Fermion Doubling in Loop Quantum Gravity

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Abstract

For the last 20 years, it has been known how to couple matter to the theory of loop quantum gravity. However, one of the most simple questions that can be asked about this framework has not been addressed; is there a fermion doubling in loop quantum gravity? This is an exceptionally important issue if we are to connect the theory to experiments. In this thesis, we will arrive at a demonstration of fermion doubling around some graphs in the large bare Λ limit. To obtain this result, we first perform a Born-Oppenheimer like approximation to the Hamiltonian formulation of loop quantum gravity to work around a theory with a fixed graph. We then make the case for identifying the energy spectrum this theory with a model of lattice gauge theory which is known to double. Appropriate reviews of fermion doubling and loop quantum gravity are provided along with an outlook of constructing a doubling-free version of LQG. Our findings suggest one should interpret matter in loop quantum gravity in a much different way.

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Contents

Chapter 1

Introduction

1.1 Basic strategy

Over the last 80 years, physicists have been struggling to create a consistent theory of quantum gravity. This is an extremely difficult quest which has led us to reconsider the basic principles of physics. To date, there are two major candidates for quantum gravity: string theory and loop quantum gravity (LQG). String theory is a natural extension of ordinary quantum field theory and is consistent with all of the particles from the standard model. However, in loop quantum gravity background independence is understood to be fundamental, making it difficult to understand the role of matter as there is no particular background to work on. In this thesis, we propose that there may be a serious conflict between loop quantum gravity and the standard model, which has a wealth of experimental evidence.

In the standard model, particles coupled to the theory are forced to be massless to preserve chiral symmetry. Instead, a particles' mass is obtained through coupling to the Higgs field. For a general class of lattice gauge theories, Nielsen and Ninomiya have proven a no-go theorem [1, 2, 3], the result of said theorem is that in order for the models to be consistent the number left and right-handed particles contained in the spectrum must be equal. This phenomenon is known as fermion doubling and naively leads to a contradiction with experiment as the weak interactions only couple to left-handed particles. Since loop quantum gravity provides a discrete space-time, one expects to observe fermion doubling. In this thesis, we will demonstrate this is indeed the case and there is a fermion doubling problem in loop quantum gravity.

The basic strategy of our analysis is the following:

1. Start with a lattice gauge theory based on a graph Γ embedded in Σ, a torus T^d. Assume Γ is regular enough for a fourier expansion of the fermion excitations to exist, allowing there to be a continuous spectrum $E = E(p)$. Verify the assumptions of the Nielsen-Ninomiya (NN) no-go theorem hold for the Hamiltonian H_Γ^{GT}.

2. Define $\mathcal{H}_\Gamma^{QG}$ to be the (spatially diffeomorphism invariant, with a physical hamiltonian coming from gauge fixing) subspace of the Hilbert space of LQG based on a graph Γ.

3. Define H^{matter} to be the Hamiltonian based on graph Γ that dominates in the Born-Oppenheimer approximation [4].

4. Use a degravitating map and dressing map to show that the spectra of H^{matter} and H_Γ^{GT} must be identical, at least in the regime of $E <<< E_{Planck}$.

5. This establishes the NN theorem for LQG.

The moral of this strategy is to carry the problem of doubling in LQG to lattice gauge theory where the NN theorem may be applied. Item 2 grants us a graph to work on. We use the Born-Oppenheimer approximation [4] to concentrate on the fermion degrees of freedom and expand around the gravitational pieces, we need the fermions to live on a background graph Γ to investigate doubling. Note our result is that there exists Γ such that fermion excitations of the corresponding background states double their spectra. We don't yet show that the fermion excitations around all states double, that would be a stronger result. In chapter 2 we will review the proof of the Nielsen-Ninomiya no-go theorem for a general class of Lattice Gauge Theories, completing item 1. Before completing items 2 to 4 in the final chapter 4, we will provide a brief review of the key elements of LQG needed for our calculation in chapter 3.

Chapter 2

Fermion Doubling in Lattice Models

2.1 Review of Chiral Symmetry

This chapter is meant to review the proof of the Nielsen-Ninomiya no-go theorem presented in [2]. Other, more rigorous proofs are listed in [1, 3]. Before getting carried away with details, we shall take a small amount of time to discuss what fermion doubling physically means.

In quantum field theory courses [5], we are taught massless solutions to the Dirac equation are eigenstates of the helicity operator

$$\hat{h} = \hat{p} \cdot \vec{S} = \hat{p}_i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \tag{2.1}$$

with σ_i being the Pauli matrices. A particle with helicity h = +/- $\frac{1}{2}$ is referred to as a right/left handed particle. In other words, a right-handed particle's momentum vector is aligned with its spin (and anti-aligned for a left-handed particle). While this is perfectly consistent and clear in the normal continuum setting, in the discrete setting a particle must come with both helicities. To see this, look at the Fourier kernel $e^{ipx/\hbar}$, and notice a minimum length corresponds to a maximum momentum. Thus the momenta p and $p + \frac{2\pi a}{\hbar}$ are identified, with a being the lattice spacing. We may add subtract off as many units of $\frac{2\pi a}{\hbar}$ as we like, resulting in a positive momenta vector being identified with a negative one. Under this identification, the helicity changes sign and we see left and right handed particles must be equivalent.

Another way to arrive at fermion doubling is to look at the energy in momentum space. Since momentum space is bounded, any functions on it must be periodic. Thus, the energy function will cross the line of zero energy up and down an equal number of times. At each crossing, we may Taylor expand the energy relationship to see an emergent low energy particle with the relativistic dispersion relation $E = pc$. Since there are an equal number of crossings with positive and negative slopes, for every emergent particle there must be a corresponding emergent particle of opposite helicity. The same property is held for all lattices with a compact (closed and bounded) momentum space. This is the key feature we will look at in our proof.

Helicity is not a meaningful observable if a particle is massive. This can be summarized quite simply, a massive particle does not travel at the speed of light, thus we can consider an observer which overtakes the particle. In this frame, the direction of the momentum is flipped, and the helicity changes sign. For massive particles, it is convenient to think of a more abstract concept, known as chirality. The chirality of a particle is determined from whether the particle transforms under the left or right-handed part of the Poincare group. From here, we may work with chirally invariant actions, where the action has a symmetry under

$$\psi_L = e^{i\theta_L}\psi_L, \quad \psi_R = e^{i\theta_R}\psi_R, \tag{2.2}$$

where we can define the left and right handed pieces using the projectors $P_L = \frac{1-\gamma_5}{2}$, $P_R = \frac{1+\gamma_5}{2}$. While chirality is a reasonably defined concept for massive particles, it is easy to show mass terms break chiral symmetry. To do this, we can write a mass term using the left and right-handed parts

$$m\bar{\psi}\psi = m(\bar{\psi}_L\psi_L + \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L + \bar{\psi}_R\psi_R). \tag{2.3}$$

We see the cross terms break chiral invariance, and a chirally invariant theory must be massless. For a massless theory, chirality is conserved and is equal to its helicity.

2.2 Nielsen-Ninomiya no-go Theorem

We are now prepared to formulate the Nielsen-Ninomiya no-go theorem:

Theorem 1. *Suppose we are given a lattice theory of free fermions governed by an action quadratic in the fields*

$$S = -i\int dt \sum_x \dot{\bar{\psi}}(x,t)\psi(x,t) - \int dt \sum_{x,y} \bar{\psi}(y,t)H(x-y)\psi(x,t), \tag{2.4}$$

with $\psi(x,t)$ an N component spinor with discrete position label x. Then fermion doubling is inevitable if the following conditions are met:
• *The underlying lattice has a well defined momentum space which is compact.*
• *The interaction H is hermitian and local, in the sense that its momentum space continuation is continuous.*
• *The charges Q are conserved, locally defined as a sum of charge densities $Q = \sum_x j^0(x) = \sum_x \bar{\psi}(x)\psi(x)$, and is quantized.*

The proof below can be generalized to a more general kinetic term, $\dot{\bar{\psi}}(y,t)T(x-y)\psi(x,t)$, so long as T doesn't vanish anywhere on the lattice. This can be seen from the equations of motion.

$$iT(x-y)\,\partial_t\psi(y,t) = H(x-y)\psi(y). \tag{2.5}$$

The equations of motion indicate the action is equivalent to one governed by the standard kinetic term along with the effective Hamiltonian H/T; if T vanishes or is nonlocal the no-go theorem does not apply. Because of this, it is often convenient to speak in terms of the Dirac operator D, defined by

$$S = a^4 \sum_x \bar{\psi} D \psi, \tag{2.6}$$

where now we add an additional assumption to the theorem: D is required to be invertible.

In addition we may couple a fixed gauge field to the fermions (such as a Yang-Mills field) without spoiling the proof, as such fields simply act as a background when working in the low-energy regime.

To characterize fermion doubling, we will look at the dispersion relation in momentum space, given by the eigenvalue equation

$$H(p)\psi_i(p) = \omega_i(p)\psi_i(p), \tag{2.7}$$

here the index i is not to be confused with the components of ψ, but rather a label of eigenvectors which is not summed on. We can choose this label such that the eigenvalues are increasing (recall they are real by the Hermiticity of H)

$$\omega_1(p) \leq \omega_2(p) \leq \cdots \leq \omega_N(p). \tag{2.8}$$

The relevant feature will be captured by the level crossings

$$\omega_i(p_{deg}) = \omega_{i+1}(p_{deg}), \tag{2.9}$$

as this corresponds to the zeroes of the energy difference in our simple picture. We will now show that the level crossings / zero energy limits correspond to left or right handed Weyl particles.

As an aside, notice that level crossings between 2 levels are generic in 3+1 dimensions, but 3 level crossings are not. To see this, consider a general 3x3 Hamiltonian spanned by the Gell-Mann matrices λ_i

$$H^{(3)}(p) = A(p)\mathbb{1} + \sum_{i=1}^{8} B_i(p)\lambda_i. \tag{2.10}$$

For a three level degeneracy, the coefficients $B_i(p)$ must all vanish. This requirement gives 8 equations for only 3 quantities, which generically has no solutions. However, two level crossings are generic in 3+1 dimensions as the relevant 2x2 piece of the Hamiltonian is spanned by three Pauli matrices.

Near a 2-level crossing at the degeneracy point p_{deg}, we may expand the relevant piece of the Hamiltonian in a Taylor Series

$$H^{(2)}(p_{\text{deg}} + \delta p) = \omega_i(p_{\text{deg}})\mathbb{1} + \delta\vec{p} \cdot \vec{A} + \delta p_k V^k_\alpha \sigma^\alpha + O(\delta p^2), \tag{2.11}$$

where α and k run from 1 to 3. To simplify the form of this Hamiltonian near p_{deg}, we will shift the momentum by

$$P_0 = H \to P_0 - \omega_i(p_{\text{deg}}) - \delta\vec{p}\cdot\vec{A} \tag{2.12}$$

$$\delta p_k \to \delta p_k \pm \delta p_\alpha V_k^\alpha, \tag{2.13}$$

where the $\pm$ sign depends on the determinant of V. The shifted Hamiltonian becomes

$$H = \vec{p}\cdot\vec{\sigma}, \tag{2.14}$$

and the corresponding eigenvalue problem is

$$\hat{\vec{p}}\cdot\vec{\sigma}U(p) = \pm p_0 U(p), \tag{2.15}$$

where $U(p)$ is the corresponding wave function. This is the Weyl equation for a Dirac fermion, and we shall identify the sign of the determinant of V with the helicity of a Weyl particle. Thus each 2 level-crossing p_{deg} is identified with a Weyl fermion in the low energy limit. It is important to note the helicity depends on the degeneracy point p_{deg} and so it now remains to identify which types of Weyl fermions emerge in the low energy limit.

To set about this task we will look at curves in the 3-D dispersion relation space (which is embedded in the full 3+1D ω-p space) defined by

$$\{(p,\omega)|\,\langle a|\omega_i(p)\rangle = 0\}, \tag{2.16}$$

with the bracket defined by

$$\langle a|\omega_i(p)\rangle = a_1\psi_1^{(i)} + a_2\psi_2^{(i)} + \cdots + a_N\psi_N^{(i)}, \tag{2.17}$$

where $|a\rangle$ is any constant N-vector. In 3 dimensions, this will turn into one complex equation for three quantities $\vec{p}$, implying the set forms a curve. These curves are of special importance because they pass through all of the degeneracy points p_{deg}. The reason for this is that at a degeneracy point, we may redefine the energy eigenstates through any superposition of the form

$$|\omega_i\rangle = \alpha\,|\omega_i\rangle + \beta\,|\omega_{i+1}\rangle \tag{2.18}$$

and choose α and β so that $\langle a|\omega_i(p_{\text{deg}})\rangle = 0$. All such curves must be closed since the Brillouin zone is compact. Thus, if the curves have an orientation, they must pass equally many times up and down through the degeneracy points, proving the theorem. In the paragraphs below we will define this orientation.

To set about this task, we will look at the phase of $\langle a|\omega_i\rangle$ on small circles wrapping around the curve near each degeneracy point. Suppose we set this small circle of radius R a distance d away from a degeneracy point at $p_z = 0$ along a curve passing through the positive p_z direction. The Weyl equation near the degeneracy point is given by eq. (2.15), with the sign being identified with helicity. The eigenvalues and eigenvectors to this equation along the mentioned circle $S^1 = \{\theta \in \mathbb{R} | (p_x, p_y, p_z) = (R\cos\theta, R\sin\theta, d)\}$ in the limit $R \ll d$ are

$$U_1 = \begin{pmatrix} 1 \\ \frac{R}{2d} e^{i\theta} \end{pmatrix}, \omega_1 = \pm(d + R);\ U_2 = \begin{pmatrix} 1 \\ -\frac{R}{2d} e^{-i\theta} \end{pmatrix}, \omega_2 = \mp(d + R). \tag{2.19}$$

In the limit $R \to 0$, we have $U_{1,2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, implying the vector $\langle a|$ along the curve is $(0, 1)$. We obtain the following for the phase rotation along the curves for both levels

$$\langle a|U_1\rangle = \pm\frac{R}{2p_z} e^{i\theta},\ \langle a|U_2\rangle = \mp\frac{R}{2p_z} e^{-i\theta}. \tag{2.20}$$

eq. (2.20) may now be used for orientation assignment. There are then two types of curves for each helicity, one crossing from $p_0 < 0, p_z < 0$ to $p_0 > 0, p_z > 0$, and one crossing from $p_0 < 0, p_z > 0$ to $p_0 > 0, p_z < 0$. Using eq. (2.20), we see the first curve is oriented in the positive (negative) p_z direction for a right (left) handed degeneracy point, and the second is oriented in the negative (positive) p_z direction for a right (left) handed degeneracy points. Since the curve must be closed and passes equally many times up and down through degeneracy points, by the orientation assignment we see the number of left and right handed particles are matched. □

2.3 Avoidance of Fermion Doubling

It may appear the problem of fermion doubling is much worse than the no-go theorem suggests; in 3+1 dimensions, the Weyl equation is

$$i\partial_t \psi(p) = \sum_i \sigma_i \sin(p_i a)\psi(p), \tag{2.21}$$

which has 8 zeroes. We can now ask ourselves if there are constructions which minimize the number of zeroes. There are several ways to reduce the number of fermions. One approach is to use staggered fermions, where the four components of a Dirac spinor get placed on different sites [6]; in the staggered fermion approach there are only four types of particles. This approach is highly non-local. It turns out we can reduce the number of fermions to just two [7] by using the Dirac operator

$$aD_C(p) = iB\gamma_0 \left(4C - \sum_\mu \cos ap_\mu\right) + iB\sum_{k=1}^{3} \gamma_k s_k(ap) \tag{2.22}$$

with

$$s_1 = \sin p_1 + \sin p_2 - \sin p_3 - \sin p_4 \tag{2.23}$$

$$s_2 = \sin p_1 - \sin p_2 - \sin p_3 + \sin p_4 \tag{2.24}$$

$$s_3 = \sin p_1 - \sin p_2 + \sin p_3 - \sin p_4. \tag{2.25}$$

Here B and C are some constants. The Dirac operator then has zeroes only at $(\tilde{p}, \tilde{p}, \tilde{p}, \tilde{p})$ and $-(\tilde{p}, \tilde{p}, \tilde{p}, \tilde{p})$, where $C = \cos \tilde{p}$. It is simple to check by looking at eq. (2.22) that the $a \to 0$ limit of this theory is not $\not{p}$; this is related to having two zeroes of the dispersion relation not located at $p = 0$. However, no construction that falls under the assumptions of the no-go theorem may have an odd number of particles.

The proof of the no-go theorem fails whenever the three main bulleted assumptions do not hold. One of the most common cures to this problem is to redefine what a chirally invariant action is. It is easy to prove that if D and γ_5 anti-commute, the action is chirally invariant. However, it may be the case that this relation should be modified in such a way that the new definition reduces to the normal one as the lattice spacing a tends to zero. This is the heart of the Ginsparg-Wilson technique, it can be shown that the lattice definition for the Dirac operator

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D \tag{2.26}$$

does not double [8].

To summarize this chapter, a general class of lattice gauge theories, namely those with chirally invariant, local, hermitian actions quadratic in the fields with local charges conserved, have equal coupling to the left and right handed sectors of the low energy theories. Since the weak interactions only couple to the left-handed part of the standard model, this provides obvious contradictions with well established particle physics. So far, many attempts at curing fermion doubling have been created. However, they all must violate one of the assumptions of the no-go theorem. We now proceed with our review of loop quantum gravity.

Chapter 3

Review of the Hamiltonian Formulation of Loop Quantum Gravity

This chapter is meant to serve as a brief review of loop quantum gravity. Those already familiar with the theory may feel free to skip this chapter and proceed into chapter 4. Derivations are provided little detail to preserve the length of this chapter, for a more thorough introduction see [9, 10, 11, 12, 13, 14] (these were chosen to be as diverse of a set of resources as possible).

Astonishingly, loop quantum gravity succeeds at constructing a mathematically well-defined, non-perturbative theory of quantum gravity without introducing any new principles outside of the well-tested GR and quantum field theory. This goal is achieved by a new choice of variables, namely the Ashtekar variables, which express GR in terms of a simple $SU(2)$ gauge theory. Due to this choice, the symmetries of GR such as background independence and diffeomorphism invariance are completely respected. We work in the canonical quantization, where it is required to smear the variables along appropriate test functions. Since the Ashtekar variables are forms, it is required to smear them along loops and surfaces, hence giving the theory its name. The simplest observables of the theory are the area and volume operators, these are what is quantized in the theory. This imposes a physical cutoff at the planck scale and results in a theory which has no ultraviolet divergences (although there are IR divergences, these are avoided through the introduction of the cosmological constant), unlike the naive quantization, which is non-renormalizable. For these reasons, the author feels LQG provides at the very least a good idea for what a quantum theory of gravity should look like, and is an important idea worth exploring.

3.1 Hamiltonian GR

In this section, we will review the Hamiltonian formulation of GR, as it is needed for the canonical quantization. For the majority of this chapter, we will work with the purely gravitational theory $T_{\mu\nu} = 0$. However, as matter is required for fermion doubling to take any meaning, we provide short review of coupling matter to LQG in section 3.6. Consider the Einstein-Hilbert action in a vacuum (setting $\frac{1}{16\pi G} = 1$ for convenience)

$$S = \int d^4x \sqrt{-g}R, \tag{3.1}$$

with R being the Ricci scalar. To determine the Hamiltonian from the Lagrangian, we need to perform the canonical transformation

$$S = \int \pi_i \dot{\phi}_i - H(\pi, \phi), \tag{3.2}$$

with ϕ_i being the appropriate fields of the theory and π_i being the conjugate momenta $\frac{\delta H}{\delta \dot{\phi}_i}$. For the Legendre Transform (3.2) to be a well-defined construct, we will be forced to work in a framework which treats time differently than space; To do this we will use the ADM variables [15], which are described below.

On a technical note, we will assume the manifold has the topology $\mathcal{M} = \mathbb{R} \times \Sigma$, with Σ a 3-dimensional spacelike manifold. This immediatly implies we can find a one-parameter family of hypersurfaces, $\Sigma_t = X_t(\Sigma)$, which foliate the manifold $\mathcal{M}$. This allows us to choose a time-parameter t to be used for evolution. This time is not an absolute time due to the diffeomorphism invariance of the action; A diffeomorphism can be used to map one foliation to another $X \to X \circ \phi$. Thus diffeomorphism invariance is equivalent to refoliation symmetry plus diffeomorphisms on the foliation.

Once we have a time parameter t, we will define the lapse, N, and shift, N_a, using the metric

$$g_{\mu\nu}dx^\mu dx^\nu = -(N^2 - N_a N^a)dt^2 + 2N_a dt dx^a + g_{ab}dx^a dx^b, \tag{3.3}$$

with a, b running over only the spatial indices. To study differential geometry on the submanifold Σ, we must rely on the induced metric, defined by

$$q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu, \tag{3.4}$$

with n_μ being the unit normal vector to Σ (which may be parametrized by the lapse/shift $(1/N,\, N^a/N)$). The role of q is to act as a projector of the full manifold $\mathcal{M}$ to the submanifold Σ. The Einstein-Hilbert action may now be re-written in terms of the lapse, shift, induced Riemann tensor $\mathcal{R}$, and extrinsic curvature K as

$$K_{\mu\nu} = q_\mu^\alpha q_\nu^\beta \nabla_\alpha n_\beta \tag{3.5}$$

$$\mathcal{R}^\mu_{\nu\rho\sigma} = q^\mu_\alpha q^\beta_\nu q^\gamma_\rho q^\delta_\sigma R^\alpha_{\beta\gamma\delta} - K_{\nu\sigma}K^\mu_\rho - K_{\nu\rho}K^\mu_\sigma \tag{3.6}$$

$$S = \int dt \int_\Sigma d^3x \sqrt{q} N(\mathcal{R} - K^\alpha_\alpha K^\beta_\beta + K_{\alpha\mu}K^{\mu\alpha}). \tag{3.7}$$

From here, it may be checked that the lapse and shift only appear in the action as lagrange multipliers, that is without any time-derivatives. For a lagrange multiplier λ, the conjugate momenta vanishes $\frac{\delta L}{\delta \lambda} = 0$, implying the lagrange multipliers λ are not dynamical variables. To perform the canonical transform, we use the momenta for the spatial components of the induced metric π_{ab} to rewrite the lagrangian as

$$S = \int dt \int_{\Sigma} \pi^{ab} \dot{q}_{ab} - N^a H_a - NH, \tag{3.8}$$

with

$$H_a = -2\nabla_b \left(\pi_a^b\right) \tag{3.9}$$

$$H = \frac{1}{\sqrt{q}} (q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}a_{cd})\pi^{ab}\pi^{cd} - \sqrt{q}\mathcal{R}. \tag{3.10}$$

Variation with respect to the lapse and shift gives the vector and scalar constraints respectively

$$H_a(q, \pi) = 0, \quad H(q, \pi) = 0. \tag{3.11}$$

The Hamiltonian is thus the sum of the contraints

$$H = \frac{1}{16\pi G} \int_{\Sigma} d^3x N^a H_a + NH. \tag{3.12}$$

At first this result should look rather surprising, as the Hamiltonian vanishes for physical solutions. Hence, there is no dynamics in the time t. This is an immediate consequence of the diffeomorphism invariance previously discussed, as the parameter t was an arbitrary label to one particular foliation of space-time. This staggering result is often referred to as the problem of time, as it clashes with our intuition from quantum theory. For instance, measurements usually take place at one instance of time and dramatically alter the state of a system. What is the role of measurement in a timeless theory? In addition, how is it possible to understand the evolution of the state ψ in "time"? Issues like these often drive physicists to interpreting the notions of background independence and diffeomorphism invariance as low energy artifacts, where GR only takes the role of an effective field theory. However, the key lesson of LQG is that the problem of time can be addressed in a background independent and diffeomorphism invariant framework of quantum gravity. To keep the review short, we refer the philosophically-minded reader to [14] for a more detailed discussion about the problem of time. However, for our calculation, we need to address the issue of evolution. The trick, carried out in detail in section 3.4, is to couple the theory to a scalar field which parametrizes the evolution of a state and generates a Schrodinger-like equation.

To understand the action of the constraints, consider the smeared versions defined by

$$H(\vec{N}) = \int_{\Sigma} d^3x \, N_a H^a, \quad H(N) = \int_{\Sigma} d^3x \, NH. \tag{3.13}$$

One can derive the infinitesimal flows in phase space

$$\{H(\vec{N}), q_{ab}\} = \mathcal{L}_{\vec{N}} q_{ab}, \quad \{H(\vec{N}), \pi^{ab}\} = \mathcal{L}_{\vec{N}} \pi^{ab}. \tag{3.14}$$

Thus the evolution under the vector constraint generates the group of spatial diffeomorphisms as the Lie derivative is an infinitesimal diffeomorphism. For the scalar constraint, one can derive

$$\{H(N), q_{ab}\} = \mathcal{L}_{N\vec{n}} q_{ab} \tag{3.15}$$

$$\{H(N), \pi_{ab}\} = \mathcal{L}_{N\vec{n}} \pi_{ab} + \frac{1}{2} q^{ab} N H - 2N\sqrt{q} q^{c[a} q^{b]d} R_{cd}. \tag{3.16}$$

On the space of solutions $H = R_{cd} = 0$ (we have not coupled any matter yet, Einstein's equations are $G_{\mu\nu} = 0$), these generate the time diffeomorphisms. It is for this reason that H^a is also referred to as the diffeomorphism constraint and H is referred to as the Hamiltonian constraint. An important thing to note is the algebra of constraints is *not* the algebra of 4-diffeomorphisms; It can not be the algebra of 4-diffeomorphisms due to the 3+1 splitting used when defining the ADM variables.

3.2 Canonical Quantization

Once we have written down the Hamiltonian as a sum of constraints we can proceed with the Dirac quantization program. A rough outline of this procedure is given below, for more detail see Dirac's book [10].

• Create a notion of a Poisson bracket $\{\,,\,\}$ between two functions on phase space. We define the Poisson bracket to be linear in both arguments, anti-symmetric, and satisfy a Leibniz $\{f_1 f_2, g\} = f_1\{f_2, g\} + \{f_1, g\} f_2$ and a Jacobi identity $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$. The Poisson bracket between any phase space function and the Hamiltonian generates a flow

$$\frac{df}{d\tau} = \{H, f\}, \tag{3.17}$$

where the parameter τ is identified with evolution in time t

• Find all constraints between momenta and positions. In other words, for momenta defined through canonical conjugation $\pi_i = \frac{\delta L}{\delta \phi(x)^i}$, determine all possible relations between π and ϕ that do not contain any time derivatives, $C_k(\phi, \pi) = 0$. Once done, one may find an additional set of secondary constraints by adding the original constraints to the Hamiltonian to create the total Hamiltonian $H_{\mathrm{T}} = H + \sum_k a_k C_k(\phi, \pi)$, and imposing the evolution of the known constraints under the total Hamiltonian is weakly zero (equal to a sum of constraints). This process may be continued until all possible secondary constraints are found. In addition, for quantum gravity all constraints are first class

$$\{C_k(\phi, \pi), C_l(\phi, \pi)\} = \sum_m b_m C_m(\phi, \pi). \tag{3.18}$$

This is not required for all theories, weakly vanishing evolution of the constraints may imply information about the a_k's.

• To proceed with quantization, promote functions on phase space to operators acting on a Hilbert space with commutation relations given by replacing the Poission bracket with commutators $\{\ ,\ \} \rightarrow \frac{[,]}{i\hbar}$. The physical states are required to satisfy the constraint equations

$$C_k(\hat{\phi}, \hat{\pi})\,|\psi\rangle = 0. \tag{3.19}$$

We will choose the following for the Poisson bracket, as it reflects the natural choice from Newtonian dynamics:

$$\{A, B\} = \int d^3x\, \frac{\delta A}{\delta q_{ab}(x)} \frac{\delta B}{\delta \pi^{ab}(x)} - \frac{\delta A}{\delta \pi^{ab}(x)} \frac{\delta B}{\delta q_{ab}}. \tag{3.20}$$

For general relativity, it may be shown that the only constraints are the ones already discussed, namely the scalar and vector constraints H and H^a. Problems start to appear with the dirac quantization procedure when we proceed to the final step, when promoting q_{ab} and π^{ab} to operators with commutation relations

$$[\hat{q}_{ab}(x), \hat{\pi}^{cd}(y)] = i\hbar \delta^c_a \delta^d_b \delta^{(3)}(x - y), \tag{3.21}$$

$$[\hat{q}_{ab}(x), q_{cd}(y)] = 0, \tag{3.22}$$

$$[\hat{\pi}_{ab}(x), \pi_{cd}(y)] = 0. \tag{3.23}$$

Perhaps the most important issue is the lack of an inner product on the space of metrics modded by diffeomorphism; There is no suitable Haar measure for general relativity. This severely limits progress of canonical quantization, as we can not verify the Hemiticity of q and π or the positivity of their spectra. There is an additional open issue, the Hamiltonian constraint is effectively impossible to solve. The corresponding operator is

$$\hat{H}\,|\psi(q_{ab})\rangle = \left[-\frac{\hbar^2}{2}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}a_{cd}) : \frac{1}{\sqrt{\det \hat{g}}} \frac{\delta^2}{\delta q_{ab} \delta q_{cd}} : -\det \sqrt{\hat{g}} R(\hat{g}) \right] |\psi(q_{ab})\rangle\,, \tag{3.24}$$

with the colon being an appropriate ordering of operators. In addition to having the product of two operators at a point (which is singular), only a semiclassical solution to the Wheeler-DeWitt equation $H\,|\psi\rangle = 0$ is known. Loop quantum gravity attempts to tackle these problems by writing GR in terms of new variables which look like an $su(2)$ gauge theory, which will be introduced in section 3.3.

3.3 The Ashtekar Variables

So far, we have attempted to canonically quantize GR, and have shown that the induced metric is a bad choice of quantization variable. We shall derive a more suitable set of variables in two steps. The first step is to define the tetrads

$$g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ} \tag{3.25}$$

to be an orthonormal frame of vectors with an internal index $I \in \{0, 1, 2, 3\}$. The splitting of a metric into tetrads is not unique, one can perform an internal Lorentz transformation of the tetrads and preserve eq. (3.25), meaning we can define spinors $e^I_\mu n^\mu$. This internal gauge theory is of crucial importance for the quantization; Loop quantum gravity works only because of this fact. To parallel transport objects with internal indices we shall define the spin connection ω

$$D_\mu \phi^I = \partial_\mu \phi^I + \omega^I_{\mu J} \phi^J \tag{3.26}$$

$$D_\mu v^I_\nu = \partial_\mu v^I_\nu + \omega^I_{\mu J} v^J_\nu - \Gamma^\rho_{\mu\nu} v^I_\rho. \tag{3.27}$$

The metricity of the standard covariant derivative $\nabla_\mu g_{\nu\rho} = 0$ implies we must have the spin connection be tetrad-compatible

$$D_\mu e^I_\nu = 0 \Rightarrow \omega^I_{\mu J} = e^I_\nu \nabla_\mu e^\nu_J, \tag{3.28}$$

giving us the spin connection in terms of tetrad variables.

To express the Einstein-Hilbert action in terms of tetrad variables, we need to write down the Riemann tensor in terms of tetrads, which can be done through

$$R^\mu_{\nu\rho\sigma} = e^I_\rho e^J_\sigma F^\mu_{\nu IJ} \tag{3.29}$$

$$R = e^I_\mu e^J_\nu F^{\mu\nu}_{IJ} \tag{3.30}$$

$$F_{\mu\nu IJ} = \partial_\mu \omega_{\nu IJ} - \partial_\nu \omega_{\mu IJ} + \omega_{\mu IK} \omega^K_{\nu J} - \omega_{\nu IK} \omega^K_{\mu J}. \tag{3.31}$$

Note the F is a commutator of derivatives, analogous to the definition of R in terms of g. The relation between the determinants of g and e is much simpler, namely

$$g = -e^2, \tag{3.32}$$

implying the Einstein-Hilbert action eq. (3.1) is

$$S_{EH} = \int d^4x \, e e^I_\mu e^J_\nu F^{\mu\nu}_{IJ} \tag{3.33}$$

in terms of the tetrad variables. Writing e and F as forms, we see

$$F^{IJ} = d\omega^{IJ} + \omega^I_K \wedge \omega^K_J \tag{3.34}$$

$$S = \epsilon_{IJKL} \int d^4x \, e^I \wedge e^J \wedge F^{KL}, \tag{3.35}$$

with ϵ the completely anti-symmetric symbol.

One striking feature of this action is that the equations of motion are identical if we vary ω and e separately, in other words

$$S(e,\omega) = \frac{1}{2}\epsilon_{IJKL}\int d^4x\, e^I \wedge e^J \wedge F^{KL}(\omega). \tag{3.36}$$

This is due to the ω equations of motion

$$\epsilon_{IJKL} e^I \wedge d_\omega e^J = 0, \tag{3.37}$$

which simply fixes $\omega(e)$ to be of the form already found by metricity eq. (3.28). The e equations of motion will recover the Einstein equations in tetrad variables.

This action is convenient for two reasons. Firstly, only first derivatives appear in the action, therefore this provides a first order description of GR. In addition, unlike the original action, it is polynomial in the fields. However, this first-order action adds some non-trivial solutions to the original Einstein-Hilbert action. Since no inverses of the matrix e appear in the action, there may exist solutions with degenerate metrics $g = e = 0$, which is not equivalent to GR. In addition, we may add a term to the action of the form

$$S(e,\omega) = \left(\frac{1}{2}\epsilon_{IJKL} + \frac{1}{\gamma}\delta_{IJKL}\right)\int d^4x\, e^I \wedge e^J \wedge F^{KL}(\omega), \tag{3.38}$$

where γ is known as the Barbero-Immerzi parameter and $\delta_{IJKL} = \delta_{I[K}\delta_{L]J}$. Remarkably, this term not only preserves Local Lorentz transformations and diffeomorphisms, but also results in the same equations of motion! This term is irrelevant in the second order theory since when eq. (3.28) holds,

$$\delta_{IJKL} e^I \wedge e^J \wedge F^{KL}(\omega) = \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}(e) = 0. \tag{3.39}$$

While this complete action reduces to GR in the classical theory , it will have nontrivial consequences on the quantum theory.

To proceed with Dirac's quantization program, we must pass from the Lagrangian into the Hamiltonian. To do this we will perform the 3+1 splitting as before, setting up a manifold $\mathcal{M} = \mathbb{R} \times \Sigma$ and defining the lapse and shift through

$$e_0^I = Nn^I + N^a e_a^I, \quad \delta_{ij} = e_a^i e_b^j g_{ab}, \tag{3.40}$$

with n being the normal to Σ and e_a^i being the spatial components of the tetrad with $i \in \{1,2,3\}$. We now need to identify the canonically conjugated variables to perform the Legendre transform. There are two main differences with the earlier derived results that will be discussed here. Firstly, the local lorentz symmetry on the internal index of the tetrads causes there to be an additional constraint. Secondly, the conjugate variables must be taken to be functions of both e and ω in the first order formalism, as opposed to just the metric g. Due to these subtleties, the constraint algebra is no longer be first class. However, it is possible to introduce a new set of variables which incorporate a part of the new constraint and reduce the remaining algebra to a first class one. These are the Ashtekar variables [16], given by

$$E_i^a = ee_i^a \tag{3.41}$$

$$A_a^i = \gamma\omega_a^{0i} + \frac{1}{2}\epsilon_{ijk}\omega_a^{jk}, \tag{3.42}$$

referred to as the densitized tetrad and Ashtekar-Barbero connection respectively. A simple calculation reveals these new variables are in fact conjugated

$$\{A_a^i(x), E_j^b(y)\} = \gamma\delta_j^i\delta_a^b\delta^{(3)}(x-y). \tag{3.43}$$

We may now express the action eq. (3.33) using these new variables as

$$S = \frac{1}{\gamma}\int dt \int_\Sigma d^3x\, \dot{A}_a^i E_i^a - A_0^i D_a E_i^a - NH - N^a H_a, \tag{3.44}$$

with

$$G_j = D_a E_i^a = \partial_a E_j^a + \epsilon_{jkl} A_a^j E^{al}, \tag{3.45}$$

$$H_a = \frac{1}{\gamma} F_{ab}^j E_j^b - \frac{1+\gamma^2}{\gamma} K_a^i G_i, \tag{3.46}$$

$$H = \left(F_{ab}^j - (\gamma^2+1)\epsilon_{ilm} K_a^l K_b^m\right)\epsilon_{ijk} E_k^a E_l^b + \frac{1+\gamma^2}{\gamma} G^i \partial_a E_i^a, . \tag{3.47}$$

Since A_0^i appears in the action with no time derivatives, there is a third set of constraints; these appear as a result of the local Lorentz invariance. The Gauss constraint generates gauge transformations, which under the choice of $\gamma = i$ are the $su(2)$ transformations. In this thesis we will work with $\gamma = i$ to simplify the Hamiltonian and recognize the constraint algebra is $su(2)$

$$H_a = -iF_{ab}^j E_j^b, \tag{3.48}$$

$$H = \epsilon_{ijk} E_k^a E_l^b F_{ab}^j. \tag{3.49}$$

. To understand the action of the new constraint G^i, we shall look at the flows generated in phase space as was done for the scalar and vector constraints. We find

$$\{E_i^a, \int \Lambda^j G_j d^3x\} = -\epsilon_{ijk} E^{aj}\Lambda^k \tag{3.50}$$

$$\{A_a^i, \int \Lambda^j G_j d^3x\} = -(\partial_a \Lambda^i + \epsilon^i_{\ jk} A_a^j \Lambda^k)\Lambda^i. \tag{3.51}$$

This is precisely the infinitesimal action of $su(2)$ on a vector and its connection.

To proceed with the quantization is is convenient to work with smeared variables. This clever trick allows us to avoid several delta functions and is crucial for quantization. Since we are working in curved spacetime, we will need to take care in defining what types of functions we smear the Ashtekar variables over (different objects will have different indices). From the definition of E given in eq. (3.41), we see E is a 2-form, thus it is natural to smear it on a surface σ

$$E_i(S) = \int_S n_a E_i^a d^2\sigma, \tag{3.52}$$

where $n_a = \epsilon_{abc}\frac{\partial x^b}{\partial \sigma_1}\frac{\partial x^c}{\partial \sigma_2}$ is the normal to σ. This quantity is referred to as the flux of E across S.

Since the connection (3.42) is a one-form, it is natural to smear it along a curve. One clever choice of variables is to use the holonomy of A

$$U_\gamma(A)[s,t] = \mathcal{P}\text{exp}\left(\int_s^t A \cdot d\gamma\right), \tag{3.53}$$

with $\mathcal{P}$ being the path ordered product and s and t specific points on the curve γ (it is custom to define $s = 0$ to be the beginning of the curve, and $t = 1$ to be the endpoint). The physical interpretation of the holonomy is that of the finite parallel transport operator, since A is the connection. The holonomy will be critical in defining the *Loop Representation* of the theory in section 3.5.

3.4 Physical Evolution: Coupling to a Clock Field

One of the issues that arises when constructing a theory of quantum gravity is how to realize physical time evolution in the presence of a vanishing Hamiltonian. A popular trick is to introduce a scalar field T and define evolution as change with respect to it [17]. We will do this by defining the Hamiltonian constraint to be

$$C = \frac{1}{2}\pi^2 + \frac{1}{2}E_i^a E^{bi}\partial_a T \partial_b T + \text{det}(E)V(T) + C_{\text{grav}} \tag{3.54}$$

with

$$C_{\text{grav}} = H + \Lambda q := C_{\text{Einst}} + \Lambda q, \tag{3.55}$$

where here we have introduced a cosmological constant to the theory and defined π the be the momentum conjugate to T . If we set the potential $V(T) = 0$, then

$$\mathcal{E} \equiv \int_\Sigma d^3x\, \pi(x) \tag{3.56}$$

is a constant of motion. This generates the symmetry $T(x) \to T(x) + const.$ as

$$\{C(N), \mathcal{E}\} = \{D(v), \mathcal{E}\} = 0. \tag{3.57}$$

We shall restrict the time by gauge fixing $\partial_a T = 0$, or equivalently we choose the gauge fixing functional

$$\mathcal{F}(x) = T(x) - \tau \tag{3.58}$$

Our gauge choice fixes the Lapse as can be seen from the bracket

$$(\dot{\partial_a T})(x) = \{C(N), \partial_a T(x)\} = \partial_a (N(x)\pi(x)) = 0, \tag{3.59}$$

thus giving

$$N(x) = \frac{a}{\pi(x)}, \tag{3.60}$$

where a is an arbitrary constant.

The interpretation of eq. (3.60) is that imposing the gauge condition fixes the infinitude of smeared Hamiltonian constraints to just one, with lapse given by eq. (3.60). However, since we must have eliminated the nonconstant piece of T by gauge fixing eq. (3.58), we must solve the constraints which have been so broken to eliminate the fields which are conjugate to them. Thus all but one of the degrees of freedom in π must be fixed by solving the Hamiltonian constraint; the remaining degree of freedom may be chosen to be the constant of motion $\mathcal{E}$. Thus π is fixed by

$$\pi(x) = \sqrt{-2C_{\text{grav}}} \tag{3.61}$$

and the remaining Hamiltonian constraint is

$$\int_\Sigma \frac{aC}{\pi} = a \int_\Sigma \frac{\pi}{\sqrt{2}} + \sqrt{-C_{\text{grav}}} = 0. \tag{3.62}$$

It is when looking at the quantum theory that we see the true power of the scalar field T. To quantize, we identify

$$\hat{\tau}\Psi = \tau\Psi, \;\; \hat{\mathcal{E}}\Psi = -i\hbar\frac{\partial}{\partial\tau}\Psi. \tag{3.63}$$

Using this prescription the Hamiltonian constraint eq. (3.62) takes the form

$$i\hbar\frac{\partial}{\partial\tau}\Psi = \int_\Sigma d^3x\sqrt{-2C_{\text{grav}}}\Psi, \tag{3.64}$$

and at last we arrive at the time evolution given by a schrodinger-like equation. This is the most fundamental equation of the quantum theory, so a few comments are needed. Firstly, we need to take care when defining the square root of an operator. One technique to remove the square root, which we apply to our calculation, is to perform an expansion in the cosmological constant Λ. However, on a more serious note, one may notice there are operator products occurring in eq. (3.64), and thus we need to define a suitable regularization process to define this operator. One needs to take care to ensure this regularized operator is still diffeomorphism invariant and background independent, even though the regularization procedure will need to explicitly break these symmetries. We carry out this process in section 3.5.

3.5 Loop Representation

In this section we will introduce a new representation for the quantum theory, namely the loop representation, given by transforming the states

$$\Psi[\gamma] = \int d\mu(A)\, T[\gamma, A]\Psi[A], \tag{3.65}$$

with $T[\gamma, A]$ being the trace of the Holonomy introduced earlier in eq. (3.53)

$$T[\gamma, A] = \text{Tr}U_{\gamma}(0,1). \tag{3.66}$$

As a side-note, this representation is general enough to be applied to any quantum field theory with a connection A [18] . One of the most interesting aspects of using this representation is that it allows us to construct exact solutions the quantum constraint equations [19]! In addition the diffeomorphism constraint has a natural representation at the quantum level. We will use the loop representation as it allows us to naturally define the regularized quantum operators. I will outline how to do the second task here and the third task in this section, for more information on the first see [12, 19].

We can write an unbroken representation of the group of diffeomorphisms in the loop representation by

$$U(\phi)\Psi[\gamma] = \Psi[\phi^{-1} \circ \gamma]. \tag{3.67}$$

The diffeomorphism constraints are represented as

$$D(v)\Phi[\alpha] = \frac{d}{dt}U(\phi_t)\Phi[\alpha], \tag{3.68}$$

with ϕ_t the family of diffeomorphisms generated by the vector v through the Lie derivative. Solving the diffeomorphism constraint is now a trivial process: solutions are given by states defined over diffeomorphism invariant classes $\Phi[\{\alpha\}]$, where $\{\alpha\}$ is the diffeomorphism invariant class of the loop α.

Often one wants to regularize a product of two operators at a point. The most basic example is the metric

$$q^{ab} = E^{ai}E^b_i. \tag{3.69}$$

For reasons we will touch upon later, the metric at a point is not a good quantum operator, but to see this we shall first introduce a regularization procedure. The goal is to introduce point splitting into loop quantum gravity, such as writing

$$q^{ab} = \lim_{x \to y} E^{ai}(x)E^b_i(y). \tag{3.70}$$

Since we want to preserve the internal $su(2)$ gauge invariance we need to construct a gauge invariant version of point splitting. To do this, we need to define a set of observables which generalize the trace of the holonomy eq. (3.66). Given a loop γ, we may select n points on it $s_1, \ldots, s_n$ to define the T^n observables

$$T^{a_1 \ldots a_n}[\gamma](s_1, \ldots, s_n) = \text{Tr}\left[E^{a_1}(\gamma(s_1))U_\gamma(s_1, s_2) \ldots E^{a_n}(\gamma(s_n))U_\gamma(s_n, s_1)\right]. \tag{3.71}$$

Before writing down the action of the T^n on a state Ψ, we shall introduce a few new pieces of notation. If there is more than one intersection point between loops α and β, we shall denote $\alpha \circ_s \beta$ to be the loop which combines α and β at the intersection point labelled by the parameter s of the loop, else $\alpha \circ_s \beta$ will reduce to the trivial loop. If the loops intersect at more than one point, we may break the two loops at more than one intersection and stiching the together in the various reroutings. We will denote the loops formed by breaking and joining α and β at the two intersections with rerouting r as $(\alpha \circ_{s_1} \circ_{s_2} \beta)_r$. The action of the T's on the states may be verified to be

$$\hat{T}^{a_1 \ldots a_n}[\gamma](s_1, \ldots, s_n)\Psi[\alpha] = \tag{3.72}$$

$$l_P^{2n} \oint dt_1 \delta^3(\gamma(t_1), \alpha(s_1))\dot{\alpha}(s_1)^{a_1} \times \ldots \oint dt_n \delta^3(\gamma(t_n), \alpha(s_n))\dot{\alpha}(s_n)^{a_n} \left(\sum_r (-1)^{q_r} \Psi\left[\gamma \circ_{t_1} \cdots \circ_{t_n} \alpha\right]\right),$$

where q_r is the number of segments along the curve that must be flipped to give the total curve a constant orientation and the factor of l_P^{2n}. To help simplify discussion, we shall refer to the points where the E's are represented on the loop of the operator as the "hands" of the operator, and that a hand acts by "grasping" a loop in the state. The effect of a grasp is to multiply the state by a distributional factor of

$$l_P^2 \int dt\, \delta^{(3)}(\gamma(s), \alpha(t))\dot{\alpha}^a(t). \tag{3.73}$$

From here we see the action of the T vanishes unless every hand grasps the loop α. The strategy of the regularization procedure is to represent q as a limit of a T^2 operator as the points where the E's act approach each other. This regulated observable will depend on a loop γ which contains these two points. The choice of loop is arbitrary, for simplicity we may choose a uniform way of defining it.

The choice of loop will be dependant on an introduction of an arbitrary flat background metric h^0_{ab} in a neighborhood of x. Given any two points y and z in the same neighborhood of x, we shall define a loop $\gamma_{y,z}$ which is a circle in the selected background metric and satisfies $\gamma_{y,z}(0) = y, \gamma_{y,z}(\pi) = z$.

In addition to this piece, we will also introduce a smearing function

$$f^\delta(x, y) = \frac{3}{4\pi\delta^3}\Theta(\delta - |x - y|) \tag{3.74}$$

to regulate distributional products. Notice f satisfies $\lim_{\delta \to 0} f^\delta(x, y) = \delta^{(3)}(x, y)$. The regularized metric is now

$$G^{ab}_\epsilon(x) = \int d^3y \int d^3z f_\epsilon(x, y) f_\epsilon(x, z) T^{ab}[y, z]. \tag{3.75}$$

As an aside to constructing the regulated Hamiltonian, we shall discuss why the metric at a point is not a valid quantum operator. We can evaluate the limit of the hands in the T^2 operator to be

$$\lim_{\delta\to 0}\left[\oint ds \oint dt\, \dot{\alpha}^a(s)\dot{\alpha}^b(t) f_\delta(x,\alpha(s)) f_\delta(x,\alpha(t))\right] \tag{3.76}$$

$$= \frac{l_P^4\sqrt{h^0(x)}}{\delta^2}\int \frac{s}{|\dot{\alpha(s)}|}\delta^{(3)}(x,\alpha(s))\dot{\alpha}^a(s)\dot{\alpha}^b(s). \tag{3.77}$$

The divergence arises naturally from attempting to multiply two distributions. To properly define the product, we must modify the definition of the limit. We shall define the renormalized observable

$$\hat{G}^{ab}_{\text{ren}}(x) \equiv \lim_{\epsilon\to 0} Z\frac{\epsilon^2}{l_P^2}\hat{G}^{ab}_\epsilon(x), \tag{3.78}$$

with Z an arbitrary renormalization constant. This limit is finite, and the action on a state is now

$$\hat{G}^{ab}_{\text{ren}}(x)\Psi[\alpha] = 6l_P^2\sqrt{h^0(x)}Z\int \frac{ds}{|\dot{\alpha}(s)|}\delta^{(3)}(x,\alpha(s))\dot{\alpha}^a(s)\dot{\alpha}^b(s)\Psi[\alpha]. \tag{3.79}$$

While this operator does represent the inverse metric in the loop representation, the operator is not background independent. The background dependence can be seen both from the factor of $|\dot{\alpha}| = \sqrt{h^0_{ab}\dot{\alpha}^a\dot{\alpha}^b}$ and the overall factor of $\sqrt{h^0}$. Since this metric was arbitrary, we've only determined q^{ab} up to a conformal factor. One could worry the breaking of background independence is a general feature of the regularization feature. However, one can construct good quantum operators, such as the Area or Volume operators. We will not do such here as it is not directly related to our results, but is an essential feature of Loop Quantum Gravity and I encourage the reader to learn more about them if not already acquainted with them. However, we shall take use of the explicit form of the volume operator, acting on a region $\mathcal{R}$,

$$V = \int_{\mathcal{R}} d^3x\sqrt{\text{det}E^{ai}(x)}. \tag{3.80}$$

Using the procedure sketched out above, we may define the regularized Hamiltonian as

$$\hat{H} = \lim_{L\to 0, A\to 0, \delta\to 0}\sum_I L^3\sqrt{-\hat{C}^{L,\delta,A}_{\text{Einst}} + \Lambda\hat{q}^L_I}, \tag{3.81}$$

with

$$\hat{C}^{L,\delta,A}_{\text{Einst}} = \frac{1}{2L^3A}\int_I d^3x\int d^3y\int d^3z f^\delta(x,y)f^\delta(x,z)\sum_{a<b}\left[\hat{T}^{\hat{a}\hat{b}}[\gamma_{xyz}\circ\gamma^A_{x\hat{a}\hat{b}}](y,z) + \hat{T}^{\hat{a}\hat{b}}[\gamma_{xyz}\circ{\gamma^A_{x\hat{a}\hat{b}}}^{-1}](y,z)\right], \tag{3.82}$$

$$\hat{q}^L_I = \frac{1}{10^6}\sum_{\hat{a}<\hat{b}<\hat{c}}\int_{I\hat{a}} d^2S_a(\sigma_1)\int_{I\hat{b}} d^2S_b(\sigma_2)\int_{I\hat{c}} d^2S_c(\sigma_3)\hat{T}^{abc}(\sigma_1,\sigma_2,\sigma_3), \tag{3.83}$$

where $\gamma^A_{a\hat{a}\hat{b}}$ is a circle with area A (defined in the background metric) based at point x in the $\hat{a}\hat{b}$ plane, and the surface integrals are over the faces (labelled by $\hat{a}, \hat{b}, \hat{c}$) of the cube I. It is simple to check eq. (3.82) reduces to the classical Hamiltonian constraint when the limits are taken. Performing this regularization, it has been shown that the action of the Hamiltonian is [17]

$$\hat{H}\Psi[\alpha] = \sum_i \sqrt{\mathcal{M}_i}\,\Psi[\alpha], \tag{3.84}$$

with the $\mathcal{M}$ being obtained through the regularization process and the sum being over intersections of the graph α. The exact details of its form are not of dramatic importance, however it turns out the $\sqrt{\mathcal{M}}$ is both background independent and diffeomorphism invariant. This section concludes all of the key aspects of LQG without matter needed for our calculations. We will talk briefly about coupling matter to the theory in the final section 3.6.

3.6 Coupling Fermions

In this section, we will briefly review the role of matter in the above calculations. For more details see [20].

The Hamiltonian and diffeomorphism constraints coupled to a free fermion (H from chapter 2 vanishes) take the form

$$C = C_{\text{grav}} + C_T + C_\psi, \quad D_a = D^{\text{grav}}_a + D^T_a + D^\psi_a, \quad G_{AB} = G^{\text{grav}}_{AB} + G^\psi_{AB}, \tag{3.85}$$

with

$$C_\psi = \pi^\alpha_A E^{aA}_{\ \ B}(D_a\psi)^B_\alpha, \qquad D^\psi_a = \pi^\alpha_A (D_a\psi)^A_\alpha, \qquad G^\psi_{AB} = \pi_{(A}\psi_{B)}, \tag{3.86}$$

where ψ is a spinor, $E^a_{AB} = E^a_i \tau^i_{AB}$ with $\tau_i = \frac{1}{2}\sigma_i$ the generators of $su(2)$, and all constraints with the superscript "grav" are the gravitational counterparts from eq.s (3.45, 3.46, 3.47). We modify the loop transform to take the form

$$\Psi[\gamma] = \int DAD\psi\, \Psi_\gamma[A,\psi]\Psi[A,\psi], \tag{3.87}$$

with (3.88)

$$\Psi_\gamma[A,\psi] = \bar{\psi}^A(\gamma_i) U^{\ B}_A[\gamma]\psi_B(\gamma_f), \tag{3.89}$$

where U is again the holonomy. We find the following variable

$$Y^a[\alpha](s) = \pi^A(\alpha_i)\, U^{\ B}_A[\alpha](0,s) E^{a\ C}_B U^{\ D}_C[\alpha](s,1)\, \psi_D(\alpha_f) \tag{3.90}$$

exceptionally useful in defining the regularized Hamiltonian.

The regularized Hamiltonian takes the form

$$H_{LA\delta\tau} = \sum_I L^3 \sqrt{-C^{L\delta A}_{\text{Einst}\,I} - C^{L\tau\delta}_{\psi\,I}}, \tag{3.91}$$

with

$$C^{L\tau\delta}_{\psi\,I} = \frac{1}{L^3}\frac{3}{\tau}\int_I d^3x d^3y\, f^\delta(0,y)\,\frac{y^a}{|\vec{y}|}\,Y^a[\gamma^\tau_{\vec{x}\vec{y}}]\left(\frac{|y|}{\tau}\right). \tag{3.92}$$

Taking the appropriate limits, the Hamiltonian may be written as

$$\hat{H} = \sum_{\substack{\text{intersections}\, i \\ \text{edges}\, e}} \sqrt{\hat{M}_i + \lambda \hat{F}_e}, \tag{3.93}$$

where λ is a free dimensionless constant emerging from the regularization procedure.

In our proof, we will need the action of C^Ψ around a fixed background state Γ, which can be shown to be

$$H^{\Psi GT}_\Gamma = \sum_{(n,\hat{a})} \pi(n)^\alpha_A\, w^a_{n,\hat{a}} \sigma^{AB}_a V(n,\hat{a})_\alpha{}^\beta\, \Psi(n+\hat{a})_{B\beta}, \tag{3.94}$$

where $\omega^a_{n,\hat{a}}$ is tangent to the edge $(n,\hat{a})$ at n and unit in the background metric

$$q^{ab} = E^{ai} E^b_i. \tag{3.95}$$

To understand this result, we look at the fermion operator Y^a. The π will act at a node, ψ acts at neighboring nodes, there's a parallel transport required to connect these, and the E expands into $\omega\sigma$ in the low energy limit since it's valued under the $su(2)$ algebra. We now have all of the resources needed to demonstrate fermion doubling in LQG.

Chapter 4

Fermion Doubling in Loop Quantum Gravity

In this chapter, we will work out the remaining items 2 to 4 of the program introduced in section 1.1. The main trick is to perform a Born-Oppenheimer like expansion around the gravitational degrees of freedom of LQG to create a background graph Γ, then show the spectrum of this theory is equivalent to that of a lattice gauge theory which doubles. To demonstrate this equivalence, we will define degravitating and dressing maps, which carry the Hilbert spaces and Hamiltonians of the two theories into each other. Two theories with identical spectra must contain an equivalent set of emergent Weyl fermions in the low energy limit. To reduce quantum gravity to lattice gauge theory, we shall work in the Hamiltonian framework of LQG, and study a model where quantum gravity is coupled to a scalar "clock" field T, a Yang-Mills theory with compact gauge group G, and a multiplet of fermion fields in a representation r of G [20]. The fermion fields are represented by two component spinors $\psi^{A\alpha}$ with conjugate momenta $\pi_{A\alpha}$, where $A = 0, 1$ labels components of a Weyl spinor and α, which may sometimes be suppressed, labels the basis of the representation r_0. For this theory, the quantum Hamiltonian constraint becomes (after gauge fixing the gauge field to constant slices $T(x) = \tau$)

$$i\hbar\frac{\partial}{\partial\tau}\Psi = \int_\Sigma d^3x\sqrt{-2[C_{\text{grav}} + C_\psi]}\Psi, \tag{4.1}$$

which is just the analog of eq. (3.64). We will define $\mathcal{W}$ to be the regulated limit of the right-hand side

$$\mathcal{W} = \lim_{L,A,\delta,\tau\to 0}\int_\Sigma d^3x\sqrt{-2[C^{LA\delta}_{\text{grav}} + C^{L\tau\delta}_{\psi}]}. \tag{4.2}$$

4.1 Classical Born-Oppenheimer Approximation

In this section, we will expand the square root in the operator (4.7) for later convenience. To do this, we separate out the cosmological constant term

$$\hat{\mathcal{C}}_{\text{grav}} = -\text{det}(E)\Lambda + \hat{\mathcal{C}}_{\text{Einst}}, \tag{4.3}$$

and perform an expansion around large Λ. Initially, this may look like a very poor expansion. Λ is the infrared cutoff [21, 22] of the theory, and expansion around a large Λ suggests we are taking the infrared cutoff to be larger than the ultraviolet cutoff. However, this is an expansion around the bare cosmological constant, which we already know to be large by the standard naturalness arguments. The particular value of Λ takes no role in our proof, as we will show in section 4.5. The expansion of $\mathcal{W}$ for large Λ is now (we will write $\epsilon \to 0$ to mean all of $L, A, \delta, \tau \to 0$)

$$\begin{aligned}
\hat{\mathcal{W}} &= \lim_{\epsilon\to 0}\int_\Sigma d^3x\sqrt{\Lambda\text{det}(E) - 2[\hat{\mathcal{C}}^\epsilon_{\text{Einst}}(x) + \hat{\mathcal{C}}^\epsilon_\psi(x)]} && (4.4)\\
&= \lim_{\epsilon\to 0}\int_\Sigma d^3x\sqrt{\Lambda\text{det}(E)\left(1 - \frac{2}{\Lambda}\frac{\hat{\mathcal{C}}^\epsilon_{\text{Einst}}(x)}{\text{det}(E)} - \frac{2}{\Lambda}\frac{\hat{\mathcal{C}}^\epsilon_\psi(x)}{\text{det}(E)}\right)} && (4.5)\\
&= \lim_{\epsilon\to 0}\int_\Sigma d^3x\sqrt{\Lambda}\left[\sqrt{\text{det}(E)} - \frac{1}{\Lambda}\frac{\hat{\mathcal{C}}^\epsilon_{\text{Einst}}(x)}{\text{det}(E)^{\frac{1}{2}}} - \frac{1}{\Lambda}\frac{\hat{\mathcal{C}}^\epsilon_\psi(x)}{\text{det}(E)^{\frac{1}{2}}} + O\left(\frac{1}{\Lambda^2}\right)\right] && (4.6)\\
&= \sqrt{\Lambda}V - \mathcal{W}^{eff}, && (4.7)
\end{aligned}$$

where the effective Hamiltonian $\mathcal{W}^{eff}$ is

$$\mathcal{W}^{eff} = \lim_{\epsilon\to 0}\frac{1}{\sqrt{\Lambda}}\int_\Sigma d^3x\frac{1}{\sqrt{\text{det}(E)}}\left(\hat{\mathcal{C}}^\epsilon_{\text{Einst}}(x) + \hat{\mathcal{C}}^\epsilon_\psi(x)\right) + O\left(\frac{1}{\Lambda^{3/2}}\right). \tag{4.8}$$

We can formulate the extension of the Nielsen-Ninomiya no-go theorem to LQG by establishing a map between the Hilbert spaces of two theories: $LQG\Psi$, loop quantum gravity with chiral fermions (and possible Yang-Mills fields), and the lattice gauge theory for fermions and Yang-Mills fields without gravity.

4.2 The Quantum Gravity Hilbert Space

In this section, we shall review some key properties about the quantization. For simplicity, we shall fix the manifold Σ to be the torus T^3. The Hilbert space of loop quantum gravity coupled to fermions and possible Yang-Mills fields will be denoted

$$\mathcal{H}^{QG\Psi}. \tag{4.9}$$

This is the spatially diffeomorphism invariant physical Hilbert space based on gauge fixing to a constant clock field T, as described in section 3.4. On this Hilbert space, we will impose the $SU(2)$/diffeomorphism and G constraints. $\mathcal{H}^{QG\Psi}$ has a physical and spatially diffeomorphism invariant inner product.

$\mathcal{H}^{QG\Psi}$ is decomposable in terms of diffeomorphism invariant classes of embeddings of a graph Γ into Σ

$$\mathcal{H}^{QG\Psi} = \oplus_{\Gamma} \mathcal{H}_{\Gamma}^{QG\Psi}. \tag{4.10}$$

Each component $\mathcal{H}_{\Gamma}^{QG\Psi}$ consists of extended spin network states, with labels corresponding to gravity, fermions, and Yang-Mills fields, diffeomorphic to Γ. One basis to the Hilbert space is the span of all spin-network states [23] of the form

$$|j, i, r, c, \psi\rangle\,, \tag{4.11}$$

where j is the spin of an irreducible representation of the gravitational $SU(2)$ associated to each edge of Γ, i is an $SU(2)$ intertwiner associated to each node of Γ, while r and c are representations and intertwiners of the Yang-Mills gauge group G associated with edges and nodes respectively. Each node of the graph carries a basis state of the Hilbert space for fermions at a site, carrying a finite dimensional representation of $SU(2) \times G$, spanned by the π and ψ operators.

(Technical point, we impose equivalence under point wise smooth diffeomorphisms, so that there are no invariant diffeomorphism invariants characterizing classes of intersections with valence 5 or greater.)

On $\mathcal{H}^{QG\Psi}$, there is a physical Hamiltonian corresponding to the classical operator $\mathcal{W}$ (given by eq. (4.7)), obtained by gauge fixing the clock field to the $T =$ constant gauge. We must now choose a regularization scheme. For simplicity we shall consider a *graph preserving regularization*, where $\mathcal{W}$ decomposes into separate actions on each graph sector

$$\mathcal{W} = \sum_{\Gamma} \mathcal{W}_{\Gamma}, \tag{4.12}$$

where each $\mathcal{W}_{\Gamma}$ acts only on the corresponding $H_{\Gamma}^{QG\Psi}$. We expect the results to be independent of this choice. We shall talk about this regularization choice in section 4.4

Also defined on each $\mathcal{H}_{\Gamma}^{QG\Psi}$ are algebras of observables, $\mathcal{O}_{\Gamma}^{QG\Psi}$, these are defined by loop and flux operators for $SU(2)$ and G, together with the fermion operators X, Y, etc, restricted to Γ.

Given a choice of graph Γ, there is also a lattice gauge theory involving the same fermion and Yang-Mills fields. The basis states are obtained by simply removing the gravitational degrees of freedom,

$$|r, c, \psi\rangle\,. \tag{4.13}$$

The lattice gauge theory Hilbert space $\mathcal{H}_{\Gamma}^{GT}$, contains an algebra of operators, $\mathcal{O}_{\Gamma}^{GT}$, an inner product, and a Hamiltonian H_{Γ}^{GT}.

To summarize, the gravitational, gauge and fermion fields are all represented by a decorated spin network Γ, whose edged are labelled by (j, r), representations of $SU(2)\times G$, and whose nodes are labelled by intertwiners coming in one of the two types. In the next section, we will discuss the mapping between these two theories.

4.3 Mapping Quantum Gravity to Lattice Gauge Theory

In this section, we shall define maps between the two theories for each Γ, with the goal being to identify the spectrum of LQG based on a graph Γ with a lattice gauge theory. The simplest map to define is the degravitating map

$$\mathcal{G}_\Gamma : \mathcal{H}_\Gamma^{QG} \to \mathcal{H}_\Gamma^{GT}, \tag{4.14}$$

which simply removes the labels corresponding to the gravitational fields

$$\mathcal{G}_\Gamma \circ |j, i, r, c, \psi\rangle = |r, c, \psi\rangle \,. \tag{4.15}$$

To go in the other direction we define dressing maps

$$\mathcal{F}_\Gamma : \mathcal{H}_\Gamma^{GT} \to \mathcal{H}_\Gamma^{QG}, \tag{4.16}$$

which takes a state of the lattice gauge theory and dresses it with gravitational fields; this is defined by a choice of amplitudes $\phi_\Gamma[j,i;r,c,\psi]$ so that

$$\mathcal{F}_\Gamma \circ |r, c, \psi\rangle = \sum_{j,i} \phi_\Gamma[j,i;r,c,\psi]\,|j, i, r, c, \psi\rangle \,. \tag{4.17}$$

The relevant choice of these amplitudes will be clear in section 4.5.

We require that

$$\mathcal{G}_\Gamma \circ \mathcal{F}_\Gamma = I \tag{4.18}$$

for the degravitaing map to be the "inverse" of the dressing map ($\mathcal{G}_\Gamma$ is onto, whereas $\mathcal{F}_\Gamma$ is into a subspace).

4.4 Regulating the Hamiltonian

We need to define regularizations of the different terms in the Hamiltonian on the fixed graph subspaces $\mathcal{H}_\Gamma^{QG}$ (these are different than those previously mentioned since we are regulating on a fixed graph).

Let us start with the fermion term in the time-gauge fixed Hamiltonian eq. (3.86)

$$H^\Psi = \int_\Sigma \frac{1}{e}\mathcal{C}^\psi = \int_\Sigma \frac{1}{e}\,\tilde{\Pi}^\alpha_A \tilde{E}^{aA}{}_B (\mathcal{D}_a\Psi)^B_\alpha \tag{4.19}$$

Acting on a state Φ_Γ in $\mathcal{H}_\Gamma^{QG}$, we want to regulate this as

$$H^\Psi \to H^\Psi_{reg\ \Gamma} = \sum_{(n,\hat{a})} \Pi(n)^\alpha_A \, E[S(n,\hat{a})]^A{}_B U(n,\hat{a})_B{}^C V(n,\hat{a})_\alpha{}^\beta \, \Psi(n+\hat{a})_{C\beta}, \tag{4.20}$$

with n being the set of nodes and $\hat{a}$ being the corresponding adjacent vertices. Here $E(S)_{AB}$ is the E^a_{AB} smeared over a two surface S with base point p defined by

$$E(S)_{AB} = \int_S d^2S(\sigma^1,\sigma^2)_a \, \tilde{E}[S(\sigma)]^a_{CD} \, U(\gamma_{p,S(\sigma)})_A^{\ C} U(\gamma^{-1}_{p,S(\sigma)})_B^{\ D}, \tag{4.21}$$

where $\gamma_{p,S(\sigma)}$ is an arbitrary, non intersecting curve in S connecting the base point p with the point $S(\sigma)$. We pick the surfaces $S(n,\hat{a})$ adopted to the graph Γ so that for each node n and adjacent edge $(n,\hat{a})$, $S(n,\hat{a})$ is a surface that crosses that edge once infinitesimally close to the node n base pointed at the intersection point. In addition, $U(n,\hat{a})_B{}^C$ is the $SU(2)$ parallel transport over the edge $(n,\hat{a})$, and $V(n,\hat{a})_\alpha{}^\beta$ is the same for the Yang-Mills gauge group G. The specific action of the curves and surfaces are crucial to the definition of the regularization.

We also have to define the fermion momentum operator. Let $\mathcal{H}^\Psi_n$ be the finite dimensional Hilbert space of fermions at the node n, which has a momentum operator π^A_n, we have, when acting on a state in $\mathcal{H}^{QG}_\Gamma$,

$$\int_\Sigma d^3x \, s^A(x)\tilde{\Pi}_a(x) = \sum_n s^A(n)\pi^A_n. \tag{4.22}$$

4.5 Lattice based Born-Oppenheimer Approximation

We now apply the tools discussed earlier in this chapter to identify loop quantum gravity with lattice gauge theory. As mentioned in the outline, we need to perform a Born-Oppenheimer approximation to expand around the gravitational degrees of freedom and work on one particular graph. To do this, we need to define the appropriate background state. Since Γ is fixed, we start by defining a class of states which have support on only on Γ

$$\Psi_\Gamma[\Delta,i,j] = \langle 0,\Gamma|\Delta,i,j\rangle = \delta_{\Gamma\Delta} f(i,j). \tag{4.23}$$

We must define the background state $\Psi_0[\Gamma]$ in this form. We must now choose the particular dependence of the state on the spins j,i. The dependence on the intertwiners i and j is chosen so that the state is an eigenvector of the Hamiltonian constraint with the smallest positive energy

$$\lim_{\epsilon\to 0} \langle 0,\Gamma| \frac{\hat{\mathcal{C}}^\epsilon_{\text{grav}}(x)}{\det(E)^{\frac{1}{2}}} |0,\Gamma\rangle = E_{\text{min}}. \tag{4.24}$$

In addition, to replicate translation invariance, we will quantize the volume operator

$$\lim_{\epsilon\to 0} \langle 0,\Gamma| \int_\Sigma \sqrt{\det(E)} |0,\Gamma\rangle = v. \tag{4.25}$$

The wave function which satisfies these properties will be referred to as $\Psi_0[\Gamma]$. As is done in atomic physics, we use this choice of background state to perform the expansion

$$\Psi[\Gamma] = e^{\imath \frac{\sqrt{\Lambda}}{c} v} \Psi_0[\Gamma] \chi[\Gamma], \tag{4.26}$$

where the eigenvalue problem for Ψ_0 fixes Γ, i, j, c is the coupling constant of the theory, and χ is a wave function over the fermionic degrees of freedom $|r, c, \psi\rangle$. We point out a subtle issue with the cosmological constant here: we may alter Λ and v in such a way as to preserve the following result. The value of v is of no physical importance to us, so instead of taking the IR cutoff to be larger than the UV cutoff, we may add appropriate powers of the UV cutoff to the cosmological constant without distrubing any of our results. To first order in the Born-Oppenheimer approximation, we neglect $\frac{\partial \Psi_0[\Gamma]}{d\tau}$, and the resulting Schrodinger equation on a background state is

$$\frac{\imath\hbar\partial\chi[\{\Gamma\}, \tau]}{\partial\tau} = H^{\text{matter}} \chi[\{\Gamma\}, \tau] \tag{4.27}$$

with

$$H^{\text{matter}} = \frac{1}{c\sqrt{\Lambda}} E_{\text{min}} + \frac{1}{c\sqrt{\Lambda}} \lim_{\epsilon \to 0} \langle 0, \Gamma | \int_{\Sigma} d^3x \, \frac{1}{\sqrt{\det(E)}} \, \hat{\mathcal{C}}^{\epsilon}_{\psi, ren}(x) |0, \Gamma\rangle \, . \tag{4.28}$$

The constant E_{min} may be subtracted without changing the results. At this stage, we perform the regularization in section 4.4. The resulting LGT Hamiltonian is

$$H^{\Psi GT}_{\Gamma} = \sum_{(n,\hat{a})} \pi(n)^{\alpha}_{A} w^{a}_{n,\hat{a}} \sigma^{AB}_{a} V(n,\hat{a})_{\alpha}{}^{\beta} \Psi(n+\hat{a})_{B\beta}, \tag{4.29}$$

where $w^{a}_{n,\hat{a}}$ is tangent to the edge $(n, \hat{a})$ at n and unit in the background metric,

$$q^{0}_{ab} = \sigma^{AB}_{a} \sigma_{b\ AB}, \tag{4.30}$$

as can be seen from the action of the Y^a operator on a fixed background. We have now demonstrated one direction of the equivalence of the LQG and LGT theories, namely the degravitating one. We must now show the other direction, the dressing map, gives the same spectrum. Once this is complete, we will have shown these two theories are equivalent, hence LQG doubles.

To perform the mapping, we need to define, for each graph subspace $\mathcal{H}^{QG}_{\Gamma}$, the appropriate degrees of freedom for the dressing map

$$\mathcal{F}^{0}_{\Gamma} : \chi \in \mathcal{H}^{GT}_{\Gamma} \to \chi \times \Phi^{0}_{\Gamma} \in \mathcal{H}^{QG}_{\Gamma}. \tag{4.31}$$

We insist the dressing state Φ^{0}_{Γ} satisfies

$$\langle \Phi^{0}_{\Gamma} | E[S(n,\hat{a})]^{AB} | \Phi^{0}_{\Gamma} \rangle = w^{a}_{n,\hat{a}} \sigma^{AB}_{a} \tag{4.32}$$

and

$$\langle \Phi^{0}_{\Gamma} | U(n,\hat{a})_{B}{}^{C} | \Phi^{0}_{\Gamma} \rangle = \delta_{B}{}^{C}, \tag{4.33}$$

In addition, we also require

$$\langle \Phi^0_\Gamma | \frac{1}{e} | \Phi^0_\Gamma \rangle = \frac{1}{e^0} = 1 \tag{4.34}$$

to ensure the volumes of cells are equal, as before. Notice we are setting a tensor density with weight one equal to a constant, thus this will only work in some frames.

We note that the $SU(2)$ Gauss's law constraint is not satisfied.

Once we have defined the dressing and degravitating maps with the above choice of amplitudes, it is easy to show the spectra between the Born-Oppenheimer expanded LQG and the corresponding Lattice Gauge theory are identical

$$\langle \chi | \langle \Phi^0_\Gamma | H^{\Psi}_{reg\ \Gamma} | \Phi^0_\Gamma \rangle | \chi \rangle = \langle \chi | H^{\Psi GT}_\Gamma | \chi \rangle \,, \tag{4.35}$$

This concludes the proof. In summary, we have performed a number of approximations to transform a theory of loop quantum gravity on a background graph into a lattice gauge theory. Upon reading this proof, one may worry about issues such as backreaction. However, it is critical to notice fermion doubling is a low energy phenomena. The left and right handed "limits" of a particle only make sense when the lattice spacing is taken to zero, in LQG the case is identical only we should be thinking in terms of the Planck Scale. What makes the proof in LQG a nontrivial exercise is the notion of a background graph Γ: this is a purely gravitational phenomena and disappears when the Planck length is taken to zero. We must use the background graph for the question of fermion doubling to take any meaning. In addition, the inner product is not the most clearly defined object here, and the concept of momentum space is rather vague. These two points justify our expansions around a background graph and hint towards a mapping to a lattice gauge theory where we know how fermion doubling arises.

4.6 Outlook

In this thesis, we have shown the spectrum of loop quantum gravity coupled to free fermions and Yang-Mills fields doubles. This provides an obvious contradiction with well known particle physics, as the weak interactions only couple to the left handed sector of the standard model. In order to connect loop quantum gravity to well-established observations, we need to modify some of the assumptions put into this work.

The main assumptions can be summarized as follows:

1. Assume loop quantization yields the physical theory of quantum gravity.

2. Assume the bare cosmological constant Λ is large enough so that the square root in the Hamiltonian constraint can be expanded.

3. The Born-Oppenheimer approximation is a reasonable assumption: The variation of gravitational fields does not effect the statement of fermion doubling.

4. Assume the interactions between the fermions are local and Hermitian.
5. Assume the momentum space is compact.
6. Assert chiral invariance has the same definition for both lattice based theories and continuum theories, namely $\{D, \gamma_5\} = 0$, with D the Dirac operator.
7. Assume the action is quadratic in the fermionic fields.

Aside from inventing a new theory of quantum gravity, there is not much that can be done about assumption one; future LQG research should be aimed at modifying the theory in a way that avoids fermion doubling. We believe the second and third assumptions are reasonable. In particular, fermion doubling is a low energy artifact of the theory, where it should be safe enough to ignore Plack-scale physics, and the cosmological constant can be added and subtracted at will. Instead, the author suspects the problem is due to the choices related to the matter sector. Below, we will discuss several possible strategies to avoid fermion doubling.

One of the requirements for the Nielsen-Ninomiya theorem to hold is that the fermion Hamiltonian must be quadratic in the fields. In standard QFT, this is a perfectly reasonable assumption to make as higher order terms are irrelevant; in 4D the coefficients of a higher order term in the fields has a positive mass dimension, hence they're irrelevant. However, in a lattice theory there is a natural cutoff provided, and in loop quantum gravity UV divergences are completely avoided, so in principle such terms can be added to the action at will. One possibility is that there may be a higher order action which does not double and is suitable for loop quantization.

Another possibility to cure the problem would be to adapt known techniques from lattice gauge theory, such as the Ginsparg-Wilson technique, among others (these include Domain-Wall Fermions, Overlap Fermions, Twisted Mass QCD, Staggered Fermions, Wilson Fermions, etc, see [6, 7, 8, 24, 25, 26, 27]). We could use the dressing map to define a Born-Oppenheimer expanded theory of quantum gravity which does not double, and use this to motivate a chiral regulation to the Hamiltonian constraint eq. (4.1).

A third possibility is to look at perfect actions [28]. In this framework a lattice theory can exactly replicate the full set of gauge symmetries the continuum limit of the theory possesses. This is done through a process of coarse-graining the fields. However, this is at the price of locality. In addition, results can currently only be obtained in a perturbative framework, which is seemingly against the motive of LQG.

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