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Effect of Inhomogeneity of the Universe on a Gravitationally Bound Local System: A No-Go Result for Explaining the Secular Increase in the Astronomical Unit

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Abstract. We will investigate the influence of the inhomogeneity of the Universe, especially that of the Lemaître–Tolman–Bondi (LTB) model, on a gravitationally bound local system such as the solar system. We concentrate on the dynamical perturbation to the planetary motion and derive the leading order effect generated from the LTB model. It will be shown that there appear not only a well-known cosmological effect arisen from the homogeneous and isotropic model, such as the Robertson–Walker (RW) model, but also the additional terms due to the radial inhomogeneity of the LTB model. We will also apply the obtained results to the problem of secular increase in the astronomical unit, reported by Krasinsky and Brumberg (2004), and imply that the inhomogeneity of the Universe cannot have a significant effect for explaining the observed $\mathrm{dAU}/\mathrm{d}t = 15 \pm 4$ [m/century].

Key words. Celestial mechanics—gravitation—cosmology—LTB model—ephemerides—astronomical unit.

1. Introduction

The advancements in astronomical and astrophysical measurement techniques, particularly those involving the solar system, has been achieved remarkable accuracy of up to 9 to 11 digits level. These technical advancements have drastically improved the accuracy of planetary ephemerides such as DE (Standish 2003), EPM (Pitjeva 2005), VSOP (Bretagnon & Francou 1988) and INPOP (Fienga *et al.* 2008) and that of various astronomical constants. With increasingly improved measurement techniques, observational models are also required to be more accurate and rigorous; for details, refer to Soffel *et al.* (2003) and the references therein.

High-precision observational data also play a crucial role in experimental relativity (Will 1993, 2006). Presently, the main parameters of parametrized post-Newtonian (PPN) approximation, β and γ are tightly constrained to the value of general relativity, i.e., $\beta = \gamma = 1$. For a more accurate verification of gravity, space tests such as LISA (Danzmann 2000), LATOR (Turyshev *et al.* 2004), and ASTROD/ASTROD i (Ni 2008) have been planned.

Thus far, theoretical developments in studies of the solar system have pertained to slow motion, slow rotation and weak field approximation, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$, where $\eta_{\mu\nu}$ is the static Minkowski metric and $h_{\mu\nu}$ is the perturbation (see Damour *et al.* 1991, 1992, 1993, 1994; Brumberg & Kopeikin 1989a, b). However, it is well known that our universe is expanding at an accelerated rate (Perlmutter *et al.* 1999). Therefore, it is natural to consider the situation that the metric tensor, instead of the Minkowskian metric, asymptotically reaches for the expanding space-time or the background Minkowskian metric $\eta_{\mu\nu}$ is replaced by the cosmological type. Several investigations have been conducted, which combine the local metric, e.g., the Schwarzschild space-time or the barycentric celestial reference system adopted by IAU, with the global cosmological comoving coordinates (McVittie 1933; Järnefelt 1940a, b, 1942; Einstein & Straus 1945, 1946; Schücking 1954; Noerdlinger & Petrosian 1972; Cooperstock *et al.* 1998; Klioner & Soffel 2004; Faraoni & Jacques 2006; Sereno & Jetzer 2007; Adkins & McDonnell 2007; Kopeikin 2007; Carrera & Giulini 2010).

The cosmological contribution to the local system has thus far been discussed based on the homogeneous and isotropic cosmological model, i.e., the Robertson–Walker (RW) model. However, inhomogeneous cosmological models have recently attracted considerable attention since these models can provide a possibility to explanation for the observed accelerated cosmic expansion without introducing the concept of dark energy. For instance, the luminosity-distance was investigated in Tomita (2000a, b, 2001a, b, c) based on the local void model and in Iguchi *et al.* (2002) using the Lemaître–Tolman–Bondi (LTB) model (Lemaître 1933; Tolman 1934; Bondi 1947; Plebański & Krasiński 2006). Moreover, Kasai (2007) reanalyzed the observed Type Ia supernovae data and proposed a phenomenological method to describe the large-scale inhomogeneity of the Universe.

Therefore, it would be significant and interesting to investigate the influence of the inhomogeneity of the Universe on the gravitationally bound local system. As far as we know, this issue has previously been examined by Gautreau (1984) and Mashhoon *et al.* (2007) . Gautreau studied the special case of the LTB model, $\mathscr{E}(r) = 0$, (see (19)) with respect to his cosmological theory of the curvature coordinates[1], while Mashhoon *et al.* investigated the cosmological contribution due to the LTB model as the tidal dynamics in the Fermi normal coordinate system.

With remarkable improvements in the observations, it has been found that there exist the unexplained phenomena in theory within the solar system; the pioneer anomaly (Anderson *et al.* 1998), the Earth fly-by anomaly (Anderson *et al.* 2008), the secular increase in the astronomical unit (Krasinsky & Brumberg 2004), and the anomalous perihelion precession of Saturn (Iorio 2009). Presently, the origins of these anomalies are far from clear. Nonetheless, they may be attributable to some fundamental properties of gravitation (see Lämmerzahl *et al.* 2008 and the references therein).

[1] Gautreau does not start from the original form of the LTB model. However, Krasiński (1997) suggested that the model by Gautreau corresponds to the sub case of the LTB model, $\mathscr{E}(r) = 0$.

Among such phenomena, the secular increase in the astronomical unit is of concern to us. From the analysis of radiometric data, Krasinsky and Brumberg (2005) discovered the positive secular trend in AU as[2]

$$\frac{\mathrm{dAU}}{\mathrm{d}t} = 15 \pm 4 \ [\mathrm{m/cy}], \tag{1}$$

see also Standish (2005). Recently, Pitjeva and Standish evaluated $\mathrm{dAU}/\mathrm{d}t \simeq 20$ [m/cy] (Pitjeva 2009). These estimated values are approximately 100 times the error of the present best-fit value of AU (Pitjeva 2005),

$$\frac{1\ [\mathrm{AU}]}{1\ [\mathrm{m}]} \equiv \mathrm{AU} = 1.495978706960 \times 10^{11} \pm 0.1. \tag{2}$$

This secular trend in AU was found by using the following relation (Krasinsky 2007)

$$t_{\mathrm{theo}} = \frac{d_{\mathrm{theo}}}{c}\left[\mathrm{AU} + \frac{\mathrm{dAU}}{\mathrm{d}t}(t - t_0)\right] [\mathrm{s}], \tag{3}$$

where t_{theo} is the computed value of the round-trip time of the light/signal (theoretical value); d_{theo}, the interplanetary distance evaluated from the lunar-planetary ephemerides in the unit of [AU]; c, the speed of light in vacuum; and t_0, the initial epoch of ephemerides. AU and $\mathrm{dAU}/\mathrm{d}t$ are, respectively, the astronomical unit and its time variation. t_{theo} is compared with the observed lapse time t_{obs}.

The time dependent term in (3) cannot be correlated with any theoretical prediction; hence, several attempts have been made to explain this phenomenon, such as the effects of the cosmological expansion (Krasinsky & Brumberg 2004; Mashhoon *et al.* 2007; Arakida 2009), mass loss of the Sun (Krasinsky & Brumberg 2004; Noerdlinger 2008), the time variation of gravitational constant G (Krasinsky & Brumberg 2004), and the influence of dark matter (Arakida 2010). However, none of these attempts have thus far been successful.

Krasinsky and Brumberg (2004) pointed out that the inhomogeneity or nonuniformity of the Universe may have a possible explanation for $\mathrm{dAU}/\mathrm{d}t$; however, they did not provide evidence to support this hypothesis. Hence, it is important to verify this indication. Further a clarification of the observational difference between homogeneous and inhomogeneous cosmological models in the local dynamics is important in the field of modern cosmology.

In this paper, we will focus on the LTB solution as the inhomogeneous cosmological model and investigate its contribution to planetary motion. In §2, we will summarize the dynamical perturbation based on the isotropic and homogeneous RW model and Robertson–McVittie (RM) model. In §3, we will derive the dynamical perturbation attributed to the LTB model. As an application of the obtained results, we will consider the secular increase in the astronomical unit, reported by Krasinsky and Brumberg (2004), in §4. Finally in §5, we will conclude the paper.

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[2] In this paper, cy refers to century as used in the study of Krasinsky and Brumberg (2004).

2. Dynamical perturbation in the RW and RM models

Before discussing the dynamical perturbation in the LTB model, let us provide a brief overview of the planetary perturbation due to the Robertson–Walker (RW) model, and subsequently, due to the Robertson–McVittie (RM) model. Without loss of generality, we first consider the flat ($k = 0$) RW metric in the standard comoving form

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)], \tag{4}$$

where $a(t)$ is the scale factor. The equation of motion of a test particle can be expressed as (Weinberg 1972; Soffel 1989)

$$\begin{aligned}\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} &= -\Gamma^i_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t} + \frac{1}{c}\Gamma^0_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}\\ &= -c^2\Gamma^i_{00} - 2c\Gamma^i_{0j}v^j - \Gamma^i_{jk}v^jv^k + \frac{1}{c}(c^2\Gamma^0_{00} + 2c\Gamma^0_{0j}v^j + \Gamma^0_{jk}v^jv^k)v^i,\end{aligned} \tag{5}$$

where t is the coordinate time; $\Gamma^\lambda_{\mu\nu}$, the Christoffel symbol; and v^i the coordinate velocity. Restricting the equatorial motion to $\theta = \pi/2$, the equations of motion for r and ϕ are given by

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = -2H\frac{\mathrm{d}r}{\mathrm{d}t}, \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) = -2Hr^2\frac{\mathrm{d}\phi}{\mathrm{d}t}, \tag{7}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. If we introduce the proper or radial length R as

$$R \equiv ra(t), \tag{8}$$

then (6) and (7) are rewritten as

$$\frac{\mathrm{d}^2R}{\mathrm{d}t^2} - R\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = \frac{\ddot{a}}{a}R, \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(R^2\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) = 0. \tag{10}$$

Hence, from the point of view of R, the motion of the test particle in RW spacetime is governed by

$$F^{(\mathrm{RW})}_R = \frac{\ddot{a}}{a}R, \quad F^{(\mathrm{RW})}_\phi = 0. \tag{11}$$

Next, in order to observe the effect of cosmological expansion on the Newtonian gravity, we adopt the RM solution (Robertson 1928; McVittie 1933),

$$\mathrm{d}s^2 = -\left[\frac{1-\frac{GM}{2c^2ra(t)}}{1+\frac{GM}{2c^2ra(t)}}\right]^2 c^2\mathrm{d}t^2 + \left[1+\frac{GM}{2c^2ra(t)}\right]^4 a^2(t)(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2), \quad (12)$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$, G is the Newtonian gravitational constant, and M is the mass of the central gravitating body, i.e., the Sun. Equation (12) is expressed in the Newtonian or first order approximation as

$$\mathrm{d}s^2 = -\left[1 - \frac{2GM}{c^2ra(t)}\right]c^2\mathrm{d}t^2 + a^2(t)(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2). \quad (13)$$

Equation (13) can be alternatively obtained from the cosmological perturbation theories (Irvine 1965; Kodama & Sasaki 1984; Tomita 1991; Mukhanov *et al.* 1992; Shibata & Asada 1995; Dodelson 2003; Adkins & McDonnell 2007),

$$\mathrm{d}s^2 = -[1 + 2\Psi(t, \boldsymbol{x})]c^2\mathrm{d}t^2 + a^2(t)[1 + 2\Phi(t, \boldsymbol{x})]\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad (14)$$

where Ψ relates to the Newtonian gravitational potential, Φ is the perturbation to the spatial curvature, and δ_{ij} is the Kronecker's delta symbol.

From (5) and (13), the equations of motion can be expressed as

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = -\frac{GM}{r^2a^3} - 2H\frac{\mathrm{d}r}{\mathrm{d}t}, \quad (15)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) = -2Hr^2\frac{\mathrm{d}\phi}{\mathrm{d}t}. \quad (16)$$

Further, using (8), the coordinates of (15) and (16) are transformed into the proper coordinates as

$$\frac{\mathrm{d}^2R}{\mathrm{d}t^2} - R\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = -\frac{GM}{R} + \frac{\ddot{a}}{a}R, \quad (17)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(R^2\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) = 0. \quad (18)$$

From (17) and (18), we find that in the Newtonian or first order approximation, the leading term of dynamical perturbation obtained from the RM model is the same as those generated from the RW model, $F_R^{(\mathrm{RW})}$ and $F_\phi^{(\mathrm{RW})}$.

3. Dynamical perturbation in the LTB model

It is generally difficult to construct any cosmological model containing a gravitating body because of the non-linearity of general relativity; however, the Robertson–McVittie (RM) model is an exception. As shown in the previous section, it may be a practical working hypothesis that the equation of motion due to both gravitating body and the cosmological effect can be phenomenologically determined by the linear combination of the Newtonian gravitational attraction and the cosmological effect evaluated in the cosmological background metric without considering the gravitating body.

With the above assumption, let us obtain the cosmological perturbations attributed to the LTB model, which can be used to replace $F_R^{(\mathrm{RW})}$, $F_\phi^{(\mathrm{RW})}$ given in the previous section.

The metric of LTB spacetime in the standard comoving form is given by (Lemaître 1933; Tolman 1934; Bondi 1947; Plebański & Krasiński 2006)

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \frac{1}{1+2\mathscr{E}(r)}\left(\frac{\partial\mathscr{R}}{\partial r}\right)^2 \mathrm{d}r^2 + \mathscr{R}^2\mathrm{d}\Omega^2. \tag{19}$$

Here, $\mathscr{R}$ denotes the functions of t and r, and

$$\mathscr{E}(r) = \frac{1}{2c^2}\left(\frac{\partial\mathscr{R}}{\partial t}\right)^2 - \frac{\mathscr{M}(r)}{\mathscr{R}} - \frac{1}{6}\Lambda\mathscr{R}^2, \tag{20}$$

$$\mathscr{M}(r) = \frac{4\pi G}{c^2}\int \rho(t,r)\mathscr{R}^2\frac{\partial\mathscr{R}}{\partial r}\mathrm{d}r, \tag{21}$$

in which Λ is the cosmological constant, $\rho(t,r)$ is the density of the cosmological pressureless particles[3], and $\mathscr{E}(r)$ and $\mathscr{M}(r)$ are the arbitrary functions of r. $\mathscr{E}(r)$ is the generalization of the curvature parameter k in the RW model, and $\mathscr{M}(r)$ is the active gravitational mass that generates the gravitational field. It may be noted that $\mathscr{R}$ has the dimension of physical length, namely, the source area distance or the luminosity distance, while r is a dimensionless coordinate value (Plebański & Krasiński 2006).

Using (5) and (19), we obtain the equations of motion for r and ϕ as

$$\begin{aligned}\frac{\mathrm{d}^2r}{\mathrm{d}t^2} &= -\left[2\frac{\partial^2\mathscr{R}}{\partial t\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial^2\mathscr{R}}{\partial r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - (1+2\mathscr{E})\mathscr{R}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\right]\frac{1}{\frac{\partial\mathscr{R}}{\partial r}} \\ &\quad + \frac{1}{1+2\mathscr{E}}\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2,\end{aligned} \tag{22}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = -\frac{2}{\mathscr{R}}\left[\frac{\partial\mathscr{R}}{\mathrm{d}t} + \frac{\partial\mathscr{R}}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t}\right]\frac{\mathrm{d}\phi}{\mathrm{d}t}, \tag{23}$$

where we ignored the $\mathscr{O}(c^{-2})$ and higher order terms. When we put the flat RW limit $\mathscr{R} \to R = ra(t)$, $\mathscr{E} \to k = 0$, (22) and (23) reduce to (9) and (10), respectively.

In order to relate r to $\mathscr{R}$ explicitly, we suppose that the background LTB spacetime is regular at the origin $r = 0$ where the central body is located, and that the test particle, such as a planet, moves around $r = 0$; hence, the cosmological redshift z in this area is sufficiently small, $z \ll 1$. Thus according to Mashhoon *et al.* (2007), we adopt the following expansion forms for $\mathscr{R}$, $\mathscr{E}$ and $\mathscr{M}$ around $r = 0$ as

$$\mathscr{R}(t,r) = ra(t)\left[1 + \frac{1}{2}\frac{1}{a(t)}\Delta(t)r + \mathscr{O}(r^2)\right], \quad \Delta(t) = \left.\frac{\partial^2\mathscr{R}}{\partial r^2}\right|_{r=0} \ll 1, \tag{24}$$

[3]In the case of the LTB model, the energy-momentum tensor is given by $T^{\mu\nu} = \rho(t,r)u^\mu u^\nu$, where $\rho(t,r)$ is the density and u^μ is the 4-velocity.

$$\mathscr{E}(r) = \frac{1}{2}\epsilon r^2 + \mathscr{O}(r^3), \quad \epsilon = \left.\frac{\mathrm{d}^2\mathscr{E}}{\mathrm{d}r^2}\right|_{r=0} \ll 1, \tag{25}$$

$$\mathscr{M}(r) = \frac{1}{6}mr^3 + \mathscr{O}(r^4), \quad m = \left.\frac{\mathrm{d}^3\mathscr{M}}{\mathrm{d}r^3}\right|_{r=0} \ll 1, \tag{26}$$

in which the scale factor $a(t)$ is defined as

$$a(t) \equiv \left.\frac{\partial\mathscr{R}}{\partial r}\right|_{r=0}. \tag{27}$$

Using these equations, (22) and (23) are rewritten as

$$\frac{\mathrm{d}^2\mathscr{R}}{\mathrm{d}t^2} - \mathscr{R}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = \mathscr{F}_{\mathscr{R}}^{(\mathrm{LTB})}, \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathscr{R}^2\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) = \mathscr{F}_{\phi}^{(\mathrm{LTB})}, \tag{29}$$

where the leading-order dynamical perturbations, $\mathscr{F}_{\mathscr{R}}^{(\mathrm{LTB})}$ and $\mathscr{F}_{\phi}^{(\mathrm{LTB})}$ are expressed as

$$\begin{aligned} \mathscr{F}_{\mathscr{R}}^{(\mathrm{LTB})} &= \left[\frac{\ddot{a}}{a} + \left(\frac{1}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}t}\right)^2\right]\mathscr{R} - \frac{2\epsilon}{\Delta}\left[\frac{\mathscr{R}\dot{a}^2}{a^2} - \frac{\dot{a}}{a}\dot{\mathscr{R}}\right] \\ &= \left[-q\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{1}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}t}\right)^2\right]\mathscr{R} - \frac{2\epsilon}{\Delta}\left[\frac{\mathscr{R}\dot{a}^2}{a^2} - \frac{\dot{a}}{a}\dot{\mathscr{R}}\right], \end{aligned} \tag{30}$$

$$\mathscr{F}_{\phi}^{(\mathrm{LTB})} = 0. \tag{31}$$

In (30), we used the standard relation in the RW model,

$$\frac{\ddot{a}}{a} = -q\left(\frac{\dot{a}}{a}\right)^2, \tag{32}$$

where q is the deceleration parameter. The first term in (30) is $F_R^{(\mathrm{RW})}$, and the second to fourth terms are corrections obtained from the LTB model. It may be considered that the second term in (30) is analogous to $F_R^{(\mathrm{RW})} = -q(\dot{a}/a)^2$.

In (30), we must evaluate ϵ and $\Delta(t)$. Since the observational cosmology indicates that our universe has a flat geometry, we can set $\epsilon = 0$. $\Delta(t)$ may be in principle obtained from the modified luminosity-redshift relation (Partovi & Mashhoon 1984),

$$d_{\mathrm{L}} = c\left[\frac{z}{H} + \frac{z^2}{2H}(1 - q - C)\right], \quad C = \frac{1}{aH^2}\frac{\mathrm{d}\Delta}{\mathrm{d}t}. \tag{33}$$

Finally, following the assumption discussed in the beginning of this section, the equation of motion, attributed to both the gravitating body and the cosmological effect due to the LTB model, can be phenomenologically given by

$$\frac{\mathrm{d}^2\mathscr{R}}{\mathrm{d}t^2} - \mathscr{R}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = -\frac{GM}{\mathscr{R}} + \mathscr{F}_{\mathscr{R}}^{(\mathrm{LTB})}, \tag{34}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathscr{R}^2\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) = \mathscr{F}_{\phi}^{(\mathrm{LTB})}. \tag{35}$$

4. Secular increase in astronomical unit

In this section, as an application of (34) and (35), let us consider the secular increase in the astronomical unit (Krasinsky & Brumberg 2004; Standish 2005; Arakida 2009). Krasinsky and Brumberg found from their analysis of planetary radar and martian orbiters/landers range data that the astronomical unit (AU) increases with respect to meters as $\mathrm{dAU}/\mathrm{d}t = 15 \pm 4$ [m/cy]. This secular trend cannot be related to any theoretical model, and thus far, the origin of this secular increase is far from clear.

Krasinsky and Brumberg suggested one possibility, that the inhomogeneity of the Universe may be an explanation for $\mathrm{dAU}/\mathrm{d}t$. We consider this possibility in terms of the LTB model. Because the current cosmological observations assert that the geometry of our Universe is flat, we choose $\epsilon = 0$. In this case, the cosmological perturbation $\mathscr{F}_{\mathscr{R}}^{(\mathrm{LTB})}$ becomes

$$\mathscr{F}_{\mathscr{R}}^{(\mathrm{LTB})} = \left[-q\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{1}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}t}\right)^2\right]\mathscr{R}. \tag{36}$$

In our approximation, $-q(\dot{a}/a)^2\mathscr{R}$ or, equivalently, $(\ddot{a}/a)\mathscr{R}$ is a dominant cosmological effect; however, its contribution is considerably negligible (Järnefelt 1940a, b, 1942; Noerdlinger & Petrosian 1972; Cooperstock *et al.* 1998; Carrera & Giulini 2010; Klioner & Soffel 2004; Sereno & Jetzer 2007; Faraoni & Jacques 2006; Adkins & McDonnell 2007; Arakida 2009). From the assumption (24), Δ can be considered as a small correction to the scale factor $a(t)$; its time variation $\mathrm{d}\Delta/\mathrm{d}t$ is also smaller. Further, it is known that the deviation of temperature in the observed cosmic microwave background (CMB) radiation is of the order of 10^{-5}; hence, we may use the following:

$$\frac{-q\left(\frac{\dot{a}}{a}\right)^2}{\left(\frac{1}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}t}\right)^2} \approx 10^{-5}. \tag{37}$$

It is, therefore, currently difficult to detect the cosmological contributions attributed not only to the RW model but also to the inhomogeneity of the Universe. The inhomogeneity in the background cosmological matter distribution does not have a detectable effect and hence cannot explain the observed $\mathrm{dAU}/\mathrm{d}t$.

5. Conclusions

We investigated the cosmological influence according to the LTB model on a gravitationally bound local system such as the solar system. We focused on planetary motion and obtained the leading-order dynamical perturbation generated from the

LTB spacetime. The obtained dynamical perturbations, especially (34), contains the contribution attributed to the RW model and also the correction terms attributed to the LTB model. Moreover, we applied the obtained results to the secular increase in the astronomical unit (Krasinsky & Brumberg 2004) and confirmed that the effect of the inhomogeneity of the Universe does not provide an explanation for $\mathrm{dAU}/\mathrm{d}t$.

In spite of several attempts (see §1), the origin of $\mathrm{dAU}/\mathrm{d}t$ is far from clear. It is now pointed out that the most plausible reason of $\mathrm{dAU}/\mathrm{d}t$ is due to either the lack of calibrations in the internal delays of radio signals within spacecrafts or the complications of the modeling of solar corona. However, none of these explanations have thus far been successful; this issue should hence be explored using all possibilities. A re-analysis of $\mathrm{dAU}/\mathrm{d}t$ incorporating new data sets is also expected.

Since the astronomical unit, as expressed in (2), is currently determined from the arrival time measurement of radar signals, we must investigate this problem in terms of light/signal propagation. To this end, we need to construct a cosmological model that combines the LTB model with a gravitating body. Of course, it may be difficult to detect the cosmological effect in the solar system. Nonetheless, since the theoretical discussions pertaining to this issue are still unresolved, it is, from theoretical point of view, interesting to develop a rigorous physical model that matches the gravitational bound local system and the cosmological models and clarify several assertions.

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Note Added in Proof

Recently Iorio (Iorio, L. 2010, *J. Cosmol. Astropart. Phys.* **6**, 4) investigated the influence of LTB model to the solar system dynamics and based on recent observational data, he evaluated the parameters of LTB metric in Fermi coordinate system K_1 and K_2.