F(y)$, $v \in T(x)$ such that

$$0 \in A_1(g(x)) - A_1(g(y)) + \rho_1[N(y,u) + M_1(g(x))],$$

$$0 \in A_2(g(y)) - A_2(g(x)) + \rho_2[S(v,x) + M_2(g(y))], \tag{5.2.1}$$

where $\rho_i > 0$ $(i = 1, 2)$ are two constants.

Some special cases:

(i) If T, F are single-valued mappings and $N(y, \cdot) = T(y)$, $S(\cdot, x) = T(x)$, then Problem (5.2.1) reduces to the problem of finding $x, y \in E$ such that

$$0 \in A_1(g(x)) - A_1(g(y)) + \rho_1[T(y) + M_1(g(x))],$$

$$0 \in A_2(g(y)) - A_2(g(x)) + \rho_2[T(x) + M_2(g(y))]. \tag{5.2.2}$$

Problem (5.2.2) is very recently considered by *Jin* [71].

(ii) If $A_1{=}A_2{=}H$, $M_1{=}M_2{=}M$, $\rho_1 = \rho$, $\rho_2 = \lambda$, $M : E \to 2^E$ be H-accretive mapping and N and S are same as in (5.2.2), then Problem (5.2.1) reduces to the problem of finding $x, y \in E$ such that

$$0 \in H(g(x)) - H(g(y)) + \rho[T(y) + M(g(x))],$$

$$0 \in H(g(y)) - H(g(x)) + \lambda[T(x) + M(g(y))]. \tag{5.2.3}$$

Problem (5.2.3) was introduced and studied by *He et al.* [62].

It is easy to see that the *system of generalized variational inclusions with* (A, η)-*accretive mappings* (5.2.1) includes many more known variational inclusions and system of variational inclusions considered and studied in recent past.

By using the resolvent operator technique associated with (A, η)-accretive mappings, we define an iterative algorithm for computing the approximate solutions of *system of generalized variational inclusions with* (A, η)-*accretive mappings* (5.2.1).

Lemma 5.2.1. (x, y, u, v), where $x, y \in E$, $u \in F(y)$, $v \in T(x)$ is a solution of *system of generalized variational inclusions with* (A, η)-*accretive mappings* (5.2.1) if and only if (x, y, u, v) satisfies

$$g(x) = J^{\rho_1,A_1}_{\eta_1,M_1}[A_1(g(y)) - \rho_1(N(y,u))], \qquad \rho_1 > 0, \tag{5.2.4}$$

where

$$g(y) = J^{\rho_2,A_2}_{\eta_2,M_2}[A_2(g(x)) - \rho_2(S(v,x))], \qquad \rho_2 > 0. \tag{5.2.5}$$

Proof. The conclusion follows directly from the Definition 1.2.24.

Based on Lemma 5.2.1 we construct the following iterative algorithm for solving the *system of generalized variational inclusions with* (A,η)*-accretive mappings* (5.2.1).

Algorithm 5.2.1. For any given $x_0, y_0 \in E$, we choose $u_0 \in F(y_0)$, $v_0 \in T(x_0)$ and $0 < \epsilon < 1$ and compute the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by iterative schemes as follows:

$$x_{n+1} = x_n - g(x_n) + J^{\rho_1,A_1}_{\eta_1,M_1}[A_1(g(y_n)) - \rho_1(N(y_n,u_n))], \tag{5.2.6}$$

where

$$g(y_n) = J^{\rho_2,A_2}_{\eta_2,M_2}[A_2(g(x_n)) - \rho_2(S(v_n,x_n))], \tag{5.2.7}$$

and choose $u_{n+1} \in F(y_{n+1})$ and $v_{n+1} \in T(x_{n+1})$ such that

$$\|u_n - u_{n+1}\| \leq D(F(y_n), F(y_{n+1})) + \epsilon^{n+1}\|y_n - y_{n+1}\|, \tag{5.2.8}$$

$$\|v_n - v_{n+1}\| \leq D(T(x_n), T(x_{n+1})) + \epsilon^{n+1}\|x_n - x_{n+1}\|. \tag{5.2.9}$$

$n = 0, 1, 2, \ldots\ldots\ldots\ldots$

Remark 5.2.1. If $\epsilon = 0$, $N(y_n, \cdot) = T(y_n)$, $S(\cdot, x_n) = T(x_n)$, then our Algorithm 5.2.1 reduces to the Algorithm 3.1 of *Jin* [71].

We prove the existence of a solution of *system of generalized variational inclusions with* (A,η)*-accretive mappings* (5.2.1) and study the convergence of iterative sequences generated by Algorithm 5.2.1.

Theorem 5.2.1. Let E be a q-uniformly smooth Banach space. Let $\eta_i : E \times E \to E$ be Lipschitz continuous mappings with constants τ_i, $A_i : E \to E$ be r_i-strongly η_i-accretive and Lipschitz continuous mappings with constants λ_{A_i} and $M_i : E \to 2^E$ be (A_i,η_i)-accretive mappings for $i = 1, 2$. Let $N, S : E \times E \to E$ be

Lipschitz continuous mappings in both the arguments with constants λ_{N_1}, λ_{N_2}; λ_{S_1}, λ_{S_2}, respectively and $F, T : E \to CB(E)$ be D-Lipschitz continuous mapping with constants λ_{D_F} and λ_{D_T}, respectively. Suppose $g : E \to E$ is Lipschitz continuous with constant λ_g and strongly accretive mapping with constant δ_g. If there exist constants $\rho_1 \in (0, \frac{r_1}{m_1})$ and $\rho_2 \in (0, \frac{r_2}{m_2})$ such that

$$[\lambda_{A_1}\lambda_g + \rho_1(\lambda_{N_1} + \lambda_{N_2}\lambda_{D_F})][\lambda_{A_2}\lambda_g + \rho_2(\lambda_{S_1}\lambda_{D_T} + \lambda_{S_2})] < \frac{[1 - \sqrt[q]{1 - q\delta_g + C_q\lambda_g^q}[\delta_g(r_1 - \rho_1 m_1)(r_2 - \rho_2 m_2)]}{(\tau_1\tau_2)^{q-1}}, \tag{5.2.10}$$

where C_q is the constant as in Proposition 1.2.2. Then the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ generated by Algorithm 5.2.1 converge strongly to x, y, u and v in E, respectively and (x, y, u, v) is a solution of *system of generalized variational inclusions with (A, η)-accretive mappings* (5.2.1).

Proof. From Algorithm 5.2.1 and using the Lemma 1.2.4, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|x_n - g(x_n) + J^{\rho_1, A_1}_{\eta_1, M_1}[A_1(g(y_n)) - \rho_1(N(y_n, u_n))] \\
&\quad -\{x_{n-1} - g(x_{n-1}) + J^{\rho_1, A_1}_{\eta_1, M_1}[A_1(g(y_{n-1})) - \rho_1(N(y_{n-1}, u_{n-1}))]\}\| \\
&\le \|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| + \|J^{\rho_1, A_1}_{\eta_1, M_1}[A_1(g(y_n)) \\
&\quad -\rho_1(N(y_n, u_n))] - J^{\rho_1, A_1}_{\eta_1, M_1}[A_1(g(y_{n-1})) - \rho_1(N(y_{n-1}, u_{n-1}))]\| \\
&\le \|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| + \frac{\tau_1^{q-1}}{r_1 - \rho_1 m_1}\|A_1(g(y_n)) \\
&\quad -A_1(g(y_{n-1})) - \rho_1(N(y_n, u_n) - N(y_{n-1}, u_{n-1}))\| \\
&\le \|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \\
&\quad + \frac{\tau_1^{q-1}}{r_1 - \rho_1 m_1}\|A_1(g(y_n)) - A_1(g(y_{n-1}))\| \\
&\quad + \frac{\tau_1^{q-1}}{r_1 - \rho_1 m_1}\rho_1\|N(y_n, u_n) - N(y_{n-1}, u_{n-1})\|.
\end{aligned} \tag{5.2.11}$$

Since g is Lipschitz continuous with constant λ_g, strongly accretive with constant δ_g and using Proposition 1.2.2, we have

$$\begin{aligned}
&\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\|^q \\
&\qquad \leq \|x_n - x_{n-1}\|^q - q\langle g(x_n) - g(x_{n-1}), J_q(x_n - x_{n-1})\rangle \\
&\qquad\qquad + C_q\|g(x_n) - g(x_{n-1})\|^q \\
&\qquad \leq (1 - q\delta_g + C_q\lambda_g^q)\|x_n - x_{n-1}\|^q. \qquad (5.2.12)
\end{aligned}$$

By the Lipschitz continuity of A_1 and g with constants λ_{A_1} and λ_g, respectively, we have

$$\begin{aligned}
\|A_1(g(y_n)) - A_1(g(y_{n-1}))\| &\leq \lambda_{A_1}\|g(y_n) - g(y_{n-1})\| \\
&\leq \lambda_{A_1}\lambda_g\|y_n - y_{n-1}\|. \qquad (5.2.13)
\end{aligned}$$

Since $N(\cdot,\cdot)$ is Lipschitz continuous in both the arguments with constants λ_{N_1} and λ_{N_2}, respectively, F is D-Lipschitz continuous with constant λ_{D_F} and using (5.2.8), we have

$$\begin{aligned}
&\|N(y_n, u_n) - N(y_{n-1}, u_{n-1})\| \\
&\qquad \leq \|N(y_n, u_n) - N(y_{n-1}, u_n)\| + \|N(y_{n-1}, u_n) - N(y_{n-1}, u_{n-1})\| \\
&\qquad \leq \lambda_{N_1}\|y_n - y_{n-1}\| + \lambda_{N_2}\|u_n - u_{n-1}\| \\
&\qquad \leq \lambda_{N_1}\|y_n - y_{n-1}\| + \lambda_{N_2}[D(F(y_n), F(y_{n-1})) + \epsilon^n\|y_n - y_{n-1}\|] \\
&\qquad \leq [\lambda_{N_1} + \lambda_{N_2}(\lambda_{D_F} + \epsilon^n)]\|y_n - y_{n-1}\|. \qquad (5.2.14)
\end{aligned}$$

Using (5.2.12)-(5.2.14), (5.2.11) becomes

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \sqrt[q]{1 - q\delta_g + C_q\lambda_g^q}\|x_n - x_{n-1}\| \\
&\quad + \frac{\tau_1^{q-1}}{r_1 - \rho_1 m_1}[\lambda_{A_1}\lambda_g + \rho_1(\lambda_{N_1} + \lambda_{N_2}(\lambda_{D_F} + \epsilon^n))]\|y_n - y_{n-1}\| \qquad (5.2.15)
\end{aligned}$$

and

$$\begin{aligned}
\|g(y_n) - g(y_{n-1})\|\|y_n - y_{n-1}\|^{q-1} &\geq \langle g(y_n) - g(y_{n-1}), J_q(y_n - y_{n-1})\rangle \\
&\geq \delta_g\|g(y_n) - g(y_{n-1})\|^q,
\end{aligned}$$

which implies

$$\begin{aligned}
\|y_n - y_{n-1}\| &\leq \frac{1}{\delta_g}\|g(y_n) - g(y_{n-1})\| \\
&\leq \frac{1}{\delta_g}\|J_{\eta_2, M_2}^{\rho_2, A_2}[A_2(g(x_n)) - \rho_2(S(v_n, x_n))]
\end{aligned}$$

$$-J^{\rho_2,A_2}_{\eta_2,M_2}[A_2(g(x_{n-1})) - \rho_2(S(v_{n-1},x_{n-1}))]\|$$

$$\leq \frac{1}{\delta_g}\left[\frac{\tau_2^{q-1}}{r_2-\rho_2 m_2}\|A_2(g(x_n)) - A_2(g(x_{n-1})) - \rho_2(S(v_n,x_n) - S(v_{n-1},x_{n-1}))\|\right]$$

$$\leq \frac{1}{\delta_g}\left[\frac{\tau_2^{q-1}}{r_2-\rho_2 m_2}\|A_2(g(x_n)) - A_2(g(x_{n-1}))\|\right] + \frac{1}{\delta_g}\left[\frac{\tau_2^{q-1}}{r_2-\rho_2 m_2}\rho_2\|S(v_n,x_n) - S(v_{n-1},x_{n-1})\|\right]. \quad (5.2.16)$$

By the Lipschitz continuity of A_1 and g, we have

$$\|A_2(g(x_n)) - A_2(g(x_{n-1}))\| \leq \lambda_{A_2}\|g(x_n) - g(x_{n-1})\| \leq \lambda_{A_2}\lambda_g\|x_n - x_{n-1}\|. \quad (5.2.17)$$

Using the same argument as for (5.2.14), we have

$$\|S(v_n,x_n) - S(v_{n-1},x_{n-1})\| \leq [\lambda_{S_1}(\lambda_{D_T}+\epsilon^n) + \lambda_{S_2}]\|x_n - x_{n-1}\|. \quad (5.2.18)$$

By (5.2.17), (5.2.18), (5.2.16) becomes

$$\|y_n - y_{n-1}\| \leq \frac{1}{\delta_g}\left[\frac{\tau_2^{q-1}}{r_2-\rho_2 m_2}[\lambda_{A_2}\lambda_g + \rho_2(\lambda_{S_1}(\lambda_{D_T}+\epsilon^n)+\lambda_{S_2})]\|x_n - x_{n-1}\|\right]. \quad (5.2.19)$$

Using (5.2.19), (5.2.15) becomes

$$\|x_{n+1} - x_n\| \leq \theta(\epsilon^n)\|x_n - x_{n-1}\|, \quad (5.2.20)$$

where

$$\theta(\epsilon^n) = \sqrt[q]{1 - q\delta_g + C_q\lambda_g^q} + \frac{(\tau_1\tau_2)^{q-1}}{\delta_g(r_1-\rho_1 m_1)(r_2-\rho_2 m_2)} \times[\lambda_{A_1}\lambda_g + \rho_1(\lambda_{N_1} + \lambda_{N_2}(\lambda_{D_F}+\epsilon^n))][\lambda_{A_2}\lambda_g + \rho_2(\lambda_{S_1}(\lambda_{D_T}+\epsilon^n)+\lambda_{S_2})].$$

Let

$$\theta = \sqrt[q]{1 - q\delta_g + C_q\lambda_g^q} + \frac{(\tau_1\tau_2)^{q-1}}{\delta_g(r_1-\rho_1 m_1)(r_2-\rho_2 m_2)} \times[\lambda_{A_1}\lambda_g + \rho_1(\lambda_{N_1} + \lambda_{N_2}\lambda_{D_F})][\lambda_{A_2}\lambda_g + \rho_2(\lambda_{S_1}\lambda_{D_T}+\lambda_{S_2})].$$

Since $0 < \epsilon < 1$, it follows that $\theta(\epsilon^n) \to \theta$, as $n \to \infty$.

From (5.2.10), we have $\theta < 1$, and consequently $\{x_n\}$ is a Cauchy sequence in E. Also it follows from (5.2.19) that $\{y_n\}$ is a Cauchy sequence in E. Since E is a Banach space, there exist $x, y \in E$ such that $x_n \to x$, $y_n \to y$ as $n \to \infty$.

We have

$$\begin{aligned}\|u_n - u_{n-1}\| &\leq D(F(y_n), F(y_{n-1})) + \epsilon^n \|y_n - y_{n-1}\| \\ &\leq \lambda_{D_F} \|y_n - y_{n-1}\| + \epsilon^n \|y_n - y_{n-1}\| \\ &= (\lambda_{D_F} + \epsilon^n)\|y_n - y_{n-1}\|,\end{aligned}$$

$$\begin{aligned}\|v_n - v_{n-1}\| &\leq D(T(x_n), T(x_{n-1})) + \epsilon^n \|x_n - x_{n-1}\| \\ &\leq \lambda_{D_T} \|x_n - x_{n-1}\| + \epsilon^n \|x_n - x_{n-1}\| \\ &= (\lambda_{D_T} + \epsilon^n)\|x_n - x_{n-1}\|\end{aligned}$$

and hence $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences in E. Let $u_n \to u \in E$ and $v_n \to v \in E$. Since g, N, S, T, F, A_i, η_i, M_i $(i = 1, 2)$ are all continuous mappings in E and by Algorithm 5.2.1, we have

$$x = x - g(x) + J^{\rho_1, A_1}_{\eta_1, M_1}[A_1(g(y)) - \rho_1(N(y, u))],$$

where

$$g(y) = J^{\rho_2, A_2}_{\eta_2, M_2}[A_2(g(x)) - \rho_2(S(v, x))].$$

Finally, we prove that $u \in F(y)$, $v \in T(x)$. In fact, since $u_n \in F(y_n)$ and

$$\begin{aligned}d(u_n, F(y)) &\leq \max\Big\{d(u_n, F(y)), \sup_{w_1 \in F(y)} d(F(y_n), w_1)\Big\} \\ &\leq \max\Big\{\sup_{w_2 \in F(y_n)} d(w_2, F(y)), \sup_{w_1 \in F(y)} d(F(y_n), w_1)\Big\} \\ &= D(F(y_n), F(y)).\end{aligned}$$

We have

$$\begin{aligned}d(u, F(y)) &\leq \|u - u_n\| + d(u_n, F(y)) \\ &\leq \|u - u_n\| + D\left(F(y_n), F(y)\right)\end{aligned}$$

$$\leq \|u - u_n\| + \lambda_{D_F}\|y_n - y\| \to 0, \text{ as } n \to \infty,$$

which implies that $d(u, F(y)) = 0$. Since $F(y) \in CB(E)$, it follows that $u \in F(y)$. Similarly, we can show that $v \in T(x)$.

Then by Lemma 5.2.1, (x, y, u, v) is a solution of Problem (5.2.1). This completes the proof.

Remark 5.2.2. For the suitable choices of operators N, S, F and T, we can easily derive Theorem 4.1 of *Jin* [71].

Chapter 6

Nonlinear Relaxed Cocoercive Generalized Variational Inclusions And Generalized Resolvent Equations

6.1. Introduction

Nonlinear relaxed cocoercive variational inclusions involving (A, η)-accretive mappings were introduced and studied by *Lan et al.* [79] in q-uniformly Banach spaces. Motivated by the work of *Lan et al.* [79], in this chapter, we generalize their problem in q-uniformly smooth Banach spaces and also we introduce and study the corresponding generalized resolvent equations.

In section 6.2, we deals with the existence and convergence of nonlinear relaxed cocoercive generalized variational inclusions with (A, η)-accretive mappings. In the last section, we introduce and study the generalized resolvent equations with (A, η)-accretive and relaxed cocoercive mappings.

6.2. Nonlinear Relaxed Cocoercive Generalized Variational Inclusions

This section deals with the study of nonlinear relaxed cocoercive generalized variational inclusions with (A, η)-accretive mappings in q-uniformly smooth Banach spaces. We define an iterative algorithm for finding the approximate solutions of this class of variational inclusions without Häusdorff metric. We also establish that the approximate solutions obtained by proposed algorithm converge to the exact solution of nonlinear relaxed cocoercive generalized variational inclusions problem.

Let $\eta, N, W : E \times E \to E$, $g, A : E \to E$ be the single-valued mappings, $B, C, D, F : E \to 2^E$ be the multivalued mappings. Let $M : E \times E \to 2^E$ be an (A, η)-accretive mapping in the first argument such that $g(E) \cap \text{dom } (M(.,.)) \neq \emptyset$. We consider the following *nonlinear relaxed cocoercive generalized variational inclusions*:

Find $u \in E$, $x \in B(u)$, $y \in C(u)$, $z \in D(u)$, $v \in F(u)$ such that

$$0 \in N(x, y) - W(z, v) + M(g(u), u). \tag{6.2.1}$$

Some special cases:

(i) If $M(g(u), u){=}M(g(u))$ and $W, D, F \equiv 0$, then Problem (6.2.1) reduces to the problem of finding $u \in E$, $x \in B(u)$, $y \in C(u)$ such that

$$0 \in N(x, y) + M(g(u)). \tag{6.2.2}$$

Problem (6.2.2) is considered by *Peng* [103].

(ii) If B and C are single-valued mappings, then Problem (6.2.2) can be replaced by finding $u \in E$ such that

$$0 \in N(B(u), C(u)) + M(g(u)). \tag{6.2.3}$$

A similar problem to (6.2.3) is considered by *Lan* [77].

(iii) If $C \equiv 0$ and $B, g = I$, the identity mapping, then Problem (6.2.3) reduces to the problem of finding $u \in E$ such that

$$0 \in N(u) + M(u). \tag{6.2.4}$$

Problem (6.2.4) is considered by *Bi et al.* [19].

We suggest a result which convert our problem *nonlinear relaxed cocoercive generalized variational inclusions* (6.2.1) into a fixed point problem.

Lemma 6.2.1. (u, x, y, z, v), where $u \in E$, $x \in B(u)$, $y \in C(u)$, $z \in D(u)$, $v \in F(u)$ is a solution of *nonlinear relaxed cocoercive generalized variational inclusions* (6.2.1) if and only if (u, x, y, z, v) satisfies

$$g(u) = J^{\rho,A}_{\eta,M(\cdot,u)}\left[A(g(u)) - \rho(N(x,y) - W(z,v))\right], \tag{6.2.5}$$

where $J^{\rho,A}_{\eta,M(\cdot,u)} = (A + \rho M(\cdot,u))^{-1}$ and $\rho \in (0, \frac{r}{m})$ is a constant.

Proof. The proof follows directly from the Definition 1.2.24.

By using the above fixed point formulation, we propose the following iterative algorithm for computing the approximate solutions of *nonlinear relaxed cocoercive generalized variational inclusions* (6.2.1) without Häusdorff metric.

Algorithm 6.2.1. For any given $u_0 \in E$, we choose $x_0 \in B(u_0)$, $y_0 \in C(u_0)$, $z_0 \in D(u_0)$, $v_0 \in F(u_0)$ and compute the sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{v_n\}$ by iterative schemes as follows:

$$g(u_{n+1}) = J^{\rho,A}_{\eta,M(\cdot,u_n)}\left[A(g(u_n)) - \rho(N(x_n,y_n) - W(z_n,v_n))\right], \tag{6.2.6}$$

$n = 0, 1, 2, \ldots\ldots..$, $\rho \in (0, \frac{r}{m})$ is a constant.

By using the definition of multivalued Lipschitz operator, we prove that the approximate solutions obtained by the proposed algorithm converge to the exact solution of *nonlinear relaxed cocoercive generalized variational inclusions* (6.2.1).

Theorem 6.2.1. Let E be a q-uniformly smooth Banach space and $\eta : E \times E \to E$ be Lipschitz continuous mapping with constant τ. Let $A : E \to E$ be r-strongly η-accretive and Lipschitz continuous mapping with constant λ_A and $M : E \times E \to 2^E$ be (A, η)-accretive mapping. Suppose $N, W : E \times E \to E$ be Lipschitz continuous mappings in both arguments with constants λ_{N_1}, λ_{N_2} and λ_{W_1}, λ_{W_2}, respectively and $B, C, D, F : E \to 2^E$ be Lipschitz continuous mappings with constants λ_B, λ_C, λ_D and λ_F, respectively. Let $g : E \to E$ be (b, ξ)-relaxed cocoercive, Lipschitz continuous with constant λ_g and strongly accretive with constant δ.

Suppose that there exist $\rho \in (0, \frac{r}{m})$ and $t > 0$ such that the following condition holds:

$$\|J^{\rho,A}_{\eta,M(\cdot,u_n)}(x) - J^{\rho,A}_{\eta,M(\cdot,u_{n-1})}(x)\| \le t\|u_n - u_{n-1}\| \tag{6.2.7}$$

for all $u_n, u_{n-1} \in E$

and

$$0 < \frac{\tau^{q-1}\lambda_A}{r-\rho m}\sqrt[q]{1-q\xi+(qb+C_q)\lambda_g^q}$$
$$+\frac{\tau^{q-1}\rho}{r-\rho m}\sqrt[q]{(\lambda_{N_2}\lambda_C+\lambda_{N_1}\lambda_B)^q-(q-C_q)(\lambda_{W_2}\lambda_F+\lambda_{W_1}\lambda_D)^q}$$
$$+\frac{\tau^{q-1}\lambda_A}{r-\rho m}+t<1 \tag{6.2.8}$$

where C_q is the constant as in Proposition 1.2.2. Then the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{v_n\}$ generated by Algorithm 6.2.1 converge strongly to u, x, y, z and v, respectively and (u,x,y,z,v) is a solution of *nonlinear relaxed cocoercive generalized variational inclusions* (6.2.1).

Proof. From Algorithm 6.2.1, Lemma 1.2.4 and (6.2.7), we have

$$\begin{aligned}
\|g(u_{n+1})-g(u_n)\| &= \|J^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A(g(u_n))-\rho(N(x_n,y_n)-W(z_n,v_n))\big]\\
&\quad -\big\{J^{\rho,A}_{\eta,M(\cdot,u_{n-1})}\big[A(g(u_{n-1}))-\rho(N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1}))\big]\big\}\|\\
&= \|J^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A(g(u_n))-\rho(N(x_n,y_n)-W(z_n,v_n))\big]\\
&\quad -J^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A(g(u_{n-1}))-\rho(N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1}))\big]\\
&\quad +J^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A(g(u_{n-1}))-\rho(N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1}))\big]\\
&\quad -J^{\rho,A}_{\eta,M(\cdot,u_{n-1})}\big[A(g(u_{n-1}))-\rho(N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1}))\big]\|\\
&\le \|J^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A(g(u_n))-\rho(N(x_n,y_n)-W(z_n,v_n))\big]\\
&\quad -J^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A(g(u_{n-1}))-\rho(N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1}))\big]\|\\
&\quad +\|J^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A(g(u_{n-1}))-\rho(N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1}))\big]\\
&\quad -J^{\rho,A}_{\eta,M(\cdot,u_{n-1})}\big[A(g(u_{n-1}))-\rho(N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1}))\big]\|\\
&\le \frac{\tau^{q-1}}{r-\rho m}\Big[\|A(g(u_n))-A(g(u_{n-1}))-\rho\big\{N(x_n,y_n)-W(z_n,v_n)\\
&\quad -(N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1}))\big\}\|\Big]+t\|u_n-u_{n-1}\|
\end{aligned}$$

$$\le \frac{\tau^{q-1}}{r-\rho m}\|A(g(u_n)) - A(g(u_{n-1}))\| + \frac{\rho\tau^{q-1}}{r-\rho m}\|N(x_n, y_n)$$

$$-N(x_{n-1}, y_{n-1}) - (W(z_n, v_n) - W(z_{n-1}, v_{n-1}))\|$$

$$+t\|u_n - u_{n-1}\|. \quad (6.2.9)$$

Since A is λ_A-Lipschitz continuous, we have

$$\|g(u_{n+1}) - g(u_n)\| \le \frac{\tau^{q-1}\lambda_A}{r-\rho m}\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\|$$

$$+\frac{\rho\tau^{q-1}}{r-\rho m}\|N(x_n, y_n) - N(x_{n-1}, y_{n-1}) - (W(z_n, v_n)$$

$$-W(z_{n-1}, v_{n-1}))\| + \Big(\frac{\tau^{q-1}\lambda_A}{r-\rho m} + t\Big)\|u_n - u_{n-1}\|. \quad (6.2.10)$$

Since g is (b, ξ)-relaxed cocoercive and λ_g-Lipschitz continuous, we have

$$\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\|^q$$

$$\le \|u_n - u_{n-1}\|^q - q\langle g(u_n) - g(u_{n-1}), j_q(u_n - u_{n-1})\rangle + C_q\|g(u_n) - g(u_{n-1})\|^q$$

$$\le \|u_n - u_{n-1}\|^q + qb\|g(u_n) - g(u_{n-1})\|^q - q\xi\|u_n - u_{n-1}\|^q + C_q\lambda_g^q\|u_n - u_{n-1}\|^q$$

$$\le \|u_n - u_{n-1}\|^q + qb\lambda_g^q\|u_n - u_{n-1}\|^q - q\xi\|u_n - u_{n-1}\|^q + C_q\lambda_g^q\|u_n - u_{n-1}\|^q$$

$$= (1 - q\xi + (qb + C_q)\lambda_g^q)\|u_n - u_{n-1}\|^q.$$

Thus, we have

$$\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| \le \sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q}\|u_n - u_{n-1}\|. \quad (6.2.11)$$

Also

$$\|N(x_n, y_n) - N(x_{n-1}, y_{n-1}) - (W(z_n, v_n) - W(z_{n-1}, v_{n-1}))\|^q$$

$$\le \|N(x_n, y_n) - N(x_{n-1}, y_{n-1})\|^q - (q - C_q)$$

$$\times\|W(z_n, v_n) - W(z_{n-1}, v_{n-1})\|^q. \quad (6.2.12)$$

Since the multivalued mappings B and C are Lipschitz continuous with constants λ_B and λ_C, we have

$$||x_n - x_{n-1}|| \leq \lambda_B ||u_n - u_{n-1}||, \tag{6.2.13}$$

$$\text{for all } x_n \in B(u_n),\ x_{n-1} \in B(u_{n-1}),$$

$$||y_n - y_{n-1}|| \leq \lambda_C ||u_n - u_{n-1}||, \tag{6.2.14}$$

$$\text{for all } y_n \in C(u_n),\ y_{n-1} \in C(u_{n-1}).$$

Since N is λ_{N_1}-Lipschitz continuous in first argument and λ_{N_2}-Lipschitz continuous in second argument, B is λ_B-Lipschitz continuous and C is λ_C-Lipschitz continuous, we have

$$\begin{aligned}
&||N(x_n, y_n) - N(x_{n-1}, y_{n-1})|| \\
&\quad = ||N(x_n, y_n) - N(x_n, y_{n-1}) + N(x_n, y_{n-1}) - N(x_{n-1}, y_{n-1})|| \\
&\quad \leq ||N(x_n, y_n) - N(x_n, y_{n-1})|| + ||N(x_n, y_{n-1}) - N(x_{n-1}, y_{n-1})|| \\
&\quad \leq \lambda_{N_2} ||y_n - y_{n-1}|| + \lambda_{N_1} ||x_n - x_{n-1}|| \\
&\quad \leq \lambda_{N_2}\lambda_C ||u_n - u_{n-1}|| + \lambda_{N_1}\lambda_B ||u_n - u_{n-1}|| \\
&\quad = (\lambda_{N_2}\lambda_C + \lambda_{N_1}\lambda_B) ||u_n - u_{n-1}||.
\end{aligned}$$

Thus,

$$||N(x_n, y_n) - N(x_{n-1}, y_{n-1})||^q \leq (\lambda_{N_2}\lambda_C + \lambda_{N_1}\lambda_B)^q ||u_n - u_{n-1}||^q. \tag{6.2.15}$$

Using the similar arguments as for (6.2.15), we obtain

$$||W(z_n, v_n) - W(z_{n-1}, v_{n-1})||^q \leq (\lambda_{W_2}\lambda_F + \lambda_{W_1}\lambda_D)^q ||u_n - u_{n-1}||^q. \tag{6.2.16}$$

Using (6.2.15) and (6.2.16), (6.2.12) becomes

$$\begin{aligned}
&||N(x_n, y_n) - N(x_{n-1}, y_{n-1}) - (W(z_n, v_n) - W(z_{n-1}, v_{n-1}))||^q \\
&\quad \leq (\lambda_{N_2}\lambda_C + \lambda_{N_1}\lambda_B)^q ||u_n - u_{n-1}||^q - (q - C_q)(\lambda_{W_2}\lambda_F + \lambda_{W_1}\lambda_D)^q ||u_n - u_{n-1}||^q \\
&\quad = \left[(\lambda_{N_2}\lambda_C + \lambda_{N_1}\lambda_B)^q - (q - C_q)(\lambda_{W_2}\lambda_F + \lambda_{W_1}\lambda_D)^q\right] ||u_n - u_{n-1}||^q.
\end{aligned}$$

It follows that

$$\begin{aligned}
&||N(x_n, y_n) - N(x_{n-1}, y_{n-1}) - (W(z_n, v_n) - W(z_{n-1}, v_{n-1}))|| \\
&\qquad\qquad \leq \sqrt[q]{(\lambda_{N_2}\lambda_C + \lambda_{N_1}\lambda_B)^q - (q - C_q)(\lambda_{W_2}\lambda_F + \lambda_{W_1}\lambda_D)^q}
\end{aligned}$$

$$\times \|u_n - u_{n-1}\|. \tag{6.2.17}$$

Combining (6.2.11) and (6.2.17) with (6.2.10), we obtain

$$\begin{aligned}
\|g(u_{n+1}) - g(u_n)\| \leq{}& \frac{\tau^{q-1}\lambda_A}{r - \rho m}\sqrt[q]{1 - q\xi + (qb + C_q\lambda_g^q)}\|u_n - u_{n-1}\| \\
&+ \frac{\rho\tau^{q-1}}{r - \rho m}\sqrt[q]{(\lambda_{N_2}\lambda_C + \lambda_{N_1}\lambda_B)^q - (q - C_q)(\lambda_{W_2}\lambda_F + \lambda_{W_1}\lambda_D)^q} \\
&\times \|u_n - u_{n-1}\| + \left(\frac{\tau^{q-1}\lambda_A}{r - \rho m} + t\right)\|u_n - u_{n-1}\| \\
={}& \Bigg[\frac{\tau^{q-1}\lambda_A}{r - \rho m}\sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q} \\
&+ \frac{\rho\tau^{q-1}}{r - \rho m}\sqrt[q]{(\lambda_{N_2}\lambda_C + \lambda_{N_1}\lambda_B)^q - (q - C_q)(\lambda_{W_2}\lambda_F + \lambda_{W_1}\lambda_D)^q} \\
&+ \frac{\tau^{q-1}\lambda_A}{r - \rho m} + t\Bigg]\|u_n - u_{n-1}\|.
\end{aligned} \tag{6.2.18}$$

By the strong accretivity of g with constant δ, we have

$$\begin{aligned}
\|g(u_{n+1}) - g(u_n)\|.\|u_{n+1} - u_n\|^{q-1} &\geq \langle g(u_{n+1}) - g(u_n), j_q(u_{n+1} - u_n)\rangle \\
&\geq \delta\|u_{n+1} - u_n\|^q
\end{aligned}$$

which implies that

$$\|u_{n+1} - u_n\| \leq \frac{1}{\delta}\|g(u_{n+1}) - g(u_n)\|. \tag{6.2.19}$$

Combining (6.2.18) and (6.2.19), we have

$$\begin{aligned}
\|u_{n+1} - u_n\| \leq{}& \frac{1}{\delta}\Bigg[\frac{\tau^{q-1}\lambda_A}{r - \rho m}\sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q} \\
&+ \frac{\rho\tau^{q-1}}{r - \rho m}\sqrt[q]{(\lambda_{N_2}\lambda_C + \lambda_{N_1}\lambda_B)^q - (q - C_q)(\lambda_{W_2}\lambda_F + \lambda_{W_1}\lambda_D)^q} \\
&+ \frac{\tau^{q-1}\lambda_A}{r - \rho m} + t\Bigg]\|u_n - u_{n-1}\| \\
={}& \theta\|u_n - u_{n-1}\|,
\end{aligned} \tag{6.2.20}$$

where

$$\theta = \frac{1}{\delta}\left[\frac{\tau^{q-1}\lambda_A}{r-\rho m}\sqrt[q]{1-q\xi+(qb+C_q)\lambda_g^q}\right.$$
$$+\frac{\rho\tau^{q-1}}{r-\rho m}\sqrt[q]{(\lambda_{N_2}\lambda_C+\lambda_{N_1}\lambda_B)^q-(q-C_q)(\lambda_{W_2}\lambda_F+\lambda_{W_1}\lambda_D)^q}$$
$$\left.+\frac{\tau^{q-1}\lambda_A}{r-\rho m}+t\right].$$

By (6.2.8), we know that $0<\theta<1$ and so (6.2.20) implies that $\{u_n\}$ is a Cauchy sequence. Thus, there exists $u \in E$ such that $u_n \to u$ as $n \to \infty$.

The Lipschitz continuity of multivalued mapping B, C, D and F implies that $x_n \to x$, $y_n \to y$, $z_n \to z$ and $v_n \to v$.

As A, η, M, N, W, B, C, D, F, g and $J_{\eta,M}^{\rho,A}$ all are continuous and by Algorithm 6.2.1, it follows that u, x, y, z and v satisfy the following relation

$$g(u) = J_{\eta,M(\cdot,u)}^{\rho,A}\left[A(g(u)) - \rho(N(x,y)-W(z,v))\right].$$

By Lemma 6.2.1, (u,x,y,z,v) is a solution of Problem (6.2.1). This completes the proof.

6.3. Generalized Resolvent Equations With (A,η)-accretive And Relaxed Cocoercive Mappings

In this section, in connection with *nonlinear relaxed cocoercive generalized variational inclusion problem* (6.2.1), we consider a generalized resolvent equations with (A,η)-accretive and relaxed cocoercive mappings in q-uniformly smooth Banach spaces. A relationship between *nonlinear relaxed cocoercive generalized variational inclusion problem* (6.2.1) and generalized resolvent equations is established. This equivalence is used to define an iterative algorithm without Häusdorff metric for solving generalized resolvent equations.

Let $\eta, N, W : E\times E \to E$, $g, A : E \to E$ be the single-valued mappings, $B, C, D, F : E \to 2^E$ be the multivalued mappings. Let $M : E \times E \to 2^E$ be an

(A, η)-accretive mapping in the first argument such that $g(E) \cap \text{dom } (M(.,.)) \neq \emptyset$. In connection with *nonlinear relaxed cocoercive generalized variational inclusions* (6.2.1), we consider the following *generalized resolvent equations:*

Find $s, u \in E$, $x \in B(u)$, $y \in C(u)$, $z \in D(u)$, $v \in F(u)$ such that

$$N(x, y) - W(z, v) + \rho^{-1} R^{\rho,A}_{\eta,M(\cdot,u)}(s) = 0, \tag{6.3.1}$$

where $\rho > 0$ is a constant, $R^{\rho,A}_{\eta,M(\cdot,u)} = I - A[J^{\rho,A}_{\eta,M(\cdot,u)}]$, I is the identity mapping and $A[J^{\rho,A}_{\eta,M(\cdot,u)}(s)] = [A(J^{\rho,A}_{\eta,M(\cdot,u)})](s)$.

We mention the following lemma which established an equivalence between the *nonlinear relaxed cocoercive generalized variational inclusions* (6.2.1) and *generalized resolvent equations* (6.3.1).

Lemma 6.3.1. The *nonlinear relaxed cocoercive generalized variational inclusions* (6.2.1) has a solution (u, x, y, z, v), where $u \in E$, $x \in B(u)$, $y \in C(u)$, $z \in D(u)$, $v \in F(u)$ if and only if the *generalized resolvent equations* (6.3.1) has a solution (s, u, x, y, z, v), where $s, u \in E$, $x \in B(u)$, $y \in C(u)$, $z \in D(u)$, $v \in F(u)$,

$$g(u) = J^{\rho,A}_{\eta,M(\cdot,u)}(s), \tag{6.3.2}$$

and

$$s = A(g(u)) - \rho\{N(x, y) - W(z, v)\}.$$

Proof. Let (u, x, y, z, v) be a solution of *nonlinear relaxed cocoercive generalized variational inclusions* (6.2.1). Then by Lemma 6.2.1, it is a solution of the following equation

$$g(u) = J^{\rho,A}_{\eta,M(\cdot,u)}[A(g(u)) - \rho\{N(x, y) - W(z, v)\}].$$

Using the fact $R^{\rho,A}_{\eta,M(\cdot,u)} = I - A[J^{\rho,A}_{\eta,M(\cdot,u)}]$ and equation (6.2.5), we have

$$\begin{aligned} & R^{\rho,A}_{\eta,M(\cdot,u)}[A(g(u)) - \rho\{N(x, y) - W(z, v)\}] \\ &= A(g(u)) - \rho\{N(x, y) - W(z, v)\} \\ &\quad - A[J^{\rho,A}_{\eta,M(\cdot,u)}[A(g(u)) - \rho\{N(x, y) - W(z, v)\}]] \\ &= A(g(u)) - \rho\{N(x, y) - W(z, v)\} - A(g(u)) \end{aligned}$$

$$= -\rho\{N(x,y) - W(z,v)\}.$$

Which implies that

$$N(x,y) - W(z,v) + \rho^{-1} R^{\rho,A}_{\eta,M(\cdot,u)}(s) = 0$$

with $s = A(g(u)) - \rho\{N(x,y) - W(z,v)\}$, i.e., (s,u,x,y,z,v) is a solution of *generalized resolvent equations* (6.3.1).

Conversely, let (s,u,x,y,z,v) be a solution of *generalized resolvent equations* (6.3.1), then

$$\rho\{N(x,y) - W(z,v)\} = -R^{\rho,A}_{\eta,M(\cdot,u)}(s) = A[J^{\rho,A}_{\eta,M(\cdot,u)}(s)] - s. \tag{6.3.3}$$

From (6.3.2) and (6.3.3), we have

$$\rho\{N(x,y) - W(z,v)\} = A[J^{\rho,A}_{\eta,M(\cdot,u)}[A(g(u)) - \rho\{N(x,y) - W(z,v)\}]]$$
$$-A(g(u)) + \rho\{N(x,y) - W(z,v)\}.$$

Which implies that

$$A(g(u)) = A[J^{\rho,A}_{\eta,M(\cdot,u)}[A(g(u)) - \rho\{N(x,y) - W(z,v)\}]]$$

and thus

$$g(u) = J^{\rho,A}_{\eta,M(\cdot,u)}[A(g(u)) - \rho\{N(x,y) - W(z,v)\}].$$

i.e., (u,x,y,z,v) is a solution of *nonlinear relaxed cocoercive generalized variational inclusions* (6.2.1).

Alternative proof. Let

$$s = A(g(u)) - \rho\{N(x,y) - W(z,v)\},$$

and from (6.3.2), we have

$$g(u) = J^{\rho,A}_{\eta,M(\cdot,u)}(s),$$

thus, we have

$$s = A[J^{\rho,A}_{\eta,M(\cdot,u)}(s)] - \rho\{N(x,y) - W(z,v)\}.$$

By using the fact that $A[J^{\rho,A}_{\eta,M(\cdot,u)}(s)] = [A(J^{\rho,A}_{\eta,M(\cdot,u)})](s)$, it follows that

$$N(x,y) - W(z,v) + \rho^{-1} R^{\rho,A}_{\eta,M(\cdot,u)}(s) = 0,$$

the required *generalized resolvent equations* (6.3.1).

We now invoke Lemma 6.2.1 and Lemma 6.3.1 to define the following iterative algorithm without Häusdorff metric for solving *generalized resolvent equations* (6.3.1).

Algorithm 6.3.1. For any given $s_0, u_0 \in E$, we choose $x_0 \in B(u_0)$, $y_0 \in C(u_0)$, $z_0 \in D(u_0)$ and $v_0 \in F(u_0)$ and compute the sequences $\{s_n\}$, $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{v_n\}$ by iterative schemes as follows:

$$g(u_n) = J^{\rho,A}_{\eta,M(\cdot,u_n)}(s_n) \tag{6.3.4}$$

and

$$s_{n+1} = A(g(u_n)) - \rho\{N(x_n, y_n) - W(z_n, v_n)\}, \tag{6.3.5}$$

where $\rho \in (0, \frac{r}{m})$ is a constant and $n = 0, 1, 2, \ldots\ldots$

Based on Algorithm 6.3.1, we give the approximation-solvability of the *generalized resolvent equations* (6.3.1) involving (A, η)-accretive mapping and cocoercive mappings in q-uniformly smooth Banach spaces. The convergence of iterative sequences generated by Algorithm 6.3.1 is also proved.

Theorem 6.3.1. Let E be a q-uniformly smooth Banach space and $A : E \to E$ be r-strongly η-accretive and Lipschitz continuous with constant λ_A, $\eta : E \times E \to E$ be Lipschitz continuous with constant τ and $M : E \times E \to 2^E$ be (A, η)-accretive mapping in the first argument. Let $N, W : E \times E \to E$ be Lipschitz continuous mappings in both arguments with constants λ_{N_1}, λ_{N_2} and λ_{W_1}, λ_{W_2}, respectively and $B, C, D, F : E \to 2^E$ be Lipschitz continuous with constants λ_B, λ_C, λ_D and λ_F, respectively. Suppose $g : E \to E$ be (b, ξ)-relaxed cocoercive and Lipschitz continuous with constant λ_g.

If there exit $\rho \in (0, \frac{r}{m})$ and $t > 0$ such that the following conditions hold:

$$\|J^{\rho,A}_{\eta,M(\cdot,u_n)}(s) - J^{\rho,A}_{\eta,M(\cdot,u_{n-1})}(s)\| \leq t\|u_n - u_{n-1}\|, \tag{6.3.6}$$

$$\text{for all } \; u_n, u_{n-1}, s \in E$$

and

$$\lambda_A \sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q} + \rho[(\lambda_{N_1}\lambda_B + \lambda_{N_2}\lambda_C) + (\lambda_{W_1}\lambda_D + \lambda_{W_2}\lambda_F)]$$
$$< \frac{(r - \rho m)[1 - (t + \sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q}) - \tau^{q-1}\lambda_A}{\tau^{q-1}}, \quad (6.3.7)$$

where C_q is the constant as in Proposition 1.2.2, then the iterative sequences $\{s_n\}$, $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{v_n\}$ generated by Algorithm 6.3.1 converge strongly to s, u, x, y, z and v, respectively and (s, u, x, y, z, v) is a solution of *generalized resolvent equations* (6.3.1).

Proof. From Algorithm 6.3.1, we have

$$\begin{aligned} ||s_{n+1} - s_n|| &= ||A(g(u_n)) - \rho\{N(x_n, y_n) - W(z_n, v_n)\} \\ &\quad -[A(g(u_{n-1})) - \rho\{N(x_{n-1}, y_{n-1}) - W(z_{n-1}, v_{n-1})\}]|| \\ &\leq ||A(g(u_n)) - A(g(u_{n-1}))|| + \rho||N(x_n, y_n) - N(x_{n-1}, y_{n-1})|| \\ &\quad +\rho||W(z_n, v_n) - W(z_{n-1}, v_{n-1})||. \end{aligned} \quad (6.3.8)$$

By the Lipschitz continuity of A, we have

$$||A(\bar{g}(u_n)) - A(g(u_{n-1}))|| \leq \lambda_A||g(u_n) - g(u_{n-1})||. \quad (6.3.9)$$

By the Lipschitz continuity of N in both the arguments and Lipschitz continuity of B and C, it follows that

$$\begin{aligned} &||N(x_n, y_n) - N(x_{n-1}, y_{n-1})|| \\ &\leq ||N(x_n, y_n) - N(x_n, y_{n-1})|| + ||N(x_n, y_{n-1}) - N(x_{n-1}, y_{n-1})|| \\ &\leq \lambda_{N_2}||y_n - y_{n-1}|| + \lambda_{N_1}||x_n - x_{n-1}|| \\ &\leq \lambda_{N_2}\lambda_C||u_n - u_{n-1}|| + \lambda_{N_1}\lambda_B||u_n - u_{n-1}|| \\ &= (\lambda_{N_2}\lambda_C + \lambda_{N_1}\lambda_B)||u_n - u_{n-1}||. \end{aligned} \quad (6.3.10)$$

By the Lipschitz continuity of W in both the arguments and Lipschitz continuity of D and F, it follows that

$$\begin{aligned} &||W(z_n, v_n) - W(z_{n-1}, v_{n-1})|| \\ &\leq ||W(z_n, v_n) - W(z_n, v_{n-1})|| + ||W(z_n, v_{n-1}) - W(z_{n-1}, v_{n-1})|| \end{aligned}$$

$$\leq \lambda_{W_2}\|v_n - v_{n-1}\| + \lambda_{W_1}\|z_n - z_{n-1}\|$$
$$\leq \lambda_{W_2}\lambda_F\|u_n - u_{n-1}\| + \lambda_{W_1}\lambda_D\|u_n - u_{n-1}\|$$
$$= (\lambda_{W_2}\lambda_F + \lambda_{W_1}\lambda_D)\|u_n - u_{n-1}\|. \tag{6.3.11}$$

Combining (6.3.9)-(6.3.11) with (6.3.8), we obtain

$$\|s_{n+1} - s_n\| \leq \lambda_A\|g(u_n) - g(u_{n-1})\| + \rho[(\lambda_{N_1}\lambda_B + \lambda_{N_2}\lambda_C)$$
$$+(\lambda_{W_1}\lambda_D + \lambda_{W_2}\lambda_F)]\|u_n - u_{n-1}\|$$
$$\leq \lambda_A\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| + \lambda_A\|u_n - u_{n-1}\|$$
$$+\rho[(\lambda_{N_1}\lambda_B+\lambda_{N_2}\lambda_C)+(\lambda_{W_1}\lambda_D+\lambda_{W_2}\lambda_F)]\|u_n-u_{n-1}\|. \tag{6.3.12}$$

Since g is (b,ξ)-relaxed cocoercive and λ_g-Lipschitz continuous, we have

$$\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\|^q$$
$$\leq \|u_n - u_{n-1}\|^q - q\langle g(u_n) - g(u_{n-1}), j_q(u_n - u_{n-1})\rangle + C_q\|g(u_n) - g(u_{n-1})\|^q$$
$$\leq \|u_n - u_{n-1}\|^q + qb\|g(u_n) - g(u_{n-1})\|^q - q\xi\|u_n - u_{n-1}\|^q + C_q\lambda_g^q\|u_n - u_{n-1}\|^q$$
$$\leq \|u_n - u_{n-1}\|^q + qb\lambda_g^q\|u_n - u_{n-1}\|^q - q\xi\|u_n - u_{n-1}\|^q + C_q\lambda_g^q\|u_n - u_{n-1}\|^q$$
$$= (1 - q\xi + (qb + C_q)\lambda_g^q)\|u_n - u_{n-1}\|^q. \tag{6.3.13}$$

It follows that

$$\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| \leq \sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q}\|u_n - u_{n-1}\|. \tag{6.3.14}$$

Combining (6.3.14) with (6.3.12), we have

$$\|s_{n+1} - s_n\| \leq [\lambda_A\sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q} + \lambda_A$$
$$+\rho[(\lambda_{N_1}\lambda_B+\lambda_{N_2}\lambda_C)+(\lambda_{W_1}\lambda_D+\lambda_{W_2}\lambda_F)]\|u_n-u_{n-1}\|. \tag{6.3.15}$$

By using Lemma 1.2.4 and condition (6.3.6), we have

$$\|u_n - u_{n-1}\| = \|J^{\rho,A}_{\eta,M(\cdot,u_n)}(s_n) - J^{\rho,A}_{\eta,M(\cdot,u_{n-1})}(s_{n-1}) - [g(u_n) - g(u_{n-1}) - u_{n-1} - u_n]\|$$
$$\leq \|J^{\rho,A}_{\eta,M(\cdot,u_n)}(s_n) - J^{\rho,A}_{\eta,M(\cdot,u_{n-1})}(s_{n-1})\| + \|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\|$$

$$= \|J^{\rho,A}_{\eta,M(\cdot,u_n)}(s_n) - J^{\rho,A}_{\eta,M(\cdot,u_n)}(s_{n-1}) + J^{\rho,A}_{\eta,M(\cdot,u_n)}(s_{n-1}) - J^{\rho,A}_{\eta,M(\cdot,u_{n-1})}(s_{n-1})\|$$

$$+\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\|$$

$$\leq \|J^{\rho,A}_{\eta,M(\cdot,u_n)}(s_n) - J^{\rho,A}_{\eta,M(\cdot,u_n)}(s_{n-1})\| + \|J^{\rho,A}_{\eta,M(\cdot,u_n)}(s_{n-1}) - J^{\rho,A}_{\eta,M(\cdot,u_{n-1})}(s_{n-1})\|$$

$$+\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\|$$

$$\leq \frac{\tau^{q-1}}{r-\rho m}\|s_n - s_{n-1}\| + t\|u_n - u_{n-1}\| + \sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q}\|u_n - u_{n-1}\|$$

$$= \frac{\tau^{q-1}}{r-\rho m}\|s_n - s_{n-1}\| + (t + \sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q})\|u_n - u_{n-1}\|,$$

which implies that

$$\|u_n - u_{n-1}\| \leq \left[\frac{\left(\frac{\tau^{q-1}}{r-\rho m}\right)}{1 - (t + \sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q}\ )}\right]\|s_n - s_{n-1}\|. \qquad (6.3.16)$$

Using (6.3.16), (6.3.15) becomes

$\|s_{n+1} - s_n\|$

$$\leq \frac{\left[\lambda_A\sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q} + \lambda_A + \rho[(\lambda_{N_1}\lambda_B + \lambda_{N_2}\lambda_C) + (\lambda_{W_1}\lambda_D + \lambda_{W_2}\lambda_F)]\right]\tau^{q-1}}{(r-\rho m)[1 - (t + \sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q}\ )]}$$

$$\times\|s_n - s_{n-1}\|$$

i.e.,

$$\|s_{n+1} - s_n\| \leq \theta\|s_n - s_{n-1}\|,$$

where

$$\theta = \frac{\left[\lambda_A\sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q} + \lambda_A + \rho[(\lambda_{N_1}\lambda_B + \lambda_{N_2}\lambda_C) + (\lambda_{W_1}\lambda_D + \lambda_{W_2}\lambda_F)]\right]\tau^{q-1}}{(r-\rho m)[1 - (t + \sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q}\ )]}.$$

From (6.3.7), we have $0 < \theta < 1$ and consequently $\{s_n\}$ is a Cauchy sequence in E. Thus, there exists $s \in E$ such that $s_n \to s$ as $n \to \infty$.

From (6.3.16), we know that $\{u_n\}$ is also a Cauchy sequence in E. Therefore, there exists $u \in E$ such that $u_n \to u$ as $n \to \infty$. Since the multivalued mappings B, C, D

and F are Lipschitz continuous, it follows that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{v_n\}$ are also Cauchy sequences, we can assume that $x_n \to x$, $y_n \to y$, $z_n \to z$ and $v_n \to v$.

Since A, g, N and W all are continuous and by Algorithm 6.3.1, it follows that

$$\begin{aligned} s_{n+1} &= A(g(u_n)) - \rho\{N(x_n, y_n) - W(z_n, v_n)\} \\ &\longrightarrow s = A(g(u)) - \rho\{N(x, y) - W(z, v)\}, \quad n \to \infty, \end{aligned} \tag{6.3.17}$$

$$J^{\rho,A}_{\eta,M(\cdot,u_n)}(s_n) = g(u_n) \longrightarrow g(u) = J^{\rho,A}_{\eta,M(\cdot,u)}(s), \quad n \to \infty. \tag{6.3.18}$$

By (6.3.17), (6.3.18) and Lemma 6.3.1, we have

$$N(x, y) - W(z, v) + \rho^{-1}[I - A(J^{\rho,A}_{\eta,M(\cdot,u)}(s))] = 0$$

i.e.,

$$N(x, y) - W(z, v) + \rho^{-1} R^{\rho,A}_{\eta,M(\cdot,u)}(s) = 0.$$

By Lemma 6.3.1, (s, u, x, y, z, v) is a solution of Problem (6.3.1). This completes the proof.

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