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Binder parameter of a Heisenberg spin-glass model in four dimensions

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We studied the phase transition of the $\pm J$ Heisenberg model with and without a random anisotropy on four-dimensional lattice $L\times L\times L\times(L+1)(L\leqslant 9)$. We showed that the Binder parameters $g(L,T)$'s for different sizes do not cross even when the anisotropy is present. On the contrary, when a strong anisotropy exists, $g(L,T)$ exhibits a steep negative dip near the spin-glass phase-transition temperature $T_{\rm SG}$ similarly to the p-state infinite-range Potts glass model with $p\geqslant 3$, in which the one-step replica-symmetry breaking (RSB) occurs. We speculated that a one-step RSB-like state occurs below $T_{\rm SG}$, which breaks the usual crossing behavior of $g(L,T)$.

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Recently, the low-temperature phase of the XY and Heisenberg spin-glass (SG) models has been attacked a great interest. It is known that vector SG models have a chiral symmetry in addition to the continuous symmetry.[1] Consequently, these models may have both SG order and chiral glass (CG) order. Kawamura and co-workers claimed that decoupling of the spin and chiral variables occurs at long distances,[2,3] and that the CG order is realized at low temperatures in three dimensions $(d=3)$, but the SG order is not.[4–7] Although the existence of the CG order has been accepted, controversy exists on the SG order. Maucourt and Grempel[8] studied the domain-wall energy $W(L)$ of the $\pm J\,XY$ model on finite $d=3$ lattices of L^3 at absolute zero temperature $(T=0)$ and speculated that $W(L)$ may increase with the linear size L. The same speculation was also given by Kosterlitz and Akino.[9] The present authors examined the stiffness of the $\pm J$ Heisenberg model at $T=0$ (Ref. 10) and $T\neq 0$,[11] and suggested that the stiffness exponent θ in $d=3$ has a positive value for $T/J\lesssim 0.19$. They also performed a Monte Carlo (MC) simulation of the same model and found that the SG susceptibility $\chi_{\rm SG}$ exhibits a divergent behavior toward the temperature of $T/J\sim 0.18$.[12] These facts strongly suggests that the SG order also occurs in the XY and Heisenberg models in $d=3$. This view of the SG order was supported in recent studies of the aging effect of the spin autocorrelation function $\langle \boldsymbol{S}_i(0)\boldsymbol{S}_i(t)\rangle$ (Ref. 13) and the nonequilibrium relaxation of $\chi_{\rm SG}$.[14] However, the Binder parameters $g(L,T)$'s for different L neither cross nor come together in both the XY (Ref. 7) and Heisenberg[6] SG models.

This fact raises an objection against the occurrence of the SG order, because it is believed that the most reliable evidence of the occurrence of the SG phase transition is the crossing of $g(L,T)$'s at the same temperature of $T_{\rm SG}$.[15–17] In fact, it was reported that the crossing of $g(L,T)$'s occurs in both the XY (Ref. 18) and Heisenberg[19] SG models in four dimensions $(d=4)$. Thus we are faced by a serious problem that *we have gotten from the opposite views of the SG order from the studies of the different quantities*.

The problem may arise from a poor knowledge of the property of $g(L,T)$ of the vector SG model. In the nonfrustrated system, the crossing of $g(L,T)$'s occurs at $T_{\rm C}$ with some positive value of $\tilde{g}$ $[\equiv g(L,T_{\rm C})]$. This property results from the fact that, in the thermodynamic limit, $g(\infty,T)=0$ for $T>T_{\rm C}$ and $g(\infty,T)=1$ for $T<T_{\rm C}$. However, the same will not always be true in the SG model, because $g(\infty,T)$ for $T<T_{\rm SG}$ will take different values due to the occurrence of the replica-symmetry breaking (RSB). In fact, it was revealed by Hukushima and Kawamura[20] that $g(\infty,T_{\rm SG}^-)$ of the infinite-range p-state Potts glass model[21] takes different values depending on the state number p, where $T^-=\lim_{\epsilon\to 0}(T-\epsilon)$. Therefore, it is crucially important to reveal the property of $g(L,T)$ in such vector SG models, in which the SG phase transition occurs at $T_{\rm SG}\neq 0$.

In this paper, we report that $g(L,T)$ of the Heisenberg SG model in $d=4$ exhibits a behavior quite different from that of the Ising SG model. We reexamined $g(L,T)$ of the $\pm J$ Heisenberg models in $d=4$ with and without a random anisotropy of magnitude D. In the case of $D=0$, when L is increased, the increment of $g(L,T)$ with decreasing temperature becomes less steep at low temperatures, and the crossing of $g(L,T)$'s which was suggested[19] for small L disappears. When $D\neq 0$, $g(L,T)$ for each L decreases, particularly around $T_{\rm SG}$, and this property is enhanced when D is increased, where $T_{\rm SG}$ is the SG transition temperature estimated from the scaling plot of $\chi_{\rm SG}$. *In particular,* $g(L,T)$ *for large* D exhibits a negative dip near $T_{\rm SG}$ which deepens with increasing L. These facts are quite interesting, because the anisotropy would stabilize the SG order.[22,23] We believe, hence, that *the absence of the crossing of g(L,T)'s for finite L says nothing about the presence of the SG order in this model.* We speculate that, even for $D=0$, $g(\infty,T_{\rm SG}^-)$ takes a small positive value (or a negative value) due to the occurrence of a one-step RSB-like state. We hope our findings will help to understand the low-temperature phase of the Heisenberg SG model in $d=3$.

We start with the anisotropic $\pm J$ Heisenberg model on a hypercubic lattice of $L\times L\times L\times(L+1)(\equiv N)$ with skew boundary conditions along three L directions and a periodic boundary condition along the $(L+1)$ direction. The Hamiltonian is described by

$$H=-\sum_{\langle ij\rangle}\left[J_{ij}\boldsymbol{S}_i\boldsymbol{S}_j+\sum_{\alpha\neq\beta}D_{ij}^{\alpha\beta}S_i^{\alpha}S_j^{\beta}\right], \tag{1}$$

where $\boldsymbol{S}_i$ is the Heisenberg spin of $|\boldsymbol{S}_i|=1$ and S_i^α is its α component $(\alpha=x,y,z)$, and $\langle ij\rangle$ runs over all nearest-neighbor pairs. The exchange interaction J_{ij} takes on either $+J$ or $-J$ with the same probability of 1/2. We assume that the anisotropy comes from pseudodipolar couplings and impose the restriction $D_{ij}^{\alpha\beta}=D_{ji}^{\alpha\beta}=D_{ij}^{\beta\alpha}$. We further assume that $D_{ij}^{\alpha\beta}$ are uniform random values between $-D$ and D. We note that the role of the anisotropy is to break the rotational symmetry of the model and to stabilize the SG order.

We performed an MC simulation of the two-replica systems of $\{\boldsymbol{S}_i\}$ and $\{\tilde{\boldsymbol{S}}_i\}$ using an exchange MC algorithm.[24] We calculated the order-parameter probability distribution $P_L(q)$ of

$$P_L(q)=[\langle\delta(q-Q)\rangle], \tag{2}$$

where $\langle\cdots\rangle$ and $[\cdots]$ mean the thermal average and the bond distribution average, respectively. Here Q is the spin overlap defined by

$$Q=\sqrt{\frac{1}{3}\sum_{\alpha,\beta}(q^{\alpha\beta})^2} \tag{3}$$

with $q^{\alpha\beta}\equiv 1/N\Sigma_{i=1}^N S_i^\alpha\tilde{S}_i^\beta$. Using $P_L(q)$, we obtained the SG susceptibility χ_{SG} and the Binder parameter $g(L,T)$ which are defined by

$$\chi_{\mathrm{SG}}=3N[\langle q^2\rangle], \tag{4}$$

$$g(L,T)=\frac{1}{2}\left(11-9\frac{[\langle q^4\rangle]}{[\langle q^2\rangle]^2}\right), \tag{5}$$

where $[\langle q^n\rangle]=\int q^n P_L(q)dq$. To examine whether the SG order occurs or not, we also calculated quantities $A(L,T)$ and $G(L,T)$ that measure the order-parameter fluctuations (OPF):

$$A(L,T)=\frac{[\langle q^2\rangle^2]-[\langle q^2\rangle]^2}{[\langle q^2\rangle]^2}, \tag{6}$$

$$G(L,T)=\frac{[\langle q^2\rangle^2]-[\langle q^2\rangle]^2}{[\langle q^4\rangle]-[\langle q^2\rangle]^2}. \tag{7}$$

Each of these quantities will exhibit a crossing behavior at T_{SG} when the SG transition occurs.[25,26] We performed the MC simulation of model (1) with various values of $D=0$, $0.1J$, $0.2J$, $0.5J$, and $1.0J$. The linear sizes of the lattice studied here are $L=3\sim9$. Equilibration was checked by monitoring the stability of the results against at least two-times longer runs. The numbers of the samples were 480 for $L=3$, 288 for $L=5$, 96 for $L=7$, and 48 for $L=9$. The details of the MC simulation will be reported in a future publication.

In Fig. 1(a), we show results of χ_{SG} in $D=0$ and $D=0.5J$. In both the cases, χ_{SG} for larger L increases rapidly as the temperature is decreased. The same was true for different values of D. The finite-size scaling analysis in each D suggested the divergence of χ_{SG} for $L\to\infty$ at a finite, nonzero temperature. An example of the scaling plot in the case

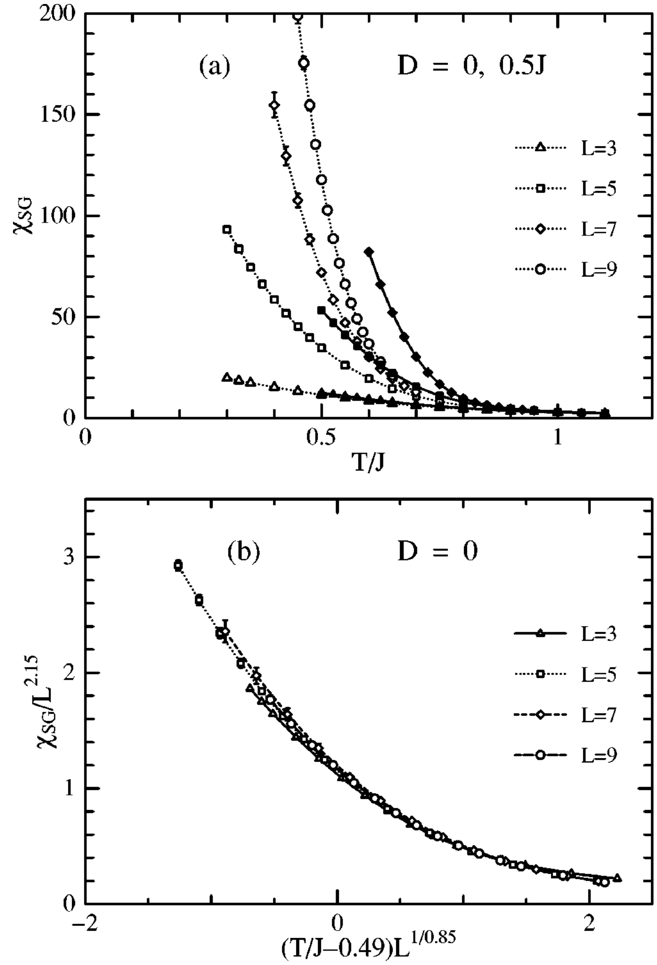

FIG. 1. (a) Temperature dependences of the spin-glass susceptibility χ_{SG} of the $\pm J$ Heisenberg model in $d=4$ for different sizes of the lattice. Open symbols represent those for $D=0$ and closed ones those for $D=0.5J$. (b) An example of the scaling plot of χ_{SG} for $D=0$.

of $D=0$ is presented in Fig. 1(b). Hereafter, we tentatively call this temperature the SG transition temperature and denote it by T_{SG}. Note that T_{SG} increases with increasing D. These results are compatible with a common belief that the SG order occurs in $d=4$ even for $D=0$, and that the SG order is stabilized by the anisotropy.[22,23]

Now we show $g(L,T)$ in different D's in Figs. 2(a)–2(c). When $D=0$, $g(L,T)$'s for $L=3$ and 5 come close to each other near T_{SG}. But they do not cross, because the increment of $g(L,T)$ for $L=5$ is suppressed below T_{SG}. This suppression does not reduce for larger L. Therefore we believe that, contrary to the previous suggestion,[19] $g(L,T)$'s for large L do not cross but converge on some nonzero value $\tilde{g}$ at $T=T_{\mathrm{SG}}$.[27] When $D\neq0$, the suppression is enhanced more. In particular, for large D, $g(L,T)$ exhibits a dip near T_{SG} which deepens as L is increased. This result is also incompatible with our naive expectation that, as D is increased, $g(L,T)$'s would tend to cross with some positive $\tilde{g}$, because the anisotropy will stabilize the SG order.

We next examined $G(L,T)$ that would exhibit a crossing behavior at T_{SG} even when $g(L,T)$ did not exhibit the cross-

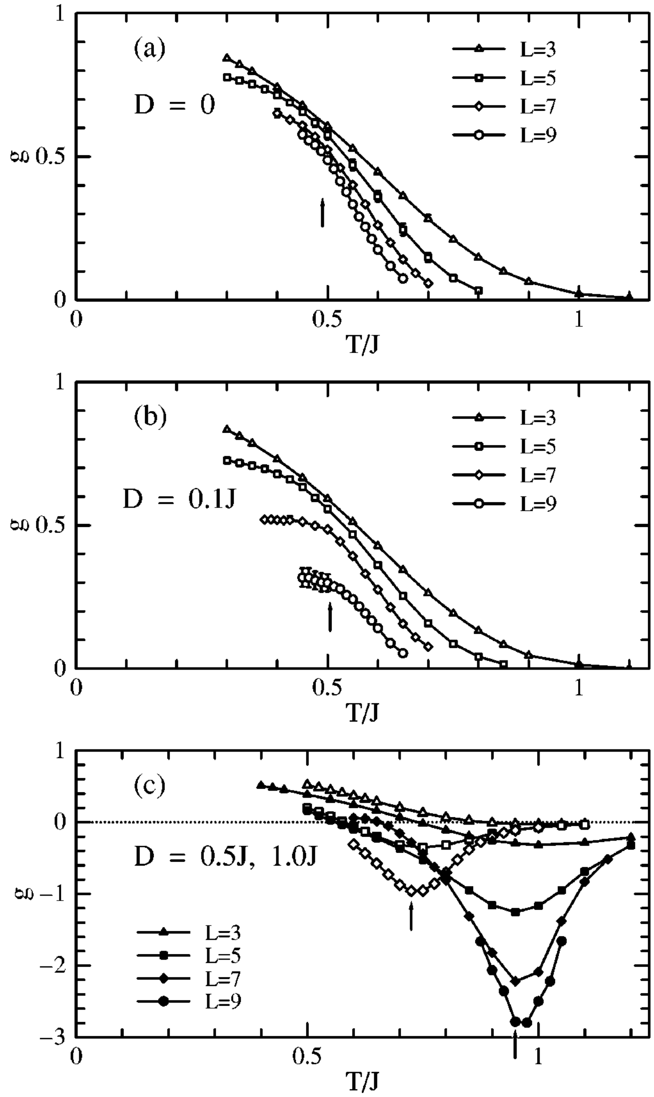

FIG. 2. Temperature dependences of the Binder parameter $g(L,T)$ of the $\pm J$ Heisenberg model in $d=4$ for different magnitude of the anisotropy D and for different sizes of the lattice; (a) $D=0$, (b) $D=0.1J$, and (c) $D=0.5J$ (open symbols) and $D=1.0J$ (closed symbols). Arrows indicate the SG transition temperature $T_{\rm SG}$ estimated from the scaling plots of the SG susceptibility.

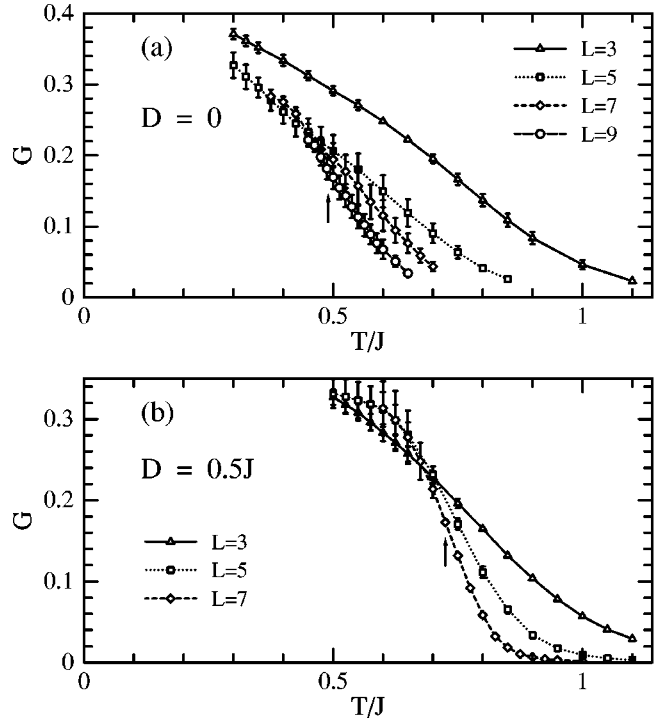

FIG. 3. Temperature dependences of the order-parameter fluctuations(OPF) $G(L,T)$ of the $\pm J$ Heisenberg model in $d=4$ for different sizes of the lattice and for different magnitude of the anisotropy; (a) $D=0$ and (b) $D=0.5J$. Arrows indicate the SG transition temperature $T_{\rm SG}$ estimated from the scaling plots of the SG susceptibility.

ing behavior.[20,26] Here, in Figs. 3(a) and 3(b), we show $G(L,T)$'s for different L in the cases of $D=0$ and $D=0.5J$, respectively. When $D=0$ (and also $D\leqslant 0.2J$), $G(L,T)$'s for large $L(\geqslant 5)$ seem to come together near $T_{\rm SG}$. This property becomes more prominent in large D, and the data for $L=3$ joins. A similar crossing has also been found in the other quantity $A(L,T)$.

We have calculated four quantities $\chi_{\rm SG}$, $g(L,T)$, $G(L,T)$, and $A(L,T)$ to examine the SG phase transition. All the quantities except for $g(L,T)$ suggested the occurrence of the SG order below $T_{\rm SG}$. Considering the facts that $G(L,T)$ and $A(L,T)$ can be used to determine the value of $T_{\rm SG}$ of such SG models in which $T_{\rm SG}$ is hardly determined by the usual crossing behavior of $g(L,T)$,[20,26] and that the L dependence of $\chi_{\rm SG}$ clearly suggests the divergence of the spin correlation length at $T_{\rm SG}$, it is natural to conclude that the SG order occurs below $T_{\rm SG}$. Therefore, the absence of the crossing of $g(L,T)$'s for finite L would say nothing about the SG phase transition in this model.

What occurs in $g(L,T)$ at $T_{\rm SG}$ for $L\rightarrow\infty$? The occurrence of a dip of $g(L,T)$ near $T_{\rm SG}$ for large D ($D=0.5J$ and $1.0J$) is suggestive. The dip deepens as L is increased implying a

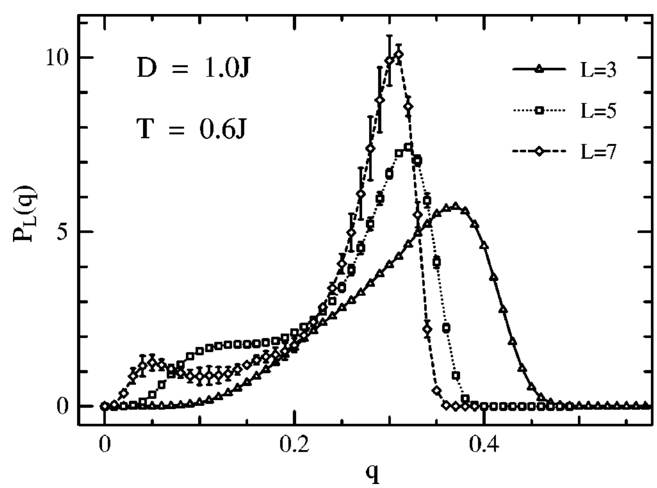

FIG. 4. The order-parameter probability distribution $P_L(q)$ of the $\pm J$ Heisenberg model in $d=4$ for $D=1.0J$ at a low temperature of $T=0.6J(<T_{\rm SG}\sim 0.95J)$.

negative divergence of $g(\infty,T_{SG}^{-})$. It is known that, in the infinite-range p-state Potts glass, the value of $g(\infty,T_{PG}^{-})$ changes continuously from 1 for $p=2$ (Ising model) to $-\infty$ for $p=4$ due to the occurrence of the one-step RSB for $p>2$.[20,21] Therefore, we suggest that $g(\infty,T)$ takes $-\infty$ or some very large negative value at T_{SG}^{-} due to the occurrence of a one-step RSB-like state. In fact, as shown in Fig. 4, the order-parameter distribution function $P_L(q)$ for $D=1.0J$ exhibits a double peak at low temperatures.[28]

On the other hand, in the case of small D, no distinct dip but a bending of $g(L,T)$ was seen around T_{SG}. We may also explain this behavior based on a plausible assumption that $g(\infty,T_{SG}^{-})$ takes a negative value or a small positive value due to the occurrence of the one-step RSB-like state. When $D=0$, the rotational symmetry recovers and $g(\infty,T)$ might exhibit some different property. We think, however, that $g(\infty,T_{SG}^{-})$ also takes a small positive value (or a negative value), because our data of Figs. 2(a)–2(c) imply the presence of no gap between the cases of $D=0$ and $D\neq 0$. Of course, we could not rule out the possibility that there exists some threshold D^*, including $D^*=0$, below which the nature of $g(\infty,T)$ changes qualitatively.

In summary, we reexamined the spin-glass phase transition of the $\pm J$ Heisenberg model in four dimensions ($d=4$) and gave a confirmation that the SG transition really occurs even when the anisotropy is absent at $D=0$. However, its transition temperature T_{SG} could not be determined from the usual crossing behavior of the Binder parameter $g(L,T)$,[19] but from the crossing of $G(L,T)$'s (and also $A(L,T)$'s), as well as from the divergence of the SG susceptibility χ_{SG}. These results show clearly that the naive extension of finite-size scaling concepts for standard phase transitions (such as Ising and Potts ferromagnets) to spin glasses do not work. We suggest that the source of the problem lies in the special properties of the order-parameter distribution function resulting from broken replica symmetry, and one urgently needs a valid theory of finite-size scaling in systems with quenched disorder.

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[27] The crossing behavior of $g(L,T)$ at $T=T_{SG}$ comes from the combination of (A) the scaling hypothesis and (B) the assumption $g(\infty,T)=0$ for $T>T_{SG}$ and $g(\infty,T)=1$ for $T<T_{SG}$. Therefore, when the SG phase transition occurs, even if the assumption (B) is not satisfied, $g(L,T)$ will converge on some nonzero value at $T=T_{SG}$. We think a large finite-size effect masks this property.
[28] Note that a double peak was found in the chirality order-parameter distribution $P_L^{(c)}(q)$ of the Heisenberg (Ref. 6) and XY (Ref. 7) SG models with $D=0$ in $d=3$, in which the one-step RSB was suggested to occur.