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Towards Constructive Description Logics for Abstraction and Refinement * †

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Abstract

This work explores some aspects of a new and natural semantical dimension that can be accommodated within the syntax of description logics which opens up when passing from the classical truth-value interpretation to a *constructive interpretation*. We argue that such a strengthened interpretation is essential to represent applications with partial information adequately and to achieve consistency under abstraction as well as robustness under refinement. We introduce a constructive version of $\mathcal{ALC}$, called $c\mathcal{ALC}$, for which we give a sound and complete Hilbert axiomatisation and a Gentzen tableau calculus showing finite model property and decidability.

1 Introduction

The successes of description logics (DLs) in the many domains of semantic information processing is based on their flexibility to strike a carefully crafted trade-off between expressiveness and implementation efficiency. DLs have their origin in knowledge representation formalisms. They aim to encapsulate semantical complexity in compact notation which is domain-specific rather than general purpose. This leverages syntax to make the handling of logical specifications both by humans as well as reasoning engines run in a much higher gear ('application-level') compared to, say, plain vanilla

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first-order logic, in which all quantification structure is made explicit ('representation-level').

Technically, DLs are related to multi-dimensional generalisations of modal logic [15, 1] and as such they are essentially guarded fragments of first-order logic. These fragments have turned out to be a breeding-ground of very well-behaved classes of logic formalisms. This work explores some aspects of yet another semantical dimension that can be accommodated within the syntax of DL which opens up when passing from the classical truth-value interpretation to a *constructive interpretation* of DL. We will argue that such a refined interpretation is essential to represent applications with partial information adequately and to achieve both consistency under abstraction as well as robustness under refinement.

1.1 When Constructiveness Matters

Knowledge representation based on description logics can be used to capture the meaning of natural language statements about specific world domains (ontologies). Often, however, such knowledge is dynamic and incomplete. Entities that make up the domain may not be fixed and tangible but abstractions of real individuals whose properties are changing and defined only up to construction. Natural language concepts rarely have a static interpretation but are subject to negotiation or context and thus require a constructive approach which is robust under refinement.

An application area where this aspect is particularly prominent and which motivates the work described here, is auditing. The digital auditing of business mass data experiences a huge increase in importance recently. Audit executives, fraud examiners and compliance professionals are pressured on all fronts to shorten audit cycles and to increase audit efficiency and quality. In particular, the efficient verification of enterprise processes is of big interest since the audit concern is getting more critical with new regulations like SOX[1], IFRS[2] and also to respond accurately to managing risks in our competitive world.

[1] Sarbanes-Oxley Act, US law of 2002 on business reporting in reaction to Enron and WorldCom scandals.

[2] International Financial Reporting Standard.

Audit statements about the validity of accounting data, absence of fraud or conformance to financial process standards must constructively take account of many dimensions of abstraction and refinement.

- First, the producers of audit data usually are ongoing business processes which the audit data can only cover a limited snapshot of. E.g., a requirement such as "each delivery order must have an associated invoice" must take into account that for some delivery order the invoice is still "in the process" and only available after refinement of the audit data.
- Second, a role like the 'legally responsible signatory' may not be fully definable once and for all but depend on the legal context. Some aspects may even deliberately be left open subject to negotiations and only refined as the auditing case progresses.
- Third, entities may be abstractions of physical individuals: The notion of the 'CEO of company X' in an audit statement is a virtual rather than concrete person who may be replaced perhaps while auditing is ongoing. The CEO which appears atomic at some level of abstraction really is a concept at a lower level where personal liability issues come in or where executive action needs to be taken.
- Forth, auditing is typically faced with vast amounts of business data. For efficiency reasons, manageable digests of the data need to be created. Such data compression may ignore purportedly irrelevant attributes of entities or scan only a subset of entities associated with a given concept. Auditing, thus, is not exact but approximated. If the quick check indicates potential irregularities then a constructive refinement of the abstracted entities and concepts must be possible to confirm or reject the case constructively. For instance, a Benford test [6] may show an abnormal distribution of digits in the sales slips of a retail chain. However, when taking into account the abstracted "irrelevant" data it may turn out that the deviation can be explained by a special promotion offer. The auditing domain demands a constructive concept of truth to take care of the potential incompleteness of knowledge.

Auditing is a prime example of a class of application domains which require the ability to express partiality and incomplete information beyond the standard open world assumption (OWA). Because the semantical meaning is context-dependent and possibly involves many levels of explication there must be a constructive notion of undefinedness which permits that concepts evolve. Classical OWA assumes that each concept is static and at the outset either includes a given entity or not. However, either option may be incorrect, if the entity or the concept is not fully defined until a later stage where lower levels of detail become available. The critical issue is that there will be entities which have the same abstract properties (such as sharing the same fillers and concepts) but still are distinct individuals at some level so that identifying them at the outset would be inconsistent.

But if OWA is not enough, how can reasoning be both correct under abstraction and sustainable under refinement? Logic offers a well-known suggestion to solve this puzzle which is to replace the traditional *binary* truth interpretation by a *constructive* notion of truth. Proof-theoretically, constructive logic is compatible with the idea of positive evidence and realisability [25]. It does not infer the presence of entities from the absence of others but insists on the existence of *computational witnesses*. Model-theoretically, constructive logic admits of an interpretation based on *stages of information* [27] so that truth is persistent under refinement[3].

1.2 Related Work

The role of intuitionistic Kripke models for knowledge representation based on partial descriptions has been highlighted in [9]. The general benefits of the Curry-Howard Isomorphism (*proofs-as-computations*) in DL have been argued in [11, 8]. In our context, more concretely, we envisage that the computational interpretation of TBox deductions as λ-terms yields verified audit tactics and that constructive ABox tableau algorithms provide engines to drive interactive games between auditee (proponent) and auditor (opponent). A third potential benefit arises from the use of DLs as a *programming type system* (see e.g., [19]) which naturally requires a constructive setting. Constructive DL concepts may not only specify the semantics of data streams in audit component interfaces but also resource requirements. This can be exploited to satisfy higher demands on robustness and efficiency in the semantic processing of mass data.

[3]One might say that classical DL is based on a *static* open world assumption (SOWA) while constructive DL supports an *evolving* open world assumption (EOWA).

In this work we discuss some of the model-theoretic aspects of the constructive interpretation of DLs, in contrast to [8] which is proof-theoretic and addresses the extraction of information terms.

The work of [9] presents an intuitionistic epistemic logic based on several refinement relations coding multiple *(partial) points of view.* Here we only consider one dimension of refinement reflecting a two-player scenario (e.g., auditor and auditee) but in a more general sense than [9]. Our refinement ordering $\preceq$ may have cycles and fallible descriptions. Such descriptive "oscillations" and "deadlocks" are intrinsic to real-world abstractions (see examples below). The notion of a simulation relation such as in [10] for semi-structured data (BDFS) can also be thought of as a refinement relation. However, it is an external meta-level concept on models and cannot be iterated. In the description logic $c\mathcal{ALC}$ proposed here refinement is internal and transitive to represent nested levels of concretisations inside a single model.

It is important to point out that the semantic dimension along which refinement takes place is implicit in $c\mathcal{ALC}$ and not coded in the syntax. This accommodates many different notions of context generically in the language of the basic description logic $\mathcal{ALC}$ [4]. The context-dependency is built into the notion of truth rather than the terminology like in other work on special cases of context such as temporal DL [2, 7, 3]. $c\mathcal{ALC}$ is meant for applications where we must be robust for several implicit notions of context-dependency but do not need to reason explicitly about some specific refinement.

Our work is to be distinguished also from many-valued DL (see e.g., [20, 17]) which is finitely valued while $c\mathcal{ALC}$ is infinitely valued and from fuzzy DL (see e.g., [24, 12, 16]) which use a quantitative notion of approximate truth whereas $c\mathcal{ALC}$ still adheres to a crisp deductive approach. Even though the envisaged application domain of auditing may use statistical analyses, at the end of the day we must cross the t's and dot the i's and be able to name the evidence.

1.3 Overview & Summary of Results

Section 2 introduces the syntax and semantics of $c\mathcal{ALC}$ and presents several examples for applications where constructivity is needed. Section 3 gives a sound and complete Hilbert axiomatisation and a Gentzen-tableau deduction system for $c\mathcal{ALC}$, showing finite model property and decidability. The computational aspects are covered by following the Curry-Howard-Isomorphism and presented in a detailed example. Finally the complexity of reasoning in $c\mathcal{ALC}$ is addressed. Section 4 discusses the semantical dimensions between $c\mathcal{ALC}$ and $\mathcal{ALC}$ and to what extent $c\mathcal{ALC}$ is a constructive weakening of classical $\mathcal{ALC}$. Finally, section 5 concludes our results and gives insights into future work.

2 Syntax and Semantics of $c\mathcal{ALC}$

Concept descriptions in $c\mathcal{ALC}$ are based on sets of *role names* N_R and *concept names* N_C and are formed as follows, where $A \in N_C$ and $R \in N_R$:

$$C, D \;\; \rightarrow \;\; A \mid \top \mid \bot \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid C \sqsubseteq D \mid \exists R.C \mid \forall R.C.$$

This syntax is more general than standard $\mathcal{ALC}$ in that it includes subsumption $\sqsubseteq$ as a concept-forming operator. The TBox statement $C \sqsubseteq D$ meaning that 'D subsumes C' is expressed as the concept identity $C \sqsubseteq D = \top$. In classical $\mathcal{ALC}$ one could use the equation $\neg C \sqcup D = \top$ to do that, essentially reducing subsumption to $\neg$ and $\sqcup$. This is no longer possible in constructive logic where these operators are independent. Being a first class operator, subsumption can be nested arbitrarily as in $((D \sqsubseteq C) \sqsubseteq B) \sqsubseteq A$. The full power of such "higher–order" subsumptions may not be needed in practice but will allow us to axiomatise the full theory of $c\mathcal{ALC}$ conveniently in the form of a Hilbert calculus. Like in $\mathcal{ALC}$ the universal concept $\top$ is redundant and codable as $\neg\bot$. Also, $\bot$ and $\neg$ can represent each other, e.g. $\bot = A \sqcap \neg A$ and $\neg C = C \sqsubseteq \bot$. Otherwise, the operators are independent.

Constructive interpretations $\mathcal{I}$ of concept descriptions extend the classical models for $\mathcal{ALC}$ by a pre-ordering $\preceq^{\mathcal{I}}$ for expressing refinement between individuals and by a notion of fallible entities $\bot^{\mathcal{I}}$ for interpreting contradiction.

Following the standard Kripke semantics of intuitionistic logic [27], entities in constructive DL are not atomic individuals but have internal structure which in general is only partially determined and thus subject to refinement. Let relation $a \preceq a'$ on entities denote that a' is *more precisely determined* than a, that a' *refines* a or that a *abstracts* a'. The relation $\preceq$ models a potential increase of information or refinement of context associated with the process of pinning down entities as real individuals. This includes the possibility that both $a \preceq b$ and $b \preceq a$, i.e., a and b have the same information content and thus are formally indistinguishable, yet still distinct $a \neq b$ because of some lower-level properties. Now, if a concept C is to be robust under refinement then $a{:}C$ and $a \preceq a'$ must imply $a'{:}C$. This is achieved by the following definition:

Definition 1. A *constructive interpretation* or *constructive model* of $c\mathcal{ALC}$ is a structure $\mathcal{I} = (\Delta^\mathcal{I}, \preceq^\mathcal{I}, \bot^\mathcal{I}, \cdot^\mathcal{I})$ consisting of

- a non-empty set $\Delta^\mathcal{I}$ of *entities*, the universe of discourse in which each entity represents a partially defined, or abstract individual;
- a *refinement* pre-ordering $\preceq^\mathcal{I}$ on $\Delta^\mathcal{I}$, i.e., a reflexive and transitive relation;
- a subset $\bot^\mathcal{I} \subseteq \Delta^\mathcal{I}$ of *fallible* entities closed under refinement, i.e., $x \in \bot^\mathcal{I}$ and $x \preceq^\mathcal{I} y$ implies $y \in \bot^\mathcal{I}$, for every fallible entity x exists a fallible filler z, i.e., $x\,R\,z \;\&\; z \in \bot^\mathcal{I}$ and all filler of a fallible entity x are fallible, i.e. $\forall z.\; x\,R\,z \Rightarrow z \in \bot^\mathcal{I}$;
- finally an interpretation function $\cdot^\mathcal{I}$ mapping each role name $R \in N_R$ to a binary relation $R^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$ and each atomic concept $A \in N_C$ to a set $\bot^\mathcal{I} \subseteq A^\mathcal{I} \subseteq \Delta^\mathcal{I}$ which is closed under refinement, i.e., $x \in A^\mathcal{I}$ and $x \preceq^\mathcal{I} y$ implies $y \in A^\mathcal{I}$.

The interpretation $\mathcal{I}$ is lifted from atomic $\bot$, A to arbitrary concepts, where $\Delta^\mathcal{I}_c =_{df} \Delta^\mathcal{I} \setminus \bot^\mathcal{I}$ is the set of *non-fallible* elements in $\mathcal{I}$:

$$
\begin{aligned}
\top^\mathcal{I} &=_{df} \Delta^\mathcal{I}\\
(\neg C)^\mathcal{I} &=_{df} \{x \mid \forall y \in \Delta^\mathcal{I}_c.\; x \preceq^\mathcal{I} y \Rightarrow y \notin C^\mathcal{I}\}\\
(C \sqcap D)^\mathcal{I} &=_{df} C^\mathcal{I} \cap D^\mathcal{I}\\
(C \sqcup D)^\mathcal{I} &=_{df} C^\mathcal{I} \cup D^\mathcal{I}\\
(C \sqsubseteq D)^\mathcal{I} &=_{df} \{x \mid \forall y \in \Delta^\mathcal{I}.\; (x \preceq^\mathcal{I} y \;\&\; y \in C^\mathcal{I}) \Rightarrow y \in D^\mathcal{I}\}\\
(\exists R.C)^\mathcal{I} &=_{df} \{x \mid \forall y \in \Delta^\mathcal{I}.\; x \preceq^\mathcal{I} y \Rightarrow \exists z \in \Delta^\mathcal{I}.\; (y,z) \in R^\mathcal{I} \;\&\; z \in C^\mathcal{I}\}\\
(\forall R.C)^\mathcal{I} &=_{df} \{x \mid \forall y \in \Delta^\mathcal{I}.\; x \preceq^\mathcal{I} y \Rightarrow \forall z \in \Delta^\mathcal{I}.\; (y,z) \in R^\mathcal{I} \Rightarrow z \in C^\mathcal{I}\}. \qquad \Box
\end{aligned}
$$

Entities in $\Delta^\mathcal{I}$ are partial descriptions representing incomplete information about individuals. Fallible elements $b \in \bot^\mathcal{I}$ may be thought of as over-constrained tokens of information, self-contradictory objects of evidence or undefined computations. E.g., they may be used to model the situation in which computing a role-filler for an abstract individual a fails, i.e., $\forall b.\, R(a,b) \Rightarrow b \in \bot^{\mathcal{I}}$, yet when a is refined to a' then a non-fallible role-filler $b' \in \Delta_c^{\mathcal{I}}$ exists with $R(a',b')$ (see Example 3 below).

Because of the abstraction fuzziness embodied by $\preceq^{\mathcal{I}}$ and $\bot^{\mathcal{I}}$ the elements of $\Delta^{\mathcal{I}}$ are *abstract* individuals or *entities* rather than *concrete* or *atomic* individuals (which are a fiction anyway, constructively speaking).

Each entity implicitly subsumes all its refinements and truth is inherited. Specifically, one can show that $x \in C^{\mathcal{I}}$ and $x \preceq^{\mathcal{I}} y$ implies $y \in C^{\mathcal{I}}$ for all concepts C. Fallible entities are information-wise maximal elements and therefore included in every concept, i.e., $\bot^{\mathcal{I}} \subseteq C^{\mathcal{I}}$ for all C.

Lemma 1. For all concepts C it holds that $\forall a \in \Delta^{\mathcal{I}}.a \in \bot^{\mathcal{I}} \Rightarrow a \in C^{\mathcal{I}}$.

Proof. The proof is by induction on the structure of C. The detailed proof is given in the appendix. □

The purpose of the present work is to show that the non-standard interpretation of Def. 1 induces a well-behaved logic, called $c\mathcal{ALC}$, which uses the same syntax but is more expressive than classical $\mathcal{ALC}$ and still admits standard TBox and ABox tableau reasoning. Before we continue expounding the theory let us look at some examples.

Example 1. Every classical interpretation $\mathcal{I}$ of $\mathcal{ALC}$ (see e.g., [4]) induces a trivial model according to Def. 1 with the *discrete* refinement relation $\preceq^{\mathcal{I}}$, i.e., the identity relation $x \preceq^{\mathcal{I}} y$ iff $x = y$ and the empty set $\bot^{\mathcal{I}} = \emptyset$ of fallible entities. These validate the formulas $C \sqcup \neg C = \top$, $\exists R.\bot = \bot$ and $\exists R.(C \sqcup D) = \exists R.C \sqcup \exists R.D$. These three axioms essentially characterise classical models (see Sec. 4). □

Example 2. Let $a = (c, d_1)$ and $b = (c, d_2)$ be two entries in a (relational) database that share the same first attribute but are distinguished in the second. If the attributes are referenced by roles \$1 and \$2 then the situation could be specified, in ABox syntax, by $a\,\$1\,c$, $a\,\$2\,d_1$, $b\,\$1\,c$, $b\,\$2\,d_2$. Now let us abstract from the second attributes and consider the pairs as partially defined entities $a^\sharp = (c, ?)$ and $b^\sharp = (c, ?)$, respectively, say in an attempt to compress information. Ignoring d_1, d_2 means that $a^\sharp$ and $b^\sharp$ carry the same information and thus can no longer be distinguished. Since the pre-order $\preceq$ measures the information content the entities $a^\sharp$ and $b^\sharp$ are mutually reachable via $\preceq$ and refine each other, i.e. $a^\sharp \preceq b^\sharp$ and $b^\sharp \preceq a^\sharp$. This cyclic refinement relationship between

$a^\sharp$ and $b^\sharp$ implies an abstract equivalence $a^\sharp \cong b^\sharp$ but not an identity $a^\sharp = b^\sharp$ keeping in mind that both have incompatible realisations $a^\sharp \preceq a$ and $b^\sharp \preceq b$, respectively.

The situation is depicted in Fig. 1. The dotted arrows correspond to refinement and solid arrows represent the attribute roles \$1, \$2. The points $a^\sharp, b^\sharp, a, b, c, d_1, d_2$ are database entities. Note that both $a^\sharp$, $b^\sharp$ have a fallible \$2 filler ($\bot$) which corresponds to a computational deadlock when selecting \$2 for $a^\sharp$ or $b^\sharp$. The concepts C, D_1 and D_2 are assumed to specify some relevant properties of our database entities.

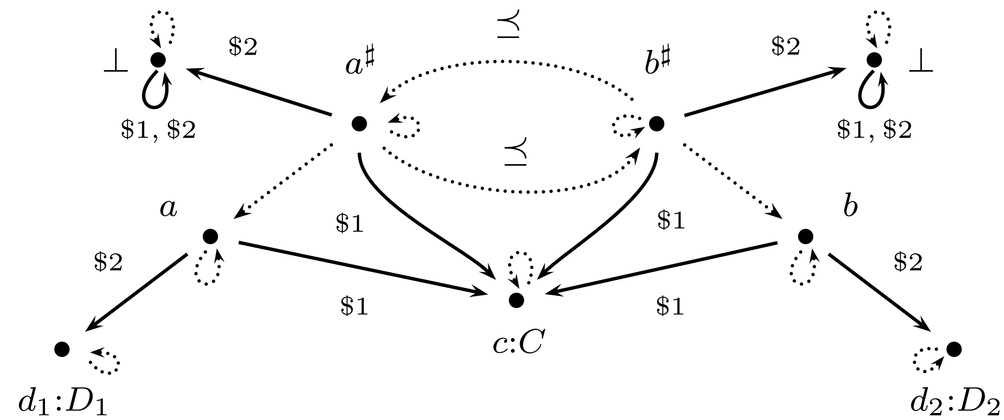

Figure 1: A simple data model with abstraction.

Since the entities $a^\sharp$, $b^\sharp$ are indistinguishable they share exactly the same concept descriptions. Formally, if $Th(x)$ denotes the set of concepts which entity x participates in, then $Th(a^\sharp) = Th(b^\sharp)$. E.g., $\exists\$1.C, \exists\$2.(D_1 \sqcup D_2) \in Th(a^\sharp)$ since every refinement of $a^\sharp$ has $c{:}C$ as filler for role \$1 and either $d_1{:}D_1$ or $d_2{:}D_2$ as a filler for \$2. The disjunction $\exists\$2.(D_1 \sqcup D_2)$ captures the choice between the two realisations of $a^\sharp$ as a concrete individual, *viz.*,(c, d_1) and (c, d_2). On the other hand, this choice cannot be resolved at the abstract level as there is no single uniform choice for the \$2-filler. This is reflected by the fact that $\exists\$2.D_i \not\in Th(a^\sharp)$ $(i = 1, 2)$ which means $\exists\$2.D_1 \sqcup \exists\$2.D_2 \not\in Th(a^\sharp)$.

Abstractions like this cannot be expressed in classical DL where existential fillers always distribute over $\sqcup$, i.e., $\exists\$2.(D_1 \sqcup D_2)$ is semantically identical to $\exists\$2.D_1 \sqcup \exists\$2.D_2$. Also, note that the Excluded Middle $\exists\$2.D_1 \sqcup \neg\exists\$2.D_1$ is not valid for $a^\sharp$. □

Example 3. Consider *hasCustomer* relationships between companies a, b, c, d. Let us assume that a has both b and c as customers, b has customer c and c has d among its customers. Further suppose that b is insolvent (concept *Insolvent*) and d solvent (concept

$\neg$*Insolvent*). Regarding possible insolvency of c nothing is known. In classical OWA we have c:(*Insolvent* $\sqcup\neg$*Insolvent*) regardless of c. This implies that a is an instance of the concept description $CW = \exists hasCustomer.(Insolvent \sqcap \exists hasCustomer.\neg Insolvent)$ specifying *credit-worthy* companies with an insolvent customer who in turn can rely on at least one solvent customer. In the first case c:*Insolvent* this customer of a is c, in case c:$\neg$*Insolvent* it is b. In a *static* world the filling customer would be unknown but fixed. However, the case analysis on c is invalid if the model arises by abstraction from a concrete taxonomy where insolvency is a *context-dependent* defect.

Fig. 2 shows an example model of the situation. Each solid edge is the relation *hasCustomer* and each dotted line codes refinement. I abbreviates the concept "Insolvent". Company c may be insolvent during some specific period of time or under some specific legal understanding of the concept *Insolvent*, represented by refinement c'. It may be solvent during another period of time or other legal regulations as represented by refinement c''. Then insolvency of c is not just unknown but *undecidable* (i.e., not fixable). The required *hasCustomer*-filler for a in concept CW cannot be obtained without contradicting one of the two directions c', c'' in which c may evolve.

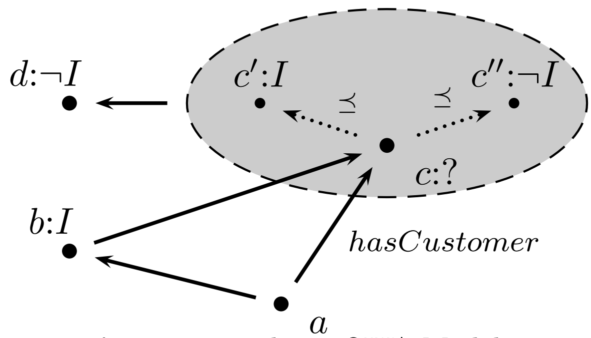

Figure 2: Evolving OWA Model.

Note that c:(*Insolvent*$\sqcup\neg$*Insolvent*) is not true in Fig. 2. The case c:$\neg$*Insolvent* conflicts with refinement $c \preceq c'$ and c':*Insolvent*, if c:*Insolvent* the refinement $c \preceq c''$ obtains c'':$\neg$*Insolvent*. Thus, neither *Insolvent* nor $\neg$*Insolvent* can be satisfied in c. In classical *static* OWA the case analysis is performed outside the model so that fillers may depend non-uniformly on the case analysis. This however requires an a-priori fixed knowledge of all problem parameters on which solvency depends. This cannot be fixed statically for c in the model of Fig. 2 once and for all.

In $c\mathcal{ALC}$ this choice is internalised and the filler of a role must be robust under case analysis. Thus, $a{:}CW$ is invalid under *Evolving* OWA because the $\exists$-filler is not realisable by a single nameable entity. $\square$

Example 4. Business data typically come in streams, e.g., as linearised database tables or time-series of financial market transactions. If streams are considered as abstract entities then DL concepts can act as a typing system to specify semantical properties of typical stream elements. To illustrate this let $\mathbb{D} = \mathbb{N} \uplus \mathbb{B} \uplus (\mathbb{N} \times \mathbb{B})$ be the discrete universe of booleans, naturals and their pairings. Consider the domain $\Delta^{\mathcal{I}} = \mathbb{D}^{\omega} = \mathbb{D}^* \cup \mathbb{D}^{\infty}$ of all finite and infinite sequences ("streams") over $\mathbb{D}$.

The refinement $\preceq^{\mathcal{I}}$ is the (inverse) suffix ordering, which is the least relation closed under the rule

$$\frac{v \in \mathbb{D}}{v \cdot s \preceq^{\mathcal{P}} s}$$

where $v \cdot s$ is the stream $s \in \mathbb{D}^{\omega}$ prefixed by value $v \in \mathbb{D}$. For instance,

$$1 \cdot (2, \mathtt{T}) \cdot \mathtt{T} \cdot \mathtt{F} \ \preceq^{\mathcal{I}} \ (2, \mathtt{T}) \cdot \mathtt{T} \cdot \mathtt{F} \ \preceq^{\mathcal{I}} \ \mathtt{T} \cdot \mathtt{F} \ \preceq^{\mathcal{I}} \ \mathtt{F} \ \preceq^{\mathcal{I}} \ \epsilon,$$

where ϵ denotes the empty stream. Under this interpretation, concepts $C^{\mathcal{I}}$, which must be closed under $\preceq^{\mathcal{I}}$, express *future projected behaviour* of streams. The empty stream has no future behaviour, it represents a computational deadlock, i.e., $\bot^{\mathcal{I}} = \{\epsilon\}$. To access the stream values let us assume that there is a distinguished (functional) role *val* which relates a stream with its first data element considered as an infinite constant stream, if such exists and the empty stream otherwise. In other words, $val(\epsilon, \epsilon)$ and $val(v \cdot s, v^{\infty})$. For instance, $val((2, \mathtt{T}) \cdot \mathtt{T} \cdot \mathtt{F}, (2, \mathtt{T})^{\infty})$ and $val(\mathtt{T} \cdot \mathtt{F}, \mathtt{T}^{\infty})$.

Let Nat and Bool be the usual programming language types considered as atomic $c\mathcal{ALC}$ concepts, i.e. $\textsc{Nat}^{\mathcal{I}} =_{df} \mathbb{N}^{\omega} = \mathbb{N}^* \cup \mathbb{N}^{\infty}$ and $\textsc{Bool}^{\mathcal{I}} =_{df} \mathbb{B}^{\omega} = \mathbb{B}^* \cup \mathbb{B}^{\infty}$, specifying streams of naturals and streams of booleans, respectively. In a similar vein, we put $(\textsc{Nat} \times \textsc{Bool})^{\mathcal{I}} =_{df} (\mathbb{N} \times \mathbb{B})^{\omega}$ to represent simple database tables as streams of data pairs. Obviously, the interpretations $\textsc{Nat}^{\mathcal{I}}$, $\textsc{Bool}^{\mathcal{I}}$, $(\textsc{Nat} \times \textsc{Bool})^{\mathcal{I}}$ are all subsets of $\Delta^{\mathcal{I}}$, closed under $\preceq^{\mathcal{I}}$ and all contain $\bot^{\mathcal{I}}$.

It is not difficult to see that in this interpretation we have the *type equivalences* $\textsc{Nat} \equiv \forall val.\textsc{Nat} \equiv \exists val.\textsc{Nat}$ and $\textsc{Bool} \equiv \forall val.\textsc{Bool} \equiv \exists val.\textsc{Bool}$. The fact that existential and universal quantification collapse under functional roles is not surprising, except perhaps for one thing: The existential typing $s \in (\exists val.\textsc{Nat})^{\mathcal{I}}$ does not imply the existence of a value $n \in \mathbb{N}$ such that $val(s, n^{\infty})$ as in classical logic since the stream s could be empty due to a non-terminating or deadlocking computation. Because these properties are undecidable for useful programming languages we cannot expect the type system to express emptiness. Otherwise it would become undecidable, too.

The indistinguishability of $\forall R.C$ and $\exists R.C$ on fallible entities is but one of the constructive, i.e., non-classical, features of the $c\mathcal{ALC}$ type system. Another one is the omission of the Excluded Middle Principle. E.g., we find that under the stream interpretation the concept $\textsc{Nat} \sqcup \neg\textsc{Nat}$ is not identical to $\top$. Take the stream $s = 0 \cdot \mathtt{T} \cdot \mathtt{T} \cdot \mathtt{T} \cdots$ which starts with value 0 and then turns into the infinite constant stream of Booleans $\mathtt{T}$. It is easy to verify that $s \notin \textsc{Bool}$ and $s \notin \neg\textsc{Bool}$. The former is obvious and the latter holds because $s \in \neg\textsc{Bool}$ would mean that s must have non-Boolean values arbitrarily late in the stream but this is not the case. Notice that we would have $\textsc{Nat} \sqcup \neg\textsc{Nat} \equiv \top$ in classical DL which is incompatible with our computational interpretation.

The other classical principle that does not hold for our streams is the distribution of existential $\exists$ over disjunction $\sqcup$, i.e., the equivalence $\exists val.(C \sqcup D) \equiv \exists val.C \sqcup \exists val.D$ which we discussed already in Example 2. Let us illustrate this in terms of an useful operation in the semantical analysis of mass data in knowledge engineering, viz. the linearisation of tables. Suppose we linearise a table $t = (n_0, b_0) \cdot (n_1, b_1) \cdot (n_2, b_2) \cdots$ of (stream) type $\textsc{Nat} \times \textsc{Bool}$ to give the flattened stream $t^{\flat} = n_0 \cdot b_0 \cdot n_1 \cdot b_1 \cdot n_2 \cdot b_2 \cdots$. What is the type of $t^{\flat}$? It is not the concept $\textsc{Nat} \sqcup \textsc{Bool}$ nor the equivalent $\exists val.\textsc{Nat} \sqcup \exists val.\textsc{Bool}$ since this would require that all elements of $t^{\flat}$ are either $\textsc{Nat}$ or all are $\textsc{Bool}$. The correct type instead is *set union* $\textsc{Nat} \cup \textsc{Bool}$ which is expressed by the concept $\exists val.(\textsc{Nat} \sqcup \textsc{Bool})$ saying that the first element of each suffix sequence is of value $\textsc{Nat}$ or $\textsc{Bool}$. The use of $\exists val$ here performs the decomposition of the stream so that the concept specification $\textsc{Nat} \sqcup \textsc{Bool}$ is applied element–wise rather than globally. In this way, the difference between concepts $\exists val.(\textsc{Nat} \sqcup \textsc{Bool})$ and $\exists val.\textsc{Nat} \sqcup \exists val\textsc{Bool}$, or between $\textsc{Nat} \cup \textsc{Bool}$ and $\textsc{Nat} \sqcup \textsc{Bool}$ for that matter, permits us to distinguish between local (dynamic) and global (static) choice. Again, in classical DL this important distinction is collapsed. Observe that an oscillating stream $s = 0 \cdot \mathtt{T} \cdot 0 \cdot \mathtt{T} \cdot 0 \cdot \mathtt{T} \cdots$ satisfies the concept $Osc =_{df} \neg\textsc{Nat} \sqcap \neg\textsc{Bool} \sqcap (\textsc{Nat} \cup \textsc{Bool})$ which says *"s is never in* $\textsc{Nat}$ *nor in* $\textsc{Bool}$ *but always in their union* $\textsc{Nat} \cup \textsc{Bool}$*"*. In fact, *Osc* specifies streams which are infinite and oscillate between $\textsc{Nat}$ and $\textsc{Bool}$. This is only possible in constructive logic which can make sense of non-atomic or non-static entities.

The flattening $t \mapsto t^\flat$ considered above, which implements a particular way of multiplexing data streams, has the functional type $\textsc{Nat}\times\textsc{Bool} \to \exists val.(\textsc{Nat}\sqcup\textsc{Bool})$. There are many other functions of this type, of course. Conversely, de-multiplexing functions taking the linearised stream $t^\flat$ back to t will have type $\exists val.(\textsc{Nat}\sqcup\textsc{Bool}) \to \textsc{Nat}\times\textsc{Bool}$. Under the Curry-Howard Isomorphism (propositions-as-types) [26, 27] the Cartesian product $C \times D$ is the constructive interpretation of conjunction $C \sqcap D$ and function spaces $C \to D$ are the constructive reading of subsumptions $C \sqsubseteq D$. In this view, multiplexing and de-multiplexing data streams would be different constructive realisations of the subsumptions

$$(\textsc{Nat} \sqcap \textsc{Bool}) \sqsubseteq \exists val.(\textsc{Nat} \sqcup \textsc{Bool}) \qquad \exists val.(\textsc{Nat} \sqcup \textsc{Bool}) \sqsubseteq (\textsc{Nat} \sqcap \textsc{Bool}).$$

The uniform flattening indicated above is nothing but a very particular translation program $(\cdot)^\flat$ of type $\textsc{Nat}\times\textsc{Bool} \sqsubseteq \exists val.(\textsc{Nat}\sqcup\textsc{Bool})$ which plays the role of a $c\mathcal{ALC}$ TBox axiom. Also note how fallibility of ϵ naturally corresponds to the polymorphism of the empty list: it can be used at any type. $\square$

It will be convenient to introduce a semantical validity relation $\models$ as follows: Write $\mathcal{I}; x \models C$ to abbreviate $x \in C^{\mathcal{I}}$ in which case we say that entity x *satisfies* concept C in the interpretation $\mathcal{I}$. Further, $\mathcal{I}$ is a *model* of C, written $\mathcal{I} \models C$ iff $\forall x \in \Delta^{\mathcal{I}}. \mathcal{I}; x \models C$. Finally, $\models C$ means $\forall \mathcal{I}. \mathcal{I} \models C$. All notions $\mathcal{I}; x \models \Phi$, $\mathcal{I} \models \Phi$ and $\models \Phi$ are extended to sets Φ of concepts in the usual universal fashion.

In typical reasoning tasks the interpretation $\mathcal{I}$ and the entity x in a verification goal such as $\mathcal{I}; x \models C$ are not given directly but are themselves axiomatised by sets of formulas, specifically a *TBox* Θ for $\mathcal{I}$ and an *ABox* Γ for $x \in \Delta^{\mathcal{I}}$. Accordingly, we write $\Theta; \Gamma \models C$ if for all interpretations $\mathcal{I}$ which are models of all axioms in Θ it is the case that every entity x of $\mathcal{I}$ which satisfies all axioms in Γ must also satisfy concept C. Formally, $\forall \mathcal{I}. \forall x \in \Delta^{\mathcal{I}}. (\mathcal{I} \models \Theta \ \& \ \mathcal{I}; x \models \Gamma) \Rightarrow \mathcal{I}; x \models C$. Here is how standard concept reasoning is covered:

- $\Theta; \{C\} \not\models \bot$ iff concept C is *satisfiable* with respect to the TBox Θ, i.e., there exists $\mathcal{I}$ with $\mathcal{I} \models \Theta$ and non-fallible $x \in \Delta_c^{\mathcal{I}}$ such that $x \in C^{\mathcal{I}}$;
- $\Theta; \{C, D\} \models \bot$ iff the concepts C and D are *disjoint* with respect to Θ, i.e., $C^{\mathcal{I}}$ and $D^{\mathcal{I}}$ do not share any non-fallible entities in all models $\mathcal{I}$ of Θ;
- $\Theta; \{C\} \models D$ iff concept C is *subsumed* by concept D, i.e., for all $\mathcal{I}$ with $\mathcal{I} \models \Theta$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$; The same can be expressed by $\Theta; \emptyset \models C \sqsubseteq D$ (by reflexivity of $\preceq$);
- $\Theta; \emptyset \models (C \sqsubseteq D) \sqcap (D \sqsubseteq C)$ iff concepts C and D are *equivalent* with respect to Θ, i.e., for all $\mathcal{I}$ with $\mathcal{I} \models \Theta$ we have $C^{\mathcal{I}} = D^{\mathcal{I}}$. We define $C \equiv D$ to be the concept description $(C \sqsubseteq D) \sqcap (D \sqsubseteq C)$.

It is easy to see that $\mathcal{I} \models C \sqcap D$ iff $\mathcal{I} \models C$ and $\mathcal{I} \models D$. It follows that all the above inferences can be reduced to concept subsumption $\Theta; \{C\} \models D$ as in classical DL. Unlike classical DL, however, we cannot reduce concept inferences to the special form $\Theta; \{C\} \not\models \bot$ of satisfiability. Instead, we need to implement the generalised satisfiability check $\Theta; \{C\} \not\models D$ for *arbitrary* D. We will see in Sec. 3.2 how to build a tableau-calculus for such generalised *constructive satisfiability*. Another difference to classical DL is that whenever $\models C \sqcup D$ then $\models C$ or $\models D$. This is known as the *Disjunction Property*, a definitive feature of constructive logic. In classical DL, we have $\models C \sqcup \neg C$ for every concept C even if neither $\models C$ nor $\models \neg C$. The Disjunction Property is the key to proof extraction for $c\mathcal{ALC}$ (See Example 5).

$c\mathcal{ALC}$ is related to the constructive modal logic CK (Constructive K) [28, 5, 18] as $\mathcal{ALC}$ is related to the classical modal system K [11]. In $c\mathcal{ALC}$ the classical principles of the Excluded Middle $C \sqcup \neg C = \top$, double negation $\neg\neg C = C$, the dualities $\exists R.C = \neg\forall R.\neg C$, $\forall R.C = \neg\exists R.\neg C$ and Disjunctive Distribution $\exists R.(C \sqcup D) = \exists R.C \sqcup \exists R.D$ are no longer tautologies but non-trivial TBox statements to axiomatise specialised classes of application scenarios (see Sec. 4). The fact that Excluded Middle, double negation and the dualities do not hold is a feature which $c\mathcal{ALC}$ has in common with standard intuitionistic modal logics such as [13, 21, 14, 23]. It is well known that these principles are non-constructive and therefore need special care. In $c\mathcal{ALC}$, however, we go one step further and refute the principle of Disjunctive Distribution (and, in fact, also the nullary version $\neg\Diamond\bot$) arguing that this principle is not consistent with abstraction. Disjunctive Distribution, which corresponds to the classical $\Diamond$-dual of the normality axiom $\Box(A \wedge B) = \Box A \wedge \Box B$, is commonly accepted for intuitionistic modal logics. In other words, as a modal logic, $c\mathcal{ALC}$ is non-normal regarding $\Diamond$ and thus proofs of decidability and finite model property for standard intuitionistic modal logics (e.g., for $\mathbf{IntK}_{\Box,\Diamond}$ [15][Chap 10]) do not directly apply.

3 Constructive Proof Systems for $c\mathcal{ALC}$

In this section we present simple Hilbert and Gentzen-style deduction systems for $c\mathcal{ALC}$ which admit a direct interpretation of proofs as computations following the Curry-Howard-Isomorphism in which the refinement relation $\preceq$ is treated implicitly. The presence of the semantic refinement structure is visible in the fact that the concept operators $\sqcap$, $\sqcup$, $\sqsubseteq$ on the one hand and $\forall R$, $\exists R$ on the other are primitive and not expressible any more in terms of each other with the help of negation as in classical DLs. This makes sense since all have different computational meaning. According to the Curry-Howard-Isomorphism concept descriptions are types so that, e.g., concept conjunction $\sqcap$ corresponds to Cartesian product $\times$, disjunction $\sqcup$ to disjoint union $+$, subsumption $\sqsubseteq$ to function spaces $\to$ (see Ex. 5).

3.1 Hilbert Calculus for $c\mathcal{ALC}$

(a)	1:	$C \sqsubseteq (D \sqsubseteq C)$
	2:	$((C \sqsubseteq (D \sqsubseteq E)) \sqsubseteq (C \sqsubseteq D) \sqsubseteq (C \sqsubseteq E))$
	3:	$C \sqsubseteq (D \sqsubseteq (C \sqcap D))$
	4:	$(C \sqcap D) \sqsubseteq C, \quad (C \sqcap D) \sqsubseteq D$
	5:	$C \sqsubseteq (C \sqcup D), \quad D \sqsubseteq (C \sqcup D)$
	6:	$(C \sqsubseteq E) \sqsubseteq ((D \sqsubseteq E) \sqsubseteq (C \sqcup D \sqsubseteq E))$
	7:	$\bot \sqsubseteq C$
(b)	$\forall K$:	$(\forall R.\,(C \sqsubseteq D)) \sqsubseteq (\forall R.C \sqsubseteq \forall R.D)$
	$\exists K$:	$(\forall R.\,(C \sqsubseteq D)) \sqsubseteq (\exists R.C \sqsubseteq \exists R.D)$
(c)	Nec :	If C is a theorem, then so is $\forall R.C$.
	MP :	If C and $C \sqsubseteq D$ are theorems, then so is D.

Note: Negation $\neg C$ can be coded as $C \sqsubseteq \bot$ and $\top$ as $\bot \sqsubseteq \bot$.

Table 1: Hilbert Calculus for $c\mathcal{ALC}$.

The Hilbert calculus is given in Table 1 (a) by the usual axioms for *intuitionistic propositional logic* [27], specifically (1)–(2) for subsumption $\sqsubseteq$, (3)–(4) for intersection $\sqcap$, (5)–(6) for disjunction $\sqcup$ and (7) for inconsistency $\bot$. Part (b) of Table 1 lists the the two *extensionality principles* $\exists K, \forall K$ for universal and existential role filling. The rules of *Modus Ponens* MP and *Necessitation* Nec are given in item (c) of Table 1.

Let the symbol $\vdash_H$ denote Hilbert deduction, i.e., $\Theta \vdash_H C$ if there exists a derivation $C_0, C_1, \ldots, C_n$ such that $C_n = C$ and each C_i ($i \leq n$) is either a hypothesis $C_i \in \Theta$, or a substitution instance of an axiom scheme from Table 1 or arises from earlier concepts C_j ($j < i$) through MP or Nec. This can be lifted to sets of concepts Φ, i.e., $\Theta \vdash_H \Phi$ by $\Phi =_{df} \bigwedge_{C \in \Phi} C$, where $\bigwedge$ is the intersection $\sqcap$ over a set of concepts.

The Hilbert calculus implements TBox-reasoning in the sense that it decides the semantical relationship $\Theta; \emptyset \models C$ which says that C is a *universal* concept in all models of TBox Θ.

Theorem 1 (Hilbert Soundness and Completeness)**.** $\Theta; \emptyset \models C$ iff $\Theta \vdash_H C$. □

Proof. Soundness and completeness follow from soundness and completeness of the Gentzen tableau system (Thm. 2), by showing that any deduction in either system can be translated into the other (see Proposition 1). □

Example 5. We reconsider the example by Brachman et.al. (1991) as reported by [8]:

$$\Theta \vdash_H \textsc{food} \sqsubseteq \exists \mathit{goesWith}.(\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine}) \tag{1}$$

in the TBox $\Theta = \{Ax_1, Ax_2\}$ where $Ax_1 =_{df} \textsc{food} \sqsubseteq \exists \mathit{goesWith}.\textsc{color}$ and $Ax_2 =_{df} \textsc{color} \sqsubseteq \exists \mathit{isColorOf}.\textsc{wine}$. The Curry-Howard-Isomorphism can be adapted to understand any Hilbert-proof of (1) as a program construction. For instance, the axiom Ax_1 can be read as a function ax_1 translating FOOD-entities f into COLOR-entities c such that $\mathit{goesWith}(f, c)$ and similarly Ax_2 is a function ax_2 from COLORS c to WINES w so that $\mathit{isColorOf}(c, w)$.

The Hilbert proof as shown in Fig. 3 then represents a $c\mathcal{ALC}$-type-directed construction of a data base program. The derivation of (1) as shown in Fig. 3 then is the construction of a uniform function from FOOD f to pairs (c, w) of COLOR c and WINE w with $\mathit{goesWith}(f, c)$ and $\mathit{isColorOf}(c, w)$. How this can be done formally has been shown by Bozzato et.al. in [8]. In the following we recall (and slightly generalise) their constructions.

1.	$\textsc{color} \sqsubseteq (\exists \mathit{isColorOf}.\textsc{wine} \sqsubseteq (\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine}))$	by IPL 3
2.	$(\textsc{color} \sqsubseteq (\exists \mathit{isColorOf}.\textsc{wine} \sqsubseteq (\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine}))) \sqsubseteq ((\textsc{color} \sqsubseteq \exists \mathit{isColorOf}.\textsc{wine}) \sqsubseteq (\textsc{color} \sqsubseteq (\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine})))$	by IPL 2
3.	$(\textsc{color} \sqsubseteq \exists \mathit{isColorOf}.\textsc{wine}) \sqsubseteq (\textsc{color} \sqsubseteq (\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine}))$	from 1.,2. by MP
4.	$\textsc{color} \sqsubseteq (\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine})$	from 3., Ax_2 by MP
5.	$\forall \mathit{goesWith}.(\textsc{color} \sqsubseteq (\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine})) \sqsubseteq (\exists \mathit{goesWith}.\textsc{color} \sqsubseteq \exists \mathit{goesWith}.(\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine}))$	by $\exists K$
6.	$\forall \mathit{goesWith}.(\textsc{color} \sqsubseteq (\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine}))$	from 4. by Nec
7.	$(\exists \mathit{goesWith}.\textsc{color} \sqsubseteq \exists \mathit{goesWith}.(\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine}))$	from 5.,6. by MP
8.	$(\exists \mathit{goesWith}.\textsc{color} \sqsubseteq \exists \mathit{goesWith}.(\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine})) \sqsubseteq (\textsc{food} \sqsubseteq (\exists \mathit{goesWith}.\textsc{color} \sqsubseteq \exists \mathit{goesWith}.(\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine})))$	by IPL 1
9.	$\textsc{food} \sqsubseteq (\exists \mathit{goesWith}.\textsc{color} \sqsubseteq \exists \mathit{goesWith}.(\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine}))$	from 7., 8. by MP
10.	$((\textsc{food} \sqsubseteq (\exists \mathit{goesWith}.\textsc{color} \sqsubseteq \exists \mathit{goesWith}.(\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine}))) \sqsubseteq (\textsc{food} \sqsubseteq \exists \mathit{goesWith}.\textsc{color}) \sqsubseteq (\textsc{food} \sqsubseteq \exists \mathit{goesWith}.(\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine})))$	by IPL 2
11.	$(\textsc{food} \sqsubseteq \exists \mathit{goesWith}.\textsc{color}) \sqsubseteq (\textsc{food} \sqsubseteq \exists \mathit{goesWith}.(\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine}))$	from 9., 10. by MP
12.	$\textsc{food} \sqsubseteq \exists \mathit{goesWith}.(\textsc{color} \sqcap \exists \mathit{isColorOf}.\textsc{wine})$	from Ax_1, 11. by MP

Figure 3: Example deduction in Hilbert system

With each concept C we associate a set of *realisers* or *information terms* $\mathrm{IT}(C)$. These realisers are then taken as extra ABox parameters so that instead of $\mathcal{I}; x \models C$ we declare what it means that $\mathcal{I}; x \models \langle\alpha\rangle C$ for a particular realiser $\alpha \in \mathrm{IT}(C)$. This so-called *realisability predicate* gives additional constructive semantics to our concepts in the sense that $\mathcal{I}; x \models \langle\alpha\rangle C$ implies $\mathcal{I}; x \models C$ while $\mathcal{I}; x \models C$ does not mean $\mathcal{I}; x \models \langle\alpha\rangle C$ for all but only for specific α if at all.

The sets $\mathrm{IT}(C)$ and refined concepts $\langle\alpha\rangle C$ are defined by induction on C. For our example we only need the following information terms:

- $\mathrm{IT}(A) =_{\mathit{df}} \{\mathrm{tt}\}$ for atomic concepts;
- $\mathrm{IT}(C \sqcap D) =_{\mathit{df}} \mathrm{IT}(C) \times \mathrm{IT}(D)$;
- $\mathrm{IT}(C \sqsubseteq D) =_{\mathit{df}} \mathrm{IT}(C) \to \mathrm{IT}(D)$;
- $\mathrm{IT}(\exists R.C) =_{\mathit{df}} \Delta^{\mathcal{I}} \times \mathrm{IT}(C)$;

- $\mathrm{IT}(\forall R.C) =_{df} \Delta^{\mathcal{I}} \to \mathrm{IT}(C)$.

Realisability is such that

- $\mathcal{I}; x \models \langle \mathrm{tt} \rangle A$ iff $x \in A^{\mathcal{I}}$;
- $\mathcal{I}; x \models \langle \alpha, \beta \rangle (C \sqcap D)$ iff $\mathcal{I}; x \models \langle \alpha \rangle C$ and $\mathcal{I}; x \models \langle \beta \rangle C$;
- $\mathcal{I}; x \models \langle f \rangle (C \sqsubseteq D)$ iff $\forall \alpha \in \mathrm{IT}(C).\ \mathcal{I}; x \models \langle \alpha \rangle C \Rightarrow \mathcal{I}; x \models \langle f \alpha \rangle D$;
- $\mathcal{I}; x \models \langle a, \alpha \rangle (\exists R.C)$ iff $(x, a) \in R^{\mathcal{I}}$ and $\mathcal{I}; a \models \langle \alpha \rangle C$;
- $\mathcal{I}; x \models \langle \alpha \rangle (\forall R.C)$ iff $\forall a \in \Delta^{\mathcal{I}}.\ (x, a) \in R^{\mathcal{I}} \Rightarrow \mathcal{I}; a \models \langle \alpha\, a \rangle C$.

One then shows that every proof $\vdash_H C$ generates, for any interpretation $\mathcal{I}$, a function $f{:}\Delta^{\mathcal{I}} \to \mathrm{IT}(C)$ such that $\forall u \in \Delta^{\mathcal{I}}. \mathcal{I}; u \models \langle f u \rangle C$. Specifically, every of the following Hilbert axioms IPL1:$C \sqsubseteq (D \sqsubseteq C)$, IPL2:$((C \sqsubseteq (D \sqsubseteq E)) \sqsubseteq (C \sqsubseteq D) \sqsubseteq (C \sqsubseteq E))$ and IPL3:$C \sqsubseteq (D \sqsubseteq (C \sqcap D))$ is realised by a λ-term: For instance, IPL1 $=_{df}$ $\lambda u.\lambda x.\lambda y.x$, IPL2 $=_{df}$ $\lambda u.\lambda x.\lambda y.\lambda z.\ (xz)(yz)$ and IPL3 $=_{df}$ $\lambda u.\lambda x.\lambda y.\ (x, y)$. Axiom $\exists K$ is the function $\exists K =_{df} \lambda u.\lambda x.\lambda y.(\pi_1 y, x(\pi_1 y)(\pi_2 y))$. Rules of MP and Nec are refined to

If $\langle \alpha \rangle C$ and $\langle \beta \rangle (C \sqsubseteq D)$ then $\langle \lambda u.(\beta\, u)(\alpha\, u) \rangle D$
If $\langle \alpha \rangle C$ then $\langle \lambda u.\lambda x.\, \alpha\, x \rangle (\forall R.C)$.

In this way, the derivation of (1) (See Fig. 3), up to reductions in the λ-calculus, corresponds to

$$prf \;=\; \lambda u.\lambda x.(\pi_1(ax_1\, x), (\pi_2(ax_1\, x), (\pi_1(ax_2(\pi_2(ax_1\, x))), \pi_2(ax_2(\pi_2(ax_1\, x))))))$$

which is an information term so that

$$\forall u.\, \mathcal{I}; u \models \langle prf\ u \rangle (\textsc{food} \sqsubseteq \exists goesWith.(\textsc{color} \sqcap \exists isColorOf.\textsc{wine}))$$

assuming that $\forall u.\, \mathcal{I}; u \models \langle ax_1\, u \rangle Ax_1$ and $\forall u.\, \mathcal{I}; u \models \langle ax_2\, u \rangle Ax_2$. Such realisers ax_1, ax_2 can be obtained from a concrete ABox [8].

Either they arise as proof terms themselves, or they are induced from a particular semantic ABox, as shown by Bozzato [8]. For instance, take the (classical) interpretation $\mathcal{I}$ described by

$$
\begin{aligned}
\Delta^{\mathcal{I}} &=_{df} \{\texttt{barolo}, \texttt{chardonnay}, \texttt{red}, \texttt{white}, \texttt{fish}, \texttt{meat}\},\\
\textsc{wine}^{\mathcal{I}} &=_{df} \{\texttt{barolo}, \texttt{chardonnay}\},\\
\textsc{color}^{\mathcal{I}} &=_{df} \{\texttt{red}, \texttt{white}\},\\
\textsc{food}^{\mathcal{I}} &=_{df} \{\texttt{fish}, \texttt{meat}\},\\
\mathit{isColorOf}^{\mathcal{I}} &=_{df} \{(\texttt{red}, \texttt{barolo}), (\texttt{white}, \texttt{chardonnay})\},\\
\mathit{goesWith}^{\mathcal{I}} &=_{df} \{(\texttt{meat}, \texttt{red}), (\texttt{fish}, \texttt{white})\}.
\end{aligned}
$$

In this interpretation (or ABox) $\mathcal{I}$ the information terms ax_1, ax_2 can be chosen such that

$$
\begin{aligned}
ax_1 &=_{df} \lambda u.\lambda x.\texttt{case}\, u\, \texttt{of}\, [\texttt{meat} \to (\texttt{red}, \mathrm{tt}) \mid \texttt{fish} \to (\texttt{white}, \mathrm{tt})]\\
ax_2 &=_{df} \lambda u.\lambda x.\texttt{case}\, u\, \texttt{of}\, [\texttt{red} \to (\texttt{barolo}, \mathrm{tt}) \mid \texttt{white} \to (\texttt{chardonnay}, \mathrm{tt})]
\end{aligned}
$$

where

$$
\begin{aligned}
&ax_1 \in \Delta^{\mathcal{I}} \to \mathrm{IT}(\textsc{food} \sqsubseteq \exists \mathit{goesWith}.\textsc{color}), \quad \text{and}\\
&ax_2 \in \Delta^{\mathcal{I}} \to \mathrm{IT}(\textsc{color} \sqsubseteq \exists \mathit{isColorOf}.\textsc{wine}).
\end{aligned}
$$

These express the constructive content of Ax_1, Ax_2 in $\mathcal{I}$.

Note that the equivalent tableau system specified in the next Sec. 3.2 would allow us to obtain *prf* more efficiently. Also, instead of reading the TBox axioms Ax_1 and Ax_2 as functions (as done in [8]) we can also interpret them constructively as *relations*, i.e., data-base tables. □

3.2 Gentzen Tableau Calculus for $c\mathcal{ALC}$

The Hilbert calculus for $c\mathcal{ALC}$ does not lend itself to efficient implementations. Much better suited for automated reasoning in practical applications are refutation or tableau calculi. Refutation or tableau calculi play an important role in automated reasoning. These combine both goal-directed proof-search and counter-model construction. In this section we will present such a tableau system for $c\mathcal{ALC}$ based on Gentzen-style sequents. In contrast to tableau systems for classical DL it is consistent with the Curry-Howard Isomorphism and thus permits proof-extraction. In contrast to natural deduction systems such as [8], Gentzen-systems not only support the constructive interpretation of proofs as λ-terms but also formalise tableau-style refutation procedures.

The tableau calculus manipulates Gentzen-style sequents $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$, where Θ, Γ, Φ are sets of concepts, not necessarily finite, and Σ, Ψ are partial functions mapping role names $R \in N_R$ to sets of concepts $\Sigma(R)$, $\Psi(R)$ which may be infinite, too. The domains of the latter functions are assumed to be finite and identical. We call $dom = dom(\Sigma) = dom(\Psi) \subseteq N_R$ the *domain* of the sequent. A sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ formalises and refines the semantic validity relationship $\Theta; \Gamma \models \Phi$ (see page 13) by extra constraints Σ, Ψ as follows: Θ is the TBox which are model assumptions. The ABox is given by the sets Σ, Γ, Φ, Ψ of the sequent. These encode information about individual entities relative to Θ. The first, Σ, Γ specify what we want an entity to satisfy and the latter Φ, Ψ what we do *not* want them to satisfy. The fact that we sandwich entities between explicit positive and negative constraints is the novel constructive aspect of the following Definition 2:

Definition 2 (Constructive Satisfiability). Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \perp^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation and $a \in \Delta^{\mathcal{I}}$ an entity in $\mathcal{I}$. We say that the pair $(\mathcal{I}, a)$ *satisfies* a sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ if $\mathcal{I}$ is a model of Θ, $\mathcal{I} \models \Theta$, and for all $R \in dom$, $L \in \Sigma(R)$, $M \in \Gamma$, $N \in \Phi$, $K \in \Psi(R)$:

- $\forall a'. \forall b. (a \preceq a' \;\&\; a' \, R \, b) \Rightarrow \mathcal{I}; b \models L$, i.e., all R-fillers of a and of its refinements a' are part of all concepts of $\Sigma(R)$;
- $\mathcal{I}; a \models M$, i.e., a and hence all its refinements are part of all concepts of Γ;
- $\mathcal{I}; a \not\models N$, i.e., a is contained in none of the concepts in Φ;
- $\forall b. \, a \, R \, b \Rightarrow \mathcal{I}; b \not\models K$, i.e., none of the R-fillers b of a is contained in any concept of $\Psi(R)$.

A sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ is *(constructively) satisfiable*, written $\Theta; \Sigma; \Gamma \not\models \Phi; \Psi$, iff there exists an interpretation $\mathcal{I}$ and entity $a \in \Delta^{\mathcal{I}}$ such that $(\mathcal{I}, a)$ satisfies the sequent. $\square$

The purpose of a *tableau* or *refutation* proof is to establish that an entity specification presented as a sequent is not satisfiable. On the other hand, if no closed tableau can be found and the calculus is complete then the failed proof search implies the existence of a satisfying entity. Our tableau calculus for $c\mathcal{ALC}$ is given by the rules seen in Fig. 4.

In all rules of Fig. 4, the hypotheses Θ, $\Sigma(R)$, Γ and conclusions Φ, $\Psi(R)$ are treated as sets rather than lists. For instance, $\Gamma, C \sqsubseteq D$ in rule $\sqsubseteq L$ is $\Gamma \cup \{C \sqsubseteq D\}$. Hence, if $C \sqsubseteq D \in \Gamma$ then Γ in the premise of $\sqsubseteq L$ is identical to $\Gamma, C \sqsubseteq D$ in the conclusion of the

$$\frac{}{\Theta \,;\, \Sigma \,;\, \Gamma, C \vdash \Phi, C \,;\, \Psi}\, Ax \qquad \frac{|\Phi \cup \Psi| \geq 1}{\Theta \,;\, \Sigma \,;\, \Gamma, \bot \vdash \Phi \,;\, \Psi}\, \bot L$$

$$\frac{\Theta \,;\, \Sigma \,;\, \Gamma, C, D \vdash \Phi \,;\, \Psi}{\Theta \,;\, \Sigma \,;\, \Gamma, C \sqcap D \vdash \Phi \,;\, \Psi}\, \sqcap L \qquad \frac{\Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi, C \,;\, \Psi \quad \Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi, D \,;\, \Psi}{\Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi, C \sqcap D \,;\, \Psi}\, \sqcap R$$

$$\frac{\Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi, C, D \,;\, \Psi}{\Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi, C \sqcup D \,;\, \Psi}\, \sqcup R \qquad \frac{\Theta \,;\, \Sigma \,;\, \Gamma, C \vdash \Phi \,;\, \Psi \quad \Theta \,;\, \Sigma \,;\, \Gamma, D \vdash \Phi \,;\, \Psi}{\Theta \,;\, \Sigma \,;\, \Gamma, C \sqcup D \vdash \Phi \,;\, \Psi}\, \sqcup L$$

$$\frac{\Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi, C \,;\, \Psi \quad \Theta \,;\, \Sigma \,;\, \Gamma, D \vdash \Phi \,;\, \Psi}{\Theta \,;\, \Sigma \,;\, \Gamma, C \sqsubseteq D \vdash \Phi \,;\, \Psi}\, \sqsubseteq L \qquad \frac{\Theta \,;\, \Sigma \,;\, \Gamma, C \vdash D \,;\, \emptyset}{\Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi, C \sqsubseteq D \,;\, \Psi}\, \sqsubseteq R$$

$$\frac{\Theta \,;\, \Sigma \,;\, \Gamma \vdash \emptyset \,;\, [R \mapsto C]}{\Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi, \exists R.C \,;\, \Psi}\, \exists R \qquad \frac{\Theta \,;\, \emptyset \,;\, \Sigma(R), C \vdash \Psi(R) \,;\, \emptyset}{\Theta \,;\, \Sigma \,;\, \Gamma, \exists R.C \vdash \Phi \,;\, \Psi}\, \exists L$$

$$\frac{\Theta \,;\, \Sigma \cup [R \mapsto C] \,;\, \Gamma \vdash \Phi \,;\, \Psi}{\Theta \,;\, \Sigma \,;\, \Gamma, \forall R.C \vdash \Phi \,;\, \Psi}\, \forall L \qquad \frac{\Theta \,;\, \emptyset \,;\, \Sigma(R) \vdash C \,;\, \emptyset}{\Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi, \forall R.C \,;\, \Psi}\, \forall R$$

$$\frac{\Theta \,;\, \Sigma \cup [R \mapsto C] \,;\, \Gamma \vdash \Phi \,;\, \Psi \qquad R \in \mathit{dom}}{\Theta, C \,;\, \Sigma \,;\, \Gamma \vdash \Phi \,;\, \Psi}\, \mathit{Hyp}_1 \qquad \frac{\Theta \,;\, \Sigma \,;\, \Gamma, C \vdash \Phi \,;\, \Psi}{\Theta, C \,;\, \Sigma \,;\, \Gamma \vdash \Phi \,;\, \Psi}\, \mathit{Hyp}_2$$

Figure 4: Gentzen-tableau rules for $c\mathcal{ALC}$.

rule. $\emptyset$ is used both as the empty set and the constant function $\emptyset(R) = \emptyset$. $[R \mapsto C]$ is the finite function with domain $\{R\}$ mapping R to the singleton set $\{C\}$ and $\Sigma \cup [R \mapsto C]$ is the union of functions with domain $\mathit{dom}(\Sigma) \cup \{R\}$ such that $(\Sigma \cup [R \mapsto C])(S) = \Sigma(S)$ for $S \neq R$ and $(\Sigma \cup [R \mapsto C])(R) = \Sigma(R) \cup \{C\}$, otherwise.

Definition 3 (Tableau and Constructive Consistency)**.** A *tableau for* $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ is a finite and closed derivation tree $\mathcal{T}$ built using instances of the rules in Fig. 4 which has $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ as its root. The sequent is *(constructively) consistent*, written $\Theta; \Sigma; \Gamma \not\vdash \Phi; \Psi$, if no tableau exists for it. □

- $\forall a'. \forall b. (a \preceq a' \;\&\; a' R\, b) \Rightarrow \mathcal{I}; b \models L$, i.e., all R-fillers of a and of its refinements a' are part of all concepts of $\Sigma(R)$;
- $\mathcal{I}; a \models M$, i.e., a and hence all its refinements are part of all concepts of Γ;
- $\mathcal{I}; a \not\models N$, i.e., a is contained in none of the concepts in Φ;
- $\forall b.\, a\, R\, b \Rightarrow \mathcal{I}; b \not\models K$, i.e., none of the R-fillers b of a is contained in any concept of $\Psi(R)$.

Our calculus is formulated in the spirit of Gentzen with left introduction rules $\sqcap L$, $\sqcup L$, $\sqsubseteq L$, $\forall L$, $\exists L$ and right introduction rules $\sqcap R$, $\sqcup R$, $\sqsubseteq R$, $\forall R$, $\exists R$ for each logical connective. These rules can be interpreted not only as tableau-style refutation steps but also have computational meaning. Specifically, the left rules correspond to input decomposition (pattern matching) and the right rules generate output information terms (data constructors). The Gentzen style presentation also lends itself to a natural game-theoretic interpretation. These features are distinct advantages over natural deduction systems such as presented in [8]. Note that there is no right intro rule $\bot R$, which is not needed. One shows for the system in Fig. 4 that whenever $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ is derivable then also $\Theta; \Sigma; \Gamma \vdash \Phi, \bot; \Psi$ is derivable which is basically $\bot L$.

It is possible to treat negated concepts directly by the following left and right introduction rules $\neg L$ resp. $\neg R$ which can be derived from the rules in Fig. 4. Since negation is encoded as $\neg C \equiv C \sqsubseteq \bot$ the rule $\neg L$ is simply a combination of the rules $\sqsubseteq L$ and $\bot L$. Rule $\neg R$ is an instance of rule $\sqsubseteq R$.

$$\frac{\Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi, C \,;\, \Psi \qquad | \Phi \cup \Psi | \geq 1}{\Theta \,;\, \Sigma \,;\, \Gamma, \neg C \vdash \Phi \,;\, \Psi}\,\neg L \qquad \frac{\Theta \,;\, \Sigma \,;\, \Gamma, C \vdash \bot \,;\, \emptyset}{\Theta \,;\, \Sigma \,;\, \Gamma \vdash \Phi, \neg C \,;\, \Psi}\,\neg R$$

Proposition 1. The Hilbert and Tableau calculi are equivalent. For any TBox Θ and set of concepts Φ we have $\Theta \vdash_H \Phi$ iff the sequent $\Theta; \emptyset; \emptyset \vdash \Phi; \emptyset$ has a tableau derivation (i.e., is inconsistent). □

Proof. This is done by showing that the tableau system can simulate the Hilbert deductions, i.e., we show that if $\Theta \vdash_H C$ then there exists a closed tableau for the sequent $\Theta; \emptyset; \emptyset \vdash C; \emptyset$. Thereof we obtain soundness of Hilbert from soundness of Gentzen (Thm. 2).

In the other direction we have to show that if there is a closed tableau for a sequent then each such sequent can be derived in the Hilbert system in closed form as an implication. The proof is by induction on the structure of a closed tableau. The details of the proof can be found in the appendix. □

Theorem 2 (Strong Soundness and Completeness)**.** A sequent is satisfiable iff it is consistent, i.e., $\Theta; \Sigma; \Gamma \not\models \Phi; \Psi \Leftrightarrow \Theta; \Sigma; \Gamma \not\vdash \Phi; \Psi$. □

Proof. For soundness we show for each derivation rule in Fig. 4 that if the conclusion is satisfiable then at least one of the premises of the rule is satisfiable, too. For the completeness direction we show that for any consistent sequent there exists a canonical constructive model that satisfies the sequent. The detailed proof is given in the appendix. □

A sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ is *finite* if it has a finite domain and for all $R \in dom$ the sets $\Sigma(R)$, $\Psi(R)$ as well as Θ, Γ, Φ are finite as well. The tableau rules in Fig. 4 induce a decidable deduction system for finite sequents. In fact, the proof of Thm. 2 shows that finite counter-models can be obtained essentially by unfolding unprovable finite end-sequents.

Theorem 3 (Finite Model Property & Decidability)**.** A finite sequent is satisfiable iff it is satisfiable in a finite interpretation. Consistency of finite sequents is decidable. □

Proof. Decidability is obtained by the simple fact that the tableau rules in Fig. 4 have the *sub-formula* property: All formulas in the premises of a rule are (not necessarily proper) sub-formulas of formulas in the conclusion. Also, the domain of a premise sequent is extended at most by a role appearing in concepts of the conclusion sequent (as in rules $\exists R$, $\forall L$ or is already part the domains). In rule Hyp_1 a role already existing in the domain of the conclusion sequent is updated. Thus, the sizes of the domains and formula sets in a tableau are bounded by the root sequent. More specifically, if we are searching for a closed tableau of a finite sequent $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle$ then we only ever need to consider tableaux with nodes formed from those (sub-)concepts and roles contained in $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle$. Since there are only a finite number of such nodes and the tableau rules are finitely branching, there are only a finite number of possible tableaux with finite root sequent $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle$. These can be enumerated and checked effectively in bounded time.

Finite Model Property follows from the completeness direction of Thm. 2 refined by showing that the canonical model which satisfies a given finite sequent is finite. □

Example 6. Auditors usually check if financial transactions expensed on different kinds of accounts are, depending on their type, in compliance with regulations and accounting standards. For example, a financial transaction *trans* may be expensed on an *Account* or by refinement on *CashBox*, say under special instructions from the manager. A corresponding ABox $\mathcal{I}$ is given in Fig. 5 with roles $N_R = \{expOn\}$ and concepts $N_C = \{\text{ACC}, \text{CASH}\}$.

There may be a TBox Θ specifying general facts about the role and concepts in $\mathcal{I}$ such as $\text{ACC} \sqsubseteq \neg\text{CASH}$ or $\forall expOn.(\text{ACC} \sqcup \text{CASH})$. The particular ABox structure of Fig. 5 can be specified by the sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ where $\Sigma(expOn) = \emptyset$, $\Gamma = \{\exists expOn.(\text{ACC} \sqcup \text{CASH})\}$, $\Phi = \{\exists expOn.\text{ACC}\}$ and $\Psi(expOn) = \{\text{CASH}\}$. Note that the ABox specification $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ is inconsistent with the classical principle of $\exists$-distributivity. E.g., if we add $\exists expOn.\text{ACC} \sqcup \exists expOn.\text{CASH}$ to Σ then the sequent becomes unsatisfiable.

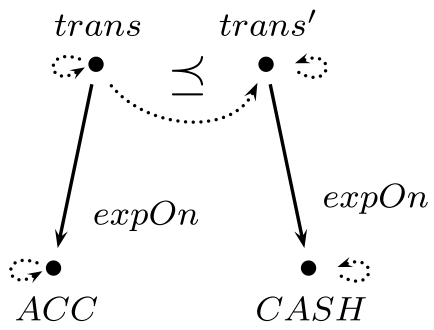

Figure 5: ABox $\mathcal{I}$

We have seen before how the following sequent specifies the ABox in Fig. 5.

$$\emptyset; \emptyset; \exists expOn.(\text{ACC} \sqcup \text{CASH}) \vdash \exists expOn.\text{ACC}; [expOn \mapsto \text{CASH}]$$

Since this sequent is satisfiable there cannot be a closed tableau for it (Thm. 2). As with classical tableaux the ABox model in Fig. 5 can be extracted systematically from the unsuccessful attempt to prove the sequent using the tableau rules.

We mentioned that satisfiability (aka unprovability) depends upon the fact that $\exists$ does not distribute over $\sqcup$ as in classical $\mathcal{ALC}$. Indeed, if we add the disjunction $\exists expOn.\text{ACC} \sqcup \exists expOn.\text{CASH}$ to the hypotheses, then there is a closed tableau. The extended sequent $\emptyset; \Sigma; \Gamma \vdash \Phi; \Psi$ is derived using $\sqcup L, \exists L, Ax$, as seen in Fig. 6. $\square$

$$\dfrac{}{\emptyset; \emptyset; \exists expOn.(\text{ACC} \sqcup \text{CASH}), \exists expOn.\text{ACC} \vdash \exists expOn.\text{ACC}; [expOn \mapsto \text{CASH}]}\ Ax$$

$$\dfrac{\dfrac{\dfrac{}{\emptyset; \emptyset; \text{CASH} \vdash \text{CASH}; \emptyset}\ Ax}{\emptyset; \emptyset; \exists expOn.(\text{ACC} \sqcup \text{CASH}), \exists expOn.\text{CASH} \vdash \exists expOn.\text{ACC}; [expOn \mapsto \text{CASH}]}\ \exists L \qquad \vdots\pi}{\emptyset; \emptyset; \exists expOn.(\text{ACC} \sqcup \text{CASH}), \exists expOn.\text{ACC} \sqcup \exists expOn.\text{CASH} \vdash \exists expOn.\text{ACC}; [expOn \mapsto \text{CASH}]}\ \sqcup L$$

Figure 6: Example of $c\mathcal{ALC}$ Tableau Deduction.

Example 7. Fig. 7 gives tableau proofs of the Hilbert axioms $\forall K$ and $\exists K$. □

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\emptyset\,;\,\emptyset\,;\,A\vdash A\,;\,\emptyset}\qquad\overline{\emptyset\,;\,\emptyset\,;\,B,A\vdash B\,;\,\emptyset}}{\emptyset\,;\,\emptyset\,;\,A\sqsubseteq B,A\vdash B\,;\,\emptyset}}{\emptyset\,;\,[R\mapsto A\sqsubseteq B,A]\,;\,\emptyset\vdash\forall R.B\,;\,\emptyset}\,\forall R}{\emptyset\,;\,[R\mapsto A\sqsubseteq B]\,;\,\forall R.A\vdash\forall R.B\,;\,\emptyset}\,\forall L}{\emptyset\,;\,\emptyset\,;\,\forall R.(A\sqsubseteq B),\forall R.A\vdash\forall R.B\,;\,\emptyset}\,\forall L}{\emptyset\,;\,\emptyset\,;\,\forall R.(A\sqsubseteq B)\vdash\forall R.A\sqsubseteq\forall R.B\,;\,\emptyset}}{\emptyset\,;\,\emptyset\,;\,\emptyset\vdash\forall R.(A\sqsubseteq B)\sqsubseteq(\forall R.A\sqsubseteq\forall R.B)\,;\,\emptyset}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\emptyset\,;\,\emptyset\,;\,A\vdash A\,;\,\emptyset}\qquad\overline{\emptyset\,;\,\emptyset\,;\,B,A\vdash B\,;\,\emptyset}}{\emptyset\,;\,\emptyset\,;\,A\sqsubseteq B,A\vdash B\,;\,\emptyset}}{\emptyset\,;\,[R\mapsto A\sqsubseteq B]\,;\,\exists R.A\vdash\emptyset\,;\,[R\mapsto B]}\,\exists L}{\emptyset\,;\,[R\mapsto A\sqsubseteq B]\,;\,\exists R.A\vdash\exists R.B\,;\,\emptyset}\,\exists R}{\emptyset\,;\,\emptyset\,;\,\forall R.(A\sqsubseteq B),\exists R.A\vdash\exists R.B\,;\,\emptyset}\,\forall L}{\emptyset\,;\,\emptyset\,;\,\forall R.(A\sqsubseteq B)\vdash\exists R.A\sqsubseteq\exists R.B\,;\,\emptyset}}{\emptyset\,;\,\emptyset\,;\,\emptyset\vdash\forall R.(A\sqsubseteq B)\sqsubseteq(\exists R.A\sqsubseteq\exists R.B)\,;\,\emptyset}$$

Figure 7: Hilbert axioms $\exists K$, $\forall K$ derived in the tableau calculus.

Example 8. The next example is based on Grimm's fairy tale "*The hare and the hedgehog*", a classical instance of fraud. A hedgehog takes a walk and runs into a hare, which he wishes a good morning. The hare, however, jokes about the hedgehog's bowlegs. Of course, the hedgehog challenges the hare for a race. The hedgehog positions his wife at the destination of the racetrack. When the race starts and the victory-certain hare near-storms, the wife of the hedgehog rises up and says: "I am already here!" Since the hare is not able to distinguish the hedgehog from its wife he looses. The hare does not accept its defeat and they repeat the race 73 times with the same result. At the 74th run the hare breaks down exhaustedly and dies.

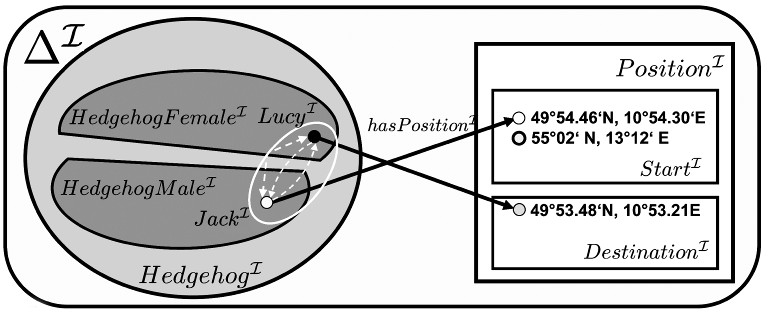

Figure 8: Model of the hedgehog couple

Fig. 8 shows a model of the situation with the individual *Jack* as the male hedgehog, i.e. $HedgehogMale^{\mathcal{I}} = \{Jack\}$, and *Lucy* representing his wife defined by $HedgehogFemale^{\mathcal{I}} = \{Lucy\}$ which are both contained in the concept *Hedgehog*. The position of *Lucy* and *Jack* is given here in terms of a role-filler over relation *hasPosition* to GPS-coordinates that belong to the concept *Start* respectively *Destination* which are subsumed by the concept *Position*. Since *Jack* and *Lucy* are not distinguishable, there is a cyclic refinement between them, i.e. they can be refined to each other.

We will use the abbreviations $Destination = A$, $Start = B$ and $hasPosition = R$ in the following. Based on our model in Fig. 8 we can show that the existential quantifier does not distribute over $\sqcup$. If $\exists R$ would distribute over $\sqcup$ as it does in classical $\mathcal{ALC}$ then this would imply that we can find a derivation for $\exists R.(A \sqcup B), \neg(\exists R.A \sqcup \exists R.B) \vdash \bot$: If the first hypothesis $\exists R.(A \sqcup B)$ of the sequent implies $\exists R.A \sqcup \exists R.B$, then this contradicts the second hypothesis $\neg(\exists R.A \sqcup \exists R.B)$ which implies $\bot$.

Now we show that the sequent cannot be derived in $c\mathcal{ALC}$. First note that $\neg(\exists R.A \sqcup \exists R.B)$ is equivalent to $\neg\exists R.A \sqcap \neg\exists R.B$. Hence we have to show that there does not exist a closed tableau for the sequent $\exists R.(A \sqcup B), \neg\exists R.A, \neg\exists R.B \vdash \bot\ ;\ \emptyset;$. This sequent models what the hare assumes to be true, viz. that for the hedgehog there exists either one R-filler to *Start* or one R-filler to *Destination*, viz. $\exists R.(A \sqcup B)$, and that the hedgehog is neither sitting in A nor in B, formulated by $\neg\exists R.A \sqcap \neg\exists R.B$. Since the hare thinks in terms of classical $\mathcal{ALC}$, he assumes that $\exists R$ distributes over $\sqcup$.

However, because of constructiveness and the fact, that *Jack* and *Lucy* are indistinguishable, the hedgehog is able to be in both positions at one time dependent upon choice of refinement. Fig. 9 shows the constructed counter model for the sequent. The sets Θ and Σ in the sequent are omitted, since we do not need them in the proof.

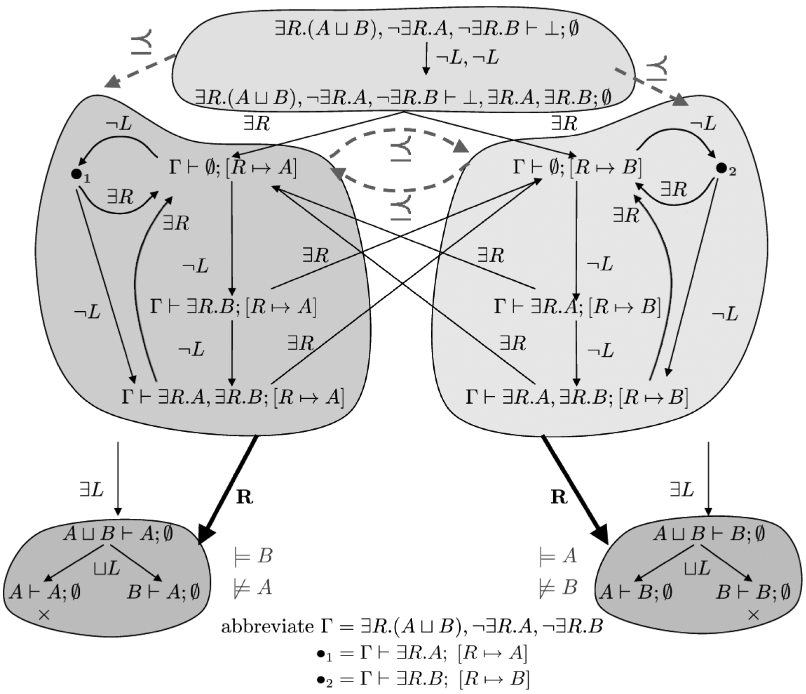

Figure 9: Proof of $\exists R.(A \sqcup B), \neg\exists R.A, \neg\exists R.B \vdash \bot\ ;\ \emptyset;$

In this case we obtain a cyclic model with two clusters of equivalent individuals which represent *Jack* and *Lucy* that can be refined to each other. The above counter model then can be collapsed to the model given in Fig. 10 that represents exactly the situation already shown in Fig. 8.

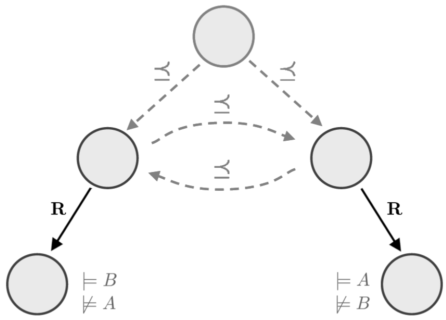

Figure 10: Simplified model of Fig. 9

□

Decidability of consistency of finite sequents is not surprising since $c\mathcal{ALC}$ can be embedded into $\mathcal{ALC}$ with transitive roles, namely $\mathcal{ALC}_{R^+}$. Therefore, the PSpace-complexity of $\mathcal{ALC}_{R^+}$ [22] forms an upper bound for satisfiability of $c\mathcal{ALC}$-concepts. On the other hand it is easy to show that concepts in negation normal form (NNF) coincide in $\mathcal{ALC}$ and $c\mathcal{ALC}$. Since all $\mathcal{ALC}$-concepts can be transformed into NNF (in linear time) and satisfiability of $\mathcal{ALC}$-concepts is PSpace, satisfiability in $c\mathcal{ALC}$ is PSpace-complete.

4 Some Specialisations between $c\mathcal{ALC}$ and $\mathcal{ALC}$

There are at least three natural dimensions in which $c\mathcal{ALC}$ is a constructive weakening of $\mathcal{ALC}$ corresponding to the axiom schemes of *Non-contradictory Fillers* $\neg\exists R.\bot$, *Disjunctive Distribution* $\exists R.(C \sqcup D) \sqsubseteq (\exists R.C \sqcup \exists R.D)$ and the *Excluded Middle* $C \sqcup \neg C$. Each of them is associated with a specific semantical restriction of interpretations which can be captured by a simple modification (strengthening) of the $c\mathcal{ALC}$ tableau calculus.

In this way, depending on the application at hand, a combination of non-classical DLs may be generated between $c\mathcal{ALC}$ and $\mathcal{ALC}$:

Interpretations without fallible elements, i.e., $\bot^{\mathcal{I}} = \emptyset$, can be axiomatised by the scheme $\neg\exists R.\bot$ which says that any entity can always be refined so it becomes fully defined for role R, i.e., all its R-fillers (if they exist) are non-fallible. In fact, the absence of axiom $\neg\exists R.\bot$ is the only effect of fallibility. It indicates the existence of entities all of whose refinements have fallible R-fillers. One can show that if an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ satisfies $\neg\exists R.\bot$ then the set $\bot^{\mathcal{I}}$ is redundant in the sense that there is a stripped interpretation $\mathcal{I}_s = (\Delta^{\mathcal{I}_s}, \preceq^{\mathcal{I}_s}, \bot^{\mathcal{I}_s}, \cdot^{\mathcal{I}_s})$ such that $\Delta^{\mathcal{I}_s} =_{df} \Delta^{\mathcal{I}}_c = \Delta^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$, $\preceq^{\mathcal{I}_s} =_{df} \preceq^{\mathcal{I}}$, $\bot^{\mathcal{I}_s} =_{df} \emptyset$ so that for all concepts C we have $C^{\mathcal{I}_s} = C^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$. This means that as long as we are only interested in non-fallible entities, $\mathcal{I}$ and $\mathcal{I}_s$ are identical. To achieve this one defines $\cdot^{\mathcal{I}_s}$ so that $A^{\mathcal{I}_s} =_{df} A^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$ for $A \in N_C$ and for all $x, y \in \Delta^{\mathcal{I}_s}$ and $R \in N_R$ we put $x\, R^{\mathcal{I}_s}\, y$ iff $x\, R^{\mathcal{I}}\, y$, or $\exists y', x'.\, x\, R^{\mathcal{I}} y' \in \bot^{\mathcal{I}}$ & $x \preceq^{\mathcal{I}} x'\, R^{\mathcal{I}}\, y$. If we want to exclude fallibility then the scheme $\neg\exists R.\bot$ can be implemented in the tableau system Fig. 4 by dropping the side-condition $|\Phi \cup \Psi| \geq 1$ from rule $\bot L$. Let us call the stronger rule without side-condition $\bot L^+$:

$$\frac{}{\Theta\ ;\ \Sigma\ ;\ \Gamma, \bot \vdash \Phi\ ;\ \Psi}\ \bot L^+$$

Using it, $\bot$ can be identified with an empty right-hand side and we get the usual right and left intro rules $\neg R$ and $\neg L$ for intuitionistic negation:

$$\frac{\Theta\ ;\ \Sigma\ ;\ \Gamma \vdash \Phi, C\ ;\ \Psi}{\Theta\ ;\ \Sigma\ ;\ \Gamma, \neg C \vdash \Phi\ ;\ \Psi}\ \neg L \qquad \frac{\Theta\ ;\ \Sigma\ ;\ \Gamma, C \vdash \emptyset\ ;\ \emptyset}{\Theta\ ;\ \Sigma\ ;\ \Gamma \vdash \Phi, \neg C\ ;\ \Psi}\ \neg R$$

Rule $\neg R$ is admissible already in $c\mathcal{ALC}$ but $\neg L$ is not. Fig. 11 shows the tableau proofs for rules $\neg\exists R.\bot$ and $\neg L$ based on $\bot L^+$. Remember that $\neg C$ abbreviates $C \sqsubseteq \bot$.

In contrast to classical and other intuitionistic logics $\exists R$ does not distribute over $\sqcup$ in $c\mathcal{ALC}$. If we add the *Principle of Disjunctive Distribution* $\exists R.(C \sqcup D) \sqsubseteq (\exists R.C \sqcup \exists R.D)$ we are essentially saying that role filling via R is *confluent* with refinement, i.e., that whenever $xR^{\mathcal{I}}y$ and $x \preceq x'$ then there exists y' such that $y \preceq y'$ and $x'R^{\mathcal{I}}y'$. In other words, if an entity x has an R-filler y, then all of its refinements, too, have an R-filler which is a refinement of y. In this case, filling and refinement are orthogonal concepts.

$$
\dfrac{\dfrac{\dfrac{}{\emptyset\ ;\ \emptyset\ ;\ \bot \vdash \emptyset\ ;\ \emptyset}\,\bot L^+}{\emptyset\ ;\ \emptyset\ ;\ \exists R.\bot \vdash \bot\ ;\ \emptyset}\,\exists L}{\emptyset\ ;\ \emptyset\ ;\ \emptyset \vdash \neg\exists R.\bot\ ;\ \emptyset}\,\sqsubseteq R
\qquad
\dfrac{\Theta\ ;\ \Sigma\ ;\ \Gamma \vdash \Phi, C\ ;\ \Psi \qquad \dfrac{}{\Theta\ ;\ \Sigma\ ;\ \Gamma, \bot \vdash \Phi\ ;\ \Psi}\,\bot L^+}{\Theta\ ;\ \Sigma\ ;\ \Gamma, \neg C \vdash \Phi\ ;\ \Psi}\,\sqsubseteq L
$$

Figure 11: Tableau proofs for $\neg\exists R.\bot$ and $\neg L$

One can show that refinement then can be assumed to be *antisymmetric*, i.e. $x \preceq y$ and $y \preceq x$ implies $x = y$. Disjunctive Distribution can be accommodated in the tableau system by strengthening rule $\exists R$ to $\exists R^+$:

$$
\dfrac{\Theta; \Sigma; \Gamma \vdash \Phi; \Psi \cup [R \mapsto C]}{\Theta; \Sigma; \Gamma \vdash \Phi, \exists R.C; \Psi}\,\exists R^+
$$

which makes it perfectly dual to $\forall R$. Fig. 12 presents the proof of Disjunctive Distribution from $\exists R^+$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\emptyset\ ;\ \emptyset\ ;\ C \vdash C, D\ ;\ \emptyset}\,Ax \qquad \dfrac{}{\emptyset\ ;\ \emptyset\ ;\ D \vdash C, D\ ;\ \emptyset}\,Ax}{\emptyset\ ;\ \emptyset\ ;\ C \sqcup D \vdash C, D\ ;\ \emptyset}\,\sqcup L}{\emptyset\ ;\ \emptyset\ ;\ \exists R.(C \sqcup D) \vdash \emptyset\ ;\ [R \mapsto C, D]}\,\exists L}{\emptyset\ ;\ \emptyset\ ;\ \exists R.(C \sqcup D) \vdash \exists R.D\ ;\ [R \mapsto C]}\,\exists R^+}{\emptyset\ ;\ \emptyset\ ;\ \exists R.(C \sqcup D) \vdash \exists R.C, \exists R.D\ ;\ \emptyset}\,\exists R^+}{\emptyset\ ;\ \emptyset\ ;\ \exists R.(C \sqcup D) \vdash \exists R.C \sqcup \exists R.D\ ;\ \emptyset}\,\sqcup R}{\emptyset\ ;\ \emptyset\ ;\ \emptyset \vdash \exists R.(C \sqcup D) \sqsubseteq (\exists R.C \sqcup \exists R.D)\ ;\ \emptyset}\,\sqsubseteq R
$$

Figure 12: Rule $\exists R^+$ implements Disjunctive Distribution

We observed that if refinement $\preceq$ is the identity relation and there are no fallible entities then the interpretation becomes classical $\mathcal{ALC}$. This corresponds to extending $c\mathcal{ALC}$ by both schemes $\exists R.(C \sqcup D) \sqsubseteq (\exists R.C \sqcup \exists R.D)$ and $\neg\exists R.\bot$ as well as the Principle of the *Excluded Middle* $C \sqcup \neg C$. But what happens if we only add the scheme $C \sqcup \neg C$ to $c\mathcal{ALC}$? It turns out that this corresponds to assuming that $\preceq$ is an equivalence relation, meaning it is *symmetric* so that $x \preceq y$ implies $y \preceq x$. In effect, then, there are no proper refinements of entities but only *clusters of indistinguishables*. In the tableau calculus we can easily implement Excluded Middle by replacing the intuitionistic rule for implication $\sqsubseteq R$ in Fig. 4 by the classical one $\sqsubseteq R^+$ which is:

$$\frac{\Theta; \Sigma; \Gamma, C \vdash \Phi, D; \Psi}{\Theta; \Sigma; \Gamma \vdash \Phi, C \sqsubseteq D; \Psi} \sqsubseteq R^+$$

The difference is that in applying $\sqsubseteq R^+$ backwards we do not lose the contexts Φ, Ψ as we do in $\sqsubseteq R$. This is the standard restriction which turns the classical into the intuitionistic sequent calculus. Fig. 13 gives the proof of the Excluded Middle from $\sqsubseteq R^+$.

$$\dfrac{\dfrac{\dfrac{}{\emptyset\ ;\ \emptyset\ ;\ C \vdash C, \bot\ ;\ \emptyset}\ Ax}{\emptyset\ ;\ \emptyset\ ;\ \emptyset \vdash C, \neg C\ ;\ \emptyset}\ \sqsubseteq R^+}{\emptyset\ ;\ \emptyset\ ;\ \emptyset \vdash C \sqcup \neg C\ ;\ \emptyset}\ \sqcup R$$

Figure 13: Rule $\sqsubseteq R^+$ implements Excluded Middle

Note that $c\mathcal{ALC}$ + Excluded Middle is properly more expressive than $\mathcal{ALC}$. Therefore, $c\mathcal{ALC}$ is not the intuitionistic analog of $\mathcal{ALC}$ in the sense of Simpson [23] but a *constructive* or *sub-intuitionistic* analog. In fact, $c\mathcal{ALC}$ + Non-contradictory Fillers yields the multi-modal version of Wijesekera's constructive modal logic [28]. However, if we further add Disjunctive Distribution and the axiom scheme $(\exists R.C \sqsubseteq \forall R.D) \sqsubseteq \forall R.(C \sqsubseteq D)$ then we obtain the multi-modal version $i\mathcal{ALC}$ of the standard intuitionistic logic of Fischer-Servi [14] known as IK [23] or **FS** [15]. See [11] for a deeper discussion of the difference between $c\mathcal{ALC}$ and $i\mathcal{ALC}$. Here it suffices to point out that $i\mathcal{ALC}$ is a special theory of $c\mathcal{ALC}$ which enforces additional relationships between role filling and refinement which may or may not be adequate for a given application.

5 Conclusion and Future Work

We presented a new constructive interpretation of DL which refines the classical one and generates a family of theories that admit computational interpretations of proofs in line with the Curry-Howard Isomorphism. This new interpretation is consistent with the idea of concepts comprising abstract entities with hidden fine-structure. It supports intensional ABox and TBox theories with semantic slack in the sense that not all aspects of the low-level structure of entities can necessarily be captured by the concept language.

This gives rise to the notion of *constructive satisfiability* and a stronger form of OWA, which we tentatively call the *Evolving Open World Assumption*.

In this work we applied this interpretation to $\mathcal{ALC}$ as the core DL obtaining $c\mathcal{ALC}$ together with sound and complete Hilbert and Tableau deduction systems. The semantics is general enough that it should be applicable to other DLs, too. It is conservative in that all constructions of $c\mathcal{ALC}$ are sound in $\mathcal{ALC}$. The point is that $c\mathcal{ALC}$ does not permit constructions which are incompatible with refinement. We have given examples where $\mathcal{ALC}$ would not be adequate. $c\mathcal{ALC}$ enjoys semantical robustness and admits decidable tableau with proof extraction and counter-model construction. Where the application supports it we can specialise $c\mathcal{ALC}$ back towards $\mathcal{ALC}$ by adding axioms or strengthen some tableau rules suitably, as discussed.

We aim to extend $c\mathcal{ALC}$ for the domain of mass data business auditing by designing specialised example ontologies. We plan to extract and automate auditing processes from proof terms by using the calculus as an interactive design and type specification system of data streams and audit component interfaces. Towards this end we will give full separation between ABox and TBox reasoning, specifically explicit representation of ABoxes in sequents. As in standard DL tableaux each node would then describe information about a full ABox rather than a single entity, which yields a more global construction.

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A Proofs for Section 3

In this appendix we give proofs of our main results. We first mention some simple auxiliary lemmas:

Lemma 1
For every concept C it holds that $\forall a \in \Delta^{\mathcal{I}}. a \in \bot^{\mathcal{I}} \Rightarrow a \in C^{\mathcal{I}}$. □

Proof. The proof is by induction on the structure of C. For the base case we assume $a \in \bot^{\mathcal{I}}$ and show

(1) $a \in \bot^{\mathcal{I}}$;

(2) $a \in A^{\mathcal{I}}$;

(3) $a \in \top^{\mathcal{I}}$.

(1) follows directly by assumption. From Definition 1 we know $\bot^{\mathcal{I}} \subseteq A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} = \top^{\mathcal{I}}$ which together with our assumption proves the cases (2)–(3).

The induction hypothesis is $\forall a \in \Delta^{\mathcal{I}}. a \in \bot^{\mathcal{I}} \Rightarrow a \in X^{\mathcal{I}}$, with $X \in \{C, D\}$. For the induction step we assume $a \in \bot^{\mathcal{I}}$ and show that the property holds for every concept constructor:

$(\neg C)^{\mathcal{I}}$ We have to show $a \in (\neg C)^{\mathcal{I}}$. By definition 1 of $(\neg C)^{\mathcal{I}}$ which only covers non-fallible refinements y of a, this is trivially true, since $\forall y. a \preceq^{\mathcal{I}} y \Rightarrow y \in \bot^{\mathcal{I}}$.

$(C \sqcap D)^{\mathcal{I}}$ Our goal this time is $a \in (C \sqcap D)^{\mathcal{I}}$. By induction hypothesis we have $a \in C^{\mathcal{I}}$ and $a \in D^{\mathcal{I}}$. This proves $a \in (C \sqcap D)^{\mathcal{I}}$.

$(C \sqcup D)^{\mathcal{I}}$ This time we have to show $a \in (C \sqcup D)^{\mathcal{I}}$. This follows directly from induction hypothesis, viz. $a \in C^{\mathcal{I}}$ and $a \in D^{\mathcal{I}}$.

$(C \sqsubseteq D)^{\mathcal{I}}$ In this case we have to prove $a \in (C \sqsubseteq D)^{\mathcal{I}}$. From assumption $a \in \bot^{\mathcal{I}}$ we have that $\forall y. a \preceq^{\mathcal{I}} y \Rightarrow y \in \bot^{\mathcal{I}}$. Then by induction hypothesis $y \in C^{\mathcal{I}}$ and $y \in D^{\mathcal{I}}$ and therefore $a \in (C \sqsubseteq D)^{\mathcal{I}}$.

$(\exists R.C)^{\mathcal{I}}$ We must show that $a \in (\exists R.C)^{\mathcal{I}}$ holds. From Definition 1 we have that $\forall y. a \preceq^{\mathcal{I}} y \Rightarrow y \in \bot^{\mathcal{I}}$ and $\exists z. y\, R\, z \;\&\; z \in \bot^{\mathcal{I}}$. By induction hypothesis $z \in C^{\mathcal{I}}$ and therefore $a \in (\exists R.C)^{\mathcal{I}}$.

$(\forall R.C)^{\mathcal{I}}$ We have to show that $a \in (\forall R.C)^{\mathcal{I}}$ holds. By Definition 1 we have $\forall y.a \preceq^{\mathcal{I}} y \Rightarrow y \in \bot^{\mathcal{I}}$ and $\forall z.y\, R\, z \Rightarrow z \in \bot^{\mathcal{I}}$, then by induction hypothesis $z \in C^{\mathcal{I}}$ which proves our goal. □

Lemma 2. Let $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ be an inconsistent sequent, i.e., there exists a closed tableau for it based on the rules in Fig. 4. Then, the following holds:

1. $|\Phi \cup \bigcup_{R \in N_R} \Psi(R)| \geq 1$;
2. For every weakening $\Theta \subseteq \Theta'$, $\Sigma \subseteq \Sigma'$, $\Gamma \subseteq \Gamma'$, $\Phi \subseteq \Phi'$ and $\Psi \subseteq \Psi'$ the sequent $\Theta'; \Sigma'; \Gamma' \vdash \Phi'; \Psi'$ is inconsistent, too;

Proof. By induction on the structure of the tableau. □

From the first condition 1. expressed in Lem. 2 it can be concluded that all sequents of the form $\Theta; \Sigma; \Gamma \vdash \emptyset; \emptyset$ are necessarily consistent. In fact, such sequents are satisfiable in any interpretation with a fallible entity.

Theorem 1 [Hilbert Soundness and Completeness]
For every concept C and set of concepts Θ we have $\Theta; \emptyset \models C$ iff $\Theta \vdash_H C$. □

Proof. Soundness and completeness follow from soundness and completeness of the Gentzen tableau system Thm. 2, proven below, if we can show that any deduction in either system can be translated into the other.

We first show that the tableau system can simulate the Hilbert deductions in the sense that if $\Theta \vdash_H C$ then there exists a closed tableau for sequent $\Theta; \emptyset; \emptyset \vdash C; \emptyset$. From this we get soundness of Hilbert $\Theta \vdash_H C \Rightarrow \Theta; \emptyset \models C$ from soundness of Gentzen (Thm. 2). The simulation amounts to verifying that the Hilbert axioms are derivable using the rules of Fig. 4 and that the Hilbert rules of Modus Ponens and Necessitation are derivable or admissible in the tableau system. Tableau derivations of the Hilbert axioms $\forall K$ and $\exists K$ in the form $\emptyset; \emptyset; \emptyset \vdash \forall R.(A \sqsubseteq B) \sqsubseteq (\forall R.A \sqsubseteq \forall R.B); \emptyset$ and $\emptyset; \emptyset; \emptyset \vdash (\forall R.A \sqcap \exists R.B) \sqsubseteq \exists R.(A \sqcap B); \emptyset$ have been given in Example 7 on page 25. The other axioms of intuitionistic logic which we list in Fig. 14 are easily constructed, too.

1: $C \sqsubseteq (D \sqsubseteq C)$
2: $((C \sqsubseteq (D \sqsubseteq E)) \sqsubseteq (C \sqsubseteq D) \sqsubseteq (C \sqsubseteq E))$
3: $C \sqsubseteq (D \sqsubseteq (C \sqcap D))$
4: $(C \sqcap D) \sqsubseteq C, \quad (C \sqcap D) \sqsubseteq D$
5: $C \sqsubseteq (C \sqcup D), \quad D \sqsubseteq (C \sqcup D)$
6: $(C \sqsubseteq E) \sqsubseteq ((D \sqsubseteq E) \sqsubseteq (C \sqcup D \sqsubseteq E))$
7: $\bot \sqsubseteq C$

Figure 14: Axioms of intuitionistic logic for $c\mathcal{ALC}$.

We only consider Necessitation and Axioms 2 and 3 here, all remaining axioms of intuitionistic $c\mathcal{ALC}$ are omitted.

Necessitation is simply an application of $\forall R$:

$$\frac{\Theta; \emptyset; \emptyset \vdash C; \emptyset}{\Theta; \emptyset; \emptyset \vdash \forall R.C; \emptyset}\ \forall R$$

Regarding Modus Ponens we resort to Thm. 2: Suppose we have a tableau for $\Theta; \emptyset; \emptyset \vdash C; \emptyset$ and for $\Theta; \emptyset; \emptyset \vdash C \sqsubseteq D; \emptyset$ but there is none for $\Theta; \emptyset; \emptyset \vdash D; \emptyset$. Then, by the completeness direction of Thm. 2 the sequent $\Theta; \emptyset; \emptyset \vdash D; \emptyset$ is satisfiable. It is straightforward to show that then at least one of $\Theta; \emptyset; \emptyset \vdash C; \emptyset$ or $\Theta; \emptyset; \emptyset \vdash C \sqsubseteq D; \emptyset$ is satisfiable, too. But this contradicts the soundness direction of Thm. 2.

Axiom 2 of intuitionistic $c\mathcal{ALC}$ can be proven by the application of the rules $\sqsubseteq R$, $\sqsubseteq R$, $\sqsubseteq R$, $\sqsubseteq L$, $\sqsubseteq L$, $\sqsubseteq L$ as shown in the following derivation.

$$\dfrac{\dfrac{}{\emptyset\,;\,\emptyset\,;\,C, D \sqsubseteq E \vdash E, C\,;\,\emptyset}\,Ax \qquad \dfrac{\dfrac{}{\emptyset\,;\,\emptyset\,;\,C, D \vdash E, D\,;\,\emptyset}\,Ax \quad \dfrac{}{\emptyset\,;\,\emptyset\,;\,C, D, E \vdash E\,;\,\emptyset}\,Ax}{\emptyset\,;\,\emptyset\,;\,C, D \sqsubseteq E, D \vdash E\,;\,\emptyset}\,\sqsubseteq L}{}\,\sqsubseteq L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\emptyset\,;\,\emptyset\,;\,C \sqsubseteq D, C \vdash E, C\,;\,\emptyset}\,Ax \qquad \emptyset\,;\,\emptyset\,;\,C \sqsubseteq D, C, D \sqsubseteq E \vdash E\,;\,\emptyset}{\emptyset\,;\,\emptyset\,;\,C \sqsubseteq (D \sqsubseteq E), C \sqsubseteq D, C \vdash E\,;\,\emptyset}\,\sqsubseteq L}{\emptyset\,;\,\emptyset\,;\,C \sqsubseteq (D \sqsubseteq E), C \sqsubseteq D \vdash C \sqsubseteq E\,;\,\emptyset}\,\sqsubseteq R}{\emptyset\,;\,\emptyset\,;\,C \sqsubseteq (D \sqsubseteq E) \vdash (C \sqsubseteq D) \sqsubseteq (C \sqsubseteq E)\,;\,\emptyset}\,\sqsubseteq R}{\emptyset\,;\,\emptyset\,;\,\emptyset \vdash (C \sqsubseteq (D \sqsubseteq E)) \sqsubseteq (C \sqsubseteq D) \sqsubseteq (C \sqsubseteq E)\,;\,\emptyset}\,\sqsubseteq R$$

Axiom 3 can be shown by the following derivation using the rules $\sqsubseteq R$, $\sqsubseteq R$, $\sqcap R$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\emptyset\,;\,\emptyset\,;\,C,D \vdash C\,;\,\emptyset}\,Ax \quad \dfrac{}{\emptyset\,;\,\emptyset\,;\,C,D \vdash D\,;\,\emptyset}\,Ax}{\emptyset\,;\,\emptyset\,;\,C,D \vdash C \sqcap D\,;\,\emptyset}\,\sqcap R}{\emptyset\,;\,\emptyset\,;\,C \vdash D \sqsubseteq (C \sqcap D)\,;\,\emptyset}\,\sqsubseteq R}{\emptyset\,;\,\emptyset\,;\,\emptyset \vdash C \sqsubseteq (D \sqsubseteq (C \sqcap D))\,;\,\emptyset}\,\sqsubseteq R$$

The proof of the remaining axioms is analogous.

In the other direction let us suppose there is a closed tableau for the sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$. Because of compactness we may assume without loss of generality that the sequent has a finite domain *dom* and all sets $\Sigma(R)$, Γ, Φ, $\Psi(R)$ involved are finite. We prove that each such sequent can be derived in the Hilbert system in closed form as an implication

$$\Theta \vdash_H (\wedge\Sigma \ \sqcap \ \wedge\Gamma) \ \sqsubseteq \ (\vee\Phi \ \sqcup \ \vee\Psi),$$

where the sub-formulas are defined as follows:

- $\wedge\Gamma =_{df} \bigwedge_{L \in \Gamma} L$, where $\bigwedge$ is the intersection $\sqcap$ over a set of concepts, e.g. if $\Gamma = \{L_1, L_2, \ldots, L_n\}$ then $\wedge\Gamma =_{df} L_1 \sqcap L_2 \sqcap \ldots \sqcap L_n$. In the special case where $\Gamma = \emptyset$ we put $\wedge\emptyset =_{df} \top$;
- $\vee\Phi =_{df} \bigvee_{M \in \Phi} M$, where $\bigvee$ is the disjunction $\sqcup$ over a set of concepts, e.g. if $\Phi = \{M_1, M_2, \ldots, M_n\}$ then $\vee\Phi =_{df} M_1 \sqcup M_2 \sqcup \ldots \sqcup M_n$. In the special case where $\Phi = \emptyset$ we put $\vee\emptyset =_{df} \bot$;
- $\wedge\Sigma =_{df} \bigwedge_{R \in dom} \forall R.\wedge\Sigma(R) = \bigwedge_{R \in dom} \forall R. \bigwedge_{K \in \Sigma(R)} K$;
- $\vee\Psi =_{df} \bigvee_{R \in dom} \exists R.\vee\Psi(R) = \bigvee_{R \in dom} \exists R. \bigvee_{N \in \Psi(R)} N$.

We will need the decomposition of $\wedge\Sigma$ in the following proof, i.e. for $dom = \{R_1, R_2, \ldots, R_n\}$ and a choice of $k \in \{1 \ldots n\}$ we can decompose $\wedge\Sigma$ into

$$\begin{aligned}
\wedge\Sigma(R) &= \bigwedge_{i \leq n} \forall R_i.\wedge\Sigma(R_i) \\
&= \forall R_1.\wedge\Sigma(R_1) \sqcap \Big(\forall R_2.\wedge\Sigma(R_2) \sqcap \big(\ldots \sqcap \forall R_n.\wedge\Sigma(R_n) \ldots\big)\Big) \\
&= \forall R_k.\wedge\Sigma(R_k) \sqcap \Big(\bigwedge_{i \leq n,\, i \neq k} \forall R_i.\wedge\Sigma(R_i)\Big).
\end{aligned}$$

The decomposition of $\vee\Psi$ proceeds analogously.

The proof is by induction on the structure of a closed tableau for $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$. For all the standard intuitionistic Gentzen rules Ax, $\bot L$, $\sqcap R$, $\sqcap L$, $\sqcup R$, $\sqcup L$, $\sqsubseteq R$, $\sqsubseteq L$ from the tableau system Fig. 4 it is well known that these can be derived in closed form from the intuitionistic Hilbert Axioms in Fig. 14 plus Modus Ponens MP. Essentially, this is the well-known encoding of combinatorial logic in λ-calculus[4].

We only give an indication below to demonstrate what is involved and then treat the quantifier cases $\exists R$, $\exists L$, $\forall R$, $\forall L$ and Hyp_1, Hyp_2.

Suppose $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ is derived by rule Ax, i.e., $\Gamma = \Gamma', C$ and $\Phi = \Phi', C$. We must show how to derive

$$\Theta \;\vdash_H\; (\wedge\Sigma \;\sqcap\; \wedge\Gamma' \sqcap C) \;\sqsubseteq\; (\vee\Phi' \sqcup C \;\sqcup\; \vee\Psi). \qquad (2)$$

The argument runs essentially like this. First let $\top =_{df} \bot \sqsubseteq \bot$. By Hilbert Axiom 7 (Fig. 14) we get $\vdash_H \top$. From Axiom 1 conclude $\Theta \vdash_H \top \sqsubseteq (C \sqsubseteq \top)$ and further with Modus Ponens $\Theta \vdash_H C \sqsubseteq \top$. Now take the instance $\Theta \vdash_H (C \sqsubseteq (\top \sqsubseteq C)) \sqsubseteq ((C \sqsubseteq \top) \sqsubseteq (C \sqsubseteq C))$ of Axiom 2. MP with the instance $\Theta \vdash_H C \sqsubseteq (\top \sqsubseteq C)$ of Axiom 1 now gives $\Theta \vdash_H (C \sqsubseteq \top) \sqsubseteq (C \sqsubseteq C)$ and another application of MP finally yields $\Theta \vdash_H C \sqsubseteq C$. At this point we exploit a general result about the intuitionistic Hilbert system (*"weakening"*), which says that if $\Theta \vdash_H C \sqsubseteq D$ then also $\Theta \vdash_H (E_1 \sqcap C \sqcap E_2) \sqsubseteq D$ and $\Theta \vdash_H C \sqsubseteq (E_1 \sqcup D \sqcup E_2)$ to argue for derivability of (2). □

All derivations assume associativity, commutativity, idempotence of $\sqcap, \sqcup$ and that eliminating neutral elements $\top, \bot$ is for free.

At this point we list some admissible rules of the intuitionistic Hilbert system which we will use in the following proofs:

AR1 *weakening* says that if $\Theta \vdash_H C \sqsubseteq D$ then also $\Theta \vdash_H (C_1 \sqcap C \sqcap C_2) \sqsubseteq D$ (left weakening) and $\Theta \vdash_H C \sqsubseteq (D_1 \sqcap D \sqcap D_2)$ (right weakening).

AR2 *currying* means that if $\Theta \vdash_H C_1 \sqsubseteq (C_2 \sqsubseteq D)$ then also $\Theta \vdash_H (C_1 \sqcap C_2) \sqsubseteq D$. The inverse direction is called *de-currying*.

[4] see Harold Simmons: *Derivation and Computation. Taking the Curry-Howard correspondence seriously.* Cambridge University Press, 2000.

AR3 *composition* is the fact that if $\Theta \vdash_H C \sqsubseteq D$ and $\Theta \vdash_H D \sqsubseteq E$ then also $\Theta \vdash_H C \sqsubseteq E$, for any C, D, E.

AR4 *monotonicity* says that if $\Theta \vdash_H C \sqsubseteq D$ then by monotonicity we also have $\Theta \vdash_H (Z_1 \sqcap C \sqcap Z_2) \sqsubseteq (Z_1 \sqcap D \sqcap Z_2)$, for any C, D, Z_1, Z_2.

• Assume the sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ is derived by rule $\exists R$, i.e., $\Phi = \Phi', \exists R.C$ and the last rule application looks like this

$$\dfrac{\dfrac{\vdots}{\Theta \; ; \; \Sigma \; ; \; \Gamma \vdash \emptyset \; ; \; [R \mapsto C]}}{\Theta \; ; \; \Sigma \; ; \; \Gamma \vdash \Phi', \exists R.C \; ; \; \Psi} \exists R$$

By induction hypothesis applied to the premise of the sequent we must have a derivation $\Theta \vdash_H (\wedge\Sigma \sqcap \wedge\Gamma) \sqsubseteq \exists R.C$. By the general weakening property in Hilbert this implies that $\Theta \vdash_H (\wedge\Sigma \sqcap \wedge\Gamma) \sqsubseteq (\vee\Phi' \sqcup \exists R.C \sqcup \vee\Psi)$ as desired.

• If the sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ is derived by rule $\exists L$ then $\Gamma = \Gamma', \exists R.C$ and the last rule application be

$$\dfrac{\dfrac{\vdots}{\Theta \; ; \; \emptyset \; ; \; \Sigma(R), C \vdash \Psi(R) \; ; \; \emptyset}}{\Theta \; ; \; \Sigma \; ; \; \Gamma', \exists R.C \vdash \Phi \; ; \; \Psi} \exists L$$

We must find a derivation

$$\Theta \quad \vdash_H \quad (\wedge\Sigma \sqcap \wedge\Gamma' \sqcap \exists R.C) \sqsubseteq (\vee\Phi \sqcup \vee\Psi). \tag{3}$$

The induction hypothesis for the premise this time yields a Hilbert derivation $\Theta \vdash_H (\wedge\Sigma(R) \sqcap C) \sqsubseteq \vee\Psi(R)$. By Necessitation we obtain $\Theta \vdash_H \forall R.((\wedge\Sigma(R) \sqcap C) \sqsubseteq \vee\Psi(R))$. From the appropriate instance of axiom $\exists K$ and MP this generates

$$\Theta \quad \vdash_H \quad \exists R.(\wedge\Sigma(R) \sqcap C) \sqsubseteq \exists R.\vee\Psi(R). \tag{4}$$

Now we claim that Hilbert derives

$$\Theta \quad \vdash_H \quad (\forall R.\wedge\Sigma(R) \sqcap \exists R.C) \sqsubseteq \exists R.(\wedge\Sigma(R) \sqcap C) \tag{5}$$

In other words, abbreviating $D =_{df} \wedge\Sigma(R)$, we must derive $\Theta \vdash_H (\forall R.D \sqcap \exists R.C) \sqsubseteq$

$\exists R.(D \sqcap C)$. By general properties of Hilbert (*"de-currying"*) it suffices to construct $\Theta \vdash_H \forall R.D \sqsubseteq (\exists R.C \sqsubseteq \exists R.(D \sqcap C))$, which can be done as follows:

1.	$D \sqsubseteq (C \sqsubseteq (D \sqcap C))$	Ax. 3
2.	$\forall R.(D \sqsubseteq (C \sqsubseteq (D \sqcap C)))$	from 1., *Nec*
3.	$\forall R.D \sqsubseteq \forall R.(C \sqsubseteq (D \sqcap C))$	from 2.,$\forall K$, MP
4.	$\forall R.(C \sqsubseteq (D \sqcap C)) \sqsubseteq (\exists R.C \sqsubseteq \exists R.(D \sqcap C))$	by $\exists K$
5.	$\forall R.D \sqsubseteq (\forall R.(C \sqsubseteq (D \sqcap C)) \sqsubseteq (\exists R.C \sqsubseteq \exists R.(D \sqcap C)))$	from 4, Ax. 1, MP
6.	$(\forall R.D \sqsubseteq \forall R.(C \sqsubseteq (D \sqcap C))) \sqsubseteq (\forall R.D \sqsubseteq (\exists R.C \sqsubseteq \exists R.(D \sqcap C)))$	from 5., Ax. 2, MP
7.	$\forall R.D \sqsubseteq (\exists R.C \sqsubseteq \exists R.(D \sqcap C))$	from 3., 6., MP.

This proves (5). By the admissible rule AR3 which says that if $\Theta \vdash_H C \sqsubseteq D$ and $\Theta \vdash_H D \sqsubseteq E$ then $\Theta \vdash_H C \sqsubseteq E$, for any C, D, E. Applying this to (5) and (4) gives us

$$\Theta \quad \vdash_H \quad (\forall R.\wedge\Sigma(R) \sqcap \exists R.C) \sqsubseteq \exists R.\vee\Psi(R). \tag{6}$$

Now observe that $\forall R.\wedge\Sigma(R)$ is a conjunctive ($\sqcap$) part of $\wedge\Sigma$ and $\exists R.\vee\Psi(R)$ is a disjunctive part of $\vee\Psi$. Hence our goal (3) follows by the weakening property of Hilbert from (6).

• Next, let the sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ be derived by rule Hyp_1, i.e., $\Theta = \Theta', C$ and the last rule application is

$$\cfrac{\cfrac{\vdots}{\Theta' \,;\, \Sigma \cup [R \mapsto C] \,;\, \Gamma \vdash \Phi \,;\, \Psi \quad R \in dom}}{\Theta', C \,;\, \Sigma \,;\, \Gamma \vdash \Phi \,;\, \Psi}\; Hyp_1$$

We must find a derivation

$$\Theta', C \quad \vdash_H \quad (\wedge\Sigma \sqcap \wedge\Gamma) \sqsubseteq (\vee\Phi \sqcup \vee\Psi). \tag{7}$$

The induction hypothesis for the premise in this case amounts to the existence of a Hilbert derivation

$$\Theta' \vdash_H \big((\bigwedge_{R' \in dom \setminus \{R\}} \forall R'.\wedge\Sigma(R')) \sqcap \forall R.(\wedge\Sigma(R) \sqcap C) \sqcap \wedge\Gamma\big) \sqsubseteq (\vee\Phi \sqcup \vee\Psi)$$

which also holds under extended assumptions as above, i.e.,

$$\Theta', C \vdash_H \big((\bigwedge_{R' \in dom \setminus \{R\}} \forall R'.\wedge\Sigma(R')) \sqcap \forall R.(\wedge\Sigma(R) \sqcap C) \sqcap \wedge\Gamma\big) \sqsubseteq (\vee\Phi \sqcup \vee\Psi).$$

To see what is going on let us abbreviate this as

$$\Theta', C \vdash_H (\wedge\Sigma' \sqcap \forall R.(D \sqcap C) \sqcap \wedge\Gamma) \sqsubseteq (\vee\Phi \sqcup \vee\Psi) \tag{8}$$

where $\wedge\Sigma' =_{df} \bigwedge_{R' \in dom \setminus \{R\}} \forall R'.\wedge\Sigma(R')$ and $D =_{df} \wedge\Sigma(R)$. Our plan is to rewrite $\forall R.(D \sqcap C)$ to $(\forall R.D \sqcap \forall R.C)$ by distributing $\forall R$ over $\sqcap$. To this end we first show how to obtain $\Theta', C \vdash_H (\forall R.D \sqcap \forall R.C) \sqsubseteq \forall R.(D \sqcap C)$:

1.	$D \sqsubseteq (C \sqsubseteq (D \sqcap C))$	Ax. 3
2.	$\forall R.(D \sqsubseteq (C \sqsubseteq (D \sqcap C)))$	from 1., *Nec*
3.	$\forall R.D \sqsubseteq \forall R.(C \sqsubseteq (D \sqcap C))$	from 2.,$\forall K$, MP
4.	$\forall R.(C \sqsubseteq (D \sqcap C)) \sqsubseteq (\forall R.C \sqsubseteq \forall R.(D \sqcap C))$	by $\forall K$
5.	$\forall R.D \sqsubseteq (\forall R.(C \sqsubseteq (D \sqcap C)) \sqsubseteq (\forall R.C \sqsubseteq \forall R.(D \sqcap C)))$	from 4, Ax. 1, MP
6.	$(\forall R.D \sqsubseteq \forall R.(C \sqsubseteq (D \sqcap C))) \sqsubseteq (\forall R.D \sqsubseteq (\forall R.C \sqsubseteq \forall R.(D \sqcap C)))$	from 5., Ax. 2, MP
7.	$\forall R.D \sqsubseteq (\forall R.C \sqsubseteq \forall R.(D \sqcap C))$	from 3., 6., MP
8.	$(\forall R.D \sqcap \forall R.C) \sqsubseteq \forall R.(D \sqcap C)$	from 7., de-currying.

Now we use the AR4 which says that if $\Theta \vdash_H X \sqsubseteq Y$ then $\Theta \vdash (Z_1 \sqcap X \sqcap Z_2) \sqsubseteq (Z_1 \sqcap Y \sqcap Z_2)$. This takes us from $\Theta', C \vdash_H (\forall R.D \sqcap \forall R.C) \sqsubseteq \forall R.(D \sqcap C)$ to

$$\Theta', C \vdash_H (\wedge\Sigma' \sqcap \forall R.D \sqcap \forall R.C \sqcap \wedge\Gamma) \sqsubseteq (\wedge\Sigma' \sqcap \forall R.(D \sqcap C) \sqcap \wedge\Gamma). \tag{9}$$

By AR3 of (8) with (9) we obtain $\Theta', C \vdash_H (\wedge\Sigma' \sqcap \forall R.D \sqcap \forall R.C \sqcap \wedge\Gamma) \sqsubseteq (\vee\Phi \sqcup \vee\Psi)$ and further, AR2 gives us

$$\Theta', C \vdash_H \forall R.C \sqsubseteq ((\wedge\Sigma' \sqcap \forall R.D \sqcap \wedge\Gamma) \sqsubseteq (\vee\Phi \sqcup \vee\Psi)). \tag{10}$$

Of course, from the assumption Θ', C we can easily conclude that $\Theta', C \vdash_H C$ and by Necessitation then $\Theta', C \vdash_H \forall R.C$. Now apply Modus Ponens to (10) which generates the derivation

$$\Theta', C \vdash_H (\wedge\Sigma' \sqcap \forall R.D \sqcap \wedge\Gamma) \sqsubseteq (\vee\Phi \sqcup \vee\Psi). \tag{11}$$

If we now unfold our definitions of $\wedge\Sigma'$ and D, (11) becomes

$$\Theta', C \quad \vdash_H \quad \big((\bigwedge_{R' \in dom \setminus \{R\}} \forall R'.\wedge\Sigma(R')) \sqcap \forall R.\wedge\Sigma(R) \sqcap \wedge\Gamma\big) \sqsubseteq (\vee\Phi \sqcup \vee\Psi)$$

which is nothing but our goal derivation (7) considering that

$$\wedge\Sigma = \bigwedge_{R' \neq R \vee R' = R} \forall R'.\wedge\Sigma(R') = (\bigwedge_{R' \neq R} \forall R'.\wedge\Sigma(R')) \ \sqcap \ \forall R.\wedge\Sigma(R).$$

• Next, let the sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ be derived by rule Hyp_2, i.e., $\Theta = \Theta', C$ and the last rule application is

$$\cfrac{\cfrac{\vdots}{\Theta' \ ; \ \Sigma \ ; \Gamma, \ C \vdash \Phi \ ; \ \Psi}}{\Theta', C \ ; \ \Sigma \ ; \ \Gamma \vdash \Phi \ ; \ \Psi} \ Hyp_2$$

We must find a derivation

$$\Theta', C \quad \vdash_H \quad (\wedge\Sigma \sqcap \wedge\Gamma) \sqsubseteq (\vee\Phi \sqcup \vee\Psi). \tag{12}$$

The induction hypothesis for the premise obtains a Hilbert derivation $\Theta' \vdash_H (\wedge\Sigma \sqcap \wedge\Gamma \sqcap C) \sqsubseteq (\vee\Phi \sqcup \vee\Psi)$ which holds under extended assumptions as well, i.e., $\Theta', C \vdash_H (\wedge\Sigma \sqcap \wedge\Gamma \sqcap C) \sqsubseteq (\vee\Phi \sqcup \vee\Psi)$. By AR2 we can derive

$$\Theta', C \quad \vdash_H \quad C \sqsubseteq ((\wedge\Sigma \sqcap \wedge\Gamma) \sqsubseteq (\vee\Phi \sqcup \vee\Psi)) \tag{13}$$

From the assumption Θ', C we get $\Theta', C \vdash_H C$ and thus from application of MP to (13) our goal (12) follows immediately.

• Next, let the sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ be derived by rule $\forall L$, i.e., $\Gamma = \Gamma', \forall R.C$ and the last rule application reads

$$\dfrac{\dfrac{\vdots}{\Theta\ ;\ \Sigma \cup [R \mapsto C]\ ;\ \Gamma' \vdash \Phi\ ;\ \Psi}}{\Theta\ ;\ \Sigma\ ;\ \Gamma', \forall R.C \vdash \Phi\ ;\ \Psi}\ \forall L$$

We have to find a derivation of

$$\Theta \vdash_H (\wedge\Sigma \sqcap \wedge\Gamma' \sqcap \forall R.C) \sqsubseteq (\vee\Phi \sqcup \vee\Psi). \tag{14}$$

The induction hypothesis for the premise yields the following Hilbert derivation $\Theta \vdash_H \big((\bigwedge_{R' \in dom \setminus \{R\}} \forall R'.\wedge\Sigma(R')) \sqcap \forall R.(\wedge\Sigma(R) \sqcap C) \sqcap \wedge\Gamma'\big) \sqsubseteq (\vee\Phi \sqcup \vee\Psi)$. Let us abbreviate this to

$$\Theta \vdash_H (\wedge\Sigma' \sqcap \forall R.(D \sqcap C) \wedge \Gamma') \sqsubseteq (\vee\Phi \sqcup \vee\Psi), \tag{15}$$

where $\wedge\Sigma' =_{df} (\bigwedge_{R' \in dom \setminus \{R\}} \forall R'. \wedge \Sigma(R'))$ and $D =_{df} \wedge\Sigma(R)$.

Consider the distribution of $\forall R$ over $\sqcap$ which has been shown in the proof of the rule Hyp_1 on page 42. This gives us the derivation

$$\Theta \vdash_H (\forall R.D \sqcap \forall R.C) \sqsubseteq \forall R.(D \sqcap C). \tag{16}$$

By AR4 applied to (16) we obtain

$$\Theta \vdash_H (\wedge\Sigma' \sqcap \forall R.D \sqcap \forall R.C \sqcap \wedge\Gamma') \sqsubseteq (\wedge\Sigma' \sqcap \forall R.(D \sqcap C) \sqcap \wedge\Gamma'). \tag{17}$$

From AR3 with (17) and (15) we get the derivation

$$\Theta \vdash_H (\wedge\Sigma' \sqcap \forall R.D \sqcap \forall R.C \sqcap \wedge\Gamma') \sqsubseteq (\vee\Phi \sqcup \vee\Psi). \tag{18}$$

We can then unfold $\wedge\Sigma'$ and D in (18) and thereby obtain

$$\Theta \vdash_H \big((\bigwedge_{R' \in dom \setminus \{R\}} \forall R'.\wedge\Sigma(R')) \sqcap \forall R.\wedge\Sigma(R) \sqcap \forall R.C \sqcap \wedge\Gamma'\big) \sqsubseteq (\vee\Phi \sqcup \vee\Psi). \tag{19}$$

Considering that

$$\wedge\Sigma = \bigwedge_{R' \neq R \vee R' = R} \forall R'.\wedge\Sigma(R') = \big(\bigwedge_{R' \neq R} \forall R'.\wedge\Sigma(R')\big) \sqcap \forall R.\wedge\Sigma(R)$$

we obtain from (19) our goal (14) as desired.

• Next, let the sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ be derived by rule $\forall R$, i.e., $\Phi = \Phi', \forall R.C$ and the last rule application reads

$$\cfrac{\cfrac{\vdots}{\Theta \,;\, \emptyset \,;\, \Sigma(R) \vdash C \,;\, \emptyset}}{\Theta \,;\, \Sigma \,;\, \Gamma' \vdash \Phi', \forall R.C \,;\, \Psi}\ \forall R$$

We have to find a derivation of

$$\Theta \vdash_H (\wedge\Sigma \sqcap \wedge\Gamma) \sqsubseteq (\vee\Phi' \sqcup \forall R.C \sqcup \vee\Psi). \tag{20}$$

The induction hypothesis for the premise yields this time a Hilbert derivation $\Theta \vdash_H \wedge\Sigma(R) \sqsubseteq C$. By Necessitation we obtain $\Theta \vdash_H \forall R.(\wedge\Sigma(R) \sqsubseteq C)$. From the appropriate instance of the Hilbert axiom $\forall K$ and MP this generates

$$\Theta \vdash_H \forall R.\wedge\Sigma(R) \sqsubseteq \forall R.C \tag{21}$$

Left- and right-weakening (AR1) of (21) then gives us

$$\Theta \vdash_H \big((\bigwedge_{R' \in dom \setminus \{R\}} \forall R'.\wedge\Sigma(R')) \sqcap \forall R.\wedge\Sigma(R) \sqcap \wedge\Gamma\big) \sqsubseteq (\vee\Phi' \sqcup \forall R.C \sqcup \vee\Psi) \tag{22}$$

which by Definition of

$$\wedge\Sigma = \bigwedge_{R' \neq R \vee R' = R} \forall R'.\wedge\Sigma(R') = \big(\bigwedge_{R' \neq R} \forall R'.\wedge\Sigma(R')\big) \sqcap \forall R.\wedge\Sigma(R)$$

gives us $\Theta \vdash_H (\wedge\Sigma(R) \sqcap \wedge\Gamma) \sqsubseteq (\vee\Phi' \sqcup \forall R.C \sqcup \vee\Psi)$ which is the goal (20) that was to be shown. □

Theorem 2 [**Tableau Soundness**]
Every satisfiable sequent is consistent, i.e., if $\Theta; \Sigma; \Gamma \not\models \Phi; \Psi$ *then* $\Theta; \Sigma; \Gamma \not\vdash \Phi; \Psi$. □

Proof. For soundness we must show for each derivation rule in Fig. 4 that if the conclusion is satisfiable then *at least one* of the premises of the rule is satisfiable.

In the special case of the axiom Ax without any premise it is trivial to see that this sequent cannot be satisfied by any interpretation $\mathcal{I}$ and entity $a \in \Delta^{\mathcal{I}}$, since it is impossible to obtain a pair $(\mathcal{I}, a)$ such that $\mathcal{I}; a \models C$ and $\mathcal{I}; a \not\models C$. This is equivalent to saying that the conclusion of Ax, namely the sequent $\Theta; \Sigma; \Gamma, C \vdash \Phi, C; \Psi$ is unsatisfiable.

The same is true for the axiom $\bot L$ with conclusion $\Theta; \Sigma; \Gamma, \bot \vdash \Phi; \Psi$. Notice, here it is crucial that the right-hand side $\Phi; \Psi$ is constrained to be non-empty.

The sequent $\Theta; \Sigma; \Gamma, \bot \vdash \emptyset; \emptyset$, for instance, is satisfiable in a fallible entity. Starting from these axioms it then follows by induction on the size of a tableau that whenever a sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ is derivable (i.e., not consistent) then it cannot be satisfiable.

To show soundness of $\sqcup L$ we assume its conclusion sequent $S_c =_{df} \Theta;\ \Sigma;\ \Gamma, C \sqcup D \vdash \Phi;\ \Psi$ is satisfiable and then show that one of its premises, namely the sequents $S_{p1} =_{df} \Theta;\ \Sigma;\ \Gamma, C \vdash \Phi;\ \Psi$ or $S_{p2} =_{df} \Theta;\ \Sigma;\ \Gamma, D \vdash \Phi;\ \Psi$, is satisfiable as well. If S_c is satisfiable then by Definition 2 of constructive satisfiability there exists a pair $(\mathcal{I}, a)$ that *satisfies* the sequent S_c, i.e. $\mathcal{I}$ is a model of $\Theta, \mathcal{I} \models \Theta$, and for all $R \in dom$, $L \in \Sigma(R)$, $M \in \Gamma \cup \{C \sqcup D\}$, $N \in \Phi$ and $K \in \Psi(R)$:

$$\forall a'. \forall b. (a \preceq a' \ \&\ a' R\, b) \Rightarrow \mathcal{I}; b \models L; \tag{23}$$

$$\mathcal{I}; a \models M, \text{in particular } \mathcal{I}; a \models C \sqcup D; \tag{24}$$

$$\mathcal{I}; a \not\models N; \tag{25}$$

$$\forall b.\, a\, R\, b \Rightarrow \mathcal{I}; b \not\models K; \tag{26}$$

- We claim that $(\mathcal{I}, a)$ satisfies S_{p1} or S_{p2}. By Definition 1 of *constructive interpretation* and from $\mathcal{I}; a \models C \sqcup D$ we see that a is contained in the union of the interpretation of the concepts C and D, i.e. $\mathcal{I}; a \models C$ or $\mathcal{I}; a \models D$. In the first case $(\mathcal{I}, a)$ satisfies S_{p1} using (23–26) and in the second case $(\mathcal{I}, a)$ satisfies S_{p2}. This proves our premises of S_c, namely S_{p1} by $\mathcal{I}; a \models C$ and S_{p2} by $\mathcal{I}; a \models D$.

- Regarding the conditions $\forall a'. \forall b. (a \preceq a' \ \& \ a' R b) \Rightarrow \mathcal{I}; b \models L$, for $L \in \Sigma(R)$, $\mathcal{I}; a \not\models N$, for $N \in \Phi$, and $\forall b. aRb \Rightarrow K$, for $K \in \Psi(R)$, nothing needs to be shown here, since the set of role mappings that are supposed to be true for all refinements of a, the set of formulas and the set of role mappings that are supposed to be false are equal in S_c and S_{pi}, $i = 1, 2$.

To show the soundness of $\sqcap R$ we assume the conclusion of the rule which is namely the sequent $\Theta;\ \Sigma;\ \Gamma \vdash \Phi, C \sqcap D;\ \Psi$ is satisfiable and show that one of its premises, namely the sequents $S_{p1} =_{df} \Theta;\ \Sigma;\ \Gamma \vdash \Phi, C;\ \Psi$ or $S_{p2} =_{df} \Theta;\ \Sigma;\ \Gamma \vdash \Phi, D;\ \Psi$, is satisfiable as well. If S_c is satisfiable then by Definition 2 there exists a pair $(\mathcal{I}, a)$ with $\mathcal{I}$ being a model of $\Theta, \mathcal{I} \models \Theta$, and for all $R \in dom$, $L \in \Sigma(R)$, $M \in \Gamma$, $N \in \Phi \cup \{C \sqcap D\}$ and $K \in \Psi(R)$:

$$\forall a'. \forall b. (a \preceq a' \ \& \ a' R b) \Rightarrow \mathcal{I}; b \models L; \tag{27}$$

$$\mathcal{I}; a \models M; \tag{28}$$

$$\mathcal{I}; a \not\models N, \text{in particular } \mathcal{I}; a \not\models C \sqcap D; \tag{29}$$

$$\forall b.\, a R b \Rightarrow \mathcal{I}; b \not\models K; \tag{30}$$

We only have to treat the condition $\mathcal{I}; a \not\models N$, for $N \in \Phi \cup \{C\}$ or $N \in \Phi \cup \{D\}$. The case $\mathcal{I}; a \not\models N$, for $N \in \Phi$ follows by assumption (29) since the set Φ is equal in S_c and S_{pi}, $i = 1, 2$. Furthermore by Definition 1 of *constructive interpretation* and from $\mathcal{I}; a \not\models C \sqcap D$ (29) we see that it is false that a is included in C and D, more precisely by pushing negation inside we get $\mathcal{I}; a \not\models C$ or $\mathcal{I}; a \not\models D$. This proves the premises of S_c, namely S_{p1} by $\mathcal{I}; a \not\models C$ and S_{p2} by $\mathcal{I}; a \not\models D$.

To show soundness of $\sqcap L$ we assume the conclusion of the rule which is the sequent $S_c =_{df} \Theta;\ \Sigma;\ \Gamma, C \sqcap D \vdash \Phi;\ \Psi$ is satisfiable and show that its premise $S_p =_{df} \Theta;\ \Sigma;\ \Gamma, C, D \vdash \Phi;\ \Psi$, is satisfiable as well.

If S_c is satisfiable then by Definition 2 of constructive satisfiability there exists a pair $(\mathcal{I}, a)$ that *satisfies* the sequent S_c, i.e. $\mathcal{I}$ is a model of $\Theta, \mathcal{I} \models \Theta$, and for all $R \in dom$,

$L \in \Sigma(R)$, $M \in \Gamma \cup \{C \sqcap D\}$, $N \in \Phi$ and $K \in \Psi(R)$:

$$\forall a'.\forall b.\,(a \preceq a' \;\;\&\;\; a'\,R\,b) \Rightarrow \mathcal{I}; b \models L; \tag{31}$$

$$\mathcal{I}; a \models M, \text{in particular } \mathcal{I}; a \models C \sqcap D; \tag{32}$$

$$\mathcal{I}; a \not\models N; \tag{33}$$

$$\forall b.\, a\,R\,b \Rightarrow \mathcal{I}; b \not\models K; \tag{34}$$

We only analyse the condition $\mathcal{I}; a \models M$, for $M \in \Gamma \cup \{C \sqcap D\}$. The case $\mathcal{I}; a \models M$, for $M \in \Gamma$ follows from (32) and due to the fact that the set Γ is equal in S_c and S_p. The second case $\mathcal{I}; a \models C \sqcap D$ is then used as follows. By Definition 1 of *constructive interpretation* and from $\mathcal{I}; a \models C \sqcap D$ we get that a is included in the intersection of the interpretations of C and D, i.e. $\mathcal{I}; a \models C$ and $\mathcal{I}; a \models D$. This immediately proves the premise S_p.

To show soundness of $\sqcup R$ we assume the conclusion of the rule $S_c =_{df} \Theta;\ \Sigma;\ \Gamma \vdash \Phi, C \sqcup D;\ \Psi$ is satisfiable and show that its premise $S_p =_{df} \Theta;\ \Sigma;\ \Gamma \vdash \Phi, C, D;\ \Psi$, is satisfiable as well. If S_c is satisfiable then by Definition 2 of constructive satisfiability there exists a pair $(\mathcal{I}, a)$ that *satisfies* the sequent S_c, i.e. $\mathcal{I}$ is a model of $\Theta, \mathcal{I} \models \Theta$, and for all $R \in dom$, $L \in \Sigma(R)$, $M \in \Gamma$, $N \in \Phi \cup \{C \sqcup D\}$ and $K \in \Psi(R)$:

$$\forall a'.\forall b.\,(a \preceq a' \;\;\&\;\; a'\,R\,b) \Rightarrow \mathcal{I}; b \models L; \tag{35}$$

$$\mathcal{I}; a \models M; \tag{36}$$

$$\mathcal{I}; a \not\models N, \text{in particular } \mathcal{I}; a \not\models C \sqcup D; \tag{37}$$

$$\forall b.\, a\,R\,b \Rightarrow \mathcal{I}; b \not\models K; \tag{38}$$

We only have to treat the condition $\mathcal{I}; a \not\models N$, for $N \in \Phi \cup \{C \sqcup D\}$. The case $\mathcal{I}; a \not\models N$, for $N \in \Phi$ follows since the set Φ is equal in S_c and S_p. By Definition 1 of *constructive interpretation* and from $\mathcal{I}; a \not\models C \sqcup D$ we conclude that $\mathcal{I}; a \not\models C$ and $\mathcal{I}; a \not\models D$ and therefore obtain $\mathcal{I}; a \not\models H$, with $H \in \{C, D\}$. This proves the premise S_p.

To show the soundness of $\exists L$ we assume that its conclusion, the sequent $S_c =_{df} \Theta;\ \Sigma;\ \Gamma, \exists R.C \vdash \Phi;\ \Psi$ is satisfiable and then show that its premise $S_p =_{df} \Theta;\ \emptyset;\ \Sigma(R), C \vdash \Psi(R);\ \emptyset$, is satisfiable as well. If S_c is satisfiable then by Definition 2 of constructive satisfiability there exists a pair $(\mathcal{I}, a)$ that *satisfies* the sequent S_c, i.e. $\mathcal{I}$ is a model of Θ, $\mathcal{I} \models \Theta$, and for all $R \in dom$, $L \in \Sigma(R)$, $M \in \Gamma \cup \{\exists R.C\}$, $N \in \Phi$ and $K \in \Psi(R)$:

$$\forall a'. \forall b. (a \preceq a' \ \ \& \ \ a' \, R \, b) \Rightarrow \mathcal{I}; b \models L; \tag{39}$$

$$\mathcal{I}; a \models M, \text{in particular } \mathcal{I}; a \models \exists R.C; \tag{40}$$

$$\mathcal{I}; a \not\models N; \tag{41}$$

$$\forall b. \, a \, R \, b \Rightarrow \mathcal{I}; b \not\models K; \tag{42}$$

In particular (40) means that a and all its refinements are part of all concepts of $\Gamma \cup \{\exists R.C\}$.

Because $\mathcal{I}; a \models \exists R.C$ there has to be a R-filler $a^\dagger$ with $aRa^\dagger$ and $\mathcal{I}; a^\dagger \models C$. We claim that $(\mathcal{I}, a^\dagger)$ satisfies the sequent S_p.

1. Regarding the condition $\mathcal{I}; a^\dagger \models L$, with $L \in \emptyset(R)$, nothing needs to be shown here, since the set of role mappings to sets of concepts that are supposed to be true is empty in the sequent S_p.

2. Next we show $\mathcal{I}; a^\dagger \models M$ for all $M \in \Sigma(R)$ and $M = C$. The first case $\mathcal{I}; a^\dagger \models \Sigma(R)$ follows from assumption (39), since $a \preceq a$, i.e. by reflexivity of refinement a refines itself, and $aRa^\dagger$ with $\mathcal{I}; a^\dagger \models \Sigma(R)$. The second case $\mathcal{I}; a^\dagger \models C$ follows from the construction of $a^\dagger$.

3. We have to show $\forall N \in \Psi(R).\mathcal{I}; a^\dagger \not\models N$ for the premise of the sequent which follows from construction of $a^\dagger$ and (42).

4. Regarding the condition $\forall K \in \emptyset.\forall b.a^\dagger Rb \Rightarrow \mathcal{I}; b \models K$ nothing needs to be shown here, since the set of role mappings to sets of concepts that are supposed to be false is empty in the sequent S_p.

To show soundness of $\exists R$ we assume that its conclusion, namely the sequent $S_c =_{df} \Theta;\ \Sigma;\ \Gamma \vdash \Phi,\ \exists R.C;\ \Psi$, is satisfiable and then prove that its premise $S_p =_{df} \Theta;\ \Sigma;\ \Gamma \vdash \emptyset;\ [R \mapsto C]$ is satisfiable as well. If the sequent S_c is satisfiable then by Definition

2 of constructive satisfiability there exists a pair $(\mathcal{I}, a)$ that *satisfies* the sequent S_c which means $\mathcal{I}$ is a model of Θ, $\mathcal{I} \models \Theta$, and for all $R \in dom$, $L \in \Sigma(R)$, $M \in \Gamma$ and $N \in \Phi \cup \{\exists R.C\}$:

$$\forall a'. \forall b. (a \preceq a' \;\&\; a' \, R \, b) \Rightarrow \mathcal{I}; b \models L; \tag{43}$$

$$\mathcal{I}; a \models M; \tag{44}$$

$$\mathcal{I}; a \not\models N, \text{in particular } \mathcal{I}; a \not\models \exists R.C; \tag{45}$$

i.e., in (43) all R-fillers of a and of its refinements a' are part of all concepts of $\Sigma(R)$, (44) says that a and hence all its refinements are part of all concepts of Γ and (45) tells us that a is not contained in any of the concepts in $\Phi \cup \{\exists R.C\}$.

Because $\mathcal{I}; a \not\models \exists R.C$ there has to be a refinement a^* with $a \preceq a^*$ and $\forall b.\ a^* Rb \Rightarrow \mathcal{I}; b \not\models C$, which means none of the R-fillers b of a^* is contained in C. We claim that $(\mathcal{I}, a^*)$ satisfies the sequent S_p.

1. We have to show that $\forall a''. \forall b. (a^* \preceq a'' \;\&\; a'' \, R \, b) \Rightarrow \mathcal{I}; b \models L$, for all $L \in \Sigma(R)$. Observe that by transitivity of $\preceq$ it follows from $a \preceq a^*$ that if $a^* \preceq a''$ then $a \preceq a''$, i.e. all a'' are refinements of a. By (43) it then follows what was to be shown.
2. Next we show $\forall M \in \Gamma.\ \mathcal{I}; a^* \models M$. Since truth is preserved under refinement, i.e. $\forall M \in \Gamma. \forall a'. a \preceq a' \Rightarrow \mathcal{I}; a' \models M$ this follows from (44).
3. Regarding the condition $\mathcal{I}; a^* \models N$ nothing needs to be shown here, since the set of formulas that is supposed to be false is empty in the sequent S_p.
4. Finally, we find that $\forall b. a^* Rb \Rightarrow \mathcal{I}; b \not\models C$, which is needed for the premise sequent S_p with $\Psi(R) = \{C\}$ follows from construction of a^*. □

We can show soundness of $\forall L$ in a similar fashion. First we assume the conclusion of the sequent $S_c =_{df} \Theta;\ \Sigma';\ \Gamma', \forall R.C \vdash \Phi;\ \Psi$ to be constructively satisfiable and from that show that its premise, namely the sequent $S_p =_{df} \Theta;\ \Sigma' \cup [R \mapsto C];\ \Gamma' \vdash \Phi;\ \Psi$ is satisfiable as well.

If S_c is satisfiable, then there exists a pair $(\mathcal{I}, a)$ that *satisfies* S_c, i.e. $\mathcal{I}, \Theta \models \Theta$, and for all $R \in dom$, $L \in \Sigma'(R)$, $M \in \Gamma' \cup \{\forall R.C\}$, $N \in \Phi$ and $K \in \Psi(R)$:

$$\forall a'. \forall b. (a \preceq a' \ \& \ a' R b) \Rightarrow \mathcal{I}; b \models L; \tag{46}$$

$$\mathcal{I}; a \models M, \text{in particular } \mathcal{I}; a \models \forall R.C; \tag{47}$$

$$\mathcal{I}; a \not\models N; \tag{48}$$

$$\forall b. a R b \Rightarrow \mathcal{I}; b \not\models K; \tag{49}$$

Because of $\mathcal{I}; a \models \forall R.C$ we have for all refinements a'' of a that $\forall b.a'' Rb \Rightarrow \mathcal{I}; b \models C$, i.e. all R-fillers of all refinements of a are contained in C.

We claim that $(\mathcal{I}, a)$ satisfies S_p.

1. We have to show that $\forall a''. \forall b. (a \preceq a'' \ \& \ a'' R b) \Rightarrow \mathcal{I}; b \models L$, for all $L \in \Sigma'(R) \cup \{C\}$. By (46) and our assumption this follows directly.
2. Next we show $\forall M \in \Gamma'. \mathcal{I}; a \models M$. This follows from (47), since $\Gamma' \subseteq \Gamma' \cup \{\forall R.C\}$.
3. Regarding the conditions $\mathcal{I}; a \not\models N$, for $N \in \Phi$, and $\forall b.aRb \Rightarrow$, for $K \in \Psi(R)$, nothing needs to be shown here, since the set of formulas and the set of role mappings that are supposed to be false are equal in S_c and S_p.

To show soundness of Hyp_1 and Hyp_2 we assume that their conclusion, namely the sequent $S_c =_{df} \Theta, C;\ \Sigma;\ \Gamma \vdash \Phi,\ \Psi$, is satisfiable and then prove that the premises $S_{p1} =_{df} \Theta;\ \Sigma \cup [R \mapsto C];\ \Gamma \vdash \Phi;\ \Psi$ of rule Hyp_1 and $S_{p2} =_{df} \Theta;\ \Sigma;\ \Gamma, C \vdash \Phi;\ \Psi$ of rule Hyp_2 with $R \in dom$ are both satisfiable as well.

If the sequent S_c is satisfiable then by Definition 2 of constructive satisfiability there exists a pair $(\mathcal{I}, a)$ that *satisfies* the sequent S_c which means $\mathcal{I}$ is a model of $\Theta \cup \{C\}$, in particular $\mathcal{I}, a \models \Theta, C$, i.e. all entities and in particular all their refinements and R-successors are contained in C and for all $R \in dom$, $L \in \Sigma(R)$, $M \in \Gamma$, $N \in \Phi$ and

$K \in \Psi(R)$:

$$\mathcal{I} \models \Theta, \text{ and } \forall c. \mathcal{I}; c \models C \tag{50}$$
$$\forall a'. \forall b. (a \preceq a' \;\&\; a' \, R \, b) \Rightarrow \mathcal{I}; b \models L; \tag{51}$$
$$\mathcal{I}; a \models M; \tag{52}$$
$$\mathcal{I}; a \not\models N; \tag{53}$$
$$\forall b. \, a \, R \, b \Rightarrow \mathcal{I}; b \not\models K; \tag{54}$$

By (50) $\mathcal{I}; a \models C$, i.e. entity a is contained in concept C. Together with (51)–(54) this proves immediately that $(\mathcal{I}, a)$ satisfies premise S_{p2} by assumption. Similarly, because of (50), we have $\forall b. a \, R \, b \Rightarrow \mathcal{I}; b \models C$ which proves that $(\mathcal{I}, a)$ satisfies S_{p1} taking into account (51)–(54).

□

In order to prove the completeness direction of Thm. 2 we first need some technical definitions and auxiliary lemmas.

Definition 4 (Saturation). A sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ is called *saturated* if the following closure conditions hold:

- $\Theta \subseteq \Gamma$ and $\Theta \subseteq \Sigma(R)$ for all $R \in \mathit{dom}$;
- If $M \sqcap N \in \Gamma$ then both $M, N \in \Gamma$;
- If $M \sqcup N \in \Gamma$ then $M \in \Gamma$ or $N \in \Gamma$;
- If $M \sqsubseteq N \in \Gamma$ then $M \in \Phi$ or $N \in \Gamma$;
- If $\forall R.M \in \Gamma$ then $M \in \Sigma(R)$;
- If $M \sqcup N \in \Phi$ then both $M, N \in \Phi$;
- If $M \sqcap N \in \Phi$ then $M \in \Phi$ or $N \in \Phi$.
- If $\bot \in \Gamma$ then $\bot \in \Sigma(R)$ for all $R \in \mathit{dom}$. □

It is technically convenient in the following to consider the partial functions Σ, Ψ as being defined for all role names, i.e., so that $\mathit{dom}(\Sigma) = \mathit{dom}(\Psi) = N_R$ but that N_R is finite. We simply put $\Sigma(R) = \emptyset$ whenever Σ does not have R in its domain. We may then lift set operations to these functions (Σ, Ψ) from role names into sets of concepts in the standard way. E.g., for every $R \in N_R$ we put $(\Sigma_1 \cup \Sigma_2)(R) =_{df} \Sigma_1(R) \cup \Sigma_2(R)$. Further, $\Sigma_1 \subseteq \Sigma_2$ holds iff for all $R \in N_R$, $\Sigma_1(R) \subseteq \Sigma_2(R)$. In this spirit we identify the empty set $\emptyset$ with the empty function $\emptyset(R) = \emptyset$.

Lemma 3. Every consistent sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ has a consistent and saturated extension $\Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi$ such that $\Sigma \subseteq \Sigma^*$, $\Gamma \subseteq \Gamma^*$ and $\Phi \subseteq \Phi^*$. □

Proof. Let $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ be a consistent sequent. We construct monotonically increasing sequences

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots \subseteq \Sigma_n \subseteq \ldots$$
$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma_n \subseteq \ldots$$
$$\Phi_0 \subseteq \Phi_1 \subseteq \Phi_2 \subseteq \cdots \subseteq \Phi_n \subseteq \ldots$$

of subsets $\Sigma_n, \Gamma_n, \Phi_n$ according to the following extension rules, starting from $\Sigma_0 =_{df} \Sigma$, $\Gamma_0 =_{df} \Gamma$ and $\Phi_0 =_{df} \Phi$:

C0. if $M \in \Theta$ and $M \notin \Sigma_n(R)$ for some $R \in N_R$ or $M \notin \Gamma_n$, then $\Sigma_{n+1} =_{df} \Sigma_n \cup [R \mapsto M]$, $\Gamma_{n+1} =_{df} \Gamma_n \cup \{M\}$ and $\Phi_{n+1} =_{df} \Phi_n$.

C1. if there is $M \sqcap N \in \Gamma_n$ but $M \notin \Gamma_n$ or $N \notin \Gamma_n$, then $\Gamma_{n+1} =_{df} \Gamma_n \cup \{M, N\}$ while $\Sigma_{n+1} =_{df} \Sigma_n$ and $\Phi_{n+1} =_{df} \Phi_n$.

C2. if there is $M \sqcup N \in \Gamma_n$ but $M \notin \Gamma_n$ and $N \notin \Gamma_n$ then put $\Sigma_{n+1} =_{df} \Sigma_n$, $\Phi_{n+1} =_{df} \Phi_n$ and

- either $\Gamma_{n+1} =_{df} \Gamma_n \cup \{M\}$ if $\Theta; \Sigma_n; \Gamma_n \cup \{M\} \vdash \Phi_n; \Psi$ is consistent,
- or $\Gamma_{n+1} =_{df} \Gamma_n \cup \{N\}$ if $\Theta; \Sigma_n; \Gamma_n \cup \{N\} \vdash \Phi_n; \Psi$ is consistent.

C3. if there is $M \sqsubseteq N \in \Gamma_n$ but $M \notin \Phi_n$ and $N \notin \Gamma_n$ then put $\Sigma_{n+1} =_{df} \Sigma_n$ and

- either $\Gamma_{n+1} =_{df} \Gamma_n \cup \{N\}$ and $\Phi_{n+1} =_{df} \Phi_n$ if $\Theta; \Sigma_n; \Gamma_n \cup \{N\} \vdash \Phi_n; \Psi$ is consistent,
- or $\Gamma_{n+1} =_{df} \Gamma_n$ and $\Phi_{n+1} = \Phi_n \cup \{M\}$ if $\Theta; \Sigma_n; \Gamma_n \vdash \Phi_n \cup \{M\}; \Psi$ is consistent.

C4. if $\forall R.M \in \Gamma_n$ but $M \notin \Sigma_n(R)$ put $\Sigma_{n+1} =_{df} \Sigma_n \cup [R \mapsto M]$ while Γ_{n+1}, Φ_{n+1}, Ψ_{n+1} remain unchanged.

C5. if there is $M \sqcup N \in \Phi_n$ but $M \notin \Phi_n$ or $N \notin \Phi_n$, then $\Sigma_{n+1} =_{df} \Sigma_n$ and $\Gamma_{n+1} =_{df} \Gamma_n$ and $\Phi_{n+1} =_{df} \Phi_n \cup \{M, N\}$.

C6. if $M \sqcap N \in \Phi_n$ but $M \notin \Phi_n$ and $N \notin \Phi_n$ then we define $\Sigma_{n+1} =_{df} \Sigma_n$, $\Gamma_{n+1} =_{df} \Gamma_n$ and

- either $\Phi_{n+1} =_{df} \Phi_n \cup \{M\}$ if $\Theta; \Sigma_n; \Gamma_n \vdash \Phi_n \cup \{M\}; \Psi$ is consistent,
- or $\Phi_{n+1} = \Phi_n \cup \{N\}$ if $\Theta; \Sigma_n; \Gamma_n \vdash \Phi_n \cup \{N\}; \Psi$ is consistent.

C7. if $\bot \in \Gamma_n$ but $\bot \notin \Sigma_n(R)$ for some R, then $\Sigma_{n+1} = \Sigma_n \cup [R \mapsto \bot]$, $\Gamma_{n+1} = \Gamma_n$ and $\Phi_{n+1} = \Phi_n$.

The steps C0–C7 may be interleaved in any order, but fairly, to generate the sequence $\langle \Sigma_i, \Gamma_i, \Phi_i \mid i \geq 0 \rangle$. The construction continues as long as one of the 'firing conditions' of C0–C7 is applicable. The fixed point is given by $\Sigma^* =_{df} \bigcup_{n<\omega} \Sigma_n$, $\Gamma^* =_{df} \bigcup_{n<\omega} \Gamma_n$ and $\Phi^* =_{df} \bigcup_{n<\omega} \Phi_n$. If the original sequent $\Theta; \Sigma; \Gamma \vdash \Phi; \Psi$ is finite then the fixed point is reached after a finite number of steps. In the infinite case we stabilise at ω given that the syntax is (recursively) enumerable. Fairness can be achieved by considering an enumeration $M_0, M_1, M_2, \ldots$ of all possible concepts with infinite repetition, which must exist, and at each stage n of the above construction checking if one of C0–C7 applies for concept M_n. Note that C0 only needs to be applied at most once for each $M \in \Theta$. The following observations are crucial:

(a) The construction produces only consistent sequents $\Theta; \Sigma_n; \Gamma_n \vdash \Phi_n; \Psi$. This means that the sequent $\Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi$, too, must be consistent by the standard compactness argument.

(b) For the fixed point Σ^*, Γ^*, Φ^* none of the conditions C0–C7 apply. It follows that all saturation conditions of Def. 4 are fulfilled.

Thus, $\Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi$ is the saturated and consistent sequent desired by Lem. 3.

Claim (a) is easily demonstrated by induction. The initial $\Theta; \Sigma_0; \Gamma_0 \vdash \Phi_0; \Psi$ is consistent by assumption and each extension step C0–C7 turns a consistent sequent $\Theta; \Sigma_n; \Gamma_n \vdash \Phi_n; \Psi$ into another consistent sequent $\Theta; \Sigma_{n+1}; \Gamma_{n+1} \vdash \Phi_{n+1}; \Psi$:

- Assume $\Theta; \Sigma_{n+1}; \Gamma_{n+1} \vdash \Phi_{n+1}; \Psi$ produced in the step C0 is inconsistent, then this implies the existence of a closed tableau for $\Theta; \Sigma_n \cup [R \mapsto M]; \Gamma_n \cup \{M\} \vdash \Phi_n; \Psi$. But by rule Hyp_1 and Hyp_2 of Fig. 4 this would give a closed tableau for

$\Theta, M; \Sigma_n; \Gamma_n \vdash \Phi_n; \Psi$ which is contradictory to the consistency of the sequent at stage n.

- For instance, if $\Theta; \Sigma_{n+1}; \Gamma_{n+1} \vdash \Phi_{n+1}; \Psi$ produced in step C1 were inconsistent we would have a closed tableau for $\Theta; \Sigma_n; \Gamma_n \cup \{M, N\} \vdash \Phi_n; \Psi$. But by rule $\sqcap L$ of Fig. 4 this would give a closed tableau for $\Gamma; \Sigma_n; \Gamma_n \vdash \Phi_n; \Psi$ which is a contradiction to the consistency of the sequent at stage n.
- Assume $\Theta; \Sigma_{n+1}; \Gamma_{n+1} \vdash \Phi_{n+1}; \Psi$ produced in the step C4 is inconsistent, then there will be a closed tableau for $\Theta; \Sigma_n \cup [R \mapsto M]; \Gamma_n \vdash \Phi_n; \Psi$. By rule $\forall L$ this would give a closed tableau for $\Theta; \Sigma_n; \Gamma_n \vdash \Phi_n; \Psi$ with $\Gamma_n \cup \{\forall R.M\} = \Gamma_n$ because $\forall R.M \in \Gamma_n$, but this is contradictory to the consistency of the sequent at stage n.
- Assume $\Theta; \Sigma_{n+1}; \Gamma_{n+1} \vdash \Phi_{n+1}; \Psi$ produced in the step C5 is inconsistent. This implies there exists a closed tableau for the sequent $\Theta; \Sigma_n; \Gamma_n \vdash \Phi_n \cup \{M, N\}; \Psi$. By rule $\sqcup R$ this would give a closed tableau for the sequent $\Theta; \Sigma_n; \Gamma_n \vdash \Phi_n; \Psi$ with $\Phi_n \cup \{M \sqcup N\} = \Phi_n$, since $\{M \sqcup N\} \in \Phi_n$. However, this is a contradiction to the assumption, namely that the sequent at stage n is consistent.
- Assume we have the sequent $S_n = \Theta; \Sigma_n; \Gamma_n \vdash \Phi_n; \Psi$ with $\bot \in \Gamma_n$ at stage n. Now suppose the sequent at stage $n+1$, namely the sequent $S_{n+1} = \Theta; \Sigma_{n+1}; \Gamma_{n+1} \vdash \Phi_{n+1}; \Psi$ produced from S_n in the step C7 is inconsistent, i.e. the sequent S_{n+1} is derivable in tableau calculus. Then $\mid \Phi_n \cup \Psi \mid \geq 1$ by Lemma 2. Now, since $\bot \in \Gamma_n$ there is a derivation

$$\frac{|\Phi_n \cup \Psi| \geq 1}{\Theta \ ; \ \Sigma_n \ ; \ \Gamma_n, \bot \vdash \Phi_n \ ; \ \Psi} \ \bot L$$

  which contradicts consistency of S_n

Nothing needs to be shown for the extension steps C2, C3 and C6, since they are consistent by their definition.

Claim (b) follows from the fact that if one of the conditions of C1–C7 is applicable for $\langle \Sigma^*, \Gamma^*, \Phi^* \rangle$, then there is $m \geq 0$ such that it is applicable in $\langle \Sigma_n, \Gamma_n, \Phi_n \rangle$ for all $n \geq m$. But this is not possible since for some $n \geq m$ the rule is eventually fired (fairness!). $\square$

Lemma 4 (Canonical Interpretation $\cdot^*$)**.**
Let Θ be a fixed TBox and Δ^* be the set of all saturated and consistent sequents $\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle$. All these sequents have Θ as their first component but may have different Σ, Γ, Φ, Ψ. We define a refinement relation $\preceq^*$ and for all role names $R \in N_R$ a filler relation R^* on Δ^* as follows:

$$\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle \ \preceq^* \ \langle\Theta;\Sigma';\Gamma'\vdash\Phi';\Psi'\rangle \quad \textit{iff} \quad \Sigma\subseteq\Sigma' \ \& \ \Gamma\subseteq\Gamma' \tag{55}$$

$$\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle \ \ R^* \ \ \langle\Theta;\Sigma';\Gamma'\vdash\Phi';\Psi'\rangle \quad \textit{iff} \quad \Sigma(R)\subseteq\Gamma' \ \& \ \Psi(R)\subseteq\Phi'. \tag{56}$$

The interpretation of atomic concepts $A \in N_C$ is given by stipulating $\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle \in A^*$ iff $A \in \Gamma$ or $\bot \in \Gamma$. A sequent $\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle$ is fallible, i.e., $\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle \in \bot^*$, iff $\bot \in \Gamma$. Then, the interpretation $* =_{df} (\Delta^*, \preceq^*, \bot^*, \cdot^*)$ is a constructive model in the sense of Def. 1 such that for every sequent $w \in \Delta^*$ the pair $(*, w)$ satisfies w according to Def. 2. More precisely, we have $* \models \Theta$ and if $w = \langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle$ then for all choices of $R \in N_R$, $L \in \Sigma(R)$, $M \in \Gamma$, $N \in \Phi$, $K \in \Psi(R)$ we have "self-satisfaction":

1. $\forall u, v \in \Delta^*.\, w \preceq^* u \ R^* \ v \Rightarrow v \models^* L$;
2. $w \models^* M$
3. $w \not\models^* N$
4. $\forall v \in \Delta^*.\, w \ R^* \ v \Rightarrow v \not\models^* K$,

where $x \models^* C$ stands for $*, x \models C$ or equivalently $x \in C^*$.

Proof. Let us first convince ourselves that $(\Delta^*, \preceq^*, \bot^*, \cdot^*)$ is a constructive model. It is trivial from Definition (55) that $\preceq^*$ is reflexive and transitive (Note it is not in general antisymmetric). Further, we claim that the interpretation A^* of basic concepts $A \in N_C$ is closed under the refinement $\preceq^*$. This is because for every pair $w \preceq^* w'$ in refinement relationship with $w = \langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle$ and $w' = \langle\Theta;\Sigma';\Gamma'\vdash\Phi';\Psi'\rangle$ we have $\Gamma \subseteq \Gamma'$ and thus whenever $w \in A^*$ then also $w' \in A^*$ by definition. Thus, A^* is closed under $\preceq^*$. Let us look at fallibility. By definition $\bot^* \subseteq A^*$ for all $A \in N_C$. Furthermore, it is obvious that if $w \preceq^* w'$ and w is fallible then w' is fallible, whence $\bot^*$ is closed under $\preceq^*$. If $w \in \bot^*$ then by definition of $\bot^*$ we have $\bot \in \Gamma$. Now we have to show that there exists a consistent and saturated sequent w' such that $\bot \in \Gamma'$ and $w \ R^* \ w'$. Consider the sequent $S = \langle\Theta;\emptyset;\Sigma(R)\cup\{\bot\}\vdash\Psi(R);\emptyset\rangle$. We claim that the sequent S is consistent. Suppose, by contradiction, $\langle\Theta;\emptyset;\Sigma(R),\bot\vdash\Psi(R);\emptyset\rangle$ is derivable. Therefore, $\Psi(R) \neq \emptyset$ by Lemma

2. This contradicts the assumption that $\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle$ is consistent (by rule $\bot L$). By Lemma 3 there exists a consistent saturated extension $S^* = \langle\Theta;\Sigma^*;\Gamma^*\vdash\Phi^*;\Psi^*\rangle$ of $\langle\Theta;\emptyset;\Sigma(R),\bot\vdash\Psi(R);\emptyset\rangle$ so that $\Sigma(R)\cup\{\bot\}\subseteq\Gamma^*$ and $\Psi(R)\subseteq\Phi^*$. Therefore $S\ R^*\ S^*$ and $S^*\in\bot^*$.

We have to show that for a consistent and saturated sequent S with $S\in\bot^*$, all R-successors S' of S are in $\bot^*$. Consider the consistent and saturated sequent $S = \langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle$ with $S\in\bot^*$. For S we have $\bot\in\Gamma$ and by Lemma 3 we know that $\bot\in\Sigma(R)$ for all R. Let $S' = \langle\Theta;\Sigma';\Gamma'\vdash\Phi';\Psi'\rangle$ be an arbitrary R-successor of S. This means by (56) that $\bot\in\Sigma(R)\subseteq\Gamma'$ so that $S'\in\bot^*$. Since S' was arbitrary all R-successors of S are in $\bot^*$ as claimed.

Thus, $*$ is a constructive model as claimed. Note that all fallible sequents $\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle\in\Delta^*$ must be such that $\Phi=\Psi=\emptyset$ by consistency, otherwise rule $\bot L$ would make a closed tableau. Hence, whenever Φ or $\Psi(R)$ for some $R\in N_C$ are nonempty we know that $\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle\in\Delta^*_c=\Delta^*\setminus\bot^*$.

Next we claim that $*$ is a model of Θ, i.e., $*\models\Theta$. This follows from the self-satisfaction condition 2., proven below, and the fact that for all $w'=\langle\Theta';\Sigma';\Gamma'\vdash\Phi';\Psi'\rangle\in\Delta^*$ we have $\Theta'=\Theta\subseteq\Gamma'$ by saturation.

It remains to show self-satisfaction, i.e., the truth conditions 1.–4. We first observe that 1. and 4. follow directly from 2. and 3., respectively, using the definition of $\preceq^*$ and R^* in the model. Let us deal with 4. first. To this end take any R^*-filler $\langle\Theta;\Sigma';\Gamma'\vdash\Phi';\Psi'\rangle$ for $\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle$ in $*$. By construction of R^*, $\Psi(R)\subseteq\Phi'$, which means that $\langle\Theta;\Sigma';\Gamma'\vdash\Phi';\Psi'\rangle\not\models^* K$ for all $K\in\Psi^*(R)$ by 3. Regarding 1. consider the scenario

$$\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle\preceq^*\langle\Theta;\Sigma';\Gamma'\vdash\Phi';\Psi'\rangle\ R^*\ \langle\Theta;\Sigma'';\Gamma''\vdash\Phi'';\Psi''\rangle$$

in the canonical interpretation $*$. Our definitions of $\preceq^*$ and R^* make sure that $\Sigma(R)\subseteq\Sigma'(R)\subseteq\Gamma''$. Thus, for all $L\in\Sigma(R)$ we get $\langle\Theta;\Sigma'';\Gamma''\vdash\Phi'';\Psi''\rangle\models L$ by 2.

In the rest of the proof we verify conditions 2. and 3. More precisely, for an arbitrary concept M we show

$$M\in\Gamma\ \Rightarrow\ \langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle\models^* M$$
$$M\in\Phi\ \Rightarrow\ \langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle\not\models^* M,$$

simultaneously by induction on M. For atomic concepts $A \in \Gamma$ and $\bot \in \Gamma$ the statement is trivial by definition of $\models^*$. Similarly, for $A \in \Phi$ we get $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \not\models^* A$ and for $\bot \in \Phi$ we get $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \not\models^* \bot$ since otherwise $A, \bot \in \Gamma$ by definition of $\models^*$ and then rule Ax would contradict consistency of $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle$.

• Suppose $M \sqcap N \in \Gamma$. Then, by saturation of the sequent $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle$ we have both $M, N \in \Gamma$ which means $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \models^* M$ and $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \models^* N$ by induction hypothesis. This implies $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \models^* M \sqcap N$.

• If $M \sqcap N \in \Phi$ then, by saturation of the sequent $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle$ this implies that $M \in \Phi$ or $N \in \Phi$ and thus $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \not\models^* M$ or $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \not\models^* N$ by induction hypothesis from which we get $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \not\models^* M \sqcap N$.

• Suppose $M \sqcup N \in \Gamma$. Then, by saturation of the sequent we have $M \in \Gamma$ or $N \in \Gamma$ which means $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \models^* M$ or $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \models^* N$ by induction hypothesis. This implies $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \models^* M \sqcup N$.

• Let $M \sqcup N \in \Phi$. Then, by saturation of the sequent $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle$ we have both $M, N \in \Phi$ which means $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \not\models^* M$ and $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \not\models^* N$ by induction hypothesis. This implies $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \not\models^* M \sqcup N$.

• Suppose $M \sqsubseteq N \in \Gamma$ and $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \preceq^* \langle \Theta; \Sigma'; \Gamma' \vdash \Phi'; \Psi' \rangle \models^* M$ where $\langle \Theta; \Sigma'; \Gamma' \vdash \Phi'; \Psi' \rangle \in \Delta_c^*$. Then, both $M \sqsubseteq N \in \Gamma \subseteq \Gamma'$ and $M \notin \Phi'$ by induction hypothesis. Since $\langle \Theta; \Sigma'; \Gamma' \vdash \Phi'; \Psi' \rangle$ is saturated this means $N \in \Gamma'$ and thus $\langle \Theta; \Sigma'; \Gamma' \vdash \Phi'; \Psi' \rangle \models^* N$ by induction hypothesis. This proves $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \models^* M \sqsubseteq N$ overall. Notice that we did not need the assumption of non-fallibility $\langle \Theta; \Sigma'; \Gamma' \vdash \Phi'; \Psi' \rangle \in \Delta_c^*$ here.

• Vice versa, suppose $M \sqsubseteq N \in \Phi$. Then consider the sequent $\langle \Theta; \Sigma; \Gamma \cup \{M\} \vdash \{N\}; \emptyset \rangle$ which must be consistent. Otherwise, if there is a tableau for $\langle \Theta; \Sigma; \Gamma \cup \{M\} \vdash N; \emptyset \rangle$ then also for $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle$ by rule $\sqsubseteq R$ of Fig. 4. Because $\langle \Theta; \Sigma; \Gamma \cup \{M\} \vdash \{N\}; \emptyset \rangle$ is consistent, Lem. 3 yields a saturated and consistent extension $\langle \Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi \rangle \in \Delta^*$ which must be non-fallible, i.e., $\langle \Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi \rangle \in \Delta_c^*$ since $N \in \Phi^*$. For this extension it holds that $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \preceq^* \langle \Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi \rangle$ because both $\Sigma \subseteq \Sigma^*$ and $\Gamma \subseteq \Gamma \cup \{M\} \subseteq \Gamma^*$. Moreover, since $M \in \Gamma^*$ and $N \in \Phi^*$ we have $\langle \Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi \rangle \models^*$

M as well as $\langle \Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi \rangle \not\models^* N$ by induction hypothesis. This demonstrates $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \not\models^* M \sqsubseteq N$ as desired.

• Now consider $\exists R.M \in \Gamma$. Then, for all sequents $\langle \Theta; \Sigma'; \Gamma' \vdash \Phi'; \Psi' \rangle \in \Delta^*_c$ with $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \preceq^* \langle \Theta; \Sigma'; \Gamma' \vdash \Phi'; \Psi' \rangle$ we have $\exists R.M \in \Gamma'$. We claim that $\langle \Theta; \emptyset; \Sigma'(R) \cup \{M\} \vdash \Psi'(R); \emptyset \rangle$ must be consistent. For otherwise, if there existed a closed tableau for $\langle \Theta; \emptyset; \Sigma'(R) \cup \{M\} \vdash \Psi'(R); \emptyset \rangle$, then by rule $\exists L$ of Fig. 4 we could obtain a closed tableau for the sequent $\langle \Theta; \Sigma'; \Gamma' \vdash \Phi'; \Psi' \rangle$ contradicting consistency. Now, since it is consistent there exists a saturated and consistent extension $\langle \Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi \rangle \in \Delta^*$ of $\langle \Theta; \emptyset; \Sigma'(R) \cup \{M\} \vdash \Psi'(R); \emptyset \rangle$ thanks to Lem. 3. By construction we have $\Sigma'(R) \subseteq \Sigma'(R) \cup \{M\} \subseteq \Gamma^*$ as well as $\Psi'(R) \subseteq \Phi^*$, so that $\langle \Theta; \Sigma'; \Gamma' \vdash \Phi'; \Psi' \rangle \; R^* \; \langle \Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi \rangle$ and $M \in \Gamma^*$. By induction hypothesis, $\langle \Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi \rangle \models^* M$ which proves $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \models^* \exists R.M$.

• Vice versa, assume $\exists R.M \in \Phi$. Let $\Psi^* =_{df} [R \mapsto M]$ be the function with domain $\{R\}$ and $\Psi^*(R) = \{M\}$. Then, $\langle \Theta; \Sigma; \Gamma \vdash \emptyset; \Psi^* \rangle$ is consistent, since any closed tableau for $\langle \Theta; \Sigma; \Gamma \vdash \emptyset; \Psi^* \rangle$ gives rise to a closed tableau for $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle$ by rule $\exists R$ from Fig. 4, contradicting the consistency assumption. So, by Lemma 3 there is a saturated and consistent extension $\langle \Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi^* \rangle \in \Delta^*_c$ which must be non-fallible because $M \in \Psi^*(R)$. The fact that $\Sigma \subseteq \Sigma^*$ and $\Gamma \subseteq \Gamma^*$ means $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \preceq^* \langle \Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi^* \rangle$. Now let any R^*-successor $\langle \Theta; \Sigma'; \Gamma' \vdash \Phi'; \Psi' \rangle$ of $\langle \Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi^* \rangle$ be given. By definition of R^* we have $M \in \Psi^*(R) \subseteq \Phi'$. Hence, by induction hypothesis, $\langle \Theta; \Sigma'; \Gamma' \vdash \Phi'; \Psi' \rangle \not\models^* M$. Overall, this proves $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \not\models^* \exists R.M$.

• Let us now tackle the case $\forall R.M \in \Gamma$. By saturation we conclude $M \in \Sigma(R)$. Then in all situations

$$\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \;\preceq^*\; \langle \Theta; \Sigma'; \Gamma' \vdash \Phi'; \Psi' \rangle \;\; R^* \;\; \langle \Theta; \Sigma''; \Gamma'' \vdash \Phi''; \Psi'' \rangle$$

it holds that $M \in \Sigma(R) \subseteq \Sigma'(R) \subseteq \Gamma''$. By induction hypothesis, $\langle \Theta; \Sigma''; \Gamma'' \vdash \Phi''; \Psi'' \rangle \models^* M$. This establishes that $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \Psi \rangle \models^* \forall R.M$. Note that the available extra assumption that $\langle \Theta; \Sigma'; \Gamma' \vdash \Phi'; \Psi' \rangle$ is non-fallible has not been needed.

• Finally, to take the other direction, let us suppose that $\forall R.M \in \Phi$. First observe that the sequent $\langle \Theta; \Sigma; \Gamma \vdash \Phi; \emptyset \rangle$ in which the last constraint is switched off is also consistent and saturated, i.e., an element of Δ^*. The former follows from weakening (see Lem. 2) and the latter is obvious from the definition of saturation which refers to the inner sequent only. The weakened sequent is a $\preceq^*$ successor, i.e., $\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle \preceq^* \langle\Theta;\Sigma;\Gamma\vdash\Phi;\emptyset\rangle$ and also non-fallible since $\Phi \neq \emptyset$. Consider the sequent $\langle\Theta;\emptyset;\Sigma(R)\vdash\{M\};\emptyset\rangle$ which is easily seen to be consistent. For if there was a closed tableau for $\langle\Theta;\emptyset;\Sigma(R)\vdash\{M\};\emptyset\rangle$ then using the rule $\forall R$ (see Fig. 4) we would also be able to construct a tableau for $\langle\Theta;\Sigma;\Gamma\vdash\Phi;\emptyset\rangle$ in contradiction to our assumptions. Take any saturated and consistent extension $\langle\Theta;\Sigma^*;\Gamma^*\vdash\Phi^*;\emptyset\rangle$ of $\langle\Theta;\emptyset;\Sigma(R)\vdash\{M\};\emptyset\rangle$ obtainable from Lem. 3. It satisfies $\Sigma(R) \subseteq \Gamma^*$ and $M \in \Phi^*$. This means $\langle\Theta;\Sigma;\Gamma\vdash\Phi;\emptyset\rangle\ R^*\ \langle\Theta;\Sigma^*;\Gamma^*\vdash\Phi^*;\emptyset\rangle$ and $\langle\Theta;\Sigma^*;\Gamma^*\vdash\Phi^*;\emptyset\rangle \not\models^* M$ by induction hypothesis. This finally proves $\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle \not\models^* \forall R.M$. □

We are now ready to tackle completeness.

Theorem 2 [Tableau Completeness]
Every consistent sequent is satisfiable, i.e., if $\Theta;\Sigma;\Gamma \not\vdash \Phi;\Psi$ *then* $\Theta;\Sigma;\Gamma \not\models \Phi;\Psi$.
□

Proof. Let $\Theta;\Sigma;\Gamma \vdash \Phi;\Psi$ be a consistent sequent. We must show that $\Theta;\Sigma;\Gamma \vdash \Phi;\Psi$ is satisfiable, i.e., that there is a constructive model $\mathcal{I}$ and an entity w in $\mathcal{I}$ which satisfy $\Theta;\Sigma;\Gamma \vdash \Phi;\Psi$. But precisely this is obtained in the canonical model $(\Delta^*, \preceq^*, \perp^*, \cdot^*)$ constructed in Lem. 4 using Θ as fixed TBox. Specifically, by Lemma 3, Δ^* contains a saturated and consistent extension $w^* = \langle\Theta;\Sigma^*;\Gamma^*\vdash\Phi^*;\Psi\rangle$ of the sequent $\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle$ so that $\Sigma \subseteq \Sigma^*$, $\Gamma \subseteq \Gamma^*$ and $\Phi \subseteq \Phi^*$. Lem. 4 now says that the pointed interpretation $(*, w^*)$ satisfies w^* and thus in particular satisfies the sequent $\langle\Theta;\Sigma;\Gamma\vdash\Phi;\Psi\rangle$ as desired. □

Theorem 3 [Finite Model Property and Decidability]
A finite sequent is satisfiable iff it is satisfiable in a finite interpretation. Consistency of finite sequents is decidable. □

Proof. Decidability is obtained by the simple fact that the tableau rules in Fig. 4 have the *sub-formula* property: All formulas in the premises of a rule are (not necessarily proper) sub-formulas of formulas in the conclusion. Also, the domain of a premise sequent is extended at most by a role appearing in concepts of the conclusion sequent (see rules $\exists R$, $\forall L$). In rule Hyp_1 a role already existing in the domain of the conclusion sequent is updated. Thus, the sizes of the domains and formula sets in a tableau are bounded by the root sequent. More specifically, if we are searching for a closed tableau of a finite sequent $\langle\Theta; \Sigma; \Gamma \vdash \Phi; \Psi\rangle$ then we only ever need to consider tableaux with nodes formed from those (sub-)concepts and roles contained in $\langle\Theta; \Sigma; \Gamma \vdash \Phi; \Psi\rangle$. Since there are only a finite number of such nodes and the tableau rules finite branching, there are only a finite number of possible tableaux with root sequent $\langle\Theta; \Sigma; \Gamma \vdash \Phi; \Psi\rangle$. These can be enumerated and checked effectively in bounded time.

Finite Model Property is argued as follows. Let Ξ be a finite set of concepts that is closed under sub-concepts. A sequent $\langle\Theta; \Sigma; \Gamma \vdash \Phi; \Psi\rangle$ is called Ξ-*bounded* if $\Theta \cup \Gamma \cup \Phi \subseteq \Xi$ and for all roles $R \in N_R$ that appear in Ξ we have $\Sigma(R) \cup \Psi(R) \subseteq \Xi$ while for all other R outside of Ξ, $\Sigma(R) = \Psi(R) = \emptyset$. Obviously, there is only a finite number of Ξ-bounded sequents. It turns out that Lem. 3 can be strengthened to say that for every consistent and Ξ-*bounded* sequent $\langle\Theta; \Sigma; \Gamma \vdash \Phi; \Psi\rangle$ there exists a consistent, saturated and Ξ-*bounded* extension $\langle\Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi\rangle$. This is easy to see by inspection of the proof of Lem. 3. All concepts added in the course of saturation are sub-concepts of concepts already in $\langle\Theta; \Sigma; \Gamma \vdash \Phi; \Psi\rangle$. Further, one then shows that the canonical interpretation in Lem. 4 can be restricted so that Δ^* only consists of Ξ-bounded sequents. The reason is simply that satisfiability of a Ξ-bounded sequent only depends on concepts in Ξ and that all existentially necessary elements of Δ^* constructed in the proof of Lem. 4 are Ξ-bounded. If we are now given a finite and consistent sequent $\langle\Theta; \Sigma; \Gamma \vdash \Phi; \Psi\rangle$ there is trivially a finite Ξ for which it is Ξ-bounded. Thus, it has an extension $\langle\Theta; \Sigma^*; \Gamma^* \vdash \Phi^*; \Psi\rangle$ in the finite interpretation Δ^* restricted to Ξ-bounded sequents and due to canonicity (Lemma 4) this extension satisfies itself.

□

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