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# The Core and Productivity-Improving Mergers in Mixed Oligopoly

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**Abstract**

We analyze productivity-improving mergers in mixed triopoly and explore stable market structures. Solely the market structures derived by the merger between a public firm and one of the two private firms with about 57 percent shares by the public firm is in the core.

*Key words*: mergers; mixed oligopoly; the core of market structures

*JEL classification*: D21; L13; L33

## 1 Introduction

The purpose of this paper is to provide a theoretical analysis on merger activities in the industry composed of one public firm and two private firms. Such an industry is usually referred to as *mixed oligopoly*, or more specifically *mixed triopoly*. The literature on mixed oligopoly can be traced back to the paper of De Fraja and Delbono (1989). The mixed oligopoly is distinguished from the oligopoly composed only of private firms especially in the objective of a public firm. In many existing works on mixed oligopoly, it is assumed that the objective of a public firm is social welfare maximization, whereas private firms aims to maximize their own profits. Since, in the real world, public firms are financed by tax revenues, it seems quite reasonable to assume that a public is devoted to improving social welfare. Although there have been many analyses of a merger in private oligopoly (e.g. Salant et al. (1983), Deneckere and Davidson (1985), and Farrell and Shapiro (1990)), not so many efforts have been carried out in studying merger activities in mixed oligopoly. Exceptions are Bárcena-Ruiz and Gárzon (2003), and Coloma (2006). Both of the papers analyzed a merger in mixed duopoly, i.e. a merger in the industry composed of a public firm and a private firm. In the paper of Bárcena-Ruiz and Gárzon, the two firms were assumed to produce heterogeneous

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products and the decision to merge by the firms was analyzed. On the other hand, in his paper, Coloma considered the case where the two firms produce homogeneous products and made welfare comparisons among possible market structures.

There are two respects in which our paper contributes to the literature on mergers in mixed oligopoly. First, although neither of the papers of Bárcena-Ruiz and Gárzon nor of Coloma considered a synergy effect of a merger, we assume that a merger yields a synergy effect to the technology of the merged firm and entails the improvement on productivity. Without assuming any synergy effects of a merger, Bárcena-Ruiz and Gárzon obtained the result that, in their setting, both of the private firm and the public firm want to merge only when the degree to which the two heterogeneous products are substitutes is sufficiently low and, moreover, the merger does not take place when the two products are perfectly substitutable. Since, in the real world, there are many examples of mergers among firms which produce highly substitutable products, this result is counterintuitive to what we would expect. In this paper, we re-examine the mergers among the firms producing homogeneous, i.e. perfectly substitutable, products in mixed triopoly under the assumption that a merger yields the improvement on productivity. It seems very natural to assume that the merger between the firms that produce highly substitutable heterogeneous products entails a synergy effect because merger participants may easily learn a strong point of each firm's production skill and/or their patents from one another. In the study of the horizontal mergers in private oligopoly, Farrell and Shapiro (1990) showed that the merger could improve social welfare if the merged firm exploits economies of scale well. In order to analyze mergers that entail the improvement on productivity, we follow McAfee and Williams (1992). In our paper, the technology of each of the three firms is identically represented by the quadratic cost function $C(q_i) = q_i^2$, where $q_i$ is the amount of the production of the firm $i$, and, as considered in the paper of McAfee and Williams, the merged firm operates the plants which were previously owned by the pre-merged firms most efficiently and thus the technology is represented by $C(q) = q^2/n$, where $q$ is the amount of the output of the merged firm and $n$ $(= 2, 3)$ is the number of the merger participants. Such a cost function of the merged firm clearly shows that a merger entails the improvement on productivity.

The other respect in which our analysis is clearly distinguished from the earlier ones is that we especially focus on the stability of market structures. We extend the usual way of analysis of mergers where solely the decision to merge by the firms is discussed. In this paper, we treat merger activities as coalition formations among the firms that are allowed to freely merge and freely break off the merger . For example, the merger between firms, say 0 and 1, with leaving a firm, say 2, standing alone can be considered as the coalition formation of $\{\{0, 1\}, \{2\}\}$. Viewing merger activities as coalition formations among the firms, to find the stable coalition formations, i.e. stable market structures, is of our interest. In order to analyze the stability of market structures, we adopt the *core*, the well-established solution concept in cooperative game theory and examine which of all possible market structures is/are stable in the sense that once any of such market structures is actually realized none of the owners of the firms wants to change this present market structure by merging with other firm or breaking off the merger.

The motivation to analyze the stability problem of merger activities perhaps needs some elaboration. In our paper, we consider the industry of mixed triopoly. In the mixed triopoly market, the variation of possible forms of a merger among the firms increases and becomes more complicated than in mixed duopoly. Consequently, it might be the case that, while the owners of some two firms, say 0 and 1, have an incentive to merge into one firm by comparing their payoffs obtained in each of the initial market structure, i.e. the coalition structure $\{\{0\},\{1\},\{2\}\}$, and the one realized after the merger, i.e. { {0,1} ,{ 2} }, the owner of the firm 0 could receive higher payoff if s/he breaks off the merger with the firm 1 and alternatively merges with the firm 2, i.e. in the structure $\{\{0,2\},\{1\}\}$, than in the case of the merger with the firm 1. In this case, if the owner of the firm 2 also has an incentive to merge with the firm 0, the merger between the firms 0 and 2 will be realized, and the merger between the firms 0 and 1 can never be realized. Therefore, in the presence of more than two firms, it is not sufficient to analyze the decision to merge in each particular case, and we should examine merger activities in terms of stable coalition formations. In the literature on mergers in private oligopoly, Barros (1998), Horn and Persson (2001), and Straume (2006) adopted the same approach. However, with the only exception of Kamijo and Nakamura (2007), there has not been any works that analyze mergers in mixed oligopoly along the approach using the core property. Among these existing works, there is a slight difference in the definitions of core property. The core property considered in this paper is the same as the one considered in Barros (1998) and Kamijo and Nakamura (2007). We refer the reader to Brito and Gata (2006) for the detailed discussion about the difference between the core property adopted by Barros (1998) and the one considered in Horn and Persson (2001) and Straume (2006). Using the core property à la Barros (1998) and Kamijo and Nakamura (2007), this paper shows that, in our mixed triopoly model, the core of market structures is non-empty and the core consists solely of the market structures derived by the merger between a public firm and one of the two private firms with about 0.57 share ratio by the public firm.

This paper is organized as follows. The next section introduces our model and presents the Cournot-Nash equilibrium for each of four regimes; mixed triopoly; merger between private firms; merger between a public firm and a private firm; and merger among all the three firms. Our results are provided in Section 3. Section 4 concludes with some remarks.

# 2 Model

## 2.1 Basic Set-up of Mixed Oligopoly

We analyze stable market structures in the industry composed of one public firm, denoted by 0, and two private firms, 1 and 2. Each firm produces a single homogeneous good and is assumed to be entrepreneurial one, i.e. the owners themselves make every managerial decision making. The public firm (resp. each of the private firms) is owned by the government (resp. a single private shareholder). In accordance with whether a merger among the firms is realized or not, we have four possible market regimes: (a) *mixed triopoly* $\{\{0\},\{1\},\{2\}\}$, (b) *merger between private firms* $\{\{0\},\{1,2\}\}$, (c) *merger between a public firm and a private firm* $\{\{0,i\},\{j\}\}$ ($i, j = 1,2,\ i \neq j$), and (d) *merger among all the three firms* $\{\{0,1,2\}\}$. Although the details of the formal descriptions of the four regimes are slightly different, we mainly introduce the set-up of the mixed triopoly. The other regimes are easily understand as an extension of the mixed triopoly.

As have been usually considered in the literature on mixed oligopoly, the inverse demand function is given as a linear function of the total output $Q$,

$$P(Q) = a - Q, \tag{1}$$

where $a$ is sufficiently large positive number. As assumed in Bárcena-Ruiz and Garzón (2003), each firm $i$ $(= 0, 1, 2)$ has an identical technology represented by the quadratic cost function

$$C(q_i) = q_i^2, \tag{2}$$

where $q_i$ is the quantity of the good produced by the firm $i$. The profit function of the firm $i$ $(= 0, 1, 2)$ is given as:

$$\Pi_i = (a - Q)q_i - q_i^2. \tag{3}$$

As usual, social welfare $W$ is measured by the sum of consumer surplus $CS = Q^2/2$, and firms' profits.

In their paper, Bárcena-Ruiz and Garzón have not discussed the case where a merger yields the improvement on productivity. The proiductivity-improving merger has been analyzed in McAfee and Williams (1992). As in the paper of McAfee and Williams and also of Nakamura and Inoue (2007) and Heywood and McGinty (2007a; 2007b), we consider that a merged firm shows the improvement on productivity. The market regimes derived by mergers, i.e. (b), (c), and (d), show the differences particularly in the forms of cost functions. If $n$ $(= 2, 3)$ firms merge into one firm, the total cost of the merged firm $C_m$ is represented as:

$$C_m(q_m) = \frac{q_m^2}{n}, \tag{4}$$

where $q_m$ is the output of the merged firm $m$. Such a cost function is supported by the assumption that the merged firm adopts the most efficient operation plan of the plants previously owned by the pre-merged firms. More precisely, the cost function in (4) corresponds to the case of the most efficient operation rates $(\lambda_1^*, \ldots, \lambda_n^*) = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ of the plants derived from the total cost minimization problem:

$$\min_{(\lambda_1,\ldots,\lambda_n)} \sum_{i=1}^{n} (\lambda_i q_m)^2, \quad \text{subject to} \sum_{i=1}^{n} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for all } i. \tag{5}$$

The profit of the merged firm is given by replacing $q_i^2$ with $q_m^2/n$ in (3).

## 2.2 Equilibrium Outcomes in the Regimes (a) to (d)

We now examine the Cournot equilibrium for each regimes. Let $U_i^r$ denote an objective function that the firm $i$ maximizes in the regime $r$ $(= a, b, c, d)$. In the rest of the paper, functions and variables with superscript $r$ $(= a, b, c, d)$ denote those considered in the regime $r$.

**(a) Mixed triopoly $\{\{\mathbf{0}\}, \{\mathbf{1}\}, \{\mathbf{2}\}\}$:** In this regime, each of the three firms has the following objective function, respectively:

$$U_0^a(q_0^a; q_1^a, q_2^a) = W^a = \frac{1}{2}\left(q_0^a + \sum_{i=1}^{2} q_i^a\right)^2 + \Pi_0 + \sum_{i=1}^{2} \Pi_i, \tag{6}$$

$$U_i^a(q_i^a; q_0^a, q_j^a) = \Pi_i, \quad (i, j = 1,2 \text{ and } i \neq j). \tag{7}$$

The first order conditions of the maximization problems give the following Cournot equilibrium:

$$q_0^{a*} = \frac{3}{13}a \text{ and } q_i^{a*} = \frac{2}{13}a, \quad (i = 1,2). \tag{8}$$

Therefore, in the Cournot equilibrium, we obtain the following equilibrium profits $\Pi_i^a$, consumer surplus, and social welfare:

$$\Pi_0^a = \frac{9}{169}a^2,\ \Pi_i^a = \frac{8}{169}a^2,\ (i = 1,2),\ CS^a = \frac{49}{338}a^2, \text{ and } W^a = \frac{99}{338}a^2.$$

Notice that, in this regime, the equilibrium profit of the public firm 0 is larger than those of the private firms 1 and 2. As has been shown in extensive literature on mixed oligopoly, in the case of quantity competition, a public firm whose objective is welfare maximization chooses the output larger than the one by a private firm, profit maximizer, because the choice of the output by the public firm is largely affected by the level of consumer surplus. Consequently, this leads to the larger market share and higher profit of the public firm. The payoffs to the owners of the firms, denoted by $V_i^a$ $(i = 0,1,2)$, are

$$V_0^a = W_0^a = \frac{99}{338}a^2, \text{ and } V_i^a = \Pi_i^a = \frac{8}{169}a^2\ (i = 1,2).$$

**(b) Merger between private firms $\{\{\mathbf{0}\},\{\mathbf{1},\mathbf{2}\}\}$:** Next, we consider the case where the two private firms merge into a new private firm denoted by 12. Let $q_{12}^b$ be the amount of the output of the merged firm 12. The objective of the merged firm 12 is to maximize its profit:

$$U_{12}^b(q_{12}^b; q_0^b) = \Pi_{12} = \left[a - \left(q_0^b + q_{12}^b\right)\right]q_{12}^b - \frac{1}{2}\left(q_{12}^b\right)^2. \tag{9}$$

The objective function of the public firm is:

$$U_0^b(q_0^b; q_{12}^b) = W^b = CS^b + \Pi_0 + \Pi_{12} \tag{10a}$$

$$= \frac{1}{2}\left(q_0^b + q_{12}^b\right)^2 + \left[a - \left(q_0^b + q_{12}^b\right)\right]q_0^b - \left(q_0^b\right)^2 + \left[a - \left(q_0^b + q_{12}^b\right)\right]q_{12}^b - \frac{1}{2}\left(q_{12}^b\right)^2. \tag{10b}$$

Note that, in the last term of its profit function, the merged firm 12 shows the improvement on productivity. In the Cournot equilibrium, we obtain the following:

$$q_0^{b*} = \frac{1}{4}a, \quad q_{12}^{b*} = \frac{1}{4}a,\ \Pi_0^b = \frac{1}{16}a^2,\ \Pi_{12}^b = \frac{3}{32}a^2,\ CS^b = \frac{1}{8}a^2, \text{ and } W^b = \frac{9}{32}a^2.$$

Let $\alpha \in [0,1]$ be a ratio of shareholding by the owner of the firm 1 in the merged firm 12. Then, the payoff to the owner of the public firm 0, $V_0^b$, and those to the pre-merged private firms 1 and 2, $V_1^b$ and $V_2^b$, are

$$V_0^b = W^b = \frac{9}{32}a^2,\ V_1^b = \alpha\Pi_{12}^b = \frac{3}{32}\alpha a^2, \text{ and } V_2^b = (1-\alpha)\Pi_{12}^b = \frac{3}{32}(1-\alpha)a^2.$$

**(c) Merger between a public firm and a private firm** $\{\{\mathbf{0},\mathbf{i}\},\{\mathbf{j}\}\}$**:** In this regime the public firm 0 and one of the private firms $i$ (= 1 or 2) merge into a new firm $0i$. Let $q_{0i}^c$ and $\Pi_{0i}^c$ denote the output and profit of the merged firm $0i$. As the objective function of the public-private merged firm $0i$, we consider the weighted average of social welfare and the profit of the merged firm:

$$U_{0i}^c(q_{0i}^c; q_j^c) = \beta W^c + (1-\beta)\Pi_{0i} \tag{11a}$$

$$= \beta\left[\frac{1}{2}\left(q_{0i}^c + q_j^c\right)^2 + \Pi_{0i} + \Pi_j\right] + (1-\beta)\Pi_{0i}, \quad (i,j = 1,2 \text{ and } i \neq j) \tag{11b}$$

where $\beta \in [0,1]$ is a ratio of shareholding by the government in the merged firm $0i$ and $\Pi_{0i}$ and $\Pi_j$ are the profit functions of the firms $0i$ and $j$, respectively, given as:

$$\Pi_{0i} = \left[a - \left(q_{0i}^c + q_j^c\right)\right]q_{0i}^c - \frac{1}{2}\left(q_{0i}^c\right)^2, \tag{12}$$

$$\left(U_j^c\left(q_j^c; q_{0i}^c\right) \equiv\right)\ \Pi_j = \left[a - \left(q_{0i}^c + q_j^c\right)\right]q_j^c - \left(q_j^c\right)^2. \tag{13}$$

The weighted average of social welfare and the profit in the objective of a public-private merged firm has first been suggested in Matsumura (1998) and also been adopted in Bárcena-Ruiz and Garzón (2003). In the Cournot equilibrium of this regime, we get:

$$q_{0i}^{c*} = \frac{3}{11-4\beta}a,\ q_j^{c*} = \frac{(2-\beta)}{11-4\beta}a,\ \Pi_{0i}^c = \frac{9(3-2\beta)}{2(11-4\beta)^2}a^2,\ \Pi_j^c = \frac{2(2-\beta)^2}{(11-4\beta)^2}a^2,$$
$$CS^c = \frac{(5-\beta)^2}{2(11-4\beta)^2}a^2, \text{ and } W^c = \frac{(68-44\beta+5\beta^2)}{2(11-4\beta)^2}a^2.$$

The payoffs to the owners of the pre-merged public firm 0 and pre-merged private firm $i$, $V_0^c$ and $V_i^c$, are

$$V_0^c = W^c = \frac{(68-44\beta+5\beta^2)}{2(11-4\beta)^2}a^2, \text{ and } V_i^c = (1-\beta)\Pi_{0i}^c = \frac{9(3-2\beta)(1-\beta)}{2(11-4\beta)^2}a^2,$$

and the one to the owner of the non-merged private firm $j \neq i$, $V_j^c$, is

$$V_j^c = \Pi_j^c = \frac{2(2-\beta)^2}{(11-4\beta)^2}a^2.$$

**(d) Merger among all the three firms** $\{\{\mathbf{0},\mathbf{1},\mathbf{2}\}\}$**:** Finally, we examine the case where all of the three firms, 0, 1, and 2, merge into one firm denoted by 012. In the similar way to the regime (c), the objective function of the merged firm is defined as follows:

$$U^d_{012}(q^d_{012}) = \gamma W^d + (1-\gamma)\Pi_{012} \tag{14a}$$
$$= \gamma\left[\frac{1}{2}\left(q^d_{012}\right)^2 + \Pi_{012}\right] + (1-\gamma)\Pi_{012}, \tag{14b}$$

where $\gamma \in [0,1]$ is a ratio of shareholding by the government in the merged firm 012 and $\Pi_{012}$ is the profit function of the merged firm given as:

$$\Pi_{012} = \left(a - q^d_{012}\right)q^d_{012} - \frac{1}{3}\left(q^d_{012}\right)^2. \tag{15}$$

Note that the merged firm in this regime shows further improvement on productivity than in the regimes (b) and (c). In the Cournot equilibrium, we obtain the following:

$$q^{d*}_{012} = \frac{3}{8-3\gamma}a,\ \Pi^d_{012} = \frac{3(4-3\gamma)}{(8-3\gamma)^2}a^2,\ CS^d = \frac{9}{2(8-3\gamma)^2}a^2,\ \text{and } W^d = \frac{3(11-6\gamma)}{2(8-3\gamma)^2}a^2.$$

The payoffs to the owners of the pre-merged firms, $V^d_i$ $(i = 0,1,2)$, are

$$V^d_0 = W^d = \frac{3(11-6\gamma)}{2(8-3\gamma)^2}a^2,\ V^d_1 = (1-\gamma)\delta\Pi^d_{012} = \frac{3(4-3\gamma)(1-\gamma)\delta}{(8-3\gamma)^2}a^2,\ \text{and}$$
$$V^d_2 = (1-\gamma)(1-\delta)\Pi^d_{012} = \frac{3(4-3\gamma)(1-\gamma)(1-\delta)}{(8-3\gamma)^2}a^2,$$

where $\delta \in [0,1]$ measures a ratio of profit distribution among the private sector, i.e. $(1-\gamma)\delta$ is a ratio of shareholding by the owner of the firm 1 in the merged firm 012.

Table 1 summarizes objective functions and payoffs of the firms in each of the four regimes.

Table 1: Firms' objectives and owners' payoffs

| regime (r) | firms' objectives: $U^r_i$ | | owners' payoffs: $V^r_i$ |
|---|---|---|---|
| regime (a) | $U^a_0 = W^a$ | | $V^a_0 = W^a(q^{a*})$ |
| | $U^a_1 = \Pi_1$ | | $V^a_1 = \Pi_1(q^{a*})$ |
| | $U^a_2 = \Pi_2$ | | $V^a_2 = \Pi_2(q^{a*})$ |
| regime (b) | $U^b_0 = W^b$ | | $V^b_0 = W^b(q^{b*})$ |
| | $U^b_{12} = \Pi_{12}$ | | $V^b_1 = \alpha\Pi_{12}(q^{b*})$ |
| | | | $V^b_2 = (1-\alpha)\Pi_{12}(q^{b*})$ |
| regime (c) | $U^c_{0i} = \beta W^b + (1-\beta)\Pi_{0i}$ | $(i = 1,2)$ | $V^c_0 = W^c(q^{c*})$ |
| | $U^c_j = \Pi_j$ | $(j \neq i)$ | $V^c_i = (1-\beta)\Pi_{0i}(q^{c*})$ |
| | | | $V^c_j = \Pi_j(q^{c*})$ |
| regime (d) | $U^d_{012} = \gamma W^d + (1-\gamma)\Pi_{012}$ | | $V^d_0 = W^d(q^{d*})$ |
| | | | $V^d_1 = (1-\gamma)\delta\Pi_{012}(q^{d*})$ |
| | | | $V^d_2 = (1-\gamma)(1-\delta)\Pi_{012}(q^{d*})$ |

## 2.3 Market Structures and the Core

Each of the four regimes, except for the mixed triopoly which itself represents the market structure, includes more than one market structure, each of which can be identified in terms of shareholding ratio in the merged firm and of merger participants, i.e. a coalition formation. For example, in the regime (c), we can find one particular market structure that is composed of the merged firm 01 with the government's shareholding $\beta = 0.5$ and the private firm 2. Which of the possible market structures will actually occur fairly depends on the managerial decision making of the three owners of the firms 0, 1, and 2: *merge*, *not to merge*, or *break off the merger*.

In the preceding subsection, given the market structure that will actually occur as a result of coalition formation among the owners, we have examined the Cournot equilibrium for each market structure. Now, a natural question to ask is which of the market structures will occur as a consequence of the owners coalition formation. This problem can be analyzed in terms of the game of coalition formation among the owners. As discussed in the introduction, in the case of more than two firms' owners, it is not sufficient to analyze the decision by the owners for each particular case, and the stability problem will be of importance. Thus, we especially focus on which market structure, or coalition formation among the owners, will be stable in the sense that once a market structure in question is realized, it never shift into any other market structure. To analyze this stability problem we invoke the *core*, the well-established solution concept in cooperative game theory. We assume that each of the owners determines the managerial decision on a merger to maximize her/his own payoff $V_i$. The reader may notice that the market structures and the payoffs in our framework corresponds to the feasible allocations and the utilities (or preferences) in the market game in exchange economy which is well-established topic in microeconomic theory (see for example, Varian (1992) pp.387-388).

To define the core of the market structures, we should start with the definition of a *blocking* market structure. A market structure $\mathcal{M}$ is said to *block* another market structure $\mathcal{M}'$ if there exists a deviant coalition of the owner(s) of the pre-marged firm(s) such that:

(i) $\mathcal{M}$ can be constructed from $\mathcal{M}'$ by solely the decision by the owner(s) in the deviant coalition, and
(ii) every owner in the coalition achieves strictly higher payoff in $\mathcal{M}$ than in $\mathcal{M}'$.

An example will help understanding the definition of blocking. Let $\mathcal{M}_{\beta=0.5}^{\{\{0,1\},\{2\}\}}$ be the market structure composed of the merged firm 01 with $\beta = 0.5$ and the private firm 2. In this case, for example, the coalition of the owners of the firms 0 and 2, $\{0, 2\}$, can construct, if they want, the new market structure that consists of the merged firm 02 with $\beta = 0.45$ and the private firm 1, denoted by $\mathcal{M}_{\beta=0.45}^{\{\{0,2\},\{1\}\}}$. If the owner of the firm 2 gains more payoff, i.e. the distributed profit, and the owner of the firm 0, i.e. the government, also achieves higher payoff, i.e. social welfare, in the new structure $\mathcal{M}_{\beta=0.45}^{\{\{0,2\},\{1\}\}}$ than in $\mathcal{M}_{\beta=0.5}^{\{\{0,1\},\{2\}\}}$, then the structure $\mathcal{M}_{\beta=0.45}^{\{\{0,2\},\{1\}\}}$ blocks $\mathcal{M}_{\beta=0.5}^{\{\{0,1\},\{2\}\}}$. Note that it is also possible that a deviant coalition consists of a single owner of a pre-merged firm. In the above example, it is possible for each of the owners of the pre-merged firms 0 and 1 to deviate from the structure by breaking off the merger and to operate their own pre-merged firms respectively, i.e. to shift into the mixed triopoly, as well. The core of the market structures is defined as:

the set of market structures that are *never blocked by any other market structure*.

We denote the core of the market structures by $\mathfrak{Co}$. If a market structure is in the core, all of the three owners of the pre-merged firms have no incentive to construct a new market structure. In this sense, the market structure in the core can be regarded as the stable one. In the next section, we examine which of the market structures is/are in the core.

# 3 Results

We now explore the core of the market structures, i.e. stable structures. Our argument proceeds through some lemmata, each of which points out the market structures which are blocked by some other market structure. Our first lemma shows that the market structure of the merged firm 012 is not in the core no matter what a ratio of shareholding by the pre-merged firms is adopted.

**Lemma 1.** *For any ratio of shareholding by the three owners of the pre-merged firms, the market structure of the merged firm* 012*,* $\mathcal{M}_{\gamma,\delta}^{\{\{0,1,2\}\}}$*, can not belong to the core, i.e.* $\mathcal{M}_{\gamma,\delta}^{\{\{0,1,2\}\}} \notin \mathfrak{Co}$*, for any* $\gamma \in [0,1]$ *and any* $\delta \in [0,1]$.

***Proof.*** The proof proceeds in two steps.

*Step 1.* Let $\mathcal{M}_{\gamma\in[0,1],\delta=\frac{1}{2}}^{\{\{0,1,2\}\}}$ be the market structure of the merged firm 012 with $\gamma \in [0,1]$ and $\delta = \frac{1}{2}$, and $\mathcal{M}_{\beta\in[0,1]}^{\{\{0,1\},\{2\}\}}$ be that of the public-private merged firm 01 and the private firm 2 with a ratio of shareholding $\beta \in [0,1]$ in the merged firm 01. We will show that the owner of the private firm 2 wants to deviate from the merger among the three firms. Since

$$\frac{dV_2^c(\beta)}{d\beta} = \frac{12(2-\beta)}{(-11+4\beta)^3}a^2 < 0, \quad \forall\beta \in [0,1], \tag{16}$$

we have

$$\min_{\beta\in[0,1]} V_2^c(\beta) = V_2^c(\beta)\big|_{\beta=1} = \frac{2}{49}a^2. \tag{17}$$

Then, solving the following equation:

$$V_2^d(\gamma,\delta)\big|_{\delta=\frac{1}{2}} = \frac{2}{49}a^2, \tag{18}$$

we obtain the result

$$V_2^d(\gamma,\delta)\big|_{\delta=\frac{1}{2}} = \min_{\beta\in[0,1]} V_2^c(\beta) \quad \text{if} \quad \gamma = \frac{93-7\sqrt{41}}{90} \approx 0.5353. \tag{19}$$

Since $\frac{d\left(V_2^d(\gamma,\delta)|_{\delta=\frac{1}{2}}\right)}{d\gamma} = -\frac{3(32-27\gamma)}{2(8-3\gamma)^3}a^2 < 0$, $\forall\gamma \in [0,1]$, we obtain

$$V_2^d(\gamma,\delta)\big|_{\delta=\frac{1}{2}} < \min_{\beta\in[0,1]} V_2^c(\beta), \ \ \forall\gamma \in \left(\frac{93-7\sqrt{41}}{90}, 1\right]. \tag{20}$$

Therefore, if $\gamma > \frac{93-7\sqrt{41}}{90}$, the owner of the pre-merged firm 2 deviates from $\mathcal{M}^{\{\{0,1,2\}\}}_{\gamma\in[0,1],\delta=\frac{1}{2}}$ and operates her/his own firm regardless of what a ratio $\beta$ is, i.e. $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta\in[0,1]}$ blocks $\mathcal{M}^{\{\{0,1,2\}\}}_{\gamma\in[0,1],\delta=\frac{1}{2}}$. In cases where $\delta \neq \frac{1}{2}$, the same conclusion also follows for one of the owners of the pre-merged private firms, 1 or 2, because one of them inevitably receives strictly less payoff than in the case of $\delta = \frac{1}{2}$.

*Step 2.* Let $\mathcal{I}$ be the interval $\left[0, \frac{93-7\sqrt{41}}{90}\right]$. To complete the proof, we have to show that $\mathcal{M}^{\{\{0,1,2\}\}}_{\gamma\in\mathcal{I},\delta\in[0,1]}$ is blocked by some other market structure. Consider the market structure of the public-private merged firm 01 and the private firm 2 with a ratio of shareholding $\beta = \gamma$, i.e. $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta=\gamma}$. We show that the coalition $\{0,1\}$ has an incentive to deviate from the merger among the three firms if $\delta = \frac{1}{2}$. Let $\beta : \mathbb{R} \to \mathbb{R}$ be such that $\beta(t) = t$. When $\gamma = \frac{93-7\sqrt{41}}{90}$, the difference between the payoffs to the owner of the firm 0 across the two market structures is

$$\left(V_0^c(\beta(\gamma)) - V_0^d(\gamma)\right)\Big|_{\gamma=\frac{93-7\sqrt{41}}{90}} = \frac{25(1024237 + 79947\sqrt{41})}{98(32396969 + 4258989\sqrt{41})}a^2 > 0. \tag{21}$$

Moreover, we have

$$\frac{d(V_0^c(\beta(\gamma)) - V_0^d(\gamma))}{d\gamma} = -\frac{3(6859 - 13655\gamma + 9324\gamma^2 - 2682\gamma^3 + 279\gamma^4)}{(8-3\gamma)^3(11-4\gamma)^3}a^2 < 0, \ \ \forall\gamma \in [0,1]. \tag{22}$$

Therefore, the government can achieve higher payoff, i.e. higher social welfare, in $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta=\gamma}$ than in $\mathcal{M}^{\{\{0,1,2\}\}}_{\gamma\in\mathcal{I},\delta=\frac{1}{2}}$. Similarly, we obtain the following results on the payoff to the owner of the pre-merged firm 1,

$$\left(V_1^c(\beta(\gamma)) - V_1^d(\gamma,\delta)\right)\Big|_{\gamma=\frac{93-7\sqrt{41}}{90},\delta=\frac{1}{2}} = \frac{(2267 + 51177\sqrt{41})}{196(309 + 14\sqrt{41})^2}a^2 > 0, \tag{23}$$

and, for all $\gamma \in [0,1]$,

$$\frac{d\left(V_1^c(\beta(\gamma)) - V_1^d(\gamma,\delta)|_{\delta=\frac{1}{2}}\right)}{d\gamma} = -\frac{3(5024 - 8031\gamma + 5460\gamma^2 - 1759\gamma^3 + 216\gamma^4)}{2(8-3\gamma)^3(11-4\gamma)^3}a^2 < 0. \tag{24}$$

Thus, the owner of the pre-merged firm 1 can gain more payoff in $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta=\gamma}$ than in $\mathcal{M}^{\{\{0,1,2\}\}}_{\gamma\in\mathcal{I},\delta=\frac{1}{2}}$. Thus, the joint deviation by $\{0,1\}$ is beneficial to each of the owners of the firms 0 and 1. The same argument as in the step 1 can be directly applied to any case of $\delta \neq \frac{1}{2}$ to show that the market structures of the merger among the three firms is blocked through the joint deviation of the government and one of the owners of the pre-merged private firms. □

The intuition behind the lemma is explained as follows. In the cases of high values of $\gamma$, the merged firm 012 sets relatively high output because of the considerable influence of the owner of the pre-merged public firm 0, and this hurts the payoffs to the owners of the pre-merged private firms. On the other hand, for low values of $\gamma$, the merged firm 012 attaches relatively high importance to its profit, then the owner of the pre-merged public firm 0 can do better by breaking off the merger. This trade-off in the owners' interests in the merged firm 012 makes the merger unstable. Indeed, as shown in the proof of the lemma, in the case of high values of $\gamma \in \left[\frac{93-7\sqrt{41}}{90}, 1\right]$, either of the two owners of pre-merged private firms, say $i$, has an incentive to deviate from the merged firm 012. On the other hand, in the case of $\gamma \in \left[0, \frac{93-7\sqrt{41}}{90}\right]$, the owners of public firm 0 has an incentive to break off the merger and to make an offer of organizing new merged firm $0i$ to the one of the two private owners 1 and 2. In both cases, the key is that the positive effect of the improvement on productivity in the merger among the three firms is relatively small to the merger between two firms.

Next, we provide our second lemma which tells that at least one of the two owners of the pre-merged private firms prefers the mixed triopoly rather than the merger between these two private firms regardless of what a ratio of the shareholding between them is adopted, i.e. the market structure of the merger between the private firms is not in the core no matter what a ratio of shareholding is in the merged firm.

**Lemma 2.** *For any ratio of shareholding $\alpha \in [0,1]$, the market structure of the merger between the private firms, $\mathcal{M}_{\alpha}^{\{\{0\},\{1,2\}\}}$, is blocked by the mixed triopoly, $\mathcal{M}^{\{\{0\},\{1\},\{2\}\}}$.*

***Proof.*** Since we have $\sum_{i=1}^{2} \Pi_i^a = \frac{16}{169}a^2 > \frac{3}{32}a = \Pi_{12}^b$, it is obvious that there exists no $\alpha \in [0,1]$ such that $\alpha\Pi_{12}^b \geq \Pi_1^a$ and $(1-\alpha)\Pi_{12}^b \geq \Pi_2^a$. □

This result is due to the strengthened market share of the public firm. It is known that two-firm mergers in the Cournot oligopoly tend often to be unprofitable, due to the aggressive response from the firms not participating in the merger (see for example, Salant et al. (1983)). Although the private merged firm 12 gets an advantage of the improvement on productivity in the current framework, the subsequent expansion of the market share of the non-merged firm, the public firm 0 which aims to maximize not profit but social welfare, becomes larger than in the case of the private Cournot oligopoly. Consequently, the profit of the merged firm 12 can not exceed the sum of the profits gained by the pre-merged private firms, and the merger between the private firms will never be beneficial to the owners of the pre-merged private firms in the current framework, either.

We now move to our third lemma. While the mixed triopoly, as stated in Lemma 2, blocks the market structures of the regime (b) and, consequently, excludes them from the core, the following lemma shows that the mixed triopoly can not belong to the core, either. To state the lemma, we let

$$\underline{\beta} = \frac{638-39\sqrt{31}}{739} \approx 0.56950 \quad \text{and} \quad \bar{\beta} = \frac{6197-39\sqrt{6001}}{5572} \approx 0.56996. \tag{25}$$

**Lemma 3.** *The mixed triopoly, $\mathcal{M}^{\{\{0\},\{1\},\{2\}\}}$, is blocked by the market structure of the public-private merged firm $0i$ and the private firm $j \neq i$, $\mathcal{M}_{\beta}^{\{\{0,i\},\{j\}\}}$, if the ratio of shareholding $\beta$ in the merged firm $0i$ is in the interval $(\underline{\beta}, \bar{\beta})$.*

***Proof.*** In the Cournot equilibrium of each of the regimes (a) and (c), we have

$$V_0^a = \frac{99}{338}a^2,\ V_0^c = \frac{(68-44\beta+5\beta^2)}{2(11-4\beta)^2}a^2,\ V_i^a = \frac{8}{169}a^2, \text{ and } V_i^c = \frac{9(3-2\beta)(1-\beta)}{2(11-4\beta)^2}a^2.$$

Thus, we obtain the following:

$$\begin{cases} 0 \leq \beta \leq \underline{\beta} \Rightarrow V_0^a \geq V_0^c \\ \underline{\beta} < \beta \leq 1 \Rightarrow V_0^a < V_0^c \end{cases} \quad \text{and} \quad \begin{cases} 0 \leq \beta < \bar{\beta} \Rightarrow V_i^c > V_i^a \\ \bar{\beta} \leq \beta \leq 1 \Rightarrow V_i^c \leq V_i^a. \end{cases}$$

Thus, each of the owners of 0 and $i$ have an incentive to jointly found the merged firm $0i$ if the shareholding ratio $\beta$ is in $\left[0,\bar{\beta}\right)\cap\left(\underline{\beta},1\right]=\left(\underline{\beta},\bar{\beta}\right)$. □

As we have just shown in the proof of Lemma 3, if the ratio of shareholding by the government is more than $\underline{\beta}$, i.e. $\beta>\underline{\beta}$, the government will agree to the merger with a private firm $i$ since she can achieve higher social welfare by the positive effect of productivity improvement. On the other hand, the owner of the pre-merged private firm $i$ can gain more payoff in the merged firm $0i$ than in the mixed triopoly whenever $\beta<\bar{\beta}$. Therefore, for any $\beta\in\left(\underline{\beta},\bar{\beta}\right)$, both of the two owners have an incentive to merge into a new public-private firm $0i$. From this observation, we immediately obtain the following lemma.

**Lemma 4.** *The mixed triopoly, $\mathcal{M}^{\{\{0\},\{1\},\{2\}\}}$, blocks the market structure of the public-private merged firm* $0i$ *and the private firm $j\neq i$, $\mathcal{M}_{\beta}^{\{\{0,i\},\{j\}\}}$, whenever the ratio of shareholding $\beta$ in the merged firm* $0i$ *is in $\left[0,\underline{\beta}\right)$ or $\left(\bar{\beta},1\right]$, i.e. $\beta\in\left[0,\underline{\beta}\right)\cup\left(\bar{\beta},1\right]$.*

***Proof.*** This lemma immediately follows from the proof of Lemma 3 where we have shown that if $\beta\in\left(\bar{\beta},1\right]$ $\left(\text{resp. } \left[0,\underline{\beta}\right)\right)$ then the owner of the private firm $i$ (resp. the owner of the public firm 0) has an incentive to deviate and change the present market structure into the mixed triopoly. □

From Lemmata 1 to 4, we now know that almost all market structures can not be in the core. The market structures that belong to any of the regimes (a), (b), and (d) are not in the core. Moreover, in the regime (c), the market structures with $\beta\in\left[0,\underline{\beta}\right)\cup\left(\bar{\beta},1\right]$ can not belong to the core, either. As a consequence, the remaining candidates that could belong to the core are the market structures of the public-private merged firm $0i$ and the private firm $j\neq i$, with the ratio of shareholding by the government $\beta\in\left[\underline{\beta},\bar{\beta}\right]$. We now state our main result, which shows that any of these market structures is in the core.

**Proposition 1.** *The market structure of the public-private merged firm 0i and the private firm $j\neq i$, $\mathcal{M}_{\beta}^{\{\{0,i\},\{j\}\}}$, is in the core whenever the ratio of shareholding in the merged firm 0i, $\beta$, is in the closed interval $\left[\underline{\beta},\bar{\beta}\right]$, i.e. $\mathcal{M}_{\beta}^{\{\{0,i\},\{j\}\}}\in\mathfrak{C}_{\mathfrak{D}}$, $\forall\beta\in\left[\underline{\beta},\bar{\beta}\right]$.*

***Proof.*** See Appendix. □

From this proposition, it can be concluded that the market structures of the public-private merged firm $0i$ and the private firm $j\neq i$ with $\beta\in\left[\underline{\beta},\bar{\beta}\right]$ are stable in the sense that any of these market structures is never blocked by the other market structures. In other words, once any of these structures is realized, it will never be replaced by any of the other market structures. It should be emphasized that the interval of the admissible ratio $\beta$ in the core $\left[\underline{\beta},\bar{\beta}\right]$ is very short, $\bar{\beta}-\underline{\beta}\approx 0.00047$. This result is fairly remarkable in that it shows a considerable contrast to the result obtained in Kamijo and Nakamura (2007). In their paper, Kamijo and Nakamura analyzed the industry composed by two symmetric private firms and a less efficient public firm. Assuming that each of the three firms has constant marginal cost of production, Kamijo and Nakamura showed that all of the four regimes, except for the regime (b), have the market structures that belong to the core. Therefore, it can be said that the stable mergers in mixed oligopoly crucially depend on the assumptions of firms' technology.

Finally, we briefly examine the case where the industry is composed only of private firms, i.e. private oligopoly, and compare the results between mixed oligopoly and private oligopoly. In the case of private oligopoly, we need to change the model summarized in Table 1 as follows: $U_0^a = \Pi_0$; $V_0^a = \Pi_0(q^{a*})$; $U_0^b = \Pi_0$; $V_0^b = \Pi_0(q^{b*})$; $U_{0i}^c = \Pi_{0i}$; $V_0^c = \beta\Pi_{0i}(q^{c*})$; $U_{012}^d = \Pi_{012}$; and $V_0^d = \gamma\Pi_{012}(q^{d*})$. Consequently, the regimes (b) and (c) become the same ones, and we let the regime (b) represent them. The Cournot equilibria of the regimes (a), (b) and (d) are obtained as:

$$(q_i^{a*}, q_1^{a*}, q_2^{a*}) = \left(\frac{a}{6}, \frac{a}{6}, \frac{a}{6}\right),\ (q_0^{b*}, q_{12}^{b*}) = \left(\frac{2}{11}a, \frac{3}{11}a\right),\ \text{and } q_{012}^{d*} = \frac{3}{8}a.$$

Then, the payoffs to the owners are determined as follows:

$$(V_0^a, V_1^a, V_2^a) = \left(\frac{a^2}{18}, \frac{a^2}{18}, \frac{a^2}{18}\right),\ (V_0^b, V_1^b, V_2^b) = \left(\frac{8}{121}a^2, \frac{27}{242}\alpha a^2, \frac{27}{242}(1-\alpha)a^2\right), \text{ and}$$
$$(V_0^d, V_1^d, V_2^d) = \left(\frac{3}{16}\gamma a^2, \frac{3}{16}(1-\gamma)\delta a^2, \frac{3}{16}(1-\gamma)(1-\delta)a^2\right).$$

Given the above payoffs, we obtain the following result.

**Proposition 2.** *In the case where the industry is composed only of private firms, none of the market structures belongs to the core, i.e.* $\mathfrak{Co} = \emptyset$.

***Proof.*** The proof is similar to those of Lemmata 1 to 4. Thus, we limit ourselves to providing the examples of blocking market structures for each market structure. For the regime (a), $\mathcal{M}^{\{\{0\},\{1\},\{2\}\}}$ is blocked by $\mathcal{M}_\alpha^{\{\{0\},\{1,2\}\}}$ whenever $\alpha \in \left(\frac{121}{243}, \frac{122}{243}\right)$. For the regime (b), $\mathcal{M}_\alpha^{\{\{0\},\{1,2\}\}}$ is blocked (i) by $\mathcal{M}^{\{\{0\},\{1\},\{2\}\}}$ if $\alpha \in \left[0, \frac{121}{243}\right)$ or $\alpha \in \left(\frac{122}{243}, 1\right]$; and (ii) by $\mathcal{M}_{\gamma,\delta}^{\{\{0,1,2\}\}}$ with $\gamma = \frac{9}{25}$ and $\frac{1}{2}$ if $\alpha \in \left[\frac{121}{243}, \frac{122}{243}\right]$. For the regime (d), $\mathcal{M}_{\gamma=\frac{1}{3},\delta=\frac{1}{2}}^{\{\{0,1,2\}\}}$ is blocked by $\mathcal{M}_\alpha^{\{\{0\},\{1,2\}\}}$ with $\alpha \in [0,1]$. By the same argument as in the proof of Lemma 1, the case of $(\gamma,\delta) = \left(\frac{1}{3}, \frac{1}{2}\right)$ is sufficient to complete the proof of the regime (d). □

The never-ending coalition formation increases transaction costs unboundedly. Furthermore, it eliminates our ability to predict which of the market structure will actually occur, which also means that it is hardly possible to prescribe economic policies in a effective way. Comparing Propositions 1 and 2, we can conclude that the presence of the public firm has a stabilizing effect in the current framework and allows us to avoid such undesirable costs.

# 4 Concluding remarks

This paper explored the stable market structures in mixed oligopoly when a single public firm and two symmetric private firms in the homogeneous good market are allowed to freely merge and freely break off the merger. We adopted the core as the solution concept to analyze the stability of market structures. We showed that the core consists solely of the market structures derived by the merger between a public firm and one of the two private firms with the shareholding ratio by the public firm, $\beta$, which is greater than $\underline{\beta} \approx 0.56950$ and less than $\bar{\beta} \approx 0.56996$. These market structures are stable in the sense that, by the definition of the core, once any of these market structures is actually realized, it never be replaced by any of the other market structures. The admissible interval of $\beta$ that ensures the stability of market structures is very short. This strong result fairly relies on the assumption that a merger yields the improvement on productivity. Without such a positive effect of a merger, our result would change and the mixed triopoly would be a unique stable market structure.

Two interesting extensions of our model remain. The first is to consider the model in which the foreign shareholders are taken into account. In the real world, some firms are foreign-owned. In this case, social welfare that the government is to maximize should not include the profits of the foreign-owned firms. Thus, the existence of the foreign shareholders will change the public firm's decision making and, consequently, the equilibrium outcomes as well. The other possible extension is to introduce the asymmetricity among the production technologies of firms in a way like $C(q) = k_i q_i^2$. In the present paper, we assumed that all the three firms have identical technologies $(k_0, k_1, k_2) = (1, 1, 1)$. It seems more natural to assume that a public firm shows inefficient performance relatively to private firms, e.g. X-inefficiency in a public firm. In the case of $(k_0, k_1, k_2) = (3, 1, 1)$, the reader may easily check that the core becomes empty in the similar method to the proofs of Lemmata 1 to 4 and Proposition 2. The analysis of more general cases of $k = (k_0, k_1, k_2)$ and the comparison among different values of the weight $k$ is left for future reseach.

## Appendix: Proof of Proposition 1

Let $\mathcal{M}^{\{\{0,i\},\{j\}\}}_{\beta\in[\underline{\beta},\bar{\beta}]}$ be the market structure of the merged firm $0i$ with a ratio of shareholding $\beta \in [\underline{\beta}, \bar{\beta}]$ and the private firm $j$ $(\neq i)$. In a series of claims below, we will show that $\mathcal{M}^{\{\{0,i\},\{j\}\}}_{\beta\in[\underline{\beta},\bar{\beta}]}$ is never blocked by any other market structure. We assume, without loss of generality, $i = 1$ and $j = 2$.

**Claim 1.** $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta\in[\underline{\beta},\bar{\beta}]}$ *is never blocked by the mixed triopoly in any case of* $\beta \in \left[\underline{\beta}, \bar{\beta}\right]$.

By Lemma 3, $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta\in[\underline{\beta},\bar{\beta}]}$ is not blocked by the mixed triopoly if $\beta \in \left(\underline{\beta}, \bar{\beta}\right)$. Moreover, in the proof of Lemma 3, we have shown that, in the case of $\beta = \underline{\beta}$, the government in $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta=\underline{\beta}}$ can achieve the same level of social welfare as in the mixed triopoly, and thus the government has no incentive to deviate from $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta\in[\underline{\beta},\bar{\beta}]}$, and also that the owner of the pre-merged firm 1 in $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta=\underline{\beta}}$ gains more payoff than in the mixed triopoly. Thus, neither of these two owners want to break off the merger. The case of $\beta = \bar{\beta}$ can be proved by the symmetric argument to the case of $\beta = \underline{\beta}$.

**Claim 2.** $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta\in[\underline{\beta},\bar{\beta}]}$ *is never blocked by the market structure of the public firm* 0 *and the private merged firm* 12 *with* $\alpha \in [0, 1]$, $\mathcal{M}^{\{\{0\},\{1,2\}\}}_{\alpha\in[0,1]}$, *in any case of* $\alpha \in [0, 1]$.

By (16), $V_2^c(\beta)$ is decreasing on $[0, 1]$, and thus we have

$$\min_{\beta\in[\underline{\beta},\bar{\beta}]} V_2^c(\beta) = V_2^c(\bar{\beta}) = \frac{(1649 + 13\sqrt{6001})^2}{1352(234 + \sqrt{6001})^2} a^2. \tag{26}$$

When $\beta = \bar{\beta}$ in $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta\in[\underline{\beta},\bar{\beta}]}$, i.e. in $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta=\bar{\beta}}$, the owner of the private firm 2 will agree with the merger between the two private firms if and only if

$$V_2^b(\alpha) > V_2^c(\bar{\beta}) \Leftrightarrow \frac{3}{32}(1-\alpha)a^2 > \frac{(1649+13\sqrt{6001})^2}{1352(234+\sqrt{6001})^2}a^2 \tag{27a}$$

$$\Leftrightarrow \alpha < \frac{401707-1768\sqrt{6001}}{621075} \approx 0.4263. \tag{27b}$$

On the other hand, we obtain the following result on the payoffs to the firm 1: for any $\alpha < \frac{401707-1768\sqrt{6001}}{621075}$,

$$V_1^b(\alpha) - V_1^c(\bar{\beta}) = \frac{3}{32}\alpha a^2 - \frac{8}{169}a^2 \tag{28a}$$

$$< \frac{3}{32}\alpha a^2\Big|_{\alpha=\frac{401707-1768\sqrt{6001}}{621075}} - \frac{8}{169}a^2 \tag{28b}$$

$$= \frac{88107-1768\sqrt{6001}}{6624800}a^2 \approx -0.0074a^2 < 0. \tag{28c}$$

Thus, by (27b) and (28c), the joint deviation by the owners of the firms 1 and 2 can not be realized if $\beta = \bar{\beta}$. Since $V_2^c$ is decreasing with respect to $\beta$, by (27a) to (28c) altogether, the joint deviation by the owners of the private firms is still impossible in any case of $\beta \in \left[\underline{\beta}, \bar{\beta}\right)$.

**Claim 3.** *$\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta\in[\underline{\beta},\bar{\beta}]}$ is never blocked by $\mathcal{M}^{\{\{0,2\},\{1\}\}}_{\beta'\in[0,1]}$ in any case of $\beta' \in [0,1]$.*

We start with the case of $\beta = \bar{\beta}$ in $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta\in[\underline{\beta},\bar{\beta}]}$. In this case, the owner of the firm 2 prefers $\mathcal{M}^{\{\{0,2\},\{1\}\}}_{\beta'\in[0,1]}$ rather than $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta=\bar{\beta}}$ if and only if the payoff, i.e. the distributed profit, in $\mathcal{M}^{\{\{0,2\},\{1\}\}}_{\beta'\in[0,1]}$ is strictly greater than the payoff, i.e. the stand-alone profit, gained in $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta=\bar{\beta}}$, i.e. the following value, $\Delta(\beta')$, must be positive:

$$\Delta(\beta') := (1-\beta')\Pi_{02}^c(\beta') - \Pi_2^c(\bar{\beta}) \tag{29a}$$

$$= \frac{9(3-2\beta')(1-\beta')}{2(11-4\beta')^2}a^2 - \frac{(1649+13\sqrt{6001})^2}{1352(234+\sqrt{6001})^2}a^2 \tag{29b}$$

$$= a^2\cdot\Xi(\beta'), \tag{29c}$$

where $\Xi(\beta') = \left[28(12134951+89440\sqrt{6001})(\beta')^2 - 2(379922845+2615912\sqrt{6001})\beta' + 328599497 + 1677091\sqrt{6001}\right] / \left[676(234+\sqrt{6001})^2(11-4\beta')^2\right]$. Solving the equation $\Delta(\beta') = 0$ subject to $\beta' \in [0,1]$, we obtain

$$\beta'^* = \frac{\left(379922845+2615912\sqrt{6001} - 39\sqrt{31920488675573+391144962052\sqrt{6001}}\right)}{339778628+2504320\sqrt{6001}} \approx 0.5151. \tag{30}$$

Since

$$\frac{d\Delta(\beta')}{d\beta'} = \frac{d((1-\beta')\Pi_{02}^c(\beta'))}{d\beta'} = -\frac{9(31-24\beta')}{2(11-4\beta')^3}a^2 < 0, \quad \forall\beta' \in [0,1], \tag{31}$$

we obtain the intermediate result that the owner of the firm 2 prefers $\mathcal{M}^{\{\{0,2\},\{1\}\}}_{\beta'\in[0,1]}$ to $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta=\bar{\beta}}$ if and only if

$$\beta' \in [0, \beta'^{*}). \tag{32}$$

On the other hand, since we have

$$\frac{dV_0^c(\beta)}{d\beta} = \frac{3(10-11\beta)}{(11-4\beta)^3}a^2 > 0, \ \ \forall \beta \in [0,\bar{\beta}] \ (\supsetneq [0,\beta'^{*}]), \tag{33}$$

the owner of the public firm 0 strictly prefers $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta=\bar{\beta}}$ rather than $\mathcal{M}^{\{\{0,2\},\{1\}\}}_{\beta'}$ if $\beta' < \beta'^{*}$ ($< \bar{\beta}$), and thus the joint deviation by the owners of the firms 0 and 2 from $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta=\bar{\beta}}$ can never be realized. Note that, from (16), the profit (or payoff to the owner) of the firm 2 in $\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta\in[\underline{\beta},\bar{\beta}]}$ is decreasing with respect to $\beta$, which in turn implies that, by (29a) to (29c) and (31), a decrease in $\beta$ leads to a decrease in $\beta'^{*}$. Thus, from the fact that $\beta'^{*} < \underline{\beta}$ and (33), the owner of the firm 0 never agrees with the joint deviation with the owner of the firm 2 in any case of $\beta \in [\underline{\beta}, \bar{\beta})$.

**Claim 4.** *$\mathcal{M}^{\{\{0,1\},\{2\}\}}_{\beta\in[\underline{\beta},\bar{\beta}]}$ is never blocked by the merger among the three firms $\mathcal{M}^{\{\{0,1,2\}\}}_{\gamma\in[0,1],\delta\in[0,1]}$ regardless of what the ratios $\gamma \in [0,1]$ and $\delta \in [0,1]$ are.*

We start with the case of $\beta = \underline{\beta}$. In this case, we have $V_0^c(\underline{\beta}) = \frac{99}{338}a^2$. Since $\frac{dV_0^d(\gamma)}{d\gamma} = \frac{27(1-\gamma)}{(8-3\gamma)^3}a^2 \geq 0$ for all $\gamma \in [0,1]$ (equality holds only in the case of $\gamma = 1$),

$$V_0^d(\gamma) - V_0^c\left(\underline{\beta}\right) = \frac{3(11-9\gamma)(-23+33\gamma)}{338(8-3\gamma)^2}a^2 > 0, \ \ \forall \gamma \in \left(\frac{23}{33}, 1\right]. \tag{34}$$

On the other hand, we obtain the following result on the payoff to the owner of the pre-merged private firm 2:

$$\frac{d(V_2^d(\gamma,\delta)|_{\delta=0})}{d\gamma} = \frac{3(32-27\gamma)}{(-8+3\gamma)^3}a^2 < 0, \ \ \forall \gamma \in [0,1] \tag{35}$$

and

$$V_2^d(\gamma,\delta)\Big|_{\gamma=\frac{23}{33},\delta=0} - V_2^c(\underline{\beta}) = \frac{4(817-260\sqrt{31})}{616005}a^2 \approx -0.0041a^2 < 0. \tag{36}$$

Note that the case of $\delta = 0$ is the most favorable case of $\delta$ for the owner of the pre-merged private firm 2. Hence, in the case of $\beta = \underline{\beta}$, by (34) to (36), the owners of the pre-merged public firm 0 and pre-merged private firm 2 can never reach an agreement about the shareholding in the merged firm 012, and thus the merger among the three firms can never be realized. Now, we examine the other cases of $\beta \in \left[\underline{\beta}, \bar{\beta}\right]$, i.e. $\beta \in \left(\underline{\beta}, \bar{\beta}\right]$. By (33), the value of $\gamma^*$ which solves the equation $V_0^d(\gamma) - V_0^c(\beta) = 0$ increases in any case of $\beta \in \left(\underline{\beta}, \bar{\beta}\right]$ than in the case of $\beta = \underline{\beta}$, i.e. $\gamma^* > \frac{23}{33}$. From

(16) and (35), we have

$$V_2^d(\gamma^*,\delta)|_{\delta=0} - V_2^c(\beta)\Big|_{\beta\in(\underline{\beta},\bar{\beta}]} < V_2^d(\gamma,\delta)\Big|_{\gamma=\frac{23}{33},\delta=0} - V_2^c(\bar{\beta}) \tag{37a}$$

$$= \frac{13739 - 221\sqrt{6001}}{828100}a^2 \approx -0.0041a^2 < 0. \tag{37b}$$

Therefore, by the same argument as in the case of $\beta = \underline{\beta}$, the merger among the three firms is impossible in any case of $\beta \in \left(\underline{\beta},\bar{\beta}\right]$.

From (33) and the fact that $\frac{dV_1^c(\beta)}{d\beta} = -\frac{9(31-24\beta)}{2(11-4\beta)^3}a^2 < 0$, $\forall\beta \in [0,1]$, any alteration on the ratio $\beta$ never improves the payoffs to both owners of the firms 0 and 1 simultaneously. Therefore, combining the assertions of the claims, we have successfully shown that $\mathcal{M}_\beta^{\{\{0,1\},\{2\}\}}$ is in the core whenever $\beta$ is in the closed interval $\left[\underline{\beta},\bar{\beta}\right]$. $\square$

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