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Valuation, Empirical Analysis, and Optimal Exercise of Open-End Turbo Certificates

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Contents

List of Abbreviations

APT Arbitrage Pricing Theory.

ARCH Autoregressive Conditional Heteroskedasticity.

CAPM Capital Asset Pricing Model.

CARA Constant Absolute Risk Aversion.

CEV Constant Elasticity of Variance.

cf. confer (*Latin*: see).

CG Conjugate Gradient.

CRRA Constant Relative Risk Aversion.

DAX Deutscher Aktienindex.

DJIA Dow Jones Industrial Average.

e.g. exempli gratia (*Latin*: for example).

EMM Efficient Method of Moments.

EONIA Euro Overnight Index Average.

EUREX European Exchange.

EURIBOR Euro Interbank Offered Rate.

EUWAX European Warrant Exchange.

FFT Fast Fourier Transform.

FSG Forward Shooting Grid.

GARCH Generalized Autoregressive Conditional Heteroskedasticity.

GDP Gross Domestic Product.

GMM Generalized Method of Moments.

i.e. id est (*Latin*: that is).

KKMDB Karlsruher Kapitalmarktdatenbank.

LGD Loss Given Default.

MCMC Markov Chain Monte Carlo.

MICD Market Index Certificate of Deposit.

ODE Ordinary Differential Equation.

OELC Open-End Leverage Certificate.

OETC Open-End Turbo Certificate.

PDE Partial Differential Equation.

PIDE Partial Integro Differential Equation.

PSOR Projected Successive Overrelaxation.

QMLE Quasi Maximum Likelihood Estimate.

SDE Stochastic Differential Equation.

SOR successive overrelaxation.

SPARQS Stock Participation Accreting Redemption Quarterly-Pay Securities.

SPIN Salomon Brothers' Standard and Poor's 500 Indexed Note.

w.r.t. with respect to.

List of Symbols

B_t Riskless asset or bond.

C_t Consumption process.

D_h Operator of the discrete first derivative with equidistant grid size h.

D_h^2 Operator of the discrete second derivative with equidistant grid size h.

F Cap of a discount certificate.

G^0 Interior of a set G.

I_d d-dimensional unity matrix.

K_t Striking price of an option or a Turbo certificate at time t.

L_t Knock-out barrier of a barrier option or certificate at time t.

$N(\cdot)$ Cumulative one-dimensional normal distribution function.

$N_d(\cdot)$ Cumulative d-dimensional normal distribution function.

P Empirical (or physical) probability measure.

Q Risk-neutral probability measure.

R_t Rebate of a Turbo certificate in the case of a knock-out.

S_f Critical stock price for American options.

T Maturity of an option or finite-lived retail certificate.

W_t d-dimensionl standard Brownian motion.

$X_{h,\delta}$ Discrete-time approximating Markov chain of stochastic process X.

Y Discount rate of a discount certificate.

$Z^{h,\delta}$ Discrete-time approximative optimal value function.

Λ Bonus level of a bonus certificate.

Ω Valuation region of an option.

Π Replicating portfolio of an option.

κ Mean-reversion speed of a mean reverting stochastic process.

λ Jump intensity in Merton (1976) model.

$\mathcal{A}(x,t)$ Control space from which a control action can be taken at state (x,t).

π Early exercise premium of an American option.

$\pi(t)$ Portfolio process.

ρ Correlation between two correlated standard Brownian motions.

σ Dispersion matrix of multidimensional stochastic process.

τ Stopping time for a stochastic process X.

θ Long-term mean of a mean reverting stochastic process.

ξ Jump size in jump diffusion models.

a Gap size of an Open-End Turbo Certificate.

b Drift vector of multidimensional stochastic process.

g Payoff function of an option or a certificate.

k Participation rate of sprint or finite-lived Turbo certificate.

q Continuously paid dividend yield.

r Riskless short-term financing rate (riskless rate of return).

z Financing rate of an Open-End Turbo Certificate.

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Chapter 1

Introduction

The German market for retail derivatives has grown dramatically to 101.7 billion euros invested in the asset class on March 31, 2012 according to Deutscher Derivate Verband (2012).[1] Compared to 2004 when there were only 48 billion euros invested, this constitutes an increase of 112%. If there had not been an ongoing financial crisis since 2007, the increase would most likely be even more pronounced, which is suggested by the record investment recorded in late 2007 of 139 billion euros and the subsequent plummeting to below 80 billions in early 2009. Since then the retail derivatives market has not fully recovered.

The general increase of the retail derivatives market volume in the past decade is in line with a general increase of privately held net assets (equity) in Germany since 2004. According to Deutsche Bundesbank (2010) these assets rose from 6,421 billion euros to 8,555 between 2004 and 2010 (a relative increase of 33%), which means

[1] These retail derivatives are also referred to as financial innovations. In Germany it is also common to call these products certificates. In this thesis, all three terms will be used interchangeably.

that retail derivatives have significantly gained market share. This poses two natural questions:

1) What is the reason for the increase in private net assets?

2) Why are retail derivatives benefiting from this development in a more pronounced way?

One possible line of explanation is put forth by Kohlert (2009) who adduces changes in social security legislation which impose a necessity for a higher degree of private responsibility which naturally attracts money to capital markets via pension funds or other types of insurance products. This is in line with the situation depicted in Deutsche Bundesbank (2010) where assets belonging to the latter classes are documented to have risen from 1,227 billion euros to 1,600. The particular development in the retail derivatives sector might be explained by special tax breaks for financial innovations in Germany until 2009. Contrary to fixed income products, capital gains on financial innovations were not taxed at all if they had been held for at least one year as it was the case with stock market investments. Secondly, the development might also have been spurred by considerably lower interest rates than in the 1980s and 1990s which rendered traditional savings accounts unfavorable.

According to Deutscher Derivate Verband (2012) 98.7% of the assets recorded in March of 2012 fall into the class of investment products, while only 1.3% are classified as leverage products.[2] Of those leverage products, the majority with 38.9% each is written either on single stocks or stock market indexes. Another 16.8% refer to commodities while interest rates and currencies play only a very minor role. In terms of absolute numbers there is about half a billion euros invested in stock market index leverage products.

[2] Contrary to leverage products, investment products oftentimes include capital guarantees and do not exhibit knock-out possibilities or any type of leverage.

Research regarding these products is naturally concerned with retail client specific questions and their ramifications for investors. The most important questions appear to be:

1) Market segmentation: Due to the fact that issuers typically are large financial institutions, primarily banks, while investors are retail clients, the two groups have different access to capital markets with respect to (w.r.t.) transaction costs, borrowing costs, limitations to short selling and so forth. This immediately leads to the second area of research.

2) Fair prices: Are retail certificates fairly priced or do issuers exploit their market power to the disadvantage of their clients?

3) Justification of potentially unfair prices: If these derivatives are not fairly priced, to which degree can this be justified in terms of structuring costs incurred by the issuers and by the offering of products retail clients could not otherwise replicate themselves as well as the enablement to access markets that would otherwise be limited to institutional investors?

The recent paper by Henderson & Pearson (2011) addresses these questions for the US American market. On the one hand they provide an account of literature stressing the beneficial effects of financial innovations. These are the generation of payoff profiles which would otherwise be unattainable for investors. Secondly, structured products were shown to reduce transaction costs for investors (Ross (1989) and Grinblatt & Longstaff (2000)) or provide tax advantages (Miller (1986)). Thus, all benefits to some extent revolve around lifting market imperfections. On the other hand financial innovations are designed specifically to the needs or perceived needs of retail investors, which do not necessarily have to coincide with a rational investor's view. If issuers engineer products with high payoffs in states considered more probable by an uninformed investor than they are in reality and vice versa this opens up selling opportunities at the expense of such uninformed investors. Using the example of SPARQS (Stock Participation Accreting Redemption Quarterly-Pay Securities) introduced by Morgan Stanley, Henderson & Pearson (2011) demonstrate that this effect can be quite sizable as those SPARQS are sold at a surcharge of at least 8%.

These results and considerations motivate to take a specific look at further structured products and markets. In this thesis we will closely examine OETCs (Open-End Turbo Certificates), the latest generation of knock-out leverage products offered to German retail investors. The two crucial and innovative features of this product are an openly communicated pricing formula by the issuers paired with a potentially endless lifetime. Considering these properties and the fact that there is only very limited research on the topic, this thesis aims at creating a comprehensive view on OETCs, which appears to be a promising direction for research.

1.1 Research Question

Based on the abovementioned research gap, we can further concretize the main intentions pursued in this thesis:

1) *Detailed discussion of the design of OETCs*: The starting point of the thesis has to be a detailed decription of the structured product which is the subject matter of this thesis.

2) *Detailed theoretical analysis of the main value drivers affecting the pricing of OETCs*: Based on the design of OETCs the next step has to be an understanding of what essentially drives the value of the product. Naturally, this has to be done from two

perspectives, the pure pricing perspective of a financial economist and the perspective of an investor who might be willing to put money into the product.

3) *Reformulation of OETCs in terms of option pricing methodology*: For the sake of the actual pricing of OETCs it is beneficial to formulate the OETC pricing problem as an American option pricing problem, for which there is a plethora of solution techniques available. These solution techniques have to be assessed and evaluated based on the previous value driver analysis. In particular, the derivation of optimal exercise prices for the American options has to be addressed at this stage.

4) *Analysis of numerical pricing techniques for the valuation of OETCs*: Once the solution technique for the OETC pricing problem has been chosen an efficient numerical solution method has to implemented to complete the pricing.

5) *Comparative statics analysis*: After choosing a pricing model to value OETCs the model has to be validated in a comparative statics analyis. In such an analysis the impact of the pricing and model parameters are assessed by varying one parameter while all other parameters are held constant. At the same time this analysis helps evaluating which parameters are most crucial to the valuation and the value OETCs create for investors.

6) *Empiricial study to assess market traded OETCs*: Finally, the relevance of the pricing methodology has to be demonstrated based on real market data for an extended period of time in an empirical analysis. This comprises both an estimation of market model parameters which are inferred from traded option prices and also an analysis for which market offered OETCs the optimal exercise threshold is breached. A second step of the empirical analysis has

to be concerned with the development of the observations over time and an economic explanation for the observations.

1.2 Outline of the Research

This thesis consists of five chapters not counting the *Introduction* (Chapter 1) and the *Conclusion* (Chapter 7). In Chapter 2 *Standard Options and Overview of Retail Certificates* we first give an overview of standard call and put options as well as popular retail certificates in the German market. In this way we highlight the versatility of these structured products in Germany and stress the necessity to assess their usefulness from an economic viewpoint. In the second part of the chapter the focus lies on Turbo certificates, the subject matter of this thesis. First we present the results on finite-lived Turbo certificates, second we introduce OETCs and provide an economic analysis under which circumstances these products should exist. Also we give an interpretation of OETCs in terms of American-style barrier options.

Chapter 3 *Option Pricing Theory* pertains to option pricing theory and is divided into four sections. Section 3.1 covers the most prominent option pricing models revolving around the Black & Scholes (1973) model, jump-diffusion models such as the one suggested by Merton (1976), stochastic volatility models like the one put forth by Heston (1993), and combinations and extensions of these models. This lays the foundation to further applying these models to the option constituted by OETCs. In Section 3.2 we shift our attention towards American-style options. Since they differ from their European counterparts in the fact that they can be prematurely exercised we first summarize the results on properties of the early exercise boundary. After that we present valuation methods for American-style options, which we classify as analytic approximations, methods based on PDEs (Partial Differential Equations) governing option prices, Monte Carlo simulations and optimal control techniques, which revolve around interpreting them as optimal stopping problems. For each of the methods we summarize the most important results. In Section 3.3 we describe barrier options, which can be activated or extinguished based on a barrier being breached, and present the most relevant results on their valuation. Section 3.4 concludes the chapter and addresses the assumptions about the market made by the respective models. In particular, this pertains to perfect and efficient markets and to which extent these assumptions are satisfied by stock markets.

In Chapter 4 *Optimal Control Theory and Optimal Stopping* we wrap up the theoretical foundations by introducing the mathematical background of optimal control and optimal stopping problems. The dynamic programming equations which govern the optimal value function of a control problem are derived in Section 4.1 as they are the main ingredient for the development of efficient numerical solution schemes. These numerical schemes revolve around approximating the underlying stochastic processes with discrete-time Markov chains, which is elucidated in Section 4.2. Section 4.3 then deals with the numerical solution methods we employ to solve the discretized optimal stopping problem.

Having the underlying option pricing theory for American and barrier options, a formulation of OETCs as such, and efficient numerical solution schemes for these types of problems in place, we carry out a comparative statics analysis in Chapter 5 *Comparative Statics Analysis of OETCs*. This analysis includes the choice of the Bakshi, Cao & Chen (1997) market model in Section 5.1. In Section 5.2 the actual comparative statics analysis is conducted which characterizes the dependence of the optimal exercise thresholds on the gap size and the financing parameter of the OETCs as well as the jump intensity and the expected jump size of the market model. In Section 5.3 we conclude the comparative statics analysis with a one-day empirical analysis of all OETCs written on the DAX and traded on July 16, 2010.

In Chapter 6 *Empirical Study for 2007 through 2009* we conduct an empirical analysis for the years 2007 through 2009, in which we analyze all OETCs on the German stock market index DAX. For this purpose we present our data set in Section 6.1. Our set consists of two parts, the actual OETCs data and the option data on the DAX from which we infer our market model parameters using a least squares fit between market and model prices. In Section 6.2 we present the results of the empirical study along with an interpretation of our main findings.

Eventually, Chapter 7 *Conclusion* summarizes the results of the entire thesis. In addition, a critical interpretation of the results from an economic point of view and in the light of the research question is provided. The thesis is then wrapped up by a discussion and an outlook on potential further research which was not addressed or only raised in this work.

Chapter 2

Standard Options and Overview of Retail Certificates

Innovative products are offered to retail customers in many countries and have long been discussed by researchers. Examples of the literature on such retail derivatives include Chen & Kensinger (1990), Wasserfallen & Schenk (1996), and Burth, Kraus & Wohlwend (2001).

Those products are usually generalizations of standard options and standard derivatives in general. Therefore, Section 2.1 introduces standard options. In Section 2.2, we provide an account of the different types of retail certificates available in the German derivatives market as well as a literature survey of previous research. In Section 2.3 we shift our attention to Turbo certificates. First, the long existing finite-time variant is discussed and eventually the relatively new open-end version is introduced.

2.1 Standard Options

In general, there are two types of options, call options and put options. Options securitize the right to either buy (call option) or sell (put option) an asset at a prespecified price in the future. This price is called the exercise or striking price (more briefly also strike) of the option. After a certain time, commonly referred to as expiry or maturity, this right perishes.

In the absence of transaction costs plain vanilla options yield the following profits P_{call} and P_{put} to their holders at maturity T depending on the underlying S and the striking price K

$$\begin{aligned} P_{call}(S,T) &= \max(S(T) - K, 0) \\ P_{put}(S,T) &= \max(K - S(T), 0), \end{aligned}$$

where $S(T)$ denotes the price of the underlying asset at time T. These payoff functions have also been illustrated graphically in Figure 2.1 for a call option with strike $K_{\text{call}} = 15$ and a put option with strike $K_{\text{put}} = 20$.

Besides, one can make distinctions as to when the option holder is entitled to exercising the option. Roughly speaking, two major types of exercise specifications have evolved. So-called European-style options are exercisable only at maturity. On the other extreme, American-style options can be exercised at any time between initiation and maturity. Of course, there are also intermediate types of options, which can, for instance, be exercised at a number of discrete times before maturity (e.g. Bermudan options or Asian options which depend on the average stock price between initiation and maturity), but classical European and, in particular, American options make up for the majority of the market share.[3]

[3] More information about the design of option markets and the various types of

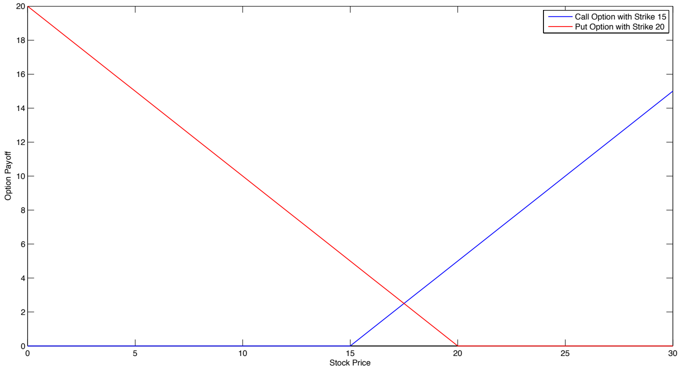

Figure 2.1: **Payoffs of Standard Call and Put Options**
This table shows the payoff functions of a standard call option with strike 15 and a standard put option with strike 20.

One very important notion is that in contrast to futures or forward contracts, in which two market participants agree on the obligation to exchange an asset for cash at a future time, options reflect the right to engage in a future transaction. This becomes relevant when at maturity the market price of the asset is less than the striking price of the option as in such a scenario no rational investor would buy an asset using the option as she could procure the asset less expensively in the market. So the option holder would forfeit her right and have the option expire worthlessly. However, such freedom can only come at a certain premium which is distinctly different from forward and futures contracts as these transactions are agreed upon so that the fair future exercise price precludes the necessity of upfront cash exchange, as is, for example, pointed out in Hull (2009).[4]

options traded in these markets can be found in various textbooks on options pricing. Among others, there are Hull (2009) and Wilmott (2006). Zhang (1998) especially deals with the more exotic types of options.

[4] In this respect fair price means the price that rules out arbitrage opportunities

Therefore, it is only a natural question to ask what a fair, arbitrage-free option premium is. Consequently, a whole string of research has developed around this question. This research can be loosely categorized in two classes. The first deals with market models trying to adequately model the underlying asset's characteristics and with the subsequent derivation of European option prices. Then, the second stage deals with the American option feature of premature exercise given a certain model. As will become clearer when discussing previous research results the additional right makes contingent claim valuation markedly more complex.

A detailed discussion of the various option pricing models is given below in Section 3.1. The intricacies of American-style options and the alterations necessary to value them are dealt with in Section 3.2. But before this detailed discussion of option pricing theory, the rest of this chapter gives an introduction to retail certificates in the German market.

2.2 Types of Retail Certificates

Since retail certificates belong to the class of structured financial products there is a virtually unlimited range w.r.t. payout and risk profiles. In the following we will give a brief overview over the most prominent (and from a sales perspective popular) varieties offered by financial institutions to private customers. In particular, we focus on:

1) Discount certificates (cf. Section 2.2.1)

2) Bonus certificates (cf. Section 2.2.2)

for any involved party. More on futures contracts can be found in the textbook by Duffie (1989).

3) Sprint certificates (cf. Section 2.2.3).

In the respective sections we will give an overview about how the type of certificate is structured, how the payoff profile works and what type of investors it might suit. The latter aspect will be discussed both from an economic point of view and from the marketing perspectives of the issuers who intend to target retail clients.

In Table 2.1 we have provided an overview of the issuers of bonus, discount, and sprint certificates and how many certificates of the respective type they have issued. It is evident that bonus and discount certificates are overwhelmingly more popular than sprint certificates with more than 100 times as many discount certificates outstanding. The data were retrieved from the website onvista.de on April 13, 2012 and depict the situation as of this particular day.

In terms of single issuers there is a mixed picture between large investment banks, such as BNP Paribas or Goldman Sachs, and German market focused banks like DZ Bank or LBBW. Most bonus certificates have been issued by BNP Paribas with Commerzbank, Deutsche Bank and DZ Bank about $1,000$ issues behind. Regarding discount certificates Deutsche Bank is the market leader with Commerzbank and the Royal Bank of Scotland ranking second and third. Besides these major competitors there are also several niche players who offer ten or fewer certificates. Among those are well known institutions like DekaBank, Landesbank Hessen Thüringen, HypoVereinsbank or the Austrian Erste Group.

2.2.1 Discount Certificates

One of the most popular classes of certificates, as evidenced by Table 2.1 are so-called discount certificates. Typically they are written

Issuer	Bonus	Discount	Sprint
Bank of America - Merril Lynch	160	265	0
Barclays Bank	505	1,366	0
BNP Paribas	3,705	1,553	72
Citigroup	49	856	30
Commerzbank	2,429	3,142	33
DekaBank	1	1	0
Deutsche Bank	2,793	3,919	37
DZ Bank	2,205	1,263	7
EFG Financial Products	1	21	0
Erste Group Bank AG	2	0	0
Goldman Sachs	1,645	703	0
HSBC Trinkaus	926	1,108	1
HypoVereinsbank	1	528	0
ING	0	1	0
JP Morgan	10	0	0
Landesbank Berlin	1	10	0
LBBW	5	180	0
Landesbank Hessen Thüringen	5	0	0
Macquarie	20	19	0
Morgan Stanley	0	108	0
Rabobank	0	1	0
Raiffeisen Centrobank	5	4	0
Societé Générale	1,154	503	0
Royal Bank of Scotland	1,348	2,033	0
UBS	73	2,773	0
Vontobel	75	547	3
WestLB	7	65	0
WGZ Bank	2	67	0
Total	17,115	21,054	183

Table 2.1: **Number of Outstanding Certificates by Type** This table shows the number of certificates written on the German stock market index DAX by type and issuer as observed on April 13, 2012. In addition, the total numbers for each type of certificates are reported.

on stock indexes or single stocks. This type of certificate revolves around the idea to mimic the behavior of underlying stock, i.e. gains and losses are supposed to coincide with those of the underlying. But the certificates are offered at a discount Y, i.e. investors have to pay less than they would have to do for the corresponding stock. In order to make up for that discount, however, the issuer imposes an upward cap F, meaning that investors will only follow the movement of the underlying as long as it does not exceed this cap. Furthermore, discount certificates usually exhibit a finite expiry time.

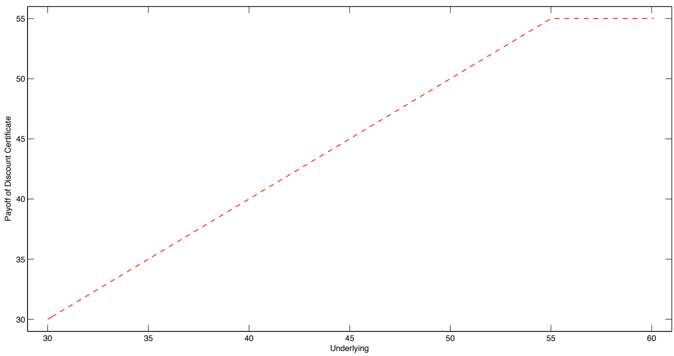

Figure 2.2: **Payoff Profile of a Discount Certificate**
This figure shows the payoff profile of a discount certificate at maturity. The certificate is written on a stock currently valued at $S_0 =$ €50.00, has a discount of $Y =$ €40.00 and is capped at $F =$ €55.00. One can see that the investment allows for a positive return if $S_T \geq Y$.

To make things more plausible we consider the following example. Assume a company currently trades at €50.00. If you buy a discount certificate with a discount $Y =$€40.00 you can obtain the stock at €40.00 rather than €50.00. In turn, there be a cap of e.g. €55.00. Then we consider the following scenarios at expiry T:

1) As long as the stock has not dropped below the discount, i.e. $S_T \geq Y$, the investor makes a profit $S_T - Y$.

2) If in turn the stock price increases to e.g. €60.00, which is beyond the imposed cap, the investor receives the cap rather than the stock price, i.e. €55.00, and has made a profit of $F - Y =$ €15.00. In relative terms this amounts to a return of 37.5%.

Figure 2.2 shows the full profit and loss profile of this exemplary investment.

From the perspective of a financial economist, discount certificates thus face a tradeoff between reduced downside risk and capped or limited upside potential. For valuation purposes it roughly speaking follows that the two levels (discount and cap) should be chosen such that the probabilities of the two events match.

For the month of November 2001 Wilkens, Erner & Röder (2003) compare price quotes of reverse convertibles and discount certificates to replicating strategies revolving around EUREX (European Exchange) traded options.[5] The cost of duplicating the claim is found to be below the issuers' price quotes which indicates mispricing in their favor. Furthermore, these deviations are related to the type of product, the issuer, the underlying, and the order flow. Baule, Rühling & Scholz (2004) also analyze the price setting behavior by issuers of discount certificates. They find that relative overpricing (compared to theoretical values) is reduced for the DAX relative to single stocks. They argue that this can be justified by the fact that EUREX traded DAX options allow for less expensive replica-

[5] Reverse convertibles are similar in style to discount certificates but focus on bond markets. Typically, there is a fixed face value on which the issuer pays interest. To make the investment attractive, this interest payment usually exceeds market rates. However, this is attained at the expense of the issuer having a special redemption right at face value and the right to settle the investment by the delivery of pre-specified stocks at maturity rather than the notional.

tion than in the single stock case. Baule, Entrop & Wilkens (2005) cover discount certificates and analyze their pricing in the context of both market and credit risk when they simultaneously appear. Empirical results suggest that the material part of pricing margins is constituted by credit risk, while the correlation between market and credit risk appears to be neglected in market prices for the most part.[6] The results are confirmed by Baule, Entrop & Wilkens (2008). In Baule (2011) the order flow of discount certificates is examined. The prevailing pattern appears to be that for tax reasons investors prefer products maturing in more than one year and that they prefer to sell back their products close to maturity.[7] It is found that a majority of issuers anticipates this order flow and prices in additional surcharges at the respective times of higher trading activity.

2.2.2 Bonus Certificates

Another very popular type of certificate with retail investors is the so-called bonus certificate. The bonus is a bonus payment Λ the investor at least receives if the certificate's underlying does not undershoot a certain and also predetermined barrier level L during its lifetime T. Regarding the payoff, we can thus distinguish between two cases:

[6] This means that because of the default risk by the issuer investors should earn a credit risk premium, which traded certificate prices due not reflect. Consequently the issuers can pick up capital below their cost of debt. To highlight this effect in terms of discount certificates, the authors compare model prices that reflect credit risk (Hull & White (1995) and Klein (1996)) to default-free model values (Black & Scholes (1973)) and test the hypothesis that market prices do not accurately reflect credit risk. They find both, a credit risk margin and a neglection of credit risk in market quotes.

[7] The reason is that investors did not have to pay taxes on the capital gains in Germany until 2009 if a product was held for more than one year or bought before 2009. Although this was changed beginning in 2009, it still applies to Baule's data set which covers the period November 2006 through December 2007.

1) If L is undershot sometime during the bonus certificate's lifetime, i.e. $S_t \leq L$ for some $t \leq T$, the value of the underlying is returned at maturity.

2) If, in turn, the barrier is not breached until maturity, the investor receives the maximum of the bonus level and the underlying price.

Consequently, the payoff of a bonus certificate is path-dependent and is given by the following functional form

$$g(S,T) = \begin{cases} \max(S_T, \Lambda) & \text{if } S_t > L \; \forall \, t \leq T \\ S_T & \text{else.} \end{cases}$$

For the ease of exposition we present an example. Let us assume that a stock trades at $S_0 =$€ 46.00 and the certificate is sold for the same price. Further assume, the barrier is fixed at $L =$ € 37.00 and the bonus level at $\Lambda =$€ 56.00. Finally, we make the assumption that the remaining time to expiry is one year. Unless the stock trades below € 37.00, which amounts to a drop of 19.6%, the investor will receive at least € 56.00. If the stock trades above € 56.00, say for example € 60.00, at expiry, the payoff will be the stock price of € 60.00.

Graphically, this has been shown in Figure 2.3 for both the case where the barrier is undershot and the scenario of the bonus level still being alive at maturity.

For advertising purposes the issuers compare the certificate to a direct investment in the underlying. The advantage of the certificate is the payment of the bonus level in the case of a favorable market development, but it comes at the price of forgoing dividend payments, a holder of the underlying would be entitled to. From the perspective of financial economics bonus certificates thus boil down to a tradeoff between the bonus level and the barrier. For the issuer

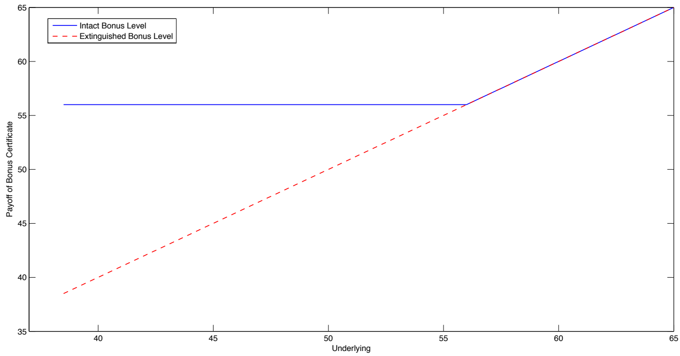

Figure 2.3: **Payoff Profile of a Bonus Certificate**
This figure shows the payoff of a bonus certificate with bonus level Λ = €56.00 and barrier L =€37.00. In particular, the payoff is depicted for both possible cases, an intact and an extinguished bonus level.

such a product is favorable if the expected bonus payment (given the probability of breaching the barrier) exceeds the expected income through dividends.

2.2.3 Sprint Certificates

Although discount and bonus certificates are clearly and without any doubt structured products, sprint certificates are more in-line with the general notion that structured products have option like features and are thus subject to much more complex market risks than direct investments in stocks. Sprint certificates in their own right are a step closer to options. This becomes manifest by the introduction of a strike K and a participation rate k.

They enter the structured payoff in a very interesting way. Up until the strike the payoff coincides with that of the underlying stock.

Above the strike issuers establish a region in which the participation in the underlying exceeds that of underlying itself by a factor k. A typical configuration of such certificate might be a participation rate of two. In such a case a €1.00 increase in the stock price translates to a €2.00 increase in the value of the payoff. However, such a seemingly beneficial participation rate comes at a premium, namely the presence of a cap C beyond which there is no further participation in the underlying. More precisely, the payoff looks as follows

$$g(S,T) = \begin{cases} S_T, & S_T \leq K \\ K + k\,(S_T - K), & K \leq S_T \leq C \\ g_{\max} & S_T \geq C, \end{cases}$$

where the maximal payoff $g_{\max}$ is given by

$$g_{\max} = K + k\,(C - K).$$

From the perspective of a financial economist, this product payoff is closely related to that of a standard European call option. Contrary to discount and bonus certificates the payoff does not depend on the path of the underlying, as it is the case for standard European call and put options. This in turn, renders valuation straightforward, as it essentially boils down to applying standard option pricing techniques to just another payoff function. This can be compared to valuing a put option instead of a call option. In Figure 2.4 we have plotted the payoff profile for sprint certificates to further highlight their behavior.

From the above characterization it is intuitive which type of investor these products suit most. As pointed out, they are essentially options and thus they are subject to much more pronounced market risk than stock market investments, for example volatility or interest

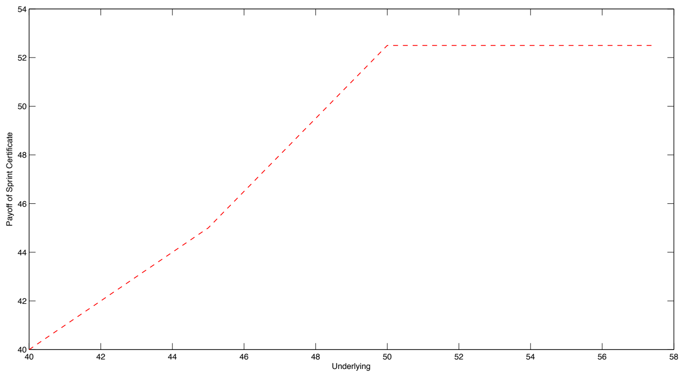

Figure 2.4: **Payoff Profile of a Sprint Certificate**
The plot shows the payoff profile of a sprint certificate with strike $K = 45$, a cap $C = 50$ and a participation rate of $k = 1.5$ in between the strike and the cap.

rate risk, or a declining time value of the product. Naturally, sprint certificates are suitable for those investors who would also invest in standard options. Furthermore, for their particular payoff profile compared to plain vanilla options to materialize, the investor should have deep insight and a thorough understanding about how she expects the market will move in the short term. If she does not do so, she might forgo her leverage and participation rate and run into the cap, in which case a plain vanilla call option would be the superior investment, or, in the case of a market decline, pay an option premium to just receive the stock which could have been obtained less expensively via a direct investment. Evidently, this product is tailored to the needs of highly informed and sophisticated investors.

2.2.4 Further Structured Products

Apart from the structured products discussed above, there are many more such products offered in the marketplace. We loosely classify them in the following way, based on the respective approaches pursued in the literature dealing with them:

1) Investment vehicles observed in the US American market

2) Derivatives dublicated using EUREX traded products

3) Products not further classified.

In addition, we also point out that there is an increasing amount of literature from the field of behavioral finance pertaining to certificates or structured products for retail clients in general.

2.2.4.1 US American Market

Chen & Kensinger (1990) deal with MICDs (Market Index Certificates of Deposit) in the US American market and derive pricing formulas and equilibrium relationships between the call and put variants.[8] Using the option component and inconsistencies in implied volatilities issuers' prices are examined with even more unfavorable situations prevailing in the put cases. Chen & Sears (1990) are concerned with the pricing of the SPIN (Salomon Brothers' Standard and Poor's 500 Indexed Note), which combines a bond with a call option.[9] Pricing is carried out using the Black & Scholes (1973)

[8] These products appeared in 1987 and offered variable interest rates to their holders. The interest rates were determined based on the performance of the stock market index S&P 500. In this way it was also possible to provide investors with long and short variants, i.e. products for which interest rates rise in increasing or falling markets respectively.

[9] According to the authors it is similar in style to the MICD but intended for longer periods and exchange-traded.

model. The valuation reveals small but discernible pricing biases in favor of the issuers which can be interpreted as structuring costs. Furthermore, it is pointed out that the issuer's risk of potentially high option payoffs at maturity can be hedged away in the futures market. The paper by Benet, Giannetti & Pissaris (2006) covers reverse exchangeable securities, structured products issued primarily in the US American market. Fair prices are obtained by replicating the claims in bonds, stocks and derivatives with established prices. Compared to these fair prices structured products exhibit mispricing in favor of the issuers. Furthermore, there appears to be a positive correlation between terminal payoffs and the issuer's financial performance, which can be used as an explanation for the credit enhancement and mispricing.

2.2.4.2 Replication via EUREX

In Burth, Kraus & Wohlwend (2001) the authors investigate securitized covered call writing in the Swiss market for structured products. They consider replicating trading strategies in the underlying markets and at EUREX and detect significant mispricing in favor of the issuer. This is attributed to the structuring costs incurred by the latter. Stoimenov & Wilkens (2005) are concerned with the pricing of equity-linked structured products. Using EUREX option prices, daily quotes can be empirically compared to these. Doing so it is revealed that conceivable premia are inherent in the price quotes in the primary market and the life cycle is found to be a significant value driver in secondary markets.[10] Grünbichler & Wohlwend (2005) deal with structured products in the Swiss market without

[10] This is in line with the life cycle hypothesis, which states that overpricing of structured products is reduced during their lifetime to earn additional profits. In this way, investors pruchase these products at a higher degree of overpricing compared to when they return the products. The difference between the levels of overpricing constitutes further profit for the issuing institution.

a capital guarantee and with option like features. Using comparable EUREX traded options the authors find mispricing and market inefficiencies which appear to be rationally exploited by the issuing institutions.

2.2.4.3 Further Literature on Structured Products

The paper by Fischer & Schuster (2002) covers index bonds with a capital guarantee which have developed in the German market. As potential interest payments of the bond are linked to the development of the stock market via various conditions the authors examine the question whether pricing by the issuers is fair. Furthermore, the potential return for risk-averse investors is dealt with. Brown & Davis (2004) investigate the Australian endowment warrants. They are long-term equity options with dividend protection features and a stochastic strike price to attain that. On the other hand significant overpricing is found and ways to mitigate that are also discussed. In the paper by Branger & Breuer (2007) portfolio selection theory is applied to portfolios of private investors. As direct investments in derivatives might be too complicated for private investors, retail certificates appear to be an attractive circumvention. In a stochastic volatility jump diffusion model applied to the German stock market index DAX it is found that for the CRRA (Constant Relative Risk Aversion) investor the annualized risk-free excess return is 14 basis points at best with discount certificates performing best and other certificates often not being held at all.[11] Baule & Blonski (2011) analyze a large sample of bank-issued warrants traded at EUWAX (European Warrant Exchange), which among others include the above mentioned discount certificates, bonus certificates, or reverse con-

[11] A CRRA investor is an investor whose utility function exhibits constant relative risk aversion. Other than this type of investor, CARA (Constant Absolute Risk Aversion) is another common assumption regarding an investor's risk aversion.

vertibles, for which margins on top of their theoretically fair prices have been reported. These margins are further investigated by the authors and found to be dependent on the order volume. Lower margins prevail for higher volume trades which is attributed to the fact that higher volume implies higher reward for conducting price comparisons.

2.2.4.4 Behavioral Finance Approach

Shefrin & Statman (1993) are concerned with the impact of behavioral finance aspects on the design of structured products. Their paper revolves around an examination of covered calls, both explicit and implicit. The study is carried out with regard to prospect theory[12], hedonic framing[13], behavioral life cycle theory[14] and cognitive errors. In Breuer & Perst (2007), the authors apply prospect theory to the pricing of discount reverse convertibles and reverse convertible bonds.[15] Standard expected utility theory yields that both types of products are of interest to investors who underestimate stock return volatility. Furthermore, such a model overestimates the demand for

[12] Prospect theory dates back to the seminal paper by Kahneman & Tversky (1979). This theory aims to explain decision making based on utility theory. But contrary to classic maximization of expected utility it includes psychological aspects. The latter leads to heuristic elements rather than rigorous maximization in the evaluation of potential gains and losses and their severity.

[13] Framing pertains to the shape in which opportunities arise, e.g. expected cashflows can be the same but exhibit different risk profiles which leads to different preferences. The term hedonic means pleasure oriented and the discipline of hedonic framing is analyzed by Thaler (1985).

[14] Shefrin & Thaler (1988) extend the life-cycle theory of saving for behavioral effects which are introduced through so-called mental rather than financial accounts: current income, current assets, and future income. Those alterations in the theory are intended to describe how agents actually behave rather than how they should rationally behave.

[15] Discount reverse convertibles and reverse convertible bonds are combinations of bonds (zero bonds and coupon carrying bonds) and short positions in put options on stocks.

the former products while underestimating the demand for the latter ones, which in turn can be explained by hedonic framing.

2.3 Turbo Certificates

Turbo certificates are commercially very successful products offered by banks on the German market. They allow for highly leveraged positions in virtually every underlying, including stocks, indexes, currencies, and commodities. This feature is strongly advertised by issuers. For this reason they are sometimes called "Leverage Certificates" by issuers.

Different product generations of Turbo certificates can be observed in the market. The first generation of Turbo certificates were European-style options with finite maturity. These certificates could be interpreted as simple down-and-out call (or up-and-out put) options with the knock-out barrier equaling the strike. This generation has been subject to extensive research including Fischer, Greistorfer & Sommersguter-Reichmann (2002), Fischer, Greistorfer & Sommersguter-Reichmann (2003) and Scholz, Baule & Wilkens (2005) who apply option pricing theory to Turbo certificates. Baule, Scholz & Wilkens (2004) and Mahayni & Suchanecki (2006) deal with semi-static hedging strategies to obtain upper and lower price bounds for underlyings with continuous sample paths. Muck (2007) considers the impact of jump risk on the pricing of Turbo certificates with finite maturity. An early overview of the different generations of discount certificates is provided by Entrop, Scholz & Wilkens (2005).

2.3.1 Finite-Lived Turbo Certificates

The first generation of Turbo certificates presents a highly leveraged investment opportunity in virtually every thinkable underlying asset class. In a certain sense they can be thought of as the next step of sprint certificates towards a completion of option features, as Turbo certificates are nothing else than a generalization of classical barrier options (cf. Section 3.3). This becomes manifest when considering the payoff of such a product

$$g(S,T) = \begin{cases} k \max(S_T - K, 0), \; S_t > L_t \; \forall t \leq T \\ R_t, \; \text{else.} \end{cases}$$

Contrary to standard barrier options Turbo certificates also have a participation rate k which can further increase or decrease the payoff. If the barrier L_t is undershot for some $t \leq T$ the certificate can become worthless. Furthermore, there is a rebate R_t which is paid in the case of a knock-out. If this factor equals zero, the Turbo certificate becomes worthless upon being knocked out. Standard barrier options are retained by setting $R_t \equiv 0$ and $k = 1$.

Of course, there are numerous variations of these certificates in practice which mostly revolve around the arrangement of the different parameters. According to Mahayni & Suchanecki (2006) two dimensions of classification prove reasonable:

1) Regarding the position of the barrier relative to the strike: The barrier can be placed below, above or equal to the strike.

2) Regarding the rebate R_t which can be zero or positive.[16]

[16] If there was a negative rebate, the investor would have to make a payment to the issuer in the case of a knock-out. For obvious reasons such products would be hard to sell to retail customers.

Wilkens, Entrop & Scholz (2001) deal with the valuation and value drivers of certificates issued on foreign currency stock indexes. As it is the case with currency and quanto options as a key such value driver the interest rate differential is identified. The paper by Fischer, Greistorfer & Sommersguter-Reichmann (2002) deals with the first generation of Turbo certificates in the German market. It uses option pricing techniques to investigate to which degree these products actually match the claims made by issuers' marketing brochures. The study rejects, that the value is independent of volatility, that the delta equals one, that the leverage equals the ratio between certificate and underlying, that the leverage is independent of the time to maturity and that the annual surcharge solely depends on the dividend yield and the riskless rate of return. Scholz, Ammann & Baule (2003) intuitively describe the first generation of Turbo certificates. They are identified as barrier options and thus their properties are compared to those of the corresponding standard options. Finally an assessment of the benefits and risks of private investors is provided. The paper by Scholz, Baule & Wilkens (2004) is concerned with short Turbo certificates on stock indexes. In particular, the paper provides economic justification for the existence and properties of this investment product by comparison to similar forward positions and variation of the knock-out barrier. In Scholz, Baule & Wilkens (2005) it is investigated whether the price quotes of certificates coincide with the published pricing formula. It is found, that price quotes are biased in favor of the issuer and that higher financing costs are charged than announced by the issuer.

Mahayni & Suchanecki (2006) explain the prevalence of down-and-out call and up-and-out put Turbo certificates in terms of applicable semi-static hedging strategies using standard options, which constitute upper and lower bounds on the value of the Turbo certificates respectively. Contrary to commonplace pricing and advertising, it is rejected that this can be independent of volatility. In the paper by Muck (2006) Turbo certificates and OTC retail derivatives from ClickOptions are investigated. They are found to be overpriced compared to fair values because of imperfect market competition, while the life cycle hypothesis mentioned in Section 2.2.4 cannot be confirmed. In Muck (2007) knock-out certificates are investigated in the Bakshi, Cao & Chen (1997) model. It is found that jumps are very important as they impose a significant gap risk on the issuers of the certificates. Furthermore, stochastic interest rates and stochastic volatility are found to play only a minor role. In a Black & Scholes (1973) model setup, Wilkens & Stoimenov (2007) empirically investigate the price quotes of finite-lived leverage certificates (barrier options in essence). They find a pronounced surcharge in bid-ask quotes compared to fair model values which at least to a certain degree can be explained as costs incurred by the issuer for market making and hedging. Since hedging is more complex and difficult in the case of short certificates, the effect is stronger for these. Entrop, Schober & Wilkens (2011) examine a data set of the trades of about $7,000$ retail costumers of a German direct bank. Analyzing the order flows for finite-lived retail certificates they confirm that certificates are overpriced using a Black & Scholes (1973) and a jump-diffusion model and confirm the life cycle hypothesis. Further, they find that the pricing policy is consistent with the overnight gap risk issuers face and the order flow as induced by the costumer.

2.3.2 Open-End Turbo Certificates

The second generation of Turbo certificates are OETCs. The main difference of the new generation is the perpetual lifetime. This implies that holders of the certificates may redeem the securities at their discretion unless the certificate is knocked out. For this reason these certificates are called OETCs. Issuers commit themselves to trading these certificates at a pre-specified price function. They are compensated by the matter of fact that strikes and barriers grow according to a pre-specified financing parameter. In order to reduce gap risk, the barrier is usually higher than the strike. These product characteristics quite naturally lead to the question of when investors should exercise their certificates. Alternatively, we may also ask how much the early exercise feature is worth. To the best of our knowledge these questions have not been addressed in the literature so far. A survey of the current literature on OETCs in provided below in Section 2.3.2.2 after a discussion of the product characteristics in Section 2.3.2.1.

2.3.2.1 Product Characteristics

OETCs are offered by a variety of financial institutions. Legally, certificates are bank bonds with a redemption value being fixed by the underlying asset. In particular, they are not subject to any public or private deposit insurance system.[17] Therefore, in the case of bankruptcy holders of certificates are treated similarly to institutional investors. For this reason the collapse of Lehman Brothers, a major issuer of Turbos and other certificates, seriously harmed many retail investors in Germany in 2008.

Table 2.2 shows the number of classic as well as open-end Turbo certificates traded in the German market.[18] Statistics cover the Eurostoxx 50, the DJIA (Dow Jones Industrial Average), and the German DAX index as examples for the underlying.

[17] In Germany there are different layers of deposit protection in the case of bank defaults. European Union (1994), European Union (1997), and European Union (2009) mandate financial institutions to protect at least € 100,000 per client and institution. In excess of this regulatory insurance, there are several protection funds by the respective banking and credit unions like private institutions and savings banks.

[18] The statistical data were retrieved from www.onvista.de for July 16, 2010.

Issuer	DAX			Eurostoxx 50			Dow Jones		
	Long	Short	Total	Long	Short	Total	Long	Short	Total
Barclays Bank	29	15	44	21	18	39	11	0	11
BNP Paribas	247	329	576	56	61	117	0	0	0
Citi Group	348	385	733	133	110	243	49	29	78
Commerzbank	368	288	656	48	33	81	101	40	141
Deutsche Bank	114	97	211	67	60	127	56	38	94
DZ Bank	96	61	157	0	0	0	0	0	0
Erste Group Bank AG	0	8	8	0	0	0	0	0	0
Goldman Sachs	263	160	423	27	9	36	25	20	45
HSBC Trinkaus	322	213	535	0	5	5	0	0	0
Lang & Schwarz	24	13	37	0	0	0	0	0	0
RBS	70	57	127	35	21	56	44	20	64
Societé Générale	0	10	10	0	8	8	0	0	0
Sal. Oppenheim	22	15	37	3	2	5	7	0	7
Vontobel	34	13	47	3	0	0	0	0	0
Total	1,937	1,664	3,601	390	327	717	293	147	440
%	53.79	46.21		54.39	45.61		66.59	33.41	

Table 2.2: **Market Overview of Turbo Certificates**
The table shows the issuers and the numbers of outstanding certificates on major stock market indexes on the German market on July 16, 2010. The sample comprises both OETCs and Turbos with finite maturity. For the number of OETCs alone please refer to the empirical study in Chapter 6 and Table 5.2.

Two types of Turbo certificates can be observed in the market: Long Turbos allow for speculation on price increases while short Turbos profit from decreases of the underlying asset. For simplicity we assume that the underlying stock does not pay any dividends throughout this study. The term *Turbo* refers to the superlinear participation in the movement of the underlying. More precisely the price function g of a long Turbo certificate is given by

$$g(S_t, t) = \max\left(S_t - K_t, 0\right), \qquad (2.3.2.1)$$

where

$$K_t = K_0 \exp\{(r+z)t\} \qquad (2.3.2.2)$$

and S_t is the level of the underlying stock, K_t the strike price, r the constant short-term refinancing rate, and z the so-called financing parameter. Conversely, the price setting formula of a short Turbo certificate reads

$$g(S_t, t) = \max\left(\hat{K}_t - S_t, 0\right), \qquad (2.3.2.3)$$

where

$$\hat{K}_t = K_0 \exp\{(r-z)t\}.$$

The issuing banks communicate these pricing formulae to their customers and promise to trade the certificates at any time according to these formulae. In this sense investors can also "exercise" their certificate by selling it to the issuing bank at the communicated price function at any time. The formulae resemble the payoff functions of standard call and put options. The main difference is that the strike price is not constant. At each instant of time it grows at the rates $r+z$ and $r-z$ respectively given an initial level K_0. Alternatively, the holder may exercise the certificate at any time according to (2.3.2.1) or (2.3.2.3). A positive payoff is only possible when the knock-out event has not taken place before. Define the knock-out barrier as

$$L_t = (1 + a)\, K_t,$$

where $a \in \mathbb{R}$ is a real number and is referred to as the gap size. For the short variant the factor $1+a$ is replaced by $1-a$. The knock-out time for the long variant τ_{long} is the first point of time t for which $S_t \leq L_t$, i.e.

$$\tau_{\text{long}} = \inf\{t \geq 0 \mid S_t \leq L_t\}.$$

Similarly, the knock-out time τ_{short} of a short Turbo certificate satisfies

$$\tau_{\text{short}} = \inf\{t \geq 0 \mid S_t \geq L_t\}.$$

In this case the investor receives a payment according to (2.3.2.1) or (2.3.2.3). The certificates become worthless thereafter. Note that (2.3.2.1) is a simplified specification of the price function. In particular, banks adjust barriers and strikes in short discrete time intervals (daily or monthly). A survey of product characteristics is presented by Entrop, Scholz & Wilkens (2009). Furthermore, the interest rate is referenced on a short term (variable) interest rate (usually one-month EURIBOR (Euro Interbank Offered Rate) or EONIA (Euro Overnight Index Average)).

Turbo certificates are frequently described by stating that they are similar to forward contracts but only allow for positive payoffs. Negative payoffs are prevented by knock-outs. More precisely, the investment could be seen as a long position in the stock which is partially debt financed by an amount equal to the strike K_0. Exercise (or sale to the issuer) of the certificate corresponds to selling the stock at S_t and paying off the debt including interest. Viewed in this way, the financing parameter z could be seen as the credit spread that the investor has to pay. Usually, for the knock-out barrier we have

$L_t \geq K_t$, i.e. in the event of a knock-out the investor still receives a positive payment.

In the case of short OETCs this interpretation does no longer hold. However, we can provide a different interpretation. Short OETCs mimic the short sale of underlyings. In this sense, these products can be viewed as a partial borrowing of the underlying (i.e. the underlying less the strike) at the issuing credit institution. The borrowed underlying is then sold in the market place and the proceeds deposited at the market interest rate. Contrary to the long variant, the investor thus receives the interest rather than paying it. However, she only receives the market interest rate less the financing parameter. The latter can again be interpreted as a fee for borrowing the underlying in the first place. Also, the knock-out barrier is less than the strike. The reason for that is that knock-outs occur through up-crossings of the barrier. In order to uphold a cushion between the barrier and the strike, it is natural that the strike is greater than the knock-out barrier.

Originally, banks offered certificates with $L_t = K_t$. However, as discussed by Muck (2007) this leads to substantial gap risk when the stochastic process of the stock price is discontinuous and therefore it might happen that $S_\tau < K_\tau$. The larger the gap size a the more unlikely this event becomes.

The following proposition addresses optimal exercise strategies and the existence of certificates when the stock price follows a continuous process. It builds on Entrop, Scholz & Wilkens (2009).

Proposition 2.3.1 (Optimal Exercise Strategy) *In a competitive financial market with no transaction costs and short selling constraints it is optimal to exercise OETCs defined by the payoff* *function (2.3.2.1) immediately if* $a > 0$, $z > 0$ *and the stock price process is continuous.*

Proof - Proposition 2.3.1:
The proof follows from no-arbitrage considerations.

Assume that the stock price process is continuous. Then an investor could build two portfolios. In portfolio one, she holds the Turbo certificate. In portfolio two, she buys a stock and borrows an amount of money equal to K_0 at an interest rate $r^* \geq r$. At an arbitrary point of time τ^* (which might be equal to the knock-out event τ) she unwinds both portfolios. From portfolio one she receives a payment equal to

$$\text{Payoff}_1 = S_t - \exp\{(r + z)\tau^*\}K_0.$$

Note that the stock price in the case of a knock-out is $S_{\tau^*} = L_{\tau^*} \geq K_{\tau^*}$. This is ensured by the fact that the process is continuous. The payoff of portfolio two is

$$\text{Payoff}_2 = S_t - \exp\{r^*\tau^*\}K_0.$$

Again by virtue of the fact that the stock price process is continuous, the payoff is nonnegative. This implies that in a competitive financial market $r^* = r$. Hence

$$\text{Payoff}_2 \geq \text{Payoff}_1. \tag{2.3.2.4}$$

Inequality (2.3.2.4) is strict unless $z = 0$.

□

Proposition 2.3.1 implies that we cannot explain any Turbo certificate with financing parameter $z > 0$ on a financial market with continuous stock price paths. In this case investors would not buy these securities.

Therefore, in order to explain these contracts we consider discontinuous stock price processes since replication of the contracts is possible with a simple buy and hold strategy. In the following we assume that the stock price is subject to jump risk. These jump risks model overnight trading halts and event risks. Especially, the latter might have an impact on certificates' prices even when $a > 0$. When there is jump risk the stock price might be less than the strike price in the knock-out event. A long position in the stock and borrowing an amount K_0 at the risk-free rate r does no longer represent a superhedge of the certificate, i.e. a position that always equals or exceeds the value of the certificate. This gap risk must be priced by the issuing institution. For the Turbo certificates with price function (2.3.2.1) this might be done by introducing a positive financing parameter z.

2.3.2.2 Literature Review

To the best of our knowledge, there are only two papers which analyze OETCs. Entrop, Scholz & Wilkens (2009) assume that holders of OETCs have a finite planning horizon T, up until which the certificate is held. In other words, OETCs are treated as if they were European-style options.[19] Furthermore, the authors provide a quasi closed-form solution if the underlying is governed by a Black & Scholes (1973) process

$$\begin{aligned} OETC_0^T &= S_0 - K_0 - K_0(\exp(zT)(1 - Q(\tau \leq T)) \\ &\quad + E^Q\left[1_{\{\tau \leq T\}} \exp(z\tau)\right] - 1). \end{aligned}$$

[19] In their paper, the authors refer to OETCs as OELCs (Open-End Leverage Certificates). In this thesis we prefer the classic name Turbo certificates, but the term leverage certificate can be used interchangeably.

Here, the probability measure Q refers to the risk-neutral measure we introduce in Section 3.2.2.5 in much greater detail. The probability $Q(\tau \leq T)$ and the risk-neutral expectation $E^Q\left[1_{\{\tau\leq T\}}\exp(z\tau)\right]$ are obtained using the methods outlined in Bielecki & Rutkowski (2002). Given this setup, a comparative statics analysis is conducted with regard to relative price deviation and profit potential, which are defined as

$$\begin{aligned} PP_t &= K_0\left(\exp((r+z)t) - \exp(rt)\right) \\ RPD_0 &= \frac{P_0 - OETC_0^T}{P_0}. \end{aligned}$$

The purpose of these two numbers is to measure the profit potential, i.e. by how much a superhedge of the price setting formula (2.3.2.1) exceeds the price at which issuers are willing to take back the certificates, and by how much fair theoretical prices deviate from the price setting formula. The two metrics are investigated for different combinations of the initial strike K_0 and the financing parameter z as well as the holding period T and the Black-Scholes volatility σ. In fact, this analysis is a purely theoretical assessment of the degree to which issuers can exploit arbitrage opportunities at the expense of the retail customer and how these opportunities depend on the parameters investigated. The reason is that according to Proposition 2.3.1 OETCs should never be held in the absence of jumps and that the presence of jumps has been well documented in past research.

In a second step, Entrop, Scholz & Wilkens (2009) include jumps in their analysis using the Boes, Drost & Werker (2007) jump model. This model distinguishes between random and overnight jumps. Using Monte Carlo simulation the authors examine the impact of jumps for three holding periods. The result is that the jump impact diminishes with increasing holding duration. However, this is the well-known average-out effect from the valuation of European options. The economic rationale behind this effect is that jumps pertain to short term shocks of the underlying, while in the long run, the diffusion is the predominant determinant of underlying prices. Consequently, the longer the holding duration, the more time there is for jumps to be offset again by diffusion. In addition, the analysis neglects that there is an early exercise premium inherent in OETCs.

Throughout their paper, Entrop, Scholz & Wilkens (2009) stress that their methodology and results only apply from the perspective of an issuer but not from the point of view of a retail client. The reason is that short-selling is not possible for OETCs and that a replicating strategy could only be set up at excessive additional costs for private investors. Put differently, exploitation of arbitrage opportunities is only possible for issuers. This situation amounts to market segmentation in the sense of Jarrow & van Deventer (1998), and thus the obtained prices can still be regarded as valid.

The second paper dealing with OETCs is the one by Rossetto & van Bommel (2009). They are the first to acknowledge that there is an option component inherent in OETCs and assuming rationality, investors act so as to maximize the option's value. Furthermore, they point out the importance of jump risk and include overnight jumps in their considerations but do not account for jumps explicitly as they assume continuous sample paths during trading hours.

In a historical simulation OETCs are examined further. The historical simulation revolves around a five-year sample of return data for DAX stocks between January 2002 and December 2006, from which returns are randomly drawn and applied to a certificate with knock-out barrier $L = 105$, a strike $K = 100$ and a financing parameter $z = 2.0\%$. They find that OETCs on average traded less than a percent above their intrinsic values in January 2007.

Despite the fact, that an option component is included in the valuation of Rossetto & van Bommel (2009) they do not apply option pricing techniques in a straightforward manner but rather resort to historical simulation of the underlying. Also they acknowledge the importance of jumps but ignore intraday jumps, which for example by Merton (1976) have been found to be a significant determinant of option prices. In fact, they do not assume any market model and do not carry out any option pricing. Instead, they draw historical returns, consider intrinsic values of the option component, and average these for all $100,000$ drawings until they match the initially assumed payoff by starting over with a new initial payoff in the case of a mismatch. In their empirical analysis, they finally do not distinguish between the return data for single stocks and the overall market index DAX. Rather all historical returns are taken from the same sample.[20] From our point of view, this renders the obtained OETCs prices debatable.

These results encourage the valuation of OETCs using option pricing techniques and considering jump risk in a consistent way.

[20]The authors argue that the single stock return data exhibited certain biases due to single extreme returns which rendered valuation results implausible.

Chapter 3

Option Pricing Theory

This chapter extensively deals with option pricing theory. It is structured in the following way: Section 3.1 gives an overview of different option pricing models w.r.t. modeling the underlying and deriving European option prices. Section 3.2 details how the early exercise feature inherent in American-style options can be dealt with. Finally, barrier options are discussed in Section 3.3 because along with most other retail certificates OETCs also exhibit a barrier, and Section 3.4 briefly summarizes the assumption of market efficiency usually made in capital market models.

3.1 Option Pricing Models

In this section the from our point of view most important classes option pricing models are discussed. The classification is made w.r.t. the sources of uncertainty and how these sources of uncertainty are modeled.

3.1.1 Black-Scholes Model

The renowned Black-Scholes model as posited in the their seminal paper Black & Scholes (1973) aims at deriving an arbitrage-free and thus fair price of a European call option. A general theory of the fair option premium for rational investors is elaborated on, developed, and formalized in great detail in Merton (1973*a*). He proves that:

1) In the absence of dividends or other disbursements to common stockholders American call options are never exercised prematurely. In consequence, they always have the same value as their European counterparts.

2) Under the same assumptions a perpetual option is worth just as much as the underlying stock.

3) If rationally determined, the option price is a convex function of the exercise price.[21]

4) If rationally determined, the option price is non-decreasing in the riskiness of the associated common stock.

One important property to prove these assertions is the notion of dominance. A security A is said to dominate security B, if in all possible states of the world, A is worth at least as much as B and at a known time in the future the return of A exceeds the return of B. Merton assumes that for rationality options must neither be dominant nor dominated securities, as such would constitute arbitrage opportunities.[22]

[21] Similar results are obtained by Jagannathan (1984), Bergman, Grundy & Wiener (1996) and Hobson (1998).

[22] In such a situation one might long A and short B and would with certainty lock in the excess return of A at the known time in the future.

Given these necessary conditions or implications for rational option pricing, Black & Scholes (1973) have to make rather restrictive additional assumptions to be able to derive their option pricing formula:

1) The underlying stock S does not pay any dividends and is governed by a geometrical Brownian motion. So the stock returns are lognormally distributed.

2) There are no transaction costs.

3) The short-term riskless interest rate r is constant and anybody can borrow or lend arbitrary amounts of money at this rate.

4) There are no restrictions to short-selling stock.

In more detail, if the stock's volatility is denoted by σ and W_t denotes a standard Brownian motion, the first assumption means that the stock price follows the stochastic process given by

$$dS_t = r\,S dt + \sigma\,S\,dW_t, \qquad t > 0. \tag{3.1.1.1}$$

The central idea put forth by Black & Scholes (1973) in order to determine the unknown price of the option is to construct a replicating portfolio Π which consists of stocks and riskless bonds.[23] For these securities the values are already known and thus they imply the value of the option. Formally, the replicating portfolio reads as follows

$$\Pi_t = a(t)S_t + b(t)B_t,$$

[23] A replicating portfolio is a portfolio which at any time has the same value as the original one although it has different constituents. To rule out arbitrage opportunities both portfolios must share an indentical value at any time. If that was not the case one could short the more expensive portfolio, long the less expensive one and upon unwinding the positions risklessly lock in the difference as a profit.

where $B_t = B_0 \exp(r\,t)$ denotes the riskless asset.[24] As the portfolio Π is assumed to be replicating the option, its changes must be identical to changes in the option price for all $t \geq 0$. If expressed in that way, however, the value of an option becomes a function of the underlying stock price and given the above assumptions on the stock price dynamics the option value C is, mathematically speaking, a function of a stochastic process, i.e. $C = C(S,t)$. Therefore, when determining the change in the option value one has to resort to Itô's Lemma, a renowned result from stochastic calculus. The lemma can be expressed as follows:[25]

Lemma 3.1.1 (Itô's Lemma) *Let $\{M_t := (M_t^1, \dots, M_t^d), \mathcal{F}_t; 0 \leq t < \infty\}$ be a vector of local martingales in $M^{c,loc}$ and $\{B_t := (B_t^1, \dots, B_t^d), \mathcal{F}_t; 0 \leq t < \infty\}$ be a vector of adapted processes of bounded variation with $B_0 = 0$ and set*

$$X_t = X_0 + M_t + B_t,$$

where X_0 is an $\mathcal{F}_0$-measurable random vector in $\mathbb{R}^d$. Let $f(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be of class $C^{1,2}$. Then $P - a.s.$ for $0 \leq t < \infty$ the following holds

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \partial_t f(s, X_s)\, ds + \sum_{i=1}^d \int_0^t \partial_i f(s, X_s)\, dB_s^i \\ &\quad + \sum_{i=1}^d \int_0^t \partial_i f(s, X_s)\, dM_s^i \end{aligned}$$

[24] The riskless asset is also referred to as a money market account or a riskless bond, thus the denotation B_t. Typically government bonds issued e.g. by the United States and Germany are considered default-free as their default risk is viewed negligible.

[25] In this basic form the lemma was first proved in Itô (1951*a*) and Itô (1951*b*). Since that time it has developed into an integral part of the field of Itô stochastic calculus. Far more information can, for example, be found in the introductory textbook by Karatzas & Shreve (2008).

$$+\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_0^t \partial_i\partial_j f(s,X_s)\,d<M^i,M^j>_s\;.$$

Proof - Lemma 3.1.1:
See, for example, Itô (1951a) or Karatzas & Shreve (2008).

□

Applying Lemma (3.1.1) the change in the option value is given by

$$dC(t) = (C_t + rSC_S + \frac{1}{2}S^2\sigma^2 C_{SS})\,dt + \sigma S_t C_S\,dW_t.$$

By equating these expressions for the changes and exploiting the fact that the stochastic components must be identical, as both portfolios have the same risk structure, one obtains the fundamental partial differential equation, also called Black-Scholes equation[26], for the option value[27]

$$0 = C_t + rSC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} - rC. \tag{3.1.1.2}$$

In this setting the price C of a standard European call option of maturity T and striking price K, i.e. the solution the above equation, is given by

$$C(S,t) = N(d_1)\,S + N(d_2)\,K\;\exp(-r(T-t)), \tag{3.1.1.3}$$

[26] As pointed out by Wilmott, Howison & Dewynne (1995) this equation is equivalent to the heat equation which is extremely well known from the field of physics. A direct solution can, for example, be found in Evans (2010).

[27] Heed that this PDE holds independently of which specific derivative security is considered as long as it can be replicated using bonds and underlying stocks. In particular, the same equation applies to put options. For different securities pricing only differs in the imposed initial and boundary conditions, i.e. the payoff function in the case of an option. This is in line with the theory of parabolic partial differential equations, in which uniqueness results are always obtained for initial boundary value problems. Details are, for instance, provided by Evans (2010).

where

$$\begin{aligned} d_1 &= \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})\,(T-t)}{\sigma\,\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\,\sqrt{T-t}. \end{aligned}$$

This result is the famous Black-Scholes formula, where $N(\cdot)$ denotes the cumulative normal distribution (i.e. $N(x)$ is the probability that a normally distributed random variable X takes on a value less or equal to x) and d_1 and d_2 are constants that only depend on market parameters and characteristics of the option. Except for the volatility parameter σ all input variables are easiliy observable in the market, thus allowing very straighforward pricing of a European call option.

Of course, the introduction of this new method of pricing options at the time instantly led to a vast amount of follow-up papers in which all properties of the new model are attended to in great detail. Black (1975) is the first to take on this subject. In his article he one by one examines the influencing parameters of the formula, i.e. the interest rate, the time to maturity, the volatility, the current stock price and the striking price. As the volatility cannot be directly observed in the market Black suggests to use historical volatilities as a proxy. Furthermore, the paper deals with possible areas of application of the model. In particular, hedging and speculative strategies involving several options are broached.[28] In addition, he discusses the effects market frictions such as taxes, transaction costs or marginal requirements in the trading accounts would have. He points out that

[28] Black illudes to the fact that options might be spread against one another. This means combining options on the same underlying with different striking prices (e.g. butterfly spread) or maturities (calendar spread) in order to synthesize desired payoff profiles which are most in line with one's beliefs of how the market will develop. A detailed discussion of the different trading strategies in provided by Hull (2009).

high taxes diminish option values and might render it better to exercise an option as opposed to a no-tax world. Similarly, combined with taxation considerations margin requirements can determine if directly shorting a stock or writing an option is more favorable. Apparently, transaction costs can make positions in options unprofitable if they have to be repeatedly rebalanced.

Cox & Ross (1976*a*) provide a survey of option pricing models in perfect markets and a review of the Black-Scholes model. In particular, they highlight qualitative properties such as the put-call-parity, which was previously derived and empirically tested by Stoll (1969). The put-call-parity is a very fundamental and model independent relationship between put and call option prices, i.e. it holds irrespective of a given model used to price options. This relationship expresses the value of a call option in terms of the corresponding put option with identical striking price and time to maturity. It reads

$$C_t + K \exp(-r(T-t)) = P_t + S_t \qquad \forall t \leq T. \qquad (3.1.1.4)$$

Researchers detected certain biases in the Black-Scholes model. By these biases they mean over- or underpricing of market prices by the model. A detailed summary of these deviations is provided by Geske & Roll (1984), who categorize the biases as follows:

1) Near-maturity bias: Shortly expiring options are found to be underpriced.

2) Exercise-price bias: In certain periods of time in-the-money options are underpriced, while out-of-the-money options are overpriced. However, these biases have been found to reverse themselves during other periods, so that there is no clear but a rather conflicting view. Among others, MacBeth & Merville (1979) and Emanuel & MacBeth (1982) have documented these biases.

3) Variance bias: Options on high variance stocks are overpriced while those on low variance stocks are underpriced. For example, the bias has been reported by Geske, Roll & Shastri (1983).

Most of these biases are particularly pronounced in combination with the application of the model to American-style options and their early exercise features.[29] Geske & Roll (1984) are able to explain the near-maturity and exercise price bias with the early exercise of American options associated to dividend payments. The variance bias can be explained by time-changing volatility and the effects of dividends alike. According to Geske & Roll (1984) uncertain and suspendible dividends can render the dependence of the option value on volatility negative, i.e. higher volatility decreases the option value.

In Merton (1973*b*) the results on put-call parity are extended to American-style options. Merton points out that for the put-call-parity to hold there must not be any rational premature exercise. Therefore, the result only applies to American-style call options, whereas for American-style put options the relationship fails.

Rendleman & Bartter (1979) introduce a two-state option-pricing model. Moreover, they are able to re-derive the Black-Scholes model as the limit of their model which they apply to options on dividend paying stocks as well as American type claims. El Karoui, Jeanblanc-Picqué & Shreve (1998) take on the question of robustness of the Black-Scholes model w.r.t. misspecification of the volatility. Since the volatility is the only parameter not readily observable in the market the question to which extent the option price changes when changes to volatility are applied immediately arises. In their article it is shown that dominance in volatility leads to dominance in

[29] Although the Black & Scholes (1973) model aims at valuing European-style options there are alterations for their American-style counterparts. Details about these methods are covered below in Section 3.2.

option prices and vice versa if the contingent claim being priced has a convex payoff.[30] For standard call and put options this condition obviously holds and the result is intuitively appealing. With increasing volatility the probability of higher payoffs rises whereas in opposite direction the downside risk is capped at zero. Thus, this increase in profit potential immediately translates to higher option premiums.

Rubinstein (1983) then proposes the displaced diffusion model as an alternative to Black & Scholes (1973). In addition to the Black-Scholes case, this model takes the capital structure of the firm into account by distinguishing between risky and riskless assets. Also, dividends are incorporated. This model is then further analyzed by Svoboda-Greenwood (2009). In the paper it is shown that the displaced diffusion model can be interpreted as an approximation to the CEV (Constant Elasticity of Variance) model.[31] This is attained by a Taylor series expansion of the functional dependence of the volatility on the asset $\sigma = \sigma(t)\, f(y)\, dW_t$. Having established option pricing formulas in this approximative setting higher order moments and conditional density functions are investigated.[32]

In addition to the literature discussing properties of the Black-Scholes model, there is another strand of literature focusing on carrying over the theory from stock options to similar options on commodities, foreign exchange or other options (so-called compound options). Black (1976) deals with futures options on commodities.

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[30] Loosely speaking, a convex function is a function with positive second derivative. A general mathematical definition can be found in Barner & Flohr (1996), Barner & Flohr (2000) or Forster (2008).

[31] The CEV model is a generalization of the Black-Scholes model where the geometric Brownian motion is replaced by the diffusion $dS_t = r\,Sdt + \sigma\, S^{\alpha}\, dW_t$ where $\alpha > 0$. According to Hull (2009) it is particularly suitable to value exotic options.

[32] According to Bauer (2001) the k-th moment μ_k of a random variable X is given is given by $E[(X - E[X])^k]$. In particular, the second moment coincides with the variance. A more detailed discussion of the economic implications of the third moment (the so-called skewness) is given in Section 3.1.3.

The key difference between stocks and commodities is the absence of a spot price, because commodities are only traded for delivery at a certain future point in time. Furthermore, commodities exhibit peculiarities such as seasonality which cannot be observed for stock prices. These seasonal effects arise out of supply and demand considerations. Obviously, the supply of agricultural goods heavily depends on the time of year with the goods being relatively scarce shortly before harvest and the opposite being true shortly thereafter.[33] Garman & Kohlhagen (1983) and Grabbe (1983) deal with options on foreign exchange (FX options). Contrary to the case of stock prices there are two riskless rates of interest to be considered, one for each currency. In these two papers appropriate adjustments to the valuation process are derived with Grabbe (1983), in addition, showing that American puts and calls are indeed more valuable than their European-style counterparts, a result well established for stock options.[34] Geske (1979*b*) introduces a concept or generalization to value compound options in the Black-Scholes model by adding the face value of debt as another parameter. Widdicks, Duck, Andricopoulos & Newton (2005) present another extension of the Black-Scholes approach to American and down-and-out barrier options. To price these options they apply perturbation theory (expansion around volatility) to the Black-Scholes equation.[35] Another application of the Black-Scholes model is made by Thomson

[33] A very detailed analysis of the additional factors (such as convenience yields) to be considered when pricing claims on commodities is given by Gibson & Schwartz (1990) and Schwartz (1997).

[34] The result follows immediately from the consideration that in an American option all the rights of the otherwise identical European option are incorporated. Thus, it must be at least as valuable.

[35] Perturbation theory is a mathematical method to obtain approximate solutions to problems not exactly solvable. It revolves around perturbing the exact solution to a related problem by a small parameter ϵ and expanding it in a power series. More details about this technique and various applications can, for instance, be found in Landau & Lifschitz (1981), Kato (1995), Smith (1985) or Fernández (2001).

(1992) who deals with forest rotation and models stumpage prices as a log-normal process.

The third very important strand of literature focuses on the aforementioned rather strict assumptions of the Black-Scholes model. In particular, this research relaxes the assumptions of no dividend payments, the absence of transaction costs, portfolio constraints in building replicating portfolios and the so-called vulnerability of options, i.e. the fact that options are subject to default risk, since the writer of an option might be incapable of paying off her counterparty at maturity. Schwartz (1977) proposes to use finite differences as a numerical solution technique if there is no closed form solution available to the option pricing problem as might be the case in the presence of discrete dividend payments.[36] Geske, Roll & Shastri (1983) then address the effect of dividend protection on option prices. They highlight that adjusting the strike on the ex-dividend day accordingly virtually rules out premature exercise of American options. Moreover, they demonstrate that market prices markedly differ from Black-Scholes prices.[37] Davis & Clark (1994) show that the classic Black-Scholes valuation fails in the presence of transaction costs. Under this assumption exact replications of the option payoffs are no longer possible, an issue circumvented by the introduction of super-replicating portfolios (i.e. portfolios which are at least as valuable as the payoff to be replicated) which involves the use of stochastic control theory. Soner, Shreve & Cvitanić (1995) prove that in the presence of transaction costs the least expensive

[36] If one is considering a stock index rather than individual stocks one can, instead of discrete dividend payments, assume the dividend to be continuously paid at a rate q. In such a scenario a closed-form solution for European options can be obtained. The key insight is to adjust the drift rate in the stock price process for the dividend payment qS. More details on the derivation and the pricing formula are presented in Hull (2009).

[37] The authors find an underpricing by the Black & Scholes (1973) model for near-maturity options as well as call options on low variance stocks, whereas calls on high variance stocks appear to be overpriced.

superhedge is the simplest one (cf. Smith (1976)), i.e. hedging a call option with one underlying share. Broadie, Cvitanić & Soner (1998) deal with portfolio constraints in the Black-Scholes model. The intuition of their approach is to construct a super-replicating option (a so-called majorant) that, with constraints, equals the original option. The economic interpretation is that this approach directly links the option price to the additional costs imposed by the constraints. Barles & Soner (1998) investigate European call option pricing in the presence of transaction costs. Since no nontrivial dominating portfolios are available in such a scenario, the authors rely on preferences introduced through utility functions condensed to one factor in the model. In terms of this single remaining factor the quality of hedging is assessed. Discrete dividends are also dealt with by Chance, Kumar & Rich (2002). They show that the Black & Scholes (1973) model is retained if the present value of the expected dividends is known. Korn & Rogers (2005) posit a discrete dividend model, which accomodates the usually assumed continuously paid dividends as the limit case.

Furthermore, many attempts have been made to incorporate default risk in the Black-Scholes model. Johnson & Stulz (1987) analyze American options and show that premature exercise might be optimal when there is default risk. A comparative statics analysis points out that the behavior of vulnerable options is markedly different from default-free options. Hull & White (1995) add a second stochastic variable to the underlying stock price. To quantify default risk they stochastically model the value of the assets of the option writer. This model is then furthermore applied to American-style options. A similar approach is taken by Rich (1996) who assumes the option writer to default when her creditworthiness violates a certain level. In particular, the author investigates different recovery scenarios and finds margin requirements to be set in excess of fair market values.

3.1.2 Jump-Diffusions

A major generalization of the Black-Scholes option pricing model was introduced by Merton (1976). This model focuses on the finding well-established by common sense and casual empiricism, that stock price movements might not be continuous. Ever observed sharp declines in stock prices based on some particular event make this seem a reasonable assumption.[38] Discontinuity in the sample paths of model stock prices is created through jumps added to the Black-Scholes process. These jumps well reflect the economic intuition that discontinuity must be imposed by sudden and sharp upward or downward movements. From a more formal point of view Merton adds a homogeneous Poisson process dN_t such that the Black-Scholes process is altered to[39]

$$dS_t = (r - \lambda\, k)\, Sdt + \sigma\, S\, dW_t \; + \xi\, SdN_t, \qquad t > 0 \qquad (3.1.2.1)$$
$$\ln(\xi) \sim N(\mu_N, \sigma_N^2).$$

For a vanishing jump intensity λ, i.e., if the probability of jumps is zero, the Black-Scholes process is retained. Furthermore, there is a normally distributed random variable ξ which is interpreted

[38] Among others Mandelbrot (1963), Press (1967) and Clark (1973) attempt to model and explain this behavior. They use a stable Paretian distribution, a superposition of normal and Poisson random variables and a subordinate stochastic process respectively to model stock price dynamics.

[39] A homogeneous Poisson process is a stochastic process, i.e. a collection of random variables $\{N(t),\, t \geq 0\}$, whose increments are Poisson-distributed. More precisely, this means that the probability of an increment of the length k is given by $P[(N(t+\tau) - N(t)) = k] = \frac{\exp(-\lambda\tau)(\lambda\tau)^k}{k!}$, $k = 0, 1, 2, \ldots$ λ is the intensity parameter which loosely speaking governs how often an event occurs. For completeness, a non-homogeneous Poisson process would be one with a time-varying intensity $\lambda = \lambda(t)$. To the best of our knowledge, though, such a process has not prevailed in the finance literature. Detailed discussions about the properties of Poisson processes and the Poisson distribution can be found in various introductory textbooks on probability theory and statistics, for example Bauer (2001) and Georgii (2001).

as the logarithmic jump size. From an economic perspective these stock price dynamics can be viewed as a differentiation between normal variations in stock prices and abnormal variations. According to Merton (1976) such fluctuations can be caused by temporary imbalances between supply and demand, the arrival of information with marginal effect on the stock price, changes in the capitalization structure or the overall economic outlook.[40] To the contrary abnormal movements modelled by the so-called jump component can be attributed to the arrival of new information with more than only marginal impact. It is characteristic for such information to appear suddenly and at random times so that choosing a Poisson process as the base model is kind of a natural selection in this regard.

In particular, it is important to heed, that the inclusion of jumps also changes the expected diffusive return of the underlying, as this return is diminished by the expected instantaneous jump size $\lambda\, k$. The economic reason for this change is that the overall expected return of the stock price still has to be the riskless rate of return r regardless, how many and which different constituents or sources of risk there are. For this reason, the expression $\lambda\, k$ is commonly referred to as the jump compensator.

The derivation of a closed-form option pricing formula is, however, more complicated because of the jumps. This is primarily due to the fact that the hedge suggested in Black & Scholes (1973) does no longer work as a complete hedge because it only eliminates the dispersion risk. In other words, applying the Black-Scholes hedge leads to a portfolio with pure jump risk. Furthermore, this jump risk cannot be hedged away at all using the stock and a riskless bond as the linearity in the possible combinations of the hedge portfolio cannot cope with the non-linear dependence of the option price on the un-

[40] A more detailed discusssion of the properties of such models and the economic causes for such behavior can, for instance, be found in Samuelson (1965), Merton (1971) or Merton & Samuelson (1974).

derlying stock. But nonetheless one can reduce the influence of the potential jumps to a level one can handle. As outlined before, jumps reflect sudden arrival of firm-specific information, which means that the associated risk is unsystematic in the sense of the CAPM (Capital Asset Pricing Model), thus not priced by the market and yielding a return identical to the one of the riskless asset. Applying this insight the option price must satisfy the following PDE

$$\begin{aligned} 0 = C_t + (r - \lambda k)SC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} - rC \\ + \lambda\, E[C(SY) - C(S)], \end{aligned} \tag{3.1.2.2}$$

where Y is the random variable percentage change in the option price if a Poisson jump event occurs and $E[\cdot]$ the corresponding expectation operator. Equation (3.1.2.2) coincides with the Black-Scholes equation (3.1.1.2) if and only if there are no jumps, i.e. $\lambda = 0$. Like in the Black & Scholes (1973) case, Merton (1976) derives a closed-form solution for his jump-diffusion model. This solution reads

$$C(t) = \sum_{n=0}^{\infty} \frac{\exp(-\lambda k\,(T-t))(\lambda k(T-t))^n}{n!}\, C_n(t),$$

where $C_n(t)$ is the price of a European call option with return and volatility parameters given by

$$\begin{aligned} \sigma_n &= \sqrt{\sigma^2 + \frac{n\sigma_N^2}{T-t}} \\ r_n &= r - \lambda k + \frac{n\ln(k)}{T-t}. \end{aligned}$$

In Smith (1976) a review of the Black-Scholes model as well as the Merton jump-diffusion model is provided. In addition, empirical verifications and applications to other contingent claims are provided.

These include valuing the equity of a levered firm[41], the effects of corporate policy[42], the valuation of dual purpose funds[43] and the risk structure of interest rates.[44] In Cox & Ross (1976*b*) several jump diffusion models are suggested and compared which include the so-called birth- and death process. Furthermore, they derive explicit valuation formulas. Carr & Wu (2003*b*) demonstrate that there are indeed continuous components and jump components in the underlying using S&P 500 index options. Furthermore, they find that the presence of jump risk varies over time, while the continuous component appears to be priced all the time. Ekström & Tysk (2007) elaborate on monotonicity and convexity results (option price w.r.t. the stock price) in the volatility, jump size and jump intensity parameters. This is done by virtue of the theory of parabolic integro-differential equations.

[41] Black & Scholes (1973) themselves pointed out that the equity value of a company can be retrieved by interpreting its value as the value of the right to buy back the company from its bondholders at the face value of outstanding debt. This, in essence, sets up an option pricing problem.

[42] Galai & Masulis (1976) assume the CAPM and the option pricing model to simultaneously hold. Doing so, they investigate the effects of changes in investment policy.

[43] A dual purpose fund is a closed-end fund with two different types of shares, income and capital shares. Holders of the former ones are entitled to whatever proceeds the fund generates including a minimum cumulative dividend and fixed maturity payment, whereas capital shares do not pay dividend payments but allow for redemption at the net asset value at maturity. Ingersoll (1976) studies dual purpose funds in detail and applies the Black-Scholes framework to find that it can be justified that such funds sell at a discount compared to their asset value.

[44] The assumption of a Modigliani & Miller (1958) world is made, i.e. the value of a firm is independent of its capital structure, if there are no transaction costs, no taxes and no costs of brankruptcy. Furthermore, Jensen & Meckling (1975) point out that the presence of agency costs would cause the total value of a firm to be a function of the debt/equity ratio. This is consistent with Long (1974) who finds that stochastic calculus can only be applied if there is no explicit reference to the value of the firm's equity or debt in the total value. In this setting Merton is able to derive the value of corporate debt given the total firm value. A valuation of a company's debt using the Black & Scholes (1973) option pricing approach is presented in Merton (1974).

Madan, Milne & Shifren (1989) posit the multinomial option pricing model as a generalization of the binomial model by Cox, Ross & Rubinstein (1979).[45] In the case of jump diffusion processes it is shown to converge to the Merton model. In Lesne, Prigent & Scaillet (2000) stochastic interest rates are added. Then the authors examine the convergence of approximations to these Merton type models.

In addition to this theory of option pricing in Merton type models, there are also extensions and applications of the models.[46] Cont, Tankov & Voltchkova (2007) deal with hedging in jump-diffusion models. The hedging strategies investigated include the underlying itself and a set of several options tailored such that the variance of the hedging error is minimized. Moreover, the performance is assessed in numerical examples. Camara (2009) presents a significant generalization of the Merton model by using two different jump terms, one for upward and one for downward jumps. Just as the Merton model, this approach captures smiles and skews which are discussed in Section 3.1.3, but in addition it is also able to explain term structures of these phenomena. Another generalization is presented by Øksendal & Sulem (2009) who consider incomplete markets.[47] They show that under risk neutrality the buyer's price is less or equal to the seller's price, while both are respectively bounded from above and below. To attain those results PDE methods and stochastic control theory are applied.

[45] The multinomial model differs from the binomial model in that it allows for n rather two branches at each node. Hence, prices along the tree are multinomially rather than binomially distributed.

[46] For example, Omberg (1988) deals with compound options in jump diffusion models. Gauss-Hermite quadrature is used and a trinomial model is employed to deal with the discontinuity in the first derivative of the option pricing formula.

[47] Formally an incomplete market is a market in which there are more different states of outcome than securities that will pay off one unit of money in these outcomes (so-called Arrow-Debreu securities), thus according to Musiela & Rutkowski (2005) a market is incomplete if there are unattainable claims.

3.1.3 Stochastic Volatility

Besides the jump diffusion approach there is another strand of literature which tries to deal with the shortcomings of the Black & Scholes (1973) model. This revolves around the observation that market volatility is not constant as assumed by Black and Scholes and in the various jump-diffusion models.[48] Because of that the volatility has been proposed to be stochastic. The most prominent such model is the one suggested by Heston (1993) which reads

$$\begin{aligned} dS_t &= r\,Sdt + \sqrt{V}\,S\,dW_t, \qquad t > 0 \\ dV_t &= \kappa\,(\theta - V)\,dt + \sigma_V\,\sqrt{V_t}\,\tilde{W}_t. \end{aligned} \tag{3.1.3.1}$$

In this model there is another source of uncertainty in the market, i.e. another so-called state variable which changes stochastically, namely the variance of the stock price process. Furthermore, these two processes do not evolve independently of each other as the two driving Brownian motions W_t and $\tilde{W}_t$ are correlated by a factor ρ. Intuitively speaking, this means that if the stock price increases or decreases by a certain percentage the volatility changes by ρ times that amount. Furthermore, the additional stochastic process is mean reverting to its long-term mean θ at a mean-reversion speed κ. The reason for that is that if V exceeds θ, the drift becomes negative. If in turn $V < \theta$ the drift is positive, which establishes gravitation back to the long-term mean for each level of volatility.

Given these market dynamics the procedure to come up with a fundamental pricing equation is very similar to the one applied in the Black & Scholes (1973) case. The only difference is the presence of the aforementioned additional state variable, which rules out repli-

[48] There is a large amount of literature dealing with this finding and at the same time pertaining to the so-called volatility smile, which is discussed in detail below in this section.

cation by just bonds and stocks. To amend this drawback, another option is added to the replicating portfolio making it look as follows

$$\Pi_t = a(t)S_t + b(t)B_t + c(t)D_t,$$

where D_t is the value of the additional option. Carrying out the same steps as before (using the multidimensional version of Itô's lemma to determine the change in the option price given the Heston (1993) market model, equating the change to the change in the replicating portfolio, collecting expressions for the respective stochastic components which must coincide again) one arrives at the following PDE[49]

$$\begin{aligned} 0 &= C_t + rSC_S + \frac{1}{2}VS^2C_{SS} + \kappa(\theta - V)C_V + \\ &\quad \frac{1}{2}V\sigma^2C_{VV} + \rho VS\sigma C_{SV} - rC. \end{aligned}$$

As it is the case with the Black & Scholes (1973) and the Merton (1976) models also the Heston (1993) model allows for a closed-form solution, albeit it is more sophisticated than the previous two. The price of a European call option is given by

$$C(t) = P_1\, S_t \; - \exp(-r(T-t))\, P_2\, K.$$

The helping variables P_1 and P_2 are obtained as

$$\begin{aligned} P_j &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Real}\left[\frac{\exp\left(-iy\ln(K)\right)\; f_j(y)}{i\,y}\right] dy \\ f_j(x) &= \exp(G(x) + H(x)V_t + i\ln(S)\,x) \\ G_j(x) &= r\,x\,i\,T + \frac{\kappa\theta}{\sigma_V^2}[(\beta_j - \rho\sigma_V yi + d_j(x))T \end{aligned}$$

[49]Note that the PDE is two-dimensional this time with the additional state variable V appearing as a second dimension. Furthermore, the correlation between the two underlying stochastic sources of risk, the two Brownian motions, translates to the mixed derivative term.

$$
\begin{aligned}
&\quad - 2\ln\left(\frac{1-g_j(x)\exp(dT)}{1-g_j(x)}\right)]\\
H_j(x) &= \frac{\beta_j - \rho\sigma xi + d_j(x)}{\sigma_V^2}\frac{1-\exp(d_j(x)(T-t))}{1-g_j(x)\exp(d_j(x)(T-t))}\\
d_j(x) &= \sqrt{(\rho\sigma_V xi - \beta_j)^2 - \sigma_V^2(2u_j xi - x^2)}\\
g_j(x) &= \frac{\beta_j - \rho\sigma_V xi + d_j(x)}{\beta_j - \rho\sigma_V xi - d_j(x)}\\
\beta_1 &= \kappa - \rho\sigma_V\\
\beta_2 &= \kappa\\
u_1 &= \frac{1}{2}\\
u_2 &= -\frac{1}{2}.
\end{aligned}
$$

How this closed form solution of the Heston (1993) model can be efficiently implemented has been demonstrated by Muck & Rudolf (2006). Despite the fact that the Heston (1993) stochastic volatility model has become the most prominent one there are several authors who have introduced and dealt with such an approach as well.

Scott (1987) analyzes option pricing when the underlying asset's volatility is stochastic. Contrary to Black-Scholes one needs two options and the underlying to form a riskless hedge and more importantly doing so does not lead to unique option prices since the underlying volatility itself is not a tradable asset. It is further argued, that volatility risk can be diversified away and that pricing depends on the risk premium associated with volatility risk.[50] To solve such option pricing problems Scott suggests Monte Carlo simulation. So is done by Johnson & Shanno (1987) who carry out a Monte Carlo simulation of a stochastic volatility model to price equity options. In their paper, Stein & Stein (1991) investigate stock price distri-

[50] The relationship between the tradability of an asset and the uniqueness of its price under a martingale measure is expounded in Joshi (2003).

butions for assets whose volatility follows an Ornstein-Uhlenbeck process.[51] In particular, emphasis is put on how stochastic volatility relates to the observation of fat tails.[52] The second aspect analyzed in this research is the implication for option pricing in such a setting. Ball & Roma (1994) examine the biases in the Black-Scholes model which are furthermore shown to be eliminated. This is done in the setting of a mean-reverting variance process as proposed by Stein & Stein (1991) as well as Heston (1993). In addition, the authors elaborate on the case where there is no correlation between the processes and find that in such a case comparatively simple power series methods for option pricing can be derived.[53] In his research, Wiggins (1987) estimates stochastic volatility processes and applies those estimates to valuing options. Doing so, he observes an overpricing by the Black-Scholes model which is even more pronounced for out-of-the money options. Schöbel & Zhu (1999) reassess the Stein & Stein (1991) paper in which an Ohrnstein-Uhlenbeck stochastic volatility model is assumed. By using a Fourier inversion method they are able to derive a closed-form solution for the European option pricing problem also involving correlation and they focus their analysis in particular on the boundary behavior near V=0 and provide numerical examples of their pricing formula.

[51] An Ornstein-Uhlenbeck process is, mathematically speaking, the only non-trivial process being Gaussian, stationary and Markov. For precise definitions of these terms, the reader is referred to Bauer (2001). In financial economics it is typically a process associated with stochastic volatility. Further properties of such processes have been thoroughly investigated in Uhlenbeck & Ornstein (1930).

[52] A fat tail is, roughly speaking, a property of a probability distribution where there is more mass associated to extreme events than there is in a normal distribution. A typical measure of this is kurtosis, which is based on the fourth moment of a distribution and given by $E[(X - E[X])^4]/\mathrm{var}(X)^2$. More information can, for example, be found in Verbeek (2008).

[53] In a power series approach the option price is assumed to be a non-linear function of the average volatility and thus the option price is expanded into a power series of the Black-Scholes price at the expected average volatility. The coefficients of the higher-order derivative terms are then the variance of the average volatility, its skewness and so forth.

Hull & White (1987) price European options in a stochastic volatility setting for both correlated and uncorrelated asset and variance processes. Their solution technique revolves around a power series approximation. Having stochastic volatility prices available, it is found that the Black-Scholes model tends to overprice options. Comte & Renault (1998) take up these findings and add a long memory feature to the stochastic volatility component of the Hull-White setting.[54] In addition to examining the effects on both processes involved (asset price and variance) the authors consider its option pricing quality by looking at implicit volatilities.

Another strand of stochastic volatility literature concentrates on the related problem of valuing currency rather than stock options. Chesney & Scott (1989) deal with European currency options on the US dollar - Swiss franc exchange rate. They do so in the Black-Scholes setting to which a stochastic variance component is added. This empirial analysis of market prices indicates some mispricing in the light of the biases w.r.t. time to maturity, striking price and volatility. Furthermore, there seems to be the possibility of arbitrage gains when trading according to the stochastic volatility model. Melino & Turnbull (1990) focus on pricing currency options on the exchange rate between the Canadian and the US dollar in a stochastic volatility framework of the underlying exchange rate process. Their empirical investigation yields that there is an improvement when it comes to how well model prices fit market prices.

In addition to the standard models there are again attempts made at relaxing the rather strict assumption of market completeness and introducing portfolio constraints to abstract from the assumption of unlimited borrowing and lending at the risk-free rate of inter-

[54] The so-called long-term feature addresses the persistence of volatility and is attained by replacing the Wiener process driving volatility by a fractional Brownian motion, which allows for correlated increments. A detailed discussion of this type of stochastic process can be found in Mishura (2008).

est as well as restrictions to short-selling assets. Cvitanić, Pham & Touzi (1999) deal with incomplete financial markets which arise because of portfolio constraints and stochastic volatility. Portfolio constraints are due to short-selling limitations for any hedger in this market. The central finding of the paper is an explicit characterization of the minimal price of replication a European contingency claim. This revolves around determining a PDE characterization of this price's representation in a shadow market. Moreover, Nicolado & Venardos (2003) highlight stochastic volatility models of the Ornstein-Uhlenbeck type and focus in particular on the incompleteness of such markets. For structure preserving martingale measures a closed-form valuation formula for European calls is derived.[55] The pricing capabilities are assessed in an empirical part. Henderson (2005) derives a comparison theorem and proves that convex option prices are decreasing in the market price of volatility risk. This allows for ordering option prices under various equivalent martingale measures.[56]

Finally, Garcia & Renault (1998) use a stochastic volatility model in connection with GARCH (Generalized Autoregressive Conditional Heteroskedasticity) and ARCH (Autoregressive Conditional Heteroskedasticity) models. The paper explains the differences between the hedging formulas of the Duan (1995) and Kallsen & Taqqu (1998) model. While the Duan model is a GARCH model, the Kallsen/Taqqu suggestion is an ARCH model and both agree on the same pricing formula but come up with different hedging formulas. The Duan formula is validated by introducing a stochastic volatility model.[57]

[55] In that context a structure preserving martingale measure is a martingale measure under which log-returns of the Barndorff-Nielsen & Shephard (2001) type of processes are also of this type.

[56] Equivalent martingale measures are treated in greater detail below in Section 3.2.2.5. The fact that they are not unique is due to the market incompleteness.

[57] ARCH and GARCH models date back to the papers by Engle (1982) and Bollerslev (1986). The objective is to model time-varying variance in time series data.

Another important justification for both jump-diffusion models and stochastic volatility is the so-called volatility smile. The volatility smile is a phenomenon observed in the market that options appear to be priced in a slightly different way than assumed by the Black & Scholes (1973) model. This was discerned by considering so-called implied Black-Scholes volatilities, i.e. the respective volatilities which have to be inserted in the Black-Scholes model to reproduce market option prices.[58] In contradiction to the model which assumes a constant volatility the resulting implied volatilities depended on both, the time to maturity and the striking price.[59] This effect was very extensively investigated by Rubinstein (1985), who found conceivably higher prices for shortly maturing out-of-the-money calls than predicted by the Black & Scholes (1973) model and a reversal of the significant striking price biases relative to Black-Scholes.

In Figure 3.1 we have plotted an exemplary smile for DAX options as it was observed in the market on December 23, 2009. The plots shows a curve for all three expiration days (January 2010, February 2010, and March 2010) that were traded on EUREX on that respective day.

This dependence structure w.r.t. the striking price can be explained in the following way: The more in-the-money an option is the relatively more probable it is that this situation does not change even if there is a strong market movement in adverse direction. Hence, market participants tend to price these options higher. This is in

This is done by having the current variance at time t depend on the error term of the previous q observations. In GARCH models there is, furthermore, also a dependence on the actual variance of the previous p observations.

[58] Lauterbach & Schulz (1990) examine the biases by caused by the Black-Scholes model and how they can be amended.

[59] A very intuitive explanation why, at least, the volatility should depend on the stock price is provided by Hull (2009). If the stock price of a company declines its leverage increases, thus rendering the equity riskier. Converserly if the stock price rises the company's leverage is reduced and consequently the equity is less risky, which in turn leads to less volatility.

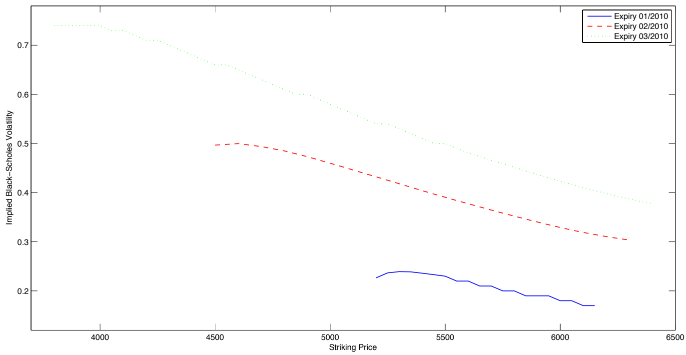

Figure 3.1: **Volatility Smile of DAX Options**
The plot shows the volatility smiles for the DAX options maturing between January and March of 2010 as it was observed on December 23, 2009. On that day the DAX closed at 5957.44 points.

line with the general notion of risk aversion, which implies that investors want to be compensated for additional risk by additional return. In the presence of jump risk, prices for in-the-money options thus increase (which in turn decreases their returns) and vice versa for out-of-the-money options. The steepness of the smile can therefore be interpreted as a measure for the degree of risk aversion.

In terms of the considered models they assign more volatility to the option than the Black-Scholes model does. Moreover, this effect is even more pronounced in the short run making the interpretation of jump risk being priced by option traders all the more intuitive. This is due to the fact that jumps are sudden and sharp (typically downward) movements, which leave the option holder with less time to recover the more short-dated the option is.[60] Thus the “safety”

[60] Xu & Taylor (1994) use foreign exchange option prices between 1985 and 1989 to retrieve the time-varying term-structure of volatility expectations. They find marked differences between long- and short-dated options.

premium placed on a short-dated option should be higher. Put differently, the later in the future an option matures, the higher its implied volatility beomes, which directly translates to the option being valued higher. This effect is, according to Hull (2009), also referred to as the average out effect because over longer periods of time market dynamics appear to be dominated by the dispersion components rather than jumps. Furthermore, these economic relationships translate from implied volatilities to return distributions, which can also be inferred from the prices of options traded in the market exploiting the fact, pointed out by Breeden & Litzenberger (1978) and Hull (2009), that the market implied distribution function is given by the second derivative of the traded security w.r.t. the striking price[61]

$$g(K) = \exp(rT)\,\frac{\partial^2 C}{\partial K^2}.$$

In this respect, $g(\cdot)$ denotes the probability density function of the stock returns. Both Rubinstein (1994) and Derman & Kani (1994) exploit this relationship between the volatility smile and the probability distribution of stock returns to infer risk-neutral probabilities, which are then applied to option pricing purposes in the context of a binomial tree. Furthermore, Jackwerth & Rubinstein (1996) also use S&P 500 options to retrieve risk-neutral probabilities, a procedure they find to better reflect the possibility of extreme events like the October 1987 stock market crash, a risk usually under-estimated by classical time-series analysis. Regarding the volatility smile Carr & Wu (2003*a*) detect that for S&P 500 index options the smile does not lose steepness with increasing time to expiry as predicted by the widely used above pricing models.

[61] A similar investigation is carried out by Melick & Thomas (1997) who examine American options on crude oil during the 1991 Gulf crisis to estimate the probability density function of the return distribution.

3.1.4 More Elaborate Models

After discussing two very prominent extensions of the Black & Scholes (1973) model, the Merton (1976) jump diffusion model and the Heston (1993) stochastic volatility model this section deals with various further generalizations.

First and foremost, of course, there is a combination of the two previously suggested extensions. This was first considered by Bates (1996). In his paper he estimates a jump-diffusion model which also includes stochastic volatility using a generalized least squares approach.[62] With this model, he is able to explain the volatility smile observed for exchange rate options between the US dollar and the Deutsche Mark. Without jumps the observed smile can only be attained for unrealistic model parameters, thus making the inherent jump risk plausible.[63] Branger & Schlag (2004), in addition, investigate the steepness of volatility smiles. It is found that for indexes the smile is steeper than for individual stocks. By considering an index to be a weighted sum of individual stocks the smiles are merely contingent on the dependence structure between the stock prices. In a stochastic volatility jump-diffusion model these differences w.r.t. the smile can be explained.

Given these theoretical advances in option pricing, Bakshi & Chen (1997) relate to the Cox, Ingersoll & Ross (1985) research. It differs in considering an explicit power utility function which enhances tractability in a Lucas (1978) exchange economy. The model is then used to derive bond and stock prices and the prices of the respective

[62] A generalized least squares approach alters the ordinary least squares approach so that the procedure is able to cope with heteroscedasticity and correlation. A detailed discussion about the method is provided in the textbook Kariya & Kurata (2004).

[63] Pan (2002) analyzes joint time series data for S&P 500 options close the maturity. This analysis is then used to explain volatility smirks for cross-sectional option price data.

contingent claims. The approach is validated by the fact that the stochastic volatility version of the stock price dynamics is capable of reconciling the volatility smile. Bakshi, Cao & Chen (1997), in addition, summarize and evaluate option pricing models to that date by assessing their pricing capabilities and hedging performance. In the most general setting this includes jumps, stochastic volatility, and stochastic interest rates at the same time. Pricing capabilities are tested using in-sample and out-of-sample data.

Furthermore, research is concerned with the distribution of asset price returns because, in essence, knowing the risk-neutral distribution and being able to price an option are inseparable. Actually, any market model makes an assumption about the distribution of the underlying asset's return. Bates (1997) investigates S&P 500 futures options w.r.t. explaining moneyness biases.[64] By deriving theoretical distribution specific constraints distributional hypotheses can be identified which are capable and which are incapable of explaining these biases. Consequently, this is a link to the observation of skewness.[65] Bates (2000) relates the observed skewness in stock price distributions (inferred from S&P 500 options) to stochastic volatility and jump models. He finds that including jumps is necessary to avoid implausible volatility of volatility but that nonetheless there are inconsistencies between sharp moves in the market and the model. Bakshi, Kapadia & Madan (2003) then aim at explaining the economic source of skewness in the returns of stock options. Furthermore, risk-aversion is related to skewness in the risk neutral

[64] The term moneyness bias refers to model and market price deviations w.r.t. how far an option is in- or out-of-the-money. Essentially such relative option prices form a direct relation to the thickness of the tail ends of return distributions.

[65] In descriptional statistics skewness is a measure of the asymmetry of a probability distribution. Whereas the normal distribution is perfectly symmetric and has vanishing skewness, asset returns appear to be assymetric with outliers more probable than assumed by the lognormality assumption. Mathematically speaking, skewness is based on the third moment of a random variable X and given by $\frac{E[(X-E[X])^3]}{\mathrm{var}(X)^{3/2}}$. More information can, for example, be found in Verbeek (2008).

density and the term-structure of this skewness is investigated. This includes identifying a systematic part and an unsystematic part of skewness risk in individual stocks.

Scott (1997) applies Fourier inversion to obtain closed-form solutions to option pricing problems when there are stochastic interest rates, stochastic volatility and jumps.[66] Properties and effects on option prices of the respective constituents are then investigated using options on the S&P 500 stock market index in an empirical study. A very important advance, in particular from a practical standpoint, is provided by Duffie, Pan & Singleton (2000). Their paper deals with option pricing in a so-called affine model setting. This is a particularly interesting set of models as, which is shown in this paper, it allows for very general closed-form solutions.[67] The practical relevance is demonstrated by an application to a jump-diffusion model with stochastic volatility. Then Chernov, Gallant, Ghysels & Tauchen (2003) provide a comparison between two model classes, loglinear models and affine jump models when it comes to modeling the underlying's distribution.[68] Furthermore, the effects on option pricing are investigated under statistical goodness-of-fit considerations and other aspects such as parsimony when it comes to computing option prices and hedge ratios.[69] A summary of var-

[66] Fourier inversion is the inversion of the famous Fourier transform, an extremely powerful mathematical tool to transform functions so that their transforms have more desirable properties. For instance, derivatives w.r.t. a certain variable transform to multiplications with that variable. A detailed theory of the Fourier transform can, for example, be found in Fourier (1822) or Bochner & Chandrasekharan (1949). A discussion of FFTs (Fast Fourier Transforms) in the field of finance is provided by Černý (2004).

[67] The term affine pertains to the dependences of the drift vector, covariance matrix and jump intensities on the state vector. Roughly and intuitively speaking, affine-linear means linear with an additional constant offset. Therefore, the Heston (1993) model, for instance, is affine because of the square root in the volatility expressions, whereas the Hull & White (1987) is not.

[68] A log-linear model is a model in which the volatilty is an exponential function of the underlying stochastic factor.

[69] Bondarenko (2003) aims at explaining the anomaly observed for put option prices. Trading strategies involving in-the-money and out-of-the-money puts

ious models and solution techniques, especially efficient numerics, can be found in the habilitation by Gerstner (2007).

An interesting new idea is presented in Boes, Drost & Werker (2007). This paper highlights the fact that in addition to the jumps observed in the market during trading hours there are also jumps resulting from overnight trading halts. The authors empirically find, that overnight jumps are responsible for about 25% of the entire jump variation so that the variation in the classic jump components is significantly reduced. Furthermore, none of the types of jumps alone can explain all option price characteristics. Another aspect is addressed by Jarrow, Protter & Shimbo (2010). The authors study the effect of asset price bubbles[70] on option pricing by assuming no arbitrage opportunities in a local martingale and incomplete market framework.[71] Their main contribution is to allow for different local martingales over time. Doing so, they apply their model to derivative pricing which yields that still American calls are not prematurely exercised, that still put-call-parity holds, that there are no

are empirically found to allow for arbitrage opportunities. This is explained by inconsistencies between market prices and classic models such as the CAPM and Rubinstein (1976) which make put prices appear too expensive. In this paper for a broad class of non-standard models it is excluded that they can contribute to explaining this.

[70] The relevance of asset price bubbles is highlighted by, for instance, the papers by Ofek & Richardson (2003), Brunnermeier & Nagel (2004), Cunado, Gil-Alana & Perez de Gracia (2005), Battalio & Schultz (2006) and Pástor & Veronesi (2006) on the technology market bubble around the year 2000, by the works on the 1929 stock market crash like White (1990), Bradford de Long & Shleifer (1991), Rappoport & White (1993), and Donaldson & Kamstra (1996), and studies concercing the US housing price bubble (Case & Shiller (2003)) or the Japanese one (Stone & Ziemba (1993)). In Scheinkmann & Xiong (2003) the development of asset price bubbles is related to Tobin's tax.

[71] A definition of local martingales can be found in the textbook by Karatzas & Shreve (2008). From an economic point of view the incompleteness of the market leads to non-uniqueness of the risk neutral measure according to Harrison & Kreps (1979), i.e. the market can choose between several ones. The contribution of Jarrow, Protter & Shimbo (2010) now is that they allow for changes of that (local martingale) measure, which can be interpreted as a regime change. Doing so, their model is able to reflect bubble birth (contrary to e.g. Camerer (1989)) without a priori including the bubble in the model.

bubbles in put prices and that the magnitudes of call price bubbles relate to the corrensponding asset price bubbles.

A more technical and mathematically demanding, but all the more promising, generalization is presented by Leippold & Trojani (2008). In this paper so-called Wishart processes are introduced to model market dynamics. Loosely speaking, these models are matrix valued affine processes (thus allowing for Fourier solution techniques) and give rise to a very general class of problems and properties to be considered. This includes stochastic correlation between underlying parameters, single- and multi-asset cases and so forth. Branger & Muck (2012), for example, discuss quanto options. With this sophisticated model they are able to simultaneously capture the stochastic correlation between assets and the time-varying volatility smile at the same time without having to continuously recalibrate the model parameters.

3.2 American Style Options

After the discussion of option pricing models in general in the previous section, this section focuses on the properties of American options. This includes both, characterizations of the early exercise boundary (cf. Subsection 3.2.1) and valuation approaches (cf. Subsection 3.2.2).

3.2.1 Characterizations and Properties of the Early Exercise Boundary

The key difference between American-style and European-style options is that the former ones provide their holders with early exercise privileges. Because of that observation it is natural and intuitive that the valuation of American options ultimately amounts to characterizing when early exercise occurs. More technically speaking, this prompts research on the shape and properties of the early exercise boundaries. The three perhaps most basic and central observations are:

1) There is always a stock price S_f, termed early exercise threshold, such that early exercise is optimal for $S > S_f$ in the case of certain call options and $S < S_f$ in the case of certain put options.
2) The early exercise threshold is a monotone increasing function in the time to maturity.
3) At the early exercise threshold the option value is tangent to the payoff function, a property referred to as the high contact condition or smooth pasting condition.[72]

As the high contact condition is of particular importance in the context of valuation using partial differential equations, a proof is presented in Section 3.2.2.3, where such a formulation is derived.

Regarding the exercise threshold below which it becomes optimal to exercise American put options we take following proposition (including the proof) from Sandmann (2009).

Proposition 3.2.1 *It can indeed become optimal to prematurely exercise American-style put options.*

Proof - Proposition 3.2.1:
Apparently, the payoff of put options increases with falling stock prices. It is thus sufficient for optimal premature exercise if the

[72] These two terms are used interchangeably and in Section 4.1.3 smooth pasting results are considered in a broader sense.

intrinsic value (compounded up to maturity) exceeds all possible payoffs at that point. At time t this means

$$(K - S_t)(1 + r)^{T-t} > K - S_T$$

As before, K denotes the strike, r the riskless rate of return and S_t and S_T the respective stock prices at times t and T. Reformulation yields

$$S_t < K\left(1 - (1 + r)^{-(T-t)}\right) + S_T(1 + r)^{-(T-t)} \qquad (3.2.1.1)$$

Thus, the stronger condition $S_t < K\left(1 - (1 + r)^{-(T-t)}\right)$ is sufficient to ensure that (3.2.1.1) holds in any case. This establishes the assertion of the proposition.

□

With regard to the monotonicity property, stated as the second item above, we follow the proof of Kwok (2008). We already know that all else held constant an option C_2 with longer maturity than option C_1 is more valuable, thus the value function of C_2 lies above that of C_1 at all times and stock prices. On the other hand both options satisfy the high contact condition at their respective early exercise thresholds. As C_2 is always greater or equal than C_1, this also holds at the point of tangency. If this was not the case there would have to be a point of intersection between the two curves which contradicts $C_2 \geq C_1$. Hence S_f is monotone increasing in the time to maturity.

In the Black-Scholes model we can also further describe the asymptotic behavior of American options in the near expiry region. For this purpose we consider call options on a dividend paying stock with continuously paid dividend yield q.[73] At expiry the value of

[73] Considering non-dividend paying stocks would be pointless as call options on such stocks are never prematurely exercised.

the American option is $C(S,T) = S - K$. The time derivative of a call option in the Black-Scholes model according to Kwok (2008) is given by

$$\frac{\partial C}{\partial T} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + (r-q) S \frac{\partial C}{\partial S} - rC.$$

Inserting $C(S,T)$ yields

$$\frac{\partial C}{\partial T}(S,T) = rK - qS, \quad S > K.$$

For an American option to be kept alive until maturity $\frac{\partial C}{\partial T}(S,T) \geq 0$ must be satisfied. If that was not the case the American option value would drop below $S-K$ just prior to expiry and thus prompting exercise. Consequently, the point $S = \frac{r}{q}K$ at which the time derivative changes sign plays a crucial role. In order to satisfy this condition, $S > K$ and $q < r$ must hold and to avoid early exercise $K < S < \frac{r}{q}K$ is required. It follows that

$$\lim_{t \to T} S_f(t) = \frac{r}{q} K.$$

This result is in line with the observation that call options on non-dividend paying assets are never exercised. In this case $(q = 0)$ we obtain $S_f(t) = \infty$ which precludes exercise at any time. If in turn the dividend yield exceeds the riskless rate of interest the situation is slightly different. From the consideration regarding the change of sign we know that the limit $S_f(T)$ is at most K. If we assume the contrary (that the option is still alive) Kwok (2008) points out that exercise is inferior to continuation as the foregone earned dividend $qS\delta t$ is more than the riskless interest $rK\delta t$ in a small time interval δt. It follows that

$$\lim_{t \to T} S_f(t) = K.$$

Jacka & Lynn (1992) examine the shape of the exercise region of general American-style contingent claims under diffusion processes and, in the one-dimensional case, find it to be up-connected if the drift of the payoff process is decreasing in the underlying.[74] Broadie & Detemple (1997) characterize the optimal exercise boundary of American options written on multiple underlying assets, e.g. the maximum of two assets or exchange options. Furthermore, they address the problem of non-convex payoffs which, for example, arises if an option is capped.[75] Detemple & Tian (2002) examine the stopping boundary of American options in a quite general model. They account for stochastic interest rates and stochastic volatiliy and identify the stopping boundary be to up-connected and state-dependent. The result also extends to capped options. By Chen, Chadam, Jiang & Zheng (2008) a rigorous proof is given for the convexity of the early exercise region of the free boundary value problem of an American put option on non-dividend paying assets. Furthermore, the result is obtained to retrieve rigorous information about the asymptotic near-expiry behavior of such options.

When approximating American option prices with numerical techniques, this also yields an approximation of the critical exercise threshold. Lamberton (1993) proves convergence of this approximate critical stock price in the case of a finite difference approximation as proposed by Brennan & Schwartz (1978). In the paper by Kuske & Keller (1998) the differential equation for the early exercise boundary is solved asymptotically.[76] Doing so the authors

[74] Loosely speaking, up-connectedness can be thought of as consisting of only one part and being an infinite interval.

[75] Capped options are options whose upward potential is limited by a so-called cap L imposed on the payoff function which is then modified to $\max((S \wedge L) - K, 0)$ in the case of call options. For $L \to \infty$ standard call options are retained. For finite L the holder of such an option can only participate in the stock price movement if prices are at most L. A study of such options is conducted by Broadie & Detemple (1995).

[76] Given the boundary value problem (3.2.2.5) presented below for the option price, Green's theorem (cf. for instance Evans (2010)) can be used to transform

obtain the optimal exercise boundary for an American put near the expiry date. In Peskir (2005*a*) many technical results about American options and stopping problems as well as the corresponding free boundary value problems are recounted. Besides, they are used to obtain a characterization of the stopping boundary by virtue of the change-of-variable formula due to Peskir (2005c). Dai & Kwok (2006) are concerned with the valuation of lookback options and Asian style options exhibiting American exercise features.[77] The characterization of the exercise region is set in a framework of variational inequalities. Göttsche & Vellekop (2011) derive an integral representation of the early exercise boundary of American put options as suggested by Kim (1990), Jacka (1991), and Carr, Jarrow & Myneni (1992). Using the Black & Scholes (1973) model the authors expand theoretical results to the more realistic case of discrete-time dividend payments.

In his appendix to Samuelson (1965), McKean (1965) provides some of the mathematical background necessary for the warrant pricing paper of Samuelson. Doing so he discusses the existence of optimal functions majorizing the discounted expectation. Afterwards, the theory is applied to the examples of a multiplicative Brownian motion and a multiplicative Poisson process. Chen (1970) takes up the results by Samuelson (1965) and McKean (1965) on warrant pricing. Using dynamic programming he establishes a relationship

it to an integral equation for the exercise boundary. The authors then provide an asymptotic solution for small times to maturity with the leading term reconciling the result by Barles, Burdeau, Romano & Samsoen (1995).

[77] According to Bowie & Carr (1994) a lookback option is an option whose strike price is the lowest price (highest price) of the underlying during the lifetime of the option in the case of the call variant (the put variant). Heynen & Kat (1995) demonstrate that discretely monitored lookback-options can be valued in semi-closed form. Further insight into lookback option valuation is obtained by Choi & Jameson (2003).

between warrant pricing and the underlying asset.[78] American securities are explicitly covered and addressed. In the paper by Kane & Marcus (1986) the wild card option inherent in treasury bond futures is investigated. Since the short position bears the right to opt for delivery until 8 PM while the price is locked in at 2 PM a put option, termed wildcard option, is constituted. Moreover, acting optimally in this environment sets up an optimal stopping problem. This problem is solved recursively in the paper. Karatzas (1989) in a survey paper is concerned with a presentation of the main results in the field of mathematical finance and continuous trading, such as optimal investment and consumption decisions as well as contingent claim pricing. In particular, he deals with the valuation of American options and points out that there is a corresponding optimal stopping problem. Carriere (1996) deals with finite time discrete Markov chains and investigates optimally stopping them.[79] Besides, it is proved that the optimal decisions are equivalent to a series of conditional expectations, which can be obtained via regression. Finally, the results are applied to American options.

3.2.2 Valuation Methods

Regarding valuation methods for American-style options, there is a plethora of different approaches. Nevertheless one can distinguish between five classes, in which these approaches can be loosely classified. In the subsequent section the five classes analytical approximations (cf. Section 3.2.2.1), lattice methods (cf. Section 3.2.2.2), PDE methods (cf. Section 3.2.2.3), Monte Carlo simulations (cf.

[78] In Section 4.1.1 it will be described what the dynamic programming equations are and their very close relationship to optimal stopping in particular and optimal control problems in general will be highlighted.

[79] This is of particular importance to the pricing of American-style contingent claims since in its discretized version, with which numerical methods to solve stochastic optimal control problems are concerned, such exercises naturally arise. Details will be covered in Section 4.2.1.

Section 3.2.2.4), and stochastic control theory (cf. Section 3.2.2.5) are discussed in detail.

3.2.2.1 Analytical Approximations

This section deals with analytical approximations to American options. Such methods have been developed since the late 1970s because researchers found resorting to numerical methods (these will be extensively covered in the subsequent sections) too onerous. The drawback of numerical solution techniques, regardless of the method, typically is that they require a lot of memory space and computational time to carry out the respective algorithms. In contrast, the benefit of analytical approximations or even exact closed-form solutions, like there are for European-style options, is the tractability of the formula's evaluation and fewer demand for computational resources.

One of the most prominent approximations is due to Geske & Johnson (1984). They deal with American put options on a dividend-paying stock in the Black & Scholes (1973) framework with the assumptions outlined in Section 3.1.1. By the same arguments given there the put option price satisfies the following PDE

$$0 = rP - rSP_S - \frac{1}{2}\sigma^2 S^2 P_{SS} - P_t.$$

Contrary to the European option counterpart, the terminal time of the valuation problem is a priori unknown because an American option can be exercised at any instance up to maturity, thus rendering the valuation exercise a so-called free boundary value problem.[80] The necessary condition at the free boundary is the payoff function

[80] Free boundary value problems are covered in detail in Section 3.2.2.3 about PDE solution techniques. Therefore, further details are omitted here.

of the option

$$P(S) \geq \max(K - S, 0).$$

Geske & Johnson (1984) derive a solution to the valuation problem with the put option being a weighted sum of the strike and the stock:

$$P = w_2 K - w_1 S, \tag{3.2.2.1}$$

where the weights are given by the following equations

$$\begin{aligned}
w_1 &= N_1(-d_1(S^*_{dt}, dt)) \\
&\quad + N_2(d_1(S^*_{dt}, dt), -d_1(S^*_{2dt}, 2dt), -\rho_{12}) \\
&\quad + N_3(d_1(S^*_{dt}, dt), d_1(S^*_{2dt}, 2dt), -d_1(S^*_{3dt}, 3dt), \\
&\quad \rho_{12}, -\rho_{13}, -\rho_{23}) + \ldots \\
w_2 &= \exp(-rdt)N_1(-d_2(S^*_{dt}, dt)) \\
&\quad + \exp(-r2dt)N_2(d_2(S^*_{dt}, dt), -d_2(S^*_{2dt}, 2dt), -\rho_{12}) \\
&\quad + \exp(-r3dt)N_3(d_2(S^*_{dt}, dt), d_2(S^*_{2dt}, 2dt), \\
&\quad -d_2(S^*_{3dt}, 3dt), \rho_{12}, -\rho_{13}, -\rho_{23}) + \ldots \\
d_1(q, \tau) &= \frac{\ln(S/q) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \\
d_2(q, \tau) &= d_1 - \sigma\sqrt{\tau} \\
\rho_{12} &= 1/\sqrt{2} \\
\rho_{13} &= 1/\sqrt{3} \\
\rho_{23} &= \sqrt{2/3}.
\end{aligned}$$

In this respect $N_d(\cdot)$ denotes the multivariate normal distribution in dimension d.[81] Unfortunately, the formula cannot be readily eval-

[81] According to Gut (2009) a random vector $Z = (Z_1, \ldots, Z_d)^T$ has d-dimensional normal distribution with expectation $\mu = (\mu_i, \ldots, \mu_d)$ and variance-covariance matrix $\Sigma = (\Sigma_{ij})_{i,j=1,\ldots,d}$ if $Z_i, i = 1, \ldots, d$ are univariately normal with expectations μ_i and variances σ^2_{ii}, $i = 1, \ldots, d$, and the covariance structure $\mathrm{cov}(Z_i, Z_j) = \Sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$, where ρ_{ij} denotes the correlation between Z_i and

uated and lacks mathematical rigor omitted in favor of palpability but nonetheless serves theoretical purposes as it can be used to compute the Greeks and thus gain further insight into the valuation mechanics.[82] Furthermore, the method can be readily extended to accomodate multiple discrete dividend payments as suggested by Roll (1977) and Geske (1979*a*), who value the dividends separately and strap the stock price process of them.

The intuition behind the derivation of the valuation formula is that, as shown in Section 3.2.1, at each instant there is a critical stock price S_f below which it is more favorable to exercise the put option. Doing so is more favorable, if the proceeds of exercise are at least as much as the option. In this way one can set up a series of considerations. At the beginning of the option's lifetime it cannot have been priorly exercised. Therefore, its value amounts to the present value of the integration of the striking price less all possible stock prices less than the critical stock price. In the second step the situation is slightly more complicated. This time around the procedure is the same except for the notion that all combiniations where there is an exercise in the first instant are excluded from the consideration. As detailedly expounded in Geske & Johnson (1984) this procedure exactly leads to (3.2.2.1).

Given this general procedure, in practice it is only possible to obtain approximate solutions. The idea behind this is to use the above representation to compute option prices with possible exercise at discrete times during the life of the option.[83] The simplest form

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Z_j and σ_i the standard deviation of Z_i. The probability density function is given by $f(x) = \frac{1}{\sqrt{2\pi}^d \sqrt{|\det(\Sigma)|}} \exp(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu))$.

[82] For details on the Greeks w.r.t. the time to maturity, the striking price and the stock price the reader is referred to the original paper by Geske & Johnson (1984).

[83] Parkinson (1977) deals with American put options which are approximated by a finite number of exercise dates. The valuation procedure then takes the expected value of a previous time step and then computes the expectation.

is an option which can only be exercised at maturity T, i.e. the European option. The second option might be the one exercisable at $T/2$ and T, the third one would allow for exercise at $T/3$, $2T/3$ and T and so forth.[84] This defines a sequence of option prices which converge to the intended American option

$$\begin{aligned} P_2 &= K\exp(-rT/2)N_1(-d_2(S^*_{T/2}, T/2)) - SN_1(-d_1(S^*_{T/2}, T/2)) \\ &\quad + K\exp(-rT)N_2(d_2(S^*_{T/2}, T/2), -d_2(K,T), -1/\sqrt{2}) \\ &\quad - SN_2(d_1(S^*_{T/2}, T/2), -d_1(K,T), -1/\sqrt{2}) \\ P_3 &= \ldots \end{aligned}$$

Naturally, the question of how to compute the limit arises. The method suggested by Geske & Johnson (1984) is Richardson extrapolation which revolves around the assumption that the error term of each particular approximation of the limit is of polynomial type in the grid length h, i.e.[85]

$$P - P_i = a_0 h^{k_0} + a_1 h^{k_1} + a_2 h^{k_2} + \ldots$$

Truncating the representation after the second order terms and applying it simultaneously to all three approximative solutions yields

To allow for the early exercise feature in every step the better of continuing and stopping is chosen, a procedure comparable to the dynamic programming equation discussed below in Section 4.1.1.

[84] Heed that here as well as in Geske & Johnson (1984) the grid of possible exercise times is equidistant. Of course, that does not have to be the case and there is a converging sequence regardless of the placement of exercise times. This is exploited by Bunch & Johnson (1992) to improve the speed of convergence.

[85] In numerical analysis Richardson extrapolation is a means to enhance the order of convergence of a converging sequence. Assume that $A(h) \overset{h\downarrow 0}{\to} A$ with order k_0, i.e. $A - A(h) = a_0 h^{k_0} + \mathcal{O}(h^{k_1}), a_0 \neq 0, k_0 < k_1$. Then for $t > 0$ we have $A - A(h/t) = a_0 (h/t)^{k_0} + \mathcal{O}(h^{k_1})$ and by subtraction and rearranging of terms $A = \frac{t^{k_0} A(h/t) - A(h)}{t^{k_0} - 1} + \mathcal{O}(h^{k_1})$. In other words, this is an approximation of A of order k_1 which is better than order k_0. More details can be found in Richardson (1911) and Richardson & Gaunt (1927) or any textbook on numerical analysis.

the following polynomial for the American price P

$$P = P_3 + \frac{7}{2}(P_3 - P_2) - \frac{1}{2}(P_2 - P_1).$$

The results by Geske & Johnson (1984) draw to a large degree on the previous results by Johnson (1983). In this paper an analytical approximation for an American put option is derived. The derivation exploits relative valuation results such as the observation that a European put with an exercise price rising at the risk-free rate of interest is more valuable than a standard American put, which in turn is more valuable than a standard European put. Blomeyer (1986) amends the Johnson (1983) approximation of American put options by allowing for one cash dividend prior to the option's expiration date. Furthermore, the method is found to perform well by assessing its efficiency in a comparison to the analytical values based on the Geske & Johnson (1984) method. With respect to the treatment of known dividends the approach takes up the findings by Roll (1977), who proposes a way to split an American option on a dividend paying stock into three European options, which can then be valued. Besides, he analyzes to what extent the Black-Scholes biases described in Section 3.1.1 are removed or maintained. Geske (1979*a*) presents an analytical solution for the value of an American call option with known dividends. However, the obtained solution is only quasi closed-form and involves multivariate cumulative normal distribution functions.

Whaley (1981) corrects misspecifications by Roll (1977) and Geske (1979*a*) in their approaches to valuing American calls with known dividends. In the summarizing paper of empirical tests by Whaley (1982) the approximation of American options using the Black approximation and a simpler one (netting the stock price of escrowed dividends) are compared to the exact valuation formula according to Whaley (1981), which is found to perform better. Besides, in a test of market efficiency using option prices and riskless hedge portfolios for the Chicago Options Exchange the hypothesis of efficiency cannot be rejected taking transaction costs into account. Omberg (1987) provides further insight and explanations into why convergence of the binomial model and compound option model by Geske & Johnson (1984) performs as well as it does. Furthermore, he suggests methods to improve convergence properties. Selby & Hodges (1987) further elaborate on this question. This piece of research covers summation identities for multivariate normal distributions as have to be computed in the Geske & Johnson (1984) approach. Exploiting these identities the number of integrals can be reduced and accordingly the efficiency of the algorithms can be enhanced. This is demonstrated in numerical examples. Bunch & Johnson (1992) significantly enhance the Geske & Johnson (1984) methodology. The improvement revolves around optimally placing the exercise points for the integral by Geske and Johnson. Doing so, the dimensionality of the integration can be reduced from four to two. Lee & Paxson (2003) are concerned with an extension of the Geske & Johnson (1984) and Geske (1979*a*) compound option approach to come up with an analytical solution to American options. It provides an exponential approximation and establishes tighter upper bounds on the price. Besides, the approximation is given for a two-factor model with stochastic interest rates.

In addition to the Geske & Johnson (1984) approximation strategy there is another strategy which revolves around a decomposition of the American option price P into the European option p and a so-called early exercise premium π, i.e.

$$P = p + \pi. \tag{3.2.2.2}$$

Kim (1990) presents an integral valuation formula for American call and put options which is based on this decomposition into a

European part and an early exercise premium. Moreover, the integral is then numerically solved. The drawback is that the analytical solution of the free boundary problem does not permit an explicit closed-form solution. Gao, Huan & Subrahmanyam (2000) deal with American-style barrier options and derive quasi-analytical expressions for the option prices and the hedge parameters. Besides, they establish relationships similar to the put-call-parity, which facilitates valuation and hedging.[86] The derivations are also based on the decomposition technique and the results are applied in numerical demonstrations. Chiarella, El-Hassan & Kucera (1999) formulate the European and American option valuation problem in terms of path integrals, which they approximate using Fourier-Hermite series expansion. In the process of doing so they are also able to obtain delta hedge ratios. In Gukhal (2001) an analytical valuation formula is presented for the value of American options based on jump-diffusions as the underlying processes. However, a shortcoming is that it cannot be readily evaluated in closed-form. Moreover, dividends are covered and a decomposition is provided which is different from the one for diffusions (European plus early exercise premium).[87] Ibánez (2003) values American put options by this decomposition. He furthermore investigates the error of the aforementioned Richardson extrapolation applied to the convergence of Bermudan options to the desired American options and improves the convergence of the Kim (1990) model. In numerical experiments he finds his approach to perform well.

[86] The authors decompose the price of American barrier options into the European counterpart and an exercise premium. The European part is then further represented as the standard option and a barrier term for which an integral representation is available. In this way, determining the hedging parameters boils down to those of the European constituent options.

[87] In the presence of jumps transition from the continuation to the exercise region might be discontinuous. This leads to adjustment terms in the decomposition, i.e. the American option equals the European option plus the expected present value of dividends received in the exercise region, less the interest paid on the strike in the exercise region, plus rebalancing costs when a jump from the exercise into the continuation region occurs.

In essence, all these methods amount to characterizing the early exercise boundary since most of the time the European counterpart of the American option to be valued can be determined in a comparatively not so onerous way. In Huang, Subrahmanyam & Yu (1996) an analytical valuation formula is presented for the value of an American put option. This formula involves the determination and approximation of the early exercise boundary and it can be applied whenever the value of the European counterpart is available. The formula is tested in numerical examples, where quanto options and stock options are valued. It is found to perform particularly well when it comes to computing hedge ratios. Ju (1998) uses the American option value representation of Kim (1990), Jacka (1991), and Carr, Jarrow & Myneni (1992). The early exercise boundary is assumed to be a piecewise exponential function on the time grid, so that the resulting integral admits a closed-form solution. Moreover, the valuation is tested for stock options and found to perform remarkably well compared to binomial methods, which are discussed in Section 3.2.2.2. In Zhu & He (2007) the Bunch & Johnson (2000) approach is revisited and an error in their derivation is found. This error is corrected leading to a new valuation formula and numerical experiments are conducted pointing out the efficiency of this new formula.

Besides, there are even further methods of approximating the American option value analytically. Sullivan (2000) takes the integral representation of an American option.[88] Since the first part of that equation yields an explicit solution only the second integral is treated numerically. The novel approach followed there is to use Gaussian quadrature with Chebychev polynomials.[89] Moreover, a study of

[88] For discretely exercisable American options the author finds a representation as $C(S,m) = \exp(-r\delta)\int_{\infty}^{\ln(B(m-\delta)/S)}(K - Se^z)f(z)dz + \exp(-r\delta)\int_{\ln(B(m-\delta)/S)}^{\infty} C(Se^z, m-\delta)f(z)dz$, where δ is the interval between two exercise points and $B(\cdot)$ the respective exercise threshold.

[89] Gaussian quadrature is an integral approximation method of the type

performance is conducted relating the speed and accuracy of this approximation to the ones of the binomial method and analytical solutions. The method is found to perform remarkably well. In Zhang & Li (2010) the critical stock price S_f of American options, i.e. the price above which call options should be exercised and the price below which put options should be exercised, is dealt with when there is first a constant and second a continuous dividend yield. The critical stock price is obtained using perturbation theory, as mentioned in Section 3.1.1, and has the form of a power series. Furthermore, numerical experiments indicate that the representation is well suited for practical use in trading and hedging. Another intriguing approximation strategy is presented by Carr (1998). It revolves around replacing the fixed maturity by a random one which eliminates time dependence.[90] The random maturity is chosen via a Poisson jump process, which is independent of the continuous stock price process. As the number of jumps tends to infinity the random option value approaches the true one.

Another very well known approximation method has been suggested by Bjerksund & Stensland (1993). They approximate the American option value by exogeneously imposing a threshold which immediately triggers exercise of the option upon being breached. Given the theoretical results outlined below in Section 3.2.2.5 that the true

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$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i g(x_i)$, where $f(x) = W(x)g(x)$, g approximately polynomial and $W(x)$ known. The method is designed to exactly integrate polynomials of degree $2n-1$ by proper choice of the weights and evaluation points. If $W(x) = \frac{1}{\sqrt{1-x^2}}$ the evaluation points are just the roots of the Chebyshev polynomials. These are orthogonal polynomials (w.r.t. the $L^2([-1,1])$ scalar product) that on the interval of orthogonality satisfy $T_n(x) = \cos(n \arccos(x))$. Details about the construction, recursive formulas and more information can be found in Freund & Hoppe (2007).

[90] In the context of portfolio optimization Cass & Yaari (1967) and Merton (1971) show that random maturity valuation problems are equivalent to infinite time valuation problems with an adjusted discount rate. So surprisingly enough, randomization entails simpler valuation formulas than the non-randomized problem.

American option value is the solution to an optimal stopping problem this approximation constitutes a lower bound. Having established a valuation formula for an arbitrary exercise threshold they proceed assessing two particular thresholds similar in style to the approximation by Barone-Adesi & Whaley (1987). In Bjerksund & Stensland (2002) this approximation method is revisited by partitioning the lifetime of the option into two intervals to which the above technique is applied separately.

As with all other models there is also a strand of literature trying to relax assumptions or attempting to generalize the model framework. Chung & Chang (2007) provide analytical upper bounds for American options, which are closed-form when the corresponding European option is closed-form. Moreover, they are independent of the underlying stock price distribution and also hold in the multidimensional cases of stochastic interest rates etc. Moreover, European and American option pricing in the presence of transaction costs is investigated by Leventhal & Skorokhod (1997). It is shown that when the stock price moves with non-vanishing probability there is only the trivial hedge (i.e. the underlying stock itself) available for American options. For European options it is proved that this condition together with stableness of the price is necessary for only having the trivial hedge available. The paper draws on the results for discrete-time models, which are significantly simpler to handle. The problem with transaction costs in continuous-time finance is that perfect hedging entails an unlimited amount of trading and thus transaction costs to be paid, while in discrete time these costs are bounded. Option prices are then obtained by replacing the replicating portfolio of Black & Scholes (1973) by a superreplicating portfolio. Such a portfolio is no longer a perfect hedge but minimizes transaction costs and in this way resolves the tradeoff between hedging accuracy and trading costs incurred. These problems are dealt with by

Bensaid, Lesne, Pagès & Scheinkman (1992), Boyle & Vorst (1992), and Edirisinghe, Naik & Uppal (1993).

Besides stock options, options written on futures are also an area of active research. Whaley (1986*a*) gives a summary about the problems associated with futures option pricing. Most notably, the difference between European options and American options is elucidated, i.e. the problem of adequately determining the early exercise premium. The article summarizes what measures have been taken to deal with this problem. In Whaley (1986*b*) pricing techniques for American futures options are reviewed and applied to testing the efficiency of the futures market. No closed-form solution to the valuation problem is provided but computationally quick analytical approximations are followed. Barone-Adesi & Whaley (1987) propose a quadratic analytical approximation method for American options. In particular, this paper covers options written on commodities and commodity futures. Furthermore, it is pointed out, that contrary to call options on stocks it might be feasible to prematurely exercise them. Also, the method is compared against binomial trees and finite differences and found to be more efficient.

3.2.2.2 Lattice Models

In their simplest form lattice models are also known as binomial models and date back to the seminal work by Cox, Ross & Rubinstein (1979). As before, it is still assumed that there are no transaction costs, taxes, or margin requirements as well as there is unlimited possibility of short selling. Contrary to the models discussed before, though, there is no continuous time stochastic process governing the dynamics of the underlying assets or economic variables. Instead it is assumed that the underlying stock price S can either move up by a factor $u > 1$ to $u\,S$ or down by a factor $0 < d < 1$ to $d\,S$ in the next time period. In addition, we always assume that $u\,d = 1$, a property that renders the tree re-combining.

If there is only one period of time until expiry of an option with strike K, its values C_u in the upward and C_d in the downward case respectively are

$$\begin{aligned} C_u &= \max(u\,S - K, 0) \\ C_d &= \max(d\,S - K, 0). \end{aligned}$$

These states can then be replicated by a cash investment B in a riskless bond, which grows at the riskless rate r and the purchase of δ stocks. This is attained by choosing

$$\begin{aligned} B &= \frac{C_u - C_d}{(u-d)(1+r)} \\ \delta &= \frac{u\,C_d - d\,C_u}{(u-d)S}. \end{aligned}$$

To rule out potential arbitrage opportunities the option value C must be identical to the value of the replicating portfolio

$$\begin{aligned} C &= \delta S + B \\ &= \frac{1}{1+r}\left[\frac{1+r-d}{u-d}C_u + \frac{u-(1+r)}{u-d}C_d\right]. \end{aligned}$$

One can furthermore define so-called upward and downward probabilities and simplify the formula accordingly[91]

$$p = \frac{1+r-d}{u-d}$$

[91] These probabilities can be interpreted as risk-neutral upward and downward probabilities. Technically they add up to one and lie strictly between zero and one. Economically the definition of p and $1-p$ allows to write the option value as the expected value (w.r.t. this probability measure) of the outcomes in the possible states discounted at the riskless rate of return.

$$\begin{aligned} 1-p &= \frac{u-(1+r)}{u-d} \\ C &= \frac{1}{1+r}\left[p\,C_u + (1-p)\,C_d\right]. \end{aligned}$$

When there are more than one, say two for the moment, valuation periods, the valuation formula can be readily carried over and generalized. In such a scenario the underlying stock can either move up twice, move down twice or move once up and once down, leading to three possible states and option values C_{uu}, C_{ud}, C_{dd} after two periods which for the first period and present time fulfill[92]

$$\begin{aligned} C_u &= \frac{1}{1+r}\left[p\,C_{uu} + (1-p)\,C_{ud}\right] \\ C_d &= \frac{1}{1+r}\left[p\,C_{ud} + (1-p)\,C_{dd}\right] \\ C &= \frac{1}{1+r}\left[p\,C_u + (1-p)\,C_d\right]. \end{aligned}$$

Thus, the option price at present time is recursively defined by repeated application of the valuation formula for the respective subperiods. As elucidated in Cox, Ross & Rubinstein (1979) this can be further generalized to the case of n periods. In this case the option value reads

$$C = \frac{1}{(1+r)^n}\left[\sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!} p^j (1-p)^{n-j} \max(u^j d^{n-j} S - K, 0)\right].$$

According to Box, Hunter & Hunter (2005) $\frac{n!}{j!\,(n-j)!} p^j (1-p)^{n-j}$ is the density of a binomial distribution with probability p and n drawings which renders it quite intuitive how this model came to be called the binomial model. In order to understand how the early exercise feature of American options is addressed within binomial

[92] Please heed, that due to our above assumption of a re-combining tree, the order of the upward and downward move does not matter, i.e. $C_{ud} = C_{du}$. In essence, this is the reason why this property is called re-combining.

models one has to recall that such a valuation works backwards in time. This means that in a first step all values at maturity are determined. Having computed those, one can move back one step in time, carry out the discounting, and obtain the option values in that time layer. If there is furthermore a privilege to prematurely exercise the discounted value is replaced by the greater of this value and the immediate exericse value. In this case the above two-step scheme is altered to

$$\begin{aligned} C'_u &= \max(C_u, S - K, 0) \\ C'_d &= \max(C_d, S - K, 0) \\ C' &= \max(C, S - K, 0). \end{aligned}$$

In this very intuitive treatment and the fact that such a scheme can readily be implemented lies much of the appeal and attractiveness of the binomial model for practical purposes.

Geske & Shastri (1985*b*) examine the convergence theory and computational efficiency for the binomial and finite difference technique (the latter is covered below in Section 3.2.2.3) when there is one underlying stochastic variable. A survey of the convergence theory between discrete time and continuous time models to that time is presented by Willinger & Taqqu (1991). In particular, emphasis is put on structure preserving properties w.r.t. no arbitrage and completeness.[93] Duffie & Protter (1992) deal with weak convergence of sequences of trading strategies and asset price processes. If both have a weak limit the authors then deal with the question under which circumstances the corresponding financial gain process converges as well.[94] In Cutland, Kopp & Willinger (1993), the authors

[93]In this context structure preserving means that the approximate discrete-time market model exhibits the same structural properties as its continuous time counterpart. If, for instance, the continuous time model is arbitrage-free, so is the discrete-time model.

[94]For a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ weak convergence to X or conver-

present a different, stronger type of convergence than the commonly discussed weak convergence for the Cox, Ross & Rubinstein (1979) model. It is called D^2 convergence and a technical generalization of L^2 convergence.[95] Contrary to weak convergence it entails convergence of hedge portfolios and contingent claims. In addition, this type of convergence is further characterized in terms of non standard analysis.[96] Amin & Khanna (1994) deal with the convergence of American option prices computed with discrete models. It is shown that if the discrete model converges weakly to the respective continuous-time one, then so do the corresponding American option prices. In Leisen & Reimer (1999) the Cox, Ross & Rubinstein (1979) model is shown to be convergent with order one. Furthermore, the authors demonstrate how convergence order two can be obtained only by adjusting the up and down factors u and d of the binomial model. In Leisen (1998) the question of obtaining the order of convergence of the binomial model for American put options is revisited as it can serve as a measure of convergence speed. Furthermore, a method is presented to reduce the initial error of such a model. The questions of when weak convergence of a sequence of models under the respective empirical measures implies weak convergence under the risk-neutral measure and thus of derivative prices, is covered in Hubalek & Schachermayer (1998). Leisen (1999) examines a binomial model with random time steps. Weak convergence to the

gence in probability, which are used interchangeably, is said to hold if for all $\epsilon > 0$ the condition $\lim_{n\to\infty} P[|X - X_n| \geq \epsilon] = 0$ is satisfied. Furthermore, for a trading strategy θ and an underlying S the financial gain process Y_t is the process given by $Y_t = \int_0^t \theta dS$ which describes the profit yielded by the trading strategy that invests θ units in S.

[95] L^2 convergence means convergence w.r.t. to the L^2 norm, i.e. a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ is said to converge to X in L^2 if $\lim_{n\to\infty} E[|X - X_n|^2] = 0$.

[96] Non-standard analysis is a branch of mathematics reformulating well known results from standard calculus in terms of infinitesimal numbers, which are an extension to real numbers. Furthermore, there is a range of applications in the field of stochastic calculus as expounded in Albeverio, Fenstad, Høegh-Krohn & Lindstrøm (1986)

Black-Scholes process as well as convergence of the corresponding European and American option prices is shown. Furthermore, the author applies extrapolation to obtain order two convergence.

The binomial model as originally developed by Cox, Ross & Rubinstein (1979) is able to value contingent claims, when there is one underlying stock price. In the market, however, there are also options traded on more than one underlying stock, e.g. basket or other exotic options.[97] Naturally a demand for valuation techniques developed which could accomodate multiple underlying economic variables. In continuous time such a multidimensional lognormal model for stock prices reads

$$dS_t = b(S_t)dt + \sigma(S_t)dW_t, \tag{3.2.2.3}$$

where $b : \mathbb{R}^N \rightarrow \mathbb{R}^N$ and $\sigma : \mathbb{R}^N \rightarrow \mathbb{R}^{N \times N}$ satisfy the following Lipschitz regularity condition for some constant $L > 0$

$$|b(x) - b(y)| + |\sigma(x) + \sigma(y)| \leq L\,|x - y| \qquad \forall x, y \in \mathbb{R}^N.$$

He (1990) proposed an N-variate $(N + 1)$-nomial process and provides a complete convergence theory of both contingent claim prices as well as the dynamic replicating portfolio strategies to the multidimensional lognormal model.[98] The starting point of his construction of such a process is the approximation of the Brownian increments,

[97] An account of various types of exotic options is, for example, provided by Zhang (1998).

[98] He (1990) very intuitively pointed out that deriving such a model is trickier than it seems at first glance. The natural approach of having two processes approximate a two-dimensional model if one does so in one dimension falls short. This is due to the fact that two processes lead to four uncertain states of the economy in the next period, while there are only three assets (two stocks and a riskless bond) to trade in. Thus, the market is incomplete. In the resulting tradeoff between convergence of the model and market completeness previous attempts of formulating such a model, like e.g. by Boyle (1988), Hull & White (1988), Boyle, Evnine & Gibbs (1989) or Madan, Milne & Shifren (1989), fell short.

which are needed for each for each of the N dimensions and for each of the n time steps, i.e. random vectors $\epsilon^k = (\epsilon_1^k, \ldots, \epsilon_N^k)^T$ for $k = 1, \ldots, n$. These random vectors are then used to further construct the actual increments of the multinomial approximation process. Assuming a sample space $\Omega = \{\omega_1, \ldots, \omega_{N+1}\}$ with an equal probability distribution He (1990) sets

$$\epsilon_j^k(\omega_s) = e_{sj}, \qquad s = 1, \ldots N+1,\ j = 1, \ldots, N,$$

where the matrix $E = (e_{sj})_{sj}$ is given by

$$e_{sj} = \sqrt{N+1}\, a_{sj}$$

with $A = (a_{sj})_{sj}$ being an orthogonal and invertible matrix whose last column is given by $\frac{1}{\sqrt{N+1}}(1, \ldots, 1)^T$. This leads to the following multinomial processes for the stocks and the bond

$$\begin{aligned} S_{k+1}^n &= S_k^n + \frac{b(S_k^n)}{n} + \sigma(S_k^n)\frac{\epsilon^k}{\sqrt{n}} \\ B_{k+1}^n &= B_k^n \left(1 + \frac{r(S_k^n)}{n}\right). \end{aligned}$$

As it is the case with the option pricing and valuation techniques, there are also papers devoted to generalization, improvement and further applications of the existing models. In Rendleman & Bartter (1980) the application of the Rendleman & Bartter (1979) approach to stock option pricing as well as an extension to bond option pricing is discussed. Hull & White (1988) provide an assessment of the control variate technique in conjunction with option pricing lattices.[99] Its usefullness is demonstrated for the cases of dividend and non-dividend paying American put options. Amin (1991) considers general path-independent discrete models such as Cox, Ross

[99] An explanation of the control variate technique is presented below in Section 3.2.2.4 on Monte Carlo simulation, where this technique is more commonplace.

& Rubinstein (1979) to value American options. In addition, the author allows for time-dependent volatility which the numerics presented by Boyle (1988) and Nelson & Ramaswamy (1990) are not capable of accomodating. In addition, multidimensional extensions are also included which ensure better accuracy. In Nelson & Ramaswamy (1990) computationally simple binomial models (models whose number of nodes grows no more than linearly in the number of time intervals) are introduced and dealt with. They are shown to converge weakly to common diffusion models applied in finance such as the CEV model. Also the authors show the convergence of the corresponding European option prices.

Broadie & Detemple (1996) introduce a new way to compute American option prices. The key aspect of the approach is that two approximations (both an upper and a lower one) are obtained. The authors then show that this method significantly, around factor 20, improves the computational speed of the method. Furthermore, an enhancement of the Cox, Ross & Rubinstein (1979) binomial model is suggested. Figlewski & Gao (1999) present a method, called adaptive mesh method, which allows for different granularity of the grid of an option pricing tree in different regions and accordingly places the majority of the nodes in those region which exhibit the highest sensitivity to variations. By this procedure, well-known from the field of numerical analysis for PDEs the computational time of established valuations is significantly reduced and other previously non-feasible and by far too computationally cumbersome valuations become possible. The applicability to practically relevant option pricing problems is demonstrated for the case of barrier options. Bally, Pagès & Printems (2005) deal with pricing and hedging of American basket options. The numerical procedure revolves around computing conditional expectations on optimal grids which are designed to minimize the projection error which is based on Graf & Luschgy (2000).

3.2.2.3 PDE-based Methods

In Section 3.1.1 a partial differential equation for the value of a European option was derived. Furthermore, it was pointed out that this PDE is valid irrespective of the contingent claim under consideration. As a consequence, it is of utmost importance how initial (or terminal) and boundary values are specified as they are the lone remaining determinants of the option premium.

In the case of a European put option the terminal value is, for instance, given by the known exercise value of the option at maturity. This has to be combined with appropriate boundary data. At the boundary $S \to \infty$ it is reasonable to assume that the option becomes worthless, as the probability that the options ends up in-the-money at maturity vanishes. For stock prices equaling zero, put-call-parity (3.1.1.4) yields an appropriate boundary condition exploiting that a call option vanishes in that case, as it becomes an option on a worthless asset. Altogether, this constitutes the following terminal boundary value problem[100]

$$\begin{aligned}
0 &= \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial P^2} + r\,S\frac{\partial P}{\partial S} - r\,V \quad S \in \mathbb{R}^+, t \in [0,T) \\
P(S,T) &= \max(K - S, 0) \qquad S \in (0,\infty) \\
P(0,t) &= K\,\exp(-r(T-t)) \qquad \forall\, t \in [0,T) \\
\lim_{S\to\infty} P(S,t) &= 0.
\end{aligned}$$

Suitable boundary data for a wider class of options such as binary, digital, cash-or-nothing options or even combinations of different kinds of options can be found in the textbook by Wilmott, Howison & Dewynne (1993). From a mathematical point of view, these are

[100] In the textbooks by Alt (2006) and Evans (2010) it is established that without proper specification of boundary and initial/terminal values uniqueness of the solution cannot be attained.

very standard problems from the field of parabolic partial differential equations, which in many cases even yield closed-form solutions. This assertion extends even to more complicated models such as multivariate models or jump diffusions as they have been discussed in detail in Sections 3.1.2 and 3.1.3, where for the Merton (1976) and Heston (1993) models closed-form solutions are provided. We shall, however, restrain ourselves to the consideration of the problem in the Black-Scholes model to ensure that the solution pattern is not obscured in cumbersome and somewhat technical considerations (for example w.r.t. notation from vector calculus) that arise when dealing with multidimensional PDEs. If no such closed-form solution exists, there is a wide array of numerical techniques which are able to very efficiently cope with initial/terminal boundary value problems numerically, most prominently the approximation techniques of finite differences and finite elements in combination with the multigrid solver for linear systems.[101]

In the case of American options the right to exercise the option at any time not only changes the valuation problem economically but it also translates to a more complicated mathematical formulation. This entirely revolves around the fact that we still know the value of the option in the case of exercise, but we do not know anymore if or when this exercise occurs. Nonetheless we are able to characterize the problem further. From Section 3.2.1 we know that there is always a stock price S_f below which it is optimal to exercise the option. If exercise of a put option is optimal for S_f then so is the case for any $S < S_f$ as this only increases the exercise value which was already valued higher than continuation. On the other hand this threshold depends on time and satisfies $0 < S_f(t) < K$.[102]

[101] For finite differences and finite elements the reader is referred to the literature mentioned below in this section. The multigrid method solves linear systems in complexity $\mathcal{O}(N)$ and is due to Hackbusch (2003).

[102] In Section 3.2.1 it is shown that S_f is monotone increasing in t. Economically, this can be interpreted as follows: If at a time t_1 it is optimal to exercise the

If we denote the valuation region of the option by Ω this consideration constitutes a disjoint partition into an exercise and a continuation region

$$\begin{aligned}
\Omega &= \mathbb{R}^+ \times [0,T] \\
\Omega_{\text{continuation}} &= \{(S,t) \in \Omega | S > S_f(t)\} \\
\Omega_{\text{exercise}} &= \{(S,t) \in \Omega | S \leq S_f(t)\}.
\end{aligned}$$

Further exploiting that, as shown below in this section, P is continuously differentiable w.r.t. S at $S = S_f$ with

$$\frac{\partial P}{\partial S}(S_f, t) = -1 \tag{3.2.2.4}$$

this sets up the following valuation problem

$$\begin{aligned}
0 = \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial P^2} + r\,S\frac{\partial P}{\partial S} - r\,V \quad & S > S_f, t \in [0,T) \\
P(S,t) = \max(K - S, 0) \quad S \leq S_f, t \in [0,T) & \qquad (3.2.2.5) \\
\lim_{S\to\infty} P(S,t) = 0 \quad t \in [0,T) & \\
P(S,T) = \max(K - S, 0) \quad S \in \mathbb{R}^+ &
\end{aligned}$$

such that P and P_S are continuous at $S = S_f$.

Intuitively speaking, condition (3.2.2.4) means that at the point where exercise becomes optimal the value function of the option touches the payoff function tangentially, i.e. they coincide and share the same derivative. Why this is the case can be made plain in a two-step argument. The first step is to show that the derivative

option at $S_f(t_1)$ then at $t_2 > t_1$ it is also optimal to exercise at $S_f(t_1)$. This is the case because, all else held constant, the time value of the option is diminished as there is less time remaining for improvement. Therefore, the exercise payout can be reduced (which corresponds to an increase in S_f) by that loss of value without losing optimality of exercise.

cannot be less than -1, which follows immediately from the fact that $P(S,t) = K - S$ where $S \leq S_f(t)$.

The reverse assertion that $\frac{\partial P}{\partial S}(S_f, t)$ cannot be greater than -1 is shown using an arbitrage argument for the case of a Black & Scholes (1973) world. Assume a portfolio consisting of a put option and one share of the underlying stock, i.e. $\Pi := P + S$. Then by application of Itô's lemma we obtain

$$d\Pi = \begin{cases} 0, \ S < S_f \\ (\partial_S P + 1)\sigma \, S dW + \mathcal{O}(dt), S \geq S_f \end{cases}$$

Assume now that $\frac{\partial P}{\partial S}(S_f, t) > -1$ and buy one share of Π by borrowing $P + S$. If $dS > 0$ this implies $d\Pi > 0$ as $(\partial_S P + 1) > 0$. Unraveling the portfolio pays off $\Pi + d\Pi > \Pi$ and paying back the borrowed money after dt prompts a cashflow of $-\Pi - \underbrace{\Pi dt}_{=\mathcal{O}(dt)}$ and thus altogether an arbitrage profit of $d\Pi > 0$. Hence, this establishes $\frac{\partial P}{\partial S}(S_f, t) = -1$.

Mathematically speaking, (3.2.2.5) is a free boundary value problem. Except for very few exceptions these problems do not yield closed-form solutions.[103] Therefore, financial economists had to resort to numerical methods. Perhaps the best known such technique for partial differential equations is a so-called finite difference approximation, which is very appealing in its intuitive approach. The computational domain is partitioned with an equidistant grid in both stock price direction (let h denote the step size) and time direction (let τ denote the step size). Then at a grid point S_i the first and second derivatives w.r.t. the stock price can be approximated

[103] Examples from the field of finance include, for instance, the infinite maturity American put option, for which Merton (1976) provided a solution, or the American call option on non-dividend paying stocks, for which it can be argued that premature exercise is never optimal so that the valuation problem reduces to a classic terminal boundary value problem.

as follows

$$\begin{aligned} D_h P(S_i) &:= \frac{S_{i+1} - S_{i-1}}{2h} \\ D_h^2 P(S_i) &:= \frac{S_{i+1} - 2S_i + S_{i-1}}{h^2}, \end{aligned}$$

where D_h and D_h^2 denote the discrete first and second derivatives. In the standard literature about numerical analysis it is well established that these approximations yield second order convergence for sufficiently smooth P.[104] Concerning the time direction, three approximations have evolved, an explicit, an implicit, and the Crank-Nicholson scheme[105]

$$\begin{aligned} \frac{P_i^{n+1} - P_i^n}{\tau} &= F_i^n(P, P_S, P_{SS}, S, t) \\ \frac{P_i^{n+1} - P_i^n}{\tau} &= F_i^{n+1}(P, P_S, P_{SS}, S, t) \\ \frac{P_i^{n+1} - P_i^n}{\tau} &= \frac{1}{2}\left[F_i^n(P, P_S, P_{SS}, S, t) + F_i^{n+1}(P, P_S, P_{SS}, S, t)\right]. \end{aligned}$$

The first two are of first order convergence while the Crank-Nicholson scheme converges with order two.[106] When inserting these approximations into the original PDE, one obtains a linear system which can then be solved to obtain the desired approximative solution to

[104] Second order convergence means that $||D_h P - \partial_S P|| + ||D_h^2 P - \partial_{SS} P|| \leq Ch^2$ for some constant $C < \infty$. The proofs use Taylor expansion and can be found, for instance, in Deuflhard & Bornemann (2002), Stoer & Bulirsch (2005) or Hanke-Bourgeois (2009). In fact, second order convergence is a desirable property as it means that refining the grid by factor two reduces the error by at least factor four.

[105] In this notation F denotes the mapping that interprets the time derivatives in terms of the spatial derivatives. Such a notation is very commonplace in the study of ODEs (Ordinary Differential Equations), which boils down to stating properties F has to satisfy. For more information the reader is referred to the textbook by Walter (2000).

[106] See the literature above for proofs and more information on these so-called one step methods. They are called one step methods as they only make use of one previous time step when approximating the current one. By using more than one previous step (multi-step methods) higher order can be attained. Examples include Runge-Kutta methods and are outlined, for example, in Butcher (2003).

the PDE. However, since the PDE is only satisfied in the continuation region, these finite difference schemes cannot be employed without further adjustments. This is why financial economists have made up workarounds that adjust the price in every time step of the numerical solution process for optimality.

Brennan & Schwartz (1976) are to the best of our knowledge the first to use finite differences in financial economics in their valuation of life insurance contracts, which are reinterpreted in terms of options. Brennan & Schwartz (1977) suggest to apply finite differences to the valuation of American put options with finite lifetime and without dividend protection as there is no closed-form solution available. The model is, moreover, applied to a set of 55 put options traded in the New York dealer market between 1966 and 1969. In the paper by Brennan & Schwartz (1978) a summary of finite difference techniques for option pricing is provided. In particular, implicit and explicit discretizations are considered.[107] Courtadon (1982) introduces a Crank-Nicholson discretization and shows that it produces higher accuracy than the implicit approximation proposed by Schwartz (1977). The article by Brenner, Courtadon & Subrahmanyam (1985) deals with the differences between options on spot or cash instruments and futures options in a PDE formulation. They find that if there was a continuous payout on the spot instrument equal to the interest rate both instruments would be valued identically. Shu, Gu & Zheng (2002) value American put options in a finite difference setting. Their discretization relaxes the commonly observed singularity problems close to the exercise boundary and expiration date.[108] In Wang & Wang (2006) finite differences

[107] The terms explicit and implicit refer to whether at time n the solution at the next time step has to be obtained with (implicit) or without (explicit) solving a linear system of equations. The benefit of using implicit schemes lies in the fact that they are unconditionally stable w.r.t. the step size, while explicit schemes carry a maximal step size to be used which might entail a significantly higher number of time steps.

[108] In the context of partial differential equations a point of singularity is a point

are applied to solve parabolic degenerate variational inequalities as they appear in the valuation of American options.[109]

A mathematically more stringent way to tackle the problem of early exercise is to directly deal with the free boundary value problem. According to Jaillet, Lamberton & Lapeyre (1990) and Zhang (1997) the free boundary value problem (3.2.2.5) can also be written as

$$
\begin{aligned}
0 &\geq \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + r\,S\frac{\partial P}{\partial S} - r\,V \\
& \quad S \in (0,\infty), t \in [0,T) \\
P(S,t) &= \max(K-S,0) =: \Psi(S) \\
& \quad S \leq S_f, t \in [0,T) \\
0 &= \left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial P^2} + r\,S\frac{\partial P}{\partial S} - r\,V\right)(P-\Psi) \\
& \quad S \in (0,\infty), t \in [0,T) \\
P(S,T) &= \Psi(S) \quad S \in (0,\infty).
\end{aligned}
$$

This version of the problem formulation provides intuition why such problems are also called linear complementarity problems. Linearity is without question as P and all respective derivatives only appear linearly. Furthermore, as pointed out above, the option value coincides with the payoff in the exercise region, where only a partial differential inequality holds, and conversely the partial differential equation holds in the continuation region where the option value exceeds immediate payoff. Such a situation is referred to as complementarity.

where the differential ceases to be well-behaved. In numerical terms this usually causes stability issues for the discretization. Since the pricing PDE only holds in the continuation region the boundary to the exercise region is a natural region for singularity issues to appear.

[109] For an explanation of variational inequalities and how they appear in the context of American options, see below in this section.

This reformulation serves as the starting point of the formulation for the American option valuation as a variational inequality which is the linchpin of treating it in a mathematically effcient and consistent way. Feng, Linetsky, Morales & Nocedal (2011) show that the option price satisfies

$$
\begin{aligned}
P(0,S) &= \psi(S) - \bar{\psi}(S) \quad x \in \Omega \\
P(t,S) &= 0 \quad t \in (0,T], S \in \partial\Omega \\
P(t,S) &\geq \psi(S) - \bar{\psi}(S) \quad t \in (0,T], S \in \Omega \\
0 &\leq (P_t, w-P) + a(P, w-P) - (\mathcal{L}\bar{\psi}, w-P) \\
& \quad \text{for all test functions } w \geq \psi - \bar{\psi},
\end{aligned}
$$

where $\Omega = [x_{\min}, x_{\max}]$, $(\cdot,\cdot) : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is the scalar product on $L^2(\Omega)$, $\bar{\psi} \in C^2(\Omega)$ such that $\psi = \bar{\psi}$ on an open neighborhood of $\partial\Omega$ and the differential operator $\mathcal{L}$ and the bilinear form $a(\cdot,\cdot) : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ are given by

$$
\begin{aligned}
\mathcal{L}P &= \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + r\,S\frac{\partial P}{\partial S} - r\,P \\
a(u,w) &= \frac{1}{2}\sigma^2 \int_{x_{\min}}^{x_{\max}} u_x w_x \, dx - (r - \frac{1}{2}\sigma^2) \int_{x_{\min}}^{x_{\max}} u_x w \, dx \\
& \quad + r \int_{x_{\min}}^{x_{\max}} uw \, dx.
\end{aligned}
$$

This is the so-called weak formulation of the free boundary value problem.[110] The boundary conditions are reflected in the choice of the test functions and the space they are taken from.[111] Just as it is the case with the strong formulation, the weak formulation does not yield a closed-form solution either. But when it comes to

[110] The term weak formulation is used because this formulation does no longer include second derivatives which were replaced by test functions that only need to be integrable and differentiable in a weak sense, i.e. that they allow for integration by parts like a strong (traditional) derivative would at least do.

[111] A very meticulous description and construction of the respective so-called Sobolev spaces with which this is attained can be found in Alt (2006).

numerical approximations these are very well studied problems as they appear in numerous fields of science.[112] The natural way to discretize both variational equations and variational inequalities is via finite elements. Not only is the corresponding exact solution less restrictive than its strong formulation counterpart, but also finite elements exhibit better numerical properties than finite differences do.[113] The reason for the advantages are that finite elements approximate the space of test functions and via the integrals of the bilinear forms and the scalar products ensure that the entire domain contributes to the approximation rather than a very limited number grid points. Furthermore, they offer much more flexibility w.r.t. the choice of approximating functions, typically polynomials.

As it was the case above, discretization of the strong formulation with finite differences and discretization of the variational inequality using finite elements according to Seydel (2009) lead to discrete linear complimentarity problems of the following type for the desired solution w and $\nu = 1, \ldots \nu_{\max} - 1$

$$
\begin{aligned}
g &:= g^{\nu+1} \\
b &:= b^{\nu} \\
Aw - b &\geq 0 \\
w &\geq g \\
(Aw - b)^T (w - g) &= 0.
\end{aligned}
$$

Such a problem can be solved numerically using an adaptation of the SOR (Successive Overrelaxation) method by Cryer (1971) for

[112] Various examples are given in Friedman (1982) and Caffarelli (1998) which include, for instance, fluid filtration in porous media and elasto-plasticity.

[113] For extensive coverage of finite elements from a mathematical standpoint, including precise definitions and further applications, we refer the reader to the textbook by Braess (2003). An introduction to finite element methods in the field of finance is provided by Topper (2005). Error estimates for the finite element discretization can, for instance, be found in Brezzi, Hager & Raviart (1977).

standard linear systems of equations. Writing $x := w-g$, $y := Aw-b$ and $\hat{b} := b - Ag$ the algorithm reads

Algorithm 3.2.1 *For $k = 1, 2, \ldots$ and $i = 1, \ldots, m-1$*

$$\begin{aligned}
r_i^{(k)} &:= \hat{b}_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - a_{ii} x_i^{(k-1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \\
x_i^{(k)} &= \max\left(0, x_i^{(k-1)} + \omega \frac{r_i^{(k)}}{a_{ii}}\right) \\
y_i^{(k)} &= -r_i^k + a_{ii}\left(x_i^{(k)} - x_i^{(k-1)}\right)
\end{aligned}$$

In this solution pattern ω is called the relaxation parameter and influences the speed of convergence. In general, though, there is no optimal such ω and it must be determined based on the problem at hand.

In addition to this basic algorithm to solve linear complementarity problems, researchers have occupied themselves with improvements and various applications in the field of financial economics. Kočvara & Zowe (1994) are concerned with the efficient numerical solution of linear complementarity problems as they arise when discretizing the free boundary value problem of American option valuation. More precisely, the authors suggest a pre-conditioned CG (Conjugate Gradient) technique with a projection step, which they compare to established solution techniques. The method is found to perform remarkably well for several differently shaped obstacle problems.[114] In the paper by Dempster & Hutton (1999) linear programming techniques such as the simplex and interior point algorithms are applied to numerically solve the linear complementarity problems

[114] In the theory of partial differential equations and numerical mathematics obstacle problems are the canonical example of free-boundary value problems. For more details and examples see for instance Friedman (1982) and Caffarelli (1998).

of plain vanilla American put and lookback options. The performance is then compared to the standard PSOR (Projected Successive Overrelaxation) technique. Siddiqi, Manchanda & Kočvara (2000) address the application of the two-step algorithm presented by Kočvara & Zowe (1994) to the valuation of American options. Furthermore, from a modeling perspective inflation and devaluation by evolution equations are taken into account. The paper by Ikonen & Toivanen (2008) deals with the pricing of American options in the Heston (1993) stochastic volatility model. In this model they compare five solution techniques for the arising linear complementarity problems, the PSOR method, a projected multigrid method, an operator splitting method, a penalty method and a componentwise splitting methods. While the accuracy of the methods is roughly comparable, it turns out that componentwise splitting is the fastest scheme. Morales, Nocedal & Smelyanskiy (2008) improve the Kočvara & Zowe (1994) algorithm by combining the PSOR method with a subspace minimization step. In numerical experiments for ill-conditioned problems the method is found to perform better than interior point and gradient projection methods.

A generalized concept of solutions are so-called viscosity solutions. This notion has been established by Crandall & Lions (1983) and has more general applications in the solution theory of optimal control problems.[115] Crandall, Ishii & Lions (1992) cover modern techniques in that context for fully non-linear PDEs. In particular, they are concerned with uniqueness and existence results for which new arguments in a more general setup are provided. Application of the results extends to the field of financial economics, for instance in the context of American option pricing. Using the dynamic programming principle Benth, Karlsen & Reikvam (2003) derive a semi-linear PDE for the value of an American option in the Black &

[115] The link between viscosity solutions and optimal control problems is further explored in Sections 4.1.1 and 4.1.2.

Scholes (1973) market model with the non-linearity stemming from the dependence on the option value itself. As there is no clear-cut solution concept to this type of equation they suggest a viscosity solution approach and prove existence and uniqueness. Roch (1997) deals with the PIDE (Partial Integro Differential Equation) governing the American option value in the stochastic volatility model by Barndorff-Nielsen & Shephard (2001). If the payoff function fulfills a Lipschitz condition, the American option value is shown to be the unique viscosity solution to the PIDE.

In addition to the valuation itself, there is also literature dealing with different types of options such as futures options or path-dependent options or the mathematical properties of the option price. Ramaswamy & Sundaresan (1985) address the research question of the optimal exercise policy of futures options. It is found that in general, premature exercise can be optimal but that the associated early exercise premium is relatively small. This is carried out in the PDE setting provided by Black (1976). Finally, the authors extend their valuation model to stochastic interest rates, one of the most important value drivers of futures contracts, and they find the option values to be rather sensitive w.r.t. deviations from the long term mean. In Geske & Shastri (1985*a*) the early exercise of American put options is investigated. The authors show that American puts are exercised either immediately after dividend payments or just prior to maturity. To carry out their considerations of optimal early exercise they apply the framework of Black & Scholes (1973) and the PDE derived therein. In the paper by Jaillet, Lamberton & Lapeyre (1990) the regularity of the pricing functions of American options is discussed.[116] Furthermore, the authors ex-

[116] In the theory of partial differential equations regularity refers to the properties of functions w.r.t. integrability and differentiability. The higher the order of differentiability the smoother a function generally is. This is of interest because more regularity in the input data, i.e. the coefficient functions etc., typically leads to improved regularity in the solutions. Examples and more precise char-

amine a numerical technique based on variational inequalities. In Barraquand & Pudet (1994) and Barraquand (1996) the FSG (Forward Shooting Grid) method is introduced which is a numerical solution technique to cope with degenerate diffusion PDEs.[117] This is applied to path-dependent contingent claim valuation where these problems naturally arise by including the path-dependent variable as a state variable. Furthermore, the FSG method is shown to be unconditionally stable and it is the first to be able to deal with the early exercise feature of American claims.[118] Kholodnyi (1997) shows that American options with general time-dependent payoffs satisfy a semilinear Black-Scholes equation. The non-linear term in this equation is economically interpreted as the cost associated with suboptimally holding the option in its exercise region. Rambeerich, Tangman & Bhuruth (2011) solve the PIDE of the American option pricing problem in an infinite activity Lévy model using exponential time integration techniques whose numerical efficiency is assessed and highlighted by comparison to a more ordinary Crank-Nicolson scheme.

3.2.2.4 Monte Carlo Simulation

Besides analytical approximations, binomial or multinomial trees and PDE-based approximation methods there is also a class of methods that directly deals with the computation of the expected value of

acterizations can be found in the aforementioned textbooks by Alt (2006) and Evans (2010).

[117] A partial differential equation is called degenerate if the coefficients in the leading derivatives tend to zero on at least parts of the domain. This can entail a change in the type of equation, for instance from parabolic to hyperbolic, which significantly impacts the numerical stability of traditional solution techniques. See, for example, Hanke-Bourgeois (2009) for more information on that effect and ways of mitigation.

[118] In numerical analysis stability refers to the error propagation of a solution algorithm across the computational domain. Roughly speaking, stability is said to hold, if the approximation error is not magnified during iteration.

the option payoff. The central part of the approximation is the simulation of the underlying, which revolves around the discretization of the SDE (Stochastic Differential Equation) assumed to govern the underlying economic dynamics. In general, assume a multidimensional process of the type (3.2.2.3)[119], i.e.

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t.$$

In discrete time, this equation can according to Hull (2009) and Duffy & Kienitz (2009) be rewritten as

$$X_i(t+\Delta t) - X_i(t) = b_i(X(t))\,\Delta t + \sigma_i(X(t))\,\epsilon_i\,\sqrt{\Delta t},$$

where $X_i(\cdot)$ denotes the i-th component of the stochastic process, ϵ_i the i-th component of a random drawing of a multivariate normal distribution with $\rho_{ik}, 1 \leq i, k \leq N$ being the correlation coefficient between the respective components and $\Delta t = \frac{T}{m}, m < \infty$.[120] Altogether, a simulation consists of n simulated paths with m steps each, in which an N-dimensional random sample of a normal distribution has to be drawn.[121] Given an initial state $X(0)$ one can then compute N sample states for $X(T)$. If g denotes the payoff function of a contingent claim C its approximative value $\hat{C}_N$ corresponding to sample size N is then obtained as

$$\hat{C}_N = \frac{1}{N}\exp(-rT)\sum_{j=1}^{N} g(X_j(T)). \tag{3.2.2.6}$$

[119] Heed that this includes the models discussed in Section 3.1 w.r.t. multiple assets as well as multiple sources of risk such as stochastic volatility.

[120] Of course, this is only one way of discretizing an SDE. An account of alternatives, which are, in fact, closely related to discretization schemes for ODEs, is provided by Kloeden & Platen (1999), Kloeden, Platen & Schurz (2003) and Duffy & Kienitz (2009).

[121] How such random samples can be generated in an efficient manner is outlined below.

Given this setup, the entire simulation boils down to the drawing of multivariate normal vectors ϵ, which in turn consists of the following three steps:

1) Generation of uniformly distributed numbers $u_1, u_2, \ldots$ in the interval [0,1]
2) Transformation of these uniformly distributed numbers to random samples of univariate normal distributions
3) Aggregation of d such univariate normal random samples to a d-dimensional random sample.

Very commonplace to perform the first step is the use of linear congruential generators which work as follows:[122]

$$\begin{aligned} x_{i+1} &= (ax_i + c) \mod m \\ u_{i+1} &= \frac{x_{i+1}}{m}. \end{aligned}$$

Here, a, c and m are predetermined integers which have to be combined with an a priori determined seed x_0 serving as a starting point for the iteration. This type of generator goes back to Lehmer (1951), who originally introduced it. Marsaglia (1972) and Knuth (1998) then investigated desirable relationships between a, c and m to achieve that the sequence of random numbers has full period.[123] Reasonable choices of the parameters are, for instance, discussed by

[122] Besides this class of estimators one might also use inverse congruential estimators or feedback shift register (see Matsumoto & Nishimura (1998) and Eichenauer-Herrmann, Herrmann & Wegenkittl (1998) respectively for more detailed information and assessment of the sampling accuracy and quality).

[123] By the very construction of the algorithm it cannot produce more than m different values as there are no more than m possible remainders when dividing by m. Furthermore, any sequence entirely repeats itself once a number appears for the second time. Thus it is a desirable property that a sequence produces all m different values before the repetition occurs which Glasserman (2004) refers to as full period.

Lewis, Goodman & Miller (1969), Fishman & Moore (1986), Park & Miller (1988), and L'Ecuyer (1988), who also suggests amending the algorithm by combining several such generators. Further improvements are provided by Wichmann & Hill (1982) and L'Ecuyer (1996).

Having obtained a sequence of uniformly distributed random numbers $u_1, u_2, \ldots, u_k$ they have to be transformed to normally distributed random numbers. To do so, two broad classes of methods have evolved. The first uses the inverse distribution function of the desired distribution[124] and the other are acceptance-rejection methods.[125] The inverse distribution function F^{-1} is then evaluated at the random numbers $u_1, u_2, \ldots, u_k$ and $F^{-1}(u_1), F^{-1}(u_2), \ldots,$ $F^{-1}(u_k)$ are normally distributed. The fact that the inverse of the cumulative normal distribution function is not available in closed form has spawned approximations of the standard normal distribution function, for instance, by Beasley & Springer (1977) and Moro (1995), who reduces the sampling error to less than 3×10^{-9} in the seven standard deviations interval around the mean.[126] As an alternative sampling method for the normal distribution one might also use the Box & Muller (1958) algorithm.

The final step consists of assembling d univariate normally distributed random numbers to a drawing of a d-dimensional normal

[124] The distribution function of a probability distribution π is given by $F(x) = P[X \leq x]$ if X has distribution π.

[125] Acceptance-rejection methods date back to von Neumann (1951) and might be more feasible if the inverse distribution function cannot be readily evaluated or if, for whatever reason, there is another function g that can be more easily sampled than f with the property $f \leq c\,g$ for some constant c. A sample ζ is then accepted as a sample of f with probability $\frac{f(\zeta)}{c\,g(\zeta)}$. The latter can be attained by drawing a U uniformly distributed on $[0, 1]$ and accepting ζ if $U < \frac{f(\zeta)}{c\,g(\zeta)}$.

[126] Heed that it is not necessary to sample a normal distribution with mean μ and standard deviation σ as such a sample is readily available by virtue of the transformation $\bar{z} = \mu + \sigma z$, where z is the sample for the standard normal distribution.

distribution. This is readily performed by the following notion provided by Glasserman (2004): if $Z_1, Z_2, \dots, Z_d$ are univariately standard normal, then $Z := (Z_1, Z_2, \dots, Z_d)^T$ has multivariate distribution $N(0, I_d)$, where I_d is the d-dimensional unity matrix. Exploiting further that AZ then has distribution $N(0, AA^T)$, the sampling problem reduces to finding a matrix A with $\Sigma = AA^T$, where Σ is the variance-covariance matrix of the multivariate normal distribution to be sampled. Using Cholesky factorization a lower triangle matrix with this property can be attained so that $\mu + AZ$ is the desired random sample of the multivariate normal distribution.[127]

Having outlined how one can obtain the single ingredients of the Monte Carlo estimator (3.2.2.6), i.e. $\hat{C}_N = \frac{1}{N}\exp(-rT)$ $\sum_{j=1}^{N} g(X_j(T))$, we have yet to make plausible that such a scheme actually converges. The argument revolves around the central limit theorem[128] which implies that

$$\frac{\hat{C_N} - C}{\sigma_C/\sqrt{N}} \overset{N\to\infty}{\longrightarrow} N(0,1),$$

i.e. convergence in distribution, where C is the true option price and the expectation of the single realizations and where σ_C is the variance of the single realizations. Put differently, the error $\hat{C_N} - C$ of the estimator is approximately normal with $N(0, \frac{\sigma_C^2}{N})$, i.e. the standard deviation of the error is $\frac{\sigma_C}{\sqrt{N}}$. In other words, the standard error of the Monte Carlo estimator tends to zero at a rate $1/2$ as N approaches infinity.

[127] Details, properties and hints on how to efficiently implement this decomposition can be found in Freund & Hoppe (2007).

[128] The central limit theorem is one of the most important and most often utilized results from probability theory. Given independent and identically distributed random variables $X_1, X_2, \dots, X_n$ with mean μ and standard deviation σ their arithmetic mean $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges in distribution to $N(\mu, \frac{\sigma^2}{n})$. The precise assumptions to be able to prove this assertion together with various generalizations can be found in any textbook on probability theory, e.g. Bauer (2001).

Given this very general outline about how Monte Carlo simulation works, it becomes quite intuitive where the advantages of this approach are in comparison to the methods decribed above. Not only is it straight forward to include multidimensional models w.r.t. both assets and other factors such as volatility but also path-dependency is readily incorporated as, for instance, needed for barrier options, which are further discussed in Section 3.3. In particular, the applicability to multidimensional problems makes Monte Carlo simulation superior to classical numerical integration techniques, because the order of convergence 1/2 is independent of the dimension, while, as for instance Glasserman (2004) points out, the convergence order of the trapezoidal rule for numerical integration amounts to $\frac{2}{d}$, where d is the dimension. In practice this means, that at least beginning with four dimensional problems, the Monte Carlo scheme converges faster.

Boyle (1977) is the first to apply Monte Carlo simulation to option pricing and he demonstrates its usefulness with several numerical examples including call options on stocks that pay discrete dividends, jump-diffusion models and the use of a control variate to enhance convergence properties. In Scott (1987) randomly changing variance is introduced via a second underlying stochastic process for the stock's variance. When deriving an option pricing formula it is pointed out that it cannot be uniquely determined as volatility is not a tradable asset. However, if the processes are assumed uncorrelated the pricing formula collapses to an integral of the Black-Scholes formula and the probability distribution of the volatility. The model is applied to market data using Monte Carlo simulation. By Tilley (1993) it is demonstrated that Monte Carlo simulation can be used to value American-style contingent claims. The differentiation between optimality of exercise or continuation is made exploiting the lattice structure of the sample paths moving backwards from expiry where the exercise boundary is known using the methodologies outlined for lattice models in Cox & Rubinstein (1985). Longstaff & Schwartz (2001) propose least squares Monte Carlo. The method deals with the valuation of American contingent claims, which cannot be directly simulated because of the early exercise feature. The approach revolves around approximating the conditional expectation of continuation using least squares regressions. The usefulness of the method is pointed out by comparison against finite differences in a multidimensional case.

The central idea of this so-called least squares Monte Carlo lies in the approximation of the continuation value of the option along all simulated sample paths. The procedure Longstaff & Schwartz (2001) suggest goes as follows:

1) Simulate sample paths of the underlying for m time steps as if the option to be valued was European-style and determine the corresponding cash-flows from exercise at expiry and denote them by X.

2) For $j = m - 1, \ldots, 1$ determine the discounted cash flows from the in-the-money paths at step $j + 1$ and denote them by Y.[129]

3) For $j = m-1, \ldots, 1$ assume that the desired conditional expectation $E[Y|X]$ is a polynomial of X and determine the expectation by regressing Y on X.

4) For each sample path determine the option value as the higher of the just determined continuation value, the conditional expectation, and the intrinsic value of the option, i.e. the payoff if it was immediately exercised.

[129] Longstaff & Schwartz (2001) argue that omitting out-of-the-money paths not only enhances efficiency but also improves the quality of the approximation.

Rogers (2002) introduces a dual Monte Carlo approach for the valuation of American options.[130] He characterizes the American option by subtracting martingales (whose supremum w.r.t. time is still integrable of class L^1) from the discounted asset price process Z_t and minimizing over that class.[131] In addition, the method is assessed in numerical examples. Stentoft (2004*b*) lays the mathematical foundation for the least squares Monte Carlo method proposed by Longstaff & Schwartz (2001). He proves its convergence in two stages, first it is shown that the approximation of the conditional expectations converges and then it is shown that the numerical option price converges to the true one. Furthermore, so is done in a multidimendional multiperiod setting.

Since the order of convergence of the Monte Carlo simulation is relatively small with 1/2 researchers have developed methods that intend to improve the speed of convergence. Among the most prominent improvements are:

- *Antithetic paths*: The antithetic path of a simulated path is the sample path with the inverse increments at any time step, i.e. the reflection at the origin of the Brownian increment. In this way one not only obtains the same number of simulations a lot faster but also the variance of the estimated option value is found to be reduced. To gain intuition suppose that u is a drawing of a uniformly distributed variable on $[0, 1]$, then so is $1 - u$ with the property that this tends to pair low and

[130] In the context of American option pricing, which is typically formulated as choosing the maximum over all possible exercise policies, duality formulation means rewriting the pricing problem as a minimization over certain classes of supermartingales. In addition to its theoretical merit, this facilitates and enables the construction of upper bounds on the true price. Loosely speaking, any approximation with a supermartingal other than the true minimizer would be biased high.

[131] In fact, the author further proves that the minimum is attained by the martingale of the Doob-Meyer decomposition (cf. Karatzas & Shreve (2008)) of $Y_t^* = \text{ess } \sup_{t \leq \tau \leq T} E[Z_\tau | \mathcal{F}_t]$.

high realizations. Due to the monotonicity of the inverse distribution function F^{-1} this translates to the simulated path. If this distribution is, moreover, symmetric about the origin, as it is the case with the normal distribution, which we are interested in, this produces values with the same magnitude but opposite sign. In a natural way this reduces the variance of the entire sample as it adds realizations which are by construction negatively correlated to the existing ones.[132]

- *Using a control variate*: The control variate technique is often used when for a simpler version of the option the correct solution and thus the error $\bar{\epsilon} := v_{\text{exact}} - v_{\text{approx}}$ of the numerical solution is known. In such a case the simulated value is adjusted by the error term of the simpler version, i.e. the simulated value V is replaced by $\tilde{V} = V + \bar{\epsilon}$.[133]

- *Importance sampling*: The idea of importance sampling is the notion that there are regions which are more impactful to the value of the option, typically those where the convexity is most pronounced, i.e. around the at-the-money point. If that is the case placing more sample paths in that area to provide a better resolution of the critical area can be attained by a change of measure that overweights the sample paths in that area. However, to ensure that the Monte Carlo estimator remains unbiased the resulting paths must be given a lower weight again provided by the Radon-Nikodym derivative of the two measures.[134]

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[132] For more theoretical background on this variance reduction technique the reader is refered to Hammersley & Handscomb (1964) and Fox (1999). Applications in the world of finance include Boyle (1977), Fishman & Huang (1983) and Rubinstein, Samorodnitsky & Shaked (1985).

[133] A detailed exposition and justification of the method is given in the textbook by Glasserman (2004).

[134] In measure theory the Radon-Nikodym derivative is a function f satisfying $\nu(A) = \int_A f d\mu$ for two measures μ and ν and measurable sets A. The result

In addition to these variance reduction techniques which according to Glasserman (2004) do not improve the order of convergence but rather the constant in the error estimates, there is also a whole set of methods which improve the order of convergence to $1 - \epsilon$ for all $\epsilon > 0$. This technique deals with the placement of the random points in the d-dimensional hypercube.[135] To attain such a filling of the hypercube, special sequences are designed and used which heavily rely on insight from the mathematical field of number theory. Furthermore, they render the entire procedure purely deterministic contrary to classic Monte Carlo in which the drawing is stochastic or at least pseudo random. Thus, calling the placements random points might be slightly unfitting. Nonetheless, the term quasi Monte Carlo has prevailed for this class of methods.

In practice, the sequences suggested by Halton (1960) and Hammersley (1960) are widely used. Their sequence in dimension d is given by

$$\begin{aligned} x_k &= (\psi_{b_1}(k), \psi_{b_2}(k), \dots, \psi_{b_d}(k)) \quad k = 0, 1, \dots \\ \psi_b(k) &= \sum_{j=0}^{\infty} \frac{a_j(k)}{b^{j+1}}. \end{aligned}$$

The bases $b_1, b_2, \dots, b_d$ are chosen relatively prime[136] and the factors $a_j(k) \in \{0, 1, \dots, b-1\}$ are obtained as the coefficients when k is represented using[137]

$$k = \sum_{j=0}^{\infty} a_j(k) b^j.$$

in its general formulation is due to Nikodym (1930). A thorough analysis and derivation can furthermore be found in Karatzas & Shreve (2008).

[135] This filling of the hypercube is essentially the replacement of drawing random samples of uniformly distributed variables on $[0, 1]$.

[136] Two integers are called relatively prime if they do not share any common divisor other than 1.

[137] If the basis is ten, this is just the representation of numbers used in everyday life with $a_j(k)$ being the digits of k. But such a representation is, of course, also possible for any other basis.

Birge (1994) examines and demonstrates the usefulness of quasi Monte Carlo methods for the case of plain vanilla call options. Barraquand (1995) deals with the arbitrage pricing of contingent claims when there are many underlying sources of uncertainty and when there is no closed-form solution available. The numerical method of choice is Monte Carlo simulation for which an error reduction method, so-called quadratic sampling, is introduced. The benefit of this method is that it can be combined with other such techniques like e.g. importance sampling and is applicable and efficient for arbitralily many sources of uncertainty as demonstrated for the case of 100. In Barraquand & Martineau (1995) the numerical valuation of American contingent claims when there are many underlying sources of risk is investigated. As lattice methods face the drawback of running out of memory in such a case, this paper uses Monte Carlo simulation together with a certain partioning of the state space, so-called stratified state aggregation.[138] In a numerical example it is demonstrated that the method can be used for up to 400 risk factors. Boyle, Broadie & Glasserman (1997) review the Monte Carlo method and related methods to enhance the efficiency of the algorithm (variance reduction and quasi Monte Carlo) as well as applications to the computation of Greeks. Ibánez & Zapatero (2004) propose a fixed point algorithm to determine the optimal exercise boundary of Bermudan options which is then used as an input for classic European Monte Carlo simulation. Stentoft (2004*a*) evaluates the Longstaff & Schwartz (2001) paper. He points out that there are superior methods with regard to the efficiency and accuracy with which the price is computed, such as using ordinary monomials rather than Laguerre polynomials.[139] Furthermore, the

[138] This is a common numerical technique for the solution of optimal control problems. For a detailed description we refer the reader to the textbook by Kushner & Dupuis (2001).

[139] Laguerre polynomials are, similarly to Chebychev polynomials, orthogonal w.r.t. the L^2 scalar product. According to Freund & Hoppe (2007) they are given by $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n)$.

method is applied to multiasset options and found to be more feasible than finite differences or binomial models in these cases. Also, in Kohler, Krzyzak & Todorovic (2010) American options are valued using Monte Carlo simulation. The conditional expectation of not exercising the option is computed with a neural network regression.[140] Furthermore, convergence rates and consistency are examined.

Besides the classical Monte Carlo simulation technique, a number of authors have developed estimators for option prices based on different simulation methodologies or at least in part involving Monte Carlo simulation. Broadie & Glasserman (1997) introduce a simulation method consisting of two estimators, one upper biased and one lower biased. A combination of both is shown to be a good and efficient approximation of the true price of the American-style contingent claim when compared to lattice and finite difference methods. In the paper by Boyle, Kolkiewcz & Tan (2003) a technique is suggested which significantly improves the benefit of using a quasi Monte Carlo rather than a Monte Carlo method in the setting of Boyle, Kolkiewcz & Tan (2000) and Boyle, Kolkiewcz & Tan (2002) who combine the stochastic mesh method with quasi Monte Carlo techniques.[141] In Andersen & Broadie (2004) upper and lower bounds for American and Bermudan options are derived. The upper bounds, in particular, are obtained by Monte Carlo simulation of the dual representation of the value function as proposed by Rogers (2002) and Haugh & Kogan (2004). In the latter paper a general method is presented to obtain upper and lower estimators of the true option price which are obtained using Monte Carlo simulation. In the case of the upper bound this is justified by virtue of the dual minimization problem. Chaudhary (2007) suggests a simula-

[140] For an overview over neural networks the reader is referred to Bertsekas & Tsitsiklis (1996) and Bertsekas (1999).

[141] In Broadie & Glasserman (2004) an assessment of the stochastic mesh method is provided in the context of high-dimensional American options.

tion method for American option valuation using the Fast Fourier Transform. It is used to compute the convolution of a transition function of the underlying asset price process, which can be fairly general as its possible processes include the variance gamma process besides classical geometric Brownian motion. The PhD thesis by Holtz (2008) deals with high-dimensional numerical integration techniques which appear in the simulations associated with the valuation of financial assets or liabilities in a very comprehensive and general manner. It can thus be viewed as a collection of state of the art simulation techniques to that date.

3.2.2.5 Control and Stopping Problems

This section covers the valuation of American contingent claims in a much more mathematically rigorous way, deriving a formulation in terms of stochastic control theory.[142] Although the procedure might appear somewhat technical at certain times, doing so is in the very nature of American options. This is due to the fact, that they entitle their holders with the right to exercise completely at their discretion during the lifetimes of the options, which essentially means in mathematical terms that their is a control variable containing information on whether or not exercise has already taken place.[143]

[142] An early introduction to the field of stochastic control theory with focus on parametric optimization and optimal stochastic control for linear systems with quadratic criteria is given in the textbook by Åström (1970).

[143] In his survey paper Samuelson (1973) summarizes the mathematical contribution made to social sciences and economics in general and financial economics in particular. Especially stressed are the fields of general equilibrium equations, constrained programming equations, theory of portfolio optimization, maximizing utility, and dynamic programming of the Bellman type, the latter of which is of importance in the context of stochastic control problems and which is dealt with in Section 4.1.1.

The derivation or (mathematically) precise and rigorous statement of the valuation problem heavily relies on the foundational theory about continuous trading put forth by Harrison & Pliska (1981) (which was later rephrased by Taqqu & Willinger (1987) in terms of probabilistic rather than functional analytic arguments) and their follow-up paper Harrison & Pliska (1983).[144] In brief, these authors establish certain terms and make several assumptions explained below:

1) There is a probability measure, under which the discounted prices of risky assets behave as if they were martingales.

2) Investors' portfolios consist of one riskless asset (thought of as a bond) and d risky assets such as stocks, which can be continuously adjusted or rebalanced. The value of the portfolio at any time is termed the value process.

3) The allocation of capital to assets is called the trading strategy.

4) Any trading strategy is assumed to be self-financing, i.e. there are no cash withdrawals or infusions and any change in the value of the portfolio is accredited to rebalancing or change in value of the held assets.

5) A trading strategy is referred to as admissible if it corresponds to a value process which is a positive martingale.

6) Besides the aforementioned assets the market is also assumed to accomodate contingent claims, i.e. integrable random variables. Those contingent claims are called attainable, which are, in the almost surely interpretation, positive random variables and at

[144] In addition, there is the famous paper by Harrison & Kreps (1979) which can be thought of as the progenitor of the other two papers as it covers very similar questions in discrete rather than continuous time formulation. Also in Kreps (1981) the non-existence of arbitrage opportunities is related to an economic equilibrium as a necessary and sufficient condition.

their expiry time T equal the value process of some admissible trading strategy.

The abovementioned assets are assumed to be governed by Itô diffusion processes given by

$$\begin{aligned} dB &= r(t)B(t)dt, \quad 0 \le t < \infty \\ dS_i &= S_i(t)\left[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\right], \quad 1 \le i \le d, 0 \le t < \infty. \end{aligned}$$

$W(t) = (W_1(t), \ldots, W_d(t))^T$ is a d-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$ w.r.t. the filtration $\mathcal{F}_t$ induced by W, the bond is denoted by B and the risky assets (stocks) by $S_1, \ldots, S_d$.

In this setting, Harrison & Pliska (1981) prove their central result, that for every attainable claim, there is a unique price precluding arbitrage. With more rigor, this is expressed in terms of measures and market completeness. In this respect an equivalent measure Q to P is deemed a martingale measure if it renders the discounted (at the risk-free rate of return) asset price processes martingales.[145] Further, a market is termed complete if every integrable contingent claim is attainable. The relationship between the price π of a contingent claim X and an equivalent martingale measure Q then is uniquely given by

$$\pi = E^Q\left[\exp\left(-\int_0^T r(s)ds\right) X\right],$$

as it was established in the same piece of research and by Harrison

[145] Two measures Q and P are considered equivalent on $(\Omega, \mathcal{F})$ if they share the same null sets, i.e. $P(A) = 0$ if and only if $Q(A) = 0$. Please heed that by the first item above, it is assumed that there exists at least one such measure. Therefore, the following theorem does not claim existence of such a measure but merely characterizes conditions for uniqueness if at least one exists.

& Kreps (1979) for discrete time. In this terminology, the central result by Harrison & Pliska (1983) regarding market completeness and arbitrage-free pricing reads:

Theorem 3.2.2 *The following assertions are equivalent:*

i) The model is complete under the reference measure Q.

ii) The set of all equivalent martingale measures is a singleton, i.e. Q is unique.

Proof - Theorem 3.2.2:
See Harrison & Pliska (1983).

□

By virtue of the relationship between equivalent martingale measures and the prices of contingent claims this theorem essentially establishes the uniqueness of an arbitrage-free price if the market is complete.

Given the notion that in the aforementioned literature contingent claim prices have well been established to be discounted values of their respective expected payoffs, the seminal work by Harrison & Pliska (1981) omits and does not tackle the question of how the prices of American options can then be understood in that setting. This question is taken up in Bensoussan (1984), who indeed provides a rigorous formulation of the American contingent claim valuation problem.

His starting point is the model setup used by Harrison & Pliska (1981), i.e. his considerations revolve around replicating or hedging contingent claims by trading in a riskless bond and d risky stocks. Bearing in mind, that American-style options can be exercised at any time, which means that their payoff, governed by a payoff function g, can be cashed in at any time, the general valuation problem is to find a portfolio or trading strategy u such that

$$\begin{aligned} u(\nu(t)) &= \nu_0(t)B(t) + \sum_{k=1}^{d} \nu_k(t)S_k(t) \\ u(t) &\geq g(t) \quad \forall t \in [0, T) \\ u(T) &= g(T), \end{aligned} \tag{3.2.2.7}$$

where $\nu_0(t), \nu_1(t), \ldots, \nu_d(t)$ are the trading strategies describing how much investment capital is allocated to the respective assets and g is the payoff function of the option to be replicated. The latter conditions reflect that, of course, at any time prior to maturity the option value must exceed or equal its intrinsic value while they have to coincide at maturity. If that was not the case at some point in time, immediate exercise would be better and the valuation problem would be trivial. In addition, Bensoussan (1984) points out that two seemingly technical conditions have to be imposed, which on the other hand serve as the entry point for most of the economic underpinning of the valuation procedure. The first condition the American option price has to satisfy is the so-called protective hedge property, which reads

$$\begin{aligned} u(t_2) - u(t_1) \leq \int_{t_1}^{t_2} [\nu_0(s)\, dB(s) &+ \sum_{k=1}^{d} \nu_k(s)\, dS_k(s) \\ &+ \sum_{k=1}^{d} \nu_k(s) z_k(s) S_k(s)\, ds], \forall t_1 \leq t_2 \leq T, \end{aligned} \tag{3.2.2.8}$$

where z_k denotes a continuously paid dividend yield on asset k. The economic interpretation is that we are contrary to the European case (cf. the derivation of the Black-Scholes formula in Section 3.1.1) no longer interested in an exact replication or hedge but rather in a superhedge. This means that the proceeds from the hedge must at any time at least outweigh the gains from the contingent claim.

The second technical condition to be imposed requires the definition of an additional stopping time

$$\hat{\theta}_t := \inf\{s \geq t | u(s) - g(s) = 0\}.$$

For any time t this is the least time s such that the claim equals the value of immediate exercise. Naturally, this time is at most T, at which they coincide under any circumstances, and can be interpreted as the waiting time until exercise becomes optimal for the first and/or next time. This notion turns out very practical in the precise distinction between the European and the American case, which becomes manifest in the following constraint

$$u(s \wedge \hat{\theta}_t) - u(t) = \int_t^{s \wedge \hat{\theta}_t} [\nu_0(s)\, dB(\lambda) + \sum_{k=1}^{d} \nu_k(\lambda)\, dS_k(\lambda) \tag{3.2.2.9}$$
$$+ \sum_{k=1}^{d} \nu_k(\lambda) z_k(\lambda) S_k(\lambda)\, d\lambda], \forall s \geq t.$$

The intuition is that prior to $\hat{\theta}_t$ it is not optimal to prematurely exercise, therefore the hedge is supposed to be exact as it is the case in the European variant. Altogether this allows for the statement of the general American option valuation solution according to Bensoussan (1984).

Theorem 3.2.3 *Given the technical conditions* (3.2.2.8) *and* (3.2.2.9) *as well as the complete market assumption there is only one possible valuation function for American contingent claims in* *the sense of* (3.2.2.7)

$$u(t) = \text{ess sup}_{t \leq \theta \leq T} E^Q \left[g(\theta) \exp\left(- \int_t^\theta r(s) ds \right) | \mathcal{F}_t \right].$$

Proof - Theorem 3.2.3:
See Bensoussan (1984).

□

The theorem only states that there cannot be any other valuation function for an American claim. But what still needs to be verified is that the suggested solution is indeed a valuation function. This is a two-step procedure, which exploits the penalization technique and follows the scheme:[146]

1) Solve the penalized problem for u_ϵ and prove that $u_\epsilon \to u$ as ϵ tends to 0.

2) If the payoff function g is an Itô process then the limes u satisfies the technical conditions (3.2.2.8) and (3.2.2.9).

The interpretation of Theorem 3.2.3 is that holders of American options behave such as to maximize the expected payoff from the option over all possible exercise times θ. This general result establishes the fact that American option valuation is an optimal stopping problem.[147]

[146] The penalization technique revolves around considering deviations from the desired solution parameterized by ϵ such that the original problem is retained as ϵ tends to zero. The benefit stems from skillfully penalizing such deviations in a way that contrary to the original problem the penalized one allows for direct solution. In this way it only remains to be shown that not only the problem formulations converge but so do their solutions.

[147] Intuitively speaking, the essential supremum rather than the classical supremum is used because the function to be maximized is an expectation which is not sensitive to changes on null sets. Therefore, the supremum only has to hold up to null sets.

The downside of these results are the rather restrictive assumptions made in Bensoussan (1984) such as boundedness and regularity conditions on the payoff function not even satisfied by standard American options, which are needed for the penalization technique to be used. For example, the payoff function of a standard American call option is not bounded by a deterministic constant on $[0, T]$, but such a constant would rather depend on T. Furthermore, in that prototypical case there is no additional positive payoff function which pays out per unit time.

This is where Karatzas (1988) takes up this strand of research and devises a reformulation of the valuation problem yielding a relaxation of the above drawbacks and validity for standard American options. His approach revolves around minimizing the initial endowment required to form a hedging portfolio. His first step is the introduction of portfolio and consumption processes $\pi(t)$ and C_t. If $\pi_i(t)$ denotes the amount of money invested in asset i at time t, a portfolio process $\pi(t) = (\pi_i(t), 0 \leq i \leq d, \mathcal{F}_t, 0 \leq t \leq \infty)$ is an $\mathbb{R}^d$-valued, adapted process satisfying the square integrability condition

$$\sum_{i=1}^{d} \int_0^T \pi_i^2(s)ds < \infty \quad \text{P a.s. }, T < \infty,$$

which is imposed for mathematical reasons. A consumption process $C = \{C_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is progressively measurable w.r.t. $\{\mathcal{F}_t\}$, takes values in $[0, \infty)$ and satisfies P-a.s., $\omega \in \Omega$

1) $C_0(\omega) = 0$

2) $t \mapsto C_t(\omega)$ is non-decreasing, right-continuous.

Intuitively, consumption starts with 0 and cannot decrease if thought of as a cumulative process. The wealth process corresponding to a pair (π, C) of portfolio and consumption processes is governed by the SDE

$$\begin{aligned} dX_t &= \left[r(t)X_t + \sum_{i=1}^{d} \pi_i(t)\left(b_i(t) + z_i(t) - r(t)\right) \right] dt - dC_t \\ &\quad + \sum_{i,j=1}^{d} \pi_i(t)\sigma_{ij}(t)dW_j(t) \quad 0 \le t < \infty. \end{aligned}$$

As before, b denotes the drift or appreciation rate, z the dividend yield and σ the dispersion matrix. The economic interpretation is that the wealth on average develops at the riskless rate plus the respective excess return (dividend yield plus appreciation rate less riskless rate) for the proportion of assets respectively invested. Further, consumption must be subtracted from the wealth as it can be thought of as negative income. Finally, the uncertainty is added using the stochastic terms. The next step is the construction of an equivalent martingale measure, which is used to state the results brought forth by Karatzas (1988). To do so, we define the helping variable θ such that

$$\sum_{j=1}^{d} \sigma_{ij}(t)\theta_j(t) = b_i(t) + \mu_i(t) - r(t), \quad 0 \le t < \infty, 1 \le i \le d.$$

Then the exponential supermartingale[148]

$$Z_t = \exp\left(- \int_0^t \theta^T(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds \right) \quad \mathcal{F}_t, 0 \le t < \infty$$

becomes a martingale and for fixed $T > 0$ and lends itself to the

[148] In general, the exponential, so-called Doléans-Dade exponential, of a semimartingale X is given as the solution to the SDE $dY_t = Y_t dX_t$ with initial condition $Y_0 = 1$. It differs from the classic exponential in the subtraction of the quadratic variation. A detailed description along with conditions under which the Doléans-Dade exponential is a martingale can be found in Revuz & Yor (2010).

definition of a probability measure by virtue of

$$\tilde{P}_T(A) := E[Z_T 1_A],\ A \in \mathcal{F}_t.$$

Applying the change of measure rule due to Girsanov (1960) we obtain:[149]

1) P and $\tilde{P}_T$ are mutually absolutely continuous[150] and
2) $\tilde{W}(t) := W(t) + \int_0^t \theta(s)\, ds, 0 \leq t \leq T$ is an $\mathbb{R}^d$-valued Brownian motion on $(\Omega, \mathcal{F}_t, \tilde{P}_T)$.

Using this newly defined measure we can now further follow the considerations of Karatzas (1988) and reinterpret the notion of hedging portfolios in terms of portfolio and consumption processes. For a given time horizon $T > 0$ and a level of initial wealth $x \geq 0$, consider an admissible pair (π, C) of portfolio and consumption processes and let X denote the corresponding wealth process. We say that (π, C) is a hedging strategy against the American contingent claim with maturity T and payout f, $(\pi, C) \in \mathcal{H}(x, T)$, if for $\tilde{P}_T$-a.s., $\omega \in \Omega$, the following requirements hold

1) $A_t(\omega) := C_t(\omega)$ is a continuous, non-decreasing function
2) $X_t(\omega) \geq f_t(\omega)\ \forall t \in [0, T]$
3) $X_T(\omega) = f_T(\omega)$
4) For the stopping time $\tau_t := \inf\{t \leq s \leq T | X_s = f_s\}$ and for fixed $t \in [0, T]$: $A_t(\omega) = A_{\tau_t(\omega)}(\omega)$.

[149] The change of measure formula states that for a Brownian motion W and a process Y with $\int_0^T Y_s\, ds < \infty$ such that the process $L_t := \exp\left(-\int_0^t Y_s dW_s - \frac{1}{2}\int_0^t Y_s^2 ds\right)$ is a martingale, the probability measure Q with density L_T with regard to P renders the process $\tilde{Y}_t := W_t + \int_0^t Y_s ds$ a standard Brownian motion.

[150] Loosely speaking, absolutely continuous means that there is a probability density function for one measure w.r.t. the other.

These conditions correspond to the conditions imposed by Bensoussan (1984) meaning that the hedge is a superhedge until maturity, exact at maturity and whenever exercise is not better than continuation. The fair price of an American contingent claim V is then considered as the minimal necessary endowment x to set up a hedge, i.e.

$$V_0 := \inf\{x \geq 0 | \, \exists(\pi, C) \in \mathcal{H}(x, T)\}.$$

With all this notation in place we can state the general result by Karatzas (1988), which furthermore recombines his notion of American contingent claim valuation with the view of Bensoussan (1984).

Theorem 3.2.4 *The fair price at $t = 0$ for the American contingent claim with maturity T and payoff f is given by*

$$V_0 = u(0) := \sup_{\tau \in \mathcal{S}_{0,T}} \tilde{E}_T \left[f_\tau \exp\left(- \int_0^\tau r(s)\, ds \right) \right].$$

Moreover, there is a strategy $(\pi, C) \in \mathcal{H}(T, u(0))$ with corresponding wealth process $X = \{X_\tau, \mathcal{F}_\tau; 0 \leq t \leq T\}$ which is continuous and satisfies for every fixed $t \in [0, T]$

$$X_t = \operatorname{ess} \sup_{\tau \in \mathcal{S}_{t,T}} \tilde{E}_T \left[f_\tau \exp\left(- \int_0^\tau r(s)\, ds \right) \right]$$

Proof - Theorem 3.2.4:
See Karatzas (1988).

□

The first condition is essentially the same characterization Bensoussan found, namely that the American option value is obtained as an optimal stopping problem. The second statement characterizes the associated value function of the hedging portfolio which in its optimization coincides with the option value.

Samuelson (1965) presents a pricing scheme for warrants like, for instance, options based on arbitrage considerations.[151] Together with McKean (1965), who proves Samuelson's assertions, exact valuation formulas for perpetual warrants on lognormal processes are derived. However, this is not possible for the early exercise feature of American options. In the paper by Dalang, Morton & Willinger (1990) stochastic processes which can be transformed to a martingale under an appropriate change of measure are studied and characterized. This translates to securities markets as the question when arbitrage gains, so-called free lunches, are precluded. In other words, the authors show that a process must be a martingale under some measure, if it precludes arbitrage. In Jacka (1991) it is shown that pricing an American put option is equivalent to solving an optimal stopping problem and gives rise to a parabolic free boundary value problem. The main contribution according to the author is the verification of the essential uniqueness of the free boundary value problem. Myneni (1992) summarizes the main findings of the American options literature to that time. It is primarily concerned with the optimal stopping formulation and its relation to free boundary value problems and variational inequalities.

Karatzas & Xue (1991) deal with incomplete markets in the context of investment consumption processes for the sake of utility maximization. The authors demonstrate that under partial observations with the asset price history being the only available information these problems are easier to solve. Cvitanić & Karatzas (1992) cover constrained portfolio optimization, i.e. maximization of utility, for assets governed by Itô processes. The control problem is solved by embedding it in an equivalent unconstrained problem. The results are then applied to incomplete markets, short-selling and spreads

[151] Although very similar to options, warrants are not identical, as they are usually much less standardized and traded over-the-counter. More information on the differences is provided in Hull (2009).

between borrowing and lending rates. The paper by Cvitanić & Karatzas (1993) is an extension of Cvitanić & Karatzas (1992) and similar to El Karoui & Quenez (1995) and focuses, in particular, on different borrowing and lending rates and incomplete markets. Cvitanić & Karatzas (1996) then focus on hedging contingent claims in the presence of proportional transaction costs. The initial wealth needed to perform such a hedge is characterized as the supremum of all discounted claim values over all probability measures rendering the wealth process a supermartingale. Furthermore, shadow prices from the corresponding dual formulation can be used to price contingent claims.[152] Option pricing in the presence of portfolio constraints, such as restrictions to short-selling or incomplete markets, is considered by Karatzas & Kou (1996) assuming that no arbitrage is permitted by the prices of contingent claims. Rather than single unique prices, arbitrage-free intervals are obtained, that include the Black-Scholes price for European options. Furthermore, the endpoints of this arbitrage-free interval can be obtained via stochastic control problems of the Cvitanić & Karatzas (1992) and Cvitanić & Karatzas (1993) types. Uniqueness inside the arbitrage-free interval is obtained by virtue of utility maximization. El Karoui, Peng & Quenez (1997) provide a broad account and summary of the theory of backward SDEs and their applications in the field of finance. Karatzas & Kou (1998) extend the general pricing theory of Bensoussan (1984) and Karatzas (1988) who established that American-style contingent claims are the solutions to properly formulated optimal stopping problems in the presence of frictions such as different borrowing and lending rates, incomplete markets, restrictions on borrowing or short-selling. The article provides explicit computations for the abitrage-free interval of American call options in a

[152] The term *shadow price* originates from the field of constrained optimization. Technically speaking, it is the value of the Lagrange multiplier at the optimal solution. Interpreted economically, it can be thought of as the marginal utility gained when relaxing the constraint by one unit.

very sophisticated manner using stochastic control theory, optimal stopping, Doob-Meyer decompositions and convex analysis.

Shepp & Shiryaev (1993) introduce a new and untraded option, called Russian option, and explicitly value it in a closed-form solution using the Black & Scholes (1973) model.[153] The option is of infinite lifetime and criteria are stated under which configuration parameters of optimal behavior change. In the paper by Duffie & Harrison (1993) explicit valuation formulas for Russian options are given by determining the optimal exercise time. This is then specified as a first passage time.[154] Besides, the authors distinguish between the dividend paying and the non-dividend paying case and find that the rational economic value of the option is only finite, when there is a dividend. Finally, the results are extended to perpetual lookback options. Peskir (2005*b*) revisits the Russian option, this time dealing with a finite time horizon. Moreover, an integral equation for the optimal exercise threshold is established.

Gerber & Shiu (1994) value perpetual American call and put options using Esscher transforms and the optional sampling theorem.[155] For geometric Brownian motion down-and-out calls and Russian options are dealt with. Furthermore, the approach and methodology is related to the high contact condition as introduced in Section 3.2.1 and related to the first order condition for optimality due to Mer-

[153] A Russian option is a potentially perpetual put option which pays the maximum stock price at which the stock has ever traded during the lifetime of the option.

[154] A first passage time is the first time at which a stochastic process ever exceeds a certain threshold. In terms of exercise policies, optimal exercise times are often the first times an underlying breaks a certain optimal exercise threshold. Darling & Siegert (1953) derive first passage times for continuous Markov processess. The results are then extended to moving barriers in the paper by Tuckwell & Wan (1984).

[155] The optional sampling theorem states conditions under which the expected value of a martingale at a stopping time equals its initial value. Esscher transforms are commonplace in actuarial sciences and were introduced to insurance pricing by Esscher (1932). For probability densities $f(x)$ their Esscher transforms $f(x,h)$ are given by $f(x,h) = \frac{\exp(hx)f(x)}{\int_{-\infty}^{+\infty}\exp(hx)f(x)dx}$

ton (1973*b*). Gerber & Shiu (1996) extend the results of Gerber & Shiu (1994) for perpetual options to the case of two underlyings. Examples for such situations include, for instance, the Mandelbrot (1978) exchange option or options on the maximum of two stocks. In Lai & Lim (2004), the authors deal with American lookback options in an optimal stopping framework. More precisely, they approximate the stopping boundary with piecewise linear functions. Furthermore, a decomposition formula is derived and applied to examining the near maturity behavior of option prices. In a local volatility model Chevalier (2005) investigates the behavior of the critical stock price, i.e. the exercise threshold, of an American put option on a dividend paying stock near maturity. Two situations can be distinguished, the dividend rate being greater than the interest rate and vice versa. In the former case an expansion of the critical stock price from the value function of the corresponding optimal stopping problem turns out to be parabolic, while the situation is less regular in the latter case. The article by Evan, Henderson & Hobson (2008) covers the question of when to optimally sell a non-traded asset if the corresponding agent has a power utility function and can invest her other assets in a complete market. The solution to this mixed optimal stopping, optimal control problem is found to be the first time the option's value exceeds a certain proportion of the entire portfolio. Carmona & Touzi (2008) deal with the valuation of swing options in the Black & Scholes (1973) model.[156] This setup can be interpreted as multiple exercise of American contingent claims which is theoretically investigated and for which existence of such exercise strategies is proved. Hussain & Shashiashvili (2010) address the delta hedging of American options. They show that for any uniform approximation of the American option value function

[156] Swing options are options that allow their holders to exercise put or call options on a number of pre-specified times. More information on their valuation and applications in the fields of commodity and electricity options can be found in Jaillet, Ronn & Tompaidis (2004) and Kluge (2006) respectively.

at equidistant rebalancing points a discrete time hedging portfolio can be set up, which in turn uniformly approximates the perfect delta hedging portfolio.[157]

A good and mathematically oriented account about derivative pricing results in the Harrison & Pliska (1981) martingale setting is provided by Kallsen (1998) in his doctoral thesis. The thesis revolves around expounding on different market models and their inherent distributional assumptions as well as different markets, such as stock and interest rate markets, under the common, established and well appreciated mathematical framework of semimartingales.

3.3 Barrier Options

Barrier options are an extension of standard options with an additional feature, the barrier. As it is the case for their standard counterparts, there are also calls and puts as well as American-style and European-style exercise rights. However, the payoff of such an option is contingent on the barrier being breached.

Naturally, this gives rise to two fundamental classes of barrier options, knock-ins and knock-outs, whose names are to a large degree self-explanatory. Knock-ins only pay off, if they are activated by breach of the barrier, while knock-outs only pay off unless they are killed during the option's lifetime.

[157] Of course, in reality a perfect delta hedge is unattainable as it would require continuous time rebalancing. In this regard, uniform approximation of a value function is understood in the usual sense, that v_h uniformly approximates v if $\sup_{x \geq 0} |v_h(x, t_k) - v(x, t_k)| \leq C\,h$ for some constant $C < \infty$ and finitely many equidistant discrete times t_k. Please note, that this uniform approximation is much stronger than pointwise approximation as the same constant applies to all stock prices.

If the barrier is denoted by L, we can define the crossing time of the barrier as

$$\tau_L = \inf\{t > 0 | S_t = L\}$$

and distinguish between up-crossings ($S_0 < L$) and down-crossings ($S_0 > L$). Intuitively, in the case of call options up-crossings correspond to knock-ins and down-crossings to knock-outs. For put options it is the other way round. This gives rise to the following four cases, where T denotes the term to maturity and K the strike:

1) A down-and-out call pays off $\max(S - K, 0)$, if $\tau_L > T,\ S_0 > L$

2) An up-and-in call pays off $\max(S - K, 0)$, if $\tau_L \leq T,\ S_0 < L$

3) A down-and-in put pays off $\max(K - S, 0)$, if $\tau_L \leq T,\ S_0 > L$

4) An up-and-out put pays off $\max(K - S, 0)$, if $\tau_L > T,\ S_0 < L$.

The conditions $S_0 \neq L$ are trivial, since $S_0 = L$ would instantly eliminate the effect of the barrier. The same holds for knock-ins starting in the knocked-in region and knock-outs starting in the knocked-out region.

In practice, however, a slightly different type of barrier breach has been established, namely knock-ins and knock-outs only being allowed at prespecified times. Such options are referred to as discrete barrier options. More precisely, we assume that the barrier is only monitored at discrete times $t_1, \ldots, t_m$, where $t_i = i\,\Delta t,\ i = 1, \ldots, m$. Further denote the underlying asset prices by $\tilde{S}_i := S_{t_i}$. In this way, the crossing time of the barrier simplifies to

$$\tilde{\tau}_L = \begin{cases} \inf\{n > 0 | \tilde{S}_n > L\} \\ \inf\{n > 0 | \tilde{S}_n < L\} \end{cases} .$$

By this change, the four barrier cases transform to:

1) A discrete down-and-out call pays off $\max(\tilde{S}_m - K, 0)$ if $\tilde{\tau}_L > m$, $S_0 > L$.

2) A discrete up-and-in call pays off $\max(\tilde{S}_m - K, 0)$ if $\tilde{\tau}_L \leq m$, $S_0 < L$.

3) A discrete down-and-in call pays off $\max(K - \tilde{S}_m, 0)$ if $\tilde{\tau}_L \leq m$, $S_0 > L$.

4) A discrete up-and-out call pays off $\max(K - \tilde{S}_m, 0)$ if $\tilde{\tau}_L > m$, $S_0 < L$.

Regarding the existence of barrier options one can put forth investors' demands. This becomes plausible by considering an example: A down-and-out call is equivalent to a standard call if the barrier is not breached. Thus the holder of such an option has no upside potential relative to the holder of a plain vanilla option, but rather only the downside risk of suffering a knock-out and losing the entire investment. On the other hand, this renders the option less expensive. Consequently, it becomes a viable alternative if an investor is convinced that the knock-out scenario is highly improbable. In this way, investors can increase returns or lower costs if the options are used for hedging purposes.

When it comes to real-world financial markets the discrete-time versions of the options are more commonplace. Once again this is because of very practical reasons. In integrated and globalized markets virtually all underlyings are traded (or at least can be traded) anywhere in the world at any time. In particular, this might even prompt a knock-out or knock-in during non-trading hours in the investor's country. Because of this it appears to be a natural wish by market participants to limit potential barrier actions to their own trading hours.

Besides these simplest continuous-time and discrete-time barrier options, options with another common feature have emerged, a so-called rebate R. This rebate is an amount of money paid in the original case of worthless expiry. More precisely:

1) A down-and-out call pays off R upon being knocked out

2) An up-and-in call pays off R if it is not activated until maturity

3) A down-and-in put pays off R if it is not activated until maturity

4) An up-and-out put pays off R upon being knocked out.

Of course, standard barrier options are only a special case of rebate barrier options, namely those with rebate $R = 0$.

Regarding valuation, one can make certain model-independent statements about standard barrier options. The first reduces the valuation problem to knock-out options as knock-ins can be replicated as a combination of a standard option and a knock-out. In detail, we have

$$C_{\mathrm{KI}} = C - C_{\mathrm{KO}} \tag{3.3.0.10}$$

$$P_{\mathrm{KI}} = P - P_{\mathrm{KO}}. \tag{3.3.0.11}$$

Here C stands for a call option and P for a put option. The subscripts KI and KO indicate knock-ins and knock-outs respectively, while no subscript indicates a plain vanilla option. These relationships can be made plain by looking at the options' payoffs. If an option has not been knocked-in, its value is zero. Conversely, the corresponding knock-out option in this case is equivalent to the standard option. If the option, however, is knocked in, it is equivalent to its standard counterpart, while the knock-out is worthless. Apparently, these considerations hold true independently of whether call or put options are investigated. By standard arbitrage arguments it follows that the replication formulas not only hold at maturity but at any point in time.

Having classified barrier options in general and outlined several of their basic properties and relations to one another, the question of how to determine the fair option premium naturally arises. This question was comprehensively answered in the paper by Kunitomo & Ikeda (1992).

Actually the authors solve a more general option pricing problem in their paper as they impose two knock-out barriers. In this way the option is nullified if it leaves the corridor between the two boundaries. Because of this property such options are also referred to as double barrier or corridor options. Furthermore, the authors deal with exponentially curved boundaries given by

$$\begin{aligned} y_1 &= B \exp(\delta_1 \, u) \\ y_2 &= A \exp(\delta_2 \, u), \end{aligned}$$

where $B \geq A > 0$ and $y_1 \geq y_2$, which implies that the two boundaries do not intersect. Obviously, by equating $\delta_1 = 0$ or $\delta_2 = 0$ the curved boundaries become flat again, as it is the case with standard barrier options. Letting $B \to \infty$ or $A \to 0$ transforms the problem to the one barrier scenario.

Although this setup appears technical at first glance or far in excess of the original interest of pricing standard barrier options, it is not without great practical relevance. This becomes manifest by abstracting from barrier option pricing and taking OETCs into account, which are the main subject matter of this thesis. As it is detailedly described in Section 2.3.2, these products exhibit exactly a barrier of this exponentially curved type. Also double barrier options are highly relevant for practical purposes.[158] And of course, being able to give a unified view and pricing of flat and curved as well as single and double barrier options is of great merit as well. This leads to consistent pricing and nests the more special cases in natural fashion.

The only drawback certainly lies the fact that Kunitomo & Ikeda (1992) only employ the Black & Scholes (1973) model (3.1.1.1) presented in Section 3.1.1. This shortcoming on the other hand is presumably caused by the complexity the pricing problem still exhibits.

As it has been established in Section 3.2.2.5 and the references therein, the option pricing problem can be stated as follows, if the two boundaries are taken into account

$$\begin{aligned} C(t) &= E\left[\exp(-rT)\max(S_T - K, 0)|S_t = S\right] \\ &= \int_K^y \exp(-rT)s(T)f(s(T))ds(T) \\ &\quad -E\left[\int_K^y \exp(-rT)f(s(T))ds(T)\right], \end{aligned}$$

where $f(s(T))$ denotes the probability density function of S_T for starting at $S_t = S$. The main ingredient to obtain a quasi-closed form solution (up to evaluations of the normal distribution in an infinite series) is the probability of the stock price not leaving the

[158] Taking up the above example of a speculator or hedger who does not believe in prices rising above or falling below the barrier, double barrier options are a natural means of extending this rationale. In this way traders who do not expect underlyings to leave a certain corridor can save more money on their investments or thus increase returns or lower the costs of hedging. If the boundaries are exponentially curved, in addition, the relative distance to the barrier can be kept constant over the course of time as the barrier then moves in unison with the underlying which, in a risk-neutral world, grows at the risk-free rate of interest. This effect further diminishes the costs of the investment.

active area of the option, which is known as the Lévy formula in probability theory for the case of standard Brownian motion.[159]

For this purpose we introduce the minimum and maximum asset prices $L(t)$ and $M(t)$ given by

$$\begin{aligned} L(t) &= \min_{0\le u\le t} S_u \\ M(t) &= \max_{0\le u\le t} S_u. \end{aligned}$$

Then we can state the following theorem brought up and proved by Kunitomo & Ikeda (1992).

Theorem 3.3.1 *Suppose* $\{S_t\}_{t\geq 0}$ *follows the geometric Brownian motion given by* (3.1.1.1) *with* $S(0) = S_0$ *and* $I \subset [A \exp(\delta_1 T),$ $B \exp(\delta_2 T)]$*. Then the probability that*

$$A \exp(\delta_1 T) < L(t) \leq M(t) < B \exp(\delta_2 T) \quad \forall t \in [0, T]$$

and $S_T \in I$ *is given by*

$$P_I = \int_I \left[\sum_{n=-\infty}^{+\infty} k_n(y) \right] \frac{dy}{y}$$

where

$$\begin{aligned} k_n(y) &= \left(\frac{B^n}{A^n}\right)^{c_{1n}} \left(\frac{A}{S_0}\right)^{c_{2n}} f_N\left(\frac{\ln(y) - \ln(\frac{S_0 B^{2n}}{A^{2n}}) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \\ &\quad - \left(\frac{A^{n+1}}{A_0 B^n}\right)^{c_{3n}} f_N\left(\frac{\ln(y) - \ln(\frac{A^{2n+2}}{B^{2n} S_0}) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \\ c_{1n} &= 2\,\frac{r - \delta_2 - n(\delta_1 - \delta_2)}{\sigma^2} - 1 \end{aligned}$$

[159]For more information and other applications of this formula the reader is referred to Anderson (1960).

$$
\begin{aligned}
c_{2n} &= 2n\,\frac{\delta_1-\delta_2}{\sigma^2}\\
c_{3n} &= 2\,\frac{r-\delta_2+n(\delta_1-\delta_2)}{\sigma^2}-1
\end{aligned}
$$

and $f_N(\cdot)$ is the density function of the standard normal distribution.

Proof - Theorem 3.3.1:
See Kunitomo & Ikeda (1992).

□

In this theorem the probability that the option does not knock out until time T has been comprehensively characterized in terms of the minimum and maximum asset prices strictly staying within the bounds imposed by the two barriers. This result, however, comes at the expense of only being available as an integral over an infinite sum, rather than a palpable closed-form expression. Nonetheless, the formula exhibits some economic intuition after all. Without the barriers, i.e. the classic Black-Scholes case, there would not be a knock-out probability, and thus $P_I = 1$. Furthermore, the growing speeds δ_1 and δ_2 of the barriers only enter the coefficients of the normal distribution density function but not the probability distribution itself. This is due to the fact, that an altered growing speed does not affect the distribution of the stock price at time T but only whether or not a prevailing state amounts to a knock-out. Therefore, the different outcomes are weighted differently in case there are other speeds in place. Finally, the growing speeds mostly enter the equation as the difference between the two speeds. As an increase in the lower barrier renders to the knock-out probability higher while this relationship is reversed for the upper barrier, it appears reasonable that the key ingredient is whether the asset price moves closer to either barrier, which is determined by the difference between δ_1 and δ_2.

A proof, why this formula holds, would exceed the scope of this thesis and distract from summarizing the key results and insights regarding barrier option valuation. Nonetheless, the proof can be found in Kunitomo & Ikeda (1992). In addition, Theorem 3.3.1 allows the statement of the general double barrier option pricing result with curved boundaries as put forth in the same piece of research:

Theorem 3.3.2 *The value of the call option at t, which is extinguished before its maturity date whenever the underlying asset price $\{S_t\}_{t\geq 0}$ reaches the upper boundary $B\ \exp(\delta_1\, u)$ or the lower boundary $A\ \exp(\delta_2 u)$ for any $u \in [t, T]$ is given by*

$$
\begin{aligned}
C(t) = S \sum_{n=-\infty}^{+\infty} \{ & \left(\frac{B^n}{A^n}\right)^{c_{1n}^*} \left(\frac{A}{S}\right)^{c_{2n}} [N(d_{1n}) - N(d_{2n})] \\
& - \left(\frac{A^{n+1}}{B^n S}\right)^{c_{3n}^*} [N(d_{3n}) - N(d_{4n})]\} \\
& - E\exp(-r\hat{t}) \sum_{n=-\infty}^{+\infty} \{ \frac{B^n}{A^n}^{c_{1n}^*-2} \frac{A}{S}^{c_{2n}} \\
& [N\left(d_{d1} - \sigma\sqrt{\hat{t}}\right) - N\left(d_{2n} - \sigma\sqrt{\hat{t}}\right)] \\
& - \frac{A^{n+1}}{B^n S}^{c_{3n}^*-2} [N\left(d_{3n} - \sigma\sqrt{\hat{t}}\right) - N\left(d_{4n} - \sigma\sqrt{\hat{t}}\right)]\}
\end{aligned}
$$

where

$$
\begin{aligned}
c_{1n}^* &= 2\,\frac{r - \delta_2 - n(\delta_1 - \delta_2)}{\sigma^2} + 1 \\
c_{3n}^* &= 2\,\frac{r - \delta_2 + n(\delta_1 - \delta_2)}{\sigma^2} + 1 \\
F &= B\ \exp(\delta_1\, T) \\
\hat{t} &= T - t \\
d_{1n} &= \frac{\ln(SB^{2n}/EA^{2n}) + (r + \sigma^2/2)\hat{t}}{\sigma\sqrt{\hat{t}}}
\end{aligned}
$$

$$
\begin{aligned}
d_{2n} &= \frac{\ln(SB^{2n}/FA^{2n}) + (r+\sigma^2/2)\hat{t}}{\sigma\sqrt{\hat{t}}} \\
d_{3n} &= \frac{\ln(A^{2n+2}/ESB^{2n}) + (r+\sigma^2/2)\hat{t}}{\sigma\sqrt{\hat{t}}} \\
d_{4n} &= \frac{\ln(A^{2n+2}/FSB^{2n}) + (r+\sigma^2/2)\hat{t}}{\sigma\sqrt{\hat{t}}}.
\end{aligned}
$$

Proof - Theorem 3.3.2:
See Kunitomo & Ikeda (1992).

□

As for Theorem 3.3.1, at first glance it is hard to find the economic intuition behind the result. Anyhow, the result can be regarded as a generalization of the Black-Scholes formula (3.1.1.3). By eliminating the barriers, i.e. $A = 0$ and $B \to \infty$, the Black-Scholes formula is retained, as in this case d_{2n}, d_{3n}, and d_{4n} vanish. If there are barriers the reflection principle implies higher order terms with $n \neq 0$ as Kunitomo & Ikeda (1992) emphasize.[160] In this way the authors have provided a comprehensive view on barrier option valuation in the Black & Scholes (1973) world which nests the classic cases without any barriers and thus is a natural and smooth extension of previous research. Therefore, their results appeal from both a mathematical and an economic point of view.

In addition to these results, there is a lot of literature on barrier options in general. In the Black & Scholes (1973) model Carr (1995) extends barrier option pricing in two ways. First, a protection period is introduced during which knock-outs are precluded and second,

[160] The reflection principle for standard Brownian motion states that the stochastic process which is reflected at time T is a standard Brownian motion as well. This result can be found in any textbook on stochastic calculus and Brownian motion, e.g. Karatzas & Shreve (2008). In our case, these reflecting times are the hitting times of the barrier and for any extinguishing path there is a non-extinguishing antithetic path, the probability of which is given as one minus the knock-out probability along the path. Therefore, we have to consider all possible hitting paths of the barriers which leads to the infinite sum.

knock-out events are contingent on a second asset touching a barrier. For both cases closed-form solutions are provided. In addition, a survey of the literature on barrier option pricing to that point is provided. Hui (1996) deals with double barrier binary options in a Black & Scholes (1973) setting. Besides, applicability of these options to trading is discussed. Roberts & Shortland (1997) deal with barrier options in an altered Black & Scholes (1973) model, in which the interest rate and volatility are deterministically time-dependent. First, the authors provide explicit valuation formulas and then apply the result of Roberts & Shortland (1995) to estimate barrier crossing times which are not available in closed-form, for example if the barrier depends on time. In the paper by Haffner & Loistl (1999) the authors provide an introduction and summary to the basic properties and fields of application for barrier options. In particular, emphasis is put on the possibilities barrier options provide when it comes to trading strategies. It is highlighted that fund managers are enabled to both, a cost reduction in risk management as hedging strategies are typically less expensive and improved investment opportunities as barrier options can increase the leverage of an investment position and are tailored more closely to the needs and demands of investors. In their follow-up article Haffner & Loistl (2000) barrier options are revisited. The main focus of this paper lies on hedging strategies for barrier options and the properties of their Greeks in relation to those of plain vanilla options without a barrier. Wystup (2002) provides an overview over the difficulties of pricing exotic options in general, and barrier options in particular, compared to the Black & Scholes (1973) model for standard options. Moreover, a plethora of valuation formulas and Greeks for various exotic options is provided.

Apart from general considerations of barrier options the question of hedging these products has spurred a lot of research as well. It can be loosely classified in two strands, static hedging and more complex hedging.

Bowie & Carr (1994) present static hedging strategies for barrier options. These hedges revolve around forming exact hedges of the exotic options by using standard options. In this way valuation boils down to the valuation of standard call and put options and hedging costs are minimized as the hedge does not require continuous rebalancing of the hedging portfolio. Only at the time when the barrier knocks in or out a one-time change is required. In the paper by Derman, Ergener & Kani (1994) approximate static replication is used for exotic options, which are decomposed into standard options. In this way hedging can be attained without continuous rebalancing of the replicating portfolio as well. Carr, Ellis & Gupta (1998) provide a static hedging strategy for barrier options which is based on a generalized put-call-symmetry.[161] The strategy allows for constant volatility and volatility being a function of the forward price. The benefit of this methodology lies in the high sensitivity to volatility changes exhibited by barrier options. This vega risk renders dynamic hedging rather expensive. Metwally & Atiya (2002) apply Brownian bridge techniques and Taylor expansion to approximate the resulting integrals in the context of Monte Carlo simulation to speed up classic Monte Carlo valuation by factor 100.[162] AitSahlia, Imhof & Leung Lai (2004) adjust classic binomial and trinomial tree methods to cope with the optimal stopping problem posed by American knock-in options. The necessity for adjustment stems from the fact that the American variant of barrier options does not allow for a decomposition of the type given in equations (3.3.0.10) and (3.3.0.11),

[161] Carr, Ellis & Gupta (1998) prove that for frictionless markets, no arbitrage and zero drift $C(K_1)/\sqrt{K_1} = P(K_2)/\sqrt{K_2}$, where the forward price F satisfies $\sqrt{K_1 K_2} = F$.

[162] According to Glasserman (2004) a Brownian bridge is a continuous-time stochastic process whose probability distribution coincides with the one of a Brownian motion contingent on $B(0) = B(1) = 0$.

since the respective suprema of exercise are generally attained at different times. Furthermore, this also renders the American knock-in option pricing problem non-Markovian which leads to a break-down of the methods established for solving free boundary value problems. Bernard, Le Courtois & Quittard-Pinon (2005) develop a numerical inverse Laplace transform technique to price Parisian options and derive their Greeks.[163] In Nalholm & Poulsen (2006), static hedges for barrier options are compared to dynamic delta hedging strategies in terms of model risk. The static hedges include standard options as hedging instruments and are found to outperform classical dynamic hedging. Furthermore, these results for the Black & Scholes (1973) model are shown to carry over to more sophisticated extensions like the CEV model, the Heston (1993) stochastic volatility model, and the Merton (1976) jump-diffusion model.

Geman & Yor (1996) draw on the double barrier option results by Kunitomo & Ikeda (1992) who derive a representation as an infinite series. In their paper, though, the Laplace transform w.r.t. to maturity of the double barrier option price is expressed in comparatively simple fashion using the methodology of Geman & Yor (1992) and Geman & Yor (1993) for Asian options.[164] Using an efficient inverse Laplace transform, the option price is readily attained. Numerical considerations suggest that practical merit lies in the observation of hedging being as efficient as pricing, although both suffer from the downsides and weaknesses of Monte Carlo simulation as discussed in Section 3.2.2.4. Pelsser (2000) is concerned with the valuation of double barrier options paying a rebate. Furthermore, double knock-

[163] These options have been suggested by Chesney, Jeanblanc-Piqué & Yor (1997) and are generalizations of standard barrier options in the fact that their payoff depends on the time the underlying spends beyond the barrier. More precisely, they are activated or knocked out once the underlying has stayed beyond the barrier for a pre-specified amount of time.

[164] According to Arendt, Batty, Hieber & Neubrander (2011) the Laplace transform F of a function f is given as the integral transformation $F(s) = \int_0^\infty \exp(-st)f(t)dt$ where $s \in \mathbb{C}$.

in barriers are discussed. The authors use Laplace transforms to option prices, but contrary to Geman & Yor (1996) they derive analytical rather than numerical solutions. To obtain their result they employ contour integration.[165] Carr & Chou (2002) value complex barrier options by hedging them with standard options, which only have to be rebalanced at certain times. The technique is applied to partial barrier options, forward starting barrier options, double barrier options, rolling options, ratchet options, and lookback options[166]. In their paper, Gobet & Kohatsu-Higa (2003) employ Malliavin calculus to the pricing of barrier and lookback options and the determination of their Greeks.[167] The improvements revolve around an additional dominating process to cope with the lack of differentiability of minimum and maximum processes required to represent these entities. In a numerical study in the Black & Scholes (1973) world, the method is shown to be superior to the classic approximations using finite differences.

Besides, there is also a strand of research concerned with discretely rather than continuously monitored barrier options, as they are more commonplace in the market but more difficult to price. Such options are treated by Broadie, Glasserman & Kou (1997). Contrary

[165] Contour integration is a technique from the fields of complex analysis and theory of functions, which deals with integrals along paths in the complex plane. More information about these integrals can be found in the textbook by Stalker (1998).

[166] According to Musiela & Rutkowski (2005) forward starting options are options which start at a pre-specified time in the future. They are comprehensively analyzed in a stochastic volatility, stochastic interest rate model by van Haastrecht & Pelsser (2011). Ratchet options or cliquet options (terms which can be used interchangeably) are a series of such forward starting options, for which Kjaer (2006) provides efficient solution schemes. The payoff of lookback options generally depends on the minimum or maximum of the underlying during the option's lifetime.

[167] Malliavin calculus is the extension of calculus of variations to stochastic processes. It can, for example, be used to define derivatives for random variables and has applications in the field of mathematical finance. In the latter it can be used to derive the Greeks of financial derivatives. More information about Malliavin calculus can, for instance, be found in di Nunno, Øksendal & Proske (2009).

to their continuously monitored counterparts these options can only knock out or in respectively at certain pre-defined points of time in the future. However, no closed-form valuation formulas exist and those for continuously monitored barriers exhibit inaccurate valuation results. This problem is tackled by a so-called continuity correction, i.e. the discrete option is subjected to a shift in the barrier and afterwards dealt with as if it were continuous. Furthermore, convergence is proved and the accuracy further assessed in numerical examples. Broadie, Glasserman & Kou (1999) price path-dependent options which are based on extremal values of the underlying in a Black & Scholes (1973) setting. For the discrete-time versions they develop correction terms applied to the continuous-time counterpart and numerical alterations to further enhance accuracy. These correction terms can be interpreted as shifts to the underlying, the barrier or the strike. Pricing examples of the methodology include lookback, barrier and hindsight options. Kou (2003) extends the results on discretely monitored barrier options by Broadie, Glasserman & Kou (1997) to all eight of their barrier option cases. Furthermore, he suggests a simpler proof which exploits methods from the field of sequential analysis as presented by Siegmund & Yuh (1982).[168] Eventually, the result is very similar to the continuity correction brought forth by Hörfelt (2003). The latter adds to the research by Broadie, Glasserman & Kou (1997) and determines pricing formulas for up-and-out calls and puts, down-and-out calls and puts as well as double barrier options, if they are discretely monitored. The results and approximations are further validated by numerical experiments.

[168] Sequential analysis is a method of statistical hypothesis testing, in which the sample size is adaptively determined as part of the calculation. This is attained by pre-defined stopping criteria once significant results are available. In this way the required sample size can be considerably reduced. Examples for sequential analysis and more information are provided in Wald (1947) and Arrow, Blackwell & Girshick (1949).

3.4 Assumption of Efficient Markets

Throughout the sections on option pricing models and valuation techniques we have always assumed perfect and frictionless markets as set out in Section 3.1.1. In the absence of frictions such as transaction costs, Oehler & Unser (2002) point out that the procurement of information does not incur any costs and thus it is rational for all market participants to gather all available information. Naturally, this entails homogeneous expectations, freedom of arbitrage and market efficiency. The latter states that all asset prices reflect all available information, which wipes out the opportunity to average excess returns above the market average.

Not only is this a somewhat idealized view of the world but to a certain degree also a debated one. If market efficiency, for which there is a plethora of literature and tests, was refuted, this would simultaneously cast doubt on the validity of the presented option pricing models. Therefore, this section provides a closer look at certain forms of market efficiency and at the extent to which market efficiency holds. Of course, market efficiency is one of the most intensively researched topics in financial economics and therefore worth numbers of theses, books, and articles solely focusing on that topic, so that this section can by no means be a complete account.

Fama (1970*a*) distinguishes between three degrees of market efficiency:

1) Weak-form tests: How well do past returns predict future returns?

2) Semi-strong-form tests: How quickly do security prices reflect public information announcements?

3) Strong-form tests: Do any investors have private information that is not fully reflected in market prices?

For all three of these types there is various literature both in favor of and suggesting evidence against market efficiency.[169] Dimson & Mussavian (1998) provide an account of the literature on market efficiency and summarize the results both in favor of the efficient market hypothesis as well as anomalies which might be interpreted as evidence against efficient markets. But as Fama (1970*a*) pointed out, every research testing for market efficiency suffers from the joint hypothesis problem. Since one investigates whether information is correctly reflected by market prices one has to choose an asset pricing model which specifies what a correct reflection means. Unfortunately, when detecting deviations it is unclear whether they stem from a flawed asset pricing model or from missing market efficiency.[170]

3.4.1 Tests for Return Predictability

According to Fama (1991) the majority of undecidedness to which extent capital markets can be deemed efficient stems from the research area of return predictability. This field covers the question of how well stock returns can be predicted from returns observed in the past or other economic variables such as price earnings ratios

[169] Jensen (1978) highlights and summarizes several early findings of anomalies (e.g. abnormal returns when applying option-implied variances, abnormal returns following disclosure information) found by researchers regarding a potential rejection of the efficient market hypothesis.

[170] Jarrow & Larsson (2012) have recently relaxed the joint-hypothesis problem by showing that an efficient market is entirely characterized by the absence of both, arbitrage and dominated securities. Furthermore, the authors draw on their result and present tests for market efficiency which do not no longer suffer from this problem.

etc. Furthermore, research can be loosely classified as focusing on short or long-term returns.

With respect to short horizons evidence is abundant that there is positive first-order autocorrelation of daily stock returns. Examples include Fama (1965) or French & Roll (1986).[171] This is borne out and detailed further by Lo & MacKinlay (1988) and Conrad & Kaul (1988). Regarding long horizons findings are more vague, but nonetheless Fama & French (1988*b*) and Poterba & Summers (1988) are able to establish negative autocorrelation for two to ten year returns which is caused by a slow reversal of temporary stationary components of prices.[172] Nonetheless, these studies tend to suffer from low statistical power. Fama (1990) aims at explaining the rational variation in stock returns by shocks to expected cash flows, by variation in returns over time in the discount rates used to price expected cash flows and by shocks to discount rates. All in all, the author is able to account for round about 60% of the total variation.

The statistical situation is better concerning the prediction power of price earnings ratios, dividend yields and default spreads for high and low-yield bonds as elaborated on by Keim & Stambaugh (1986), Campbell & Shiller (1988), Fama & French (1988*a*) and Fama & French (1989). The studies by Campbell (1987) and Chen (1991)

[171] In particular, the authors find that stock prices are more variable during trading hours. They believe this is caused by noise trading, as examined by Black (1986), which leads to pricing errors caused by uninformed investors. These are then reversed which becomes manifest in short-term negative autocorrelation observed in daily returns.

[172] If short-term deviations from the efficient market price prevail, then in an efficient market those deviations should be overturned in the long run. Therefore, it should be possible to observe negative autocorrelation for time horizons between two and ten years. In their extremly long time series studies ranging back to 1871 such autocorrelation could only be detected when including data from the time of the Great Depression in the 1930s, certainly a time of extra-ordinary market conditions. Therefore, these results tend to be supportive of the efficient market hypothesis.

focus further on term spreads.[173] By Fama (1976*a*) it is examined to which degree forward rates serve as good predictors of future spot rates. According to Fama (1975) and Fama (1976*b*) uncertainty in forward rates predominantly stems from the uncertainty about future inflation rates. In particular, it is shown that the market reacts in accordance with the observed monthly negative autocorrelation and its time variance, which is in line with the hypothesis of efficient markets. The paper by Fama & Bliss (1987) is further concerned with the forecast power of forward rates for future spot rates. It is found that with increasing horizon the forecasting power increases as well which is attributed to slow mean reversion in interest rates.

Apart from testing market efficiency directly the joint hypothesis problem lends itself to the possibility of testing asset pricing models instead. If such models, which assume efficient markets, describe market prices and returns in a sufficiently accurate way, then this might strengthen the efficient market hypothesis. In this strand of literature Fama (1991) identifies three classes of models, the CAPM by Sharpe (1964), Lintner (1965) and Black (1972), multifactor models and consumption-based models as suggested by Rubinstein (1976), Lucas (1978) and Breeden (1979).

With respect to the CAPM there is a plethora of literature devoted to assessing and mostly refuting the explanatory power of market betas for cross-sectional expected returns. Jensen, Black & Scholes (1972) and Fama & MacBeth (1973) find anomalies with zero beta portfolios, which appear to expect higher returns than the risk-free rate of interest. Roll (1977) and Stambaugh (1982) adduce that stock market indexes are bad proxies for the market portfolios.

[173] More precisely, in Chen (1991) the author investigates variables that are correlated with macro-economic circumstances, such as aggregate production growth or dividend yields, are thus priced in an equilibrium due to Fama (1970*b*) and can serve as forecast variables for future stock returns. The latter is based on the fact that past and future growth of the gross national product are related to the stock market premium in excess of the risk-free rate of interest.

Chan, Hamao & Lakonishok (1991) and Fama & French (1992) find a size effect that expected returns depend on the book-to-market ratio. In general, though, this evidence is viewed as a shortcoming w.r.t. the CAPM rather than market inefficiency. According to Fama (1991) this is caused by the fact that anomalies are time persistent and can at least in part be explained by rational multifactor models.

Naturally, multifactor models have been developed which try to explain the observed expected returns via linear regression on multiple factors. Most prominently, Ross (1976) introduced the APT (Arbitrage Pricing Theory). The subsequent debate about how many factors to include (Roll & Ross (1980), Shanken (1982), Chen (1983), Lehmann & Modest (1988), Roll & Ross (1984), Dhrymes, Friend, Gultekin & Gultekin (1984) and Trczinka (1986)) could not find a definite and clear answer. Chen, Roll & Ross (1986) explicitly look for economic variables which are correlated with returns. It turns out that the industrial production growth rate and the spread between high and low-grade bonds is most important while unexpected inflation and term spreads play only a minor role. Furthermore, these variables are shown by Chan, Chen & Hsieh (1983) to resolve the size effect. Fama (1981) deals with real returns of stocks, i.e. returns adjusted for inflation, to further elaborate on the surprising finding of a negative relation between inflation rates and stock prices. It is found, that real returns of stocks prices are positively related to real economic activity such as capital expenditures and that inflation serves as a proxy for that. This is further interpreted in terms of money demand theory and the quantity theory of money.[174]

The paper by Fama & French (1996) shows that most anomalies observed in stock returns (dependence on size, past earnings, book-to-

[174] The quantity theory of money establishes a direct, proportional relationship between prices and money supply.

market value ratios etc.) disappear when using the Fama & French (1992) three-factor model rather than the CAPM. Fama (1998) challenges inferences of market inefficiency based on long-term anomalies. It is argued that on average there is neither an over- nor an under-reaction to the arrival of information but both occur about equally frequently. Furthermore, most such anomalies are not persistent w.r.t. a change of methodology, i.e. they can only be detected under very distinct circumstances. All in all, the author suggests to uphold the efficient market hypothesis due to the lack of consistent alternatives.

3.4.2 Event Studies

With respect to event studies Fama (1991) points out that financial economists have emphasized three sorts of events which are of particular interest to researchers regarding market efficiency:

1) News about dividends,

2) Issue of new common stock,

3) Corporate control transactions.

Naturally, these events are predominantly questions of corporate finance as in the stock markets the investigated market values of corporations are determined.

If there is an unexpected change in dividends, this typically prompts a change in the stock price of the same direction. This is reported by Charest (1978*a*), Charest (1978*b*), Aharony & Swary (1980) and

Asquith & Mullins (1983) and means that an increase in dividends is good news for stock holders.[175]

It was further found by Asquith & Mullins (1986) that issuing new common stock is bad news for the stock prices while redemptions, according to Dann (1981) and Vermaelen (1981), are good news.[176]

Concerning corporate control transactions it is empirically well established that mergers lead to large capital gains for the stock holders of the target company. Examples of such studies include, for instance, Mandelker (1974), Dodd & Ruback (1977), Bradley (1980), Dodd (1980) or Asquith (1983). For management buyouts the same is confirmed by Dodd & Warner (1983) and Kaplan (1989).

The question then is how this relates to market efficiency. Since in efficient capital markets prices reflect available information, the reaction time to the arrival of new information (event) is a natural measure for market efficiency. As Fama (1991) points out, event studies contrary to other studies do not face the joint hypothesis problem as the (potentially) abnormal returns following announcements by far exceed the average daily returns (15% instead of 0.04%). Thus the majority of studies is supportive of market efficiency.[177]

[175] From an economic perspective, this is interpreted in terms of signaling to the market that the company is in good enough shape to not only maintain its level of expenditures but also to disburse additional earnings to its shareholders.

[176] The economic intuition behind this phenomenon is information asymmetry. When stocks are overvalued issuing new stock is more profitable. Similarly, when using free cash flows to redeem stock, agency costs are lower.

[177] Nonetheless, there are also studies which argue differently, when evaluating stock prices of acquiring firms, as it is e.g. done by Asquith (1983), Roll (1986) or Franks, Harris & Titman (1991). But when considering slow adjustments the joint-hypothesis problem arises again.

3.4.3 Tests for Private Information

Scholes (1972) shows that (in line with common sense) corporate insiders indeed have access to information not publically available. When testing for private information three strands of literature have emerged. The first is concerned with insider trading, the second with securities analysts' information and the third with professional portfolio management. Regarding insider trading, Jaffe (1974) uses the CAPM to show that there are insider profits in the market which can even further be exploited by non-insiders after publication. Seyhun (1986) agrees on that but not on the persistency of the opportunities once they become public and attributes it to a size effect, that small stocks tend to have high returns in the CAPM (see Banz (1981)). Lloyd-Davies & Canes (1978) and Stickel (1985) provide evidence that some analysts have private information that translates to statistically significant price movements upon publication. This means that they are compensated for the costs they incur to obtain the beneficial information rendering the market less than perfectly efficient. However, all investors are still rational and in line with the noisy rational expectations hypothesis by Grossman & Stiglitz (1980).[178] Regarding professional portfolio management there is a mixed picture.[179] Jensen (1969a) and Jensen (1969b) find that private information is atypical for professional fund managers. Henriksson (1984) and Chang & Lewellen (1984) find that there can be enough private information to offset management fees. Negative evi-

[178] Loosely speaking, the rational expectations hypothesis due to Lucas (1972) states that the expectation of an asset price and the ex post realized one coincide.

[179] Fama (1972) is concerned with methodologies of assessing investment performance, i.e. the ability of fund managers to pick stocks from general benchmark portfolios to which they allocate more funds because of higher return prospects. Doing so serves as a measure of how well fund managers can materialize on information not yet reflected in market prices. The notion of risk and return used to assess performance is based on the capital market theory put forth by Sharpe (1964) and Lintner (1965), i.e. the CAPM.

dence, i.e. negative excess returns, for pension plans and endowment funds are reported by Beebower & Bergstrom (1977) and Ippolito & Turner (1987). The findings are further upheld if challenged for biases in the Sharp-Lintner model. Chen, Roll & Ross (1986), Chan & Chen (1991) and Fama & French (1992) present multifactor models that mitigate the anomalies of the Sharpe-Lintner model (size effect, book-to-market value anomalies).

All in all, because of the joint hypothesis problem it is hard to reject strong market efficiency but nonetheless the tendency is that there are abnormal returns which can be attributed to insider information.

3.4.4 Behavioral Aspects

Over recent years another branch of finance literature has developed aiming to explain what is left unexplained by the tradional asset pricing models such as the CAPM, the APT and multi-factor models generalizing the aforementioned ones. Contrary to the established models, this branch does not assume fully rational investors but bounded rationality instead. MacKinlay (1995) deals with two possible sources for missing explanatory power of the CAPM, missing risk factors on the one hand and the presence of irrational investors on the other hand. Furthermore, it is found that the latter empirically serves better to fill the gap of explanation.

This discernment has spawned numerous literature to explore the field of not fully rational finance. Barberis, Shleifer & Vishny (1998) present a model that is able to capture both over- and underreactions observed in empirical findings.[180] The idea is that investors

[180] It is observed in the markets that upon the arrival of good news, for instance earnings announcements, prices only slowly reflect this and thus underreact (see for example Cutler, Poterba & Summers (1991)). If, however, there is a series of good announcements prices seem to overreact in the long run (see for instance Fama & French (1988*b*) or Poterba & Summers (1988)).

study time series and tend to neglect deviations from the seemingly prevailing model. In the paper by Daniel, Hershleifer & Subrahmanyam (1998), under- and overreactions in securities markets are investigated. The authors determine two sources to explain the phenomena, investor overconfidence and biased self-attribution. Hong & Stein (1999) add to the approaches pursued by Barberis, Shleifer & Vishny (1998) and Daniel, Hershleifer & Subrahmanyam (1998). However, the focus lies on the interaction between market participants rather than cognitive biases that bound the assumed investor rationality. The model is applied to investigate underreaction in the short run, caused by gradual diffusion of information, and overreaction in the long run implied by too simple models used by those chasing this trend. Campbell (2006) covers the relatively little explored field of private households and analyzes their actual behavior compared to what they should rationally do. Contrary to professional investors or corporations private households are subject to more factors and aspects from behavioral finance as they face higher market frictions and exhibit a much wider spread between educated investors and, how the author calls it, naive investors.

In more recent literature Barberis & Xiong (2012) develop a model of realization utility[181] and are able to explain among others the disposition effect[182] and the relatively low investment success of individuals. The paper by Fryman, Barberis, Camerer, Bossaerts & Rangel (2010) employs neural data in addition to a behavorial model to explain investors' behavior. The results confirm the realization utility hypothesis and lend themselves to explaining a commonly observed disposition effect. Empirically overcondidence is found to produce

[181] The term realization utility describes the observation that investors appear to gain or lose utility from merely realizing gains or losses on the their investments in addition to that based on the current level of wealth. Examples include Shefrin & Statman (1985) and Barberis & Xiong (2009).

[182] The disposition effect refers to the often observed behavior that investors are willing to realize gains after stocks increased while they are reluctant to realize losses after a drop in stock prices.

negative autocorrelation in the long run and additional volatility. Biased self-attribution in turn causes positive short-run autocorrelation.

Chapter 4

Optimal Control Theory and Optimal Stopping

In this chapter we give an account of optimal control theory and optimal stopping w.r.t. the aspects we need below in Chapters 5 and 6 to numerically deal with OETCs. For this purpose the chapter is structured in the following way: Section 4.1 presents the theoretical formulation of optimal control problems and corresponding results in continuous time, Section 4.2 deals with discretizations of the problems and Section 4.3 concludes with the actual solution methods.

4.1 Optimal Control Problems

In this section we present formulations and theoretical results for optimal control problems. Section 4.1.1 sheds light on the dynamic programming equations, which are the focal point of optimal control theory as they govern the optimal value function. Section 4.1.2 deals with Pontryagin's Minimum Principle, which is an important theoretical reformulation of optimal control theory. In Section 4.1.3 the theory of the previous two sections is applied to optimal stopping problems, which are a special case of optimal control problems.

4.1.1 Dynamic Programming Equations

This section establishes the dynamic programming equations, which are satisfied by the value function of a stochastic control problem in continuous time.[183] This process consists of three parts:

1) Precise formulation of the optimal control problem and its ingredients
2) Establishing the principle of optimality
3) Deduction of the Hamilton-Jacobi-Bellman equations using stochastic calculus.

The starting point is the mathematical definition of a discount function for a vector-valued process on the time interval I, which can be thought of as the option's lifetime, i.e. $I = [0, T]$.

[183] In this thesis we only present the continuous time formulation as we throughout the work assume a continuous time market model for the dynamics of the underlying stochastic variables. Furthermore, we would like to point out that the term dynamic programming equations is used interchangeably with Hamilton-Jacobi-Bellman equations in continuous time and Bellman equations in discrete time. In this thesis we also adhere to this convention.

Definition 4.1.1 *A discount function $\beta : I \times \mathbb{R}^N \to \mathbb{R}^N$ has the following properties:*

i) $\beta(0, x) = x \quad \forall\, x \in \mathbb{R}^N$

ii) $\beta(t, \beta(\bar{t} - t, x)) = \beta(\bar{t}, x) \quad \forall\, t, \bar{t} \in I, x \in \mathbb{R}^N$

iii) $\beta(t, s\,x + y) = s\,\beta(t, x) + \beta(t, y)$

iv) $\beta(t_1 + t_2, x) = \beta(t_1, x)\,\beta(t_2, x)$.

Though appearing technical at first glance, these properties are quite intuitive. The first means that there is no instantaneous discount, the second means that intermediate discount is irrelevant, the third means linearity in the claim to be discounted, i.e. claims can be discounted separately, and the fourth pertains to the exponential form of the discount factor. It is readily seen that these properties are indeed satisfied by the discount function $\beta(t, x) = \exp(-rt)\,x$, where r is the riskless rate of return and the product is to be interpreted componentwise.

Besides, the notion of a control variable has to be properly defined, for which we follow Kushner & Dupuis (2001).

Definition 4.1.2 *A control is a mapping $u : I \times \Omega \to \mathcal{A}(x, t)$ which assigns a control action for any time t and state x from the set of admissible actions $\mathcal{A}(x, t)$ for that combination of state and time, which is to be specified separately. In the case of explicit dependence on x the control is referred to as feedback control.*

In this definition Ω is the state space of the stochastic process X equipped with a filtration, as it is defined in Section 3.2.2.5. However, to establish a formulation as a stochastic control problem we have to allow for dependence of the coefficient functions on the control variable and we have to impose several technical conditions.

We assume the control function $u : I \times \mathbb{R}^d$ to be progressively measurable, the coefficient functions $b : I \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : I \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ to be continuous and for fixed third arguments (the control action) α differentiable with the following bounds on the derivatives

$$\begin{aligned} ||b_t|| + ||\nabla b|| &\leq C \\ ||\sigma_t|| + ||\nabla \sigma|| &\leq C \\ ||b(t,x,\alpha)|| &\leq C\,(1 + ||x|| + ||\alpha||) \\ ||\sigma(t,x,\alpha)|| &\leq C\,(1 + ||x|| + ||\alpha||). \end{aligned}$$

In this setting, $C < \infty$ is a suitable constant. For open and bounded sets $\mathcal{R}_\alpha \subset \mathbb{R}^d$ and $\mathcal{R}_X \subset \mathbb{R}^d$ admissibility of the control is specified via

$$\mathcal{A}(t,x) := \{u : I \times \mathbb{R}^d \to \mathbb{R}^d \,|\, u(s, X(s)) \in \mathcal{R}_\alpha, X(s) \in \mathcal{R}_X \forall s \in [t,T]\}.$$

Define now the functional J which is to be maximized as

$$\begin{aligned} J(t,x;u) \quad := \quad & E[\int_t^T \beta(s-t, g_1(s, X(s), u(s, X(s))))\, ds \\ & + \beta(T-t, g_2(T, X(T), u(T, X(T)))) \,|\, X(t) = x] \end{aligned}$$

and the corresponding value function as

$$v(t,x) := \max_{u \in \mathcal{A}(t,x)} J(t,x;u). \tag{4.1.1.1}$$

In Section 3.2.2.5 it has been established that the value of an American option is given by the solution of an optimal stopping problem, which has now been retained in a stochastic control problem formulation. Here, the function g_1 plays the role of a continuous payout, which is not present in our application, i.e. $g_1 \equiv 0$, and $g_2 =: g$

is the terminal payoff.[184] In Karatzas & Shreve (2010) it is shown that this v is indeed well-defined.[185] The purpose of this section is to further characterize this optimal control problem.

The Bellman principle or principle of optimality, as stated by Kushner & Dupuis (2001), allows for a major modification

$$v(t,x) = \max_{u \in \mathcal{A}(t,x)} E[\int_t^T \beta(s-t, g_1(s, X(s), u(s, X(s))))\, ds + \beta(t-\bar{t}, v(\bar{t}, X(\bar{t}))) \mid X(t) = x]. \tag{4.1.1.2}$$

The intuition behind this principle is that regardless of what happens now an optimal policy must be optimal from the next point on. From a practical point of view this is of utmost importance as it gives rise to a reduction of the maximization problem of the entire time interval to a maximization over the next small time interval, which will below allow for a solution technique backwards in time.

The derivation of the dynamic programming equation (which dates back to the original work by Bellman (1954)) is carried out in two stages:

1) Derivation without discounting, i.e. $\beta(t,x) = x$.

2) Modification for the presence of a discount factor.

According to Karatzas & Shreve (2010) the assumptions of Itô's lemma (cf. Lemma 3.1.1) are satisfied by the value function v. Ap-

---

[184] An example where both types of payouts occur, might be a cost minimization problem of running a production machine. In this situation g_1 can be thought of as the immediate cost of running the machine and g_2 would be the terminal cost of replacing the machine.

[185] Being well-defined in this context means that the involved integrals, expectations and maxima actually exist.

plication yields

$$\begin{aligned}
v(\bar{t}, X(\bar{t})) &= v(t, X(t)) + \int_t^{\bar{t}} v_t(s, X(s))ds \\
&\quad + \int_t^{\bar{t}} \sum_{k=1}^d v_{X_k}(s, X(s))dX_k(s) \\
&\quad + \frac{1}{2} \sum_{k,l,m=1}^d \int_t^{\bar{t}} \sigma_{km}(s, X(s), u(s, X(s))) \\
&\quad \sigma_{lm}(s, X(s), u(s, X(s))) v_{X_l X_k}(s, X(s), u(s, X(s)))\, ds.
\end{aligned}$$

For the ease of notation we introduce $\bar{\sigma} := \sigma\sigma^T$, i.e. $\bar{\sigma}_{kl} = \sum_{m=1}^d \sigma_{km}\sigma_{lm}$. On the other hand, by virtue of Bellman's principle (4.1.1.2) we have

$$\begin{aligned}
&v(t, X(t)) - v(\bar{t}, X(\bar{t})) \\
&\qquad = \max_{u \in \mathcal{A}} E\left[\int_t^{\bar{t}} g_1(s, X(s), u(s, X(s)))\, ds \mid X(t) = x\right].
\end{aligned}$$

Adding up establishes

$$\begin{aligned}
0 &= \max_{u \in \mathcal{A}} E[\{g_1(s, X(s), u(s, X(s))) \\
&\quad + v_t(s, X(s)) \\
&\quad + \frac{1}{2} \sum_{k,l=1}^d \bar{\sigma}_{kl}(s, X(s), u(s, X(s))) v_{X_l X_k}(s, X(s))\}ds \\
&\quad + \sum_{k=1}^d \int_t^{\bar{t}} v_{X_k}(s, X(s))dX_k(s) | X(t) = x].
\end{aligned}$$

Inserting the assumptions about X gives us

$$0 = \max_{u \in \mathcal{A}} \{E[\int_t^{\bar{t}} \{g_1(s, X(s), u(s, X(s))) + v_t(s, X(s))$$

$$
\begin{aligned}
&+\sum_{k=1}^{d} b_k(s, X(s), u(s, X(s))) v_{X_k}(s, X(s)) \\
&+\frac{1}{2} \sum_{k,l=1}^{d} \bar{\sigma}_{kl}(s, X(s), u(s, X(s)))\, v_{X_l X_k}(s, X(s))\} ds \mid X(t) = x] \\
&+E[\int_t^{\bar{t}} \sigma_{km}(s, X(s), u(s, X(s)))\, v_{X_k}(s, X(s)) dW_m(s) \mid X(t) = x]\}.
\end{aligned}
$$

According to Evans (1983) the last summand vanishes. By division through $\bar{t} - t$ we can apply the mean value theorem for integration, which yields a $\xi \in [t, \bar{t}]$ such that

$$
\begin{aligned}
0 \quad = \quad & \max_{u \in \mathcal{A}} E[\{g_1(\xi, X(\xi), u(\xi, X(\xi))) + v_t(\xi, X(\xi)) \\
&+\sum_{k=1}^{d} b_k(\xi, X(\xi), u(\xi, X(\xi)))\, v_{X_k}(\xi, X(\xi)) \\
&+\frac{1}{2} \sum_{k,l=1}^{d} \bar{\sigma}_{kl}(\xi, X(\xi), u(\xi, X(\xi)))\, v_{X_l X_k}(\xi, X(\xi)) \mid X(t) = x].
\end{aligned}
$$

Letting $\bar{t} \to t$ and exploiting the fact that everything is known at time t so that the expectation operator can be dropped, finally establishes

$$
\begin{aligned}
0 = v_t(t, x) + \max_{u \in \mathcal{A}} [g_1(t, x, u(t, x)) + b(t, x, u(t, x)) \cdot \nabla v(t, x) \\
+ \frac{1}{2} \mathrm{tr}(\bar{\sigma} \,\mathrm{Hess}\; v(t, x))]
\end{aligned}
\tag{4.1.1.3}
$$

with terminal condition $v(T, x) = g_2(T, x)$. This equation can be regarded (and solved) as a maximization w.r.t. u depending on x, b and $\bar{\sigma}$ and the v-derivatives and afterwards with the obtained value for u as a PDE for v.

To adjust for a discount factor we assume that $\beta : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is given in multiplicative form, e.g. $\beta(t,x) = \exp(-rt)\,x$, and its restriction to $[0,T]$ be differentiable in $(0,\gamma)$ for fixed $x \in \mathbb{R}^d$ and constant w.r.t. varying x such that[186]

$$-\infty < \lim_{\xi \to 0} \frac{\beta'(\xi)}{\beta(\xi)} =: \zeta < \infty.$$

This prompts adjustment of the value function (4.1.1.1) to

$$\begin{aligned} v(t,x) &= \max_{u \in \mathcal{A}} E\Big[\int_t^T \beta(s-t)\, g_1(s, X(s), u(s, X(s)))\, ds \\ &\quad + \beta(T-t)\, g_2(T, X(T), u(T, X(T))) \mid X(t) = x\Big] \end{aligned}$$

and of Bellman's principle (4.1.1.2) to

$$\begin{aligned} v(t,x) &= \max_{u \in \mathcal{A}} E\Big[\int_t^{\bar{t}} \beta(s-t)\, g_1(s, X(s), u(s, X(s)))\, ds \\ &\quad + \beta(\bar{t}-t)\, v(\bar{t}, X(\bar{t})) \mid X(t) = x\Big]. \end{aligned}$$

We now choose $\bar{t}$ such that $\bar{t} - t < \gamma$ and repeat the previous calculations. The only difference is that Itô's lemma is to be applied to $\beta(\bar{t}-t)\, v(\bar{t}, X(\bar{t}))$ so that the time derivative v_t of v is replaced by

$$v_t(s, X(s)) \to \beta'(s-t)\, v(s, X(s)) + \beta(s-t)\, v_t(s, X(s)),$$

while all other terms are just multiplied with $\beta(s-t)$. As it also appears as the discount of g_1 we can divide by $\beta(s-t)$ and adjust equation (4.1.1.3) to

$$0 = \zeta v(t,x) + v_t(t,x) + \max_{u \in \mathcal{A}} [g_1(t, x, u(t,x))$$

[186] In a slight misuse of notation we denote the restriction of β by β as well. However, as the meaning ought to be self-explanatory whenever this substitution is made everything else would unnecessarily complicate readability.

$$+ b(t,x,u(t,x)) \cdot \nabla v(t,x) \tag{4.1.1.4}$$
$$+ \frac{1}{2}\mathrm{tr}(\bar{\sigma}\mathrm{Hess}\ v(t,x))]$$

with the terminal condition $v(T,x) = g_2(x,T)$ as before. For our discount function $\beta(t) = \exp(-rt)$ we can also readily obtain the coefficient ζ as

$$\zeta = \lim_{\xi \to 0} \frac{\beta'(\xi)}{\beta(\xi)} = -r.$$

So, in essence, discounting only introduces an additional reaction term $-r\,v(t,x)$ to equation (4.1.1.4), which is the desired dynamic programming equation or Hamilton-Jacobi-Bellman equation.

Furthermore, the Hamilton-Jacobi-Bellmann equation can be viewed as a necessary condition for the solution to the optimal control problem. From Fleming & Soner (2010) we take the following proposition which establishes existence and uniqueness of solutions:

Proposition 4.1.3 *Let $\mathcal{A}$ be compact, $b, \bar{\sigma}$ and g_1 together with their partial derivatives continuous and bounded and $g_2 \in C^3$. If $\bar{\sigma}$ is uniformly parabolic, i.e.*

$$\sum_{k,l=1}^{d} \bar{\sigma}_{kl}(t,x,u)\,\xi_k\xi_l \geq c\,||\xi||^2 \quad \forall \xi \in \mathbb{R}^d, (t,x,u) \text{ and some } c > 0,$$

then there is a unique solution to equation (4.1.1.4).

Proof - Proposition 4.1.3:
See Fleming & Soner (2010).

□

Although this establishes existence and uniquess, it reveals drawbacks of studying the Hamilton-Jacobi-Bellman equation with PDE methods when setting up suitable numerical techniques. Whereas uniform parabolicity is quite standard in such a context and ensures

non-singul-
arity of the matrix across the domain, the regularity assumptions and boundedness might be rather hard to achieve in practice.

On the other hand the characterization of optimal control problems via the Hamilton-Jacobi-Bellman equation is comprehensive as the following sufficient condition shows:

Proposition 4.1.4 *Let $w \in C^{1,2}$ be a solution to equation* (4.1.1.4) *satisfying*

$$|w(t,x)| \leq K\left(1 + ||x||^m\right)$$

for some constants K, m. If a maximizing strategy u^* in (4.1.1.4) *is admissible we have*

$$w = J(\cdot, \cdot; u^*) = v.$$

Proof - Proposition 4.1.4:
See Fleming & Soner (2010).

□

4.1.2 Pontryagin's Minimum Principle

In addition to the dynamic programming equations there is also a deep theory pertaining to a further characterization of the optimal control variate. The key element in this theory is the use of an adjoint variable. As the solutions to dynamic programming equations often lack the regularity required by a solution in the classical sense Fleming & Soner (2010) present a characterization in terms of a generalized notion of solutions.

Theorem 4.1.5 (Pontryagin's Minimum Principle) *Let $u^*(\cdot)$ be an optimal control at (t, x), which is right-continuous at each $s \in [t, t_1)$, and $P(s)$ be the (so-called) adjoint variable satisfying* *i)*

$$\frac{d}{ds}P_j(s) = -\sum_{l=1}^{n} \partial_j b_l(s, x^*(s), u^*(s))P_l(s) - \partial_j L(s, x^*(s), u^*(s))$$

ii)

$$P(s)\cdot b(s, x^*(s), u^*(s)) + L(s, x^*(s), u^*(s)) = -H(s, x^*(s), P(s))$$

iii)

$$P(t_1) = Dg_2(x^*(t_1))$$

Then for each $s \in [t, t_1)$

$$(H(s, x^*(s), P(s)), P(s)) \in D^+v(s, x^*(s)),$$

where $D^+v(t,x)$ *is the set of superdifferentials of* v *at* (t,x) *and* $L = g_1 + \frac{1}{2}tr(\bar{\sigma}\,\mathrm{Hess}\ v(t,x))$ *and the Hamiltonian* H *is given by*

$$H(t,x,p) = \sup_{v\in\mathcal{A}} \left(-b(t,x,v)\cdot p - L(t,x,v)\right).$$

Proof - Theorem 4.1.5:
See Fleming & Soner (2010).

□

A detailed proof of the Theorem 4.1.5 is omitted because we consider the details of this intricate result outside the scope of this thesis which is not intended to focus on the mathematical details and subtleties but rather to provide the mathematics necessary to accurately formulate and solve the pricing problems from the field of financial economics.

Before giving an interpretation of this result we are due an explanation of superdifferentials as they appear in the above theorem:

Definition 4.1.6 *Let $W \in C^0(\bar{Q})$ and $(t,x) \in Q$. The set of superdifferentials of W at (t,x), denoted by $D^+W(t,x)$ is the collection of all $(q,p) \in \mathbb{R} \times \mathbb{R}^n$ which satisfy*

$$\begin{aligned}(q,p) &= (w_t(t,x), \nabla w(t,x)) \\ (t,x) &\in \arg\,\max\{(W-w)(s,y)|(s,y) \in \bar{Q}\}\end{aligned}$$

for some $w \in C^1(Q)$.

The benefit of the above theorem, which goes back to the work by Pontryagin, Boltyanskii, Gamkrelidze & Mishchenko (1998), is that it relates the minimization of the Hamiltonian to the set of superdifferentials. More precisely, it states that the trajectory from taking optimal control action at any time indeed renders the corresponding Hamiltonian a superdifferential of the optimal value function. Furthermore, this concept also covers the often missing regularity of both solutions and ingredient components (such as assumptions about market behavior) in a generalized setting.

From even another perspective it can be interpreted as a kind of verification of the dynamic programming equation. Before, dynamic programming was seen as a means of characterizing optimality and thereby retrieving optimal solutions. Now we state that if optimal trajectories are inserted in the Hamiltonian, it is indeed the differential that describes the optimal value function.

Another notion of Pontryagin's principle is given by Fleming & Rishel (1975). The authors relate it to the method of characteristics known from the theory of hyperbolic and non-linear partial differential equations.[187] They point out that the partial differen-

[187] The method revolves around a reduction of the PDE to a family of ODEs, from which the solution is obtained by integration of the initial data. Further information about this method is provided by Evans (2010).

tial equation involved in the minimum principle is same as the one governing the characteristic strip, i.e. the family of ODEs.

4.1.3 Application to Optimal Stopping

Given the general characterization of optimal control problems in Sections 4.1.1 and 4.1.2, researchers have focused on the particular control problem of optimally stopping a stochastic process. As established in the aforementioned, these problems are quite commonplace not only in financial economics but in business in general. As pointed out by Kushner & Dupuis (2001), optimal stopping problems are the simplest optimal control problems, since there is only the choice between continuation and stopping. Hence, research has been very fruitful with regard to characterization of solutions.

Dynkin (1963) characterizes the optimal stopping times and value functions of general discrete-time Markov processes and provides an extension to continuous time processes. In Taylor (1968) the author deals with optimal stopping problems based on three different kinds of reward or payoff functions, fixed reward at T, fixed reward and continuously paid reward at and up to T, as well as average reward. Most of the results are special cases of the optimal stopping problems considered by Dynkin (1963). Fakeev (1970) investigates the optimal stopping problem from a technical mathematical perspective. In this regard, it is characterized under which circumstances the optimum can be attained and how it can be approximated. Moreover, it is proved that it does not make any difference if one takes the supremum over all stopping times or over the restrained set of bounded stopping times.[188] In the paper by Bather (1970) the optimal stopping of Brownian motion processes is addressed. In this

[188] More precisely, the author maximizes the expected reward from the underlying stochastic process over all Markov times satisfying $P[t \leq \tau < \infty] = 1$ and $E[X_{\bar{\tau}}] < \infty$ and shows that the resulting optimal value function v coincides

context he re-establishes the fact, that an optimal value function satisfies the heat equation with appropriate boundary conditions and furthermore, regularity assumptions regarding exit times from open sets are relaxed compared to previous work, e.g. by Dynkin (1963). Fakeev (1971) extends the results of Fakeev (1970), in which the optimal reward is characterized as the essential supremum and the smallest super-martingale majorizing the underlying process. Here in a Markov case a characterization in terms of excessive functions is provided.[189] van Moerbeke (1974) discusses the solution of an optimal stopping problem with a linear reward or gain function and accurately describes the stopping boundary. van Moerbeke (1976) extends the results and minutely characterizes the stopping boundary in dependence on the reward. Besides, he addresses regularity questions concerning the stopping boundary and the reward and states several examples of optimal stopping problems, e.g. from warrant pricing as introduced by Samuelson (1965) and McKean (1965). In Bismut & Skalli (1977) previous results by Fakeev (1970) are revisited by virtue of proving existence of optimal stopping times and completely characterizing them. This is done employing the methods and results suggested by Bismut (1977). Jacka (1993) presents a new result on the known fact that the essential supremum S of the expectation of a stochastic process X is the minimal supermartingale dominating the underlying process X if S is a semimartingale. He then applies this result to optimal stopping and obtains a smooth pasting result in the sense of Section 3.2.1.

---

with the solution v^*, i.e. $v^* = v$, where the considered Markov times only satisfy $P[\tau \geq t] = 1$.

[189] The term excessive is defined by comparing functions with discounted expectations of that function. Excessiveness is then said to hold if the function exceeds the discounted expected value even for a vanishing interest rate when the discounting effect is minimal. Fakeev (1971) shows that the optimal value function is the smallest excessive majorant of the payoff function.

Besides this theory there is also work on more general processes and more advanced results in the characterization of optimality. Shepp (1969) is concerned with optimally stopping modified Wiener processes. In particular, the optimal exercise rule for the process $W(t)/(a+t), a > 0$, is shown to be the first time a certain threshold is reached. Furthermore, the paper extends the characterization of optimally stopping discrete-time processes $(X_n)_n$ where the X_n share a common distribution and the payoff is given by the average of the first n observations. The paper by El Karoui & Karatzas (1991) deals with the Skorokhod problem of reflecting a process at a moving boundary.[190] It is shown that the value function can be described by the integration of the solution to the corresponding stopping problem, i.e. stopping when first reaching the moving boundary. Dubins, Shepp & Shiryaev (1993) are concerned with the optimal stopping of Bessel processes.[191] In particular, the stopping rules for maximum gain functions are investigated and those maximal inequalities, i.e. inequalities for the maximum of stochastic processes, are derived which provide upper bounds for the expected maximum of the process until a given stopping time. Pham (1998) deals with an optimal stopping problem of a controlled jump diffusion process. It is proved that the value function (supremum of expected discounted gain) is a viscosity solution of the PIDE corresponding to the dynamic programming equation. Besides, comparison and maximum principles are investigated.

[190] The Skorokhod problem was originally introduced by Skorokhod (1961). A recent summary and extensions are provided by Reed, Ward & Zhan (2011).

[191] According to Øksendal (2010) Bessel processes are the solutions to the SDE $dX_t = dW_t + \frac{n-1}{2}\frac{dt}{X_t}$ where W_t is an n-dimensional Brownian motion.

4.2 Approximation Methods

In this section we introduce a class of approximation or discretization methods for optimal control problems revolving around Markov chains. The discretization process generally consists of two steps:

1) the derivation of the discrete-time Markov chain control problem (cf. Section 4.2.1) and

2) the verification that the chain actually approximates the continuous time problem (cf. Section 4.2.2).

4.2.1 Discrete-Time Markov Chain

In Section 4.1.1 we have established the dynamic programming equations (4.1.1.4) which govern and characterize the solution to stochastic optimal control problems in continuous time. For the sake of tractability we recall that they read

$$\begin{aligned} 0 &= -r\, v(t,x) + v_t(t,x) + \max_{u\in\mathcal{A}}[g_1(t,x,u(t,x)) \\ &\quad +b(t,x,u(t,x))\cdot\nabla v(t,x) + \\ &\quad \frac{1}{2}\mathrm{tr}(\bar{\sigma}\,\mathrm{Hess}\, v(t,x))]. \end{aligned}$$

The subject matter of this section is now to describe a discretization scheme which allows us to solve such an equation numerically. This is done according to the following steps:

1) Establish a discrete time Markov chain $X_{h,\delta}$, which approximates the underlying stochastic process X in a suitable sense. What this means is further specified in Section 4.2.2 below.[192]

[192] For a very general discussion of approximating random processes we refer the

2) Establish the corresponding discrete time dynamic programming equations and ensure that their solution converges to the solution of the continuous time formulation.

3) Establish a numerical solution technique which actually solves the discrete time version of the dynamic programming equations.

This section deals with the first and second items, before the next section covers the approximating properties and the actual solution scheme is presented in Section 4.3.

The way in which an approximating Markov chain is constructed might appear slightly bohemian as it is a mixture of stochastic approximation techniques and classic PDE solution techniques. The idea goes back to the works by Kushner (1990) and Kushner & Dupuis (2001), who detailedly describe this procedure. It revolves around a finite difference discretization of the Hamilton-Jacobi-Bellman equation, but only to obtain a Markov chain on the corresponding grid and not to numerically solve the PDE itself. Given the relatively poor regularity mentioned above in Section 4.1.1, the latter approach does not appear to be promising at all.

The first step is to discretize equation (4.1.1.4) using the following implicit finite difference approximations

$$
\begin{aligned}
\partial_i v(x,t,\gamma) \quad &\rightarrow \quad v_i^{h,\delta}(x,t,\gamma) := \begin{cases} \frac{v(x+e_i h_i,t,\gamma)-v(x,t,\gamma)}{h_i} & \text{if } b_i \geq 0 \\ \frac{v(x,t,\gamma)-v(x-e_i h_i,t,\gamma)}{h_i} & \text{else} \end{cases} \\
\partial_t v(x,t,\gamma) \quad &\rightarrow \quad v_t^{h,\delta}(x,t,\gamma) := \frac{v(x,t+\delta,\gamma)-v(x,t,\gamma)}{\delta} \\
\partial_{ii} v(x,t,\gamma) \quad &\rightarrow \quad v_{ii}^{h,\delta}(x,t,\gamma)
\end{aligned}
$$

reader to the textbook by Kushner (1984). In his book he studies the approximation by weakly convergent diffusion and jump diffusion processes and presents examples of applications from economically important fields other than finance, such as operations research or engineering.

$$:= \frac{v(x + e_i h_i, t, \gamma) + v(x - e_i h_i, t, \gamma) - 2v(x, t, \gamma)}{h_i^2}$$

$$\partial_{ij} v(x, t, \gamma) \quad \to \quad v_{ij}^{h,\delta}(x, t, \gamma)$$

$$:= \begin{cases} \frac{2v(x,t,\gamma)+v(x+e_ih_i+e_jh_j,t,\gamma)+v(x-e_ih_i-e_jh_j,t,\gamma)}{2h_ih_j} \\ \quad -\frac{v(x+e_ih_i,t,\gamma)+v(x+e_jh_j,t,\gamma)}{2h_ih_j} \\ \quad -\frac{v(x-e_jh_j,t,\gamma)+v(x-e_ih_i,t,\gamma)}{2h_ih_j} \\ \quad \text{if } \bar{\sigma}_{ij} \geq 0 \\ \\ -\frac{2v(x,t,\gamma)+v(x-e_ih_i+e_jh_j,t,\gamma)+v(x+e_ih_i-e_jh_j,t,\gamma)}{2h_ih_j} \\ \quad +\frac{v(x+e_ih_i,t,\gamma)+v(x+e_jh_j,t,\gamma)}{2h_ih_j} \\ \quad +\frac{v(x-e_jh_j,t,\gamma)+v(x-e_ih_i,t,\gamma)}{2h_ih_j} \\ \quad \text{else.} \end{cases}$$

These approximations correspond to the suggestions made in Kushner & Dupuis (2001).[193] In particular, an upwind scheme is used for the mixed derivatives, which makes the algorithm numerically more stable.[194] In order to come up with a discrete-time Markov chain, we insert the finite difference approximations into the PDE corresponding to the Bakshi, Cao & Chen (1997) model with logarithmic prices, our most general model from Section 3.1.4, replace t by n/δ, multiply with δ and obtain

$$\left(1 + \frac{\delta}{h_1}|\mu| + \frac{\delta}{h_2}\kappa|\theta - V| + \frac{\delta}{h_1^2}V + \frac{\delta}{h_2^2}V\sigma_V^2 - |\rho|\sigma_V V\frac{\delta}{h_1 h_2} + \delta r\right)$$
$$v(x, n\delta, u)$$

[193] An example of early work on this subject is Quadrat (1989), who finds that for discrete time Markov chains and quadratic cost functionals the Hamilton-Jacobi-Bellman equation collapses to a Riccati equation.

[194] In general, any other finite difference approximation can be used as well. The resulting Markov chain would be different but it would still be an approximation. The convergence properties of such other schemes are an area of open research. For a recent example see Koutsoukos (2005).

$$
\begin{aligned}
= \quad & v(x, n\delta + \delta, u) + \frac{\delta}{h_1}\mu^+ \, v(x + h_1 e_1, n\delta, u) \\
& + \frac{\delta}{h_1}\mu^- \, v(x - he_1, n\delta, u) + \kappa(\theta - V)^+ \frac{\delta}{h_2} \, v(x + h_2 e_2, n\delta, u) \\
& + \kappa(\theta - V)^- \frac{\delta}{h_2} \, v(x - h_2 e_2, n\delta, u) + \frac{1}{2} V \frac{\delta}{h_1^2} \, v(x + h_1 e_1, n\delta, u) \\
& + \frac{1}{2} V \frac{\delta}{h_1^2} \, v(x - h_1 e_1, n\delta, u) + \frac{1}{2} V \sigma_v^2 \frac{\delta}{h_2^2} \, v(x + h_2 e_2, n\delta, u) \\
& + \frac{1}{2} V \sigma_v^2 \frac{\delta}{h_2^2} \, v(x - h_2 e_2, n\delta, u) \\
& + \frac{1}{2} \sigma_V V \frac{\delta}{h_1 h_2} \rho^+ \, v(x + h_1 e_1 + h_2 e_2, n\delta, u) \\
& + \frac{1}{2} \sigma_V V \frac{\delta}{h_1 h_2} \rho^- \, v(x + h_1 e_1 - h_2 e_2, n\delta, u) \\
& + \frac{1}{2} \sigma_V V \frac{\delta}{h_1 h_2} \rho^+ \, v(x - h_1 e_1 - h_2 e_2, n\delta, u) \\
& + \frac{1}{2} \sigma_V V \frac{\delta}{h_1 h_2} \rho^- \, v(x - h_1 e_1 + h_2 e_2, n\delta, u) \\
& + \frac{1}{2} \sigma_V V \frac{\delta}{h_1^2} (-\rho^+ - \rho^-) \, v(x + h_1 e_1, n\delta, u) \\
& + \frac{1}{2} \sigma_V V \frac{\delta}{h_1^2} (-\rho^+ - \rho^-) \, v(x - h_1 e_1, n\delta, u) \\
& + \frac{1}{2} \sigma_V V \frac{\delta}{h_2^2} (-\rho^+ - \rho^-) \, v(x + h_2 e_2, n\delta, u) \\
& + \frac{1}{2} \sigma_V V \frac{\delta}{h_2^2} (-\rho^+ - \rho^-) \, v(x - h_2 e_2, n\delta, u).
\end{aligned}
$$

So far we have only discretized the PDE and rearranged the equation but we can already obtain the desired discrete-time Markov chain. In the next step we identify transition probabilities by looking at the coefficients of the respective values at the grid points of the value function v. Because of the discretization the equation at each point only involves neighboring states.[195] In addition, we exploit the

[195] This is a very desirable property. Its consequence is that the iteration matrices below are rendered sparse, i.e. there are only five diagonals filled in the matrices. By virtue of that property, iteration speed is significantly enhanced compared

relationships

$$
\begin{aligned}
a^+ + a^- &= |a| \\
a^+ - a^- &= a,
\end{aligned}
$$

where a^+ is the positive part and a^- the negative part of a real number a given by

$$
\begin{aligned}
a^+ &= \max(a, 0) \\
a^- &= -\min(a, 0).
\end{aligned}
$$

Doing so yields the following diffusive transition probabilities and interpolation interval

$$
\begin{aligned}
&\hat{p}_D^{h,\delta}((x, n\delta), (x + h_Y e_Y, n\delta)|\gamma) \\
&\quad = \frac{\left(r - \lambda^Q \mu^Q - \frac{1}{2} V_t\right)^+ \frac{\delta}{h_Y} + \frac{1}{2} V_t \frac{\delta}{h_Y^2} - \frac{1}{2} \sigma_V V_t |\rho| \frac{\delta}{h_Y h_V}}{Q^{h,\delta}(x)} \\
&\hat{p}_D^{h,\delta}((x, n\delta), (x - h_Y e_Y, n\delta)|\gamma) \\
&\quad = \frac{\left(r - \lambda^Q \mu^Q - \frac{1}{2} V_t\right)^- \frac{\delta}{h_Y} + \frac{1}{2} V_t \frac{\delta}{h_Y^2} - \frac{1}{2} \sigma_V V |\rho| \frac{\delta}{h_Y h_V}}{Q^{h,\delta}(x)} \\
&\hat{p}_D^{h,\delta}((x, n\delta), (x + h_V e_V, n\delta)|\gamma) \\
&\quad = \frac{\kappa(\theta - V_t)^+ \frac{\delta}{h_V} + \frac{1}{2} V \sigma_V^2 \frac{\delta}{h_V^2} - \frac{1}{2} \sigma_V V_t |\rho| \frac{\delta}{h_Y h_V}}{Q^{h,\delta}(x)} \\
&\hat{p}_D^{h,\delta}((x, n\delta), (x - h_V e_V, n\delta)|\gamma) \\
&\quad = \frac{\kappa(\theta - V_t)^- \frac{\delta}{h_V} + \frac{1}{2} V \sigma_V^2 \frac{\delta}{h_V^2} - \frac{1}{2} \sigma_V V_t |\rho| \frac{\delta}{h_V h_Y}}{Q^{h,\delta}(x)}
\end{aligned}
$$

to a full matrix. If we were to employ an alternative finite difference scheme involving more than the directly neighboring states this would lead to additional diagonals in the interation matrix being filled.

$$\begin{aligned}
\hat{p}_D^{h,\delta}((x,n\delta),(x+h_Y e_Y+h_V e_V,n\delta)|\gamma) &= \frac{\frac{1}{2}\sigma_V V_t \frac{\delta}{h_Y h_V}\rho^+}{Q^{h,\delta}(x)} \\
\hat{p}_D^{h,\delta}((x,n\delta),(x+h_Y e_Y-h e_V,n\delta)|\gamma) &= \frac{\frac{1}{2}\sigma_V V_t \frac{\delta}{h_Y h_V}\rho^-}{Q^{h,\delta}(x)} \\
\hat{p}_D^{h,\delta}((x,n\delta),(x-h_Y e_Y+h_V e_V,n\delta)|\gamma) &= \frac{\frac{1}{2}\sigma_V V_t \frac{\delta}{h_Y h_V}\rho^-}{Q^{h,\delta}(x)} \\
\hat{p}_D^{h,\delta}((x,n\delta),(x-h_Y e_Y-h_V e_V,n\delta)|\gamma) &= \frac{\frac{1}{2}\sigma_V V_t \frac{\delta}{h_Y h_V}\rho^+}{Q^{h,\delta}(x)} \\
\hat{p}_D^{h,\delta}((x,n\delta),(x,n\delta+\delta)|\gamma) &= \frac{1}{Q^{h,\delta}(x)} \\
\Delta\hat{t}^{h,\delta}(x,\gamma) &= \frac{\delta}{Q^{h,\delta}(x)},
\end{aligned}$$

where $Q^{h,\delta}(x)$ is defined by

$$\begin{aligned}
Q^{h,\delta}(x) &= 1+\frac{\delta}{h_Y}\left(\left|r-\lambda^Q\mu^Q-\frac{1}{2}V_t\right|\right)+\frac{\delta}{h_V}\left(\kappa|\theta-V|\right)+ \\
&\quad V\delta\left(\frac{1}{h_Y^2}+\frac{\sigma_V^2}{h_V^2}-\frac{|\rho|\sigma_V}{h_Y h_V}\right).
\end{aligned}$$

Due to this choice of the finite differences, which only involve neighboring states for discretization at each point, a chain is constructed which can also only change to neighboring states in a diffusion model. In the presence of jumps these diffusive transition probabilities have to be altered in order to account for movements to non-neighboring states. The adjusted transition probabilities are obtained by the following convex combination[196] of the diffusive part and the yet to be determined jump probability involving the jump intensity λ

$$p(x,y|\gamma) = (1-\lambda\,\Delta\hat{t}^{h,\delta}(x,\gamma))\,\hat{p}_D^{h,\delta}(x,y|\gamma)+$$

[196] By virtue of this convex combination we can treat the approximation of diffusion and jump risk separately. In case there is no jump risk, i.e. the a vanishing jump intensity λ^Q, the following approximation will collapse to the already established diffusion part.

$$\lambda\, \Delta \hat{t}^{h,\delta}(x,\gamma)\, \Pi\{\rho | q_h(x,\rho) = y - x\}.$$

The jump probability can be computed given the assumption of the normal distribution of jumps in logarithmic prices. Denoting the cumulative distribution function of the normal distribution by Φ_{N} the jump probability then is computed as

$$\Pi\{\rho | q_h(x,\rho) = y - x\} = \left(\Phi_{\mathrm{N}}(y_1 - x_1 + \frac{1}{2}h) - \Phi_{\mathrm{N}}(y_1 - x_1 - \frac{1}{2}h) \right).$$

These values are readily available by numerical integration. Define the Markov chain $\{\xi_n^{h,\delta}\}_n$ as the chain with the transition probabilities given above and the states given by the grid S_h. In Section 4.2.2 local consistency is established.

What remains to be done is the replacement of the dynamic programming equations for the optimal value function by the ones for its discrete-time counterpart $Z^{h,\delta}$. In the latter case the expected value is the probability weighted average of the neighboring states to which traversal of the chain is possible. At the same time we can incorporate the decision between stopping and continuing in a $0-1$-parameter γ. The discrete version of the dynamic programming equations reads

$$\begin{aligned} Z^{h,\delta}(x, n\delta) &= \max(g(\xi_n^{h,\delta}), E_{x,t}^{h,\delta} Z^{h,\delta}(\xi_n^{h,\delta})) \\ &= \max_\gamma [\sum_y \hat{p}^{h,\delta}(x, n\delta; y, n\delta | \gamma)\, Z^{h,\delta}(y, n\delta) \\ &\quad + \exp(-r\delta)\, \hat{p}^{h,\delta}(x, n\delta; x, n\delta + \delta | \gamma)\, G^{h,\delta}(x, n\delta + \delta)], \end{aligned}$$

where $\gamma \in \{0, 1\}$ represents the choice between stopping and continuing and $\hat{p}^{h,\delta}$ are the probabilities of movement to the respective states. The intuitive interpretation of the procedure is that we decide between the better of two alternatives: immediate exercise and the expected value of another step on the grid.

Moreover, we face the usual problem of having to truncate the computational domain because we can treat an infinite domain in neither the stock price dimension nor the volatility direction. This truncation comes along with adding an obligatory stopping condition at the truncation points in stock price direction and a reflecting boundary in the volatility direction due to mean reversion. Such compulsory stopping means that we immediately exercise the option if we ever exceed a pre-determined maximal stock price $S_{\max}$. This obligatory stopping threshold can be chosen such that the numerical solution is virtually independent of it. Technically speaking we can deal with this by adding a vector which can be interpreted as the expected additional exercise value when exceeding the domain. If $x \in S_h$ denotes the present state of the chain on the computational grid this vector reads

$$\begin{aligned}\hat{C}(x, n+1, \gamma) \quad = \quad & \sum_{y \in \partial S_h} \hat{p}^{h,\delta}(x, n\delta; y, n\delta|\gamma)\, g(y, n\delta) + \\ & \exp(-r\delta)\, \hat{p}^{h,\delta}(x, n\delta; x, (n+1)\,\delta|\gamma) \\ & Z^{h,\delta}(x, (n+1)\,\delta).\end{aligned}$$

The value function is replaced by the payoffs if there is a step onto the stopping boundary. They can be considered as additional artificial payoffs which are imposed by and associated with obligatory stopping. If we further introduce a transition matrix $R_\gamma^{h,\delta} = \hat{p}^{h,\delta}(x, n\delta; y, n\delta|\gamma)$ we can rewrite the dynamic programming equations in the following compact matrix-vector notation

$$Z^{h,\delta}(x, n\delta) = \max_\gamma \left[R_\gamma^{h,\delta}\, Z^{h,\delta}(x, n\delta) + \hat{C}(x, n+1, \gamma) \right]. \quad (4.2.1.1)$$

Intuitively, the approximative chain traverses the grid, which is governed by the transition matrix $R_\gamma^{h,\delta}$ and at each step decides whether or not to continue. If the chain ever exceeds the boundary or if the time horizon is reached, immediate stopping is imposed, which is governed by the additional payoff vector $\hat{C}$.

4.2.2 Local Consistency

By $\Delta\xi_n^h := \xi_{n+1}^h - \xi_n^h$ we define the increment of the approximating Markov chain. Assume further that the following conditions hold regarding the expectation and covariance structure of the approximating chain

$$
\begin{aligned}
E_n^h[\Delta\xi_n^h] &= \Delta t^h(x,u)b(x,u) + \\
&\quad \mathcal{O}(h^\alpha \Delta t(x,u)) \\
E[(\Delta\xi_n^h - E[\Delta\xi_n^h])(\Delta\xi_n^h - E[\Delta\xi_n^h])^T] &= \sigma(x)\sigma(x)^T\Delta t^h(x,u) + \\
&\quad \mathcal{O}(h^\alpha \Delta t(x,u)) \\
|\xi_{n+1}^h - \xi_n^h| &= \mathcal{O}(h).
\end{aligned}
$$

These conditions are introduced by Kushner (1990) and referred to as local consistency conditions as the approximating chain locally mimics the behavior of the underlying stochastic process. These approximating properties play a crucial role in the convergence proofs of the payoff function. The exposition below is based on Kushner (1990) and Kushner & Dupuis (2001) and is carried out for the previously mentioned interpretation of minimizing a cost functional.[197] From now on assume that there is a compact set G such that the process is forced to stop by the time it leaves the interior G^0 of G,

[197] Without difficulty one sees that minimization and maximization problems are equivalent by virtue of $\max f(x) = \min(-f(x))$ for every real valued function f.

i.e.

$$\tau = \inf_t \{X(t) \notin G^0\},$$

unless it has been stopped previously. The purpose now is to find the stopping time $\bar{\tau} \leq \tau$ such that

$$v(x) = \inf W(x, \bar{\tau}).$$

To proceed we have to make several assumptions regarding the underlying stochastic process:

1) X has a unique solution in the weak sense as described by Karatzas & Shreve (2008) and Øksendal (2010).[198]
2) $b(\cdot)$ and $\sigma(\cdot)$ are bounded and continuous
3) $q(\cdot)$ is measurable and bounded, $q(\cdot, \rho)$ is continuous for each ρ
4) $g_1(\cdot)$ and $g_2(\cdot)$ are bounded and continuous[199]
5) G is compact and the closure of its interior.[200]

Theorem 4.2.1 *Under the above assumptions and if* $\inf_{x \in G} g_1(x) = k_0 > 0$, *there exists an optimal stopping time* τ' *and*

$$E_x[\tau'] \leq \max_{y \in G} \frac{g_2(y)}{k_0}.$$

[198] Essentially this amounts to the drift, dispersion and jump parameters satisfying uniform Lipschitz conditions, as given in the same textbooks. Since we confine our process to a compact set these conditions are, indeed, satisfied.

[199] As before, the conditions in items ii) through iv) follow from the confinement to a compact set.

[200] According to Bredon (2010) the smallest closed set F with $A \subset F \subset Y$ is called the closure of the set A in a topological space Y.

Proof - Theorem 4.2.1:
See Kushner & Dupuis (2001).

□

This theorem reduces the set of stopping times that must be considered as optimal ones will always have a bounded first moment. Hence, others can be neglected. Given the discrete time version of the dynamic programming equation the convergence theorem can formulated as follows:

Theorem 4.2.2 *If* $\sup_{x\in G} E_x[\tau'] < \infty$ *replaces the strict positivity of* g_1 *or under the conditions of Theorem 4.2.1, where* $\tau' = \inf\{t|X(t) \notin G\}$*, we have* $V^h \to V$*.*

The following proof is based on the ones given in Kushner (1990) and Kushner & Dupuis (2001).

Proof - Theorem 4.2.2:
Let $(\psi^h(\cdot), \bar{\rho}_h)$ denote the continuous parameter approximating chain and its optimal stopping time respectively and define $w^h(\cdot)$ and $N^h(\cdot)$ as

$$\begin{aligned} w^h(t) &= \int_0^t D_h^+(s) P_h'(s)\, dM^h(s) \\ &\quad + \int_0^t (I - D_h(s) D_h^+(s))\, d\tilde{w}(s) \\ N^h(t, H) &= \sum_{n:\nu_n^h \le t} I_{\rho_n \in H}. \end{aligned}$$

Then the sequence $(\psi^h(\cdot), w^h(\cdot), N^h(\cdot), \bar{\rho}_h)$ is tight and we can assume that $\bar{\rho}_h$ satisfies the upper bound from Theorem 4.2.1 for all h and $x \in G^0$. By virtue of the Markov property it follows for integral k that

$$\limsup_h E_x[\bar{\rho}_h]^k < \infty,$$

which in turn renders the sequence of stopping times uniformly integrable. Let $(x(\cdot), w(\cdot), N(\cdot), \rho)$ denote the limit of a weakly convergent subsequence. Then the limit process satisfies

$$X(t) = x + \int_0^t b(X(s))ds + \int \sigma(X(s))\, dW(s) \qquad (4.2.2.1)$$
$$+ \int_0^t \int_\Gamma q(x(s^-), \rho)\, N(ds\, d\rho)$$

and there is a filtration $\mathcal{F}_t$ such that $w(\cdot)$ is an $\mathcal{F}_t$-Wiener process, $N(\cdot)$ is an $\mathcal{F}_t$-Poisson measure, ρ is an $\mathcal{F}_t$-stopping time and $X(\cdot)$ is adapted to $\mathcal{F}_t$. By the uniform integrability and the weak convergence we have

$$W^h(x, \bar{\rho}_h) = v^h(x) \to W(x, \rho) \geq v(x).$$

In order to obtain the reverse inequality we apply an ϵ-optimal stopping rule. For this purpose, assume $\epsilon > 0$ and note that there are $\delta > 0$ and $T < \infty$ such that the stopping times for (4.2.2.1) only take the values $\{n\delta, n\delta \leq T\}$ with increasing the cost functional by at most ϵ. Assume further that ρ_ϵ is a stopping time for that restrained problem. Then this ϵ-optimal stopping time can be viewed as defined by functions $F_n(\cdot)$ which are continuous in w for each of the other variables and such that the probability law of ρ_ϵ is determined by $P[\rho_\epsilon = 0]$ and for $n > 1$

$$P[\rho_\epsilon = n\delta \mid w(s), N(s), s \leq n\delta, \rho_\epsilon > n\delta - \delta]$$
$$= F_n(w(p\theta), N(p\theta, \Gamma_j^q), j \leq q, p\theta \leq n\delta).$$

Let ρ_h denote the stopping time for $\psi^h(\cdot)$ which is analogous to ρ_ϵ. Define σ_n^h by

$$P[u^h(\sigma_n^h) = \alpha \mid u^h(\sigma_i^h), i < n, \psi^h(s), w^h, N^h(s), s \leq n\delta]$$
$$= F_n(\alpha, u^h(\sigma_i^h), i < n, w^h(p\theta), N^h(p\theta, \Gamma_j^q),$$

$$j \leq q, p\theta \leq \sigma_n^h)$$

and let the probability law of ρ_h be determined by $P[\rho_h = 0] = P[\rho_\epsilon = 0]$ and for $n > 1$

$$\begin{aligned}P[\rho_h = \sigma_n^h \mid \psi^h(s), w^h(s), N^h(s), s \leq \sigma_n^h, \rho_h > \sigma_{n-1}^h] \\ = F_n(w^h(p\theta), N^h(p\theta, \Gamma_j^q), j \leq q, p\theta \leq \sigma_n^h).\end{aligned}$$

By weak convergence and the uniqueness of (4.2.2.1) the proof is completed.

□

Finally, we point out that the approximating Markov chain indeed satisfies the above local consistency conditions. This was first shown by Kushner & DiMasi (1978).

4.3 Numerical Solution Methods

Given the discretized version (4.2.1.1) of the dynamic programming equations as derived in Section 4.2.1 they have to be solved numerically. For this purpose many of the iterative solution techniques used for linear systems of equations, such as the Jacobi iteration or the Gauß-Seidel iteration, can be used in accordingly adjusted form.

For these methods to be applied, we have to impose a set of further assumptions suggested by Kushner & Dupuis (2001) for the numerical procedure:

1) R and $\hat{C}$ are continuous in the control for each pair x and y from the state space.

2) There exists an admissible feedback control $u_0(\cdot)$ such $R(u_0)$ is

a contraction and the infima of the costs over all admissible controls are bounded from below.

3) $R(u)$ is a contraction for any feedback control $u(\cdot)$ for which the associated cost is bounded.

4) If the cost associated with the sequential use of the feedback controls $u^1(\cdot), \ldots, u^n(\cdot), \ldots$, is bounded, then $R(u^1)\ldots R(u^n) \stackrel{n\uparrow\infty}{\to} 0$

Because of the discounting involved in the setup of the iteration matrix the contraction properties are satisfied. Furthermore, as we assume a compact computational domain boundedness follows immediately where required, as continuous functions take on there maxima and minima on compact sets. This allows us to formulate the Jacobi iteration.

Proposition 4.3.1 (Jacobi Iteration) *Let $u(\cdot)$ be an admissible feedback control such that R_γ is a contraction and make the above assumptions. Then for any vector Z_0 the sequence recursively defined by*

$$Z_{m+1}(x) = \max_\gamma \left[\sum_y R_\gamma Z_m(y,n) + \hat{C}(x, n+1, \gamma) \right], \qquad (4.3.0.2)$$
$$n = N, N-1, \ldots, 0$$

converges to the unique solution of the optimal stopping problem.

Proof - Proposition 4.3.1:
Let $u^m(\cdot)$ be minimizing in (4.3.0.2) at step m then

$$\begin{aligned} Z_1 &= R(u^1)Z_0 + C(u^1) \\ Z_m &= R(u^m)\ldots R(u^1)Z_0 + \sum_{i=1}^m R(u^m)\ldots R(u^{i+1})C(u^i). \end{aligned}$$

This is the m-step cost for the policy which uses $u^i(\cdot)$ when there are still i steps to go and with terminal cost Z_0. The maximizing property in (4.3.0.2) yields that for any other admissible feedback control sequence $\{\tilde{u}^m(\cdot)\}$ for which the payoff is bounded

$$Z_{m+1} \geq R(\tilde{u}^{m+1})V_m + C(\tilde{u}^{m+1}).$$

Iterating the last inequality yields

$$Z_{m+1} \geq R(\tilde{u}^{m+1})\ldots R(\tilde{u}^1)Z_0 + \sum_{i=1}^{m+1} R(\tilde{u}^{m+1})\ldots R(\tilde{u}^{i+1})C(\tilde{u}^i),$$

which is the payoff for an $m+1$-step process under the controls $\{\tilde{u}^i(\cdot)\}$ and terminal payoff Z_0. Thus, Z_m is indeed the asserted maximal m-step payoff.

Since there is a unique solution to the discrete time formulation of the dynamic programming equations (4.2.1.1) we can assume $\tilde{u}(\cdot)$ to be a maximizer therein. Then we have

$$\begin{aligned} R(u^{m+1})V_m + C(u^{m+1}) &= Z_{m+1} \\ &\leq R(\tilde{u})Z_m + C(\tilde{u}) \\ R(\tilde{u})V + C(\tilde{u}) &= Z \\ &\geq R(u^{m+1})Z + C(u^{m+1}), \end{aligned}$$

which implies that

$$\begin{aligned} R(\tilde{u})(Z - Z_m) &\geq (Z - Z_{m+1}) \\ &\geq R(u^{m+1})(Z - Z_m). \end{aligned}$$

By iterating this latter set of inequalities we obtain

$$R^{m+1}(\tilde{u})(Z - Z_0) \geq Z - Z_{m+1}$$

$$\geq \quad R(u^{m+1}) \dots R(u^1)(Z - Z_0).$$

The boundedness of $\{Z_m\}$ follows from the contraction property of $R(\tilde{u})$. Thus by $R(u^m) \dots R(u^1) \stackrel{m\uparrow\infty}{\longrightarrow} 0$ it follows that $Z_m \stackrel{m\uparrow\infty}{\longrightarrow} Z$.

□

Kushner & Dupuis (2001) point out that the algorithm can be enhanced by making use of new iterates or values for $Z^n(x)$ once they become available and not only when they are available for all x. This slight adjustment is called Gauß-Seidel iteration and can be formulated as follows:

Proposition 4.3.2 (Gauß-Seidel Iteration) *Let $u(\cdot)$ be an admissible feedback control such that R_γ is a contraction and make the above assumptions. Then for any vector Z_0 the sequence recursively defined by*

$$\begin{aligned} Z_{m+1}(x) \quad &= \quad \max_\gamma [\sum_{y<x} R_\gamma Z_{m+1}(y,n) + \sum_{y \geq x} R_\gamma Z_m(y,n) \\ & \qquad + \hat{C}(x, n+1, \gamma)], n = N, N-1, \dots, 0 \end{aligned}$$

converges to the unique solution of the optimal stopping problem.

Proof - Proposition 4.3.2:

A proof of the theorem as well as elucidations of further properties can be found in Kushner & Kleinman (1971) and Kushner (1972).

□

Schweitzer, Puterman & Kindle (1985) present an aggregation-disaggregation procedure to numerically solve equations as they arise in semi-Markov reward processes. Applications include queueing systems and controlling multiproduct inventories.[201] The proposed iterative algorithm is found to be superior to alternative successive approximations. Chernoff & Petkau (1986) show that certain classes of sequential decision problems collapse to optimal stopping problems. These problems can than be numerically tackled by solving the corresponding free-boundary value problems for the heat equation via further simplified optimal stopping problems. The authors assess these approximations as good rough estimates of the exact solution which can be exploited for the construction of continuity corrections for the original problems. In the working paper by Tsitsiklis & van Roy (1997) the authors investigate optimal stopping problems for discrete time ergodic Markov processes.[202] They show existence and uniqueness of solutions to Bellman's equation and provide a stochastic approximation which converges almost surely.

Kushner (1999) extends the Markov chain approximation methods introduced by Kushner (1990) to diffusion processes with a control-dependent variance. The key to efficient numerical solutions is to allow for transitions to non-neighboring states. These states are chosen based on the state and the control. Besides, convergence of the suggested approximation is proved. In his PhD thesis, Engel (2009) gives a comprehensive account of constrained optimization problems. It is particularly concerned with PDE constraints and quadratic functionals. For such problems a multigrid solver is developed, which includes smoothing and pre-conditioning. In addition,

[201] Queueing theory or the theory of queueing systems deals with the study of waiting lines from a mathematical perspective. Economically relevant applications among others include production processes, telecommunications and traffic engineering. For an introduction the reader is referred to Gross, Shortle, Thomson & Harris (2008) and Zukerman (2011).

[202] For a detailed explanation of the term ergodic we refer the reader to the introductory textbook by Norris (1997) on Markov chains.

active set strategies are included that deal with sequential quadratic programming.[203]

[203] Quadratic programming, similar to linear programming, deals with the mathematical optimization of quadratic functions under constraints. According to Nocedal & Wright (2006) active sets are those constraints in mathematical programming for which equality rather than inequality holds.

Chapter 5

Comparative Statics Analysis of OETCs

In Section 2.3.2 we introduced OETCs and pointed out that they can be interpreted as American-style options with a moving striking price. American options can in turn be understood as optimal stopping problems. The former was expounded on in Section 3.2.2.5, while Section 4.2 provided solution techniques for these optimal stopping problems. In this chapter we will now combine these results, rephrase the valuation of OETCs as an optimal stopping problem, and carry out a comparative statics analysis.

This comparative statics analysis of OETCs contributes to the literature in several aspects. First, we analyze optimal exercise thresholds. These thresholds depend on jump risk as well as product characteristics such as the financing and gap parameters defined above in Section 2.3.2. The amount of jump risk is positively related to early exercise thresholds. Secondly, we argue that in the absence of jumps the existence of most OETCs cannot be explained at all. Instead, even retail investors could generate more attractive payoff structures by borrowing money and buying the underlying stock. Thirdly, in a situation with a fixed investment horizon we identify the early exercise premium as the difference between European and American-style Turbo certificates. It turns out that the early exercise premium significantly impacts the price. Finally, in an empirical study we investigate a sample comprising all $1,345$ long OETCs traded in the market on July 16, 2010. On the observation day, all but one (a knock-out) of the certificates should be exercised. This result suggests that investors do not exercise certificates optimally, which in turn leads to profits for the issuers.

This section builds on two strands in the literature. The first strand is the growing literature on investment certificates mentioned above in Section 2.3. Similarly to other authors, we take an option pricing approach to Turbo certificates. Option pricing models have been extensively discussed and analyzed in Section 3.1.

The second strand is the literature on American-style options for which various solution techniques have been developed. These techniques have been outlined above and include analytical approximations (cf. Section 3.2.2.1), binomial models (cf. Section 3.2.2.2), finite differences (cf. Section 3.2.2.3), Monte Carlo simulation (cf. Section 3.2.2.4), and optimal control theory (cf. Section 3.2.2.5).

5.1 Market Model

In Section 2.3.2 we introduced the mechanics, fundamentals and characteristics of OETCs. In particular we pointed out in Proposition 2.3.1 that the existence of these certificates can only be rationally explained if the underlying is subject to jump risk. The intuitive explanation for this is that in pure diffusion models financing costs $r + z$ always exceed the expected rate of return r of the underlying.

In the presence of discontinuous stock price processes the Turbo certificate could be replicated by investors who buy a stock and borrow an amount of money equal to K_0. However, the replication would only be exact if the interest rate applicable for borrowing money is equal to $r + z$ and the LGD (Loss Given Default) in the knock-out case is 100% (independently of the amount by which the strike is undershot). In other words, the replication argument assumes that all investors have the same credit quality and that they have no further funds in order to cover losses in the event of a jump related knock-out. Certainly, these are very restrictive assumptions.[204] In particular, investors with a better credit quality and consequently lower credit spreads might find certificates attractive because they allow for speculation on stock price increases while offering protection against downward jumps.[205] This protection against downward jumps provides option value to the owner of a certificate. The question naturally arises under which conditions it is favorable to own the certificate. The cost from the financing rate that is potentially higher than the credit spread (or LGD of less than 100%) has to be compared to the benefit of protection against downward jumps.[206]

[204] Despite the fact that these assumptions appear rather restrictive at first glance, they are still natural to a certain degree. This is because on the one hand the issuers of OETCs do not take the creditworthiness of buyers into account in a way more sophisticated than assuming the same credit spread for every investor. On the other hand, as Entrop, Scholz & Wilkens (2009) pointed out, this setup allows the issuers to superhedge an OETC as long as z is greater than the issuer's own credit spread. Thus credit quality does not necessarily have to be modeled explicitly.

[205] In this way, these investors are charged a higher credit spread than the one corresponding to their respective creditworthiness. On the other hand this can be justified to a certain extent by protection against downward jumps below the striking price of the OETC.

[206] The latter further stresses the above explanation that only discontinuous stock price processes provide option value to the holders of OETCs. If the stock price

For simplicity and in order to analyze the impact of the protection against downward jumps we assume that investors and issuers cannot default, i.e. both can borrow and lend at the risk free rate of interest.[207] Furthermore, there exists a risk-neutral measure Q. Investors face an optimal stopping problem in which they want to maximize the present value of the expected payoff

$$\begin{aligned} g^*(S_t) &= \max_T E^Q\left[\exp(-r\,T)\,g(S_T,T)\right] \\ &= \max_T E^Q\left[\exp(-r\,T)\,\max(S_T-K_T,0)\right] \end{aligned}$$

under the risk neutral measure. Investors need to determine the optimal stock price threshold at which it is optimal to exercise the certificate. This is similar to the problem faced by the holder of an American-style call option. The main differences are that strike and barrier are not constant and that certificates have infinite maturity. In the following we assume that the model proposed by Bakshi, Cao & Chen (1997) applies, i.e.

$$\begin{aligned} dX_t = \begin{pmatrix} dY_t \\ dV_t \end{pmatrix} &= \begin{pmatrix} r-\lambda^Q\mu_X^Q-\frac{1}{2}V_t \\ \kappa(\theta-V_t) \end{pmatrix} dt \\ &+ \sqrt{V_t}\begin{pmatrix} 1 & 0 \\ \rho\sigma_V & \sqrt{1-\rho^2}\,\sigma_V \end{pmatrix} dW_t \\ &+ \begin{pmatrix} \xi^Y dN_t^Y \\ 0 \end{pmatrix}, \end{aligned}$$

process was continuous and there could thus be no jumps, downward jump protection would be moot and investors would be charged positive credit spreads z for a worthless option. In this case the whole investment would also boil down to a simple leveraged investment in the underlying. In order to materialize those investments require that the expected rate of return of the underlying (the riskless rate of return r for risk-neutral investors) is higher than the financing costs $r+z$.

[207] Potential default of issuers has a significant pricing impact as well. This is demonstrated in Baule, Entrop & Wilkens (2005).

where $Y_t = \ln(S_t)$ and

$$\xi^Y \sim N(\mu_X^Q, \sigma_X^2)$$

under the risk neutral measure. The vector of state variables contains Y_t and V_t which denote the log-stock price and its variance respectively. Log-stock prices are exposed to jump and diffusion risk, the variance is affected by diffusion risk only. By W_t we denote a two-dimensional Wiener process, by N_t^Y a one-dimensional Poisson process which can be interpreted as the jump counter, and ξ^Y can be viewed as the jump size which is normally distributed. The correlation of diffusion risk between both state variables is ρ. The variance has a long term mean θ to which it gravitates at a mean reversion speed κ. The volatility of variance is σ_V. The risk neutral jump intensity is λ^Q, i.e. intuitively speaking the risk-neutral probability of a jump happening during a small time interval Δ is roughly $\lambda^Q\Delta$. By standard arguments we assume that the jump intensity and the expected jump size as well as mean reversion speed and level need to be adjusted when moving from the empirical measure P to the risk neutral measure Q. The model nests the models of Black & Scholes (1973), Heston (1993) and Merton (1976) as special cases. In Table 5.1 we have summarized for which values of the parameters the respective models are retrieved.

5.2 Presentation of Comparative Statics

The objective of a comparative statics analysis is two-fold. On the one hand it aims at validating the chosen market model as suitable for the problem at hand. This is attained by demonstrating that the key parameters of OETCs, the optimal exercise threshold and the early exercise premium, depend on the parameters of the model

Parameter	Description	Black & Scholes (1973)	Merton (1976)	Heston (1993)
λ^Q	Jump intensity	0	-	0
μ_X^Q	Expected jump size	0	-	0
κ	Mean reversion speed	0	0	-
θ	Mean reversion level	0	0	-
ρ	Diffusive Correlation	0	0	-
σ_V	Volatility of volatility	0	0	-
ξ^Y	Jump size random variable	0	-	0

Table 5.1: **Models Nested in the Bakshi, Cao & Chen (1997) Model**
This table shows how the Bakshi, Cao & Chen (1997) model nests the model by Black & Scholes (1973), the pure stochastic volatility model by Heston (1993) and the pure jump-diffusion model by Merton (1976). A zero in the table indicates that the respective parameter has to be set to zero in order to collapse the model to a simpler one. Similarly, '-' denotes that the parameter is present in the model and has to be estimated from market data.

in an economically meaningful, explicable and intuitive way. Naturally, these considerations are of a more qualitative than quantitative character. On the other hand, the comparative statics analysis provides insight to which extent the optimal exercise threshold and the early exercise premium depend on the parameters of the model.

In our comparative statics analysis we investigate the pricing of OETCs. Especially, the choice of the jump parameters is important since jump risk is the main driver for optimal exercise thresholds. In the base case scenario, we choose the following specification

$$\begin{aligned}
\lambda^Q &= 0.7 \\
\mu_X^Q &= -0.14 \\
\sigma_J &= 0.08 \\
\kappa &= 3.4 \\
\theta &= 0.05 \\
\rho &= -0.77 \\
r &= 4\% \\
K &= 4,800.
\end{aligned}$$

This generic selection is based on Eraker (2004) and Breuer (2008) who conduct parameter estimations for various stock markets. Moreover, we assume that the Turbo certificate is written on a non-dividend paying stock and that the financing and gap parameters are given by

$$\begin{aligned}
a &= 1.5\% \\
z &= 3.0\%.
\end{aligned}$$

The choice of the gap parameter a implies $K_t = \frac{L_t}{1+a}$ for the strike. These are typical contract specifications traded in the market, e.g.

for Turbo certificates on the German stock market index DAX to be analyzed empirically below in Section 5.3 and Chapter 6.

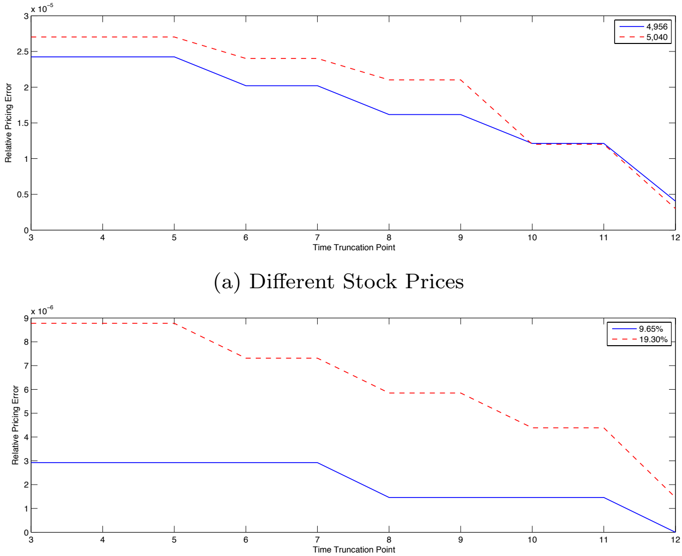
The outline of this section is as follows. First, we investigate the effect of truncation of the computational domain. Second, we analyze early exercise thresholds. Third, we follow Entrop, Scholz & Wilkens (2009) and consider a fixed investment horizon. This allows us to derive early exercise premiums.

5.2.1 Truncation of the Computational Domain

In order to compute numerical prices we have to truncate the computational domain in the time direction without significantly influencing the results.[208] In Figure 5.1 we have plotted the relative pricing errors for several levels of stock prices, volatilities, and truncation points in relation to the solution with a time horizon of 15 years. This truncation can be thought of as a comparative static of the imposed maturity. In other words, we force stoppage at a fixed maturity and examine how prices behave when changes to this maturity are made.

In Figure 5.1, we observe that for examined stock prices and volatilities the relative pricing error decreases when longer maturities are taken into account. However, we notice that the overall level of error is extremely small suggesting convergence of the algorithm: In the stock price case the pricing error amounts to less than 0.003% and to less than 0.001% when considering different volatilities. Note

[208] The solution methods for optimal stopping problems presented in Section 4.3 work backwards in time and start at maturity. Therefore, we have to impose an artificial maturity (which amounts to a fixed investment horizon) in a way that does not alter results significantly. Taking into account that infinite-lived derivatives have a vanishing theta risk, it becomes plausible that such a truncation point is available. In contrast to the fixed investment horizon considered by Entrop, Scholz & Wilkens (2009) we still allow for premature exercise and do not render the valuation problem European-style.



(a) Different Stock Prices

(b) Different Volatility

Figure 5.1: **Comparative Statics Time-to-Maturity**
This figure shows the relative pricing error compared to a 15-year time horizon. The scenario considered is the above base case scenario with gap size $a = 1.5\%$, financing parameter $z = 3.0\%$, jump intensity $\lambda^Q = 0.73$ expected jump size of $\mu_X^Q = -0.1437$, and jump volatility $\sigma_J = 0.0822$. The initial strike is $4,800$.

that the plot takes the shape of steps due to stepwise changes in maturity. Interpolating with a different function is highly debatable, since we lack knowledge of the exact asymtotics of truncation. All in all, changing the truncation point is virtually irrelevant and we can we henceforth assume that the selection of maturity $T = 5$ yields meaningful results.

5.2.2 Exercise Thresholds

Figure 5.2 shows optimal thresholds in various scenarios. Note that these thresholds are the maximum stock prices at which investors ought to hold the certificates. Intuitively, this can be explained by the matter of fact that investors compare the option value from protection against downward jumps with the financing parameter z. If the option value is higher than the costs incurred by increasing strike and barrier prices, investors are attracted by the certificate. The option value is negatively related to stock prices because higher stock prices render jumps below the strike level less likely. Therefore, rational investors will hold the certificates until the underlying stock price reaches the threshold level and then exercise it immediately. As a result, we can identify situations in which it is not rational to hold the certificates.

Figure 5.2a shows the relationship between the optimal exercise threshold and the gap size a for various parameters of diffusion volatilities $\sigma_{\text{diff}} = \sqrt{V}$. We make two observations: First, there is a negative relationship between the threshold and the gap size. The larger the gap size the less probable a beneficial jump is and in consequence holding the certificate is less attractive. Therefore, the option should be exercised earlier. Second, diffusion volatility has a positive impact on Turbos. Intuitively, high volatility increases the probability of movements in the direction of the knock-out barrier and thus into a region in which jumps below the strike are likely.

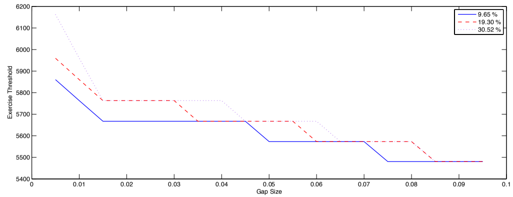
In Figure 5.2b, we analyze the impact of the financing parameter on thresholds. The relationship is negative since a financing parameter is unattractive to investors. This is due to the fact that there is higher cost associated with continuation and hence the threshold lowers. Alternatively, we might state that a high financing parameter leads to less demand of the products due to smaller continuation regions. Again, volatility has a positive impact on certificates' thresholds. This can be explained by a similar argument as above.

Figures 5.2c and 5.2d depict the relationship between thresholds and jumps. Generally speaking, they confirm that thresholds increase as negative jumps become more likely. In this case the protection against downward jumps is more valuable and the continuation region grows. In Figure 5.2c more negative jumps are the result of an increased jump intensity. In Figure 5.2d, decreases of expected jump sizes shift the jump probability distribution to the left. The impact of diffusion volatility can be interpreted as in Figures 5.2a and 5.2b.

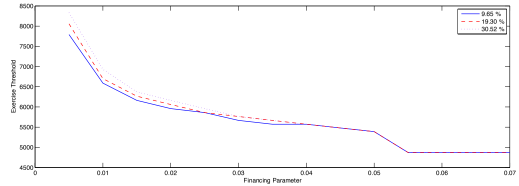
Finally, the thresholds also allow investors to identify mispriced certificates: As mentioned above issuers communicate the price function (2.3.2.1) to the market. This price function reflects the intrinsic value of the certificate. In the exercise region the intrinsic value is larger than the continuation value, i.e. the certificate is offered at a premium. Conversely, in the continuation region the no-arbitrage price of the certificate is higher than the intrinsic value. Issuers offer certificates at a discount to the no-arbitrage price.

5.2.3 Exercise with Fixed Investment Horizon

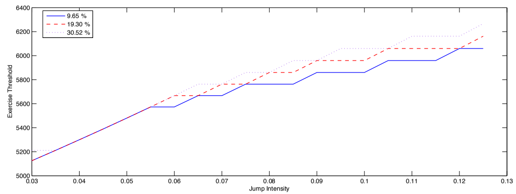
In contrast to Entrop, Scholz & Wilkens (2009) we find that jump risk has a non-negligible impact on the valuation of Turbo certificates. This can be explained by the different valuation approach. Entrop, Scholz & Wilkens (2009) price certificates as if they were European-style options. They assume that investors have fixed investment horizons and hold certificates until either maturity or knock-out. In contrast, the application of optimal stopping theory allows us to take early exercise opportunities into account.



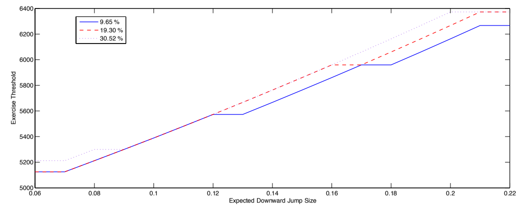
(a) Gap Size



(b) Financing Parameter



(c) Jump Intensity



(d) Expected Jump Size

Figure 5.2: **Dependence of Optimal Exercise Thresholds on the Parameters of OETCs**
This figure shows the effect of the gap size (cf. Panel 5.2a), the financing parameter (cf. Panel 5.2b), the jump intensity (cf. Panel 5.2c), and the expected downward jump size (cf. Panel 5.2d) on the optimal early exercise threshold for different levels of diffusive volatility. The scenario considered corresponds to the base case scenario, i.e. $a = 1.5\%$, $z = 3.0\%$, $\lambda^Q = 0.73$, $\mu_X^Q = -0.1437$, and $\sigma_J = 0.0822$. The initial strike is $4,800$.

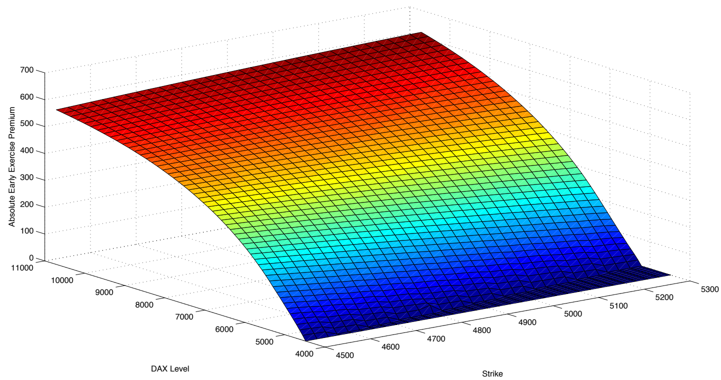
In order to analyze the right of early exercise we follow Entrop, Scholz & Wilkens (2009) and consider a fixed investment horizon. The early exercise premium π is the difference between the value of an American-style Turbo certificate Z_{American} and an otherwise identical European-style counterpart Z_{European}. More precisely, we define

$$\pi := Z_{\text{American}} - Z_{\text{European}}.$$

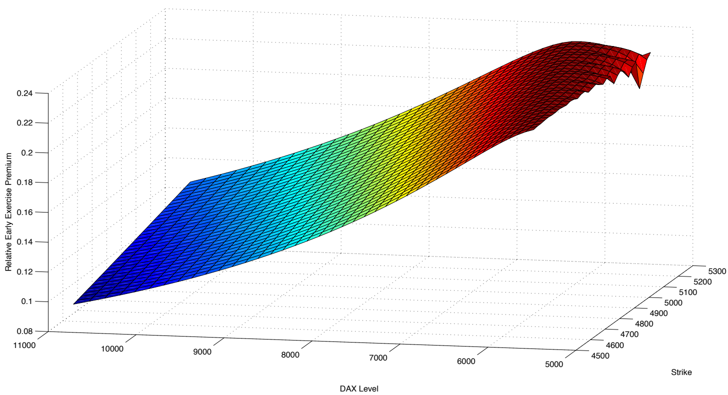
Entrop, Scholz & Wilkens (2009) take into account investment horizons between 0.01 years (roughly three and a half days) and one year. They report that gap risk is significant for very short investment horizons only. However, this might as well be due to the effect that jumps "average out" for long investment horizons. This effect is also known for standard (European) options. In order to investigate this issue in greater detail we consider a rather long-term investment horizon of five years.

Figure 5.3 shows the size of the early exercise premia. It turns out that the impact is significant. For most strikes and spot prices, the size of the premium is far more than 10% which exceeds bid-offer spreads by far.[209] In addition to that the impact of the model and product characteristics were analyzed in similar fashion as in Section 5.2.2. Sensitivities support the results, i.e. the probability of large negative jumps is positively related to the early exercise premium. Furthermore, the impact of the gap size a and the financing parameter z is negative. For the sake of brevity we do not report them in greater detail here.

[209] Bid-offer spreads are usually very small for Turbo certificates. See also the discussion below in Section 5.4.



(a) Absolute Early Exercise Premium



(b) Relative Early Exercise Premium

Figure 5.3: **Surface Plot of the Early Exercise Premium**
This figure shows the early exercise premium with respect to spot and strike price. Model parameters correspond to the base case scenario, i.e. $a = 1.5\%$, $z = 3.0\%$, $\lambda^Q = 0.73$, $\mu_X^Q = -0.1437$, and jump volatility $\sigma_J = 0.0822$.

5.3 Empirical Study for a Single Day

For a first rough and brief empirical assessment of OETCs we consider certificates on the German stock market index DAX. This empirical assessment is limited to the consideration of a single day. A detailed empirical study over a three-year period will follow below in Chapter 6. Nonetheless, such a one-day snapshot is suitable to sharpen one's view of what to look for in a detailed investigation.

Conducting an empirical analysis of Turbo certificates written on the DAX presupposes the availability of a market model and its corresponding parameters, which is attained by means of estimation. In light of the empirical study presented in Chapter 6 we present the estimation procedure of how to infer market and model parameters from options traded at EUREX in Section 5.3.1. This is done in a sufficiently general way to cover the situation of Chapter 6, but in a more general way than needed for the discussion of the one-day snapshot of OETCs written on the DAX taken on July 16, 2010 in Section 5.3.2.

5.3.1 Estimation Procedure

In order to be able to apply the above optimal stopping framework to real-world market data we first have to choose a market model. Our model of choice is the Merton (1976) jump diffusion model (3.1.2.1) which reads

$$\begin{aligned} dS_t &= (r - \lambda^Q\, k)\, S dt + \sigma_D\, S\, dW_t \, + \xi\, S\, dN_t, \qquad t > 0 \\ \ln \xi &\sim N(\mu_X^Q, \sigma_J^2). \end{aligned}$$

Compared to the optimal stopping theory and the numerical approximations and solution techniques outlined in Sections 4.2 and

4.3, this is a simplification as these models also allow for stochastic volatility and jumps in volatility. However, this simplification is readily justified both from an economic perspective and from a computational perspective.

In economic terms, we have demonstrated in Section 5.2 that optimal exercise thresholds are far more sensitive to the stochastic behavior in stock price dimension than to stochastic modeling of stock price volatility. From a computational perspective we can argue that the valuation time per certificate reduces from over thirty minutes to about thirty seconds. Taking into account that in the empirical study of Chapter 6 there are just under $200,000$ certificates quotes in our sample, it becomes clear that it is not feasible to consider both jumps and stochastic volatility at the same time.

According to the market model (3.1.2.1) we thus have to estimate four parameters from market data, the diffusive volatility σ_D, the jump intensity λ^Q, the expected jump size μ_X^Q and the standard deviation of the expected jump size σ_J. In addition, we need to know the risk-free rate of interest and the DAX price of the trading day. Both are taken directly from market quotes. For the interest rates we use the EONIA overnight index to proxy the riskless rate of return and for stock prices the closing price of the respective trading day is employed.[210]

The Merton (1976) model parameters are inferred from the prices of traded call options written on the underlying of the OETC for each trading day.[211] To this base sample of estimation data we then apply the criteria put forth by Bakshi, Cao & Chen (1997):

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[210] Interest rates and closing prices of stocks are both obtained from Thomson Reuters Data Stream Advance.

[211] Bakshi, Cao & Chen (1997) infer market model parameters from both call and put option prices on the S&P 500 stock market index. They find that results are of very similar quality and conclude, that it suffices to merely take call option prices into account. We follow the same approach here.

- All options with less than six days to expiry are removed to account for liquidity related biases
- All options with prices less than €0.35 are excluded to limit the impact of price discreteness
- All options violating the arbitrage condition (5.3.1.1) below are excluded.

From an economic point of view we observe that the market model we employ is a continuous model not only in time but also in stock price dimension.[212] However, real world capital markets exhibit minimum tick sizes, thus rendering stock price changes discrete. As long as stock prices are large enough though, this is not a problem as the tick size is negligible compared to the stock price. However, for small stock prices the tick size can become substantial. Therefore, we adopt the abovementioned threshold. With regard to time to expiry, options just before maturity exhibit higher liquidity and trading volume than other options. Consequently, they are more susceptible to price changes as option sensitivities are particularly pronounced for these types of options.[213]

In order to work with arbitrage-free option prices we further impose the following arbitrage condition

$$C_i(\lambda^Q, \mu_X^Q, \sigma_D, \sigma_J, K_i, T_i, S_t, r) \geq \max(0,\ S_t - K_i). \qquad (5.3.1.1)$$

Here C_i refers to the observed i-th option of a day's sample, K_i

[212] Please not that the expression continuous here does not refer to the sample paths of the model, which can, of course, be discontinuous in a jump diffusion model, but rather to the state space of the model, i.e. the values the price process can take on.

[213] Chiang (2009) documents a monthly seasonal pattern of option liquidity peaking on the third Friday of each month, which is when listed options mature. Thus, removing options with less than a week to expiry amounts to dropping the most liquidly traded options.

and T_i to the respective strike and time to maturity, r to the risk-free rate of interest and S_t to the stock price of the respective day. The parameters $\lambda^Q, \mu_X^Q, \sigma_D, \sigma_J$, we mentioned above, are inherently priced by the market but not directly observable, which is why we have to estimate them below. The reasoning behind the arbitrage condition goes as follows: Option prices should always be positive and greater than their intrinsic value $S_t - K_i$, which is the profit the option holder could immediately cash in upon exercising the option. The latter is due to the fact, that in addition to the intrinsic value each option bears a time value which represents the positive probability that the intrinsic value increases until maturity.

Now denote by $j \in \{1, 2, \ldots D\}$ the D days on which we wish to estimate model parameters and by D_j the number of option prices used for estimation on day j. The corresponding option prices observed in the market on day j are referred to as $C_i^j, i = 1, 2, \ldots D_j, j \in \{1, 2, \ldots D\}$. Furthermore, we have Merton (1976) model prices for each observed option. We count them in the same way as the market prices but add a hat to indicate that they are model prices. Thus, on day j we denote by $\hat{C}_i^j, i = 1, 2, \ldots D_j, j \in \{1, 2, \ldots D\}$ the jump diffusion model price of the i-th option in the respective day's cross section.

In order to estimate the Merton (1976) model parameters we employ the least squares method suggested by Bakshi, Cao & Chen (1997). For this methodology the model option prices $\hat{C}_i^j$ of a given day j are assumed to be a function of the market parameters to be estimated, i.e.

$$\hat{C}_i^j = \hat{C}_i^j(\lambda^Q, \mu_X^Q, \sigma_D, \sigma_J).$$

During the estimation process for each $j \in \{1, 2, \ldots D\}$ we pick the particular set of parameters $((\hat{\lambda}^Q)^j, (\hat{\mu}_X^Q)^j, \hat{\sigma}_D^j, \hat{\sigma}_J^j)$ among all possible combinations of $((\lambda^Q)^j, (\mu_X^Q)^j, \sigma_D^j, \sigma_J^j)$ which minimizes the sum of squared errors between model and market prices on day j. More precisely, we set

$$((\hat{\lambda}^Q)^j, (\hat{\mu}_X^Q)^j, \hat{\sigma}_D^j, \hat{\sigma}_J^j) = \arg\min \left[\sum_{i=1}^{D_j} (\hat{C}_i^j - C_i^j(\lambda^Q, \mu_X^Q, \sigma_D, \sigma_J, K_i^j, T_i^j))^2 \right]. \tag{5.3.1.2}$$

Heed that we have to make extensive use of indexes by always stating j in order to indicate that all estimations are done separately for each j and thus all the results depend on j as intended, so that for every estimation day we obtain a new set of estimated model parameters. Furthermore, all estimations also depend on the strike K_i^j and the term to expiry T_i^j for option i on day j. With our choice of the minimization function, which considers absolute rather than relative errors, we overweight large option prices compared to small option prices. If we used relative errors instead, the situation would be the other way round. Following Bakshi, Cao & Chen (1997) it is good practice to choose (5.3.1.2) for the minimization and live with the over- or underweighting dilemma.

With regard to the estimation technique employed here, we have to point out that, while relatively simple, straighforward and easy to implement it exhibits certain shortcomings. Eraker, Johannes & Polson (2003) point out four of these drawbacks and suggest to amend them using MCMC (Markov Chain Monte Carlo) methods. The downsides mentioned there are:

1) MCMC also provides estimates of the unobservable latent volatility, jump times, and jump sizes,

2) MCMC also accounts for estimation risk,

3) MCMC methods exhibit superior sampling properties compared to competing methods,

4) MCMC methods are computationally effcient, thus allowing to check the accuracy of the method using simulations.

Not only would an MCMC estimation approach tackle the shortcomings of the minimization of squared option pricing errors, but also those of the EMM (Efficient Method of Moments) method (see e.g. Durham & Gallant (2001) or Brandt & Santa-Clara (2002)), simulated maximum likelihood and the implied-state GMM (Generalized Method of Moments) (see e.g. Pan (2002)). Furthermore, Eraker, Johannes & Polson (2003) point out that these improvements are further pronounced if the estimated model includes stochastic volatility or even jumps in volatility. In respect of this observation Jacquier, Polson & Rossi (1994) find MCMC to be superior to QMLE (Quasi Maximum Likelihood Estimate) using simulation, which Andersen, Chung & Sørensen (1999) extend to EMM.

Naturally, the mentioned studies spawned more research using the MCMC methodology to estimate more sophisticated market models and analyze their relevance for option pricing.

Eraker, Johannes & Polson (2003) find that the inclusion of jumps in both the stock price dimension and the volatility component is important and omitting jumps in volatility leads to misspecification because it ignores the conditional volatility of returns, which is shown to be quickly changing. In terms of option pricing, the authors show that jumps in volatility significantly increase implied volatility for deep in-the-money and deep out-of-the-money options. Eraker (2004) picks up the void left by Eraker, Johannes & Polson (2003) who did not include option prices in their estimations. Using joint options and returns data jump and volatility risk premia are estimated. Furthermore, complex jump specifications are found to help simultaneously explaining options and returns data while not substantially improving option pricing. The results for the US

American market are extended by Breuer (2008) to seven major stock market indexes across the world by including, for instance, the Japanese Nikkei 225 or the German DAX. Using MCMC it is confirmed that simultaneous jumps in returns and volatility improves data fitting. In addition, estimations are provided for both the empirical and risk-neutral measure. Rodrigues & Schlag (2009) employ MCMC to analyze to which degree jumps of stock market indexes can be traced back to jumps in the underlying individual stocks. It is found that not necessarily a large number of jumps in individual stocks coincides with an index jump.

Given these advantages of MCMC techniques, we have to explain why for our purposes we deemed it sufficient to use a least squares approach. On the one hand such a method is by far more tractable and straightforward. On the other hand the focus of our study does not lie on evaluating different market models for which it would be essential to capture even small nuances in the parameters with the estimation techniques. In addition, the comparative statics analysis of Section 5.2 demonstrates that while these parameters are crucial to the valuation of OETCs and determining their optimal exercise thresholds, exercise thresholds appear insensitive enough to justify living with the drawbacks of a least squares approach. Another aspect strengthening this view is, that our model does not include stochastic volatility and would thus not be able to fully exploit the advantages of MCMC anyway.

In Section 5.3.2 we will apply this estimation procedure to determine market model parameters and employ them to deal with a one-day snapshot of real-world OETCs written on the German stock market index DAX.

5.3.2 Snapshot of July 16, 2010

In this section we are going to apply the optimal stopping methodology described above and already applied in the comparative statics analysis of Section 5.2 to treat real-world OETCs on the German stock market index DAX as observed on July 16, 2010. In particular, we have to estimate market model parameters and apply them to derive optimal exercise thresholds.

The DAX is a portfolio of 30 large German stocks. As a performance index, the DAX assumes that dividends are reinvested. Hence, we do not have to address potential future dividends explicitly, which is in line with our above assumption of a non-dividend paying underlying. In order to calibrate our model to the market we follow Bakshi, Cao & Chen (1997) and infer Merton (1976) model parameters from the option market as outlined in Section 5.3.1. Standard European-style calls and puts are liquidly traded at EUREX.

For our snapshot of July 16, 2010, we are looking for estimates $\hat{\lambda}^Q$ (jump intensity), $\hat{\mu}_X^Q$ (expected jump size), $\hat{\sigma}_D$ (fixed diffusive volatility), and $\hat{\sigma_J}$ (standard deviation of the jump size).

The estimation is based on $D_1 = 93$ observations of call option prices from Thomson Reuters Datastream Advance which were observed on July 16, 2010.[214] These option prices meet the elimination criteria outlined above and the arbitrage condition (5.3.1.1).

The interest rate is the one-month EURIBOR on that day ($r = 0.556\%$). We obtain the following estimates

$$\begin{aligned}\hat{\sigma}_D &= 0.23\\ \hat{\lambda}^Q &= 0.11\end{aligned}$$

[214]Please note, that in the notation of Section 5.3.1 there is only one estimation day, i.e. $D = 1$. Consequently, there are also no D_j's other than D_1.

$$\begin{aligned} \hat{\mu}_X^Q &= -0.05 \\ \hat{\sigma}_J &= 0.20. \end{aligned}$$

These parameters are applied to compute model certificate prices and optimal exercise thresholds. The data for our study has been manually obtained from the website www.onvista.de on July 16, 2010. It comprises a sample of $1,345$ OETCs which meet the product characteristics outlined in Section 2.3.2.1. The sample further includes certificates from eleven major issuers with each certificate applying to 0.01 DAX contracts. Bid-offer spreads are tight. Where applicable, they mostly lie below € 0.02 and the majority are even as low as € 0.01. In line with the issuers' prospectuses we assume that the financing parameter is fixed at 3% for the entire sample. Conversely, the gap size for each OETC is taken from the sample data and fixed as well. Note that these choices are somewhat debatable since these parameters are usually subject to change by the issuer in times of extra-ordinary market conditions. See also Entrop, Scholz & Wilkens (2009) for a detailed overview over product characteristics.

Table 5.2 shows the details on the certificates and the results and groups them by issuer. Optimal thresholds can be compared to DAX levels. The opening and closing prices of the index were $6,165$ and $6,040$ points respectively on July 16, 2010. The DAX varied between $6,018$ points (low) and $6,205$ points (high) on the observation day. Gap sizes range from 0.43% in the case of Vontobel to 2.11% for Barclays and Lang & Schwarz. The overall average gap size is found to be 1.39% which corroborates our assumption of 1.5% for the comparative statics analysis in Section 5.2. For the sake of palpability we define the following relative gap measures

$$D_{\text{DAX-KO}} := \frac{\text{DAX}_{\text{close}} - L_t}{L_t}$$

Issuer	Number of Certificates	Average Data Certificates					
		Strike	Gap	Barrier	Distance DAX to KO	Distance DAX to threshold	Distance threshold to KO
DZ Bank	89	5,113.46	0.68%	5,205.36	16.89%	15.77%	1.14%
Vontobel	64	5,220.48	0.43%	5,242.15	16.04%	14.78%	1.11%
Deutsche Bank	93	4,333.96	0.93%	4,374.22	48.61%	46.90%	1.15%
Citigroup	187	4,086.28	1.18%	4,125.04	66.30%	64.12%	1.21%
The Royal Bank of Scotland	47	4,522.66	1.45%	4,587.87	34.14%	33.00%	0.85%
BNP Paribas	208	4,498.47	1.11%	4,547.98	37.88%	36.47%	1.04%
HSBC Trinkaus	158	4,131.73	1.07%	4,175.88	60.59%	58.99%	1.02%
Commerzbank	292	3,969.28	2.02%	4,040.87	64.21%	62.38%	1.05%
Goldman Sachs	172	4,841.40	2.01%	4,938.02	25.89%	24.75%	0.93%
Barclays	19	4,586.56	2.11%	4,683.16	30.84%	29.64%	0.91%
Lang & Schwarz	16	4,886.08	2.11%	4,988.13	22.57%	21.59%	0.78%
Total	1,345	4,397.42	1.39%	4,457.75	46.58%	45.03%	1.05%

Table 5.2: **Descriptive Statistics for OETCs on July 16, 2010**
This table shows the descriptive statistics by issuer of our sample of OETCs on the DAX traded in the market on July 16, 2010. It lists the average striking price, the average gap between the strike and the barrier, the average barrier and the average distances to a potential knock-out based on the day's closing quote of the DAX and the computed optimal exercise threshold. Heed that the distance to a potential knock-out is in relation to the barrier and does not imply that the DAX would have to jump by that amount to trigger a knock-out.

$$
\begin{aligned}
&= \text{Distance DAX to KO}_t \\
D_{\text{Threshold-KO}} &:= \frac{\text{Threshold}_t - L_t}{L_t} \qquad (5.3.2.1)\\
&= \text{Distance Threshold to KO}_t \\
D_{\text{DAX-Threshold}} &:= \frac{\text{DAX}_{\text{close}} - \text{Threshold}_t}{\text{Threshold}_t} \\
&= \text{Distance DAX to Threshold}_t.
\end{aligned}
$$

By the first distance between the DAX and the knock-out barrier we measure how far away and thus how probable a knock-out of the certificate is. From an investor's point of view this number might be used as an indicator of the riskiness of the certificate.[215] The second distance between the optimal exercise threshold and the knock-out barrier serves as a means to quantify the region of optimality for OETCs. This is due to the observation that for DAX levels below the knock-out barrier the certificate is practically worthless and beyond the optimal exercise threshold it is suboptimal to hold the certificate. Therefore, this number can be seen as the size of the region where certificates should be held. From an investor's perspective, the larger this number is the longer the optimal holding duration of the certificate is, because with a decreasing distance the probability of traversal across either boundary of the region of optimality increases. The third distance measures the distance between the DAX and the optimal exercise threshold. A positive distance indicates that the certificate is suboptimal to hold, on the other hand a negative one indicates optimality. Moreover, this distance describes how far away from optimality a certificate is or, if optimal, how far away from a change of this status the OETC is.

[215] The distance between the DAX and the knock-out barrier is also similar in style to the moneyness of options, which indicates the distance between the underlying and the strike. Because of the constant gap size between the strike and the knock-out barrier, $D_{\text{DAX-KO}}$ can be interpreted as a moneyness based on a shifted strike.

Before moving on, we would like to point out that reporting a distance between the DAX and the knock-out barrier of say $D_{\text{DAX-KO}} = 5\%$ does not mean that the DAX has to drop by as much as 5% to trigger a knock-out. The fact that the certificate is still alive implies that $L_t < \text{DAX}_{\text{close}}$ and thus

$$\begin{aligned} \frac{\text{DAX}_{\text{close}} - L_t}{\text{DAX}_{\text{close}}} &= \frac{L_t}{\text{DAX}_{\text{close}}} D_{\text{DAX-KO}} \\ &< D_{\text{DAX-KO}}. \end{aligned}$$

This means that we have to multiply the percentage distance between the DAX and the knock-out barrier with the ratio of the two numbers to obtain by how much the underlying has to drop to cause a knock-out. The root cause for this behavior is that $D_{\text{DAX-KO}}$ is measured in terms of the knock-out barrier rather than the underlying. Naturally, the same applies to the other two distances as well and we can put forth similar arguments.

The average distance between the exercise threshold and the knock-out level of outstanding certificates is 1.05%. The average distance between the DAX and the knock-out barrier (46.58%) turns out to be substantially larger in relation to the distance between the optimal exercise threshold and the knock-out level.[216] Differences are observed between major issuers for $D_{\text{DAX-KO}}$. For instance, the DAX overshoots the barrier of Vontobel certificates by 16.04% in the mean while the corresponding number for Citigroup is 66.30%. These numbers are compared to the distance between the optimal exercise threshold and the knock-out level of Vontobel and Citigroup with mean value of 1.11% and 1.21% respectively. Looking at individual certificates it turns out that except for one certificate (WKN

[216] We would like to emphasize that these distances are computed as percentage numbers with regard to the barrier. They do not imply that the DAX has to plummet by 46.58% for a knock-out to occur. The required jump size depends on the single certificate and would be significantly lower.

DZ1FQN) issued by DZ Bank all certificates should have been exercised on that particular day. This observation holds for open and closing as well as high and low values of the DAX. However, this one certificate had a barrier of $6,101.10$ (strike of $6,174.31$) and was knocked out on that day.

Results suggest that theoretically, it is possible to purchase a certificate at at discount because issuers commit themselves to trade according to the exercise value (2.3.2.1). This is the case when the continuation value is larger than the exercise value. However, for the day under consideration it turns out that none of these certificates is available for trade. Investors can only buy and hold certificates which should be exercised. Thus, the continuation values of these securities are smaller than the exercise values. As we know from other American-style options the financial disadvantage from non-optimal exercise behavior is the profit of the issuer. Of course, from the perspective of an issuer there is always the risk that a certificate is issued that might become attractive for investors. However, this risk is limited. Those certificates that are potentially attractive for investors have barriers that are close to the DAX. These certificates are thus very likely to be knocked out and these investment opportunities hence disappear from the market. Certainly, in the context of knock-outs issuers face gap risk. Nevertheless, they should be compensated for this risk as long as enough investors continue to hold enough certificates in the exercise region.

5.4 Interpretation of the Results

The existence of OETCs can only be explained when underlying stock prices are subject to jump risk. Otherwise, rational investors could replicate these products by a very simple strategy of buying stock and borrowing money. In a pure diffusion model this debt would be riskless and hence the risk-free rate of interest should apply yielding more attractive payoffs than certificates with a financing rate $z > 0$. Therefore, we apply the model of Bakshi, Cao & Chen (1997). Based on optimal stopping theory we derive exercise thresholds, i.e. prices of the underlying stock beyond which investors should sell their certificates. These thresholds result from the trade-off between the financing parameter z and the gap size a on the one hand and downward jump protection on the other. Furthermore, we follow Entrop, Scholz & Wilkens (2009) and consider fixed investment horizons. We show that early exercise premia are economically significant even when very long investment horizons are considered. Finally, in a one-day empirical application we analyze the prices of $1,345$ certificates on the DAX traded on July 16, 2010. It turns out that the optimal exercise thresholds of all but one of them (a knock-out during the trading day) are overshot by the DAX. From a theoretical point of view these certificates should not be held at all. In summary, we can argue that OETCs in certain situations allow for beneficial outcomes of investments. However, such situations are based on the underlying hovering between the knock-out barrier and the optimal exercise threshold. In our case this is a relatively small and narrow band. Because of the general movement of the underlying it is though highly likely that this beneficial region is exited within short periods of time, thus wiping out potential investment opportunities. Nevertheless, the situation might be slightly different in an environment of market stress like the sovereign debt crises currently observed in Europe and the USA.

Eventually, this first assessment of market traded OETCs only pertains to a single day. Although the obtained results are very intuitive and economically explicable, the choice of a single observation day always bears some randomness which might influence the results in an unforeseeable direction. Such impact can manifest itself in biased estimates of market data as well as data of the OETCs themselves. This might be the case because of extra-ordinary market movements on that respective day, particularly high or low trading volume by market participants on that day, a holiday in other major markets such as the US or Japan which regularly influences trading in Europe as well, or various other reasons. Although none of this appears to be case the empirical study should still be extended to covering a longer period of time.

Apart from those general reasons, there are also very OETC-specific reasons suggesting longer empirical studies. For example, we mentioned above that for OETCs to be optimally held the underlying has to hover around a small band between the knock-out barrier and the optimal exercise threshold. This naturally entails the question of how long this seemingly unlikely scenario is sustained in reality. In a longer empirical study, such situations could be identified and monitored.

Therefore, Chapter 6 picks up that gap and addresses the questions of how OETCs, the associated optimal exercise behavior, and holders of OETCs behave over the course of time.

Chapter 6

Empirical Study for 2007 through 2009

This chapter takes up the key research gap left in Chapter 5. The empirical analysis is extended from one day to three years which will allow for obtaining a much clearer picture about whether the observations in that chapter have to be attributed to the movements of a certain trading day or whether they are part of an overall trend. The general items addressed in this chapter are:

1) Are there more OETCs which can be rationally held in the long run?

2) Do potential investment opportunities only sustain for short periods of time?

3) Are there any trends in the observed long distances between the DAX and the knock-out barrier and in the narrowness of the continuation region?

The chapter is organized in two sections: Section 6.1 presents the data for the study and Section 6.2 the results.

6.1 Data for the Study

In this section we describe the data for our study which consists of the certificates data (cf. Section 6.1.1) on the one hand and the option data to estimate market parameters (cf. Section 6.1.2) on the other hand.

6.1.1 Certificates Data

For our empirical study, we consider a sample of up to $1,097$ long OETCs on the German stock market index DAX. The sample consists of all such certificates that were alive on July 10, 2010 and that were traded on the Frankfurt stock exchange. For the entire sample we have traded prices for the years 2007 through 2009, which totals $196,743$ price quotes.[217] The data were taken from the Karlsruhe capital market data base KKMDB (Karlsruher Kapitalmarktdatenbank), which provides capital market data for teaching and research purposes.

The certificates in our sample are issued by eleven different financial institutions, which play a significant role in the German market. On December 30, 2009, the last trading day of our three-year period, our sample looks as shown in Table 6.1.

[217] From this sample we have already removed the very few obvious misquotes. Of course, we do not want such quotes to negatively impact our study. However, we do not examine the prices themselves or to which degree they coincide with the price setting formula (2.3.2.1) (according to which issuers have committed themselves to trade OETCs), but rather use these quotes together with the certificates' parameters to assess whether exercise or holding is the preferable strategy.

Issuer	Number of Certificates	Average Strike	Average Gap	Average Barrier
DZ Bank	29	4,607.245	0.015	4,673.594
Vontobel	22	4,741.027	0.010	4,787.901
Deutsche Bank	87	4,050.965	0.010	4,091.340
Citigroup	156	3,733.197	0.012	3,768.322
The Royal Bank of Scotland	58	3,972.098	0.019	4,041.904
BNP Paribas	183	4,265.961	0.011	4,312.831
HSBC Trinkaus	145	3,853.815	0.011	3,897.243
Commerzbank	265	3,688.316	0.021	3,760.311
Goldman Sachs	122	4,485.712	0.021	4,580.419
Barclays Bank	18	4,456.341	0.021	4,550.431
Lang & Schwarz	12	4,648.949	0.021	4,745.465
Total	1,097	4,013.895	0.016	4,072.560

Table 6.1: **Descriptive Statistics for OETCs on Dec. 30, 2009** The table provides information about the number of outstanding OETCs on December 30, 2009 by issuer and on an aggregate level. In addition, the average strike, the average gap size, and the average barrier are provided.

Considering only those certificates that were alive on July 10, 2010 implies that our results potentially exhibit a certain survivorship bias. This is due to the fact that OETCs which were knocked out before that date do not appear in the sample. Thus an evaluation of the OETCs in our sample might overestimate the actual distance between the DAX and the knock-out threshold, since shortly before their extinction knock-outs are most likely situated closer to the barrier than survivors. Nonetheless, our assessment in Chapter 5 makes it plausible that a potential bias is rather small. Knock-outs appear to be very rare events (1 in $1,345$ on a day with a sizable range between the low and high price of the DAX). Assuming that such market movements happen more seldom than once a week, this suggests that over 99.9% of the quotes should be correctly recorded.

Consequently, we cannot rule out that there is a survivorship bias but we can expect its effect to be most likely insignificant in terms of distances between the underlying, the knock-out barrier and the optimal exercise threshold.

Furthermore, we might underestimate the number of certificates that should be rationally held. This is due to the fact that those certificates are naturally closer to the knock-out barrier than others. Therefore, unrecorded knock-outs might have been in the continuation region shortly before the knock-out event. On the one hand our above argument indicates that this should be a rare scenario (thus keeping the bias small) and on the other hand such a bias would corroborate our rationale that optimal holding opportunities are usually very short lived. Either way, it appears that there cannot be a sizable negative influence on our results.

As described in Section 2.3.2.1 each certificate is characterized by its strike, its barrier, and the constant gap size between them. For illustrative purposes Table 6.1 displays the average values of these parameters by issuer.

The table reveals that once again there are substantial differences. The gap size ranges between 0.010 for Vontobel and Deutsche Bank and 0.021 for Commerzbank, Goldman Sachs, Barclays, and Lang & Schwarz, i.e. the range amounts to a factor of two. This means that it takes jumps of twice the length to undershoot both the barrier and the strike. But also the average striking prices differ considerably across the sample. We observe the lowest value for Commerzbank with roughly $3,688$ points and the highest for Vontobel with about $4,741$ points.[218] Below in this section, this is intuitively explained by the market conditions at the time when the bulk of these certificates

[218] Since the relative gap size between the strike and the barrier is constant for OETCs the average barriers behave in the same way as the average striking prices and are not further discussed here.

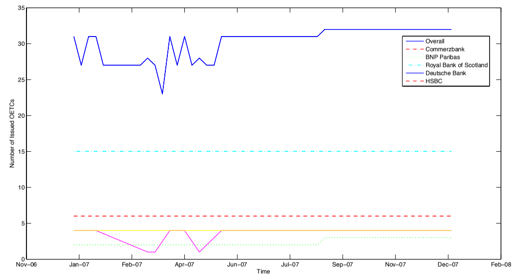
was issued. This observation gives rise to the question of how the number of OETCs in our sample and their parameters developed over time. Figure 6.1 shows the number of certificates in our sample for the years 2007 through 2009.[219]

We discern that the number of certificates is relatively constant throughout the year 2007 with several fluctuations (cf. Panel 6.1a) due to the fact, that on several days there are no quotes or misquotes that have been removed from the sample. The overall number only moves from 31 to 32. Commerbank, HSBC and the Royal Bank of Scotland constantly have six, four, and 15 respectively. The slight increase of one is due to BNP Paribas.

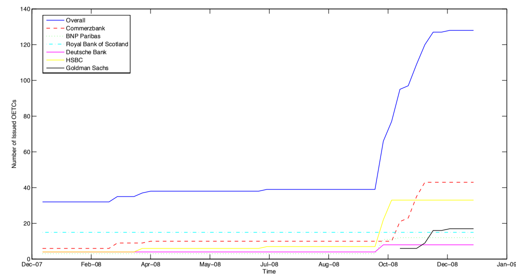
The situation stays the same until about October 2008, when the number of overall certificates more than triples from 39 to 128. Except for the Royal Bank of Scotland all issuers contribute to this development with Goldman Sachs appearing as a new issuer. They issued 17 OETCs between October and December 2008. The strongest increase can be observed for HSBC and Commerzbank. For Commerzbank the number of issued certificates rises from 10 to 43 in this period. For HSBC the increase is from 7 to 33. For BNP Paribas the increase is from three to twelve in absolute terms.

In the calendar year 2009 there is again about an eight- to nine-fold increase in the overall number of issued OETCs from 128 to $1,097$. The majority of this increase has to be attributed to Commerzbank (43 to 256), BNP Paribas (12 to 183), HSBC (33 to 145), and Citigroup (0 to 156). For the Royal Bank of Scotland and Goldman Sachs there is a less significant increase from 15 to 58 and 17 to 122 respectively. The same holds for Deutsche Bank with a rise from 8

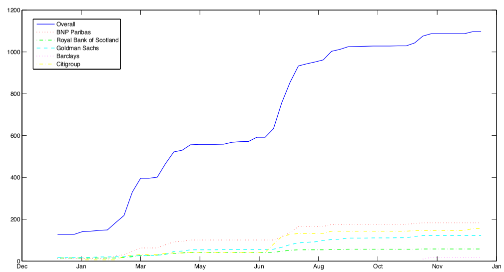
[219] In this figure, we have divided the data for the year 2009 into two slices for the sake of improved readability as we deem twelve graphs in the same plot rather confusing. The grouping into the two slices was done randomly without any economic interpretation to it.



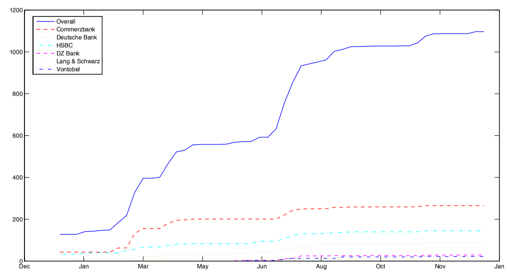
(a) Number of Outstanding Certificates 2007



(b) Number of Outstanding Certificates 2008



(c) Number of Outstanding Certificates 2009 - Slice 1



(d) Number of Outstanding Certificates 2009 - Slice 2

Figure 6.1: **Time Series Plot of the Number of Issued OETCs**
In this Figure we depict the number of outstanding certificates by issuer for the years 2007 through 2009. Panel (6.1a) shows the data for 2007 (with several removed misquotes which prompt the fluctuations), panel (6.1b) for 2008 and panels (6.1c) and (6.1d) for 2009. For the sake of readability the data for 2009 is divided into two slices.

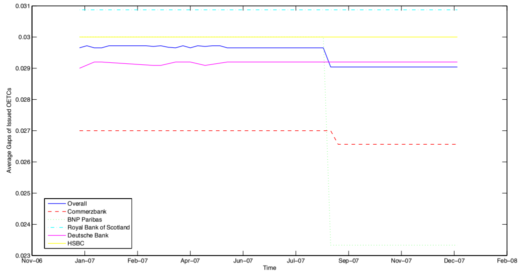
to 87. Besides Lang & Schwarz, DZ Bank, Vontobel and Barclays appear as new issuers. Until the end of the year they issued twelve, 29, 22, and 18 OETCs respectively.

Besides the total number of OETCs, we also consider the average gap sizes of the outstanding products. Although for each single certificate the gap size is constant, the average gap sizes change over the course of time with new OETCs being issued. This has been illustrated in Figure 6.2.

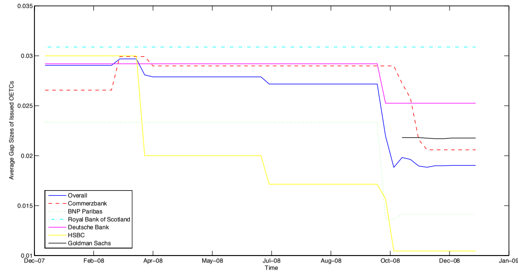
For the year 2007, we observe mostly constant gap sizes. Of course, this is due to fact, that the number of certificates was almost constant throughout the year. All changes naturally coincide with new certificates being issued.[220] Interestingly, these changes typically lower the average gap sizes which means that the newly issued OETCs have a smaller gap than the existing ones.

In 2008 this trend mostly continues. For Deutsche Bank, HSBC and BNP Paribas we see that average gap sizes decrease when new certificates are offered. In the case of Commerzbank, though, there is a mixed picture. In February they issue new certificates with higher gap sizes but in April and October the newly issued certificates have lower gap sizes again. As mentioned above, Goldman Sachs appears as a new issuer in October 2008. The average gap size of their certificates, however, is slightly higher than the overall market average (0.020 vs. 0.022). In addition to this relative behavior it is worthwhile discussing the absolute gap sizes. The overall average amounts to 0.029 at the beginning of the year but is reduced to 0.019 at the end of 2008. Among issuers there are significant differences. The average gap size is highest for the Royal Bank of Scotland with 0.031 throughout the entire year. For HSBC the average gap size goes down from 0.03 to 0.01 in several steps. Starting in April of

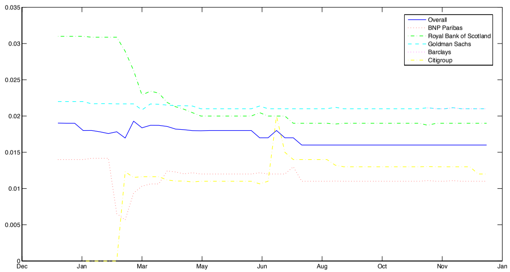
[220] Apart from these real changes, there are also minor fluctuations until April 2007 resulting from misquotes or missing quotes we removed from our sample.



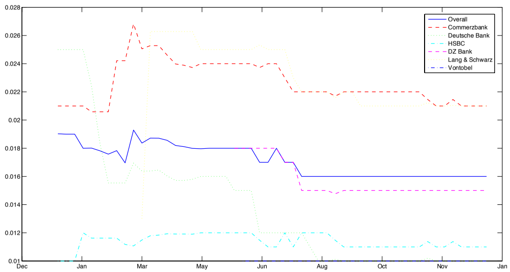
(a) Average Gap Size of Outstanding Certificates 2007



(b) Average Gap Size of Outstanding Certificates 2008



(c) Average Gap Size of Outstanding Certificates 2009 - Slice 1



(d) Average Gap Size of Outstanding Certificates 2009 - Slice 2

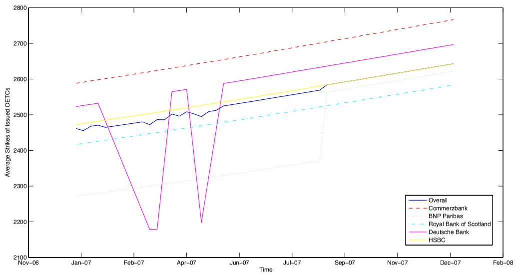
Figure 6.2: **Time Series Plot of the Average Gap Sizes of Issued OETCs**
In this Figure we plot the average gap sizes of outstanding certificates by issuer for the years 2007 through 2009.

2008 HSBC also exhibits the lowest average in our sample. The values for the other issuers lie between 0.02 and 0.03 for the entire year.

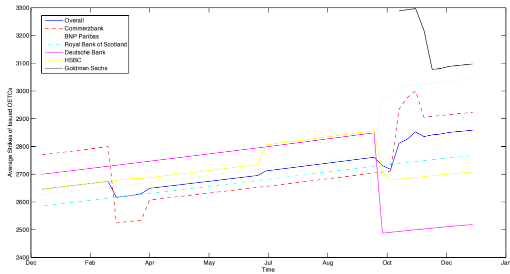
In the year 2009 the tendency of declining overall gap sizes is maintained but it is not as sharp as before, since there are two significant increases when the average gap size for Citigroup increases from 0.0 to 0.012 and 0.011 to 0.020. For the Royal Bank of Scotland we observe a decrease from 0.031 to 0.019, which is no longer the highest value in the sample. At the end of the year this position is taken on by Commerzbank and Goldman Sachs with 0.021. The lowest values in turn are observed for HSBC, BNP Paribas, Vontobel and Deutsche Bank with 0.011, 0.011, 0.010 and 0.010 respectively. In addition to the already mentioned sharp upward moves for Citigroup we observe similar moves for Commerzbank (0.021 to 0.024) and Lang & Schwarz (0.013 to 0.026).

In addition to the number of outstanding certificates and average gap sizes we also consider the average striking prices of the issued OETCs, as plotted in Figure 6.3.

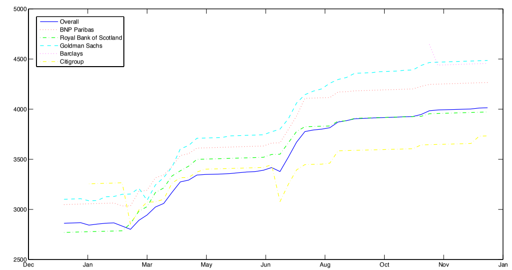
Unless there is an issue of a new certificate the average striking price grows at the risk-free interest rate as given by equation (2.3.2.2). This is confirmed by the market data for the year 2007. Except for Deutsche Bank and BNP Paribas the average strike constantly grows, which is in line with the abovementioned fluctuations for Deutsche Bank and the additional issue by BNP Paribas. In absolute terms there are significant differences again. The lowest starting value is observed for BNP Paribas with roughly $2,272$ and the highest for Commerzbank with about $2,588$. When BNP Paribas issued a new OETC the average strike makes a sizeable jump from round about $2,442$ to $2,620$.



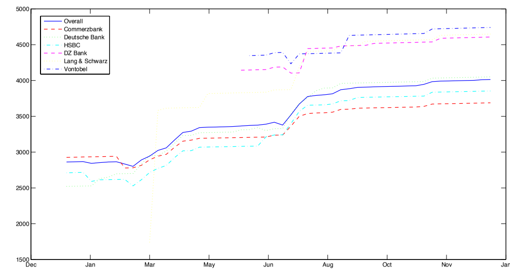
(a) Average Strike of Outstanding Certificates 2007



(b) Average Strike of Outstanding Certificates 2008



(c) Average Strike of Outstanding Certificates 2009 - Slice 1



(d) Average Strike of Outstanding Certificates 2009 - Slice 2

Figure 6.3: **Time Series Plot of the Average Strikes of Issued OETCs**
In this Figure we show the average strikes of outstanding certificates by issuer for the years 2007 through 2009.

Naturally, this behavior continues in 2008 with all jumps in the average striking prices being induced by the issuing of new products. The most significant move can be observed for Commerzbank. At the beginning of 2008 they exhibit the highest average strikes, but in March they drop to the lowest values from $2,942$ to $2,778$. Until October they stayed at this end of the spectrum but then jumped back up close to the top ($2,711$ to $2,935$), only behind BNP Paribas and Goldman Sachs ($2,975$ and $3,288$ respectively). A similar upward jump at about the same time can be observed for BNP Paribas. Jumps in the opposite direction are exhibited by HSBC ($2,856$ to $2,678$) and Deutsche Bank ($2,849$ to $2,487$).

In 2009 there are so many, almost continuous, new issues of OETCs that we can no longer observe a constant growth as in 2007 and 2008. However, the strikes generally increase for all issuers. Except for two downward jumps for Citigroup ($3,266$ to $2,850$ and $3,446$ to $3,078$) in the first half of the year there are no sharp moves either. Furthermore, the overall increase is very substantial. During the year 2009 the overall average strike increased from $2,862$ to $4,014$. The highest values in the sample are even as high as $4,500$ with Goldman Sachs, Lang & Schwarz, DZ Bank and Vontobel at $4,486$, $4,649$, $4,607$ and $4,741$. For Lang & Schwarz we also observe the most significant increase over the year 2009 starting from $1,742$. On the other end of the spectrum, we discern the lowest average strikes for Commerzbank and Citigroup ($3,688$ and $3,733$) to end the year.

Apart from the fact that the strikes grow at the risk-free rate of interest plus the financing rate, we might argue that the increased strikes are in line with a general rise of the market which is reflected in the parameters the new issues.

6.1.2 Estimation of Market Model Parameters

For all OETCs in our sample described in Section 6.1.1 we want to compute optimal exercise thresholds. How this can be done using the methodology of optimal stopping has been outlined in Section 5.2. Moreover, in Section 5.3.1 we have chosen the Merton (1976) jump-diffusion model and explained how its parameters can be inferred from option markets.

Since there are 784 trading days in our sample of Section 6.1.1 it is plausible that we cannot use the same set of parameters for the entire three-year period. On the other hand we cannot re-estimate the parameters on each trading day either, as this would be too onerous. Therefore, we conduct a new estimation for the parameters every week. In this way the number of estimations is reduced from 784 days to 157 weeks.

In addition, this is the same approach taken by Dumas, Fleming & Whaley (1998). Furthermore, they suggest to use Wednesdays for the estimations because on Wednesdays there are the fewest holidays. If there is a holiday on a Wednesday after all, we use the preceding trading day. In case this is a holiday as well, the subsequent business day is used. Therefore, for all Wednesdays between January 2007 and December 2009 (or their respective replacement days) we consider all traded call option prices.[221]

In addition to the elimination and arbitrage criteria outlined in Section 5.3.1, we also remove obvious misquotes from the cross section as we did with the sample of OETCs in Section 6.1.1. The estimations themselves are carried out using Matlab functionality for

[221] The option data were retrieved from Thomson Reuters Data Stream Advance. Since options trading is forward looking and because of the fact that a certain set of maturities expires every month, our estimation sample includes options expiring between January 2007 and March 2010. Typically, i.e. except on very few occasions, option prices with up to three months maturity are available.

No.	Reporting Period	Table with Estimated Parameters
1	January through April 2007	Table A.1
2	May through August 2007	Table A.2
3	September through December 2007	Table A.3
4	January through April 2008	Table A.4
5	May through August 2008	Table A.5
6	September through December 2008	Table A.6
7	January through April 2009	Table A.7
8	May through August 2009	Table A.8
9	September through December 2009	Table A.9

Table 6.2: **Reporting of Estimated Market Parameters**
For reporting purposes and the sake of readability, we have grouped the estimated market parameters in time periods of four months each. This table shows in which table the results for the respective periods are reported.

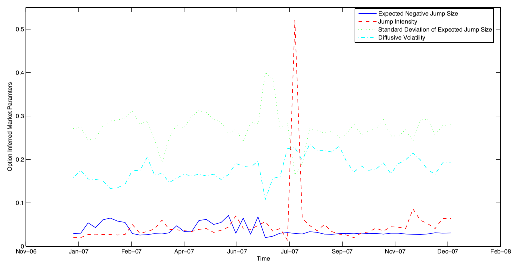
minimization problems. We group the results of the estimation procedure in clusters of four months each and for the sake of readability we report them in Appendix A as indicated by Table 6.2. To visually highlight the development of the parameters over time, Figure 6.4 provides a plot of the estimated diffusive volatilities, the expected downward jump sizes together with their standard deviations, and the jump intensities for each of the years 2007, 2008, and 2009.

Before moving on to the application of these parameters to value OETCs and determine their optimal exercise thresholds, we should evaluate the estimated parameters and shed light on their economic context. Our main observation is that there appear to be comparatively low expected downward jump sizes. For 2007 we find an average of -3.7%, for 2008 -2.63%, and for 2009 -2.42%. Compared to our base case scenario, which was based on Eraker (2004) and Breuer (2008), with roughly -14%, these estimates are markedly lower. Furthermore, they also seem to decline even further from year to year. A similar picture can be drawn for the jump intensities. The above base case scenario reports an intensity of 0.73, i.e. 0.73 jumps a year or one jump about every 16 months. In our sample we observe significantly lower values: 0.049 for 2007, 0.132 for 2008, and 0.114 for 2009. The opposite holds for the standard deviation of the jump sizes. The base case scenario exhibits round about 8%, while our sample shows 27.3% for 2007, 19.3% for 2008, and 24.0% for 2009.

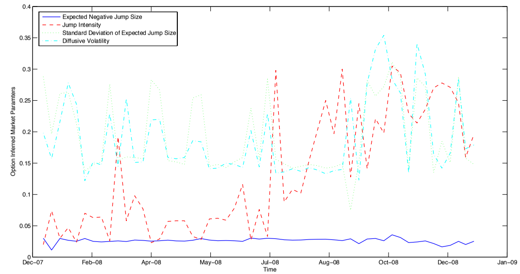
From an economic perspective, this means that market participants expect shorter jumps, which occur a lot more rarely. On the other hand there is far more uncertainty regarding the size and frequency of an actual jump.

A possible line of explanation for these observations, particularly in the fall of 2008, might be the financial crisis of the years 2008 and 2009. Those years saw a very sharp decrease of the DAX by more than 50% (and a subsequent partial recovery making up for roughly half the losses) which was among other factors induced by a decline of the price-adjusted GDP (Gross Domestic Product) of 4.7% for Germany in 2009 (cf. Statistisches Bundesamt (2011)). This makes it very likely that market participants were already extremely pessimistic regarding the future economic outlook and thus priced the constituent single stocks of the DAX accordingly. Doing so means that there should not be any further negative surprises which could adversely impact stock prices. Since such a negative surprise would be the equivalent of a jump, the explicit jump risk appears to have decreased during this crisis.

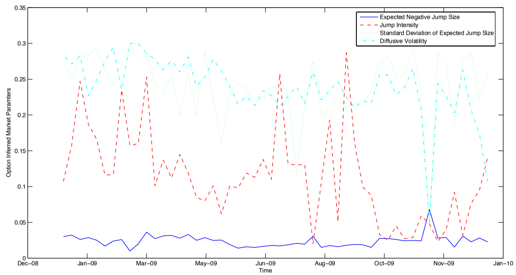
Apart from these high-level considerations, at the peak of the financial crisis in October 2008 we nonetheless observe increased jump intensity, downward jump sizes and corresponding standard deviations, and diffusive volatilities in the short run. This becomes plain



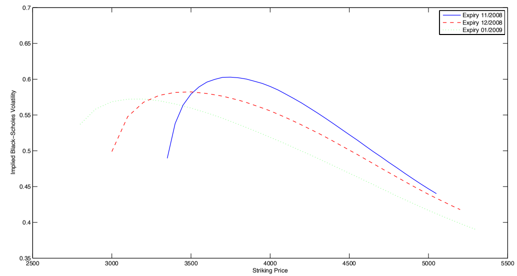
(a) Estimated Market Parameters 2007



(b) Estimated Market Parameters 2008



(c) Estimated Market Parameters 2009



(d) Volatility Smile on October 22, 2008

Figure 6.4: **Time Series Plot of the Estimated Market Parameters and Option Smiles**
This figure depicts how the option implied expected downward jump size, the jump intensity, the standard deviation of the expected jump size, and the diffusive volatility have evolved over the years 2007 through 2009. Furthermore, the option smiles for October 22, 2008 are shown.

by comparing the values to those of adjacent months and weeks, which indicates a local peak during the month of October of 2008. In economic terms, this can be interpreted as increased uncertainty in capital markets or even panic following the bankruptcy of Lehman Brothers just one month before. When comparing the expected jump sizes to actual market returns we, moreover, find that single day returns in excess of the expected value were absolutely commonplace. On the 23 trading days in October 2008 there were returns of less than -4.9% six different times (three of these days even showed returns of -6.5% and -7%). On the other hand there were two days with +11% or more and only three days on which the DAX changed by less than a percent in any direction. Furthermore, this goes in line with a reported increase in diffusive volatility for our sample which is as high as 35%.

The uncertainty in the market is further corroborated by Panel 6.4d which shows the option smiles of October 22, 2008. Compared to usual option smiles (e.g. those depicted in Figure 3.1), which indicate that longer-term options are more relatively expensive than short-term options and that in-the-money options are relatively more expensive than out-of-the-money options to account for the fact that they are less susceptible to jump risk, this day's smile is somewhat reversed. We discern that out-of-the-money options exhibit higher implied volatilities (meaning they are relatively more expensive) in the short term. Economically speaking, this means that market participants seem to perceive them as less risky than medium- or long-term options, which most likely indicates that there is a lot of uncertainty about the future market development in the longer time horizon. To make this more plausible and palpable, the uncertainty appears to even outweigh the short-term jump risk which typically disappears in the long run, but it appears to be persistent enough to render short-term out-of-the-money options relatively more expensive than their medium-term counterparts.

In terms of the valuation of the OETCs in our sample, the comparative statics analysis of Chapter 5 gives a rather clear indication of what that means. The presence of jump risk in the shape of higher jump intensities and higher expected downward jump sizes is beneficial for OETC investors as it increases their option values and widens their continuation regions. If in turn there is less such jump risk priced by the market, OETCs lose attractiveness which manifests itself in lower optimal exercise thresholds and presumably more certificates being held inefficiently. The extent to which this holds will be explored below in Section 6.2.

6.2 Results of Empirical Study

In this section we present and evaluate the results of our empirical study, in which we have computed the optimal exercise thresholds for OETCs on the DAX for the years 2007 through 2009. In order to be able to better interpret the thresholds we relate them to the DAX and the knock-out barriers of the certificates. We do so by recalling the three distances (5.3.2.1) defined in Section 5.3

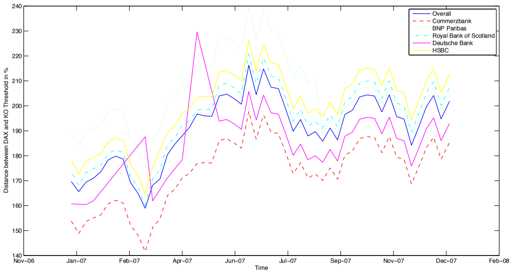
$$\begin{aligned}
D_{\text{DAX-KO}} &:= \frac{\text{DAX}_{\text{close}} - L_t}{L_t} \\
&= \text{Distance DAX to KO}_t \\
D_{\text{Threshold-KO}} &:= \frac{\text{Threshold}_t - L_t}{L_t} \\
&= \text{Distance Threshold to KO}_t \\
D_{\text{DAX-Threshold}} &:= \frac{\text{DAX}_{\text{close}} - \text{Threshold}_t}{\text{Threshold}_t} \\
&= \text{Distance DAX to Threshold}_t.
\end{aligned}$$

In the following, we will plot, closely examine, and interpret all three distances over the years 2007, 2008, and 2009. In particular, we extend the one-day glimpse provided in Chapter 5 to a period of three years. In this way we will be able to identify tendencies or patterns among the characteristics of OETCs and derive suggestions about when and for how long OETCs should be held by investors. Compared to a one-day snapshot this allows for an elimination of potential effects of the choice of day.

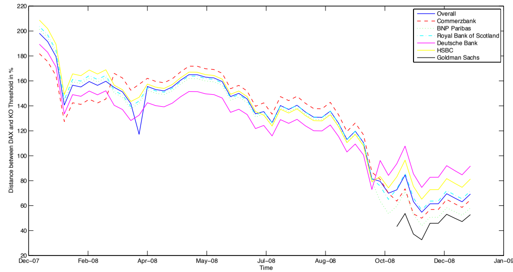
In Figure 6.5 we display the distances between the DAX and the knock-out barriers of the certificates by issuer and on an aggregate level across all issuers for the years 2007 through 2009. In general we observe a relatively parallel movement among issuers. However, w.r.t. the absolute level, there are substantial differences between them.

The overall level is highest for the year 2007, as shown in Panel 6.5a where distances range between roughly 140% and 240%, meaning that the DAX exceeds the extinguishing barrier by that amount. During the year of 2007 we also observe, that the ranking of the issuers in that area is relatively constant with BNP Paribas and HSBC usually exhibiting the highest distances to the knock-out barrier and Deutsche Bank and Commerzbank typically being at the lower end of the spectrum. The difference between the upper and the lower end of the spectrum is round about 40 percentage points throughout the entire year.

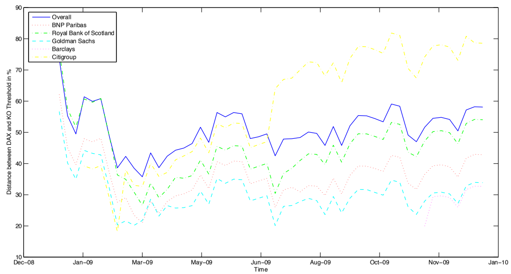
In 2008 the picture changes as shown in Panel 6.5b. Except for the months of November and December the gap between the highest and lowest values is down to approximately 25 percentage points. However, contrary to 2007 the ordering among the issuers is shuffled during the year. At the beginning, Deutsche Bank and Commerzbank exhibit the lowest values again, then Commerzbank moves to the top for about March to August and to the lower end again in the last third of 2008. On the other hand Deutsche Bank appears at



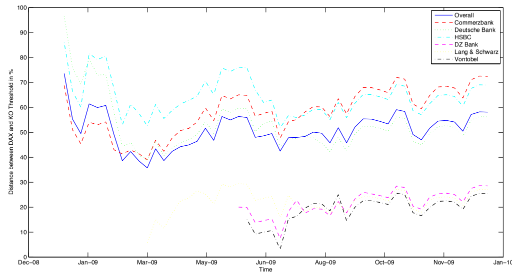
(a) Distance of DAX to KO Threshold 2007



(b) Distance of DAX to KO Threshold 2008



(c) Distance of DAX to KO Threshold 2009 - Slice 1



(d) Distance of DAX to KO Threshold 2009 - Slice 2

Figure 6.5: **Time Series Plot of the Distance between the DAX and the KO Barrier**
This figure depicts the development of the average distance between the DAX and the KO threshold of the certificates for the years 2007 through 2009 by issuer.

the top at the end of the year. Almost constantly at or close to the top is HSBC again. During the month of October Goldman Sachs appears as a new issuer. All the way through to the end of the year they exhibit the lowest distance between the DAX and the knock-out barriers of the certificates. Furthermore, the overall level of the distances decreases significantly from a range of about 180 to 220 down to about 40 to 80 percentage points.

For the year 2009, there are almost twice as many issuers. For the sake of readability, the corresponding graphs have been distributed across two plots, Panels 6.5c and 6.5d. During 2009 the ranking among issuers is relatively constant again. As before, HSBC is at the upper end of the spectrum and Goldman Sachs usually ranks at the lower end. Furthermore, the trend established by Goldman Sachs that new issuers immediately fill in at the lower end continues with the appearance of Barclays, Lang & Schwarz, Vontobel and DZ Bank. The distances for the new issuers even lie as low as 10 to 20 percentage points. On the other hand the overall level of the distances does not further decrease nor does it increase again, at least not significantly.

These observations can be explained in a very intuitive way. Since the knock-out barrier increases at the constant rate of $r + z$ the DAX has to increase at the same pace to keep the distances up at the same level. However, in 2008 the DAX sharply declined from its all-time high in 2007 of $8,105.69$ points to $4,127.41$ points during 2008 and to as low as $3,666.41$ in 2009.[222] After hitting the low mark, the DAX started to recover and increased to $5,957.43$ closing the year 2009. This devolopment has been graphically represented in Figure 6.6. Furthermore, the market saw a major drop in the

[222] Please note, that these numbers are always closing prices. Therefore, it is possible that during the day even higher or lower prices were observed. But for us, these further details would not add any more value.

riskless rate of interest beginning in 2008.[223] Whereas the EONIA interest rate even increased from 3.69% to 3.92% in 2007 to as high as 4.47% on September 23, 2008 it then decreased to 2.35% closing the calendar year. The trend continued throughout 2009 when the EONIA rate even fell to 0.299% during the month of December. A graphical representation is also provided in Figure 6.6.

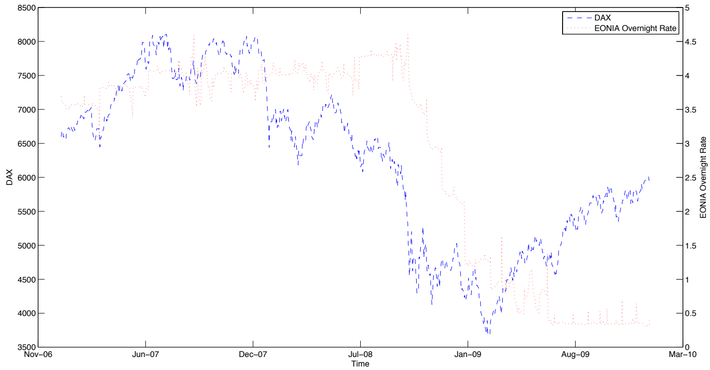


Figure 6.6: **Time Series Plot of the German Stock Market Index DAX and the EONIA Overnight Rate between 2007 and 2009**
This figure depicts the development of the DAX between 2007 and 2009. It shows the all-time high during 2007, the substantial decline during 2008 and the beginning recovery during 2009. Furthermore, the unheard of drop in interest rates from about 4% to almost zero over the same period is depicted.

The development of the year 2009 can be explained as the superposition of two different effects. On the one hand, the DAX significantly increased during the recovery from the financial crisis of the year 2008 which implies larger distances between the DAX and the knock-out barriers. On the other hand, 2009 sees the introduction of

[223] The most important reason for the decline in interest rates was that central banks lowered interest rates to support the heavily plummeting economy following the financial crisis of the years 2008 and 2009.

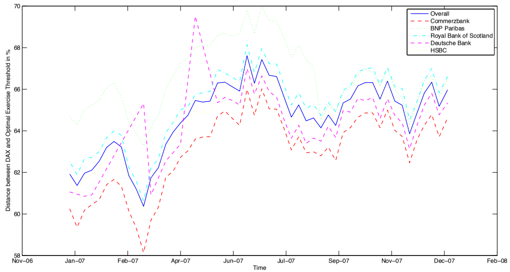
a plethora of new OETCs, which typically appear with lower such distances. Therefore, it is reasonable to draw the conclusion that the increase in the distances for existing OETCs is offset by the lower distances for newly introduced ones.

In Figure 6.7 we present the distance between the DAX and the optimal exercise thresholds for the year 2007 (cf. Panel 6.7a), the year 2008 (cf. Panel 6.7b) and the year 2009 (cf. Panels 6.7c and 6.7d). Intuitively speaking, this measures the degree of suboptimality of OETCs which should not be held.

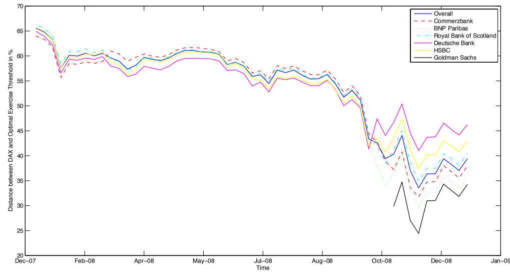
For the year 2007 we observe a general but fluctuating rise in the distance from about 62% to roughly 66%. Between June and July the distance peaks at about 68% but in February it is as low as approximately 60%. For Commerzbank and the Royal Bank of Scotland we observe a practically parallel movement to the overall average with the former being slightly under the average (going down to about 58%) and the latter slightly above. The same holds for HSBC but it appears to be randomly above or below the average, though relatively close all the time. For BNP Paribas the movement is also mostly parallel but peaks are more pronounced reaching about 70% between June and July. Deutsche Bank exhibits certain spikes up to about 69% in the first half year while it moves almost strictly in parallel to and below the overall average in the second half of the year.

The behavior can be explained by the movement of the DAX, which is roughly mimicked by the distances. The reason why the distance changes is that the time dependent optimal exercise threshold reacts more slowly or sublinearly to changes in the underlying.[224]

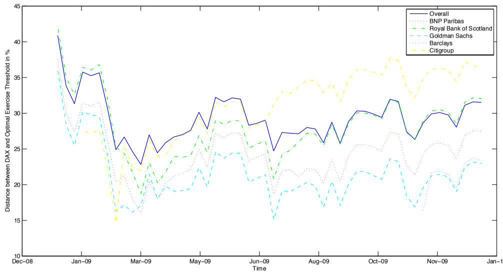
[224] From an economic perspective it is quite plausible that there is is a sublinear relationship between the underlying and the optimal exercise thresholds, which represent the tradeoff between the possibility of downward jumps below the strike and the financing costs. So to speak the threshold is the DAX level where downward jumps finally become too improbable. As it takes very long jumps to



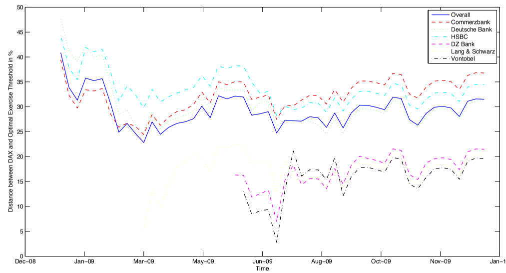
(a) Distance of DAX to Optimal Exercise Threshold 2007



(b) Distance of DAX to Optimal Exercise Threshold 2008



(c) Distance of DAX to Optimal Exercise Threshold 2009 - Slice 1



(d) Distance of DAX to Optimal Exercise Threshold 2009 - Slice 2

Figure 6.7: **Time Series of the Distance between the DAX and the Optimal Exercise Threshold**
This figure depicts the development of the average distances between the DAX and the optimal exercise thresholds of the certificates for the years 2007 through 2009 by issuer.

In 2008 the overall trend reverses. Along with a decreasing DAX, the overall average distance between the DAX and the optimal exercise thresholds decreases from round about 65% to roughly 40%. Until about October, when there are various new issues of OETCs the observed distances for all issuers move very closely and in parallel. Furthermore, there are hardly any changes (with the exception of Commerzbank) in whether an issuer is below or above the average. In March Commerzbank moves from below to above the average, where it stays until October. Apart from that, Deutsche Bank and HSBC generally appear to be below the average while the Royal Bank of Scotland and BNP Paribas are above.

In October the whole situation changes. Most likely, this is due to the fact that many new issues of certificates take place and issuers differently fix the parameters of the new issues.[225] Between October and December we observe a very parallel movement of the distance between the DAX and the optimal exercise threshold as well as a very distinct order among issuers. From the highest to the lowest values the order is: Deutsche Bank, HSBC, Royal Bank of Scotland, Commerzbank, BNP Paribas, and Goldman Sachs. Starting at about 40% in October, the values bifurcate to a range between

go below the strike, these jumps are from the tail end of the jump distribution, which is relatively flat. Consequently, an increased DAX level hardly affects the probability of beneficial jumps but it directly increases the distance between the threshold and the DAX.

[225] Not only is this quite plausible and intuitive but it can also be very well verified with the example of the Goldman Sachs issues at the end of October 2008. On October 23 and 24 there are three issues each so that all six first appear in our sample for October 29. The average distance between the DAX and the knock-out level is 43.2% on that day, which is by far the lowest for all issuers with BNP Paribas exhibiting the second lowest value at 59.6%. This further translates to the distance between the DAX and the optimal exercise threshold. Although none of the six certificates should be rationally held they can be viewed as less suboptimal than the rest as they only exceed the threshold be 29.9%. Again BNP Paribas ranks second with 36.5%. Thirdly, the band of optimality is also the widest for Goldman Sachs with 0.35% and Commerzbank ranking second at 0.21%.

35% and 45%. Upward and downward peaks go to 50% for Deutsche Bank and 25% for Goldman Sachs.

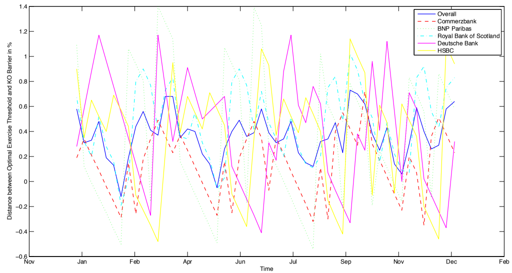
In addition to the issuing of new certificates we can bring forth the peak of the financial crisis in October 2008 with the Lehman Brothers collapse of September 2008 only one month removed. As discussed above, this prompted a significant drop of the DAX together with increased jump risk which manifested itself in higher jump intensities and higher expected downward jump sizes compared to adjacent months. Since exercise thresholds and jump risk are positively related the two effects combined can explain the decreased distances between exercise thresholds and the DAX across all issuers.

In the year 2009 we observe many fluctuations of the overall average with a minor decrease from over 40% to about 32%. For the most part of the year the movement of the distances is roughly parallel, in particular between April and December. Between January and March fluctuations are pronounced. For Citigroup, Commerzbank, and the Royal Bank of Scotland we discern a traversal from below the overall average to above the overall average. A shift in the opposite direction is only observed for Deutsche Bank. Moreover, we see a general increase in the range of values, similar to the one documented for October 2008. The values are highest for Commerzbank, Citigroup, and HSBC with about 35% and 36%. In turn, Vontobel and DZ Bank exhibit the lowest values with roughly 20%. This observation is in line with 2008, when Goldman Sachs appeared as a new issuer, as do Vontobel and DZ Bank, and instantly took the position of lowest values. Other issuers who are generally below the overall average are Goldman Sachs, Barclays and BNP Paribas.

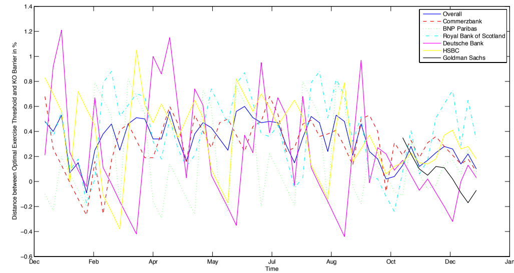
From an economic perspective we observe that all averages are by far positive. Although we observe an average of below 5% for Vontobel in July 2009, the averages typically are in excess of 20% with values as high as 70% in 2007. In turn, this means that OETCs are generally suboptimal to hold by the given percentage number. Thus, on an aggregate level they should not be held at all. However, we also observe a general decrease in the distance between the optimal exercise thresholds and the DAX. On the one hand this can be explained by the movement of the DAX as above. Another argument we might put forth is increased competition among issuers. Between 2007 and 2009 we observe a rise from about 30 to about 1,000 outstanding certificates in the marketplace, which makes this argument plausible. Due to the increased competition issuers might have to make their products more attractive. This can be attained by reducing the sub-optimality of the OETCs. The observation that new issuers often offer the certificates with the lowest distance between the DAX and the optimal exercise threshold further corroborates this rationale.

In Figure 6.8 we have plotted the observed distances between the optimal exercise thresholds and the knock-out barriers of the OETCs for the years 2007 (cf. Panel 6.8a), 2008 (cf. Panel 6.8b), and 2009 (cf. Panels 6.8c and 6.8d). Intuitively speaking, this measures the width of the continuation region, in which holding the certificate is rational.

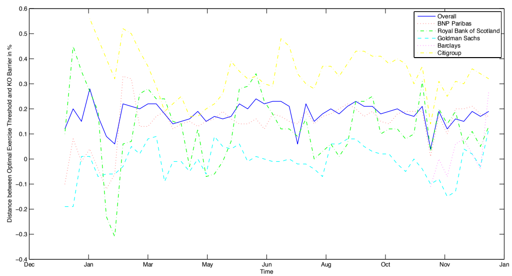
For the year 2007 we observe sharp fluctuations of the distance. The intuitive interpretation why this distance behaves in such a fashion is the stable growth of the barrier on the one hand and the sensitivity of the optimal exercise threshold w.r.t. stock price movements, the most important determinant of optimality, on the other hand. Consequently, the relative difference between optimal exercise thresholds and barriers is highly susceptible to stock price price changes. The values of this distance range from 1.4% to -



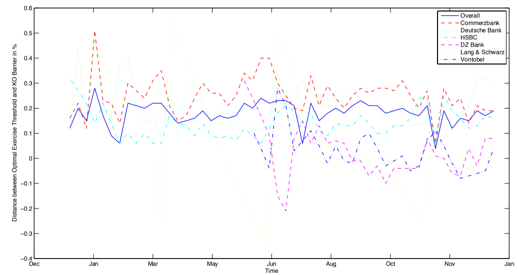
(a) Distance of Optimal Exercise Threshold to KO Barrier 2007



(b) Distance of Optimal Exercise Threshold to KO Barrier 2008



(c) Distance of Optimal Exercise Threshold to KO Barrier - Slice 1



(d) Distance of Optimal Exercise Threshold to KO Barrier - Slice 2

Figure 6.8: **Time Series Plot of the Distance between the Optimal Exercise Threshold and the Knock-out Barrier**
This figure depicts the development of the distance between the optimal exercise threshold and the knock-out barrier of the certificates for the years 2007 through 2009 by issuer.

0.5%, both attained by BNP Paribas.[226] HSBC and Deutsche Bank exhibit the second highest peaks in both directions with about 1.2% and -0.4%. The overall average moves between 0.6% and -0.1%.

In general, it is not possible to detect a certain pattern apart from the fact that peaks in one direction appear to be immediately followed by peaks in the opposite direction. Naturally, the effect is least pronounced for the overall average because averaging has a certain effect of attenuation. We furthermore stress, that this behavior is in line with the fact that during the year 2007, there were the fewest, namely two, OETCs outstanding by BNP Paribas, and four respectively by HSBC and Deutsche Bank. Even less fluctuations we observe for Commerzbank with six traded certificates and the Royal Bank of Scotland with 15. In addition, there is a tendency, that the certificates issued by the Royal Bank of Scotland almost on average exhibit a higher distance virtually throughout the entire year. This is explained by the gap sizes of roughly 3.1% and 2.7%. The larger the gap size is, the less probable it becomes that an investor gets knocked out without any rebate. Therefore, higher gaps allow for a longer expected holding duration which manifests itself by a higher optimal exercise threshold.

In 2008 the trend of very high fluctuations continues with Deutsche Bank again showing the most pronounced such behavior. For them the average distance between the optimal exercise thresholds and the knock-out barriers varies between 1.2% and -0.4%, while the overall average only ranges from roughly 0.5% to -0.1%. The overall

[226] Please note, that we observe negative distances between the optimal exercise threshold and the knock-out barrier. This means that the exercise threshold lies between the strike and the knock-out barrier. Although counter-intuitive at first glance, this is a valid case because of the fact that OETCs allow for a rebate in the case of a knock-out between knock-out barrier and strike. Given the characteristics of the certificate and the market conditions (especially stock price level and downward jump risk) at a certain time, the best possible average outcome for an investor might be a knock-out with a rebate.

average only very slightly diminished from 0.37% to 0.32%. Similarly sharp movements can be observed for HSBC (1.0% to -0.4%) and BNP Paribas (0.8% to -0.2%). The Royal Bank of Scotland and Commerzbank rank somewhere in the middle. Furthermore, we identify a reduced level of fluctuations beginning in October except for the Royal Bank of Scotland. This is directly attributable to the fact that the Royal Bank of Scotland did not issue new OETCs at that time as did the other issuers in our sample for 2008. This also includes Goldman Sachs who first appeared as an issuer at that time. Regarding the distance between the optimal exercise thresholds and the knock-out barrier they fill in at the lower end of the spectrum and clearly below the overall average with a trend of a declining distance until the end of the year 2008.

There are two lines of reasoning for this observation. First the average gap size for Goldman Sachs certificates amounts to about 2.2% compared to an overall average of 1.9%, which would suggest the opposite behavior given our argument above. At the same time we observe a reduced expected jump length compared to the calendar year 2007 (-2.5% vs -3.7%).[227] As pointed out in Section 5.2 about the comparative statics w.r.t. the expected downward jump size this has a negative effect on the value of the certificate and on the optimal exercise threshold. This effect is then even further strengthened by the reduced jump variation (27.7% vs 22.2%) which makes large downward jumps less probable. From a competitive market point of view we can argue that the closer the distance between the optimal exercise threshold and the knock-out barrier is, the more beneficial the product becomes for its issuer, because there are simply more states of the market in favor of the issuer. Consequently, issuer bene-

[227] At first glance, this is a very unintuitive observation. However, the average jump risk and the short-term jump risk was actually increased compared to adjacent months. Furthermore, this period of time saw very strong movements of the DAX on almost every trading day so that we can expect the estimation results to depend more strongly on the observation day than usual.

fits seem to outweigh market competition in favor of retail costumers at that point in time.

In 2009 the distance between the optimal exercise threshold ranges between roughly 0.3% and 0.1% with an average of 0.18%, the latter of which is a significant reduction from the previous years' values. This is in line with the above observation that for a new issuer the distance is less than the overall average, as a very large number of new issues took place during 2009. Consequently, those new issues show more favorable parameters for their issuers. Also the fluctuations are reduced which is due to the larger sample sizes in 2009.

The observed new issues by DZ Bank, Lang & Schwarz as well as Vontobel corroborate the above rationale as those OETCs mostly and instantly appear below the overall average, although for Lang & Schwarz we identify larger fluctuations due to the smaller sample size compared to the other two. Apart from that, the trend of some issuers constantly ranking above the average (Citigroup, Commerzbank) and others below the average (Royal Bank of Scotland, Goldman Sachs, BNP Paribas) is reinforced. However, there is no clear pattern which allows us to relate this behavior to the observed gap sizes for the certificates. Therefore, we suggest that these issuers follow different strategies with their products. Narrowing the distance between the optimal exercise thresholds and the knock-out barriers might be indicative of a profit maximization by the issuer, while a widening of this gap could be based on the consideration to offer more beneficial products to attract new costumers.

In addition to an examination of the observed distances between the DAX, the knock-out levels, and the optimal exercise thresholds, which naturally takes place on an aggregate level it is also worthwhile considering which OETCs are not in the exercise region but should rather be held by investors according to our analysis. There are twenty such certificates in our sample, which have been summarized in Table 6.3. Put differently, more than 99.5% of the examined certificates should not be rationally held according to our analysis. For the other twenty certificates we report the above distances together with the observation day, the corresponding strike, gap and exercise thresholds. The former are reported to better put those certificates and their characteristics in perspective to the overall situation, which has been discussed before at length.

In general, we have two factors that influence optimal exercise thresholds. Diffusive volatility is bad for investors (as it entails a higher risk of diffusive knock-out) and thus lowers the exercise threshold, jump risk is good for investors and increases the exercise threshold. Furthermore, optimality of exercise depends on time and the interplay of these factors with the DAX level itself. Therefore, we explore all these directions for possible explanations why certain OETCs appear to become rational to hold.

Right away, we can make two major observations in Table 6.3. First, all certificates which can be rationally held by investors appear in a two-month period between July and September of 2009 and second they are all marketed by two issuers, Goldman Sachs and Citigroup. Naturally, since it is optimal to hold rather than exercise them (i.e. the DAX does not exceed the exercise threshold), they exhibit negative distances between the DAX and the optimal exercise thresholds. However, all these distances are below 1% so that a future change of that status (either through a knock-out or by becoming suboptimal) is highly likely. This is further corraborated by the narrowness of the exercise region measured in terms of the distance between the optimal exercise threshold and the knock-out level. For this distance we observe an average value of 0.42%, while the overall average for all certificates in our sample is 0.18% between July and Spetember 2009. This is in line with our argument that a larger

No.	Day	WKN	Issuer	Strike	Gap	Threshold	$D_{\text{DAX-KO}}$	$D_{\text{DAX-Th.}}$	$D_{\text{Th.-KO}}$
1	8-Jul-2009	CG5ELP	Citigroup	4,497.53	1.6%	4,596.25	0.04%	-0.52%	0.56%
2	8-Jul-2009	CG8143	Citigroup	4,538.98	0.0%	4,596.25	0.74%	-0.52%	1.26%
3	8-Jul-2009	GS3YT6	Goldman Sachs	4,492.22	2.2%	4,596.25	-0.38%	-0.52%	0.13%
4	8-Jul-2009	GS5YJQ	Goldman Sachs	4,513.77	2.1%	4,596.25	-0.80%	-0.52%	-0.29%
5	8-Jul-2009	GS8Y8A	Goldman Sachs	4,501.70	2.2%	4,596.25	-0.59%	-0.52%	-0.08%
6	8-Jul-2009	GS8YDF	Goldman Sachs	4,469.82	2.0%	4,596.25	0.25%	-0.52%	0.77%
7	8-Jul-2009	GS8YDG	Goldman Sachs	4,468.22	2.1%	4,596.25	0.25%	-0.52%	0.77%
8	22-Jul-2009	GS8Y8L	Goldman Sachs	5,032.93	2.1%	5,168.38	-0.32%	-0.91%	0.59%
9	19-Aug-2009	GS8Y8R	Goldman Sachs	5,141.68	2.1%	5,255.73	-0.31%	-0.45%	0.15%
10	19-Aug-2009	GS8Y96	Goldman Sachs	5,128.26	2.1%	5,255.73	-0.12%	-0.45%	0.33%
11	19-Aug-2009	GS8YFD	Goldman Sachs	5,137.60	2.1%	5,255.73	-0.31%	-0.45%	0.15%
12	19-Aug-2009	GS8YFE	Goldman Sachs	5,118.06	2.2%	5,255.73	0.06%	-0.45%	0.52%
13	19-Aug-2009	GS8YFG	Goldman Sachs	5,138.54	2.1%	5,255.73	-0.31%	-0.45%	0.15%
14	19-Aug-2009	GS8YFR	Goldman Sachs	5,135.91	2.2%	5,255.73	-0.31%	-0.45%	0.15%
15	19-Aug-2009	GS8YFU	Goldman Sachs	5,127.19	2.2%	5,255.73	-0.12%	-0.45%	0.33%
16	2-Sep-2009	CG5NKH	Citigroup	5,230.60	1.4%	5,344.56	0.31%	-0.46%	0.78%
17	2-Sep-2009	CG8498	Citigroup	5,301.97	0.0%	5,344.56	0.34%	-0.46%	0.80%
18	2-Sep-2009	GS8Y97	Goldman Sachs	5,201.06	2.2%	5,344.56	0.13%	-0.46%	0.59%
19	2-Sep-2009	GS8Y98	Goldman Sachs	5,201.32	2.1%	5,344.56	0.13%	-0.46%	0.59%
20	2-Sep-2009	GS8Y9G	Goldman Sachs	5,231.86	2.1%	5,344.56	-0.42%	-0.46%	0.04%
Avg.	-	-	-	4,930.46	1.9%	5,022.01	-0.09%	-0.50%	0.42%

Table 6.3: **Overview of OETCs, which could be rationally held**
This table displays the OETCs which could be rationally held along with their base data (issuer, strike, gap) and their evaluation data (optimal exercise threshold, distances). Since it appears rational to hold these certificates they exhibit negative distances between the DAX and the optimal exercise threshold.

such distance is beneficial for investors. In addition, the rather small distance between the knock-out barrier and the DAX supports the suggestion of Chapter 5 that rationally held OETCs are very close to being knocked-out and thus such investment opportunities are highly likely to disappear rather quickly.[228]

To validate these results we explore whether they are caused by particular market circumstances at the time of the observations. For this purpose we examine the estimated market parameters for our sample. With the exception of July 22, 2009, we always detect a higher jump intensity compared to the neighboring days in the sample. This suggests that optimality is mostly determined by the short-term level of jump risk rather than its overall level observed during our three-year sample, since the estimated parameters for the respective days are generally lower than those estimated for years prior.

Besides the jump intensity, it is worthwhile investigating whether similar effects can be observed for the other impact factors for optimality. Regarding the expected downward jump size, our sample does not exhibit any hint that the optimality of exercise might have been caused by a short-term peak in that variable since the estimated values of the respective days do not show any special behavior compared to their neighbors. The same holds true for the standard deviation of the expected jump size. If there is no change in the expectation an increase in the variablity might still have increased the probability of longer downward jumps. Yet, we neither

[228] In our sample we also observe several cases where there is a negative distance between the DAX and the knock-out barrier which would be indicative of a knock-out event. However, these values can be explained without a knock-out event by unpublished changes to the gap sizes, which would contradict our assumption of constant gap sizes. Although issuers are allowed to do so in extra-ordinary circumstances, they communicate in their prospectuses that gap sizes are mostly held constant. Therefore, constancy of gap sizes is still a valid assumption.

find a special increase nor a decrease, which would be different from normal fluctuations.

Finally, the Merton (1976) diffusive volatility has to be considered. Contrary to jumps, diffusion has a negative impact since it increases the likelihood of diffusive knock-outs which can never occur below the strike. A reduced diffusive volatility in our sample thus could also explain the existence of OETCs, which can be rationally held by investors. However, we cannot confirm such a behavior based on our estimated parameters.

All in all we draw the conclusion that on the days in our sample the optimality appears to be solely induced by pronounced jump intensity in the short run. We do not find substantial support, that optimality might be influenced by product characteristics. For certificates issued by Goldman Sachs the average gap size is in line with the average gap size of all their OETCs. For Citigroup, there are certificates with both higher and lower than average gap sizes and yet both appear optimal. Furthermore, we find evidence for our suggestion in Chapter 5 that, optimal investment opportunities are not time persistent and hence disappear rather quickly, since none of the OETCs detected appears optimal on more than one observation day.

Chapter 7

Conclusion

This chapter concludes the thesis by summarizing the main results in Section 7.1 and giving an outlook to promising further questions not addressed in this piece of research in Section 7.2.

7.1 Summary and Discussion of Results

In this thesis we address the valuation of OETCs, a popular retail derivative in the German private investment market. The contribution to existing literature and research on this investment vehicle can be classified in the following categories:

1) Methodological advances in terms of mathematical problem formulation
2) Theoretical advances in terms of the employed market model,
3) Comparative statics analysis,
4) Empirical analysis.

The first contribution of this thesis lies in the area of *methodological advances* and revolves around the discernment that OETCs are essentially American-style options with a moving strike price, which continuously grows at the risk-free rate of interest plus a financing spread. In this regard, exercise of the option amounts to returning it to the issuer at the openly communicated redemption price (2.3.2.1), for which the respective institution is willing to take back the product at any time. Therefore, it appears quite natural and straightforward to apply American option valuation techniques.

With regard to these valuation approaches we identified five classes of problem formulation: analytical approximation techniques, lattice methods, PDE-based valuation, Monte Carlo simulation, and formulation as a stochastic control problem. In Sections 3.2.2.1, 3.2.2.2, 3.2.2.3, 3.2.2.4, and 3.2.2.5 they have been discussed at length in terms of their strenghts, weaknesses and applicability to real-life option pricing problems. The development of these five methods is closely related to the fact that American option prices hardly ever allow for a closed-form solution (not even in the simplest case of a plain vanilla American put option) so that resorting to approximation techniques is inevitable. Nonetheless, on a very abstract mathematical level all these methods are equivalent again as they are approximations of the same valuation problem which has a unique solution, the fair price of the OETCs.

In the case of OETCs it is almost natural to pursue a stochastic control approach. This is because of the various goals pursued in this thesis. From a theoretical standpoint this formulation is intuitive and tractable and allows for gaining economic insight. From a practical perspective, optimal control problems happen to allow for very powerful numerical solution techniques, which readily incorporate various models and more sophisticated features such as jumps in the same setting with only very slight adjustments, which is in sharp contrast especially to PDE-based methods and analytical approximations. In the case of PDEs jumps can even alter the type of the equation from parabolic to hyperbolic which would require totally different numerical approximations. Compared to Monte Carlo simulation, the numerics of stochastic control problems are parsimonious in terms of required computational resources to carry out simulation.

The valuation problem is eventually tackled by applying the stochastic approximation techniques proposed by Kushner & Dupuis (2001). Again, a lot of merit of this approach lies in its intuition in respect of the construction of the approximating Markov chain. Thus, the methodological advance of this thesis lies in the fact that to the best of our knowledge we are the first to apply these approximation techniques to a problem from the area of financial economics.

The second area of contribution of this thesis is of economic nature and covers *theoretical advances.* The pivotal point of these advances is Proposition 2.3.1, in which we have demonstrated that in frictionless markets OETCs should not exist, unless asset prices are discontinuous and exhibit jumps. This observation entails two substantial consequences:

1) Since it is well documented that asset prices do not follow continuous sample paths, there is a justification for these products to be traded and held in the marketplace. Of course, it becomes a meaningful question, under which circumstances rational investors would hold OETCs.

2) Economically meaningful results can only be obtained by considering asset prices, which allow for discontinuity.

The first item is addressed in the empirical study of Chapter 6, the resuls of which are summarized below. The second item is in very sharp contrast to existing literature, in particular the paper by Entrop, Scholz & Wilkens (2009), who carry out a comparative statics analysis in the Black & Scholes (1973) world with continuous sample paths and ignore the early exercise premium of OETCs in connection with a jumping model. Naturally, they observe the average-out effect of jumps in option valuation. This effect describes that short-term shocks like jumps are dominated by the on average steadily growing diffusive part in the long run. This establishes that the Merton (1976) model should be used as the minimum level of sophistication in connection with OETCs.

Furthermore, we apply the Bakshi, Cao & Chen (1997) model with stochastic volatility to our setting. In this way we obtain two noteworthy results:

1) Stochastically modeling asset price volatility adds little explanatory power to both the prices of OETCs and the rational exercise behavior investors should follow.

2) Issuers' claims that OETC prices are indepedent of volatility on account of the volatility independent redemption price (2.3.2.1) can be unequivocally refuted.[229]

Both results are perfectly in line with previous research. The former is a well-known result for the finite-time counterparts of OETCs and was first reported by Muck (2007). The latter is well established for all types of certificates and it has to be considered surprising that issuers still use this wrong claim to advertise option-like products.

The third class of insights established in this thesis pertains to the *comparative statics analysis* of OETCs. The pivotal point of this

[229] Bear in mind that the payoff functions of plain vanilla stock options do not depend on the asset price volatility either. Nonetheless it is beyond any question that volatility is the main value driver of stock options.

analysis has been to identify how the optimal exercise thresholds of OETCs depend on their value drivers. Since we characterized these certificates as a means to invest in the trade-off between financing costs for the leveraged position in the underlying on the one hand and capped downside risk as a result of the issuers' gap risk on the other, we have closely investigated the four parameters associated to that: the financing cost z and the gap size a as parameters of the certificate as well as the jump intensity λ and the expected jump size μ as market observables governing the jump components of the underlying.

Our study yields very unambiguous results:

1) Increasing the financing parameter lowers the optimal exercise threshold, as higher financing costs are negative for investors and thus they should only hold the product for a shorter period of time.

2) Increasing the gap size also lowers the optimal exercise threshold, as larger gaps provide more protection against gap risk for the issuers. Thus it is bad for investors and again they should only hold OETCs for shorter durations.

3) Increasing the jump intensity raises the optimal exercise threshold, as jumps occur more often then and in particular beneficial jumps for the investor are rendered more likely.

4) Increasing the expected downward jump size increases the optimal exercise thresholds, as on average jumps reach further downward and the situation is more favorable for the investor so that OETCs ought to be held for a longer duration.

In this way we have demonstrated that our model captures the features of OETCs as they are predicted and can be explained by the above economic rationale. At the same time this in a certain way shows the merit of our approach (model choice and solution method) as the results are economically viable.

Fourthly, we have conducted an *empirical study* of OETCs written on the DAX for the years 2007 to 2009, which significantly extends the one-day snapshot also investigated in Chapter 5. In this study we have employed the Merton (1976) model to determine optimal exercise thresholds for $1,097$ certificates and identified whether or not they should be exercised or rationally held. Our study reveals that the overwhelming majority of these certificates cannot be held rationally. The only exceptions we find are 20 quotes between July and September 2009.

Motivated by this observation, we explore the pricing of OETCs in the exercise region. This is done using the relative distances between the optimal exercise threshold and the knock-out barrier, the DAX and the knock-out barrier and the DAX and the optimal exercise threshold. It is very intuitive that a large distance between the optimal exercise thresholds and the DAX measures by how far OETCs are tailored to the issuers' benefit. The same holds true for the distance between the DAX and the knock-out level, which can be viewed as the security blanket for the issuer against a knock-out, in which case gap risk might materialize. Thirdly, the distance between the optimal exercise threshold and the knock-out barrier measures the width of the area in which holding the certificate would be rational.

Considering these distances it has been revealed that in general the situation has continuously improved for investors over the observed time period. Nonetheless, this means that holding OETCs is still mostly irrational although less pronounced. For this finding three explanations can be put forth. On the one hand this is caused by downward market movement in 2008 as a result of the financial crisis which naturally reduces the distance between the DAX and the knock-out barrier and thus the extent by which OETCs are in-the-money. This goes along with increased jump risk during this crisis which is also beneficial for OETC investors. On the other hand we have suggested that increased competition among issuers can be adduced as a cause for declining distances. This is based on and corroborated by the observation that newly issued certificates typically exhibit smaller distances. In this way, the large increase in the number of outstanding certificates beginning in October of 2008, which further intensified throughout 2009, has led to reduced distances.

7.2 Implications for Future Research

This thesis extends the existing research in the various ways outlined above. Nonetheless, the lines of advances mentioned there also indicate, in which direction future studies might head.

From a mathematical point of view it is probably worthwhile implementing even more recent numerical solution schemes. A very promising methodology could be the multigrid method suggested by Kushner & Dupuis (2001), which might considerably reduce computational effort for the solution through better approximation error reduction in each iterative step and improved sparsity properties of the involved matrices. Confidence about that can be taken from the fields of partial differential equations and numerical simulation, in which multigrid methods have prevailed as the most common and state-of-the-art solution technique because of their $\mathcal{O}(N)$ complexity compared to $\mathcal{O}(N^2)$ of standard solvers. Furthermore, better numerical efficiency can be regarded as a prerequisite to incorporate more advanced market models.

More advanced market models to be incorporated might be the Eraker (2004) model, which in addition to stochastic volatility also allows for jumps in the volatility component. Although stochastic volatility has been found to be only of minor influence to the optimal exercise thresholds of OETCs, the findings and explanations of Eraker (2004) suggest that this might be different for volatility jumps. In the paper, it is argued that stock price jumps tend to cluster over time, i.e. large jumps are typically followed by further sharp market movements. However, that stylized fact contradicts the assumption of independently arriving jumps in the Merton (1976) and Bakshi, Cao & Chen (1997) models. If volatility was allowed to jump, this would lead to periods of a significantly increased level of volatility which in turn can explain subsequent sharp market movements. Combining this with the fact that jumps are the reason why OETCs exist in the market makes it plausible that our jump-diffusion models might underestimate jump risk and thus predict too small optimal exercise thresholds. Further recent research that suggests the relevance of jumps in volatility is provided by Huang & Wu (2004) and Broadie, Chernov & Johannes (2007) who use Lévy processes to document that jumps in volatility help explain the behavior of S&P 500 options and futures options. Cont (2006) suggests a quantitative measure to assess model uncertainty in option prices which might help to further quantify the potential impact of such a more advanced model.

Furthermore, our assumption of basically default-free issuers is debatable as the bankruptcy of the renowned investment bank Lehman Brothers in 2008 emphatically shows. Certainly, buyers of certificates are not default-free either. The first shortcoming might be taken on by using vulnerable options methodology as suggested for instance by Johnson & Stulz (1987) or Hull & White (1995). The second issue of positive credit spreads for individual investors appears to be more difficult. On the one hand such a feature is built-in in OETCs by virtue of the financing parameter z, but on the other hand this factor is not investor-specific. Consequently, investors with credit spreads $\bar{z} > z$ would always fare better by investing in the certifcate than by a leveraged stock position since the payoff $\max(S_t - K_0 \exp((r + \bar{z})t))$ is always less than the one of the OETCs. In other words, such investors would have an exercise threshold of infinity.[230] As a result of these considerations the creditworthiness of every single individual has to be known and aggregate supply and demand have to be condensed to equilibrium prices. Moreover, the effect of individual credit spreads should not be overestimated as both issuers and investors have credit spreads. This leads to a situation where both spreads cancel out each other to a certain extent. Thus, the important value driver of optimal exercise thresholds should be the net credit spreads between issuers and investors. In that context, another line of research worthwhile pursuing is the one by Elkamhi, Ericsson & Wang (2012) who show that put features in bonds provide protection against liquidity, interest rate and default-risk. Thus the effect of default risk on the exercise thresholds of OETCs might be limited by the presence of this option. Nonetheless, it should be left to future research to carefully examine these effects and decide which prevails in which situation.

Regarding the comparative statics analysis, one might investigate the short variant of OETCs. This would be an interesting exercise and complete the view on these certificates, although one might

[230] From an economic point of view this can be intuitively explained. Defaultable investors who are not charged with their appropriate risk premia and credit spreads are attracted by this favorable situation for them. Consequently, they would always hold the certificate at a discount compared to their individually fair price. Without adjustment of the financing rate which reflects the investors default risk, this observation is independent of time and would uphold forever, thus an infinite exercise threshold.

expect an even less favorable situations for retails investors as the average jump sizes are negative while it takes upward jumps to knock out short OETCs.

Furthermore, the comparative statics analysis could be conducted afresh using the suggested model extensions mentioned above. In this way one would be able to reassess the influence of jumps together with their parameters on the tradeoff with the financing costs and the gap size. This would also shed light on whether jump risk is potentially over- or underestimated. Another step might be the application of these models to single stock OETCs. Contrary to single stocks, stock market indexes are well-diversified. This should lead to pronounced and potentially unsystematic jump risk. Taking further into account, that jumps are the phenomenon that creates the value of OETCs it should be worthwhile investigating whether single stock OETCs provide better investment opportunities for retail investors.

Eventually we suggest to extend the empirical study along the same lines mentioned above, i.e. also consider short OETCs, include certificates on single stocks, and employ more sophisticated models. Apart from that the empirical study might be extended to a longer period of time. In particular the fact that markets are under additional distress due to the European sovereign debt crisis, might render a study interesting, which models defaultable OETCs.

Finally, it could be worthwhile following the issuers' marketing prospectuses and tap into commodity markets, as OETCs are often marketed as a means for retail investors to enter commodity markets such as crude oil, natural gas, precious metals or agricultural goods and diversify their portfolios in this way. From a numerical and mathematical point of view, this could be done in the same setting presented in this thesis, but from an economic point of view that would require totally different market models which reflect the intricacies of commodity markets such as seasonality, the cost of carry concept along with the convenience yield, and the effect of backwardation and contango which heavily impacts rollover costs. These rollover costs are incurred at the expiry of the front month future when the issuer has to set up a new hedge. Models capable of dealing with these properties are, for example, the ones suggested by Gibson & Schwartz (1990) for oil, Cortazar & Schwartz (1994) for contingent claims on copper, Schwartz (1997), or Trolle & Schwartz (2009).[231] Fama & French (1987) evaluate the theory of storage which aims at explaining differences between futures and spot prices in terms of interest rates, convenience yield and costs incurred for warehousing. They find evidence for spot price changes based on these explanatory factors.

[231] A model for the dynamics of forward curves in the presence of seasonality is presented by Borovkova & Geman (2007).

Appendix A

Estimated Market Model Parameters

In this section we present the results of the estimation of the Merton (1976) market model from option prices conducted in Section 6.1.2. In particular, we estimated the following four parameters for each of the $D = 157$ re-estimation days of our sample:

1) Diffusive volatility $\hat{\sigma}_D$
2) Jump intensity $\hat{\lambda}^Q$
3) Expected jump size $\hat{\mu}_X^Q$
4) Standard deviation of the expected jump size $\hat{\sigma}_J$.

In this respect the hat always indicates that the parameters are not observed in the market but rather estimated and, where applicable, the superscript shows that the parameter is to be understood w.r.t. to the risk-neutral measure Q rather than the empirical measure.

In addition to the four estimated parameters we also report several further information. In the column titled *Options* we provide the total number of options available on the given day. Column N gives the number of options actually used, i.e. those that satisfy the arbitrage condition (5.3.1.1), the term to maturity criterion and the price criterion mentioned above. In column Δ we then give the number of options excluded based on these criteria. Finally in columns *Interest Rate* and *Close Price* we report the risk-free rate of interest used for the given day and the respective day's closing price for the DAX.

We obtained the following estimation results:

Sample Description					Estimated Parameters				Market Data	
No.	Day	Options	N	Δ	$\hat{\sigma}_D$	$\hat{\lambda}^Q$	$\hat{\mu}_X^Q$	$\hat{\sigma}_J$	Interest Rate	Close Price
1	3-Jan-2007	242	240	2	0.158	0.020	-0.0293	0.271	0.0354	6,691.32
2	10-Jan-2007	242	235	7	0.175	0.020	-0.0301	0.274	0.0347	6,566.56
3	17-Jan-2007	214	116	98	0.155	0.027	-0.0540	0.245	0.0351	6,701.70
4	24-Jan-2007	237	235	2	0.154	0.028	-0.0430	0.249	0.0351	6,748.37
5	31-Jan-2007	241	172	69	0.151	0.027	-0.0610	0.276	0.0353	6,789.11
6	7-Feb-2007	213	147	66	0.133	0.027	-0.0650	0.288	0.0347	6,915.56
7	14-Feb-2007	215	168	47	0.135	0.026	-0.0580	0.291	0.0351	6,961.18
8	21-Feb-2007	244	208	36	0.143	0.027	-0.0550	0.295	0.0351	6,941.66
9	28-Feb-2007	251	244	7	0.175	0.050	-0.0294	0.311	0.0354	6,715.44
10	7-Mar-2007	270	257	13	0.174	0.030	-0.0259	0.281	0.0349	6,617.75
11	14-Mar-2007	273	198	75	0.205	0.034	-0.0268	0.289	0.0375	6,447.70
12	21-Mar-2007	198	153	45	0.164	0.040	-0.0292	0.247	0.0374	6,712.06
13	28-Mar-2007	198	139	59	0.168	0.060	-0.0284	0.191	0.0375	6,816.89
14	4-Apr-2007	271	171	100	0.147	0.039	-0.0312	0.252	0.0377	7,073.91
15	11-Apr-2007	271	269	2	0.157	0.038	-0.0474	0.280	0.0376	7,152.83
16	18-Apr-2007	274	213	61	0.166	0.036	-0.0339	0.273	0.0376	7,282.34
17	25-Apr-2007	260	218	42	0.162	0.034	-0.0335	0.299	0.0376	7,343.08

Table A.1: **Estimated Merton (1976) Parameters for January through April 2007**
The table provides information about the estimation of the Merton (1976) model option pricing parameters between January and April of 2007. This includes the number of option prices available for the respective trading days, the number of options used and excluded for the estimation based on the arbitrage criteria, the observed closing price of the DAX and the observed interest rate in the market. Furthermore, the obtained model parameters $\hat{\sigma}_D$, $\hat{\lambda}^Q$, $\hat{\mu}_X^Q$, and $\hat{\sigma}_J$ are displayed for each estimation day.

Sample Description					Estimated Parameters				Market Data	
No.	Day	Options	N	Δ	$\hat{\sigma}_D$	$\hat{\lambda}^Q$	$\hat{\mu}_X^Q$	$\hat{\sigma}_J$	Interest Rate	Close Price
18	2-May-2007	272	272	0	0.166	0.039	-0.0593	0.312	0.0377	7,455.93
19	9-May-2007	283	232	51	0.162	0.041	-0.0620	0.308	0.0363	7,475.99
20	16-May-2007	283	178	105	0.166	0.032	-0.0503	0.293	0.0376	7,481.25
21	23-May-2007	281	226	55	0.154	0.037	-0.0548	0.285	0.0376	7,735.88
22	30-May-2007	282	227	55	0.165	0.044	-0.0710	0.261	0.0375	7,764.97
23	6-Jun-2007	296	292	4	0.191	0.070	-0.0301	0.269	0.0361	7,730.05
24	13-Jun-2007	296	193	103	0.184	0.042	-0.0650	0.241	0.0400	7,680.76
25	20-Jun-2007	196	196	0	0.182	0.039	-0.0278	0.286	0.0398	8,090.49
26	27-Jun-2007	210	210	0	0.196	0.048	-0.0680	0.282	0.0400	7,801.23
27	4-Jul-2007	315	272	43	0.108	0.057	-0.0201	0.400	0.0393	8,075.26
28	11-Jul-2007	319	318	1	0.156	0.035	-0.0232	0.386	0.0399	7,898.54
29	18-Jul-2007	264	264	0	0.161	0.041	-0.0303	0.271	0.0399	7,893.61
30	25-Jul-2007	264	257	7	0.225	0.014	-0.0315	0.284	0.0399	7,692.55
31	1-Aug-2007	345	292	53	0.227	0.520	-0.0297	0.165	0.0397	7,473.93
32	8-Aug-2007	359	327	32	0.198	0.064	-0.0283	0.190	0.0401	7,605.94
33	15-Aug-2007	303	276	27	0.235	0.048	-0.0337	0.272	0.0393	7,445.90
34	22-Aug-2007	374	343	31	0.221	0.036	-0.0321	0.266	0.0395	7,500.48
35	29-Aug-2007	376	296	80	0.221	0.050	-0.0281	0.261	0.0397	7,439.18

Table A.2: **Estimated Merton (1976) Parameters for May through August 2007**
The table provides information about the estimation of the Merton (1976) model option pricing parameters between May and August of 2007. This includes the number of option prices available for the respective trading days, the number of options used and excluded for the estimation based on the arbitrage criteria, the observed closing price of the DAX and the observed interest rate in the market. Furthermore, the obtained model parameters $\hat{\sigma}_D$, $\hat{\lambda}^Q$, $\hat{\mu}_X^Q$, and $\hat{\sigma}_J$ are displayed for each estimation day.

Sample Description					Estimated Parameters				Market Data	
No.	Day	Options	N	Δ	$\hat{\sigma}_D$	$\hat{\lambda}^Q$	$\hat{\mu}_X^Q$	$\hat{\sigma}_J$	Interest Rate	Close Price
36	5-Sep-2007	378	340	38	0.217	0.034	-0.0275	0.264	0.0449	7,588.03
37	12-Sep-2007	378	298	80	0.231	0.028	-0.0292	0.251	0.0403	7,472.99
38	19-Sep-2007	280	259	21	0.195	0.026	-0.0297	0.258	0.0401	7,750.84
39	26-Sep-2007	280	207	73	0.171	0.020	-0.0289	0.282	0.0420	7,804.15
40	3-Oct-2007	363	307	56	0.185	0.031	-0.0304	0.257	0.0377	7,955.30
41	10-Oct-2007	363	220	143	0.175	0.033	-0.0292	0.265	0.0387	7,986.57
42	17-Oct-2007	285	182	103	0.178	0.042	-0.0296	0.271	0.0395	7,985.41
43	24-Oct-2007	352	332	20	0.192	0.035	-0.0281	0.293	0.0395	7,828.96
44	31-Oct-2007	354	229	125	0.167	0.045	-0.0301	0.254	0.0405	8,019.22
45	7-Nov-2007	366	247	119	0.190	0.044	-0.0301	0.254	0.0396	7,799.62
46	14-Nov-2007	295	264	31	0.199	0.041	-0.0283	0.268	0.0396	7,783.11
47	21-Nov-2007	358	321	37	0.215	0.086	-0.0276	0.243	0.0396	7,518.42
48	28-Nov-2007	358	326	32	0.199	0.060	-0.0273	0.291	0.0396	7,723.66
49	5-Dec-2007	378	340	38	0.176	0.052	-0.0284	0.293	0.0372	7,944.77
50	12-Dec-2007	378	337	41	0.166	0.041	-0.0312	0.256	0.0396	8,076.12
51	19-Dec-2007	247	238	9	0.192	0.064	-0.0304	0.278	0.0380	7,837.32
52	27-Dec-2007	249	237	12	0.192	0.064	-0.0307	0.281	0.0377	8,038.60

Table A.3: **Estimated Merton (1976) Parameters for September through December 2007**
The table provides information about the estimation of the Merton (1976) model option pricing parameters between September and December of 2007. This includes the number of option prices available for the respective trading days, the number of options used and excluded for the estimation based on the arbitrage criteria, the observed closing price of the DAX and the observed interest rate in the market. Furthermore, the obtained model parameters $\hat{\sigma}_D$, $\hat{\lambda}^Q$, $\hat{\mu}_X^Q$, and $\hat{\sigma}_J$ are displayed for each estimation day.

Sample Description					Estimated Parameters				Market Data	
No.	Day	Options	N	Δ	$\hat{\sigma}_D$	$\hat{\lambda}^Q$	$\hat{\mu}_X^Q$	$\hat{\sigma}_J$	Interest Rate	Close Price
53	2-Jan-2008	334	321	13	0.198	0.020	-0.0302	0.288	0.0371	7,949.11
54	9-Jan-2008	339	225	114	0.157	0.074	-0.0114	0.197	0.0409	7,782.71
55	16-Jan-2008	276	250	26	0.216	0.030	-0.0298	0.261	0.0396	7,471.57
56	23-Jan-2008	281	256	25	0.278	0.047	-0.0268	0.263	0.0392	6,439.21
57	30-Jan-2008	341	307	34	0.244	0.024	-0.0253	0.213	0.0405	6,875.35
58	6-Feb-2008	341	284	57	0.122	0.070	-0.0297	0.137	0.0395	6,847.51
59	13-Feb-2008	287	238	49	0.152	0.063	-0.0252	0.148	0.0396	6,973.67
60	20-Feb-2008	287	231	56	0.147	0.064	-0.0243	0.152	0.0394	6,899.68
61	27-Feb-2008	354	288	66	0.229	0.026	-0.0251	0.276	0.0393	6,997.85
62	5-Mar-2008	358	247	111	0.145	0.192	-0.0259	0.154	0.0393	6,683.71
63	12-Mar-2008	359	251	108	0.253	0.058	-0.0249	0.160	0.0396	6,599.37
64	19-Mar-2008	254	234	20	0.151	0.098	-0.0272	0.159	0.0410	6,361.22
65	26-Mar-2008	254	230	24	0.152	0.077	-0.0267	0.157	0.0410	6,489.26
66	2-Apr-2008	254	239	15	0.219	0.023	-0.0255	0.283	0.0396	6,777.44
67	9-Apr-2008	351	265	86	0.220	0.029	-0.0261	0.267	0.0394	6,721.36
68	16-Apr-2008	279	230	49	0.159	0.057	-0.0271	0.154	0.0393	6,702.84
69	23-Apr-2008	340	304	36	0.157	0.058	-0.0259	0.152	0.0390	6,795.03
70	30-Apr-2008	340	248	92	0.159	0.058	-0.0256	0.148	0.0412	6,948.82

Table A.4: **Estimated Merton (1976) Parameters for January through April 2008**
The table provides information about the estimation of the Merton (1976) model option pricing parameters between January and April of 2008. This includes the number of option prices available for the respective trading days, the number of options used and excluded for the estimation based on the arbitrage criteria, the observed closing price of the DAX and the observed interest rate in the market. Furthermore, the obtained model parameters $\hat{\sigma}_D$, $\hat{\lambda}^Q$, $\hat{\mu}_X^Q$, and $\hat{\sigma}_J$ are displayed for each estimation day.

Sample Description					Estimated Parameters				Market Data	
No.	Day	Options	N	Δ	$\hat{\sigma}_D$	$\hat{\lambda}^Q$	$\hat{\mu}_X^Q$	$\hat{\sigma}_J$	Interest Rate	Close Price
71	7-May-2008	340	290	50	0.186	0.033	-0.0269	0.254	0.0397	7,076.25
72	14-May-2008	269	224	45	0.184	0.028	-0.0296	0.259	0.0396	7,083.24
73	21-May-2008	318	280	38	0.141	0.061	-0.0272	0.145	0.0391	7,040.83
74	28-May-2008	324	279	45	0.142	0.062	-0.0262	0.148	0.0400	7,033.84
75	4-Jun-2008	326	277	49	0.150	0.059	-0.0265	0.143	0.0392	6,965.43
76	11-Jun-2008	332	280	52	0.148	0.078	-0.0261	0.152	0.0396	6,650.26
77	18-Jun-2008	220	191	29	0.143	0.116	-0.0252	0.159	0.0394	6,728.91
78	25-Jun-2008	220	195	25	0.202	0.027	-0.0304	0.238	0.0388	6,617.84
79	2-Jul-2008	220	190	30	0.144	0.076	-0.0288	0.156	0.0385	6,305.42
80	9-Jul-2008	220	186	34	0.229	0.033	-0.0301	0.284	0.0420	6,386.46
81	16-Jul-2008	165	118	47	0.134	0.298	-0.0294	0.156	0.0420	6,155.37
82	23-Jul-2008	215	180	35	0.136	0.088	-0.0277	0.149	0.0421	6,536.09
83	30-Jul-2008	219	176	43	0.141	0.108	-0.0271	0.142	0.0421	6,460.12
84	6-Aug-2008	219	179	40	0.137	0.102	-0.0273	0.145	0.0422	6,561.39
85	13-Aug-2008	160	143	17	0.143	0.154	-0.0282	0.148	0.0421	6,422.19
86	20-Aug-2008	208	173	35	0.139	0.202	-0.0285	0.146	0.0419	6,317.80
87	27-Aug-2008	208	146	62	0.133	0.250	-0.0286	0.142	0.0421	6,321.03

Table A.5: **Estimated Merton (1976) Parameters for May through August 2008**
The table provides information about the estimation of the Merton (1976) model option pricing parameters between May and August of 2008. This includes the number of option prices available for the respective trading days, the number of options used and excluded for the estimation based on the arbitrage criteria, the observed closing price of the DAX and the observed interest rate in the market. Furthermore, the obtained model parameters $\hat{\sigma}_D$, $\hat{\lambda}^Q$, $\hat{\mu}_X^Q$, and $\hat{\sigma}_J$ are displayed for each estimation day.

Sample Description					Estimated Parameters				Market Data	
No.	Day	Options	N	Δ	$\hat{\sigma}_D$	$\hat{\lambda}^Q$	$\hat{\mu}_X^Q$	$\hat{\sigma}_J$	Interest Rate	Close Price
88	3-Sep-2008	208	170	38	0.138	0.197	-0.0276	0.144	0.0422	6,467.49
89	10-Sep-2008	208	101	107	0.140	0.300	-0.0264	0.149	0.0421	6,210.32
90	17-Sep-2008	110	9	13	0.254	0.128	-0.0292	0.075	0.0433	5,860.98
91	24-Sep-2008	237	190	47	0.123	0.245	-0.0215	0.152	0.0417	6,052.87
92	1-Oct-2008	238	198	40	0.278	0.141	-0.0289	0.282	0.0411	5,806.33
93	8-Oct-2008	263	236	27	0.332	0.221	-0.0298	0.257	0.0441	5,013.62
94	15-Oct-2008	228	194	34	0.354	0.198	-0.0261	0.272	0.0363	4,861.63
95	22-Oct-2008	268	200	68	0.284	0.305	-0.0356	0.311	0.0351	4,571.07
96	29-Oct-2008	278	260	18	0.262	0.294	-0.0313	0.287	0.0347	4,808.69
97	5-Nov-2008	395	355	40	0.134	0.229	-0.0232	0.142	0.0338	5,166.87
98	12-Nov-2008	397	314	83	0.341	0.214	-0.0242	0.286	0.0309	4,620.80
99	19-Nov-2008	303	239	64	0.292	0.236	-0.0259	0.272	0.0288	4,354.09
100	26-Nov-2008	388	336	52	0.164	0.270	-0.0217	0.135	0.0289	4,560.50
101	3-Dec-2008	388	312	76	0.142	0.278	-0.0164	0.185	0.0288	4,567.24
102	10-Dec-2008	388	303	85	0.166	0.271	-0.0188	0.151	0.0232	4,804.88
103	17-Dec-2008	247	228	19	0.285	0.246	-0.0252	0.288	0.0222	4,708.38
104	23-Dec-2008	247	240	7	0.173	0.159	-0.0201	0.158	0.0223	4,629.38
105	30-Dec-2008	246	234	12	0.179	0.196	-0.0253	0.148	0.0224	4,810.20

Table A.6: **Estimated Merton (1976) Parameters for September through December 2008**
The table provides information about the estimation of the Merton (1976) model option pricing parameters between September and December of 2008. This includes the number of option prices available for the respective trading days, the number of options used and excluded for the estimation based on the arbitrage criteria, the observed closing price of the DAX and the observed interest rate in the market. Furthermore, the obtained model parameters $\hat{\sigma}_D$, $\hat{\lambda}^Q$, $\hat{\mu}_X^Q$, and $\hat{\sigma}_J$ are displayed for each estimation day.

Sample Description					Estimated Parameters				Market Data	
No.	Day	Options	N	Δ	$\hat{\sigma}_D$	$\hat{\lambda}^Q$	$\hat{\mu}_X^Q$	$\hat{\sigma}_J$	Interest Rate	Close Price
106	7-Jan-2009	367	293	74	0.284	0.107	-0.0301	0.274	0.0216	4,937.47
107	14-Jan-2009	267	232	35	0.271	0.158	-0.0323	0.253	0.0209	4,422.35
108	21-Jan-2009	323	263	60	0.283	0.247	-0.0261	0.281	0.0149	4,261.15
109	28-Jan-2009	323	250	73	0.227	0.187	-0.0288	0.284	0.0120	4,518.72
110	4-Feb-2009	354	255	99	0.249	0.166	-0.0251	0.294	0.0119	4,492.79
111	11-Feb-2009	356	280	76	0.275	0.116	-0.0170	0.260	0.0124	4,530.09
112	18-Feb-2009	296	218	78	0.295	0.117	-0.0240	0.240	0.0127	4,204.96
113	25-Feb-2009	360	239	121	0.235	0.235	-0.0260	0.270	0.0129	3,846.21
114	4-Mar-2009	362	217	145	0.300	0.157	-0.0100	0.300	0.0127	3,890.94
115	11-Mar-2009	366	225	141	0.300	0.160	-0.0200	0.270	0.0084	3,914.10
116	18-Mar-2009	250	184	66	0.286	0.253	-0.0364	0.285	0.0088	3,996.32
117	25-Mar-2009	252	197	55	0.278	0.101	-0.0273	0.244	0.0097	4,223.29
118	1-Apr-2009	363	287	76	0.263	0.137	-0.0312	0.231	0.0096	4,131.07
119	8-Apr-2009	363	299	64	0.276	0.112	-0.0318	0.253	0.0085	4,357.92
120	15-Apr-2009	306	209	97	0.260	0.145	-0.0280	0.200	0.0092	4,549.79
121	22-Apr-2009	368	272	96	0.281	0.119	-0.0331	0.258	0.0095	4,594.42
122	29-Apr-2009	368	319	49	0.240	0.085	-0.0250	0.200	0.0051	4,704.56

Table A.7: **Estimated Merton (1976) Parameters for January through April 2009**
The table provides information about the estimation of the Merton (1976) model option pricing parameters between January and April of 2009. This includes the number of option prices available for the respective trading days, the number of options used and excluded for the estimation based on the arbitrage criteria, the observed closing price of the DAX and the observed interest rate in the market. Furthermore, the obtained model parameters $\hat{\sigma}_D$, $\hat{\lambda}^Q$, $\hat{\mu}_X^Q$, and $\hat{\sigma}_J$ are displayed for each estimation day.

Sample Description					Estimated Parameters				Market Data	
No.	Day	Options	N	Δ	$\hat{\sigma}_D$	$\hat{\lambda}^Q$	$\hat{\mu}_X^Q$	$\hat{\sigma}_J$	Interest Rate	Close Price
123	6-May-2009	373	308	65	0.253	0.080	-0.0287	0.289	0.0053	4,880.71
124	13-May-2009	294	220	74	0.278	0.101	-0.0249	0.221	0.0073	4,727.61
125	20-May-2009	323	199	124	0.261	0.062	-0.0253	0.161	0.0092	5,038.94
126	27-May-2009	328	233	95	0.240	0.101	-0.0190	0.230	0.0100	5,000.77
127	3-Jun-2009	335	274	61	0.215	0.098	-0.0140	0.245	0.0058	5,054.53
128	10-Jun-2009	335	185	150	0.226	0.119	-0.0160	0.235	0.0079	5,051.18
129	17-Jun-2009	215	191	24	0.213	0.113	-0.0150	0.218	0.0085	4,799.98
130	24-Jun-2009	215	149	66	0.234	0.138	-0.0165	0.224	0.0138	4,836.01
131	1-Jul-2009	215	131	84	0.227	0.110	-0.0178	0.240	0.0034	4,905.44
132	8-Jul-2009	221	154	67	0.212	0.258	-0.0172	0.236	0.0036	4,572.65
133	15-Jul-2009	160	93	67	0.226	0.132	-0.0187	0.230	0.0034	4,928.44
134	22-Jul-2009	198	114	84	0.240	0.130	-0.0207	0.118	0.0037	5,121.56
135	29-Jul-2009	219	159	60	0.216	0.131	-0.0196	0.211	0.0035	5,270.32
136	5-Aug-2009	221	203	18	0.260	0.020	-0.0306	0.274	0.0034	5,353.01
137	12-Aug-2009	169	120	49	0.221	0.106	-0.0150	0.199	0.0034	5,350.09
138	19-Aug-2009	169	107	62	0.234	0.193	-0.0177	0.213	0.0034	5,231.98
139	26-Aug-2009	207	186	21	0.247	0.051	-0.0160	0.222	0.0034	5,521.97

Table A.8: **Estimated Merton (1976) Parameters for May through August 2009**
The table provides information about the estimation of the Merton (1976) model option pricing parameters between May and August of 2009. This includes the number of option prices available for the respective trading days, the number of options used and excluded for the estimation based on the arbitrage criteria, the observed closing price of the DAX and the observed interest rate in the market. Furthermore, the obtained model parameters $\hat{\sigma}_D$, $\hat{\lambda}^Q$, $\hat{\mu}_X^Q$, and $\hat{\sigma}_J$ are displayed for each estimation day.

Sample Description					Estimated Parameters				Market Data	
No.	Day	Options	N	Δ	$\hat{\sigma}_D$	$\hat{\lambda}^Q$	$\hat{\mu}_X^Q$	$\hat{\sigma}_J$	Interest Rate	Close Price
140	2-Sep-2009	217	149	68	0.225	0.287	-0.0180	0.210	0.0033	5,319.84
141	9-Sep-2009	217	121	96	0.210	0.161	-0.0193	0.216	0.0033	5,574.26
142	16-Sep-2009	105	79	26	0.220	0.099	-0.0187	0.214	0.0037	5,700.26
143	23-Sep-2009	231	140	91	0.216	0.088	-0.0151	0.235	0.0034	5,702.05
144	30-Sep-2009	231	186	45	0.256	0.033	-0.0277	0.271	0.0053	5,675.16
145	7-Oct-2009	334	298	36	0.256	0.025	-0.0274	0.282	0.0034	5,640.75
146	14-Oct-2009	287	168	119	0.228	0.045	-0.0261	0.253	0.0037	5,854.14
147	21-Oct-2009	336	294	42	0.238	0.027	-0.0243	0.255	0.0035	5,833.49
148	28-Oct-2009	339	297	42	0.265	0.029	-0.0248	0.294	0.0035	5,496.27
149	4-Nov-2009	342	281	61	0.207	0.059	-0.0242	0.232	0.0033	5,444.23
150	11-Nov-2009	342	223	119	0.051	0.048	-0.0680	0.049	0.0034	5,668.35
151	18-Nov-2009	280	243	37	0.244	0.023	-0.0283	0.286	0.0035	5,787.61
152	25-Nov-2009	327	222	105	0.227	0.040	-0.0289	0.271	0.0035	5,803.02
153	2-Dec-2009	327	195	132	0.203	0.092	-0.0155	0.184	0.0035	5,781.68
154	9-Dec-2009	328	278	50	0.263	0.032	-0.0308	0.276	0.0035	5,647.84
155	16-Dec-2009	205	139	66	0.207	0.076	-0.0228	0.286	0.0035	5,903.43
156	23-Dec-2009	209	139	70	0.174	0.095	-0.0282	0.223	0.0030	5,957.44
157	30-Dec-2009	209	144	65	0.107	0.141	-0.0228	0.259	0.0032	5,957.43

Table A.9: **Estimated Merton (1976) Parameters for September through December 2009**
The table provides information about the estimation of the Merton (1976) model option pricing parameters between September and December of 2009. This includes the number of option prices available for the respective trading days, the number of options used and excluded for the estimation based on the arbitrage criteria, the observed closing price of the DAX and the observed interest rate in the market. Furthermore, the obtained model parameters $\hat{\sigma}_D$, $\hat{\lambda}^Q$, $\hat{\mu}_X^Q$, and $\hat{\sigma}_J$ are displayed for each estimation day.

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This dissertation analyzes Open-End Turbo Certificates (OETCs), a popular class of retail derivatives. OETCs can be exercised at any time at the investor's discretion. In order to explain the existence of the certificates jump risk must be considered. We propose and implement an optimal stopping approach to price these securities, which further allows for determining optimal exercise thresholds. They result from the trade-off between benefits from downward jump protection and financing costs. We show that early exercise right has a significant impact on their values. In an empirical analysis pertaining to the years 2007 through 2009 it turns out that certificates which could be rationally held are very rare, although the degree by which the underlying exceeds the optimal exercise thresholds continually declines over the considered period. We suggest three lines of explanation: general market movement, jump risk perception by the market, and increased competition among issuers.

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