House Price Risk in Mortgage Contracts

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Research has shown that mortgage default is closely related to house prices. When house prices fall the borrower has an incentive to default. Since default incurs substantial cost to the lender, the borrower and many other market participants, as well as the society, house price is a risk in a mortgage contract. This was clearly demonstrated during the 2007-09 financial crisis. In this thesis I discuss some (potential) measures to manage mortgage default risk arising from low house prices.

Chapter 1 is on mortgage insurance. Mortgage insurance is commonly used by lenders to transfer mortgage default risk to an insurer. After a brief introduction of the mortgage and mortgage insurance markets in US and Canada, I specify a simple mortgage insurance contract and a multiple state model for mortgage termination. The contract is then priced under the model.

I explore the possibility of hedging house price risk in Chapter 2. If we assume a perfect market where house price risk can be traded exists, and a mortgage contract is a contingent claim on house prices, then the classic delta hedging is useful in hedging house price risk.

In chapter 3 I discuss an innovative type of mortgage contract – property index-linked mortgage. The purpose of this contract design is to reduce the borrower's propensity to default when house price declines. In particular, when house price declines, the mortgage balance and payment are reduced. I analyze this contract from the borrower's perspective and find that such contracts are effective in reducing default incentives and as a result, the lender may also be better off due to lower deadweight default cost.

The last chapter focuses on house price basis risk. House price basis risk refers to the situation where the value of an individual property appreciates differently from the index. This may lead to problems such as suboptimal hedging, underpricing and lower efficiency for financial products involving house price index. In this chapter I develop a basis risk model that can be used to simulate reasonable individual house prices for a given index.

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Chapter 1

Mortgage Insurance

In this chapter, we introduce the basic concepts of mortgage and mortgage insurance. We also introduce a model for mortgage termination that can be used to price a mortgage insurance contracts. The concepts and model provide a platform for further development of mortgage default risk management in later chapters. We start with strong assumptions and simple models to demonstrate the basic principle behind mortgage insurance contract evaluation. In practice more elaborate models would be required. Since we do not have access to mortgage insurance data, we consider a representative contract and use parameter values from the actuarial review to the mutual mortgage insurance fund operated by Federal Housing Administration.

The first section introduces the US and Canadian mortgage and mortgage insurance markets, including their sizes, typical types of contracts, regulations and recent performance. A simple mortgage insurance contract is specified in section 2. In section 3 we propose a continuous-time semi-Markov multiple state model for mortgage termination. We derive transition probabilities and discuss relevant risk factors. Section 4 presents the actuarial pricing formulas for a mortgage insurance contract written on a newly originated loan. Numerical examples are given in section 5. Section 6 concludes.

1.1 Introduction

A residential property could be the largest single expense over one's life. Most people cannot afford it when they need one without external financial support. One of the most common ways to finance a residential property is through a mortgage loan. In a typical residential mortgage loan, the lender provides the borrower with the additional amount of money needed to purchase a property after the borrower makes a down payment. The property becomes the collateral that is subject to the lender's acquisition if the borrower fails to meet repayment commitments. After loan origination, the borrower repays the loan according to a pre-agreed schedule until the outstanding balance reduces to zero. The repayment schedule depends on factors including the mortgage contract rate and the amortization term.

There are two risks the lender faces after loan origination. One is default risk and the other is prepayment risk. During the repayment period the borrower may default on the loan by stopping making payments or prepay the loan by repaying the full amount of the outstanding balance before maturity. The reasons behind could be life events happening to the borrower such as unemployment, or economic considerations such as low house price and low interest rate.

Default and prepayment are risks to the lender because they cause losses if the borrower defaults when the collateral, i.e. the house, is worth less than the outstanding loan balance, or if the borrower prepays when the market interest rate is lower than the contract rate. On the other side, the borrower is exposed to the risk of losing the property and the equity invested in it if timely payments cannot be arranged. Despite these risks, mortgage loans are widely used in many developed countries including the US, the UK, Canada, Australia, France, Germany, Japan, etc. In this thesis we focus on the default risk caused by low house prices, and how this risk can be managed.

1.1.1 Mortgage Markets in the US and Canada

Both the US and Canada have active housing and mortgage markets. By the end of 2014, the total value of real estate owned by households in the US was 20.7 trillion, of which 9.4 trillion consisted of mortgage loans.[1] More than 7000 institutions reported mortgage lending activities in 2014 and about 6 million loans were originated.[2] The Canadian market is relatively small compared to the US market. The residential mortgage credit outstanding was 1.2 trillion in May 2014, and 75% of them were held by chartered banks.[3]

The two markets are quite different in lending standards and mortgage contracts. In the US, the long-term fixed-rate mortgage (FRM) is the most popular mortgage contract.[4] Such a contract locks in an interest rate and amortizes the initial loan amount over 15

[1]Source: Federal Reserve Z.1 Financial Accounts of the United States Second Quarter 2015, Table B.101 Balance Sheet of Households and Nonprofit Organizations(1).

[2]Source: Federal Reserve Bulletin, The 2014 Home Mortgage Disclosure Act Data.

[3]Source: Canadian Housing Observer 2014.

[4]See for example McDonald and Thornton[63] and deRitis[32].

or 30 years so that the borrower makes equal monthly payments. Another major type of mortgage contract in the US market is the adjustable-rate mortgage (ARM). In this contract, the amortization rate varies periodically (from daily to annually) according to some index, such as the one-year LIBOR. The ARM borrower tends to refinance the loan into a FRM when mortgage rate falls in order to reduce payments and eliminate interest rate adjustment. A hybrid ARM is a combination of FRM and ARM. It has an initial fixed-rate period before converting to variable rates. These contracts were attractive before the 2007-09 financial crisis: 32% of mortgages originated between 1999 and 2006 were hybrid mortgages, and they were used heavily by subprime borrowers.[5]

As highlighted in Allen[1], "The Canadian mortgage market is relatively simple and conservative, particularly when compared with its U.S. counterpart." The majority of Canadian residential mortgages are rollover loans amortized over 25 years with a contract term of 5 years. Interest rates are renegotiable every 5 years. The reasons Canadians prefer a short contract term are discussed in Kiff[58]. Variable-rate mortgages are also available in Canada. A 2014 survey showed that about 20% of borrowers chose variable-rate mortgages.[6]

Canadian and US lenders take different approaches to manage default and prepayment risks. Prepayment affects the lender's cash flow and reduces the interest to be received if the market rate at the time of prepayment is below the contract rate. Canadian lenders set explicit prepayment penalties, which can be the higher of interest lost and three month interest plus a fixed reinvestment fee.[7] FRMs in the US typically do not carry explicit prepayment penalties. The cumulative prepayment rate by 2014 for FHA 30-year fixed-rate mortgages originated in 2009 was 42%.[8] The US ARMs may have prepayment penalties of several months' interest in the first 3 or 5 years. Although FRMs are free prepayable, points charged at loan origination implicitly offset this advantage. Kiff[58] made a numerical comparison of loan origination cost in the US and Canada, indicating that the extra refinancing cost paid by the US borrowers may exceed the prepayment penalty paid by Canadian borrowers.

The 2007-09 credit crisis has focused people's attention on mortgage default risk. There are many reasons for borrowers to walk away from the debt. For example, borrowers are not able to make timely payments because of unexpected job loss/illness or ARM interest rate resetting. Significant property value depreciation also motivates a borrower to stop making payments since the property is worth less than what the borrower owes. Although

[5]Source: Barth et al.[9].

[6]Source: Canadian Housing Observer 2014.

[7]Kiff[58] and Lea[60].

[8]Source: Actuarial Review for FHA MMI Fund Forward Loans for FY2014 [50].

collateralized by a property, lenders could lose 30-60% of the loan balance following a foreclosure.

Lenders and mortgage investors are taking active measures to mitigate default risk. Among those measures, setting high lending standards is effective in the loan origination phase. The two commonly used indicators of default risk are initial LTV and credit score. High LTV loans, also called high ratio loans, are risky because borrowers do not put much equity in the property. A decline in property value easily leads to negative equity. In the US, lenders usually require the borrower to purchase mortgage insurance (MI) if LTV is above 80%, while in Canada this requirement is mandatory by law. As a result, over 50% of the mortgages held by Canadian financial institutions are insured. MI will be discussed in detail shortly.

FICO score is a commonly used credit score. In 2006, 45% of subprime borrowers had scores below 620 and 93% of prime borrowers had scores above 620.[9] In 2014, the average credit score were 680 for FHA borrowers,[10] and 731 for CMHC borrowers.[11] This indicated that the FHA loan portfolio faced greater credit risk. Besides LTV and credit score, there are other risk indicators such as debt-to-income ratio, loan-to-income ratio, type of mortgage contract, interest rate, loan purpose, source of down payment, etc.

Based on these indicators, the lender determines whether the borrower, or the corresponding loan, is prime or subprime. A prime loan typically has a lower contract rate. In the US, subprime lending constituted about 20% of total loan originations during 2004-2006 but only about 1% in mid 2008.[12] A loan is delinquent if the borrower has missed a few payments but not defaulted. In the first quarter of 2014, the average 90-day delinquency rates for all mortgages in the US and Canada were 2.39% and 0.31%.[13] Delinquency rates differ substantially between prime and subprime loans in the US during the financial crisis. They are 2.23% for prime loans and 17.85% for subprime loans in 2008.[14] The Financial Stability Board attributed the low delinquency rates experienced by Canadian lenders to "conservative loan underwriting standards".[15] Lea[60] arrived at a similar conclusion that "the lack of subprime lending and less use of limited or no documentation lending were major factors" of low delinquencies. This is true in Canada as the subprime market is less

[9] Source: Barth et al.[9]. Each lender has its own definition for prime and subprime borrower. FICO score is likely to be one criterion but may not be the only one.

[10] Source: Annual Report to Congress Regarding the Financial Status of the Mutual Mortgage Insurance Fund Fiscal Year 2014.

[11] Source: CMHC Mortgage Loan Insurance Business Supplement 2014.

[12] Source: Barth et al.[9].

[13] Source: Canadian Housing Observer 2014.

[14] Source: Barth et al.[9].

[15] Canadian Housing Observer 2012.

than 5% and Alt-A[16] market is also small.

1.1.2 Mortgage Insurance Markets in the US and Canada

Mortgage Guaranty Insurance Model Act[68] defines mortgage insurance(MI) to be “The insurance against financial loss by reason of nonpayment of principal, interest or other sums agreed to be paid under the terms of any note or bond or other evidence of indebtedness secured by a mortgage, ...”. An MI policy is a contract between the insurer and the lender of a mortgage loan, specifying how the insurer compensates the lender should the borrower default on the loan. The insurance premium is paid by the borrower and may be capitalized into the mortgage loan. In North America, MI plays an important role in stimulating and expanding mortgage markets. It lowers the down payment requirement and the mortgage rate the lender charges, since default risk is partially transferred to the insurer. As a result, borrowers with MI can afford the down payment and monthly payments that otherwise cannot and lenders are more willing to make investment in real estate markets. The regulations governing mortgage insurers in Canada and the US are very different.

There is one public and two private mortgage insurers in Canada. The federal crown corporation Canada Mortgage and Housing Corporation (CMHC) has a market share of about 50%, and the other two (Genworth Canada and Canada Guaranty) share the rest. All three insurers are regulated by Office of the Superintendent of Financial Institutions (OSFI) and backed by the Canadian government. MI in Canada is mandatory for all mortgages with LTV above 80%. The insurance premium is charged upfront and may vary with initial LTV.

By the end of 2014, CMHC's total insurance-in-force was 543 billion.[17] For non-portfolio loans, 38% had initial LTV higher than 80%, and the average borrower equity was 34%.[18] The national average of borrower equity was 54%. This large difference was expected since high ratio loans were required to be insured. However the average delinquency rate was 0.52% – only 0.21% higher than the national average.

There are a few public and private mortgage insurers in the US. The largest government-backed insurer is Federal Housing Administration (FHA), founded in 1934 with the objective to stabilize the real estate market after the Great Depression. FHA's market share

[16]Alt-A loans are for borrowers who cannot provide required income documentation but have a relative good credit score. Such loans are riskier than prime but less risky than subprime loans.

[17]Source: CMHC 2014 annual report. The legislative limit for total insurance-in-force is 600 billion.

[18]Source: CMHC Mortgage Loan Insurance Business Supplement 2014.

declined from 18% in 2009 to 10% in 2014.[19] Private insurers' expansion and contraction could be a main reason behind these fluctuations. Particularly during the 2008 credit crisis, two major private insurers (PMI and RMIC) were under orders of supervision and prohibited from writing new business.

FHA insures the full loan amount as Canadian insurers do but charges upfront plus annual premiums. The premium rates depend on LTV, loan term and loan amount. For example, the annual premium for a 15-year FRM with loan amount less than or equal to 625,500 and LTV 95% is 0.70% of outstanding loan balance and is charged throughout the entire loan term. The upfront premium is 1.75% of initial loan amount for all mortgages.[20] The FHA total insurance-in-force by the end of 2014 was 1155 billion and the 2014 FHA actuarial review[52] estimated the economic value of its insurance fund to be about 6 billion.[21]

The US private mortgage insurers suffered great losses during the 2007-09 crisis. Default rate was as high as 15% in 2010 and gross claims paid was 12.9 billion, but they were only 5% and 2.8 billion in 2007.[22] The US private mortgage insurers, unlike FHA, only guarantee the lender 20-30% of the outstanding loan balance. The Model Act[68] suggested a maximum coverage limit of 25%, which was normally accepted by state regulators. The insurer usually has two options after receiving a claim: 1) pay the lender full amount of outstanding loan balance and acquire the mortgaged property; or 2) pay the covered amount and let the lender retain the property. The insurer needs to consider the market value of the property, the loan balance and other reasonable expenses before making a decision.

1.2 A Simple Mortgage Insurance Contract

In this scetion, we specify a simple mortgage insurance (MI) contract. See [39, 80] for more detailed information.

The contract is based on a fully amortized FRM loan secured by a property. The insurer guarantees the lender the full amount of loss caused by default, in exchange for an insurance premium paid by the borrower.

[19]Source: Annual Report to Congress Regarding the Financial Status of the Mutual Mortgage Insurance Fund Fiscal Year 2014.

[20]Source: US Department of Housing and Urban Development, Mortgagee Letter 2015-01 and its appendix published on January 9, 2015

[21]Economic value is defined by available capital resource less expected present value of future cash flows.

[22]Source: Mortgage Insurance Companies of America 2012-2013 Fact Book and Member Directory.

- Term of the insruance contract: the insurance contract expires as soon as the mortgage loan is terminated.
- Premium: (i) a single upfront premium; or (ii) upfront plus annual premiums.
- If the loan is repaid in full, all the insurer's obligation is discharged and no premium is rebated. If the loan has defaulted, the insurer pays the lender at the time of default the amount specified below.
- Claim amount: outstanding loan balance less the sale price of the property, plus related expenses.

We need the following assumption to make claim calculation feasible at the moment of default.

Assumption 1.1 (immediate claim payment). At the time of default, the lender acquires and sells the mortgaged property; files a claim to the insurer; the claim is processed by the insurer and the claim amount is calculated and paid. In short, there is no delay between default and claim payment.[23]

The amount of claim depends on many factors. Suppose a loan defaults t years after origination. Denote the claim amount by L_t, the property value by H_t, the outstanding loan balance by U_t all at the time of default, then

$$L_t = g(t, H_t, U_t, \cdots).$$

The function $g(\cdot)$ is complex in general but we can impose a simple form. For example, if we assume the default expense is a proportion ξ_t of the property value, and the only recovery the lender can obtain is the proceeds from the sale of the property, then

$$L_t = \max[U_t - (1 - \xi_t)H_t, 0].$$

Alternatively, if the net loss is assumed to be a proportion γ_t of the loan balance, then

$$L_t = \gamma_t U_t.$$

1.3 Model for Mortgage Termination

Mortgage insurers need to estimate the incidence and severity of losses in order to set up an appropriate reserve. In this section we model the incidence of default and prepayment.

[23] In practice, there are delays between default and property sale. The insurer is responsible for the interest accrued during that period which could be as long as one year.

Our model is very simple and ignores several institutional features.

Mortgage termination has been studied extensively. Kau, et al.[57] use a contingent claim model which assumes that decisions made by rational borrowers only depend on property value and market mortgage rate. Deng[30] and Calhoun and Deng[19] use a competing risks model with proportional hazard rates for prepayment and default risks. Taylor and Mulquiney[67] consider a discrete-time Markov process for loan status, while Ji[53] proposed a semi-Markov multiple state model for reverse mortgage loan status. Our multiple state model is adapted from the status transition model used in the 2011 and later actuarial reviews of FHA mortgage insurance fund.

There are three sets of factors affecting loan termination: economic conditions, mortgage contracts and borrower characteristics. Important economic variables include mortgage rate, property value, unemployment rate, etc. The mortgage contract contains loan level information such as LTV, contract rate, loan amount and amortization schedule. Borrower characteristics include borrower specific information such as credit score and debt-income ratio.

1.3.1 Transition between Loan Status

Compared to competing risks models, multiple state models are more intuitive and flexible. A multiple state model is defined by states and transitions. The specification of states and transitions is based on observation and experience rather than mathematical formulation, and different perspectives or modelling purposes may lead to different designs.

From the lender's perspective, the status of a mortgage loan is determined by the payments from the borrower. Ideally it should also depend on other factors that affect the probability of repayment and how the loan may be repaid. However some of those factors, such as the borrower's willingness and ability to repay, are not directly observable, and some others, such as house price indices and mortgage rates, affect the performances of all loans. Loan payments on the other hand can be directly observed for each loan without cost, and they serve as a good instrument of the unobservable factors. For example, if regular loan payments are made on a loan then the lender has no incentive to investigate the borrower regarding making future payments. If one or two payments have been missed on a loan, then the lender is aware that the loan is in trouble and may take action to minimize loss. This is because for whatever reason the borrower misses a few payments, he/she is likely to miss more payments in the future.

The model we propose is shown in Figure 1.1. There are four states, represented by the four rectangles, and five possible transitions, represented by the five arrows. A loan is in one of the four states depending on its status and the loan status depends on the payments received by the lender. It is in Active if regular payments are being made; in Delinquency if some payments are missed; in Prepay if it has been prepaid; and in Default if it has defaulted.

The two states Active and Delinquency divide current loan into to categories. They are treated differently by the lender and have different characteristics. Active loans are healthy and require little the lender's attention because the regular payments indicate the borrower is capable and willing to repay the debt. Delinquent loans typically require lender's actions such as sending notifications of payments being missed or a negotiation with the borrower. The risk of default for delinquent loans are higher than for active loans, and are more likely resulting in loss. Prepay and Default are the absorbing states indicating the termination of the mortgage contract. Each loan starts at Active and is terminated through one of the following three ways: (i) entering Prepay, (ii) entering Default, (iii) not entering Prepay or Default before the end of loan term, i.e. repaying the loan in full as scheduled.

An active loan can be prepaid or be delinquent, but it cannot be defaulted without being delinquent for some time. This is due to the common practice that a loan is treated as default only when a certain number of payments have been missed. This is a more stringent requirement than delinquency and make delinquency an early indicator of default. Since the lender of a delinquent loan takes measures (e.g. loan modification) to mitigate losses, it is possible that the borrower becomes capable of making payments again afterwards. If this is the case then the loan is cured and returns to the Active state. It is also possible that the borrower misses a few payments but be able to prepay the loan, for example, by selling the house or refinancing.

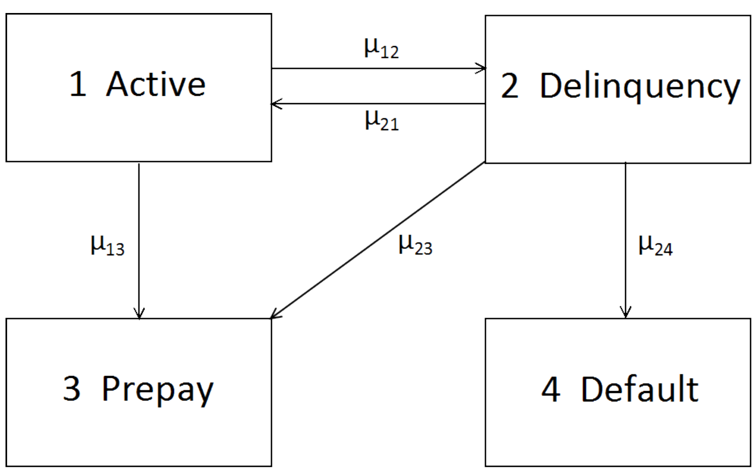

Figure 1.1: The multiple state model for mortgage termination

The μ_{ij} attached to each arrow represents the intensity of that transition, for example μ_{12} is the transition intensity from Active to Delinquency. This type of intensity-based, continuous-time modelling of transitions was introduced by Hoem[47, 48] and summarized by Waters[84]. It has been widely used in actuarial science, see for example Dickson, et al.[33] and Jones[55]. Intuitively, intensity is a measure of the propensity for a loan to change its status. To study this model in detail, we first introduce some notation:

- S_t: loan status at time t, $S_t \in \{1, 2, 3, 4\}\ \ t > 0$. For example, "$S(t) = 2$" means the loan is in Delinquency at t.
- D_t: the duration of the current stay in state S_t, i.e. the last transition was into S_t and it occurred at $t - D_t$.
- $p_{\overline{11}}(x, t)$: the conditional probability that the loan stays in Active in $[x, x + t]$ given it is active at time x, i.e. $P[S_{x+s} = 1, \forall s \in [0, t] | S_x = 1]$
- $p_{\overline{22}}(x, u, t)$: the conditional probability that the loan stays in Delinquency in $[x, x+t]$ given it has been delinquent since time $x-u$, i.e. $P[S_{x+s} = 2, \forall s \in [0, t] | S_x = 2, D_x = u]$, $u \leq x$
- $p_{1j}(x, t)$: the conditional probability that the loan is in state j at time $x + t$ given it is in Active at x, i.e. $P[S_{x+t} = j | S_x = 1]$, $j \in \{1, 2, 3, 4\}$
- $p_{2j}(x, u, t)$: the conditional probability that the loan is in state j at time $x + t$ given it has been in Delinquency since $x - u$, i.e. $P[S_{x+t} = j | S_x = 2, D_x = u]$, $u \leq x$, $j \in \{1, 2, 3, 4\}$

By law of total probability, $\sum_j p_{1j}(x, t) = \sum_j p_{2j}(x, u, t) = 1$. These conditional probabilities can be derived from transition intensities. We define transition intensities $\mu_{ij}(j \neq i)$ by

$$\mu_{1j}(x) = \lim_{\Delta t \to 0^+} \frac{p_{1j}(x, \Delta t)}{\Delta t}, \quad \text{for } \ j = 2, 3$$

and

$$\mu_{2j}(x, u) = \lim_{\Delta t \to 0^+} \frac{p_{2j}(x, u, \Delta t)}{\Delta t}, \ \ u \leq x, \quad \text{for } \ j = 1, 3, 4.$$

Here are some remarks:

1. This is a semi-Markov model because μ_{2j} depends on past information, i.e. the duration of stay in state 2. It is possible that μ_{1j} also depends on duration of stay, however we do not model it because the dependency of μ_{2j} on the duration of stay is more closely

related to default, the event of our interest. As mentioned earlier, there is no general rule of specifying a multiple state model, as long as the specification is reasonable. For example, it might be more appropriate to specify the dependency of μ_{1j} on the duration of stay if an insurer wants to model time to delinquency.

2. Transition intensities are functions of current time, duration of stay and any other factors that affect transition probabilities, such as property value, outstanding loan balance and market mortgage rate. We only choose some significant and representative ones, which will be discussed in detail later.

3. From the above definition, we have the following approximations for transition probabilities

$$\begin{aligned} p_{1j}(x, \Delta t) &= \mu_{1j}(x) \cdot \Delta t + o(\Delta t) \quad j \neq 1 \\ p_{2j}(x, u, \Delta t) &= \mu_{2j}(x, u) \cdot \Delta t + o(\Delta t) \quad j \neq 2, \end{aligned}$$

where

$$\lim_{\Delta t \to 0^+} \frac{o(\Delta t)}{\Delta t} = 0.$$

That is, the probability of transition in a small time interval can be approximated by the product of transition intensity and the length of that interval.

Assuming the probability of multiple transitions within $[x, x + \Delta t]$ is $o(\Delta t)$ for any x.[24] We have the following formulas for "stay" probabilities (see Waters[84]):

$$\begin{aligned} p_{\overline{11}}(x, t) &= \exp\left[- \int_0^t \mu_{12}(x+s) + \mu_{13}(x+s) ds \right] \\ p_{\overline{22}}(x, u, t) &= \exp\left[- \int_0^t \mu_{21}(x+s, u+s) + \mu_{23}(x+s, u+s) + \mu_{24}(x+s, u+s) ds \right]. \end{aligned}$$

By conditioning on the time of the first transition, we can derive integral expressions for transition probabilities,

$$p_{11}(x, t) = \int_0^t p_{\overline{11}}(x, s) \mu_{12}(x+s) p_{21}(x+s, 0, t-s) ds + p_{\overline{11}}(x, t) \tag{1.1}$$

$$p_{21}(x, u, t) = \int_0^t p_{\overline{22}}(x, u, s) \mu_{21}(x+s, u+s) p_{11}(x+s, t-s) ds. \tag{1.2}$$

[24] This is a common assumption in a multiple state model framework, see for example Dickson et al.[33] and Hoem[48].

The initial conditions are $p_{11}(x, 0) = 1$ and $p_{21}(x, 0, 0) = 0$. Appendix A provides a procedure to solve $p_{11}(x, t)$ and $p_{21}(x, 0, t)$ numerically, which will be used later. Transition probabilities between other states can be derived similarly but are not shown here since they will not be used. Integral expressions for transition probabilities are not unique. For example, to compute $p_{11}(x, t)$, we may condition on the times of the last transition to state 2 and the last transition back to state 1:

$$p_{11}(x,t) = \int_0^t \int_0^{t-s} p_{11}(x,s)\mu_{12}(x+s)p_{\overline{22}}(x+s,0,w)\mu_{21}(x+s+w,w)$$
$$p_{\overline{11}}(x+s+w, t-(s+w))dwds + p_{\overline{11}}(x,t).$$

1.3.2 Intensity Estimation

The next step is to determine transition intensity functions $\mu_{1j}(x)$ $j \in \{2, 3\}$ and $\mu_{2j}(x, u)$ $j \in \{1, 3, 4\}$ for all possible x, t, u. We estimate these functions using the results published in the 2011 Actuarial Review of the Federal Housing Administration Mutual Mortgage Insurance Fund Forward Loans [50].[25] It contains in its appendix A a detailed list of risk factors and their effects on transition probabilities, which provides us with a useful base for estimating intensities.

The 2011 review adopts a multinomial logit model for quarterly conditional probabilities. Let $\pi_{ij}(t)$ denote the conditional probability of transition into state j and π_{ii} the conditional probability of stay at state i, both at time $t + \frac{1}{4}$, given the loan is in state i at t. Then

$$\pi_{12}(t) = \frac{e^{\alpha_{12}+X_{12}(t)\beta_{12}}}{1 + e^{\alpha_{12}+X_{12}(t)\beta_{12}} + e^{\alpha_{13}+X_{13}(t)\beta_{13}}}$$
$$\pi_{13}(t) = \frac{e^{\alpha_{13}+X_{13}(t)\beta_{13}}}{1 + e^{\alpha_{12}+X_{12}(t)\beta_{12}} + e^{\alpha_{13}+X_{13}(t)\beta_{13}}}$$
$$\pi_{11}(t) = \frac{1}{1 + e^{\alpha_{12}+X_{12}(t)\beta_{12}} + e^{\alpha_{13}+X_{13}(t)\beta_{13}}},$$

where α_{ij} and β_{ij} are to be estimated, and $X_{ij}(t)$ are vectors of risk factors. Transition probabilities from state 2, π_{2j}, $j \in \{1, 2, 3, 4\}$, are defined similarly. Note that using our probability notations, $\pi_{1j}(t) = p_{1j}(t, \frac{1}{4})$. From remark 3 following the definitions of

[25] The review can be retrieved at http://portal.hud.gov/hudportal/HUD?src=/program_offices/housing/rmra/oe/rpts/actr/actrmenu. I do not use results from later versions because this chapter was written in 2012 and there was not sufficient time to revise the numbers before this thesis is submitted.

transition intensities, we can estimate intensities by

$$\mu_{1j}(x) \approx 4\pi_{1j}(x) \quad j \neq 1$$
$$\mu_{2j}(x,u) \approx 4\pi_{2j}(x,u) \quad j \neq 2.$$

The review considers eighteen risk factors and a total of fifty (dummy) explanatory variables. One of the reasons they choose to use a large amount of categorical dummy variables is "not satisfied with either the stability or interpretation" of the "application of continuous versions of the dynamic explanatory factors".[26] For simplicity, we have selected six important risk factors. Each unselected risk factor is assumed to be static and assigned a value. This value is then added to the intercept. The six selected factors are:

- Initial loan-to-value (LTV): the ratio of loan size (U_0) to the purchase price of mortgaged property(H_0) at loan origination, denoted by R_0.
- FICO score: a score assigned to each borrower representing his/her credit history.
- Default incentive: called probability of negative equity in the review. It indicates the borrower's propensity to default. We denote it by DI.
- Prepayment incentive: called mortgage premium in the review. It indicates the borrower's propensity to prepay. We denote it by PI.
- Mortgage age: this is the variable t, time since loan origination.
- Duration of delinquency: called default duration in the review. This is the variable u, time since loan became delinquent.

We assume LTV and FICO score are fixed at loan origination and do not change throughout the loan term. Mortgage age and duration of delinquency are essential to our semi-Markov model as they are part of the definitions of transition intensities. DI and PI are two stochastic variables connecting house prices and mortgage rates to default and prepayment.

We define DI by

$$\mathrm{DI}_t = \Phi\left(\frac{\log U_t - \log(H_0 \cdot I_t/I_0)}{\sigma_t}\right),$$

where I is the house price index, which represents the change in house prices, U is the outstanding loan balance, σ is hour price volatility, and the subscript t denotes mortgage age. The function $\Phi(\cdot)$ is the standard normal distribution function. It is clear that a lower index, i.e. house price, leads to a higher DI.

[26] See page A-12 of the review.

We define PI to be the relative spread between the contract rate $r_c = r_m(0)$ and the current market mortgage rate(MR) $r_m(t)$:

$$\mathrm{PI}_t = \frac{r_c - r_m(t)}{r_c}.$$

A positive PI indicates a lower market rate than the contract rate. In this case prepayment through refinancing reduces future loan payments. We assume there is only one unique mortgage rate available to the borrower.

It is clear from the definitions for DI and PI that their values depend on I_t and $r_m(t)$. In a scenario analysis, each economic scenario is represented by a pair of deterministic I_t and $r_m(t)$; while in a stochastic analysis both I_t and $r_m(t)$ are assumed to follow stochastic processes. We can expand the $X_{ij}(t)\beta_{ij}$ terms by

$$\begin{aligned}
X_{1j}(t)\beta_{1j} &= \beta_{1j}^1 R_0 + \beta_{1j}^2 \mathrm{FICO} + \beta_{1j}^3 \mathrm{PNE}_t + \beta_{1j}^4 \mathrm{PI}_t + \beta_{1j}^5 t \\
X_{2j}(t,u)\beta_{2j} &= \beta_{2j}^1 R_0 + \beta_{2j}^2 \mathrm{FICO} + \beta_{2j}^3 \mathrm{PNE}_t + \beta_{2j}^4 \mathrm{PI}_t + \beta_{2j}^5 t + \beta_{2j}^6 u.
\end{aligned}$$

1.4 Contract Pricing

In this section we determine the actuarial fair premium(s) for a mortgage default insurance policy specified in section 1.2. The termination of the underlying mortgage loan follows the multiple state model from the previous section. By the actuarial equivalence principle, the expected present value (EPV) of the premiums equals the EPV of claims. Since the multiple state model assumes default can occur at any time, it is more convenient to consider a fully continuous situation where

Assumption 1.2 (fully continuous). All payments, including loan repayment and annual premiums (if any), are made continuously.

The following assumption simplifies our analysis by assuring that the lender receives a constant flow of payments before loan termination disregard the status of the loan.

Assumption 1.3. When a loan is delinquent, loan repayment and annual premiums (if any) are paid as scheduled as if it was active. Moreover, any loan that has not been prepaid or defaulted before the end of loan term is treated as being repaid in full and the insurance contract expires without claim.[27]

[27] Insurers may take actions such as payment reduction/suspension to prevent delinquent loans from defaulting.

This assumption is not consistent with the reality where the borrower may not make or make reduced payments during delinquency. However if we assume no payment during delinquency, then either loan term needs to be extended or loan payments need to be raised. This complicates our analysis by making loan term, loan balance and/or loan payments stochastic and contingent on the time and duration of the loan being delinquent.

First consider the case of a single upfront premium P. It can be calculated by

$$P = \int_0^T L_t v_t \lambda_{14}(t)dt, \tag{1.3}$$

where T is the loan term, L is the claim amount, v is discount factor and $\lambda_{14}(t)dt$ approximates the probability of loan default occurs in $(t, t+dt)$. More precisely, by conditioning on the times of the last transition into state 2 and the transition into state 4,

$$\lambda_{14}(t) = \int_0^t p_{11}(0,s)\mu_{12}(s)p_{\overline{22}}(s,0,t-s)\mu_{24}(t,t-s)ds. \tag{1.4}$$

To see the intuition, rewrite $\lambda_{14}(t)dt$ in the following order

$$p_{11}(0,s)\mu_{12}(s)ds \cdot p_{\overline{22}}(s,0,t-s)\mu_{24}(t,t-s)dt,$$

where the first term is the probability the last transition into state 2 occurs in $(s, s+ds)$ and the second term is the conditional "stay" probability at state 2 until t when the transition into state 4 occurs, given a loan is at state 2 at s.

Another type of premium structure requires an upfront premium plus annual premiums.[28] Define the upfront premium rate p_0 to be the percentage of initial loan size charged immediately at origination, and the annual premium rate p_t to be the percentage of outstanding loan balance charged per annum. Assume a constant annual premium rate, i.e. $p_t \equiv p$, and it is charged until the loan balance reaches a certain proportion of the original property value, say αH_0. For a mortgage amortized continuously over T years with initial loan size U_0 and contract rate r_c, the outstanding balance at time t is

$$U_t = U_0 \frac{e^{r_c T} - e^{r_c t}}{e^{r_c T} - 1}.$$

[28]This was used by FHA, see page B-3 to B-8 of the 2011 review [50].

The maximum premium period t_m is such that $U_{t_m} = \alpha H_0$; thus we have

$$t_m = T + \frac{1}{r_c} \log\left(1 - \frac{\alpha}{R_0}\left(1 - e^{-r_c T}\right)\right),$$

where $R_0 = U_0/H_0$ is the initial LTV. The EPV of premium is given by the following summation:

$$p_0 U_0 + \int_0^{t_m} \left(\int_0^t p U_s v_s ds\right)\left[\lambda_{14}(t) + \lambda_{13}(t)\right] dt + \left(1 - \int_0^{t_m} \left[\lambda_{14}(t) + \lambda_{13}(t)\right] dt\right) \int_0^{t_m} p U_s v_s ds,$$

where

$$\lambda_{13}(t) = p_{11}(0,t)\mu_{13}(t) + \int_0^t p_{11}(0,s)\mu_{12}(s) p_{\overline{22}}(s,0,t-s)\mu_{23}(t,t-s) ds.$$

The first term $p_0 U_0$ is the upfront premium; the second and third terms calculate the EPV of annual premiums conditioning on the mode of loan termination. Equate this summation to the EPV of claims which is given by the right hand side of Eq.(1.3), we have the following explicit expression for p:

$$p = \frac{\int_0^T L_t v_t \lambda_{14}(t) dt - p_0 U_0}{\int_0^{t_m} \left(\int_0^t U_s v_s ds [\lambda_{14}(t) + \lambda_{13}(t)]\right) dt + \left(1 - \int_0^{t_m} \lambda_{14}(t) + \lambda_{13}(t) dt\right) \int_0^{t_m} U_s v_s ds}. \tag{1.5}$$

Our goal is to determine P in Eq.(1.3) and p in Eq.(1.5) for a given p_0.

1.5 Numerical Examples

In this section, we consider two numerical examples. The first example consists of some deterministic scenarios, each represented by a path of house prices and a path of mortgage rates. The second example consists of stochastic house prices and mortgage rates. The values for static parameters are shown in Table 1.1. Since the majority of loans insured by FHA in 2011 was between 90%-97%, we set initial LTV to 95%. The average FICO score for FHA borrowers in 2011 was about 700 and we assume the borrower in our examples has a slightly higher score of 725. The combination of the 1% upfront premium rate and the 78% lower bound for annual premiums was used by FHA in 2011 for 15-year mortgages.

Discount factors v_t at integer years are given on page B-12 of the 2011 review [50]. Values between integer years are approximated by linear interpolation. We assume the

Parameter	Value	Description
H_0	400,000	Initial property value
R_0	95%	Initial loan-to-value
U_0	380,000	Initial loan size ($= H_0R_0$)
T	15	Loan term (in year)
FICO	725	Borrower's credit score
p_0	1%	Upfront premium rate
α	78%	Annual premiums stop when loan balance reaches 100α percent of initial property value

Table 1.1: Parameter specification

claim severity is

$$L_t = \max[U_t - 0.7H_0I_t/I_0, 0].$$

To calculate DI we also need house price volatility. House price volatility refers to the dispersion of house prices from the index. In the process of constructing repeat sales index, it is found that the deviation of house price from the index grows with the duration between the sales. To capture and remove the effect of time-varying variance, Calhoun[18] suggests the variance be a quadratic function of duration. This approach has been adopted by Federal Housing Finance Agency (FHFA) to construct the national house price index. FHFA also publishes the fitted coefficients of duration and duration squared. According to the 2011 results, the coefficients are 0.0061076 and -0.00006416, which implies the volatility at time t is

$$\sigma_t = \sqrt{0.0061076t - 0.00006416t^2}.$$

1.5.1 Scenario Analysis

We first consider the scenario shown in Figure 1.2. It mimics the base-case curves starting from 2007 quarter 3 in Exhibit D-2 and D-3 of the review. The first four years are actual experience and the rest are from Moody's forecasts. The index will drop from 375 to 320 in 4 years and stay at that level for 2 years before moving up. The mortgage rates will also decline in the first 3 years but then rise and remain at the initial level of 6.55%. Both curves are piecewise linear.

DI and PI can be computed based on these information. Exhibit A-3.2 on page A-23 of the review shows estimated values of coefficients (intercept and β's) we need to estimate transition intensities. All integrations involved in the calculation of transition probabilities

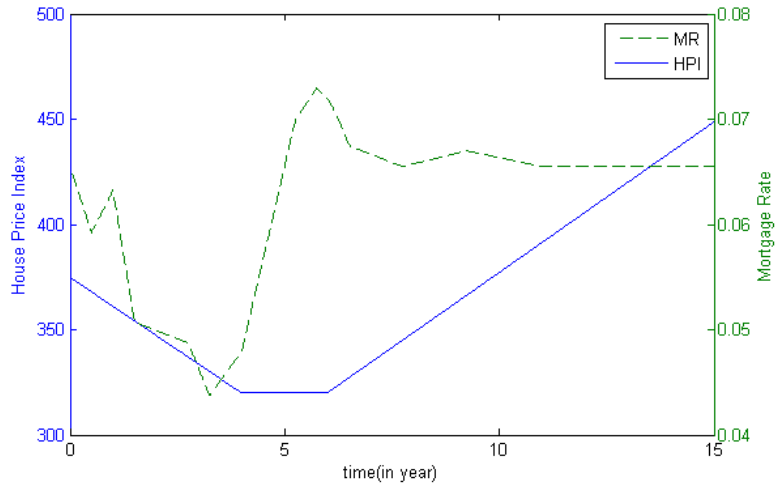

Figure 1.2: Scenario 1: HPI & MR

and EPVs are approximated by composite trapezoidal rule with mesh size $h = 1/120$, see Appendix A for detail.

An evaluation of Eq.(1.3)&(1.5) gives $P = 11889$ (3.13% of U_0) and $p = 0.74\%$. The cumulative probability of default is shown in Figure 1.3. The probability rises sharply in the first five years due to low house prices relative to loan balances, and then levels off when the index increases and loan balance decreases.

The premium will be very different if we change our scenario. Now consider the one shown in Figure 1.4. They are both based on Moody's base-case forecasts for 2011 quarter 3 and after. The index increases and so does the mortgage rate. The single upfront premium decreases substantially to $P = 2766$ (0.73% of U_0). The annual premium rate is not calculated since the upfront premium rate is below 1%.

We can also adjust static parameters to investigate their effects on premiums. For example if we use the scenario 1 index and mortgage rate but assume a FICO score of 600 and an LTV of 97.5%, the loan becomes riskier and higher premiums should be charged: $P = 31921$ (8.18% of U_0) and $p = 2.42\%$.

1.5.2 Stochastic Analysis

Now we assume the house price index and mortgage rate move stochastically and the EPV of future claim payments, denoted by P, becomes a random variable. The distribution of

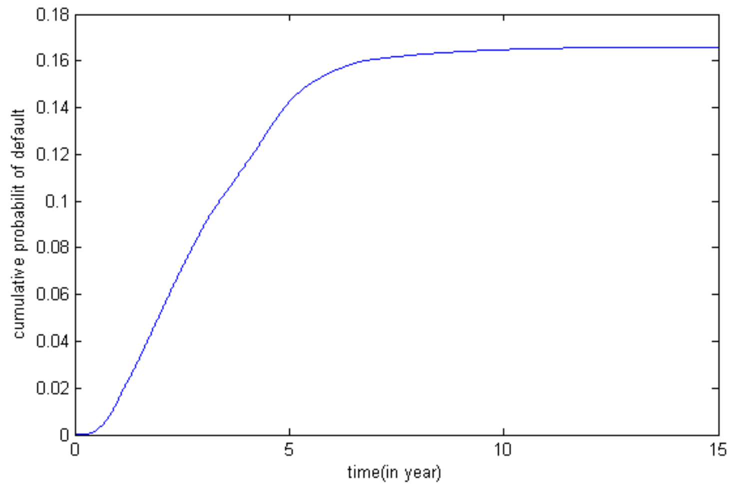

Figure 1.3: Scenario 1: cumulative probability of default

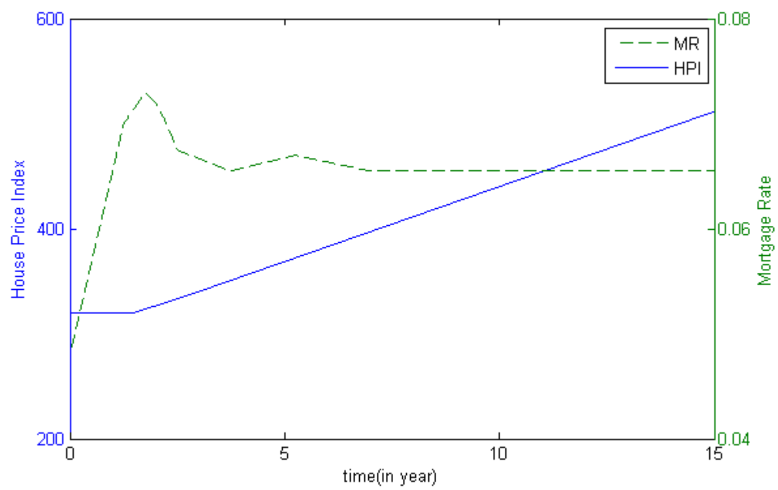

Figure 1.4: Scenario 2: HPI & MR

P depends on the model for index and mortgage rate. We use simulation to estimate the distribution and its statistics.

A few models for house price index and mortgage rates have been proposed. Kau et al.[57] model house values by a geometric Brownian motion and interest rates by a mean reverting process:

$$\frac{dH}{H} = (\alpha - s)dt + \sigma_H dz_H$$
$$dr = \gamma(\theta - r)dt + \sigma_r \sqrt{r} dz_t$$
$$dz_H \cdot dz_t = \rho dt$$

Chang et al.[27] complement the GBM by adding a Poisson process to capture the jump risk.

Recognizing the significant serial correlation of house price returns, Capozza et al.[23] propose the following reverting-to-fundamental model

$$\Delta P_{kt} = \Big[\alpha + \sum_i \alpha_i (Y_{kit} - Y_i^*)\Big] \Delta P_{k,t-1} + \Big[\beta + \sum_i \beta_i (Y_{kit - Y_i^*})\Big] \big(P_{k,t-1}^* - P_{k,t-1}\big) + \gamma \Delta P_{kt}^*,$$

where P_{kt} is the logarithm of house value at time t in city k, Δ denotes the value difference with respect to time, Y_i are independent variables, Y_i^* is the mean value of Y_i, and P_{kt}^* is the logarithm of fundamental house value. The authors in particular consider the effects of information costs, construction costs and backward-looking expectations on serial correlation and fundamental reversion.

Hanewald and Sherris[44] compare a time series model(ARIMA(3,1,1)) with other regression models for postcode level house price indices in Sydney. The regressors under consideration include postcode, season, economic variables (GDP growth, interest rate, inflation rate, etc.) and socio-demogrphic variables (income, unemployment rate, median age, household size, etc.). Li et al.[62] identify a ARMA(1,3)-EGARCH(1,1) model for UK Nationwide House Price Index. The model successfully captures autocorrelation, conditional heteroskedasticity and leverage effects in the index log return series.

The 2012 FHA actuarial review[51] used a joint autoregressive model for Treasury rate, mortgage rate and house price index.[29] The log return of the index follows

$$\begin{aligned} Y_t = \mu + \alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_3 + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \beta_3 Y_{t-3} + \beta_4 r_{1,t} \\ + \beta_5 r_{1,t-1} + \beta_6 s_{10,t} + \beta_7 s_{10,t-1} + \beta_8 s_{m,t} + \beta_9 s_{m,t-1} + \epsilon_{h,t}, \end{aligned} \tag{1.6}$$

where
D_1, D_2, D_3 are dummy variables for seasons spring, summer, fall,
$r_{1,t}$ is the 1-year Treasury rate,
$s_{10,t}$ is the spread between 10-year and 1-year Treasury rate,

[29] See Appendix G.

HPI coef.	Estimate	MR coef.	Estimate
μ	-0.001	α	0.005
$\alpha_1, \alpha_2, \alpha_3$	0.005	β_{1m}	-0.179
β_1	0.613	β_{2m}	0.169
β_2	-0.153	β_{3m}	0.053
β_3	0.398	β_{4m}	0.742
β_4	-0.596	γ_{0m}	2.35×10^{-7}
β_5	0.600	γ_{1m}	0.128
β_6	-0.770	γ_{2m}	0.795
β_7	0.748		
β_8	-0.649		
β_9	0.581		
γ_0	1.2×10^{-5}		
γ_1	0.955		

Table 1.2: Coefficients estimate of the HPI/MR model, copied from Exhibit G-9 & G-10 of the 2012 FHA actuarial review[51].

$s_{m,t}$ is the spread between MR and 10-year Treasury rate,
$\epsilon_{h,t}$ is a normal random variable with mean zero and variance

$$\sigma_{h,t}^2 = \gamma_0 + \gamma_1 \epsilon_{h,t-1}^2.$$

The mortgage rate spread follows

$$s_{m,t} = \alpha + \beta_{1m} r_{1,t} + \beta_{2m} r_{1,t-1} + \beta_{3m} s_{10,t} + \beta_{4m} s_{m,t-1} + \epsilon_{m,t}$$

and

$$\sigma_{m,t}^2 = \gamma_{0m} + \gamma_{1m} \epsilon_{m,t-1}^2 + \gamma_{2m} \sigma_{m,t-1}^2$$

We use this model for our stochastic analysis. The parameters have been calibrated using US historical data (see Table 1.2[30]). All parameters are significant at 90% level.

We generate 1000 independent scenarios of house price index and mortgage rate and Eq.(1.3) is evaluated for each scenario. The mean of the EPV of claim payments is 7154 (1.88% of initial loan size) and the standard deviation is 3463. The distribution is right skewed with skewness of 1.04, indicating large claims in some scenarios. The 10% and 90% quantiles are 3427 and 11685. If the premium is set to be mean value plus one standard

[30]It is copied from Exhibits G-9 & G-10 of the review.

deviation, i.e. 10617 based on this set of simulation, then the probability of negative NPV (loss) is 14.1%. The 90% value at risk(VaR) and conditional tail expectation(CTE) are estimated to be 1068 and 4080, both in terms of loss.

It is generally believed that declining house prices accompanied by rising mortgage rates raise delinquency and default rates. Figures 1.5 and 1.6 show two scenarios with low and high EPVs of claims. Note that the index in Figure 1.6 is higher than that in Figure 1.5 at the end of year 15.

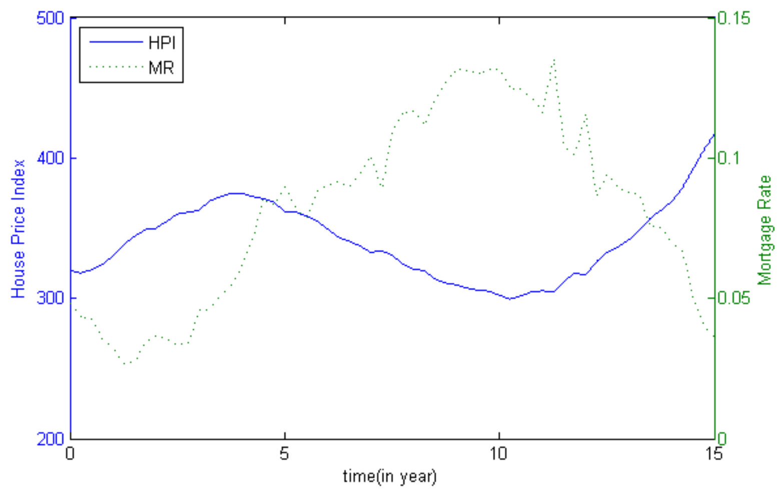

Figure 1.5: A scenario with low EPV of claims: $P = 1327$

To find the main determinant of claims, we divide the 15-year period into three 5-year periods and regress EPVs on the average MR and the average annual index returns within each 5-year interval. Table 1.3 listed the five significant variables with their estimated coefficients and t statistics. It is evident from the coefficients that the index and MR in the first 5 years have the greatest effect on claims, and their signs are expected. Five years after loan origination, the outstanding balance reduces to 70.63% of the initial house price. Therefore the probability of default, as well as the loss given default, is low. Refinancing also becomes less beneficial.

1.6 Conclusion

Mortgage insurance is used by mortgage lenders to transfer default risk to an insurer. Both the US and Canada have mature mortgage insurance industry. In this chapter, we

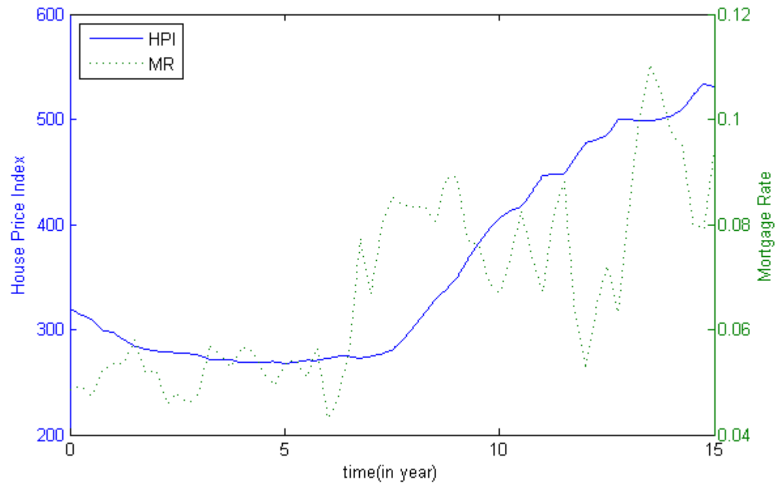

Figure 1.6: A scenario with high EPV of claims: $P = 17211$

Average of	Coefficient($\times 10^3$)	t statistic	p-value
HPI return year 0-5	-77.88	-27.52	< 0.001
HPI return year 6-10	6.31	3.29	0.001
HPI return year 11-15	-4.01	-2.40	0.017
MR year 0-5	72.77	6.34	< 0.001
MR year 6-10	-11.123	-2.38	0.018
Intercept	4.17	8.120	< 0.001

Table 1.3: Linear regression results. Dependent variable: EPV. Explanatory variables: average index returns and mortgage rates within each 5-year period. $R^2 = 0.517$

specify a simple mortgage insurance contract and price the insurance premium(s) based on a multiple state model of loan termination. We assume the transition of loan status is a continuous process and depend on various risk factors. We make strong assumptions on the relation between house price and loan default, as well as on loss severity.

Our numerical examples are implemented using results from the 2011 and 2012 FHA actuarial reviews. Results show that the insurance premium depends critically on the scenario. The premium will be high if a distressing market is expected, and be low if the market is promising. This is because low house prices increase the probability of default, which increases the expected claim. The stochastic analysis shows that a major determinant of claim is the house values in the beginning years of a mortgage loan.

Besides mortgage insurance contracts, the multiple state model developed in this chapter can be used to evaluate mortgage contracts, as well as securities with payoffs contingent on mortgage terminations. In Chapter 4, we will use it to price a new type of mortgage – property index-linked mortgage.

Chapter 2

Hedging House Price Risk

In this chapter, we discuss the possibility of hedging house price risk in mortgage contracts. Hedging is not considered as a main default risk mitigation strategy in the current mortgage industry because real estate related financial instruments are either not closely correlated with the prices of mortgaged properties or have illiquid markets and constrained trading. In this chapter, we assume there exists an ideal market where house price risk can be traded without friction through a suitable security. Such a market may be developed in the future as there are increasing hedging and speculating needs.

We model the relation between house prices and mortgage defaults in a contingent claim framework. We consider the classic delta hedging strategy and analyze its effectiveness to a mortgage insurer. The intuition behind delta hedging is to trade an appropriate amount of hedging security such that if the value of the insurer's liability under the insurance contract is linear in house price, then the insurer is immune to house price risk. The access to a suitable hedging security would improve the operation of the insurer as the insurer would be able to choose how much risk to retain.

The first section introduces some existing financial instruments that are related to house prices. Section 2 presents the assumptions and models that are important to our hedging strategy, which is constructed in section 3. We provide a numerical example in Section 4. Section 5 concludes.

2.1 Financial Instruments on House Prices

Residential properties are not only for dwelling, but are also for investment. As a special class of investment asset, real estate has some features that are different from financial assets such as securities and derivatives. First, real estate properties have heterogeneous characteristics and an individual property may not be traded for decades. This means the market prices of properties cannot be observed frequently. Second, the transaction costs associated with purchase and sale is ranging from 5-8% of the transaction price. They include agent commission, legal fees, sometimes restoration and insurance. Third, it is difficult to invest in part of a property or short selling one. With these unfavorable features, real estate is not attractive to investors, if the only way to engage is trading properties directly.

To overcome these disadvantages, indices that represent the average change in property values have been created. Consider the US S&P/Case-Shiller Home Price Indices as an example.[1] The indices are based on repeat sales prices of residential houses in 20 defined areas. Different weights are attached to each area to construct the composite 10, composite 20 and national indices. The indices are calculated monthly and published with a two-month lag. The base month is January 2000 and all indices are set to 100. Figure 2.1 shows the composite 10 index from January 1987 to August 2015.[2] The composite 10 index for December 2014 was 187.59, implying that the average house prices in the 10 selected areas increased by 87% in the period between 2000 and 2014. The index declined by one third during the 2007-09 financial crisis.

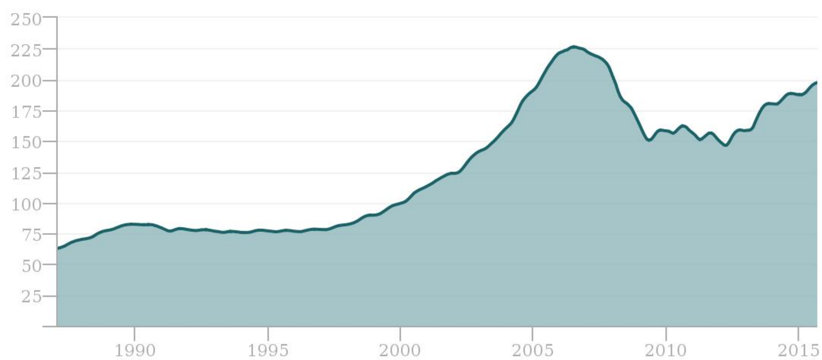

Figure 2.1: S&P Case-Shiller Composite 10 index, January 1987 to August 2015.

Due to limited number of transactions within an area, a property index, or a house price index, is not updated as frequently as a stock price index. Some property indices are published with lags. We should also note that these indices only represent the average

[1]See [78] for a full technical detail.

[2]Source: S&P Dow Jones Indices http://ca.spindices.com/indices/real-estate/sp-case-shiller-10-city-composite-home-price-index.

performance of the housing market, but not the value change of any individual property. The changes in individual house prices may be very different from the change in index value.[3]

Once an index is defined, we can construct various index-contingent derivatives to trade house price risk. For example, index futures are one of the most actively traded property derivatives compared to other standardized derivative contracts. Futures based on S&P Case-Shiller house price indices have been traded on Chicago Mercantile Exchange since 2006. The futures are on composite 10 and regional indices, with maturities ranging from 18 to 60 months.[4]

Nevertheless this market is currently not practical for hedging. First, trading volumes are low in general.[5] Second, the prices of these futures contracts are not predictable and they do not have a stable relation with the underlying index values.[6] This could be a consequence of market illiquidity and low trading volume. It could also be caused by the mismatch between hedging and speculative demands. Despite these disadvantages, property index futures is still a potential choice for hedging tool, as suggested by Case and Shiller[25].

Other types of index based contracts are mostly traded over-the-counter. For example, a total return swap is a series exchange of cash flows, where the party on the long position pays a fixed percentage of a notional amount, and in return, the counter-party on the short position pays the return of an index. The fixed percentage reflects the investor's expectation on index return. Suppose the market price for a one year swap on some index is 1%, then the long side will make profits if the average return on that index is higher than 1%. A property index note is a variable rate bond that pays coupons based on the returns of a property index. It may have features such as a specific participation rate, or a non-negative return guarantee. Similar to swaps, the long side is exposed to house price risk. More details can be found in commercial brochures such as [38, 83].

Among non-index based instruments, a publicly traded real estate investment trust (REIT) may be another potential choice of hedging tool. A REIT is an investment company specialized in real estate and real estate mortgages. In most countries REITs enjoy different tax rules from other companies, but a company has to satisfy certain conditions to be qualified as a REIT. To investors who want to gain exposure on real estates but do not wish to buy properties, publicly traded REITs are good choices since their stock prices should to some extent reflect the performance of real estate and mortgage markets.

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[3] See Chapter 4 for a discussion of house price basis risk.

[4] See [28] for contract specifications.

[5] See Shiller[74] for a discussion of reasons.

[6] See Jud and Winkler[56], Leventis[61] and Syz[82] for some analysis.

Compared to index based derivatives, REIT stocks are more liquid and have less short selling constraints. Investors can choose from a large number of combinations of REIT stocks and their derivatives. However there are disadvantages. First, REITs typically invest in commercial properties but not residential properties, and mortgaged properties are usually owner-occupied. The commercial and residential real estate markets may differ in the risk factors affecting the property values, hence the two markets may not have a strong correlation. Second, the investment portfolios hold by REITs are usually not geographically concentrated. Shorting their stocks may not effectively protect the downward movement of house prices in a specific area. Third, REIT stocks are publicly traded and their prices are influenced by the entire stock market but not only the real estate market. Fourth, there may not be enough REIT shares available for short selling.[7]

2.2 Models and Assumptions

In this section, we make assumptions about default and the house price index. We consider both continuous and discrete cases.

Assumption 2.1a (discrete). Suppose each time period is h year. The borrower repays the mortgage loan at the end of each period, and only default at the times of payment due.[8]

Let H_0, U_0 be the property value and loan amount at loan origination and r_c be the continuously compounded contract rate, then the amount of payment is

$$M^d = U_0 \frac{e^{r_c h} - 1}{1 - e^{-r_c T}},$$

and the before-payment outstanding loan balance on payment days are

$$U^d_{ih^-} = U_0 \frac{e^{r_c h} - e^{r_c(ih-T)}}{1 - e^{-r_c T}} \quad i = 1, 2, \ldots, \frac{T}{h}.$$

It then follows that

$$U^d_t = \begin{cases} U^d_{t^-} - M^d & t = h, 2h, \ldots, \frac{T}{h}h \\ U^d_{\lfloor \frac{t}{h} \rfloor h} \cdot e^{r_c(t - \lfloor \frac{t}{h} \rfloor h)} & \text{otherwise} \end{cases} .$$

[7]See Case et al.[26] for a detailed discussion.

[8]For example, $h = \frac{1}{12}$ means monthly payments.

Note that U_t^d is a right continuous function with jumps at payment days, $U_{T^-}^d = M^d$ and $U_T^d = 0$. Since borrowers default before making the payment due, we more precisely have $p_t \equiv p_t(H_t, U_{t^-}^d)$ and $L_t \equiv L_t(H_t, U_{t^-}^d)$ for $t = h, 2h, 3h, \cdots$.

Assumption 2.1b (continuous). The borrower repays the mortgage loan continuously and may default at any time.[9]

Based on loan amortization the repayment rate is

$$M^c = U_0 \frac{r_c}{1 - e^{-r_c T}},$$

and

$$U_t^c = U_0 \frac{1 - e^{r_c(t-T)}}{1 - e^{-r_c T}}.$$

In the analysis followed, we will make it clear which one of the above assumptions is in force. We keep assuming immediate claim payment (Assumption (1.1)).

2.2.1 Default Decision

The borrower's propensity to default depends on many factors such as current property value, outstanding loan balance, market/contract mortgage rate, financial status, income, marital status, etc. To avoid complexity, we assume that.

Assumption 2.2. The borrower's propensity to default depends on two factors only: property value (H_t) and loan balance (U_t). The relationship between default propensity and H_t and U_t is deterministic and known at loan origination.

The mathematical expressions for default propensity are different for continuous and discrete cases. Let T_d be the default time random variable and T be the loan term.

Assumption 2.3a (probability of default). Suppose the borrower makes discrete payments. On each payment day, his/her propensity to default is expressed as a conditional probability p_t defined by

$$p_t = \text{Prob}(T_d = t | T_d \geq t) \begin{cases} \geq 0 & \text{if} \quad t = h, 2h, 3h, \cdots, T \\ = 0 & \text{otherwise} \end{cases}$$

[9]This is similar to assumption 1.2.

By Assumption 2.2, p_t is a function of H_t and U_t, i.e. $p_t \equiv p_t(H_t, U_t)$ and this is a deterministic and known function. We further assume p_t is at least twice differentiable with respect to H_t.

It then follows that the probability for a loan to survive beyond time t, denoted by Y_t^d, is a function of p_s for $s \leq t$:

$$Y_t^d = \prod_{i=1}^{\lfloor \frac{t}{h} \rfloor} \left[1 - p_{ih}(H_{ih}, U_{ih})\right], \tag{2.1}$$

where $\lfloor x \rfloor$ denotes the integer part of x. By convention $Y_t = 1$ if the lower limit of the product is strictly larger than the upper limit. Note that Y_t is a right continuous, non-increasing step function having jumps at $h, 2h, 3h, \cdots$. The jump sizes depend on $p_h, p_{2h}, p_{3h}, \cdots$, which in turn depend on $(H_h, U_h), (H_{2h}, U_{2h}), (H_{3h}, U_{3h}), \cdots$.

Assumption 2.3b (intensity of default). Suppose the borrower repays continuously. His/Her propensity to default at time t is expressed as an intensity μ_t defined by [10]

$$\mu_t = \lim_{\Delta t \to 0^+} \frac{\text{Prob}(t \leq T_d < t + \Delta t | T_d \geq t)}{\Delta t}.$$

Similarly by Assumption 2.2, $\mu_t \equiv \mu_t(H_t, U_t)$ is a deterministic and known function. We further assume μ_t is at least twice differentiable with respect to H_t.

In this case,

$$Y_t^c = \exp\left[-\int_0^t \mu_s(H_s, U_s) ds\right]. \tag{2.2}$$

For both discrete and continuous cases, H_t is the single risk factor because U_t is known and fixed. We will use p_t, $p_t(H_t)$ and $p_t(H_t, U_t)$ interchangeably and the same for μ_t, $\mu_t(H_t)$ and $\mu_t(H_t, U_t)$. These two functions should be non-increasing in H_t, because intuitively default is negatively related to property value holding other factors the same.

For simplicity we assume there is no prepayment and basis risk. They will be considered in chapters 3 and 4.

Assumption 2.4 (no prepayment). There is no loan prepayment.

[10] This definition is consistent with the definition of transition intensity in chapter 1 section 1.3.1. The default process here can be viewed as a simplified multiple state model with only two states: Active and Default, and one transition: Active to Default.

Assumption 2.5 (**pooling**). An insurer issues a large number of homogeneous mortgage default insurance policies. The prices of all mortgaged properties are perfectly correlated. All borrowers have the same propensity to default but make default decisions independently.

Under Assumptions 2.3 & 2.5, the insurer can predict the exact[11] number of defaults in a pool of policies at all times, as long as the property values H_t are given. Assumptions 2.3 & 2.4 also suggest that there is no need to model mortgage interest as it has no impact on default and prepayment is prohibited. From now on, we base our analysis on one policy in a large portfolio of policies. It reflects the average performance of the portfolio.

In the discrete case, the expected claim amount per policy at payment date ih is

$$X^d_{ih} = Y^d_{(i-1)h} p_{ih} L_{ih} \qquad i = 1, 2, 3, \cdots, \frac{T}{h}, \tag{2.3}$$

and in the continuous case, the expected claim rate (dollar/year) at time t is

$$X^c_t = Y^c_t \mu_t L_t \qquad t \geq 0, \tag{2.4}$$

where the claim size $L_t \equiv L_t(H_t, U_t)$ is assumed to be a smooth function of H_t and U_t. In both cases, the insurance policy can be treated as a contingent claim on H_t. For example in the discrete case, claim payments are equivalent to a portfolio of T/h European type path-dependent options with payoffs given by (2.3). Therefore European option pricing and hedging theories can be applied.

2.2.2 House Prices

The market value of an individual property is not always observable. However, as pointed out in the previous section 2.1 we can use a property index as a proxy for property values. The next two assumptions connect financial instruments and property values to an index.

Assumption 2.6. There exists an index I_t that is perfectly correlated with the value of all mortgaged properties. Without loss of generality, we assume $I_t \equiv H_t$.

Assumption 2.7. There exists a publicly traded security whose price S_t is perfectly correlated with the index I_t that satisfies Assumption 2.6. Without loss of generality, we assume $S_t \equiv I_t$. We further assume this security can be traded without friction.[12]

[11] This excludes randomness in the number of defaults for a given H_t path.
[12] This implies no transaction costs.

Under these two assumptions, $H_t \equiv S_t$ and we will replace H_t by S_t to make it clear that it is the security we are trading but not the property.

Assumption 2.8. The security prices follow the geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{2.5}$$

where the constants μ and σ represent the expectation and volatility of the returns on the security, and W_t is a standard Brownian motion.

Figure 2.2 shows a diagram elaborating Assumptions 2.6–2.8. To complete the market, we assume there is a risk-free account where borrowing and lending are carried out at a constant continuously compounded interest rate r.

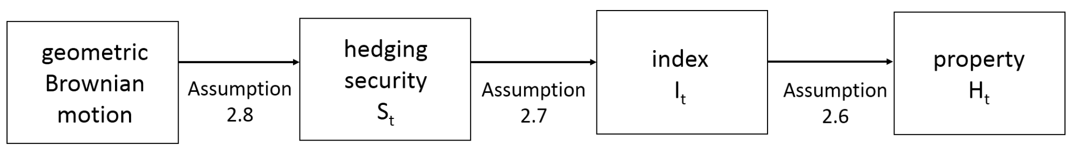


Figure 2.2: The assumed relation among property value, index and hedging security.

2.3 Hedging

In this section, we construct our hedging strategy based on the assumptions we made in the previous section. In the discrete case the insurance policy can be viewed as a portfolio of path-dependent European put options, and in the continuous case the insurance policy is a contingent claim with continuous payoff. We can use delta hedging as the hedging strategy in both cases. Delta hedging is first articulated in Black and Scholes[15] and Merton[64]. They show that under certain conditions and assumptions (they are all satisfied here), European options can be perfectly replicated by the underlying asset and the risk free asset. Bergman[11] extends the results to path dependent options. We rely on Bergman's results to construct the hedging strategy specific to our insurance policy in the discrete case.

2.3.1 Discrete Case

Treat the insurance policy as a portfolio of T/h contingent claims maturing at $h, 2h, \ldots, T$ with payoffs X_{ih} defined in (2.3) respectively. We first consider replicating the ith one.

Under Assumptions 2.6 & 2.7, equations (2.3) & (2.1) can be rewritten as

$$X_{ih}^d(S_{ih}, Y_{(i-1)h}^d) = Y_{(i-1)h}^d p_{ih}(S_{ih}) L_{ih}(S_{ih}), \tag{2.6}$$

and

$$Y_{jh}^d \equiv Y_{jh}^d(S_{jh}; S_h, S_{2h}, \cdots, S_{(j-1)h}) = \prod_{k=1}^{j} \left[1 - p_{kh}(S_{kh})\right] \quad 0 < j < i. \tag{2.7}$$

It is then clear that the claim payment depends on some historical values of S_t summarized by $Y_{(i-1)h}^d$, as well as the current value. Equation (2.7) only gives the expression for Y_t^d where it has jumps, but this is sufficient to characterize a right continuous step function. It is evident that $dY_t^d = 0$ when $t \neq jh$. Since p_t is assumed to be twice differentiable with respect to S_t,

$$\begin{aligned} dY_t^d &= \frac{\partial Y_t^d}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 Y_t^d}{\partial S_t^2} (dS_t)^2 \\ &= \mu_t^Y dt + \frac{\partial Y_t^d}{\partial S_t} \sigma S_t dW_t, \end{aligned}$$

when $t = jh, 0 < j < i$. The drift μ_t^Y is irrelevant to our hedging portfolio. Hence for $0 \leq t < ih$ we can more compactly write

$$dS_t = \mu^0 dt + \sigma^0 dW_t \tag{2.8}$$

$$dY_t^d = \mu^1 dt + \sigma^1 dW_t, \tag{2.9}$$

where

$$\begin{aligned} \mu^0 &= \mu S_t \qquad \sigma^0 = \sigma S_t \\ \mu^1 &= \begin{cases} \mu_t^Y \\ 0 \end{cases} \qquad \sigma^1 = \begin{cases} \dfrac{\partial Y_t^d}{\partial S_t} \sigma S_t & t = jh, 0 < j < i \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

Apply the characterization theorem in Bergman[11], we obtain

$$\Delta_t^i \equiv \Delta_t^i(S_t, Y_t^d) = \frac{\partial V_t^i}{\partial S_t} + \frac{\sigma^1}{\sigma^0} \frac{\partial V_t^i}{\partial Y_t^d} \quad 0 \leq t < ih, \tag{2.10}$$

where $V_t^i \equiv V_t^i(S_t, Y_t^d)$ is the no arbitrage value of the ith contingent claim satisfying the following partial differential equation (PDE):

$$\frac{\partial V_t^i}{\partial t} + \frac{1}{2}\left(\sigma^{0^2}\frac{\partial^2 V_t^i}{\partial S_t^2} + \sigma^{1^2}\frac{\partial^2 V_t^i}{\partial {Y_t^d}^2} + 2\sigma^0\sigma^1\frac{\partial^2 V_t^i}{\partial S_t \partial Y_t^d}\right) \\ + rS_t\frac{\partial V_t^i}{\partial S_t} + \left(\mu^1 - \frac{\mu^0 - rS_t}{\sigma^0}\sigma^1\right)\frac{\partial V_t^i}{\partial Y_t^d} - rV_t^i = 0 \tag{2.11}$$

for $0 \leq t < ih$ with terminal condition

$$V_{ih}^i = X_{ih}^d(S_{ih}, Y_{(i-1)h}^d). \tag{2.12}$$

Therefore our delta hedging strategy is: hold Δ_t^i shares of stock at time t and lend/borrow the rest $V_t^i - \Delta_t^i S_t$ using the risk-free asset. Since Δ_t^i is continuously changing, the strategy requires continuous trading of the underlying stock, which is impractical. The numerical example in the next section assumes discrete rebalancing and analyzes the errors. The PDE (2.11) with boundary condition (2.12) may not have an explicit solution, but the solution admits the Feynman-Kac representation (see for example Theorem 1.3.17 in Pham[71]):

$$V_t^i(S_t, Y_t^d) = E^{\mathbb{Q}}\left[e^{-r(ih-t)}X_{ih}^d \middle| S_t, Y_t^d\right], \quad t < ih \tag{2.13}$$

where $\mathbb{Q}$ is the probability measure under which W_t^* is a standard Brownian motion and

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

With this representation, V_t^i can be estimated using simulation.

Since the insurance policy is a portfolio of contingent claims, the policy value is simply the sum of the values of each contingent claim that has not matured, i.e.

$$V_t^d = \sum_{i=\lfloor \frac{t}{h} \rfloor + 1}^{T/h} V_t^i, \tag{2.14}$$

and similarly the delta for the policy is the sum of the deltas

$$\Delta_t^d = \sum_{i=\lfloor \frac{t}{h} \rfloor + 1}^{T/h} \Delta_t^i = \frac{\partial V_t^d}{\partial S_t} + \frac{\sigma^1}{\sigma^0}\frac{\partial V_t^d}{\partial Y_t^d}. \tag{2.15}$$

2.3.2 Continuous Case

A continuous-payoff contingent claim with maturity T can be characterized by its payoff rate X_t^c, which is defined by

$$X_t^c = \lim_{\epsilon \to 0^+} \frac{C[t, t+\epsilon)}{\epsilon},$$

where $C[a, b)$ denotes the total payoff amount in interval $[a, b)$, or equivalently

$$C[t, t+\epsilon) = X_t^c \epsilon + o(\epsilon).$$

Note that X_t^c may depend on any value of S_u for $u \leq t$.

Assumption 2.9. X_t^c is bounded and continuous.

This assumption guarantees that X_t^c is integrable. Let the pair (V_t^c, Δ_t^c) denote a self-financing continuously adjusted portfolio strategy that holds Δ_t^c shares of stock and lends/borrows $V_t^c - \Delta_t^c S_t$ dollars using the risk-free asset at time t. It is straightforward to show by no arbitrage arguments that if a contingent claim can be replicated by a self financing portfolio strategy, then its value must be equal to the value of the replicating portfolio at any time.

Proposition 2.1. The time t value of the mortgage insurance policy with expiration T and claim rate X_t^c given by (2.4)&(2.2) can be expressed as the following expectation

$$V_t^c(S_t, Y_t^c) = E^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)} X_s^c ds \middle| S_t, Y_t^c \right] \quad t \in [0, T]. \tag{2.16}$$

The self-financing replicating portfolio strategy is (V_t^c, Δ_t^c), where

$$\Delta_t^c(S_t, Y_t^c) = \frac{\partial V_t^c(S_t, Y_t^c)}{\partial S_t}. \tag{2.17}$$

Proof. [13] Based on the above arguments, if we can find a self-financing portfolio strategy that pays X_t^c dollars per year at time t and has zero value at maturity T, then the policy value must be equal to the value of the portfolio. Mathematically, if there exists a strategy (V_t^c, Δ_t^c) satisfying

$$\begin{cases} dV_t^c = \Delta_t^c dS_t + (V_t^c - \Delta_t^c S_t) r dt - X_t^c dt \\ V_T^c = 0 \end{cases}, \tag{2.18}$$

[13] The logic behind this proof follows Björk[14]

then the policy can be replicated by (V_t^c, Δ_t^c) and its time t value equals V_t^c. Our goal is to find such a strategy. Let $V_t^c \equiv V_t^c(S_t, Y_t^c)$ be a process that is twice differentiable in S_t and Y_t^c, once differentiable in t, and solves the following PDE:

$$\frac{\partial V_t^c}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t^c}{\partial S_t^2} + rS_t \frac{\partial V_t^c}{\partial S_t} - Y_t^c \mu_t \frac{\partial V_t^c}{\partial Y_t^c} - rV_t^c + X_t^c = 0 \quad 0 \leq t < T \tag{2.19}$$

with terminal condition[14]

$$V_T^c = 0. \tag{2.20}$$

Define

$$\Delta_t^c = \frac{\partial V_t^c(S_t, Y_t^c)}{\partial S_t}.$$

We claim that (V_t^c, Δ_t^c) defined above satisfies condition (2.18), hence is a replicating portfolio strategy for our insurance policy.

To verify, we first apply Ito's Lemma to $V_t^c(S_t, Y_t^c)$:

$$dV_t^c = \frac{\partial V_t^c}{\partial t}dt + \frac{\partial V_t^c}{\partial S_t}dS_t + \frac{\partial V_t^c}{\partial Y_t^c}dY_t^c + \frac{1}{2}\frac{\partial^2 V_t^c}{\partial S_t^2}(dS_t)^2 + \frac{1}{2}\frac{\partial^2 V_t^c}{\partial Y_t^{c2}}(dY_t^c)^2 + \frac{\partial^2 V_t^c}{\partial S_t \partial Y_t^c}dS_t dY_t^c.$$

From (2.2), we have

$$dY_t^c = -Y_t^c \mu_t dt.$$

Substitute in the differentials for S_t and Y_t^c and combine stochastic and non-stochastic terms, we obtain

$$dV_t^c = \left[\frac{\partial V_t^c}{\partial t}dt + \mu S_t \frac{\partial V_t^c}{\partial S_t} - Y_t^c \mu_t \frac{\partial V_t^c}{\partial Y_t^c} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t^c}{\partial S_t^2}\right]dt + \left[\sigma S_t \frac{\partial V_t^c}{\partial S_t}\right]dW_t.$$

By definition of Δ_t^c and PDE (2.19)

$$\begin{aligned} dV_t^c &= \left[rV_t^c + (\mu - r)S_t\Delta_t^c - X_t^c\right]dt + \sigma S_t \Delta_t^c dW_t \\ &= \Delta_t^c\left[\mu S_t dt + \sigma S_t dW_t\right] + (V_t^c - \Delta_t^c S_t)rdt - X_t^c dt \\ &= \Delta_t^c dS_t + (V_t^c - \Delta_t^c S_t)rdt - X_t^c dt. \end{aligned}$$

This is exactly the differential in condition (2.18). Another condition $V_T^c = 0$ is guaranteed by (2.20). Therefore, the V_t^c that solves (2.19) & (2.20) is the value of our insurance policy.

[14] Boundary conditions for $S_t = 0$ and $S_t = \infty$ are not specified here as we assume they are always satisfied by the solution.

By Feynman-Kac representation,

$$V_t^c(S_t, Y_t^c) = E^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)} X_s^c ds \middle| S_t, Y_t^c\right] \quad t \in [0, T].$$

$\square$

2.4 A Numerical Example

In this section we demonstrate the delta hedging strategy using a numerical example. The example is based on discrete default and discrete rebalancing. We initially assume the prices of the security, which equals the house prices, follow the geometric Brownian motion we have defined and simulate the prices accordingly. Since we hedge in the model, the result will be a verification of the delta hedging theory and should produce small hedging error. We later consider an alternative model for security prices. We do not use real data at this stage.

We first specify the conditional default probability function $p_t(S_t, U_{t^-}^d)$, the loss severity function $L_t(S_t, U_{t^-}^d)$, and other parameter values. We then determine the numerical methods for calculating $V_t^d(S_t, Y_t^d)$ and $\Delta_t^d(S_t, Y_t^d)$. At last we present and analyze our results.

2.4.1 Functions and Parameter Values

As suggested by Case and Shiller[25], the probability of default can be modelled as a function of current loan-to-value (CLTV) ratio. We denote CLTV by R, i.e.

$$R_t = \frac{U_{t^-}}{H_t} = \frac{U_{t^-}}{S_t},$$

and

$$p_t(R_t) = p_t\left(\frac{U_{t^-}}{S_t}\right) = p_t(S_t).$$

Note that R_t is left continuous. If we further assume the function is time invariant[15], then

$$p_t(S_t) = p(S_t).$$

[15]The review[50] reports that mortgage age is a significant risk factor of default probabilities after accounting for the effect of current loan-to-value. Our time invariance assumption is only for simplification.

Parameter	Value	Description
$H_0(S_0)$	\$400,000	initial house price (security price)
R_0	95%	initial loan-to-value
r_c	6%	contract rate
T	15	loan term (in year)
h	1/12	monthly payment
r	5%	risk-free rate
μ	6.4%	expected return on the security
σ	20%	volatility of security price
Default probability parameters		
α_0	3	
β_0	-7.0 for $R \leq 1.2$ -3.4 for $R > 1.2$	intercept
β_1	3.0 for $R \leq 1.2$ 0.0 for $R > 1.2$	coefficient of current loan-to-value

Table 2.1: Benchmark parameter values

The review[50] uses a logistic model for default probability. In our simplified version, we assume

$$p(R_t) = \frac{e^{\beta_0+\beta_1 R_t}}{\alpha_0 + e^{\beta_0+\beta_1 R_t}}.$$

We assume the loss is equal to the difference between outstanding loan balance and property value (stock price), if positive; that is

$$L_t = \max(U_{t^-} - H_t, 0) = \max(U_{t^-} - S_t, 0)$$

To implement the delta hedging strategy numerically, we need a benchmark insurance contract as well as plausible parameter values. The values for default probability related parameters are adapted from 2011 FHA actuarial review. The benchmark parameter values are summarized in Table 2.1. In this example, we use a relatively high volatility for the security to study the effectiveness of discrete rebalancing in a volatile market.

Figures 2.3 shows a randomly generated scenario of security prices. We simultaneously plot current LTV and loan survival probabilities. The house prices fall by more than 25% in the first two years, causing the current LTV rises to above 1.3. The borrower becomes underwater and the probability of default increases sharply, indicated by the fast decline of survival probabilities. House prices increase later and then decline again but the effect

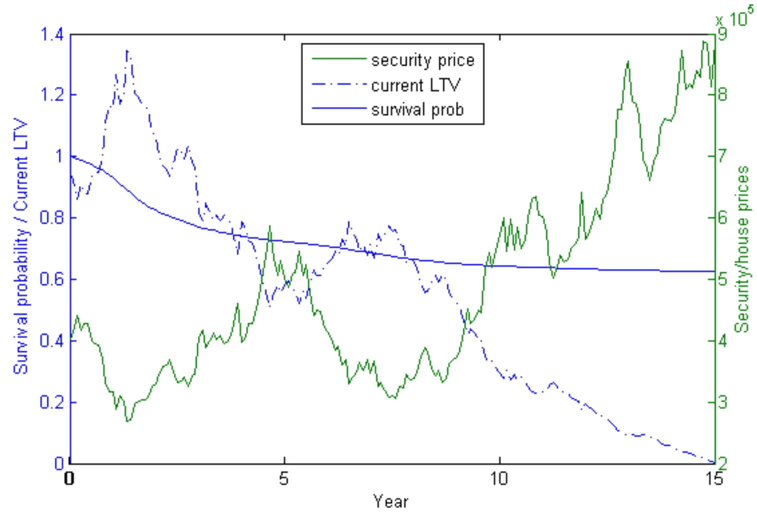


Figure 2.3: A randomly generated scenario of security prices and the corresponding current LTV and survival probabilities.

of this decline on default is less strong than the effect of the decline in the beginning years. This is because by the time of the second house price decline the borrower has already accumulated some equity in the house such that the current LTV is not as high as in the beginning years. It is also expected that the current LTV and house prices move in opposite directions. In the numerical example followed, we simulate 1000 scenarios for security prices.

2.4.2 Calculating Policy Value and Delta

To construct the hedging portfolio we need to calculate V^d and Δ^d. If t is a rebalancing time (not necessarily a payment time), then follow the previous analysis,

$$
\begin{aligned}
V_t^d(S_t) &= \sum_{i=\lfloor \frac{t}{h} \rfloor + 1}^{T/h} e^{-r(ih-t)} E^{\mathbb{Q}}\left[X_{ih}^d \middle| S_0, S_h, \ldots, S_{\lfloor \frac{t}{h} \rfloor h}, S_t \right] \\
&= E^{\mathbb{Q}}\left[\sum_{i=\lfloor \frac{t}{h} \rfloor + 1}^{T/h} e^{-r(ih-t)} X_{ih}^d \middle| S_0, S_h, \ldots, S_{\lfloor \frac{t}{h} \rfloor h}, S_t \right]
\end{aligned}
$$

where X_{ih}^d is defined in (2.3). We use the sequence $S_0, S_h, \ldots, S_{\lfloor \frac{t}{h} \rfloor h}$ instead of Y_t^d to emphasize the dependence of policy value on stock prices. The process Y_t^d does not appear explicitly in the numerical calculation. We use Monte Carlo simulation to estimate this expectation. If we write

$$V_t^d(S_t) = E^{\mathbb{Q}}[W] \quad \text{and} \quad v = var^{\mathbb{Q}}(W),$$

and n paths of stock price are simulated, then by central limit theorem, the estimated policy value, $\hat{V}_t^d(S_t)$, asymptotically follows a normal distribution with mean $V_t^d(S_t)$ and variance v/n. The formula for Δ_t^d is given by Eq.(2.15). If we substitute in the values of σ^0 and σ^1, the right hand side of (2.15) can be treated as the total derivative of policy value with respect to stock price. We approximate it by forward difference

$$\Delta_t^d(S_t) = \frac{\partial V_t^d(S_t)}{\partial S_t} \approx \frac{\hat{V}_t^d(S_t + \epsilon) - \hat{V}_t^d(S_t)}{\epsilon} = \hat{\Delta}_t^d(S_t). \tag{2.21}$$

Both $\hat{V}_t^d(S_t + \epsilon)$ and $\hat{V}_t^d(S_t)$ are calculated based on the same set of random paths. We want to choose an ϵ to minimize the difference between actual and estimated Δ_t^d. Follow Glasserman[41],

$$\epsilon^* = \left(\frac{2v}{[V_t^{d''}(S_t)]^2 n} \right)^{\frac{1}{3}}$$

minimizes the mean square error of $\hat{\Delta}_t^d(S_t)$. For simplicity, we set $\epsilon = (2v/n)^{1/3}$. Whenever $V_t^d(S_t) < 0.01$, both $V_t^d(S_t)$ and $\Delta_t^d(S_t)$ are set to zero until T.[16] We use 1000 simulations to estimate policy values V^d whenever needed.

2.4.3 Results and Analysis

As shown in section 2.3.1, the delta hedging strategy is perfect in a frictionless market where securities can be traded continuously without transaction cost. This means with an initial premium equal to the time-zero policy value being charged at loan origination and the corresponding hedging portfolio being constructed and continuously rebalanced over time, the insurer's net position in the policy (i.e. the value of the hedging portfolio) is precisely zero at the end of loan term after all claims have been paid.

[16] This usually happens between year 6 and 8. We assume in this case the policy value becomes zero and no more hedging is needed. In addition, the calculation of $\hat{\Delta}_t^d(F_t^T)$ is difficult and inaccurate when $V_t^d(S_t)$ is small.

Frequency	Mean(SE)	Std	Min	10%	Median	90%	Max
weekly	18(35)	1111	-3835	-1372	62	1381	4082
biweekly	80(47)	1499	-6760	-1833	141	1870	5919
monthly	108(63)	1989	-8569	-2358	174	2492	7798

Table 2.2: Statistics of HEs based on 1000 independent scenarios. $U_0 = 380,000, V_0^d \approx 5551(1.46\%$ of $U_0)$. SE stands for standard error.

However in practice, as well as in our implementation, continuous trading is not feasible and policy values and deltas are approximated numerically. Consequently the value of the hedging portfolio at the end of loan term deviates from zero. Discount this value to the time of loan origination using risk-free rate and call it hedging error (HE). Since HE depends on security prices, we can obtain the distribution and statistics of HE using the 1000 security price paths we have simulated. We consider different rebalancing frequencies, modifications of hedging strategies, alternative parameter values and security price models.

Rebalancing Frequency

The policy value at origination as a percentage of loan size U_0 is 1.46%, with 95% confidence interval of $(1.4553\%, 1.4660\%)$.[17] For each scenario of security prices, we consider monthly, biweekly and weekly rebalancing. The statistics of HE are shown in Table 2.2.

The mean HEs are all positive but tend to zero as frequency increases. The results are expected and confirm the validity of delta hedging theory. In the case of weekly rebalancing, the p-value for the null hypothesis "mean HE is zero" is 0.62, indicating that zero mean HE cannot be rejected. The large absolute values of minimum and maximum imply ineffectiveness of discrete rebalancing in some scenarios.

To visualize how the hedging portfolio tracks policy value, we pick one scenario and simultaneously plot portfolio and policy values in the first 8 years of the mortgage term. Figure 2.4 shows that weekly rebalancing performs much better than biweekly and monthly rebalancing and generates a small positive HE. It is also clearly demonstrated in this figure that policy values move in opposite direction as security prices. Unless otherwise specified, we use weekly rebalancing for the rest of this section.

[17] We do not use the same initial value and delta for all scenarios but estimate them independently. The difference is caused by Monte Carlo error. This should not have a significant effect on the analysis of HR.

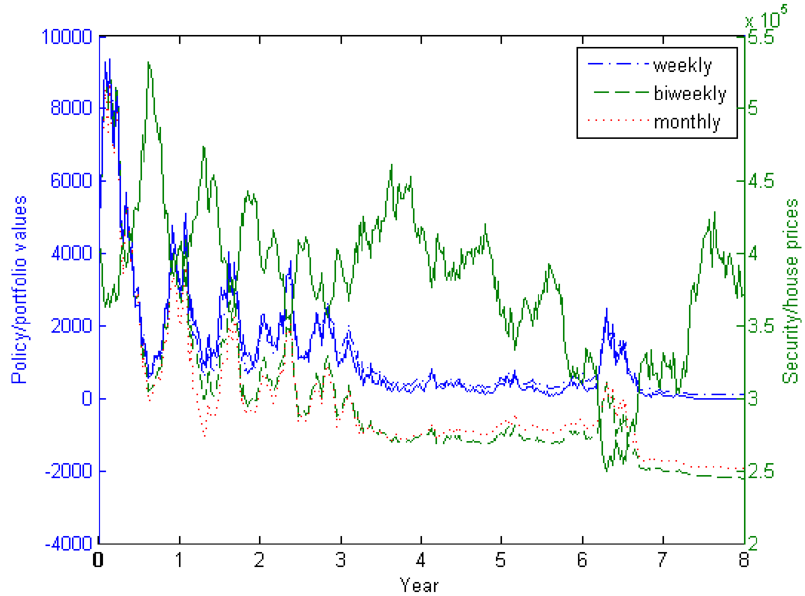


Figure 2.4: A scenario where weekly rebalancing provides a good hedge.

Hedging Period and Proportion

We now consider some modifications of delta hedging, including hedging only in the first 3 years, hedging 50% of the policies, and no hedge. The results are shown in table 2.3.

We can make the following observations: 1) Full Hedge produces the mean HE closest to zero, and HEs are about symmetric around zero. HEs from other strategies are more skewed with mean deviated significantly from zero. 2) All modified delta hedging strategies produce positive mean HE but have a heavy lower tail. This implies the insurers benefits in most cases but faces greater risk of extreme loss. These strategies are vulnerable in a

Hedging strategy	Mean(SE)	Std	10%	Median	90%	Skewness
full hedge	18(35)	1111	-1372	62	1381	-0.09
first-3-year	330(113)	3564	-1636	378	3345	-2.54
50% of policies	795(138)	4367	-3923	2412	3278	-3.40
no hedge	1572(272)	8608	-7523	5100	5791	-3.48

Table 2.3: Statistics of HEs based on 1000 independent scenarios. $U_0 = 380,000, V_0^d \approx 5551 (1.46\% \text{ of } U_0)$. SE stands for standard error.

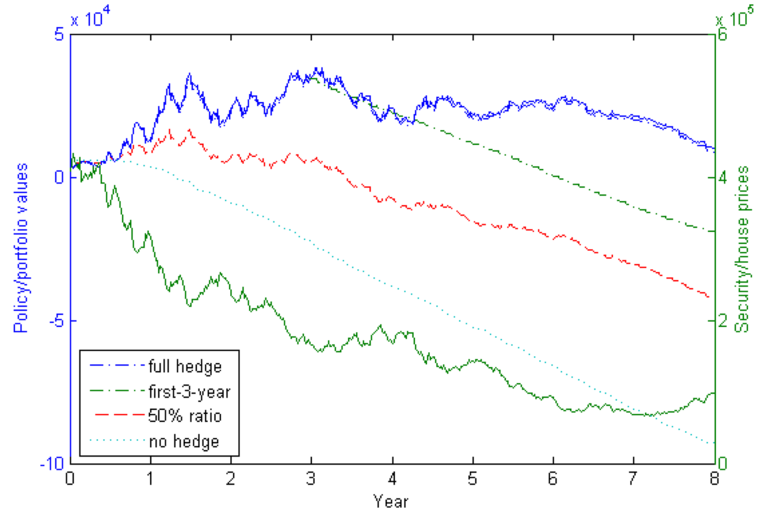


Figure 2.5: A scenario where modified hedging strategies fail.

distressing market where protection is most needed. Figure 2.5 shows such a scenario. 3) HEs from no hedge are positive in 804 scenarios, 307 among which produce HEs equal to the initial policy value, indicating zero claim. One may conclude that under this particular model no hedge generates more profit than full hedge in most scenarios (806 out of 1000), but this does not necessarily imply no hedge is a better strategy than full hedge from a risk management perspective. Section 2.3.4.4 discusses this issue in more detail.

Test for Robustness

We consider alternative values for contract rate r_c, initial LTV R_0, volatility of security price σ and sensitivity of default to current LTV β_1 to test the robustness of delta hedging under discrete rebalancing. One parameter is changed in each case. Table 2.4 indicates that initial policy value is sensitive and increasing in all four parameters. None of the HE means is significantly different from zero at a 95% level based on a student t test.

Full Hedge vs. No Hedge

The goal of hedging is to mitigate house price risk. The non-zero HE of full hedge is a result of discrete trading; it can be affected by house price fluctuations but the dependency is different from and weaker than the dependency of HE from no hedge on house prices. In the benchmark, we have a bullish market where property value increases in most of the

	V_0^d(% of U_0)	HR Mean(SE)[% of V_0^d]	HR Std[% of V_0^d]
Benchmark	5551 (1.46)	18(35)[0.32]	1111[20.00]
$r_c = 0.1$	7204 (1.90)	40(39)[0.55]	1218[16.91]
$R_0 = 0.85$	2528 (0.74)	6(23)[0.22]	725[28.69]
$\sigma = 0.15$	2060 (0.54)	16(18)[0.75]	581[28.19]
$\beta_1 = 3.5$	7974 (2.10)	-65(50)[-0.82]	1579[19.81]

Table 2.4: $U_0 = 380,000$, the last two columns are mean and standard deviation of HEs. Benchmark: $r_c = 0.06$, $R_0 = 0.95$, $\sigma = 0.20$, $\beta_1 = 3.0$

	V_0^d	Full hedge			No hedge			out of
		Mean(SE)	VaR	CTE	Mean(SE)	VaR	CTE	1000
Benchmark	5551	18(35)	-1372	-2009	1572(272)	-7523	-20197	194
$\mu = 0.03$	5530	-18(37)	-1416	-2165	-1965(429)	-19062	-37131	310
$\mu = 0$	5531	-33(41)	-1558	-2436	-6625(540)	-32441	-47623	458
$\sigma = 0.30$	16316	-4(63)	-2549	-3657	2685(634)	-25637	-46282	287
$\sigma = 0.40$	29849	-67(97)	-3968	-5739	3779(957)	-43258	-64209	340
RSLN 1	1624	-324(72)	-3017	-5895	-900(220)	-6088	-18158	232
RSLN	1877	-433(80)	-3627	-6712	-1635(278)	-9446	-23823	271
RSLN 2	2770	-487(97)	-4788	-7401	-3395(402)	-16420	-36193	386

Table 2.5: Both VaR and CTE are at 90% level. The last column is the number of scenarios where full hedge has a larger HE than no hedge. Monthly rebalancing for RSLN model. Benchmark: $\mu = 0.064$, $\sigma = 0.20$.

world and no hedge tends to outperform full hedge. To compare the two strategies in less optimistic markets, we change the model parameters μ and σ and calculate the mean, 90% value at risk (VaR) and conditional tail expectation (CTE) of HEs, as well as the number of scenarios where full hedge generates higher HE than no hedge.

The results are shown in the first five rows of Table 2.5. Full hedge becomes more advantageous when μ is reduced or σ is increased, implied by the relative small absolute values of VaR and CTE. For no hedge, an increase in volatility increases mean HE but also magnifies tail risk. VaR and CTE from no hedge are ten times of those from full hedge.

In addition to geometric Brownian motion, we consider a regime switching lognormal (RSLN) model for security prices. A two regime RSLN model consists of regime 1, representing a thriving environment with higher drift and lower volatility and regime 2, representing an opposite depressing environment. Prices follow geometric Brownian motion in each regime with its specific parameters. To simplify simulation, we assume the regime switching process is observable.

The market under a RSLN model is incomplete (see Hardy[45]) because the regime switching process is not replicable. It follows that there does not exist a perfect hedging strategy.[18] For the purpose of demonstration, we keep using delta hedging. Policy value is calculated by equations (2.14) & (2.13) and delta is approximated using (2.21). Due to market incompleteness, there are many choices of risk neutral measures. We choose a simple $\mathbb{Q}$ such that the drifts in both regimes equal risk-free rate. Note that $\mathbb{Q}$ is used for policy pricing and delta calculation, and $\mathbb{P}$ is used for simulating scenarios of security prices. Figure 2.6 shows our RSLN model with assumed parameter values. To save computational cost, we assume monthly rebalancing and regimes can only be switched at the end of a month. Transition probabilities at month ends are denoted by p_{12} and p_{21} and are independent of stock prices.

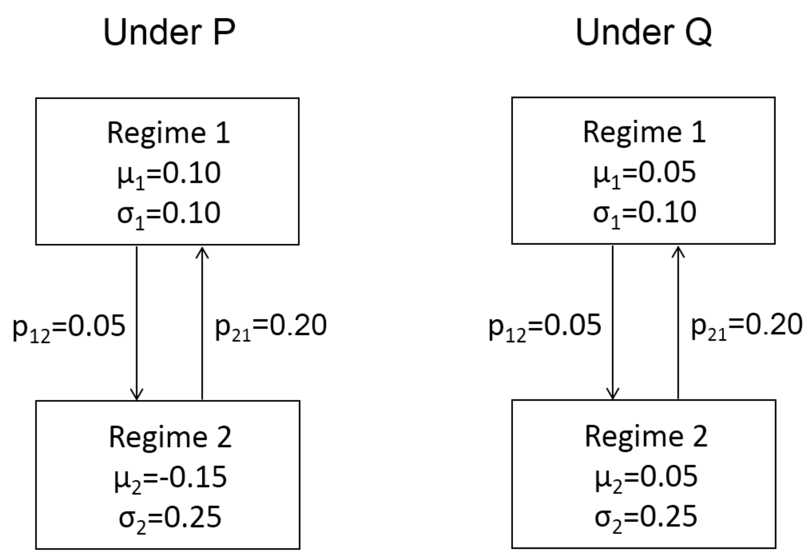


Figure 2.6: Two regime RSLN.

The last three rows of Table 2.5 show the statistics of HEs when security price follows RSLN model starting from regime 1, the stationary distribution of regimes and regime 2. Mean HE deviates significantly from zero, suggesting the imperfectness of delta hedging in these cases. Although no hedge is more likely to generate a higher return than full hedge, its average return is lower and it faces severer losses in extreme scenarios.

[18] There are hedging strategies dealing with incomplete market, for example mean variance hedging and utility based hedging, but none of them can perfectly replicate the payoff.

2.5 Conclusion

In this chapter, we examine the use of a traded security to hedge the default risk in a mortgage insurance contract. In the framework of our simple theoretical model and very strong assumptions, we show the ability of hedging in mitigating default risk arises from house price fluctuation. Ignoring prepayment and basis risk, we find that hedging errors are not significantly different from zero, and are mainly caused by discrete rebalancing and numerical calculation.

There are several reasons why hedging is not considered as a risk mitigation in the current mortgage and mortgage insurance industries. First, there is not a suitable hedging instrument in the current market. The existing property index derivatives markets are not liquid and the derivative prices are not predictable. Second, the relation between house price and insurance claim is complex. The changes in individual house prices are different from the changes in the index due to basis risk. Third, there are other risks other than house price risk embedded in a mortgage or mortgage insurance contract. For example, the prepayment risk which is closely related to interest rates rather than house prices.

The benefit of our study is to disentangle the factors that currently make hedging impractical. To the extent that future innovations and development in the housing finance market remove or reduce the role of these factors, hedging will become feasible.

Chapter 3

Property Index-Linked Mortgage

In this chapter, we discuss a new design of mortgage contract – Property Index-Linked Mortgage (PILM). A notable feature of PILM is that mortgage balances and payments are linked to an appropriate property index so that the borrower repays less when the value of the mortgaged property declines. The intuition behind is to reduce the borrower's propensity to default when he/she has a negative equity position in the mortgage loan. The approach we take to analyze PILM is different from those taken by other researchers. We take the borrower's perspective and analyze how the borrower's incentives change under a PILM compared to a standard FRM. The borrower's utility, as well as the lender's cash flow, then depends on the action the borrower takes regarding the mortgage. The borrower may choose to continue to repay, default or refinance. Our multiperiod utility optimization model is a simplified life cycle model. The results show that within a wide range of plausible parameter values, the borrower and the lender of a PILM can both be better off than those of a FRM.

Section 1 provides some background on PILM, including motivation, recent proposals and a discussion of the existing studies. Section 2 specifies the three mortgage contracts that we will study and compare in detail in later sections. They are one standard FRM and two designs of PILM. In section 3, we setup the borrower's problem mathematically and provide a method to solve it. We present the baseline numerical results in section 4 and use alternative parameter values in section 5. Section 6 concludes.

3.1 Motivation, Literature Review and Discussion

This section discusses the causes and cost of default. They are the motivation for PILM. We expect PILM to be a more efficient design than FRM when negative equity is a triggering event of default and default is costly to the lender and/or the borrower. We then review three recent proposals of PILM and discuss the limitation of these studies.

3.1.1 Motivation

The effectiveness of PILM is based on two assumptions: 1) low (negative) equity leads to default, and 2) default is costly. When both of the assumptions hold, reducing the borrower's obligation at adverse times should lead to a lower default propensity and hence a lower default cost. The use of PILM is justified if the savings from default cost exceed the reduction in mortgage payment. We start with a discussion of "default triggers".

The Causes of Default

Researchers have been trying to identify the causes of default since 1990s. Studies[1] employing different frameworks, statistical methods and data sets draw different conclusions, but the majority of them focus on two factors: negative equity and financial illiquidity. Negative equity refers to the situation where a borrower owes more than the mortgaged property is worth. It is usually caused by a significant decline in property value. Financial illiquidity directly affects a borrower's ability to repay the loan. This could be a result of adverse life events such as loss of job or illness. These events may exhaust not only a borrower's income and savings, but also the ability to refinance. It is beyond the scope of this thesis to study in detail which of the two factors is more influential in causing mortgage default, but summarizing some existing results is necessary since if negative equity is irrelevant to default, reducing loan balance could be useless in preventing defaults.

Kau et al.[57] and Ambrose and Buttimer[2] establish a contingent claim framework based on option pricing theory, assuming the borrower treats default as a financial option with property value and mortgage rate being two underlyings. For a given mortgage rate, the borrower defaults as soon as the property value is below some boundary.

Deng[30] incorporates the option view into a competing risk model and estimates the explanatory power of some important factors such as option values and unemployment

[1] Gerardi et al.[40] list a series of studies that are related to the causes of defaults.

rate. The results show that the value of the put option, which increases when property value declines, is statistically significant in explaining the probability of default. Order[69] summarizes some studies using different data sets and models, showing that both equity and FICO score (indicator of adverse events) were strong indicators of default propensity. By using credit card utilization and combined loan-to-value, Elul et al.[35] conclude that both negative equity and illiquidity are significantly associated with default and the two factors interact with each other.

The 2011 FHA actuarial review[50] follows the approach of Deng et al.[31] and finds that negative equity is a significant factor in determining loan status transition. Factors measuring a borrower's liquidity, such as unemployment rate and debt-to-income ratio, are not statistically significant until 2012. In addition to raising default propensity, negative equity also hinders voluntary cure of delinquent loans, as Goodman[42] argues "..., becoming delinquent can trigger a re-evaluation of financial circumstances. At that point, curing becomes an economic decision...".

Despite the traditional view that negative equity is a necessary but not sufficient condition for default,[2] the notion of strategic default starts attracting people's attention.[3] Strategic default refers to default at the time when the borrower has the ability to make payments but does not choose to. Bhutta et al.[12] conclude that borrowers do not voluntarily walk away until their properties were deeply underwater, say, the loan balance is 150% of the value of the property. Guiso et al.[43] analyze strategic default using survey data. They find that 26.4% (35.1%) of defaults were strategic in 2009 (2010), and suggest that "strategic defaults represent an important fraction of defaults when home equity is negative". They also find that the borrower's propensity to default strategically increases with the size of equity shortfall and there is a risk of social contagion that knowing someone had defaulted strategically increases the homeowner's willingness to default strategically by 51%. This means when borrowers are increasingly aware of strategic default, controlling the extent of negative equity becomes important in reducing defaults.

Cost of Default

There are both monetary and nonmonetary costs associated with default. To a borrower, monetary cost includes 1) a worse credit leading to higher financing cost in the future, 2) recourse in states with deficiency statutes, 3) moving, relocation and searching for new housing, and 4) giving up the potential of property value increase in the future.

[2] See for example Foote et al.[37].

[3] Also known as "ruthless default", or "walk-away default". The term "strategic default" was first searched on Google in 2009.

Nevertheless the borrower enjoys the benefit of free rent during foreclosure period, which may last for 6-12 months. Nonmonetary cost, such as sentimental attachments to homes, the bad feeling of doing something immoral and social stigma, is difficult to quantify and varies among borrowers.

Lenders would consider the use of PILM only when they face a high cost of default or foreclosure. Brueggeman and Fisher [17] list various costs associated with default, and Ambrose and Capone[5] conclude that foreclosure is the most expensive default alternative the lender. Explicit cost incurred during the foreclosure process includes 1) property taxes and hazard insurance, 2) property management, 3) interest accrued on loan balance, 4) legal and administrative fees and 5) sales cost. Furthermore, the process of foreclosure can take more than one year. Ambrose et al.[4] show that a long delay of foreclosure provides the borrower incentives to default and live rent free. Foreclosure alternatives may reduce the total cost, but lenders could be exposed to moral hazard because of revealed information; and if the negotiation of an alternative fails the total cost just increases.

There are different methods to sell the foreclosed property and the lender always has the opportunity to purchase it. Public auction is a widely used method. If the lender wins the auction,[4] the property then becomes a real estate owned (REO) property.[5] It is well known that a foreclosed property or REO is likely to be sold at discount. This is a substantial cost of foreclosure faced by the lender. Pennington-Cross[70] finds that on average foreclosed properties appreciate 22% less than other properties, and Campbell et al.[22] estimate a price discount of 27% in the presence of foreclosure. One explanation of this discount is the property's physical depreciation during foreclosure period, caused by lack of maintenance or vandalism. Also the lender has the incentive to sell an REO as quickly as possible, since REO is a nonperforming asset that requires constant input (maintenance) while generating no output (interest or rent). As a result, the lender has to accept a discount in the illiquid housing market. All costs combined, the lender could lose 30% to 60% of the outstanding loan balance in a foreclosure.

Researchers have also been concerned about the negative externalities of foreclosure. Besides the social contagion of strategic default mentioned in the previously, foreclosure tends to lower the prices of nearby houses. Campbell et al.[22] estimate that a typical foreclosure in 2008 reduces the total prices of neighboring houses by $477,000. Anenberg and Kung[8] use listing data to show that properties located close to a foreclosed property

[4] The data used in Campbell et al.[22] indicate that lenders acquire properties in 82% of the cases.

[5] Stark[79] describes typical foreclosure processes in jurisdictions that either require a judicial sale or permit a non-judicial sale. Johnson[54] provides a hypothetical example illustrating the detail of a foreclosure process and explains the reason behind certain phenomenon, for example, why the lender bids at loan balance, and why in many cases no one bids higher even the property is worth more.

sell at a 1.6% discount on average. Using foreclosure law as an instrument, Mian and Sufi[65] calculate that "house prices declined by 1.0 percentage points for every 1 percent of home owners going into foreclosure between 2007 and 2009". What is worse is that lower house prices lead to more defaults since borrowers in the same neighborhood are deeper underwater. This vicious cycle creates a downward spiral of house prices.

Overall, the cost of foreclosure is considerable. To prevent default, the lender should try to keep the borrower motivated in making payment. This could be done by reducing the borrower's loan balance and payment when house prices decline. The house price risk is then shared between the borrower and the lender, as the lender receives less payment when house prices fall. Since individual house prices are not readily observable without expensive ongoing appraisals, the borrower's payments are linked to a suitable property index rather than the exact value of the mortgaged property. This type of mortgage contract is called a property index linked mortgage (PILM). Next we review some recently proposed designs of PILM.

3.1.2 Recent Proposals

The three proposals to be reviewed are adjustable balance mortgage (ABM) by Ambrose and Buttimer[3], continuous workout mortgage (CWM) by Shiller et al.[77] and shared-responsibility mortgage (SRM) by Mian and Sufi[65].

Adjustable Balance Mortgage

According to Ambrose and Buttimer[3], an ABM is fully amortized at origination using a fixed contract rate and term. The mortgage balance is adjusted according to the return of a local property index capped by the originally scheduled balance. Let H_0 denote the initial property value and

$$H_t = \frac{I_t}{I_0} H_0$$

be the property value at time t calculated based on the index I_t. Let U_t be the originally scheduled balance at time t. Then at a predetermined adjustment date, the balance is adjusted to

$$U_t^* = \min[H_t, U_t]$$

and loan payments are revised by re-amortizing U_t^* over the remaining term. By design U_t^* is capped by U_t, hence the revised loan payment is never higher than the initial level but has the potential to be lower. This suggests that the ABM be sold with a premium over the corresponding FRM that has the same contract rate and term as the ABM.

Ambrose and Buttimer price this premium by assuming the ABM either has a higher contract rate or a higher initial points. They model the value of a mortgage loan as a contingent claim on mortgage rate and property value (H_t). The value of the mortgage is then solved backwards. Their results show that for a 95% LTV mortgage, quarterly balance adjustment reduces the value of default option to zero while only raises contract rate by 42 basis points, or equivalently, raises the initial points by 71 basis points. They also find that when lenders face high default cost, they would even be willing to offer the contract rate lower than the corresponding FRM.

The authors qualitatively analyze the pros and cons of using a local property index as an approximation to the property value. Using an index eliminates the costly and time-consuming independent appraisals, as well as the risk of moral hazard. If instead the individual property value is used, borrowers may have the incentive to intentionally reduce the property value or alter the appraisal result in order to reduce loan payment.

One major drawback of using an index is basis risk. In the case where the return of the index is lower than that of the property, the mortgage balance may be over adjusted and the lender bears all the risk. The authors argue that this risk could be mitigated through portfolio effects and hedging, if the index well captures the average movement of house prices and the lender has a portfolio of ABMs that reflect this average. In the opposite case where the return of property is lower, the balance may be adjusted higher than the actual property value and the borrower does not get full benefit from the ABM. Nevertheless the borrower is no worse off than under a standard FRM and still has the option to default.

Continuous Workout Mortgage

The basic idea of CWM proposed in Shiller et al.[77] is similar to ABM – sharing house price risk between the lender and the borrower. Rather than linking mortgage balance to an index, CWM directly links payments to the index. The balance is then revised to be the expected present value of future cash flows under some probability measure.

This approach simplifies calculation in some circumstances and admits closed form pricing formulas. It is also more flexible in facilitating additional features. For example, under continuous repayment assumption, the mortgage repayment rate of a CWM with partial protection and protective threshold is given by

$$M_t^c(H_t) = \rho \min\left\{1, 1 - \kappa\left(1 - \frac{H_t}{K}\right)\right\},$$

where M_t^c is the continuous repayment rate, ρ is the cap of repayment that could be equal to or above the originally scheduled level, κ is the proportion protected and K is the threshold below which the adjustment kicks in. CWM can be priced by finding a set of contractual parameters (contract rate, cap of repayment rate, etc.) that equates the time zero value of cash flows and the original loan amount.

The authors have also mentioned prepayments due to non-financial reasons and its rate is modelled by a constant intensity which is independent of housing market and balance adjustment. Prepayment in this sense is a risk to long term investors because it induces mismatch of asset and liability. The risk is also borne by standard FRM contracts and can be mitigated by imposing a prepayment penalty. However, it seems prepayment risk could be more serious in a PILM when mortgage balance or payment is adjusted downwards. This is discussed shortly.

Shared-Responsibility Mortgage

The SRM proposed by Mian and Sufi[65] is similar to CWM that mortgage payment is linked to property index. The difference is that when property value declines, both mortgage payment and loan balance are adjusted downwards proportionally from the originally amortized values. To charge for this downward protection, the authors suggest that the lender takes a small share, say 5%, of the capital gain from the property when it is sold or refinanced if the appraised value is higher than its initial price.

The authors set their discussion of SRM in a broad economy. First, it prevents foreclosure. The current LTV is never lower than the initial LTV by design so borrowers would rather sell the property than being evicted. Less foreclosures mitigates the vicious cycle of house price decline. Second, it maintains household consumption. Households tend to cut down consumption when their net worth reduces. By impeding house price decline and shifting a significant portion of equity loss from borrowers to lenders who have a low marginal propensity to consume, SRM helps borrowers keep their consumption. Third, it prevents job loss by retaining household consumption. The saved jobs further contribute to consumption and more job saving. Last but not least, SRM may even help prevent the housing bubble. Since SRM lenders share more housing risk, they would be more wary about the housing market. If they believe the prices are going to fall they may raise mortgage rate to compensate for the risk, which results in less loan origination and reduction in housing demand.

3.1.3 Discussion

Although the three designs of PILM are proposed, some issues are untouched or need more extensive analysis. This subsection discusses qualitatively prepayment risk, basis risk, and modelling.

Prepayment Risk

Borrowers prepay mortgages for various reasons: some are non-financial, such as job relocation, and some are financial, such as taking advantage of a lower rate. To lenders, prepayment may cause mismatch between asset and liability, and a reduction in expected return when market interest rate falls, hence is treated as a risk. This risk is borne in both FRM and PILM contracts, but PILM could suffer more. The prepayment risk we discuss here is referred to those borne exclusively by PILM – risks that borrowers prepay because of (over) adjustment of loan balance/payment. This type of prepayment is "strategic", for it is financially motivated and takes advantage of balance adjustment.

In contrast to principal write-down, PILM mortgage adjustment is not permanent – payment reverts to the original level when index rises. Consider the scenario in Figure 3.1 where U_t is the originally amortized balance and H_t is the property value estimated from the index. Let t_1, t_2 and t_3 be three adjustment dates. According to the balance adjustment scheme of ABM, balances are adjusted to $U^*_{t_1} = H_{t_1}$, $U^*_{t_2} = H_{t_2}$ and $U^*_{t_3} = U_{t_3}$ at t_1, t_2, t_3. The lender's cost is then the reduction of payments between t_1 and t_3, which is typically smaller than either $U_{t_1} - U^*_{t_1}$ or $U_{t_2} - U^*_{t_2}$. However a prepayment at t_2 could turn the temporary adjustment to a permanent one and increases the cost to $U_{t_2} - U^*_{t_2}$. Such strategic behavior reduces the efficiency of PILM because it substantially increases the lender's cost. The risk of strategic prepayment can be exacerbated by the presence of basis risk.

We therefore conjecture that the effect of balance adjustment is two folded. The borrower's propensity to default is reduced, and the propensity to prepay is increased. The increase in propensity to prepay is also pointed out in Shiller[75]: "people might strategically prepay their mortgages at a time when home prices start rising strongly". Low default propensity is desirable but the high prepayment propensity is a negative side effect. It is difficult to estimate how strong this side effect could be, for PILM has not been put into

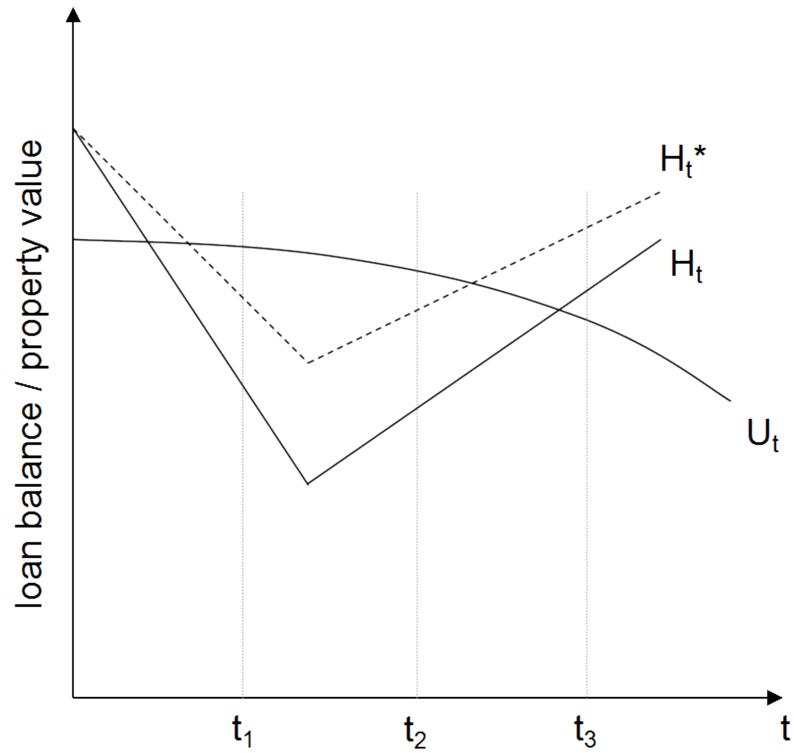


Figure 3.1: Basis risk of PILM: actual property value (H_t^*) is higher than index estimated value (H_t). The original balance is U_t and the adjusted balance is the minimum of U_t and H_t.

practice yet. It is suggested by Shiller et al.[77] that the lender can impose a prepayment penalty to mitigate this potential risk.

Basis Risk

Basis risk is discussed in Ambrose and Buttimer[3], but I think it is worth a more extensive analysis. In the context of PILM, basis risk refers to the risk arising from the situation that the value of a property is not perfectly correlated with the property index. This is the case in reality because the value of a property depends on many individual specific factors such as property type, household utilization, maintenance, etc., hence is different from one to another. We can further assume that borrowers as property occupiers have more detailed information about their properties than lenders, so the actual values of the properties are known to them but not to lenders.[6] Basis risk is not a problem to FRM lenders but may become one to PILM lenders, as it can lead to borrower strategic behavior.

We use the following example to demonstrate the effect of basis risk on PILM. Assume the contract is an ABM and the property and the index start with the same initial value H_0. Suppose at an adjustment date t, $H_t < U_t$ and the loan balance is adjusted to H_t. The existence of basis risk implies that rather than the property and the index having the same time-t value of H_t, they have different values. We assume the actual property value is H_t^* and it is only known to the borrower. As a result, if $H_t^* < H_t$ the borrower may opt for strategic default because he/she is still underwater; if $H_t^* > H_t$ the borrower may choose to prepay to realize an extra gain of $H_t^* - H_t$. These behaviors are at the cost of the lender and may reduce the efficiency of PILM.[7]

When considering mitigating basis risk, a lender should not worry much about the situation of $H_t^* < H_t$, as long as the difference is not too large. Although the borrower has negative equity after adjustment, they at least get partial benefit from PILM and their propensity to strategic default is reduced. The efficiency loss in this situation is the reduction in benefits due to under-adjustment. If the borrower defaults, the lender's position is not worse than the corresponding FRM lender, because in most cases the lender's claim on the collateral (equals loan balance plus default cost) is more than what can be recovered.

A lender should be more concerned about the situation of $H_t^* > H_t$. In this case loan balance is over adjusted. The loss to the lender is limited as long as the loan stays current, since the adjustment is only temporary. However the borrower may behave strategically against the lender. Consider again the scenario in Figure 3.1 where the index depreciates faster than the property. If an ABM borrower does not prepay or default between t_1 and t_3, the loss to the lender is the extra reduction in repayment between t_1 and t_3, which depends on $H_{t_1}^* - H_{t_1}$ and $H_{t_2}^* - H_{t_2}$ but is smaller than both. If the borrower defaults, the extra loss due to basis risk is positive when the net recovery from foreclosure is higher than H_t. This does not happen unless the property value is deviated substantially from the index and the foreclosure cost is mild. The worst scenario to the lender is that the borrower prepays at some time between t_1 or t_3. The lender incurs an instant extra loss of $H_t^* - H_t$ due to over-adjustment. In addition, refinancing becomes easier to the borrower as his/her

[6] Borrowers can have their properties appraised but lenders may not be able to appraise all properties in their mortgage portfolios.

[7] Similar risk also exists in loan modification. Foote et al.[36] define "Type I error" as a lender fails to modify a loan and the borrower defaults, and "Type II error" as a lender modifies a loan but the borrower would not have defaulted without the modification. Both errors induce losses. Type I and II errors correspond to the situations of $H_t^* < H_t$ and $H_t^* > H_t$ respectively.

equity position reverses from negative to positive. Consequently, basis risk exacerbates prepayment risk.

To summarize, basis risk reduces the efficiency of PILM and over-adjustment is especially unfavorable to the lender. There is not much can be done by the lender to mitigate this risk but the analysis above shows that the loss may not be large, and prepayment penalty may be helpful in preventing strategic prepayment due to over-adjustment. Basis risk is discussed in more detail in Chapter 4.

Modelling

Contingent claim models have been used in mortgage literature for decades. It is appealing because its mathematical formulation is relatively simple, usually in the form of stochastic partial differential equations. The solution sometimes admits closed form expression. There are also well developed numerical methods for solving differential equations.

However these models have drawbacks. First, they typically require a complete market where all risks are tradable. The two important risks are house price risk and interest rate risk. Unfortunately there is no financial instrument currently available for effective house price hedging. The derivative market is not liquid and REIT stocks are not closely correlated with house price indices.

Second, these models cannot account for borrower heterogeneity. Borrowers with different risk aversions, patience in consumption, willingness to repay may default under different market conditions. But the decision rule derived from contingent claim models are identical for all borrowers.

Third, it is not easy to incorporate non-strategic default into a contingent claim model. For example, financial illiquidity is usually caused by an accidental event that is beyond the control of the borrower. This means that default may occur when economic boundary conditions are not satisfied.

Alternatively, a multiple state model, such as the one introduced in chapter 1, describes a borrower's behavior more flexibly. For any given levels of property value and mortgage rate, there are positive probabilities for the borrower to default, prepay and stay current. Using probabilities instead of boundaries captures the heterogeneous behavior among borrowers. The model has no requirement on financial market and the probabilities have accounted for default and prepayment due to non-financial reasons.

Another choice of model is utility optimization model. We can assume the borrower maximizes his/her expected utility of life time consumption by choosing among the mortgage related actions (default, prepay, etc.). Such model is also flexible enough to analyze the effect of different borrower characteristics on default.

3.2 Contract Specification

FRM contract has long been the dominant type of contract in the US mortgage market. The payment under a FRM is fixed at loan origination and independent of the value of the mortgaged property. The PILM contracts we will specify have the feature that their payments and balances are contingent on the property value – it is reduced when property value declines (significantly).

In this section we first give a general discussion about some designing issues of PILM contract. We then specify, with mathematical terms, one FRM contract and two different designs of PILM contracts that we will study in the rest of this chapter. At last we provide a comparison between the two PILMs using a numerical example.

3.2.1 PILM Design

As suggested in the existing proposals, there are two choices of what to adjust: balance and payment. For balance adjustment, lenders may reset the balance such that the CLTV never exceeds a predetermined target level. In ABM, the level is 100%. A higher target level could be a better choice since empirical studies have found that borrowers do not default until the loans are deep underwater, and in addition it may reduce prepayment risk as discussed in the previous section.[8] For payment adjustment, it can be adjusted in proportion to the change in property value. The lender can set a cap on adjustment to allow the borrower to accumulate equity when housing market recovers. The originally amortized balance or payment can serve as this cap. The amount of adjustment should be a function of property value, which is estimated from an index. We assume there is no lag in index updating.

In the previous section we discussed mitigating the risk of strategic prepayment. One way is to stipulate a prepayment penalty so that borrowers cannot lock in the reduced balance through prepayment after a downward loan adjustment. If possible, the penalty

[8]For a borrower facing both illiquidity and negative equity, balance reduction may not be enough to prevent the mortgage from default. The lender should also consider granting loan forbearance to help the borrowers get through financial hardship. Combined with principal reduction, loan forbearance could be more effective.

could be prepayment-type dependent. For example, prepayment through refinancing from the same lender with a similar PILM is exempt from penalty, and prepayment through property sale or FRM refinancing does incur a penalty.

Since a PILM lender shares with the borrower house price risk, PILM may be sold at a higher price than the corresponding FRM. The higher price can be in the form of

- A higher initial point. This amount is fixed at loan origination and independent of future housing market.
- A higher contract rate. This increases payments and balances at all times. If the adjustment cap is the originally scheduled balance, it is also increased.
- Shared appreciation. This means the lender has a claim on the property if its value rises above its initial level at the time of loan termination. For example if the property is sold at a higher price than the purchase price, the lender can receive a proportion of the difference between these two prices. If there is not a sale, the lender can have the property appraised.[9]
- A higher cap for balance adjustment. Under this form, a PILM borrower may face a higher payment than a FRM borrower when the property index rises above the initial level.[10]
- A higher prepayment penalty. The borrower can avoid the penalty by not prepaying but this is desired by the lender in most cases.

In what follows, we use H_t, M_t and B_t to denote the time-t values of the mortgaged property, the mortgage payment and the outstanding loan balance, respectively. The formulas for M_t and B_t presented below are derived under the assumption that the borrower does not prepay prior to time T, the term of the mortgage.

[9] One disadvantage is that it suffers from prepayment: a lender may never get the share if the borrower prepays before house price rises. The problem is that shared appreciation is not front-loaded and a borrower can avoid paying the surcharge by terminating the contract at any time. Using the evidence from life insurance, Hendel and Lizzeri[46] finds that front-loading of premiums is efficient in a long-term unilateral commitment contract. The situation is similar in a mortgage loan. Shared appreciation may further induce adverse selection when there is basis risk. Borrowers having high-valued properties are more likely to prepay, leaving the lender a pool of mortgages with low collateral values. On the other side, shared appreciation makes borrowers reluctant to refinance at good times in order to avoid losing the share, hence prevents equity withdraw when property value is high.

[10] A drawback of a higher adjustment cap is that it takes longer for a borrower to move into a positive equity position after experiencing a depressing market.

3.2.2 Fixed Rate Mortgage (FRM)

In a FRM contract, the loan is fully amortized at origination such that the borrower repays the same amount at each time point. The mortgage payment due at time t is given by

$$M_t = \alpha H_0 \frac{\mathrm{ER}_0}{1-(1+\mathrm{ER}_0)^{-T}}, \qquad t = 1, \ldots, T,$$

where α is the initial loan-to-value (LTV) ratio and ER_0 is the mortgage contract rate for a FRM contract originated at $t = 0$. Note that for a FRM contract, the value of M_t is determined at origination and does not change over time. The amount of the outstanding balance declines to zero over the term of the loan from the initial amount of αH_0. It is a deterministic function of time and thus does not depend on the current property value. There is no prepayment penalty.

3.2.3 Property Index Linked Mortgage: Version One (PILM1)

In a PILM1 contract the outstanding balance on the mortgage is adjusted to reflect house price changes as measured by an appropriate property index. Immediately after the mortgage payment at time t, the unpaid principal balance is adjusted so that the current LTV (calculated using the house price implied by the property index) does not exceed the target LTV. If the price decline is severe enough the loan balance is adjusted downwards. The mortgage payment then drops as it is based on the revised loan balance. In this way the borrower's payment goes down when there is a severe house price decline. On the other hand if house prices rise, the loan balance may be adjusted upwards, but the post-adjustment balance is capped by an amount calculated at loan origination.[11]

We first specify the reference balance and payment. The reference balance is the balance the borrower faces when the loan is not adjusted, and it also represents the upper bound of post-adjustment loan balance if an adjustment is triggered. The reference payment serves a similar purpose. Let U_t be the time-t reference balance and $\bar{M}$ be the (constant) reference payment. The values of U_t and $\bar{M}$ are determined at loan origination by amortizing the initial loan amount over the loan term with contract rate $\mathrm{ER}_0 + \mathrm{RS1}$, where RS1 is the rate spread of PILM1 over the corresponding FRM; that is,

$$U_t = \begin{cases} \alpha H_0, & t = 0 \\ U_{t-1}(1+\mathrm{ER}_0+\mathrm{RS1}) - \bar{M}, & t = 1, \ldots, T \end{cases},$$

[11] The advantages of capping the post-adjustment loan balance is discussed by Ambrose and Buttimer [3].

and

$$\bar{M} = \alpha H_0 \frac{\mathrm{ER}_0 + \mathrm{RS1}}{1 - (1 + \mathrm{ER}_0 + \mathrm{RS1})^{-T}}.$$

During the term of the loan, the actual loan balance B_t may deviate from U_t due to adjustments that ensure the target LTV, $0 < \bar{\alpha} < 1$, is maintained. Mathematically, the post-adjustment loan balance of the PILM1 at time t can be expressed as

$$B_t = \min\left(\bar{\alpha} H_0 \frac{I_t}{I_0}, U_t\right), \quad t = 1, \ldots, T.$$

For simplicity, we assume that there is no basis risk, so that the actual property value is always identical to the property value implied by the index throughout the term of the mortgage; that is,

$$H_t = \frac{I_t}{I_0} H_0, \qquad 0 \leq t \leq T.$$

Under this assumption, the formula for calculating the post-adjustment loan balance can be simplified to

$$B_t = \min(\bar{\alpha} H_t, U_t), \quad t = 1, \ldots, T.$$

We refer readers to Ambrose and Buttimer [3] for a qualitative discussion on basis risk and to Chapter 4 for a possible method to quantify the effect of basis risk on lenders' cash flows.

The payment due at the end of the next period is then calculated by amortizing B_t over the remaining term of the loan:

$$M_{t+1} = B_t \frac{\mathrm{ER}_0 + \mathrm{RS1}}{1 - (1 + \mathrm{ER}_0 + \mathrm{RS1})^{t-T}}, \quad t = 1, \ldots, T-1.$$

Note that $M_1 = \bar{M}$ because by design a loan adjustment at $t = 0$ is impossible. Also note that the current mortgage payment depends on the property value at the end of the previous period, so it takes one period for loan adjustment to take effect.

Should the borrower choose to prepay at time t, he/she would be subject to a prepayment penalty PP_t that equals a fraction of the difference between U_t and B_t; that is,

$$\mathrm{PP}_t = \mathrm{PRE1}(U_t - B_t), \quad t = 1, \ldots, T-1,$$

where $0 \leq \mathrm{PRE1} \leq 1$ is a constant parameter.[12] Upon prepayment the sum of the post-

[12] The purpose of the penalty is to avoid strategic prepayments. A prepayment at the time of a deep

adjustment balance B_t and the prepayment penalty PP_t becomes the initial amount of the new loan. Note that if an adjustment is not triggered at time t, then $B_t = U_t$ and hence the prepayment penalty is zero.

3.2.4 Property Index Linked Mortgage: Version Two (PILM2)

Under a PILM2 contract the mortgage payment is reduced if there is a decline in the house price relative to its value at loan origination. The mechanism used in this case is different from that used in PILM1 contracts. In a PILM1 the loan balance is adjusted to reflect house price declines and this induces a reduction in the mortgage payments, but in a PILM2 contract the mortgage payment is adjusted first and the outstanding loan balance is computed based on the revised mortgage payment. As with PILM1 contracts, the loan adjustments under PILM2 contracts are capped and are linked to a suitable property index. We also assume here that there is no basis risk.

Let us consider a PILM2 with a rate spread of RS2 over the corresponding FRM and thus a contract rate of $\mathrm{ER}_0 + \mathrm{RS2}$. Its reference payment and time-t reference balance are respectively given by

$$\bar{M} = \alpha H_0 \frac{\mathrm{ER}_0 + \mathrm{RS2}}{1 - (1 + \mathrm{ER}_0 + \mathrm{RS2})^{-T}}$$

and

$$U_t = \begin{cases} \alpha H_0, & t = 0 \\ U_{t-1}(1 + \mathrm{ER}_0 + \mathrm{RS2}) - \bar{M}, & t = 1, \ldots, T \end{cases}.$$

The mortgage payment would be reduced by a fraction of a percentage point per a percentage point reduction in the house price (relative to the house price at loan origination). The actual amount M_t due at time t is determined by the value of the associated property index at time $t - 1$; that is, for $t = 0, 1, \ldots, T - 1$,

$$M_{t+1} = \min\left\{\bar{M}, \bar{M}\left[1 - \kappa\left(1 - \frac{I_t}{I_0}\right)\right]\right\} = \min\left\{\bar{M}, \bar{M}\left[1 - \kappa\left(1 - \frac{H_t}{H_0}\right)\right]\right\},$$

where $0 \leq \kappa \leq 1$ is the 'workout proportion', which indicates how many percentage points the mortgage payment would reduce per a percentage point decrease in the property

adjustment would allow the borrower to lock in the low balance, thereby causing a substantial loss to the lender. It should be emphasized that a loan adjustment is not equivalent to a principal write down; for the former, the balance would revert to the reference balance when the real estate market rebounds; for the latter, the reduction in principal is permanent.

index.[13] The second equality in the equation above is due to our assumption that there is no basis risk. Note that we also have $M_1 = \bar{M}$ for the PILM2.

The post-adjustment loan balance[14] at time t is defined to be the present value of all future payments, assuming they all equal M_{t+1}; that is,

$$B_t = M_{t+1}\frac{1-(1+\mathrm{ER}_0+\mathrm{RS2})^{t-T}}{\mathrm{ER}_0+\mathrm{RS2}}, \quad t = 1, \ldots, T-1.$$

The specification of the prepayment penalty is the same as that for PILM1 contracts, except that PRE1 is replaced with another parameter PRE2 that lies between 0 and 1.

3.2.5 Comparison of PILM1 and PILM2

The two PILM designs have the same main objective: to reduce the homeowners' mortgage payments in a depressed housing market. However, the adjustment mechanisms are different when there is a drop in house prices. Under a PILM1 the first step is to reduce the outstanding balance and the payment is adjusted in the next step. Under a PLIM2 the initial step is to reduce the mortgage payment and the loan balance is revised in the subsequent step on the basis of the reduced payment. This section compares these two mortgage designs and uses a numerical example to illustrate the main differences.

By definition, the balance adjustment for PILM1 depends on both the current property value H_t and the reference balance U_t. Towards the end of the mortgage term, U_t becomes small so it is more difficult to trigger an adjustment and even if an adjustment is triggered it tends to be mild. In sharp contrast, for PILM2, an adjustment is triggered as long as the current property value H_t is lower than the initial value H_0. Because the reference payment $\bar{M}$ is constant over time, the chance for triggering remains even throughout the loan term.

We use a simple numerical example to demonstrate the aforementioned differences. Assume that the property value is $H_0 = 400$ initially, falls to 300 before the first payment is due, and remains constant thereafter. Assume further that $T = 30$, $\alpha = 0.95$, $\bar{\alpha} = 1$, $\kappa = 0.5$ and the contract rate for all mortgages is 0.05.

[13]The adjustment scheme and the term 'workout proportion' are taken from Shiller et al. [77].

[14]Determining the loan balance for a PILM2 contract is operationally difficult, but we need a definite formula for B_t in order to compute prepayment penalties. Shiller et al. [77] define the loan balance of a continuous workout mortgage as the expected present value of all future payments under a certain risk-neutral measure, whereas Mian and Sufi [65] define the loan balance for a shared-responsibility mortgage as the balance specified in the amortization schedule established at loan origination.

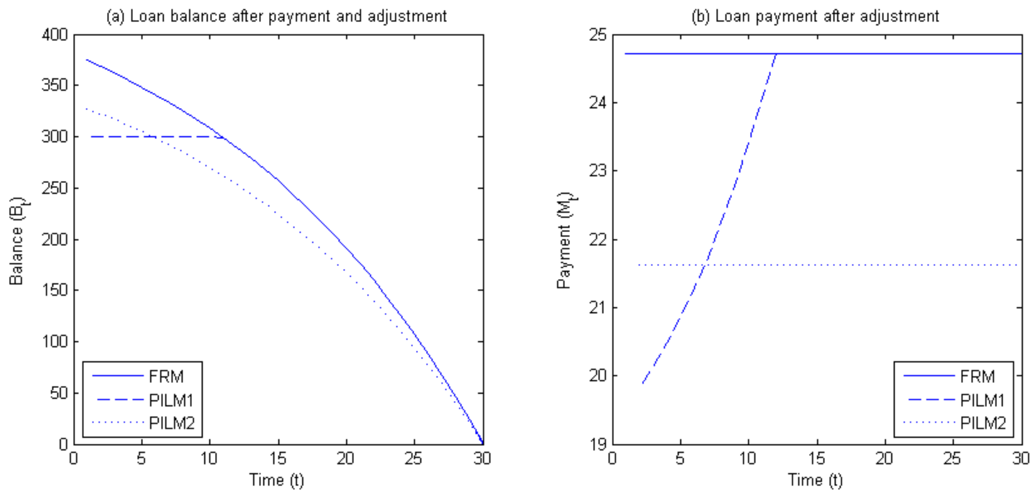


Figure 3.2: A comparison of the loan balances (the left panel) and mortgage payments (the right panel) among the three contracts under consideration. The values are based on the hypothetical situation that the property value is $H_0 = 400$ initially, falls to 300 before the first payment is due, and remains constant thereafter. It is assumed that the amortization term is 30 years, the contract rate for all three contracts is 0.05, the initial LTV is $\alpha = 0.95$, the target LTV is $\bar{\alpha} = 1$ and the workout proportion is $\kappa = 0.5$.

Figure 3.2 displays the loan balances and mortgage payments for all three contracts over time. Note that all contracts have the same first payment, because it takes one period for any PILM adjustment to realize and a loan adjustment at $t = 0$ is impossible. For the FRM contract, the mortgage payment is constant over the entire mortgage term. For the PILM1 contract, there is a sharp reduction in payment at $t = 2$, but the payment then increases gradually until it reaches its original level at $t = 12$. The reason behind the increasing trend in payment is that at each time point from $t = 1$ to $t = 10$ the PILM1 balance is adjusted to 300 and the loan is re-amortized over the remaining term. As t increases, the re-amortization period shortens, causing the payment to increase. The increase ends at $t = 12$, because starting at $t = 11$ the values of U_t are lower than 300 and hence no adjustment is triggered. For the PILM2 contract, the payment does not change after $t = 2$ because the adjustment depends solely on the property value, which is assumed to remain constant after dropping to 300.[15] This example demonstrates that given a one-time drop in house price in early mortgage years, the payment adjustment under a PILM1 contract is intense initially but weakens as the loan ages, while the payment adjustment under a

[15]Under a PILM1 contract, the payment reduction (as a proportion of the reference payment) at $t = 2$

PILM2 contract (with $\kappa < 1$) is less intense initially but tends to last longer.[16]

3.3 The Borrower's Problem and Its Solution

We compare the three contracts from the borrower's perspective. The differences in mortgage balances and payments under these contracts leads to different borrower incentives, therefore we analyze PILM through borrower behavior. Our approach is similar to that of Campbell and Cocco [20, 21]. We assume a representative borrower maximizes his/her expected lifetime utility of consumption by taking mortgage-related actions. Consumption depends on the choice of actions and the housing market. The borrower needs to determine the optimal actions under all possible conditions he/she may face. We then simulate different economic scenarios and compute the utility and cash flow realized by the borrower and the lender, assuming the borrower behaves optimally. To study PILM we compare the borrower's optimal action and utility, as well as the lender's cash flow under a PILM with those under a FRM. In what follows we define, describe and solve the borrower's problem step by step. We start with premises and assumptions.

3.3.1 Premises and Assumptions

To capture the effect of PILM on borrower's behavior, it is necessary to constrain our discussion within a simple world. The following premises is the start point of our discussion:

is the same as the loan balance reduction as a proportion of the reference balance at $t = 1$:

$$\frac{\alpha H_0(1 + \mathrm{ER}_0 + \mathrm{RS1}) - \bar{M} - fH_0}{\alpha H_0(1 + \mathrm{ER}_0 + \mathrm{RS1}) - \bar{M}},$$

which approximately equals to $\frac{\alpha - f}{\alpha}$ when T is large, where f is the current to initial house price ratio that is small enough to trigger an adjustment. In our hypothetical scenario, $f = 0.75$ and the payment reduction at $t = 2$ (calculated using the approximation) is 21%. Under a PILM2 contract, the payment reduction at $t = 2$ is exactly $\kappa(1 - f)$, which equals 12.5% in our hypothetical scenario. If $\alpha = 1$, the PILM2 payment reduction would be smaller than the PILM1 payment reduction by a fraction of κ.

[16] Given the same contract rate, a PILM2 with a full workout ($\kappa = 1$) dominates a PILM1 with a target LTV $\bar{\alpha}$ no smaller than the initial LTV α. Consider the situation when the payment adjustments of both PILMs are triggered. On the basis of the adjustment formulas, the balance of the PILM1 would be adjusted to $\bar{\alpha} H_t$, while the balance of the PILM2 with $\kappa = 1$ would be adjusted to $U_t \frac{H_t}{H_0}$. Since $U_t \leq \alpha H_0 \leq \bar{\alpha} H_0$, the balance of the PILM2 never exceeds that of the PILM1. For a meaningful comparison between the two contract types, we do not consider PILM2s with a full workout.

1. The borrower has decided to purchase a property and take out a mortgage loan. We do not consider the choice between buying and renting. However the borrower may enter the rental market if he/she defaults on the loan.

2. The main purpose of purchasing a property is to enjoy the utility of living in it. There exists a rental market providing the same living utility.

3. The borrower is forward looking. A decision is made based on the current condition and the expectation of the future, but not on the past.

4. The borrower makes decision at each time point, or equivalently the end of each period.

5. The borrower is rational when making decisions, and all decisions are made based on economic reasoning.

The following are some simplifying assumptions.

Assumption 3.1 (no basis risk). The property index is updated timely and it is perfectly correlated with the value of the mortgaged property.

Assumption 3.1 has been stated in the previous section when we specify PILM contracts. We state it here again for clarity. This assumption ensures that the PILM lender never over- or under- adjusts the loan.

Assumption 3.2. The borrower has no risky investment opportunities except the mortgaged property.

Assumption 3.3. The borrower receives a constant and non-storable endowment L at each time point.

Assumptions 3.2 & 3.3 are not inappropriate because we are not interested in comparing investment in real estate and other risky assets, nor do we consider income risk.[17] Our focus is on the effect of PILM on default prevention. Assuming riskless and perishable income avoids many real world complexities. For example, both FRM and PILM borrowers would face risks from non real estate markets and negative life events if they were considered. Now we formalize the optimization problem by first defining the borrower's objective function.

[17] Income risk can be analyzed within our framework.

3.3.2 The Borrower's Objective

Mathematically, the borrower's problem at loan origination can be expressed as

$$\max_{\{A_1,\ldots,A_T\}} U(C_0) + E_0\left[\sum_{t=1}^{T} \beta^t U(C_t(A_t)) + \beta^T V(H_T, A_T)\right], \tag{3.1}$$

where

- The maximization is taken over actions at all time points. Action is the only variable controlled by the borrower.
- T is the mortgage contract term. A mortgage can be terminated by the borrower before T (e.g. through default or refinancing).
- $E_t[\cdot]$ is the expectation with respect to all information available up to time t.
- C_t is the consumption at time t. It depends on the borrower's action and the housing market.
- β is the utility discount factor representing the borrower's time preference of consumption. $\beta \in [0, 1]$.
- $U(\cdot)$ is the time independent utility function with relative risk aversion $\gamma \neq 1$

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}.$$

- $V(\cdot, \cdot)$ is the bequest function. It represents the time-T value of the total utility of all consumptions beyond T.

The solution to this problem is a decision rule and its associated maximized utility. The rule specifies at any possible state of the world the action the borrower should take in order to maximize his/her expected utility. The maximized utility is an indicator of borrower welfare. Each mortgage contract has its own optimal decision rule. To solve the problem, we first define all variables and their transitions, and then describe how they are related to the borrower's utility.

3.3.3 Variables and Bequest

There are two sets of variables. One set is controlled by the borrower. In our case it has only one variable action A_t. The other set of variables describes the conditions the borrower faces, such as house price, market mortgage rate, etc. Based on these conditions, the borrower makes decision by choosing an action. The rule of choosing the action that maximizes the expected utility is the borrower's optimal decision rule. Hence the rule is a function mapping a state of the world to an action.

We first define our control variable. At each time point after the borrower receives the endowment L, he/she takes action A_t. In particular, the borrower can choose one of the following three actions before the mortgage loan is terminated:

- Continue to repay ($A_t = P$)

  The borrower may choose to pay the amount due M_t and receive the ownership benefit sH_t, where s denotes the constant benefit rate.[18] The consumption from the ownership benefit is in addition to the consumption from the income.

- Refinance ($A_t = R$)

  The borrower may choose to refinance the existing loan into another loan with an initial balance that equals the post-adjustment balance B_t of the original loan plus the prepayment penalty PP_t and a contract rate that equals the current market mortgage rate MR_t. We permit the borrower to refinance at any time and as many times as he/she wishes.[19] For simplicity, we assume that the new loan is always a FRM and that there is only one prevailing market rate which applies to all FRM loans, regardless of the loan term. If this action is chosen, the borrower would pay the amount due M_t, pay the refinancing cost RC and receives the ownership benefit sH_t.

- Default ($A_t = D$)

  The borrower may choose to default, so that the mortgage contract is terminated and the property is foreclosed. If this action is chosen, the borrower would not pay the amount due, but incur a default cost DC and receive no ownership benefit.[20] For

[18] The ownership benefit can be interpreted as the extra utility arising from owning the property, in addition to the benefit of living in the property. This benefit could be pecuniary and/or psychological and increases with the value of the mortgaged property.

[19] This means that there is no requirement on the property value at the time of refinancing.

[20] In our set-up, both DC and RC are not retained by the lender. We assume that the lender has no recourse on the borrower's income.

the remaining of his/her life, the borrower would have to seek housing in the rental market. The rental property is assumed to provide the same living utility as the foreclosed property, but give no ownership benefit.

If the borrower defaults, then he/she has to rent a property and pay the rent Y_t. The action of renting a property is denoted by $A_t = N$. Given the aforementioned assumptions, we see that after defaulting, the borrower would no longer need to make decisions as N is the only possible action. Moreover, in a feasible sequence of actions $\{A_t, t = 1, 2, \ldots, T\}$ there can be at most one D, and if $A_t = D$ then $A_{t+1} = A_{t+2} = \ldots = A_T = N$.

Since the endowment is assumed to be non-storable and the assumed utility function is strictly increasing, it is optimal for the borrower to consume all income left after his/her housing expenditure. It follows that for $t = 1, 2, \cdots, T$,

$$C_t(A_t) = \begin{cases} L - M_t + sH_t & \text{if} \quad A_t = P \\ L - M_t + sH_t - \mathrm{RC} & \text{if} \quad A_t = R \\ L - \mathrm{DC} & \text{if} \quad A_t = D \\ L - Y_t & \text{if} \quad A_t = N \end{cases}.$$

The rent Y_t is assumed to be an increasing function of property value at the previous time point:

$$Y_t = \min(\bar{Y}, cH_{t-1}),$$

where $0 < c < 1$ is a constant and $0 < \bar{Y} < L$ is the cap of the rent.[21] The cap is imposed to ensure that default is always 'affordable'. The time-0 consumption is given by

$$C_0 = L - (1 - \alpha)H_0 - \mathrm{IP} \cdot \alpha H_0,$$

where IP denotes the upfront costs to the borrower as a percentage of the loan size.

State variables characterizes the state of the world the borrower faces. We let ER_t be the effective rate at which the current FRM mortgage payment M_t is calculated. The value of ER_t depends on the borrower's action. If the borrower chooses to refinance at time t, then the subsequent mortgage payments would be (re-)calculated on the basis of

[21] We acknowledge that the actual relationship between house prices and rents is more complicated. The historical values of the S&P/Case-Shiller 10-City Composite Home Price Index and the Owners' Equivalent Rent of Primary Residence Index indicate that rents increased with house prices from 1992 to 2005, but after the burst of the real estate bubble in 2006 home prices fell by about 1/3 while rents kept rising. The values of the home price and rent indexes can be retrieved at http://ca.spindices.com/indices/real-estate/sp-case-shiller-10-city-composite-home-price-index and http://research.stlouisfed.org/fred2/series/CUUR0000SEHC01, respectively.

the prevailing market mortgage rate MR_t. If the borrower chooses to continue repaying the current loan, then ER_t would simply be the same as ER_{t-1}. If the borrower chooses to default or has already defaulted, then ER_t becomes irrelevant to the borrower's problem and we set it to 0 for convenience. Hence, we have

$$\mathrm{ER}_{t+1}(A_t) = \begin{cases} \mathrm{ER}_t & \text{if} \quad A_t = P \\ \mathrm{MR}_t & \text{if} \quad A_t = R \\ 0 & \text{if} \quad A_t = D, N \end{cases}$$

for $t = 1, 2, \cdots, T-1$ and $\mathrm{ER}_1 = \mathrm{ER}_0$.

Depending on the contract type, the mortgage payment responds to the borrower's action differently. For FRMs,

$$M_{t+1}(A_t) = \begin{cases} M_t & \text{if} \quad A_t = P \\ M_t \dfrac{1-(1+\mathrm{ER}_t)^{t-T}}{\mathrm{ER}_t} \dfrac{\mathrm{MR}_t}{1-(1+\mathrm{MR}_t)^{t-T}} & \text{if} \quad A_t = R \\ 0 & \text{if} \quad A_t = D, N \end{cases};$$

for PILM1s,

$$M_{t+1}(A_t) = \begin{cases} B_t \dfrac{\mathrm{ER}_0 + \mathrm{RS1}}{1-(1+\mathrm{ER}_0+\mathrm{RS1})^{t-T}} & \text{if} \quad A_t = P \\ (B_t + \mathrm{PP}_t) \dfrac{\mathrm{MR}_t}{1-(1+\mathrm{MR}_t)^{t-T}} & \text{if} \quad A_t = R \\ 0 & \text{if} \quad A_t = D, N \end{cases};$$

and for PILM2s,

$$M_{t+1}(A_t) = \begin{cases} \min\left\{\bar{M}, \bar{M}\left[1-\kappa\left(1-\dfrac{H_t}{H_0}\right)\right]\right\} & \text{if} \quad A_t = P \\ (B_t + \mathrm{PP}_t) \dfrac{\mathrm{MR}_t}{1-(1+\mathrm{MR}_t)^{t-T}} & \text{if} \quad A_t = R \\ 0 & \text{if} \quad A_t = D, N \end{cases},$$

where $t = 1, 2, \cdots, T-1$. Note that $M_1 = \alpha H_0 \frac{\mathrm{ER}_0}{1-(1+\mathrm{ER}_0)^{-T}}$ for FRMs and $M_1 = \bar{M}$ for PILMs.

The housing market is characterized by two variables: the property value H_t and the market mortgage rate MR_t. We assume that these variables follows the Markovian processes below:

$$\begin{cases} \log H_{t+1} = \log H_t + \left(\mu_H - \frac{1}{2}\sigma_H^2\right) + \sigma_H Z_H \\ \log \mathrm{MR}_{t+1} = \log \mathrm{MR}_t + \left(\mu_{\mathrm{MR}} - \frac{1}{2}\sigma_{\mathrm{MR}}^2\right) + \sigma_{\mathrm{MR}} Z_{\mathrm{MR}} \end{cases}, \tag{3.2}$$

where μ_H and μ_{MR} are the rates of appreciation, σ_H and σ_{MR} are the volatilities, and (Z_H, Z_{MR}) follows a standard bivariate normal distribution with a correlation coefficient ρ. Non-Markovian processes may more realistically capture the dynamics of house prices and market mortgage rates, but will substantially increase the number of state variables and hence computational burden.[22] In principle, there could be two sets of parameters. One could reflect the market perceived by the borrower, while the other the real market. If the two sets of parameters coincide, then the borrower behaves optimally; otherwise he/she does not.

Based on the above specifications, the state variables for FRMs are $t, H_t, \mathrm{MR}_t, \mathrm{ER}_t, M_t$, and for PILMs are $t, H_t, \mathrm{MR}_t, M_t$. The variables t, H_t, MR_t do not depend on the borrower's decisions, but ER_t and M_t do. We do not need to include ER_t as a state variable when modeling a PILM. This is because if the borrower continues to repay at time t then ER_t is identical to ER_0, and if not then the PILM contract is terminated. However, we still need to include M_t when modeling a PILM because it is related to H_{t-1}.

Finally, we specify the bequest function $V(H_T, A_T)$, which represents the total utility, discounted to $t = T$, of the consumption after (but not including) T. Assuming that the property value remains constant after time T and that the borrower has a non-random lifespan of T_L measured from loan origination, we have

$$V(H_T, A_T) = \begin{cases} \sum_{t=1}^{T_L-T} \beta^t U(L + sH_T) & \text{if} \quad A_T = P, R \\ \sum_{t=1}^{T_L-T} \beta^t U\big(L - Y_{T+1}\big) & \text{if} \quad A_T = D, N \end{cases}.$$

In this expression, condition '$A_T = D, N$' implies that $A_t = D$ for some $1 \leq t \leq T$ and thus the borrower no longer owns the property at time T; condition '$A_T = P, R$' implies that default has never occurred over the term of the mortgage and therefore the borrower owns the property at time T.

[22]Campbell and Cocco [21] also use a geometric Brownian motion for modeling property values but a first order autoregression for modeling interest rates. More sophisticated approaches to modeling house prices and market mortgage rates are considered by Kau et al. [57], Hanewald and Sherris [44], Li et al. [62] and the FHA actuarial review [52].

It is clear that owning the property always derives a higher bequest and that how much home ownership affects bequest depends on the end-of-term property value. From the expressions for $V(H_T, A_T)$ and $C_t(A_t)$ we can infer that each mortgage decision depends critically on bequest, ownership benefit and rent. According to our set-up, when the property value declines, the expected bequest decreases, the current ownership benefit drops and renting a property becomes more attractive. The linkage of mortgage decisions to these three quantities allows us to capture the potential relationship between the property value and the propensity to default.

Here are a few remarks and implications followed from the above formulation:

1. The borrower faces house price risk before T whether the loan is defaulted or not. Ownership benefit, bequest and rent all depend on property value.

2. Given default occurs, a borrower's post-default utility increases when property value falls because rent falls with property value. As a result, the borrower may prefer defaulting on the mortgage when property value is extremely low to keeping a high valued property with debt. This could be the case in reality because when property value is low and is expected to remain so in the future, a borrower may choose to default on the current loan and purchase a similar property later.

3. Since there is no decision to make after default, the post-default utility can be directly computed by taking the expectation without optimization.

4. People may make mortgage decisions based on two considerations: the equity position and consumption. The former depends on the value of the property and the loan balance, and the later depends on loan payment. In our model the borrower's utility of consumption is related to both.

3.3.4 Solving the Optimization Problem

We solve the borrower's problem by dynamic programming (similar to the method used in Campbell and Cocco [20, 21] and Cocco et al. [29]). Define the time-t value of a mortgage contract as the optimized expected discounted utilities of future consumption starting at time t; that is,

$$\max_{\{A_t, A_{t+1}, \cdots, A_T\}} U(C_t) + E_t\left[\sum_{i=t+1}^{T} \beta^{i-t} U(C_i(A_i)) + \beta^{T-t} V(H_T, A_T)\right].$$

We use J_t^k, where $k \in \{\text{FRM}, \text{PILM1}, \text{PILM2}\}$ and $t = 1, \ldots, T$, to represent the time-t value of mortgage contract k. The value of J_t^k depends on all state variables that are applicable to the specific contract type. We further define the value of an action as the optimal utility derived from taking the action; that is,

$$I_t^k(A_t) = \begin{cases} U(C_t(A_t)) + \beta E_t[J_{t+1}^k] & \text{if} \quad A_t = P, D, N \\ U(C_t(A_t)) + \beta E_t[J_{t+1}^{\text{FRM}}] & \text{if} \quad A_t = R \end{cases},$$

for $t = 1, 2, \cdots, T-1$, and

$$I_T^k(A_T) = U(C_T(A_T)) + V(H_T, A_T).$$

The expectations in the above expression are taken with respect to $(H_{t+1}, \text{MR}_{t+1})$. It follows that the contract value is the maximum over all feasible actions:

$$J_t^k = \begin{cases} \max\left\{I_t^k(P), I_t^k(R), I_t^k(D)\right\} & \text{if} \quad M_t > 0 \\ I_t^k(N) & \text{if} \quad M_t = 0 \end{cases},$$

for $t = 1, 2, \cdots, T$. Note that in the above expression, condition '$M_t = 0$' is equivalent to the condition that default occurs prior to time t.

Since refinancing at time T is never optimal,[23] the terminal contract value can be simplified to

$$\begin{aligned} J_T^k &= \begin{cases} \max\left\{I_T^k(P), I_T^k(D)\right\} & \text{if} \quad M_T > 0 \\ I_T^k(N) & \text{if} \quad M_T = 0 \end{cases} \\ &= \begin{cases} \max\left\{U(L - M_T + sH_T) + V(H_T, P), U(L - \text{DC}) + V(H_T, D)\right\} & \text{if} \quad M_T > 0 \\ U(L - Y_T) + V(H_T, N) & \text{if} \quad M_T = 0 \end{cases}, \end{aligned}$$

which can be computed for all possible states. Therefore, we can compute the values of J_t^k for $t = 1, \ldots, T-1$ and the associated optimal actions by a backward recursion which starts at $t = T$. As we reach the end of the backward recursion, we obtain the contract value at loan origination:

$$J_0^k = U(C_0) + \beta E_0[J_1^k].$$

This optimization problem is solved numerically and some detail of the implemention can be found in Appendix B.

Recall the assumption that a PILM can only be refinanced into a FRM. Because the value of a PILM refinancing depends on the corresponding FRM's contract value, we need

[23] After making a payment of M_T, the loan balance becomes zero. Refinancing at time T therefore brings no benefit but incurs a cost of RC.

to perform the recursion for a FRM first before that for the corresponding PILMs. One should also note that the utility realized by a borrower by following the optimal decision rule may not equal the contract value, since the realized utility also depends on the housing market which is beyond the borrower's control. This implies that for a particular scenario the optimal decision rule does not guarantee an optimal realization of utility.

3.4 Baseline Results and Analysis

This section implements the model we developed in the last section and discusses the results. We start with a description of the parameter values used in our baseline calculation and discuss the rationale for the values selected. Then we analyze the borrower's optimal behaviour under different economic conditions. In particular we examine how the incentives to default, maintain the current mortgage and refinance depend on mortgage rates and property values across the three contracts. We are especially interested in the differences between the two PILM designs. Our analysis includes the borrower's behaviour at a fixed time point as well as over time. Finally we compare both the borrower's and the lender's welfare under the three contracts by simulation.

3.4.1 The Baseline Parameter Values

The baseline parameter values used in the analysis are displayed in Table 4.2. We now discuss these choices and explain our justification.

Parameters related to the mortgage contracts

We consider 30-year mortgages with an initial LTV of $\alpha = 0.95$ for a property valued $H_0 = 400$ at time 0. Each time period corresponds to 1 year. The assumed upfront cost (as a percentage of the loan amount) is IP = 2%, which includes the initial points charged by the lender (approximately 0.7% over the period of 2009 to 2013[24]), commissions, legal fees and appraisal fees. The initial FRM contract rate ER_0 is set to the market mortgage rate MR_0 at time 0. In the baseline calculations, we assume that PILM lenders do not demand a rate spread (i.e., RS1 = RS2 = 0). The chosen values of $\bar{\alpha}$, κ, and PRE1, PRE2 are arbitrary, but will be sensitivity tested in Section 3.5.

[24]Source: Freddie Mac Primary Mortgage Market Survey for Conventional Conforming 30-year FRM.

Parameter	Description	Value
Contract		
T	Mortgage term (years)	30
α	Initial LTV	0.95
IP	Upfront cost (as a fraction of the loan amount)	0.02
ER_0	Initial FRM contract rate	0.05
$\bar{\alpha}$	Target LTV for PILM1	1
RS1	Rate spread for PILM1	0
PRE1	Prepayment penalty parameter for PILM1	1
κ	Workout proportion for PILM2	0.5
RS2	Rate spread for PILM2	0
PRE2	Prepayment penalty parameter for PILM2	1
Housing market (resale and rental)		
H_0	Initial property value	400
MR_0	Initial market mortgage rate	0.05
μ_H	Rate of appreciation of property values	0
σ_H	Volatility of property values	0.1
μ_{MR}	Rate of appreciation of mortgage rates	0
σ_{MR}	Volatility of mortgage rates	0.1
ρ	Correlation between the log-returns on H_t and MR_t	0
DC	Default cost	10
RC	Refinancing cost	6
s	Ownership benefit (as a fraction of the property value)	0.02
$\bar{Y}$	Upper bound of rents	25
c	Rent-to-price ratio	0.05
Borrower's characteristics		
L	Annual income	70
β	Intertemporal utility discount	0.96
γ	Coefficient of relative risk aversion	3
T_L	Lifespan measured from loan origination (years)	60

Table 3.1: The baseline parameter values used in analyzing borrowers' behaviour.

Parameters related to the housing market (resale and rental)

We set MR_0 to 5%, because the average Federal Housing Administration (FHA) 30-year FRM mortgage rate was 5.23% in 2010 and has been reducing since then.[25] We

[25]Source: FHA Mortgage Insurance Single-Family 30-Year Fixed Interest Rates, available at http:

assume a refinancing cost of 6, which is lower than the upfront cost of the original loan ($400 \times 0.95 \times 0.02 = 7.6$), on the grounds that the loan balance at the time of refinancing is lower than that at time 0. We assume a default cost of 10 and an ownership benefit of 2% of the property value. These two values vary among borrowers, depending on various nonpecuniary factors including credit degradation, social stigma and the desire for homeownership.

In the baseline calculations, we assume that the borrower's subjective view on the housing market coincides with the actual market (i.e., only one set of μ_H, σ_H, μ_{MR}, σ_{MR} and ρ is used). We also assume that property values and market mortgage rates are volatile ($\sigma_H = \sigma_{\mathrm{MR}} = 0.1$) but possess no trend ($\mu_H = \mu_{\mathrm{MR}} = 0$). What we are interested is a period of turmoil where default rates of the traditional FRM contract rise. The experience of the housing and mortgage markets in the past decade provides us with a good example. Based on the S&P/Case-Shiller 10-city composite home price index from March 2004 to February 2014, the estimated house price appreciation rate is almost 0 and the estimated volatility is 0.11. These estimates are close to our choices of baseline values. In Section 3.5 we will examine how the results change if the parameters in the processes for H_t and MR_t are calibrated to longer history.

Over the period of October 1992 to February 2014, the sample correlation between the annual log-returns on the S&P/Case-Shiller 10-city composite home price index and the annual log-returns on the average FHA 30-year FRM rate is negative, but the sample correlation becomes positive if the conventional conforming 30-year FRM rate is considered instead. Because the sample correlation takes no definite sign, we set ρ to zero in our baseline calculations.

The time-t rent is a fraction $c = 0.05$ of the house price at time $t-1$, subject to a maximum of $\bar{Y} = 25$. Campbell and Cocco [21] permit the fraction to vary with the prevailing interest rate, but for simplicity we assume that it is the same as the initial mortgage rate MR_0 and remains constant throughout the term of the mortgage.

Parameters related to the borrower's characteristics

Given the assumed values of α, H_0 and MR_0, the annual mortgage payment for a 30-year FRM is 24.72. The assumed borrower's income $L = 70$ implies that the ratio of mortgage payment to income is 35.31%. This ratio is in between FHA's limit on the housing-related expense to income ratio, which is about 31%, and FHA's limit on

//portal.hud.gov/hudportal/HUD?src=/program_offices/housing/rmra/oe/rpts/rates/irmenu.

the total recurrent expense to income ratio, which is about 43%.[26] We assume that the borrower will live $T_L = 60$ years after the inception of the mortgage.[27] Following Cocco et al. [29], we set the utility discount β to 0.96. Following Campbell and Cocco [20] and Kung [59], we assume that the coefficient of relative risk aversion γ is 3.

3.4.2 Optimal Borrower Choices at Time One

We now examine how the borrower's choice depends on different housing market environments at $t = 1$. To do so we assume a borrower takes out a mortgage loan at $t = 0$ and examine the borrower's optimal decision one year after loan origination under a range of house prices and market mortgage rates. This enables us to better understand how the borrower's decision is impacted by the prevailing market conditions. We carry out the analysis for our three mortgage contracts. For the FRM contract the results are consistent with the historical experience. If interest rates fall it is optimal to refinance the loan at a lower rate, if house prices have dropped it is optimal to default, and if neither happens it is optimal to continue with the current loan. For the PILM contracts we find the same broad patterns but the results are more nuanced. For example, we find that under some conditions a PILM borrower will optimally continue where a FRM borrower would either default or refinance.

In Figure 3.3 we show, for each contract, the optimal actions at $t = 1$ under different combinations of log property value and log mortgage rate. We use 'o' to indicate that continuing to repay is the optimal action, '+' to indicate refinancing is optimal and 'x' to indicate default is optimal. The black dot represents the time-0 log property value (5.99) and log mortgage rate (-3.00), so it is easy to read from the figure that how a deviation of time one market condition from time zero condition impacts the borrower's decision.

The optimal actions for the three contracts are roughly the same when the property value and mortgage rate are extremely high and/or low: (1) when both the property value and mortgage rate are very high, continuing to repay is optimal because both the refinancing and default options are deep out-of-the-money; (2) when the mortgage rate is very low but the property value is very high, refinancing is optimal as it reduces loan payment and a PILM adjustment is not likely to be triggered; (3) when the property value is very low, default becomes optimal. However, the three contracts are quite different

[26]Source: http://portalapps.hud.gov/FHAFAQ/controllerServlet?method=showPopup&faqId=1-6KT-1040.

[27]According to the Human Mortality Database (www.mortality.org), the life expectancy of a 22-year-old U.S. female in 2010 is approximately 60 years.

in terms of the default boundary, which refers to the highest property value at which the borrower finds it optimal to default. It is clear that the PILMs have lower default boundaries, because they provide payment reductions at times when the property value is low.[28] To obtain a better understanding of the differences between the three contracts, in what follows we study Figure 3.3 in greater depth.

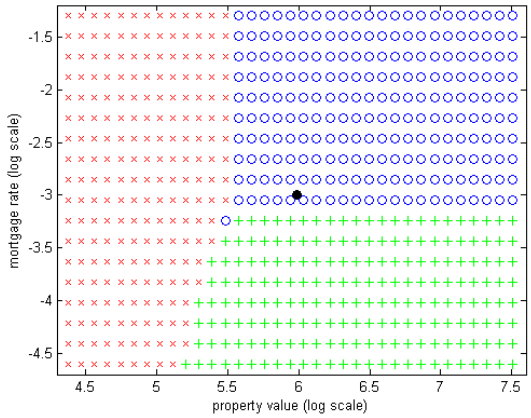
We first compare the FRM with the two PILMs. For the FRM, default and refinancing compete with each other in the low-property-value-low-mortgage-rate region. The competition arises because the borrower's propensity to default increases when the value of his/her home equity drops but the amount of mortgage payment also decreases when the loan is refinanced at a lower rate. In this region, the borrower defaults if the effect of a low property value is stronger than that of a low mortgage rate, and refinances otherwise. For the PILMs, continuing to repay also plays a role in the competition due to the possibility of a loan adjustment. When the property value declines, default is less attractive to a PILM borrower than to a FRM borrower, because PILMs offer a partial compensation through a reduction in mortgage payments. For this reason, the PILMs have lower default boundaries in comparison to the FRM. Due to the assumption that a PILM can only be refinanced into a FRM, the PILM borrowers lose the opportunity for loan adjustments as soon as they refinance. Because the benefit from the current and/or future loan adjustment(s) may outweigh that from refinancing, refinancing also becomes less attractive even when the market interest rate is low. Overall, the continuation region is larger in the diagrams for the PILMs than in the diagram for the FRM.

Next, we compare the two PILMs. We can account for the differences between the two PILMs by considering the following factors which determine their values:

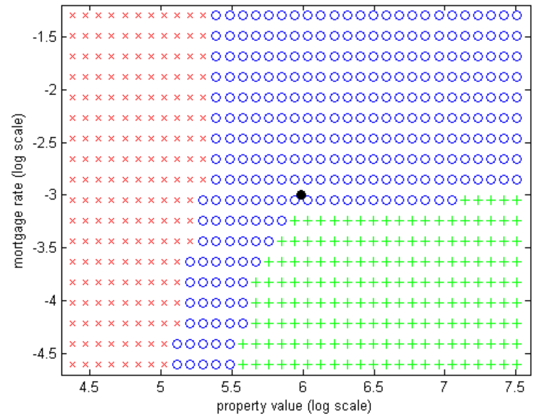
1. the current loan adjustment (realized at $t = 2$);
2. the opportunity to receive future loan adjustments;
3. the option to refinance;
4. the option to default.

From Figure 3.3 we observe that the most striking distinction between the two PILMs is that it is more likely for the PILM1 borrower to continue when both property value and mortgage rate are low. This outcome is a consequence of the two contracts' adjustment

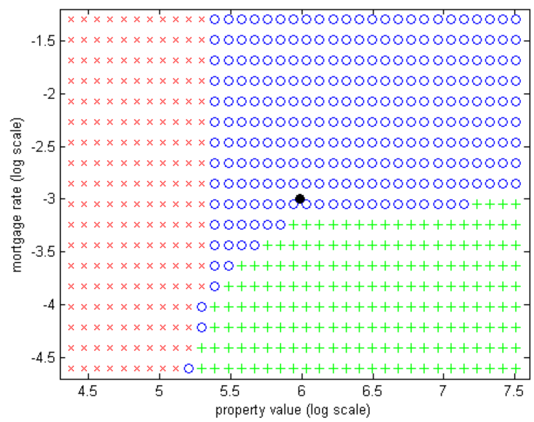
[28] This implies that the PILM borrower maintains the incentive to default, though it is significantly lower than the FRM borrower's. Our results are different from Ambrose and Buttimer's [3] due to the different models we use.



(a) FRM



(b) PILM1



(c) PILM2

Figure 3.3: The optimal actions at $t = 1$ under different combinations of $\log(H_1)$ and $\log(\mathrm{MR}_1)$. We use 'o', '+' and 'x' to represent continuing, refinancing and default, respectively. We use a black dot to indicate the time-0 log property value (5.99) and log mortgage rate (-3.00).

schemes, which we have illustrated in Section 3.2.5. When the property value plummets in early mortgage years, the loan adjustment provided by a PILM1 tends to be stronger at the outset but fades away over time. In a low interest rate environment, a PILM1 borrower can maximize benefit by continuing to repay for some time and then refinancing to lock in a low mortgage rate when the loan adjustment is no longer significant. In this way, the PILM1 borrower can enjoy the benefits from both the current loan adjustment and the option to refinance. By contrast, in the identical situation, the loan adjustment provided by a (partial workout) PILM2 tends to be milder but lasts longer. If a PILM2 borrower chooses to refinance, then a substantial amount of future loan adjustments would be lost; but if he/she chooses to repay, then the benefit of locking in a low mortgage rate would be forfeited. Thus, the PILM2 borrower can only benefit from loan adjustments or a low mortgage rate lock-in, but not both. It follows that in the first several years after mortgage origination, for some combinations of property value and mortgage rate under which the PILM1 borrower chooses to continue, the PILM2 borrower chooses not to.

In Figure 3.4 we compare the contract values of the two PILMs under different combinations of property value and mortgage rate. We use '$\Box$' to indicate that PILM1 has a higher contract value, '$\triangledown$' to indicate that PILM2 has a higher value and '$\Diamond$' to indicate that the two PILMs have the same value. The black dot represents the time-0 log property value (5.99) and log mortgage rate (-3.00). Note that upon default or refinancing, the two PILM borrowers have the same future consumption as they pay the same rents or make the same mortgage payments.[29] It immediately follows that

$$I_t^{\mathrm{PILM1}}(D) = I_t^{\mathrm{PILM2}}(D) \quad \text{and} \quad I_t^{\mathrm{PILM1}}(R) = I_t^{\mathrm{PILM2}}(R),$$

for $t = 1, 2, \cdots, T$. Hence, the difference between the contract values of the two PILMs must arise when the two PILM borrowers have different optimal choices or when both PILM borrowers continue to repay. With this fact in mind, the pattern of the diagram in Figure 3.4 can be explained readily as follows:

<u>Low mortgage rates and high property values; both borrowers refinance</u>

When the market mortgage rate is very low, refinancing becomes the optimal action for both borrowers, which means the contract value is simply the value of refinancing. Because $I_1^{\mathrm{PILM1}}(R) = I_1^{\mathrm{PILM2}}(R)$, both PILMs have the same contract value.

<u>High mortgage rates and high property values; both borrowers continue</u>

When the mortgage rate and property value are both high, both borrowers continue to repay even though there is currently no loan adjustment. At a high current property value, the chance of a future PILM2 adjustment is higher than that of a future PILM1 adjustment. This is because a PILM2 adjustment will be triggered once the property value becomes lower than the initial property value H_0, which is a fixed constant, whereas a PILM1 adjustment will be triggered only if the property

[29]We assume that if a PILM is refinanced, it must be refinanced into a FRM. Also, because we assume that PRE1 = PRE2 and RS1 = RS2 = 0, the balance upon refinancing must be U_t, whose value is the same for both PILMs.

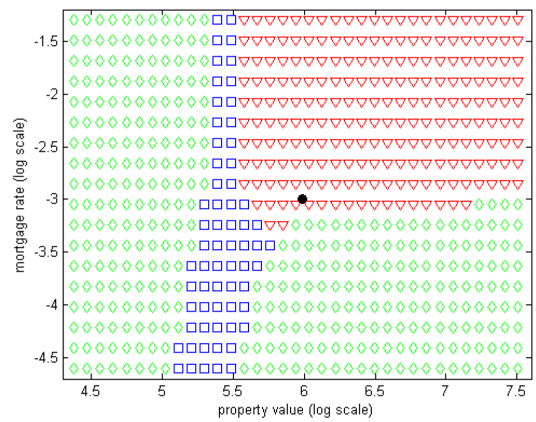


Figure 3.4: A comparison of the contract values of the two PILMs at $t = 1$. We use '$\square$' for the situation when the PILM1 has a higher value, '$\triangledown$' for the situation when the PILM1 has a lower value, and '$\Diamond$' for the situation when the two PILMs have the same value. We use a black dot to indicate the time-0 log property value (5.99) and log mortgage rate (-3.00).

value is lower than the reference balance U_t, which gradually diminishes to 0 over time. Therefore, the value of the opportunity to receive future loan adjustments for the PILM2 is higher, leading to $I_1^{\text{PILM1}}(P) < I_1^{\text{PILM2}}(P)$ and hence $J_1^{\text{PILM1}} < J_1^{\text{PILM2}}$.

Low property values; both borrowers default

When the property value is very low, default becomes the optimal action for both borrowers, which means the contract value is simply the value of default. Because $I_1^{\text{PILM1}}(D) = I_1^{\text{PILM2}}(D)$, both PILMs have the same contract value.

Low property values; the borrowers behave differently

Note that the default boundary for the PILM1 is lower than or the same as that for the PILM2. When the time-1 property value is in between the default boundaries of the two PILMs, the PILM1 borrower continues to repay but the PILM2 borrower defaults. In such a situation, PILM1 has a higher contract value, because

$$J_1^{\text{PILM1}} = I_1^{\text{PILM1}}(P) > I_1^{\text{PILM1}}(D) = I_1^{\text{PILM2}}(D) = J_1^{\text{PILM2}}.$$

Using a similar argument, we can explain why the PILM1 has a higher contract value in situations when the PILM1 borrower chooses to continue while the PILM2 borrower chooses to refinance.

Low property values; both borrowers continue

When the property value is low but still higher than the default boundaries of the PILMs, both borrowers continue to repay provided that the market interest rate does not justify refinancing. In this situation, as the property value approaches the default boundaries, the PILM1 has a higher contract value. This phenomenon may be explained by the fact that if the drop in the property price is very significant and occurs during early mortgage years, then the loan adjustment provided by a PILM1 is more pronounced than that provided by a PILM2.

3.4.3 Optimal Borrower Choices Over Time

In this sub-section we study how the borrower's behaviour vary with time. The borrower's propensity to default and refinance can be examined through default and refinancing boundaries. A low default boundary means the borrower defaults only when property value is low, implying a low propensity to default. Similarly a low refinancing boundary indicates a low propensity to refinance as the borrower only refinances when the market mortgage rate is low. These boundaries change over time and are different under different mortgage contracts.

Let us first focus on the trends in the default boundaries. In Figure 3.5 we show the default boundaries, calculated at a fixed market mortgage rate of 5%, over the entire mortgage term for the three mortgage contracts.

First, the default boundaries for the two PILMs are lower than or the same as that for the FRM, indicating that overall the PILM borrowers have a lower propensity to default. This result is expected, because the PILM borrowers are partially compensated by a loan adjustment in the event of a substantial drop in house price.

Second, the default boundary for the PILM2 is higher than that for the PILM1 during the first four years, and reduces steadily over time. The former outcome can be explained by the fact that in early mortgage years a PILM1 loan adjustment tends to be more pronounced. The latter outcome can be attributed to the fact that towards the end of the mortgage term, the number of outstanding payments is small but the likelihood of a PILM2 loan adjustment does not diminish (as it comes into effect whenever the property value is lower than H_0).

Third, the default boundary for the PILM1 increases with time in the first 12 years and subsequently converges to the FRM's default boundary. The variation in the PILM1's default boundary over time arises from the following two forces that are acting in opposite directions: (1) as the borrower builds equity in the property, he/she is less likely to default;

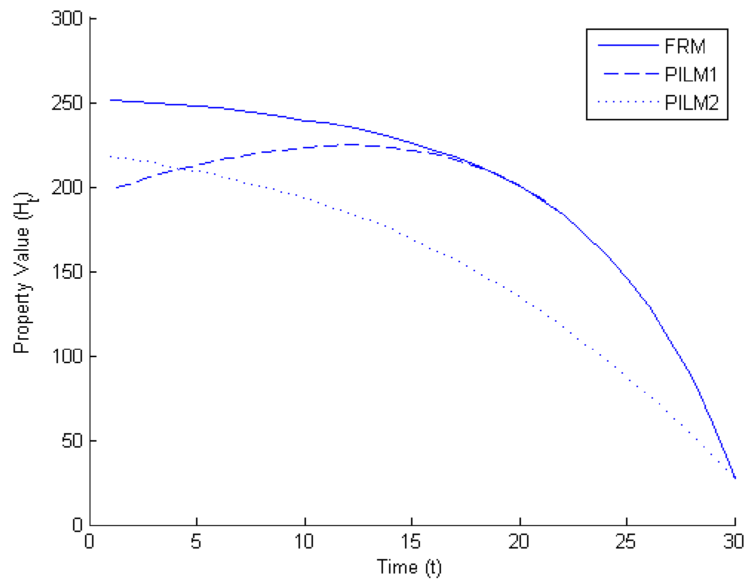


Figure 3.5: The default boundaries of the three mortgage contracts over time, calculated under the assumptions that $\mathrm{MR}_t = \mathrm{MR}_0 = 0.05$, $\mathrm{ER}_t = \mathrm{ER}_0 = 0.05$ and $M_t = 24.72$ for all $t > 0$.

(2) the default option becomes more attractive than the other two options as the loan balance reduces, because the opportunity to receive loan adjustments becomes smaller (a PILM1 loan adjustment is triggered only when H_t is less than U_t, which decreases over time) and the option to refinance is less valuable. In this example, the effect of (2) is greater than that of (1) over the first 12 mortgage years, leading to a rise in the PILM1's default boundary. Approaching the end of the mortgage term, a PILM1 adjustment becomes more difficult to be triggered and even if it is triggered the extent tends to be small. For this reason, as $t \to T$, the PILM1 becomes literally identical to the FRM and thus has a default boundary that is close to the FRM's.

In a similar manner, we can also study a mortgage contract's refinancing boundary, the highest mortgage rate at which the borrower finds it optimal to refinance. The refinancing boundary for a PILM depends on the current and potential future loan adjustments. The higher the adjustments are, the lower market mortgage rate is required to make refinancing sufficiently attractive, thereby resulting in a lower refinancing boundary. In Figure 3.6 we show the refinancing boundaries, calculated at a fixed house price of 300, for the three mortgage contracts over time. It can be seen that the refinancing boundaries have similar patterns as the default boundaries. The explanations to the patterns of the default

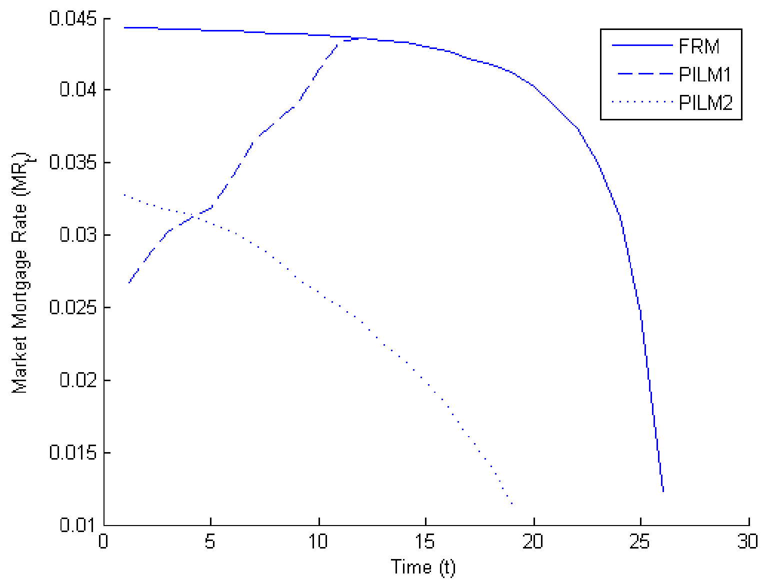


Figure 3.6: The refinancing boundaries of the three mortgage contracts over time, calculated under the assumptions that $H_t = 300$, $\mathrm{ER}_t = \mathrm{ER}_0 = 0.05$ and $M_t = 24.72$ for all $t > 0$.

boundaries also apply to the patterns of the refinancing boundaries.

3.4.4 Lender's Valuation

Thus far we have just considered the borrower's perspective. We have seen that the PILM contracts lead to lower mortgage payments when house prices fall and hence a lower default propensity relative to the standard FRM. Indeed this is the justification for the PILMs in the first place. However for the PILM contracts to be viable they need to be offered in the market place. Hence we need to value these contracts from a lender's perspective. To do so we project the cash flows assuming optimal borrower behaviour and include the deadweight costs suffered by the lender upon default. These costs are described in detail by Brueggeman and Fisher [17] and Qi and Yang [73]. We obtain the lender's valuation of the loan by taking the expected net present value of its cash flows under a given mortgage contract at an appropriate discount rate. We use Monte Carlo simulation to estimate these expected values.

We rank the different contracts from the viewpoint of each party. In the case of the

	FRM	PILM1	PILM2
Cumdef (%)	15.76	12.87	4.38
EC	56.84	57.16	57.48
$\mathrm{WG^B}$ (%)	–	0.57	1.12
APV	380	379.88	382.13
$\mathrm{WG^L}$ (%)	–	−0.03	0.56

Table 3.2: Summary of results of default incidence, mortgage valuation and welfare gains. 'Cumdef' represents the cumulative probability of default at the end of the loan term, 'EC' represents the equivalent constant consumption, '$\mathrm{WG^B}$' represents the borrower's welfare gain, 'APV' represents the average present value of the lender's cash flows, and '$\mathrm{WG^L}$' represents the lender's welfare gain.

borrower we compute the expected utility under each contract and convert it to a level stream of consumption. The borrower will prefer a higher consumption stream to a lower one. In the case of the lender we calculate the discounted value of its expected cash flows under the mortgage. We find that for a range of plausible parameters both parties prefer the PILM contract so the contract design is Pareto improving in these circumstances.

We also show that the cumulative defaults are significantly lower under the PILM mortgages than under the traditional FRM mortgage. Mortgage defaults have spillover effects (Mian et al. [66], Bradley et al. [16]), impose social costs (Andritzky [7]), and thus to the extent that these new designs reduce defaults this is another advantage.

We generate 10,000 economic scenarios, each of which contains one path of property values and one path of market mortgage rates. For each simulated scenario, we derive the borrower's optimal actions, compute the borrower's utility and calculate the borrower's consumption and the lender's cash flows at different time points. The results are reported in Table 3.2.

Consistent with our earlier results, we find that a payment adjustment feature can significantly reduce the cumulative probability of default at the end of the loan term (Cumdef). We refer a PILM default to as a 'direct' default if refinancing has never occurred before and an 'indirect' default if it happens after the mortgage is refinanced into a FRM. In our simulation, about 38% of the PILM1 defaults and 40% of the PILM2 defaults are indirect. The default probabilities reported in Table 3.2 incorporate both direct and indirect defaults.

We define the equivalent consumption (EC) for contract k as the constant level of consumption C^k that yields a time-0 utility that equals J_0^k; that is,

$$\sum_{t=0}^{T_L} \beta^t U(C^k) = J_0^k, \quad k \in \{\text{FRM}, \text{PILM1}, \text{PILM2}\}.$$

From the simulations, we can estimate J_0^k and hence C^k. We interpret the percentage difference between the ECs of a PILM and the corresponding FRM as the borrower's welfare gain (WG^B) should he/she choose the PILM instead of the FRM. The values of WG^B are positive for both PILM contracts. This result is expected, because it is assumed that the loan adjustment feature is provided to the PILM borrowers at no cost (RS1 = RS2 = 0).

We calculate the average present value (APV) of the lender's cash flows, using a varying discount rate that equals the prevailing market mortgage rate less 0.88%. The spread of 0.88% is chosen in such a way that the average present value of the FRM lender's cash flows equals the initial loan amount. It is assumed in the calculations that the lender recovers a net amount of RR = 50% of the (reference) loan balance upon default.[30] We interpret the percentage difference between the APVs of a PILM and the corresponding FRM as the lender's welfare gain (WG^L) should he/she write the PILM rather than the FRM. The value of WG^L for the PILM1 is close to zero, while that for the PILM2 is positive. This result indicates that the saving arising from the reduction in default and hence foreclosure costs is (more than) enough to cover the cost arising from the potential loan adjustments.

3.5 Robustness Checks to Input Parameters

In this section we examine the sensitivity of the results to different input parameters. This exercise provides deeper insight into the model and enables us to assess the importance of different parameters. It proves convenient to categorize the baseline parameters into four groups. These groups correspond to the borrower's characteristics, mortgage and rental markets, mortgage contract characteristics and housing and interest rate dynamics. We compute the various items of interest in Table 3.2 under a range of different input parameters by varying the baseline assumptions one at a time. For each of our three mortgage contracts we compute the cumulative default probability, the equivalent level consumption and the lender's valuation. This information enables us to rank the contracts

[30] The recovery rates of FHA loans are in between 40% and 60%. Source: Exhibit E-2 of the FHA Actuarial Review [52].

from both the borrower's and lender's perspectives and also provides a comparison with the baseline results.

Table 3.3 reports, for each mortgage contract, the values of Cumdef, EC and APV that are calculated on the basis of alternative parameter values. This table also shows the welfare gains to the borrower and lender when a PILM is used instead of a FRM. To ensure comparability against the baseline results, here we also discount the lender's cash flows at the prevailing market mortgage rate less 0.88%.

When interpreting the results, readers should keep in mind that EC and APV are only measures of the overall behaviour and welfare. That being said, even if the values of EC and APV under an alternative assumption are very close to their baseline values, the patterns of the borrower's behaviour, the borrower's consumption and/or the lender's cash flows could still be very different.

Category	Alternative values	FRM			PILM1					PILM2				
		Cumdef	EC	APV	Cumdef	EC	$\mathrm{WG^B}$	APV	$\mathrm{WG^L}$	Cumdef	EC	$\mathrm{WG^B}$	APV	$\mathrm{WG^L}$
Baseline		15.76	56.84	380.00	12.87	57.16	0.57	379.88	−0.03	4.38	57.48	1.13	382.13	0.56
Borrower's characteristics	$\beta = 0.92$	35.23	54.05	356.67	31.76	54.29	0.44	362.50	1.63	21.01	54.44	0.72	368.35	3.27
	$\gamma = 4$	18.79	56.20	377.50	15.88	56.54	0.60	377.98	0.13	5.94	56.85	1.16	381.27	1.00
	$L = 85$	13.69	72.27	381.83	10.84	72.58	0.43	381.17	−0.17	3.69	72.89	0.86	382.46	0.16
	$T_L = 70$	13.08	57.02	382.73	10.11	57.53	0.89	381.98	−0.20	3.04	57.86	1.47	383.03	0.08
Mortgage & rental markets	$s = 0.03$	7.89	60.21	388.56	5.41	60.60	0.65	385.48	−0.79	1.24	60.97	1.26	384.33	−1.09
	DC=20	13.25	56.81	382.61	10.33	57.13	0.56	381.88	−0.19	2.83	57.47	1.16	383.11	0.13
	RC=12	18.33	56.45	383.71	13.83	56.89	0.78	384.69	0.26	3.44	57.28	1.47	387.51	0.99
	$Y = 65$	14.70	56.85	381.78	12.53	57.16	0.55	380.29	−0.39	4.28	57.48	1.11	382.25	0.12
	$\mathrm{ER}_0 = \mathrm{MR}_0 = c = 0.04$	18.17	58.83	381.44	15.39	59.08	0.42	382.47	0.27	5.05	59.42	1.00	385.53	1.07
	RR = 0.7	N.A.		386.61	N.A.			384.19	−0.63	N.A.			383.82	−0.72
Contracts	$\alpha = 0.9$	11.58	52.78	384.65	9.35	52.93	0.28	384.46	−0.05	2.46	53.27	0.93	383.52	−0.29
	$\bar{\alpha} = 1.05$, $\kappa = 0.4$	N.A.			13.52	57.05	0.37	380.64	0.17	7.31	57.30	0.81	382.43	0.64
	$\bar{\alpha} = 0.95$, $\kappa = 0.6$	N.A.			12.05	57.34	0.88	378.21	−0.47	2.10	57.69	1.50	380.10	0.03
	RS1 = RS2 = 0.01	N.A.			16.56	56.22	−1.09	386.01	1.58	13.96	56.26	−1.02	388.02	2.11
	PRE1 = PRE2 = 0	N.A.			8.06	57.42	1.02	376.13	−1.02	4.87	57.58	1.30	375.82	−1.10
Dynamics of house prices and mortgage rates	$\mu_H = 0.0449$, $\sigma_H = 0.0930$, $\mu_{\mathrm{MR}} = -0.0226$, $\sigma_{\mathrm{MR}} = 0.1225$, $\rho = 0.2463$	2.50	56.64	397.37	1.92	57.12	0.85	389.76	−1.92	0.50	57.41	1.36	385.91	−2.88
		0	63.54	413.60	0	63.60	0.09	413.11	−0.12	0	63.61	0.11	412.88	−0.17

Table 3.3: Results that are based on alternative parameter values. 'Cumdef' represents the cumulative probability of default at the end of the loan term, 'EC' represents the equivalent constant consumption that derives the contract value, '$\mathrm{WG^B}$' represents the borrower's welfare gain, 'APV' represents the average present value of the lender's cash flows, and '$\mathrm{WG^L}$' represents the lender's welfare gain. For the last category, the first row shows the results for the situation when the borrower's view on the dynamics of property values and mortgage rates is captured by the alternative parameters but the actual dynamics are governed by the baseline parameters; the second row shows the results for the situation when the actual market and the borrower's view are characterized by the same set of alternative parameters.

3.5.1 Sensitivity to the Borrower's Characteristics

We begin with the borrower's time preference of consumption, represented by the utility discount parameter (β). A smaller β implies that the borrower is less patient as future consumption is discounted more heavily. When β decreases from 0.96 to 0.92, the default probabilities for all contracts are more than twice their corresponding baseline estimates. We can explain the change in default propensity by considering the benefit and cost of default. Since a necessary condition for default is that the prevailing market rent is cheaper than the current loan payment, default increases the borrower's current consumption as it substitutes the current loan payment with a relatively cheaper rent. On the other hand, default decreases the borrower's future consumption, because after default the borrower has to pay rent until he/she dies. At the baseline default boundary, the benefit and cost arising from default exactly offset each other. However, when β becomes smaller than its baseline value, the benefit outweighs the cost, because the borrower weights his/her current consumption more than his/her future consumption. In other words, a borrower with a smaller β should prefer to default at the baseline default boundary, implying that the default boundary for him/her is higher than that for the baseline borrower. Hence, regardless of the contract type, a less patient borrower has a higher propensity to default.

As the primary objective of the loan adjustment feature in a PILM is to reduce the borrower's incentive to default, it is reasonable to conjecture that from the lender's perspective the advantage of PILMs over the corresponding FRM is more profound when the borrower (in the absence of any loan adjustment feature) has a higher propensity to default. This conjecture is confirmed in the sensitivity analysis. When β reduces from 0.96 to 0.92, the FRM borrower has a significantly higher propensity to default (Cumdef increases from 15.76% to 35.23%). At the same time, the values of WG^{L} for both PILM1 and PILM2 increase significantly, indicating that the advantage of the loan adjustment feature to the lender becomes more prominent.

When γ increases from 3 to 4, the values of Cumdef for all contracts increase, suggesting that a higher risk aversion means a higher propensity to default. This outcome can be understood by comparing the levels of risk that borrowers and renters face. According to our set-up, the only risk that a renter faces is the uncertainty associated with rent. This uncertainty is related exclusively to house price risk and is in principle limited due to the cap $\bar{Y}$ imposed. By contrast, all borrowers bear the risk associated with ownership benefit plus the uncertainty arising from potential default and refinancing in the future. Also, a PILM borrower is subject to the added uncertainty arising from potential loan adjustments. Hence, compared to a renter, a borrower is subject to more sources of risk. Consequently, as risk aversion increases, one would be more likely to default and rent than to repay or refinance. We also observe that the increase in Cumdef (due to the increase in

γ) comes with an increase in $\mathrm{WG^L}$. This observation is again in line with the conjecture that the benefit of loan adjustments to the lender is more prominent when the borrower has a higher propensity to default.

For all three mortgage contracts, an increase in income (L) raises the borrower's consumption at every future date by a fixed amount, thereby providing the borrower with a higher level of expected utility irrespective of whether he/she chooses to default or not. This fact is the major reason for the increase in the equivalent consumption (EC) of all borrowers as L increases from 70 to 85. We also observe that when L increases, the default probabilities for all three contracts decrease. This observation indicates that an increase in L benefits those who choose not to default more than those who choose to. It also agrees with our intuition that a person with a higher income is less likely to default. Furthermore, $\mathrm{WG^L}$ becomes lower as the default propensity reduces, in line again with the previously mentioned relationship between default propensity and the advantage of the loan adjustment feature to the lender.

When the borrower's remaining lifespan (T_L) increases from 60 to 70, his/her default propensity reduces. This outcome can be explained as follows. A larger T_L increases the number of periods (beyond time T) over which an individual can consume, thereby increasing the bequests for both homeowners and renters. However, because homeowners always enjoy higher consumption than renters after time T, the increase in the bequest for homeowners are higher than that for renters. As a result, homeownership becomes more attractive and the incentive to default reduces.

3.5.2 Sensitivity to the Mortgage and Rental Market Parameters

We next examine the sensitivity of the results to the mortgage and rental market parameters. When the ownership benefit rate (s) increases from 2% to 3%, the total defaults for all contracts are significantly reduced. We obtain this result because an increase in s raises homeowners' consumption in all states of the world but not the renters', thereby making default (i.e., forfeiting homeownership) less attractive. When s is increased to 3%, the FRM borrower's default propensity becomes so low that the provision of loan adjustments is no longer advantageous to lenders (the values of $\mathrm{WG^L}$ for both PILMs are significantly below zero). This result means that when the ownership benefit rate is high, the reduction in foreclosure costs due to lower default rates may not be sufficient to cover the cost of loan adjustments. In this case, the PILM lender may consider using a less extensive adjustment scheme or charging a spread over the corresponding FRM.[31]

[31] The impact of charging a positive spread is detailed in Section 3.5.3.

	PILM1			PILM2		
	Direct	Indirect	Total	Direct	Indirect	Total
RC = 6	799	488	1287	264	174	438
RC = 12	1118	265	1383	291	53	344

Table 3.4: The number of direct defaults, the number of indirect defaults and the total number of defaults (out of the 10,000 simulated scenarios) for each of the PILM contracts. The values are calculated at a refinancing cost of 6 (the baseline assumption) and 12 (the alternative assumption).

When the default cost (DC) increases from 10 to 20, the default probabilities for all contracts decrease. This result can be attributed to the fact that a higher DC reduces the consumption when default takes place. The increase in DC also comes with a decrease in WG^{L}, because, as we explained earlier, the benefit of loan adjustments to the lender should be less substantial when the borrower has a smaller propensity to default.

The effect of a change in the refinancing cost (RC) depends on the contract design. For a FRM contract, a higher RC decreases the values of refinancing and the option to refinance, but has no direct effect on the value of default. Consequently, default becomes more attractive and thus the borrower's propensity to default increases. This relationship is confirmed in our results, which show that when RC increases from 6 to 12, the value of Cumdef for the FRM increases from 15.76% to 18.33%. For a PILM contract, an increase in RC has two opposing effects on the borrower's propensity to default. On one hand, it makes direct default more attractive. On the other hand, it reduces the borrower's propensity to refinance into a FRM, which has a higher default boundary compared to a PILM, thereby leading to a reduction in indirect defaults. The relative strengths of these two offsetting effects determine the change in the overall default probability. In this analysis, the former effect is stronger for the PILM1 but weaker for the PILM2 (see Table 3.4). As a result, the value of Cumdef for the PILM1 increases but that for the PILM2 decreases.

It is interesting to note that the when RC increases, the values of APV for all contracts increase noticeably, even though the default probabilities for the FRM and PILM1 contracts are higher. This interesting outcome may be attributed to the reduced propensity to refinance. From Table 3.5 we observe that when RC increases from 6 to 12, the borrowers refinance in fewer scenarios and the total number of refinancing for each contract is substantially reduced. Because a necessary condition for refinancing is lower mortgage payments, the less often the borrowers refinance, the larger amount of payments would be received by the lenders. If the effect of a reduced propensity to refinance outweighs the (opposite) effect of a larger propensity to default, then the cash flows to the lender and hence the value of APV would increase. From this example, we can tell that for a given

	FRM		PILM1		PILM2	
	# of scenarios	Total #	# of scenarios	Total #	# of scenarios	Total #
RC = 6	7249	14312	7286	14245	5183	9561
RC = 12	5107	6604	5299	6657	3198	3858

Table 3.5: "# of scenarios" shows the number of scenarios (out of the 10,000 simulated scenarios) in which the borrower refinances at least once and "Total #" shows the total number of refinancing in all simulated scenarios for each of the three contracts. The values are calculated at a refinancing cost of 6 (the baseline assumption) and 12 (the alternative assumption).

contract, a higher default probability does not necessarily imply a lower APV.

When the rent cap ($\bar{Y}$) increases from 25 to 65, the default probabilities for all three contracts decrease slightly. The decrease in default probabilities is expected, because a higher $\bar{Y}$ means that renters are subject to more risk and hence default becomes relatively less attractive. The decrease in default probabilities is quite small, because the property values at the default boundary are much lower than the property values that would result in a rent of $\bar{Y}$.

Next we consider the situation when the initial market mortgage rate (MR_0), the initial effective rate (ER_0) and the rent-to-price ratio (c) are all reduced by 1%. The reduction in these parameters has two opposite effects on the borrower's propensity to default. First, a lower effective rate reduces the borrower's mortgage payments and hence makes continuing to repay more attractive. Second, a lower rent-to-price ratio reduces rents hence increases the attractiveness of default. The results indicate that the latter effect is stronger, as the values of Cumdef for all three contracts become higher. Because the reduction in MR_0, ER_0 and c increases consumption regardless of homeownership, the values of EC for all three contracts are higher.

Finally, when the recovery rate (RR) increases from 50% to 70%, the default costs are reduced and hence the average present value of the cash flows to the lender increases. The increase is the most significant to the FRM lender, because it faces the highest default probability. As RR increases, the benefit arising from the loan adjustment feature that aims to reduce default propensity becomes less significant. When RR = 70%, the values of WG^{L} for both PILMs become negative, indicating that with this recovery rate the benefit cannot cover the cost of loan adjustments. Note that a change in the recovery rate has no impact on the borrower's behaviour or utility.

3.5.3 Sensitivity to the Mortgage Contract Parameters

This subsection examines the sensitivity of the results to the mortgage contract parameters. We start with the initial LTV (α). When α reduces from 0.95 to 0.9, all borrowers have a lower propensity to default. This is because a lower α corresponds to a smaller initial loan balance, which leads to smaller mortgage payments for all contracts.[32] However, a change in α has no impact on rents. Hence, when α decreases, default becomes relatively less attractive and the propensity to default reduces. Furthermore, in line with our conjecture that the benefit of loan adjustments to the lender is less substantial when the borrower has a lower propensity to default, we observe that the values of WG^{L} for both PILMs become smaller (and negative) as α reduces.

When α reduces from 0.95 to 0.9, all borrowers have lower equivalent consumptions. The decrease in EC can be attributed to two reasons. First, as α becomes smaller, there is a shift of consumption from $t = 0$ to $t = 1, \ldots, T$, thereby resulting in a decrease in EC as intertemporal utility discount applies to the consumptions beyond time 0. Second, because the initial loan balance reduces, the borrower's potential benefits arising from refinancing, default and loan adjustments are less significant.[33]

Next, we consider parameters $\bar{\alpha}$ and κ in the PILM adjustment schemes. According to the specifications of the adjustment schemes, a lower target LTV ($\bar{\alpha}$) makes a PILM1 loan adjustment easier to be triggered, while a higher workout proportion (κ) increases the extent of a PILM2 loan adjustment if it is triggered. When loan adjustments are easier to trigger and/or stronger if triggered, PILM borrowers should have a lower propensity to default and higher welfare. The relationship is confirmed in our simulation results. For the PILM1, when $\bar{\alpha}$ rises from 1 to 1.05, Cumdef increases but EC decreases; for the PILM2, when κ increases from 0.5 to 0.6, Cumdef falls but EC rises. The exact opposite is true when $\bar{\alpha}$ reduces from 1 to 0.95 and κ decreases from 0.5 to 0.4. To the lenders, a lower $\bar{\alpha}$ or a higher κ means more benefits arising from the reduction in default costs but higher costs of loan adjustments. Table 3.3 shows that the values of APV for both PILM lenders become lower when $\bar{\alpha}$ decreases and κ increases, indicating that the latter effect outweighs the former.

We next move to sensitivity testing the mortgage rate spreads between the PILMs

[32] A reduction in α implies a decrease in FRM and PILM2 mortgage payments in all states of the world, because FRM payments are proportional to α and are fixed at origination while PILM2 loan payments (both adjusted and unadjusted) are proportional to $\bar{M}$ (which is in turn proportional to α). A reduction in α implies a decrease in PILM1 mortgage payments at times when no loan adjustment is triggered.

[33] In practice the borrower may prefer a higher down payment as it lowers the borrower's contract rate and/or mortgage insurance premium. These benefits may outweigh the utility loss due to consumption shifting and reduced option values, but are not considered in our partial equilibrium framework.

	FRM	PILM1			PILM2		
		Direct	Indirect	Total	Direct	Indirect	Total
RS1 = RS2 = 0	1576	799	488	1287	264	174	438
RS1 = RS2 = 0.01		348	1308	1656	343	1053	1396

Table 3.6: The number of defaults (out of the 10,000 simulated scenarios) for each of the three contracts. For each PILM, the specific numbers of direct and indirect defaults are also shown. The values are calculated at RS1 = RS2 = 0 (the baseline assumption) and RS1 = RS2 = 0.01 (the alternative assumption)

	FRM	PILM1	PILM2
RS1 = RS2 = 0	5252	4817	2966
RS1 = RS2 = 0.01		9021	8211

Table 3.7: The number of scenarios (out of the 10,000 simulated scenarios) in which the borrower refinances at least once in the first 5 years of the mortgage term for each of the three contracts. The values are calculated at RS1 = RS2 = 0 (the baseline assumption) and RS1 = RS2 = 0.01 (the alternative assumption),

and the FRM. When the rate spreads RS1 and RS2 increase from 0% to 1% (i.e., the contract rate of the PILMs increases from 5% to 6%), the PILMs' default probabilities rise dramatically. From Table 3.6, which displays the breakdown of PILM defaults under the baseline and alternative rate spread assumptions, we observe that the increase in default probabilities is due primarily to the increase in indirect defaults. Moreover, it can be seen from Table 3.7 that at the higher rate spread most of the PILM borrowers refinance during the first 5 years of the mortgage term. As the PILM borrowers tend to refinance (into a FRM) in early years, it is not surprising that the default probabilities for the PILM borrowers become close to that of the FRM borrowers. It is also worth noticing that WG^{B} becomes negative while WG^{L} becomes strictly positive. We may therefore view a positive rate spread as a means to transfer welfare from borrowers to lenders.

Finally, we examine the consequences of setting the prepayment penalty parameters (PRE1 and PRE2) to zero. The effects of having a zero prepayment penalty are threefold. First, when the prepayment penalty is waived, the values of refinancing and the option to refinance increase, because the PILM borrowers are able to lock in the low post-adjustment loan balance at no cost if they refinance. As a result, the values of continuing and refinancing increase, leading to a reduction in direct defaults. Second, as the borrower's propensity to refinance rises, the probability of an indirect default tends to increase. Third, without any prepayment penalty, the borrower is able to lock in the loan balance at a time when the loan adjustment is very substantial. A smaller loan balance at the moment of refinancing tends to reduce the probability of an indirect default. The overall impact on the default probability depends on the relative strengths of these three effects. Table 3.3 shows that reducing the prepayment penalty to zero results in a substantial decrease in the PILM1 default probability but a slight increase in the PILM2 default probability. Again, we may regard the prepayment penalty as a means to transfer welfare between lenders and borrowers, since the removal of prepayment penalty increases the borrowers' welfare but decreases the lenders'.

In this sensitivity analysis, we see that a change in the value of contract-related parameter tends to have opposite effects on the borrower's and lender's welfare. Because the values of these parameters are negotiable, they can be used to adjust the allocation of the welfare gain arising from the loan adjustment feature between the borrower and lender. If we assume that the mortgage market is perfectly competitive and that all lenders are risk-neutral, then the PILM1 (PILM2) contract that would be offered in the market should be the one with parameters $\bar{\alpha}$, RS1 and PRE1 (κ, RS2 and PRE2) that would result in $\mathrm{WG}^{\mathrm{L}} = 0$. The problem of PILM pricing then boils down to the calculation of such parameters.

3.5.4 Sensitivity to the Parameters in the House Price and Mortgage Rate Processes

We conclude this section by analyzing the impact of changes in the parameters that are associated with the dynamics of house prices and mortgage rates. We first estimate the parameters in the processes for H_t and MR_t on the basis of the historical annual log-returns on the monthly S&P/Case-Shiller 10-City Composite Home Price Index and the conventional conforming 30-year FRM mortgage rates, respectively.[34] The estimation results ($\mu_H = 0.0449, \sigma_H = 0.0930, \mu_{\mathrm{MR}} = -0.0226, \sigma_{\mathrm{MR}} = 0.1225, \rho = 0.2463$) indicate that over the calibration window, house prices were rising but mortgage rates were decreasing on average.

Throughout the following analysis, we assume that each borrower's view on the property and mortgage markets is always consistent with the calibrated parameters. The borrowers

[34] We consider data over the period from Oct 1992 to Feb 2014. The house price index is available at http://ca.spindices.com/indices/real-estate/sp-case-shiller-10-city-composite-home-price-index. The mortgage rates are obtained from the Freddie Mac Primary Mortgage Market Survey, available at http://www.freddiemac.com/pmms/pmms_archives.html.

considered here are more optimistic compared to the baseline borrowers, since an upward expected trend in property values (characterized by a positive value of μ_H) increases ownership benefit and bequest while a downward trend in mortgage rates (captured by a negative value of μ_{MR}) reduces mortgage payments if the loan is refinanced. The realized behaviour and welfare of a borrower depend on the real property and mortgage markets. Two different cases concerning the real markets are considered.

In the first case, we assume the real markets are represented by the baseline processes for H_t and MR_t (with $\mu_H = \mu_{\mathrm{MR}} = 0$). This assumption implies that the borrowers are mistakenly optimistic, which in turns leads to suboptimal behaviour. Because the overly optimistic borrowers do not default when it is actually optimal to default, the default probabilities for all contracts become very low. As default risk reduces, the welfare of all lenders (represented by APV) increases. The PILM lenders' welfare increases less significantly than the FRM lender's. This is because as the borrower's propensity to default reduces, the loan adjustment feature in a PILM becomes more likely to be triggered and hence more costly, thereby offsetting the benefit arising from the reduction in default risk.

In the second case, we assume that both the real markets and the markets perceived by the borrowers are characterized by the calibrated parameters. Under this assumption, the default probabilities further reduce (to zero). This is because under the current assumption H_t and MR_t are more likely to drift to values at which the action of default is not optimal, but the borrowers' decision rules (which are based on their perceived markets) remain unchanged. The borrowers' welfare (represented by EC) increases, because of an upward trend in the actual property values and a downward trend in the actual mortgage rates. In terms of Cumdef, EC and APV, the three mortgage contracts look similar to one another. The similarity is in part because when $\mu_H = 0.0449$ the simulated property values are so high that PILM loan adjustments are seldom triggered, and in part because when $\mu_{\mathrm{MR}} = -0.0226$ the simulated mortgage rates are so low that most PILM borrowers refinance their loans (into FRMs) in early years.

3.6 Conclusion

The 2008-2009 financial crisis saw a dramatic increase in mortgage defaults. The once thought to be the safest mortgage instrument – fixed rate mortgage (FRM) – experienced a 5% delinquency rate in 2009. Such a high rate indicates that the threat of foreclosure is not enough to deter default when house prices decline and the borrowers are deep under water. It is well known that default imposes substantial deadweight costs to both borrowers and lenders. Recent studies have also pointed out that foreclosures have negative social externalities. To remedy the weakness of FRM and save the deadweight costs of default, property index linked mortgages (PILM) have been proposed. These contracts link the balance and payment of the mortgage loan to a property index so that when property values decline, the loan is adjusted correspondingly. Consequently, the borrowers repay less and their incentives to default weaken.

This chapter considers two different designs of PILM. The first design (PILM1) aims at controlling the current LTV ratio below a certain level, so the borrower will not be deep under water. The second design (PILM2) aims at compensating the borrower whenever the property is worth less than its purchase price. Despite the difference in loan adjustment schemes, they both require the lender to protect the borrower's equity in the mortgaged property.

We analyze the two PILM designs from the borrower's perspective. Under the baseline assumptions, both PILM designs reduce the borrower's propensity to default and increase his/her utility. The PILM lenders may achieve a higher expected present value of cash flows than the FRM lender, although we assume they do not charge for the equity protection they provide. This finding suggests that the PILMs could be Pareto improving relative to the FRM in the borrower-lender economy. The main reason behind this improvement is that the cost of loan adjustments is covered by the savings of foreclosure costs arising from the lower default probability. Comparing PILM1 with PILM2, we find that PILM2 has a lower default probability and derives higher welfare for both the borrower and lender.

We also analyze how the results change when alternative parameter values are assumed. The highlights of our findings are as follows. First, in all cases except when the PILM lenders charge a higher mortgage rate or the housing market is favorable to the borrowers, the PILM borrowers have lower default probabilities and higher welfare compared to the corresponding FRM borrower, but the PILM lenders' welfare could be higher or lower than that of the corresponding FRM lender.

Second, for a given contract, we find that a lower default probability is usually associated with higher borrower welfare. However, this relation does not hold in a few cases. For example, when the initial LTV is smaller (i.e., the down payment is increased), all borrowers have lower default probabilities and lower welfare compared to the baseline borrowers. A higher refinancing cost has a similar effect on PILM2.

Third, in the cases where the FRM borrower has a low default probability, the PILM lenders tend to have smaller, or even negative, welfare gains over the FRM lender. For example, when the ownership benefit increases or the borrower expects a bullish housing market, the FRM default probabilities drop considerably from the baseline value and both PILM lenders have smaller expected present values of cash flows compared to the FRM lender. In these cases, the borrower's propensity to default is low without equity protection, so the savings from foreclosure costs is insufficient to cover the cost of PILM loan adjustments, leading to lower expected present values of cash flows.

Lastly, since the contract-related parameters typically have opposite effects on the welfare of the borrower and lender, they can be used to adjust the welfare allocation between the borrower and lender. In our model, we can characterize a PILM1 contract by ($\bar{\alpha}$, RS1, PRE1) and a PILM2 contract by (κ, RS2, PRE2). These contracts can be priced through these parameters.

Finally we point out some limitations of our model. We assume no basis risk so that the values of all properties are perfectly correlated with the underlying house price index. We also assume a constant non-storable income for simplicity. These two assumptions can be relaxed and incorporated into our model by adding a basis risk model (see Chapter 4) and a stochastic income model (see Cocco et al. [29]), respectively. Our model can be further enriched by considering uncertainty in the borrower's lifespan.

Regarding mortgage decisions, we rule out some common practices in the real world, such as equity withdraw refinancing and loan modification. We also simplify our analysis by only allowing the PILM borrowers to refinance the loan into a FRM, and by prohibiting the borrowers from purchasing a property after default. These simplifications help us avoid complications while enable us focusing on comparing different mortgage designs.

Another strong assumption we make is that the housing market is exogenous to the borrower's choices and actions. This assumption may not hold when we consider a large number of borrowers. If PILMs do prevent default and default drags down property values during a recession, then PILMs do not only reduce default costs but also stabilize the housing market. Furthermore, PILMs may increase housing demand because households that do not want to bear the full house price risk and do not own property might consider homeownership through PILM financing. To resolve these issues we need an equilibrium model (see Kung [59]), which is beyond the scope of this paper.

Chapter 4

House Price Basis Risk

House price basis risk refers to the risk arising from the difference between the changes in individual property values and the changes in a house price index. This difference is inevitable because an index only captures the changes of average house prices, while properties have different characteristics and hence different appreciation rates. Basis risk plays a role in many cases where house price index is involved, for example hedging house price risk using property derivatives, pricing index linked mortgages, pricing home equity release products, etc. Ignoring or underestimating basis risk may lead to mispricing or suboptimal risk management. However the literature on house price basis risk is scarce. In this chapter, we try to model this risk and calibrate parameters using empirical data. We then apply this model to analyze the effect of basis risk on mortgage portfolios.

In section 1 we introduce house price basis risk by emphasizing its importance in house price index based contracts. We also review the related literature and discuss criterion for a good basis risk model. Our basis risk model is specified in section 2. In section 3 we calibrate our model by historical data and then use the calibrated model to simulate house prices. In section 4, we use the simulated house prices to analyze the effect of basis risk on different mortgage portfolios. Section 5 concludes.

4.1 Introduction

A house price index (HPI) reflects the changes in average prices of houses that are located within a region or a country. It is computed based on property sales prices using statistical methods. HPI is an important indicator of the real estate market and is useful to many market players, since most of the real estate properties are not frequently traded and their prices are not easily observable. However when an index provides people with a broad view of the aggregate market, it filters out information about individual properties. Each individual property has its unique attributes, such as location, size, age, style, etc., hence its appreciation in value is different from others'. This also means individual properties appreciate at different rates from the index, and the actual property value deviates from the value estimated from an index.

Basis risk refers to the risk arising from the deviation of an individual property value from the corresponding index estimated value, or equivalently from the difference in the appreciation rates of an individual property and the HPI. This risk is worth discussing because more and more HPI related financial instruments have been used and proposed. For example, the futures on S&P/Case-Shiller Home Price Indices have been traded in Chicago Mercantile Exchange since 2006. More recently, the property index-linked mortgage has been proposed to mitigate mortgage default risk. Basis risk may affect the usefulness or effectiveness of these products.

The idea of using HPI-based derivatives to hedge mortgage default risk has been proposed and discussed since 1990's (Case et al. [26], Case and Shiller [25], Shiller [74]). Mortgage investors and insurers usually hold or insure a portfolio of mortgages. When they hedge against default risk using index derivatives, it is ideal to construct a hedging position for each loan and then sum over all individual positions to obtain the portfolio hedging position. This is because the mortgaged properties have different values and hence the mortgagors' default propensities have different sensitivities to the changes in the index. Ignoring basis risk implicitly assumes all property values are the same, which leads to a uniform hedging position for all loans. In general the sum of different individual hedging positions is different from the sum of the uniform one. Therefore ignoring basis risk may result in suboptimal hedging strategies.

Basis risk is also an important element in pricing mortgage insurance premiums and home equity release products. Pu et al. [72] show under their simple house price model that an increase in the correlation among house prices increases the variance of a mortgage insurer's payout. This causes an increase in insurance premium if the insurer adopts a variance premium pricing principle. Andrews and Oberoi [6] realize that basis risk plays a role in the pricing of the non-negative equity guarantee embedded in a home equity release product. They suggest this risk being borne and managed by a specialized agent, who charges the borrower a basis risk premium. In both cases premiums would be inadequate if basis risk was underestimated.

To mitigate default risk, researchers have recently proposed a new design of mortgage contract that links loan balance and payments to a property index (Ambrose and Buttimer [3], Shiller et al. [77], Mian and Sufi [65]). Although using a public index instead of property value avoids costly appraisals and reduces the risk of moral hazard, it introduces basis risk. The loan balance and payments are over (under) adjusted if the property depreciates less (more) than the index.[1] This risk is recognized and briefly discussed by Ambrose and Buttimer [3], but none of the literatures so far has analyzed the effect of basis risk on index-linked mortgage quantitatively. In this chapter we try to quantify this effect as a demonstration of the usefulness of our basis risk model.

Given the importance of house price basis risk in pricing, hedging and risk management, it is surprising that the literature on it is very limited. One approach used to characterize the dispersion of individual house prices is to assume the difference between the log house price and the log index follows a Gaussian random walk plus an error term (see, e.g. Calhoun [18]). This approach is used to construct repeat sales index from paired sales prices. Another approach is to model both index and individual house prices by geometric Brownian motions, and model the dependency by a constant correlation coefficient (see, e.g. Pu et al. [72], Duarte and McManus [34]).

Our model is different from these. Follow the 2012 FHA actuarial review [51], we model the log-return of the index by an AR-ARCH process. We assume the log-return of individual house prices follows an AR process of the same order as the index, but the distribution of its innovation is conditional on the innovation of the index. In particular, the conditional mean is equal to the adjusted index innovation and the conditional variance depends on a parameter and a standard zero-mean distribution. The conditional variance controls the degree of the basis risk. We calibrate the parameters and the distribution using regional HPIs, and then simulate individual house prices according to the calibrated model. The simulated price paths are satisfactory in the sense that they meet the following three criteria for a good basis risk model.

First, the average of individual house prices is close to the index at any time. This assures the representativeness of the index. Second, the standard deviation of individual house prices increases over time but the annualized standard deviation decreases over time. This is observed from empirical data. Andrews and Oberoi [6] find that the annualized return difference between individual homes and HPI becomes less variable over time. Third, the individual house price paths do not fluctuate rapidly, which is a natural result of the standard assumption that the quality of any house remains unchanged.

Using our model one can fix an index path first, and then simulate house price paths. This procedure allows us to isolate the effect of changes in basis risk from the effect of simultaneous change in basis risk and the index. If two sets of individual house prices are simulated first then it is unlikely to have identical indices, i.e. averages of house prices.

[1] A similar problem exists in the design of home equity insurance policies (Shiller and Weiss [76]).

As an example, we will use our model to estimate default rates and values of different mortgage portfolios under different degrees of basis risk, while keeping the index fixed.

4.2 The Basis Risk Model

In this section we specify our basis risk model. We first model the index by a stochastic process and then specify the distribution of the individual house prices conditional on the index. This allows us to simulate house price paths after an index path is fixed. We focus more on the conditional distribution than on the index process since the conditional distribution has larger impact on the dispersion of house price paths.

Various models for HPI have been proposed and analyzed (Li et al. [62], Hanewald and Sherris [44], FHA actuarial review [51], etc.). For simplicity, we assume the quarterly log return of HPI follows an AR(r)-ARCH(q) process:

$$\begin{aligned} Y(t) &= \mu + \beta_1 Y(t-1) + \beta_2 Y(t-2) + \cdots + \beta_r Y(t-r) + \epsilon_t \\ \sigma_t^2 &= \gamma_0 + \gamma_1 \epsilon_{t-1}^2 + \gamma_2 \epsilon_{t-2}^2 + \cdots + \gamma_q \epsilon_{t-q}^2 \end{aligned} \tag{4.1}$$

where

$$Y(t) = \log \frac{H(t+\frac{1}{4})}{H(t)}$$

is the quarterly log return of the index $H(t)$ for the quarter starting at time t and σ_t^2 is the variance of the innovation ϵ_t. Coefficients $\mu, \beta_i's, \gamma_i's$ are to be calibrated.

To ensure house prices do not fluctuate too much, we assume the log return of individual house prices follows the same AR(r) process with the same coefficients as the index. Denote the time-t value of the ith house by $H^i(t)$ and its log return by $Y^i(t)$, then

$$Y^i(t) = \log \frac{H^i(t+\frac{1}{4})}{H^i(t)},$$

and

$$Y^i(t) = \mu + \beta_1 Y^i(t-1) + \beta_2 Y^i(t-2) + \cdots + \beta_r Y^i(t-r) + \epsilon_t^i. \tag{4.2}$$

For convenience, we normalize the initial prices of all houses to equal to the initial index, i.e. $H^i(0) = H(0)$ for all i.

The distribution of individual innovation ϵ_t^i is the key to our basis risk model. First, it must be related to the index innovation ϵ_t, otherwise we cannot guarantee the average of house prices being close to the index. Second, the variance of ϵ_t^i controls the degree of basis risk. A larger variance generates more dispersed returns and hence higher basis risk. We assume the distribution of ϵ_t^i conditional on ϵ_t is

$$\epsilon_t^i|\epsilon_t \sim \epsilon_t - \delta + \omega Z, \tag{4.3}$$

where δ and ω are two parameters and Z follows some standard zero-mean distribution. The δ is an adjustment factor that is necessary for the average of house prices to equal the index.[2] It is important to note that δ, ω, Z are essential to the basis risk. In the next section when we calibrate the model we focus on identifying these quantities. Also note that the conditional distribution of ϵ_t^i is independent of the main model and the variance structure of ϵ_t. One can choose another set of coefficients or other ARMA-GARCH specifications when modelling and simulating the index while retaining the basis risk structure (4.3).

4.3 Model Calibration and House Price Simulation

In this section, we calibrate the basis risk model parameters δ, ω and the distribution of Z to the historical regional HPIs. We then use the calibrated model to simulate house price paths under a given index path. Our model provides a good fit for the historical HPIs and the simulated paths meet the three criteria mentioned in the introduction.

4.3.1 Calibration

Due to data limitation, we use regional HPIs to calibrate the parameters. The dispersion of regional indices could have a similar pattern as the dispersion of individual house prices, since each region has its own characteristics like each house does. In addition, using indices has the advantage that we know the paths rather than only discrete pairs of prices if repeat sales data were used. The regional HPIs we use are the S&P/Case Shiller (SPCS) Home Price Indices for 20 Metropolitan Statistical Areas in the US and the calibration window is from January 2000 to December 2013.[3] The indices are updated monthly. The index

[2]Since we are modelling log returns, the arithmetic average of house prices without the adjustment factor δ would be larger than the index (but the geometric average would equal the index). To see this, note that property values are conditionally log-normally distributed. If ϵ_t^i has a conditional mean of ϵ_t with a positive conditional variance, i.e. $E[\epsilon_t^i|\epsilon_t] = \epsilon_t$, $\mathrm{Var}(\epsilon_t^i|\epsilon_t) = v^2 > 0$, then $E\left[e^{\epsilon_t^i}\middle|\epsilon_t\right] = e^{\epsilon_t + v^2/2} > e^{\epsilon_t}$. Furthermore, the magnitude of the adjustment should depend on the conditional variance.

[3]Data can be retrieved at http://ca.spindices.com/index-family/real-estate/sp-case-shiller.

values are normalized to 100 in January 2000. Figure 4.1 shows all 20 index paths and their unweighted average. We will treat these regional indices as individual house prices and their average as the index.

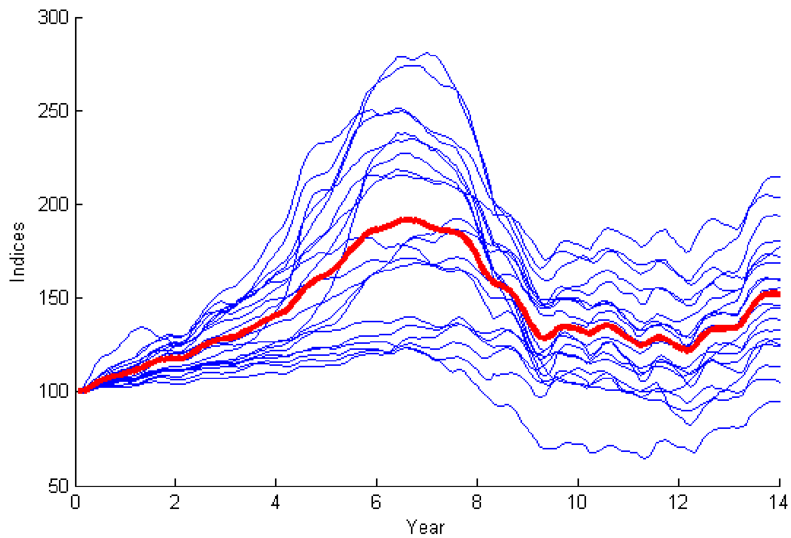


Figure 4.1: S&P/Case-Shiller Home Price Indices for 20 MSAs and their unweighted average (the thick line), Jan 2000 – Dec 2013.

We use a two stage calibration procedure. In the first stage we convert the monthly data to quarterly returns and calibrate the returns of the average of regional indices to an AR(3)-ARCH(1) process.[4] We then obtain index innovations ϵ_t. Note that these AR-ARCH coefficients are not directly related to the basis risk parameters, but they are necessary for obtaining ϵ_t and ϵ_t^i.

In the second stage we calibrate the parameters δ, ω and the distribution of Z. Using the fitted coefficients from the first stage, we can infer the values of the individual innovations ϵ_t^i by assuming AR(3)-ARCH(1) process for all 20 regional index paths. Consider centralized

[4]The fitted coefficients are $\mu = 3.9060 \times 10^{-4}$, $\beta_1 = 2.3560$, $\beta_2 = -2.1160$, $\beta_3 = 0.7228$, $\gamma_0 = 1.4562 \times 10^{-5}$, $\gamma_1 = 0.4283$. We choose $r = 3$ and $q = 1$ following the 2012 FHA actuarial review [51]. The estimated standardized residuals do not pass the Ljung-Box Q-test but do pass the Engle-ARCH test. The lag of 3 not being enough to remove autocorrelation may be due to the trend and seasonality in the return series. For simplicity we impose that $r = 3$. The order of $q = 1$ successfully removes the ARCH effect.

individual residuals of the form

$$\tilde{\epsilon}_t^i = \epsilon_t^i - (\epsilon_t - \delta) \sim \omega Z.$$

Aggregating the 20 paths, we choose δ such that the average of $\tilde{\epsilon}_t^i$ across all paths and all time points is 0. This gives $\delta = 1.4376 \times 10^{-5}$. To determine the distribution of Z and the value of ω, we examine the sample distribution of $\tilde{\epsilon}_t^i$. Figures 4.2 shows the quantile-quantile(QQ) plots of $\tilde{\epsilon}_t^i$ against normal distribution (left panel) and against student t distribution with degree of freedom 4 (right panel). From these plots we see that t distribution provides a better fit than normal distribution except for the right tail. Assuming Z follows a t_4 distribution, we choose ω to match the sample variance of $\tilde{\epsilon}_t^i$ to the theoretical variance of ωZ.

The same estimation procedure is applied to the FNC residential price indices[5] that cover 30 MSAs. Using the same calibration window, we obtain similar results, despite FCN indices are hedonic. The first two columns of Table 4.1 show the calibrated distribution and parameter values.

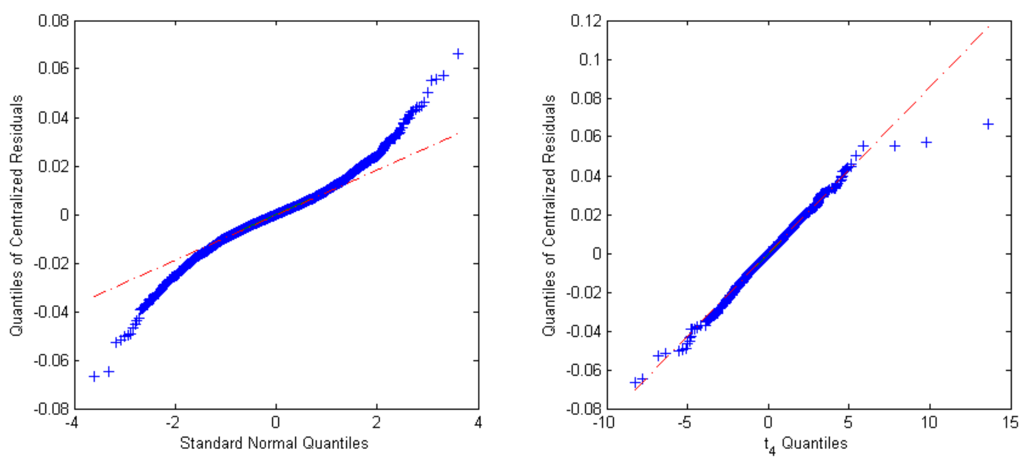


Figure 4.2: QQ plot of centralized residuals($\tilde{\epsilon}_t^i$) against standard normal (left panel) and student t distribution with 4 degrees of freedom (right panel) based on 3240 samples.

[5]Data can be retrieved at http://www.fncrpi.com/tables.aspx.

	SPCS	FNC	simulated data	
Distribution of Z	t(df=4)	t(df=8)	t(df=4)	normal
ω	0.0083	0.0176	0.0085	0.0085
$\delta(\times 10^{-5})$	1.44	2.59	36	25
Values at the end of year 14				
Index	608	610	298	
Mean	608	610	296	299
Std	131	124	134	90

Table 4.1: The distribution and parameter values of the basis risk model. "Mean" and "Std" are the mean and standard deviation of property values at the end of year 14 when initial values are normalized to 400. They indicate the representativeness of the index and the dispersion of property values. Simulation runs to year 15, we report the values at year 14 in order for them to be comparable with the SPCS & FNC data.

4.3.2 Simulation

Empirical data only contain very limited number of "house price paths", which is usually not enough for an analysis where one party is exposed to house price risk from many houses. Therefore to use our basis risk model effectively, we need to simulate house price paths. By simulation we can also fix an index path we want and control the level of basis risk.

Assume there are $N = 1000$ houses ($i = 1, 2, \ldots, 1000$). We first fix an index path as shown in Figure 4.3. It is intentionally chosen to be a depressing scenario to facilitate our later analysis about property index-linked mortgage. The index falls by about 40% from its initial level at the end of year 5 and then gradually rises.

We re-estimate the AR-ARCH coefficients by fitting the quarterly index return to an AR(3)-ARCH(1) process,[6] and then use the fitted coefficients, together with the selected δ and ω, to simulate property values.[7] The columns under "simulated data" in Table 4.1 show some statistics of the simulated house price paths.

Using t distribution with 4 degrees of freedom for Z with a similar value of ω identified previously, we need to select a larger δ to adjust the average to the index. The standard

[6] Coefficients are: $\mu = 0.0024$, $\beta_1 = 0.6878$, $\beta_2 = -0.3966$, $\beta_3 = 0.5169$, $\gamma_0 = 3.0202\times 10^{-5}$, $\gamma_1 = 0.9248$. Similar to model calibration, the autocorrelations are not completely removed by the AR(3) terms but the ARCH effect is removed.

[7] We need three initial values for Y and Y^i, they are $Y_{-2} = 0.011042, Y_{-1} = -0.007145, Y_0 = -0.028071$. These values have insignificant effect on simulation.

deviation at the end of year 14 is similar to the empirical data but the basis risk is relatively higher because the index at the end of year 14 in our depressing scenario is much lower than in the empirical data. We in addition consider a lower level of basis risk by using a normal distribution for Z.

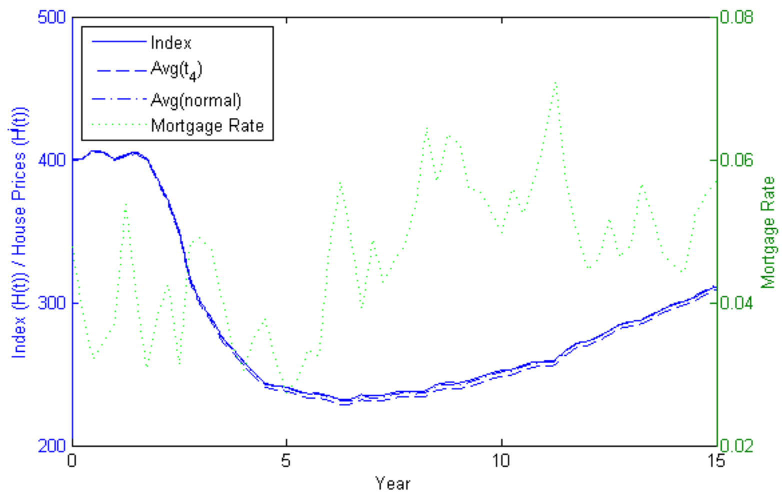


Figure 4.3: The fixed HPI and mortgage rates. The two dashed lines are averages of simulated house prices assuming Z follows t_4 and normal distributions.

To visualize what simulated house price paths look like, we also plot their averages in Figure 4.3. They stick close to the index over time, which indicates that the first-simulated index is representative. Figure 4.4 plots 20 house price paths under each assumed distribution for Z. Paths intersect the index and each other, but do not exhibit high-frequency fluctuation. For the same value of ω, paths are more dispersed when Z has a larger variance. Standard deviations of house prices are shown in Figure 4.5. They increase over time but the annualized standard deviations (standard deviation divided by time) decrease over time. This is consistent with the assumption about house price volatility in Calhoun [18] and the empirical findings in Andrews and Oberoi [6].

Numerical results in the next section are all based on these two sets of simulations. Since we simulate house prices quarterly, linear interpolation is used to compute the house prices within a quarter. The case of $Z \sim N(0,1)$ will be referred to as low basis risk and $Z \sim t_4$ as high basis risk. In addition we will consider the case of no basis risk where all house prices equal the index at all times.

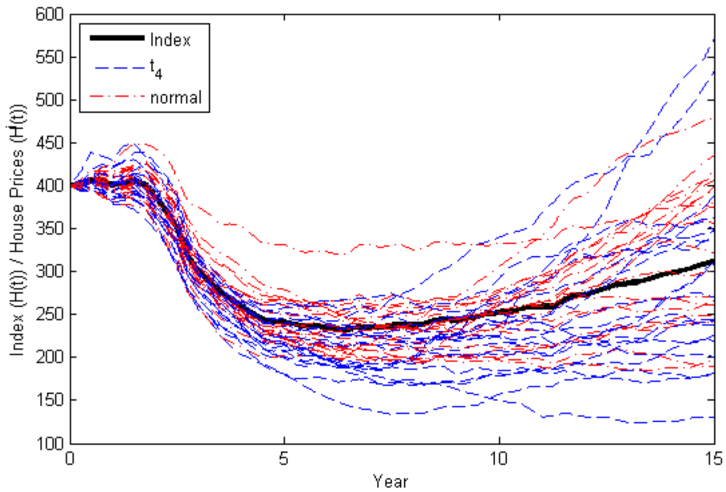


Figure 4.4: Twenty house price paths are plotted under each assumed distribution for Z.

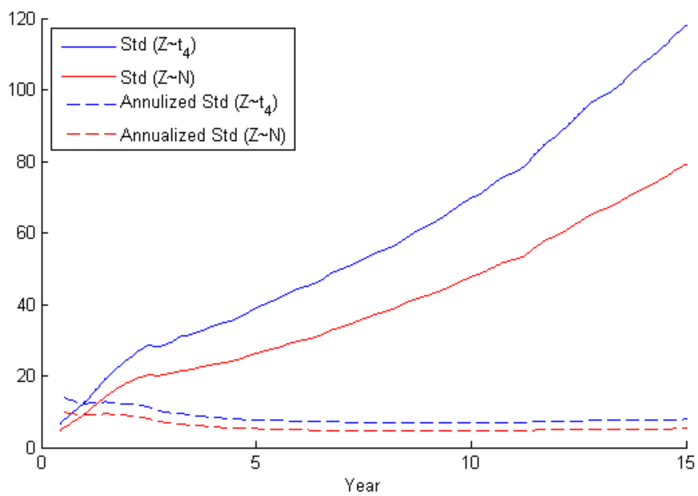


Figure 4.5: The standard deviations and annualized standard deviations of house prices over time.

4.4 The Effect of Basis Risk on Mortgage Portfolios

In this section we use the simulated house prices to analyze the effect of basis risk on portfolios of different mortgage contracts. We consider two types of mortgages: the traditional fixed rate mortgage (FRM) and the property index-linked mortgage (PILM) introduced in Chapter 3, and three levels of basis risk: no, low and high. We assume a lender holds a portfolio of mortgages (either all FRMs or all PILMs) with homogeneous characteristics, except that the values of the mortgaged houses evolve differently according to the simulated paths. The potential effect of basis risk on PILM is qualitatively analyzed in section 3.1.3.2. In this section we try to quantify this effect by comparing portfolio default rates and values across different levels of basis risk and different contract types.

4.4.1 Mortgage Contracts and Their Values

We first specify the FRM and PILM contracts. They are very similar to the FRM and PILM1 contracts specified in Chapter 3 except that we use a continuous framework.[8] The value of a contract to the lender depends on the timing and mode of mortgage termination, which in turn depends on house prices and the index (for PILM). We use the multiple state model specified in Chapter 1 for mortgage termination.

FRM

In a FRM contract, the loan is fully amortized at origination such that the borrower repays the same amount within any time interval of the same length. The rate of mortgage payment at time t (in year) is

$$M(t) = \alpha H(0) \frac{r_c}{1 - e^{-r_c T}},$$

where r_c is the continuously compounded contract rate for the FRM, α is the initial loan-to-value (LTV) ratio and T is the loan term. The value of $M(t)$ is determined at loan origination and does not change over time. The loan balance at time t is

$$B(t) = \alpha H(0) \frac{1 - e^{-r_c(T-t)}}{1 - e^{-r_c T}}.$$

There is no prepayment penalty.

[8] There are two reasons we consider continuous framework here. First, it has been well established in Chapter 1. Second, we only simulate house prices at the end of each quarter but to the lender loan default and prepayment can occur within a quarter, say at month ends.

PILM

Following the specification of PILM1 in section 3.2.3, we define the reference balance to be the balance determined at loan origination by amortizing the initial loan amount over the loan term with continuously compounded contract rate r_c, which is the same as the FRM contract rate. Denote the reference balance at time t by U_t, then

$$U(t) = \alpha H(0) \frac{1 - e^{-r_c(T-t)}}{1 - e^{-r_c T}}$$

The actual loan balance $B(t)$ may deviate from $U(t)$ if the loan is adjusted due to a significant drop in HPI. We assume that at the beginning of each quarter, the loan balance is adjusted so that the current LTV (value indicated by HPI, not necessarily the actual value) does not exceed a target. Mathematically the post-adjustment loan balance is

$$B(t) = \min[\bar{\alpha} H(t), U(t)], \quad \text{for } t = 0, \frac{1}{4}, \frac{2}{4}, \cdots, \frac{4T-1}{4},$$

where $0 < \bar{\alpha} < 1$ is the target LTV. The payment rate is then calculated by amortizing $B(t)$ over the remaining term of the loan:

$$M(t) = B(t) \frac{r_{c1}}{1 - e^{-r_c(T-t)}}.$$

There is no balance adjustment within each quarter, so for $t \in [\frac{i}{4}, \frac{i+1}{4}), i = 0, 1, 2, \ldots, 4T-1$,

$$B(t) = B\Big(\frac{i}{4}\Big) \frac{1 - e^{-r_c(T-t)}}{1 - e^{-r_c(T-\frac{i}{4})}},$$

and $M(t) = M\big(\frac{i}{4}\big)$.

Should the borrower choose to prepay at time t, he/she is subject to a prepayment penalty of

$$\mathrm{PP}(t) = \theta(U(t) - B(t))$$

due at the time of prepayment, where $0 \le \theta \le 1$ is a constant prepayment. Upon prepayment the lender receives the sum of the post-adjustment balance $U(t)$ and the prepayment penalty $\mathrm{PP}(t)$.

Contract and Portfolio

We consider two portfolios of loans. One consists of 1000 FRMs and the other 1000 PILMs. These loans and the borrowers of the loans have homogeneous characteristics, except that the values of the mortgaged houses evolve differently over time. We assume the portfolios are well diversified in the sense that the underlying houses represent the real estate market and their average values approximately equal the index. We further assume that conditional on the house values, mortgage terminations are mutually independent.

We define the contract value of a mortgage loan to be the present value of the cash flow from that loan. This is a random variable since it depends on when and how the loan is terminated, and also on the market condition if the loan is a PILM. We denote the value of the i-th loan of type k by V_i^k, where $i \in \{1, 2, \ldots, 1000\}, k \in \{\text{FRM}, \text{PILM}\}$. Then the value of a portfolio is $V^k = \sum_{i=1}^{N} V_i^k$.

In general it is very difficult to derive analytically the distribution of V^k since it is the sum of a sequence of random variables. We have two additional difficulties: 1) the distributions of V_i^k are not known, and 2) V_i^k have different distributions for different i when basis risk is present. Therefore we approximate these distributions by simulating loan terminations.

We assume the process of loan status follows the multiple state model specified in Chapter 1,[9] and the values of the mortgaged house underlying the i-th loan in both portfolios follow the i-th house price path generated previously. The mortgage rates are shown in Figure 4.3. Basis risk affects loan portfolios through the effect of house values on loan status transitions, since transition intensities depend on loan balances and house values.

We denote the j-th realized present value of cash flows from the i-th loan of type k by v_{ij}^k. It is a random sample of V_i^k, and is calculated as follows

$$v_{ij}^{\text{FRM}} = \begin{cases} \int_0^t d(s)M(s)ds + d(t)U(t) & \text{if prepaid at } t \\ \int_0^t d(s)M(s)ds + d(t)\gamma U(t) & \text{if defaulted at } t \\ \int_0^T d(s)M(s)ds & \text{if not prepaid or defaulted before } T \end{cases},$$

[9]For each mortgage we simulate the times and destination states of all transitions occurred from origination to termination. Transition times are simulated using "stay" probabilities $p_{\overline{11}}$ and $p_{\overline{22}}$. Conditioning on the time of each transition, the probability that a state is selected as the destination is proportional to the corresponding transition intensity. Simulation ends as soon as state "Prepay" or "Default" is reached. We then record the ending time as loan termination time, and the ending state as the mode of termination.

and

$$v_{ij}^{\text{PILM}} = \begin{cases} \int_0^t d(s)M(s)ds + d(t)(B(t) + \text{PP}(t)) & \text{if prepaid at } t \\ \int_0^t d(s)M(s)ds + d(t)\min[B(t), \gamma U(t)] & \text{if defaulted at } t \\ \int_0^T d(s)M(s)ds & \text{if not prepaid or defaulted before } T \end{cases}.$$

The function $d(t)$ is the time varying discount factor, and the parameter $0 < \gamma < 1$ represents the fraction of (reference) loan balance the lender can recover following loan default. We simulate 50,000 samples for each mortgage in both FRM and PILM portfolios, i.e. $j = 1, 2, \cdots, 50,000$. The j-th realized value for a portfolio is

$$v_j^k = \sum_{i=1}^{1000} v_{ij}^k.$$

This is a random sample of V^k.

The cumulative default rate of a mortgage portfolio is the proportion of mortgage loans that have defaulted by the end of loan term T. This is also a random variable and we denote it by D^k. We can easily obtain an estimation for default rate from simulated loan terminations. Let d_{ij}^k indicate whether the j-th simulation of the i-th loan of type k is terminated through default; that is, $d_{ij}^k = 1$ if default occurs and $d_{ij}^k = 0$ if not. Then the default rate for the portfolio k based on the j-th simulation is $d_j^k = \frac{1}{1000}\sum_{i=1}^{1000} d_{ij}^k$, and the average of d_j^k (over j) is an estimation of the expected value of D^k.

4.4.2 Results and Analysis

We consider three levels of basis risk: 1) no basis risk – all house prices are perfectly correlated with the index, 2) low basis risk – Z follows a normal distribution, and 3) high basis risk – Z follows a t distribution with 4 degrees of freedom. When there is no basis risk, all mortgages within a portfolio have the same stochastic process for its loan status, hence they have the same probability of default and the same distribution for their values. We call a mortgage with underlying house value being the same as the index "index mortgage", and a portfolio consisting of index mortgages "index portfolio". With the presence of basis risk, each mortgage loan follows a stochastic process of status transition different from another.

The specified parameter values are shown in Table 4.2. We assume a 15 year loan term with initial LTV of 95%. The PILM balance is adjusted as soon as the index falls below the reference balance, but the PILM lender does not charge for this adjustment provision.

Upon prepayment, the borrower has to repay the reference balance and upon default, the lender can recover half of the reference balance from foreclosure.

Parameter	Description	Value
$H(0)$	Initial hosue price	400
r_c	FRM contract rate	0.048
α	Initial LTV	0.95
T	Mortgage term	15
$\bar{\alpha}$	PILM target LTV	1
θ	Prepayment penalty parameter	1
γ	Recovery rate	0.5

Table 4.2: Parameter specification.

Default Rate

Table 4.3 shows the default rates. For both FRM and PILM portfolios, default rate increases with the level of basis risk.[10] According to our loan termination model, lower house values increase the probability of defaults. When a mortgaged house has a value below the index, the corresponding mortgage has a higher probability to default than the index mortgage. On the other hand if the house value is above the index, the corresponding mortgage is less likely to default than the index mortgage.

Default(%)	No	Low	High
FRM	31.33	35.67	38.46
PILM	19.99	24.08	27.84

Table 4.3: Default rates for FRM and PILM portfolios under different levels of basis risk (no, low, high).

Since the average of house prices approximately equals the index, the fact that default probability increases with the level of basis risk suggests that the effect of negative deviation of house prices from the index on default propensity is stronger than the effect of positive house price deviation. It is also clear that in all cases the PILM portfolio has a lower default rate than the FRM portfolio. This shows that loan adjustment is effective in preventing default in a depressing scenario, even with the presence of basis risk.

[10]The PILM default rate is not zero because we are using the multiple state model that incorporates defaults due to any reason.

Value of a Mortgage Contract

We have obtained 50,000 random samples ($v_{ij}^k, j = 1, 2, \ldots, 50000$) for the present value random variable (V_i^k) of the i-th loan in the portfolio k under all three levels of basis risk. These samples can be used to examine the distribution of V_i^k. Figure 4.6 shows the density estimation of the values of FRM and PILM index mortgages. In this case the distributions of V_i^k are the same for all i so we randomly pick one and use its 50,000 samples.[11] The two densities are irregular and do not belong to any standard family. They have multiple modes and heavy left tails. The sample mean and sample standard deviation (std) for the FRM index mortgage are 370.70 and 66.81, for the PILM mortgage are 378.86 and 56.57. In this depressing scenario, PILM is superior to FRM in terms of risk and return when there is no basis risk.

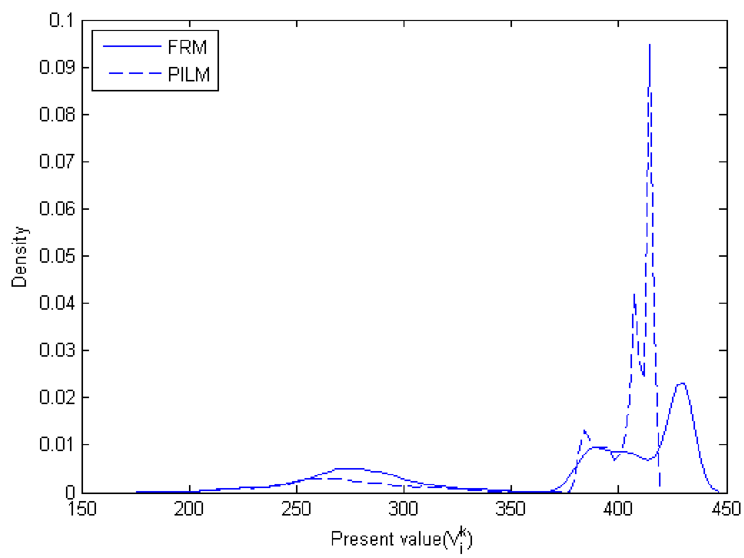


Figure 4.6: Density estimation of the present value random variables of the FRM and PILM index mortgages based on 50,000 samples.

When there is basis risk, V_i^k have different distributions for different i because the value of each mortgaged property evolves differently. We plot in Figure 4.7 the sample means and sample stds for all $V_i^k, i = 1, 2, \cdots, 1000, k = \mathrm{FRM}, \mathrm{PILM}$, under high basis risk. The figure shows that for a given index, our basis risk model effectively disperses mortgage values on the mean-std plane. We also find that in the high sample mean region, PILMs

[11]We could have used all 1000×50000 samples but that would not make much difference.

have lower values than FRMs. However this does not imply PILMs have lower average values, as there are fewer PILMs having low sample means.

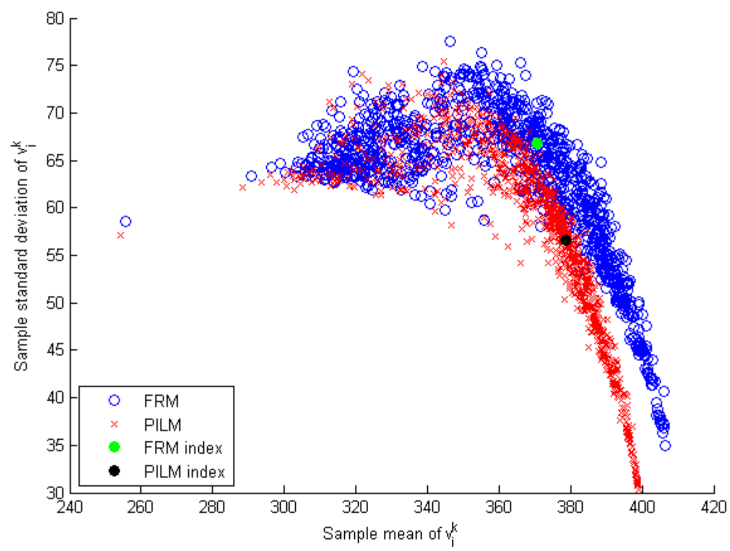


Figure 4.7: A scatter plot of the sample means and sample stds of $V_i^k, i = 1, 2, \cdots, 1000, k = \text{FRM}, \text{PILM}$ under high basis risk. The two dots represent the index FRM and the index PILM.

Value of A Mortgage Portfolio

The value of a portfolio is the sum of the values of all mortgage contracts in that portfolio, i.e. $V^k = \sum_{i=1}^{N} V_i^k$. When there is no basis risk, $V_i^k, i = 1, 2, \ldots, N$ are independent and identically distributed. By the classical central limit theorem,

$$\frac{\sum_{i=1}^{n} V_i^k - n\mu^k}{\sqrt{n(\sigma^k)^2}} \to N(0,1) \quad \text{in distribution as} \quad n \to \infty,$$

provided that the mean μ^k and variance $(\sigma^k)^2$ of V_i^k are finite. This suggests that the value of an index portfolio can be approximated by a normal distribution when N is large:

$$V^k = \sum_{i=1}^{N} V_i^k \quad \text{approx} \sim N(N\mu^k, N(\sigma^k)^2). \tag{4.4}$$

The classical central limit theorem is not applicable to the cases of low and high basis risk because the distributions of V_i^k are not identical. Nevertheless we can use a more general version of central limit theorem that only requires independency among individual random variables.[12] The theorem states that for a sequence of random variables $X_i, i = 1, 2, \ldots$ each with mean 0 and variance $\sigma_i^2 < \infty$, if the following Lyapounov's condition

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{n} E\left[|X_i|^{2+\delta}\right] = 0 \tag{4.5}$$

holds for some positive δ, where $s_n^2 = \sum_{i=1}^n \sigma_i^2$, then

$$\frac{\sum_{i=1}^n X_i}{s_n} \to N(0,1) \quad \text{in distribution as} \quad n \to \infty.$$

This theorem implies that when N is large,

$$V^k = \sum_{i=1}^{N} V_i^k \quad \text{approx} \sim N\left(\sum_{i=1}^{N} \mu_i^k, \sum_{i=1}^{N} (\sigma_i^k)^2 \right), \tag{4.6}$$

where μ_i^k is the mean of V_i^k and $(\sigma_i^k)^2$ is the variance. When $\mu_i^k = \mu^k$ and $(\sigma_i^k)^2 = (\sigma^k)^2$ for all i, (4.6) reduces to (4.4).

To use normal approximation, we first need to verify that the Lyapounov's condition (4.5) holds. We do that numerically. Let $\tilde{V}_i^k = V_i^k - E[V_i^k]$, and $\tilde{v}_{ij}^k = v_{ij}^k - \frac{1}{50000}(v_{i1}^k + v_{i2}^k + \cdots + v_{i50000}^k)$ be samples of $\tilde{V}_i^k$. We take $\delta = 1$ and approximate the third moment and variance of $\tilde{V}_i^k$ using $\tilde{v}_{ij}^k$. The left hand side of (4.5), with X_i being substituted with V_i^k, is calculated for each n from 1 to 1000, and is plotted in Figure 4.8 for the case of high basis risk. The tendency towards zero is clear. We have similar results for the low basis risk case.

Next we need to check whether $N = 1000$ is large enough to achieve good approximations shown in (4.4) and (4.6). We use QQ plot as an examination tool. Figure 4.9 shows the QQ plot of v_j^{PILM} from high basis risk case against standard normal. It strongly supports normality. The QQ plots for v_j^{FRM}, and under low and no basis risk cases are similar.

Based on the theory and the evidence just presented, we can assume the distribution of portfolio value is normal for both FRM and PILM portfolios and under all levels of basis

[12] See for example Theorem 27.3 of Billingsley[13]. The mortgage contract values V_i^k are independent when house price paths are fixed.

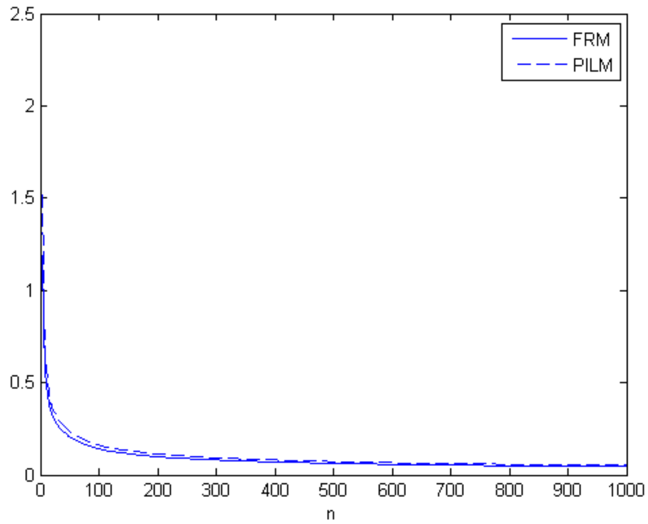


Figure 4.8: Numerical verification of Lyapounov's condition (4.5) based on centralized samples from high basis risk case.

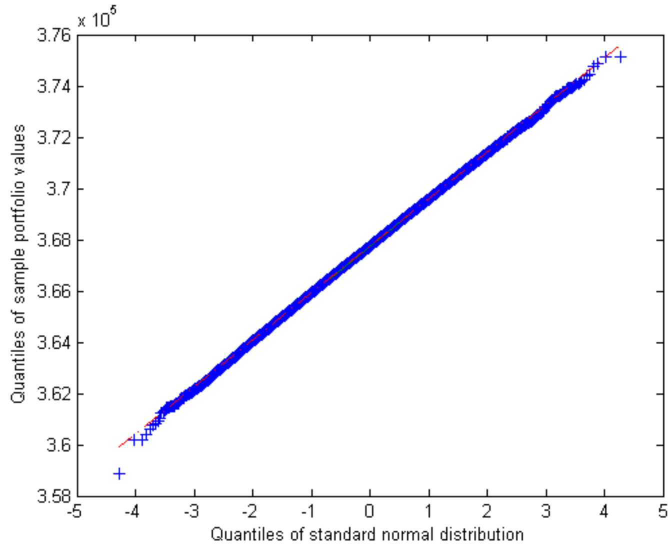


Figure 4.9: The normal QQ plot of v_j^{PILM} from high basis risk case.

risk. Hence we only need to estimate and report the mean and std of portfolio values, as they fully characterize normal distributions. Given the random samples v_{ij}^k, there are two approaches for estimation. One is to use the sample mean and sample std of v_j^k as the estimators. The other is to estimate the means and stds of the values of all mortgage contracts in a portfolio and then use central limit theorems ((4.4) and (4.6)) to obtain the estimates of the mean and std of portfolio values. The two approaches produce the same estimation for the mean portfolio values because

$$\frac{1}{50000}\sum_j v_j^k = \frac{1}{50000}\sum_j \Big(\sum_i v_{ij}^k\Big) = \sum_i \left(\frac{1}{50000}\sum_j v_{ij}^k\right).$$

The estimates of std could be different but based on unreported calculation they are very close to each other in all cases. Table 4.4 shows the means and stds of portfolio values estimated using the first approach.

Portfolio	No		Low		High	
value	Mean	Std	Mean	Std	Mean	Std
FRM	370.70	2.11	363.65	2.04	359.17	2.01
PILM	378.86	1.79	373.00	1.83	367.74	1.84

Table 4.4: Sample means and sample stds of $v_j^k, j = 0, 1, \cdots, 50000$ under different levels of basis risk. The distributions of portfolio values are approximately normal so only means and stds are estimated. All values are in thousand ($\times 10^3$). The initial portfolio loan size is $380,000$.

It is clear from the table that the mean portfolio values decrease as basis risk becomes larger, but the stds are not affected much. We have previously demonstrated that negative deviation of house prices from the index has larger impact on default probability than positive deviation. Similarly we see here that negative shocks of house values have larger impact on mortgage values than positive shocks with similar magnitudes.

We can alternatively interpret these observations by the "concavity/convexity" of our model. To see this, let $H^i(t)$ be the value of the i-th house at time t, and denote the average house value by $E[H^i(t)]$, which approximately equals $\frac{1}{N}\sum_{i=1}^N H^i(t)$. Taking $H^i(t), t \in [0,T]$ as input, our model outputs the mean of the value of the corresponding mortgage contract ($\mu_i^k \approx E[V_i^k]$). If we denote our model by an operator f, then f should be "increasing", and the expected portfolio value with and without basis risk can be expressed by $\sum_{i=1}^N f(H^i(t))$

and $Nf(E[H^i(t)])$ respectively.[13] Our numerical results imply that

$$Nf(E[H^i(t)]) > \sum_{i=1}^{N} f(H^i(t)),$$

i.e.

$$f(E[H^i(t)]) > \frac{1}{N}\sum_{i=1}^{N} f(H^i(t)) \approx E[f(H^i(t))].$$

This would be the Jensen's inequality for a concave function if $H^i(t)$ were real numbers. Similarly if we let g be the operator mapping property value path to the probability of default, then g is "decreasing" and "convex", according to the results in Table 4.3. The effect of basis risk on the changes of default rates and portfolio values may depend on many factors, including the degrees of "concavity" and "convexity" of operators f and g, the fixed index, the deviation of house prices from the index, the pattern of house price dispersion, etc.

Tables 4.3 and 4.4 show that basis risk has similar effects on FRM and PILM portfolios. The PILM portfolio has lower default rates and higher expected value than the FRM portfolio under all levels of basis risk under discussion, despite that the PILM lender does not charge for the loan adjustment. These results suggest that PILM could be preferred to FRM even when basis risk is present and the individual house values are not known to the lender. Note that our results are based on a fixed depressing scenario. In thriving scenarios where loan adjustments are seldom or never triggered, FRM and PILM portfolios should have similar default rates and values.

Symmetric Information

In the previous analysis, we assume the PILM lender adjusts loan balances based on the index. This is because we assume the lender does not know the actual house prices. We also assume the borrower knows the house prices so the transition of loan status depends on the house prices. In this section we assume information is symmetric between the borrowers and the lender, hence the PILM lender adjusts loan balance according to the house values rather than the index. That is, the balance of the i-th PILM becomes

$$B^i(t) = \min[\bar{\alpha} H^i(t), U(t)], \quad \text{for } t = 0, \frac{1}{4}, \frac{2}{4}, \cdots, \frac{4T-1}{4}.$$

[13] When there is no basis risk, the value of each property follows the index, which equals the average of property values with basis risk($E[H^i(t)]$).

The reference balance $U(t)$ is the same for all PILM loans since they all start with the same initial loan amount and contract rate. The simulation of loan status is the same as before except that the loan balances are different. Portfolio values can also be approximated by normal distributions. Results are shown in Table 4.5.

	No	Low	High
Default(%)			
PILM	19.99	24.08	27.84
PILM(SI)		17.58	17.28
Mean portfolio value($\times 10^3$)			
PILM	378.86	373.00	367.74
PILM(SI)		379.46	377.45
Std portfolio value($\times 10^3$)			
PILM	1.79	1.83	1.84
PILM(SI)		1.69	1.66

Table 4.5: The effect of symmetric information on the default rate and value of the PILM portfolio under different levels of basis risk. "SI" denotes symmetric information.

Compared to the case of asymmetric information, default rates decrease significantly, mean portfolio values increase and the stds of portfolio values decrease. These results show that adjusting loan balance according to actual house prices is superior to adjusting according to the index. They also imply that knowing house prices increases the advantage of PILM over FRM. Comparing the cases with and without basis risk, we do not find much difference in mean portfolio values, but default rates are lower when basis risk becomes stronger.

4.5 Conclusion

As house price indices are being used more often in housing related financial products, house price basis risk becomes an important consideration for designing, pricing, valuing and managing these products. Ignoring or underestimating house price basis risk may lead to under pricing or ineffective hedging. Therefore an appropriate model for house price basis risk is needed.

In this chapter, we propose a basis risk model that have the following desired properties. First, it is relatively independent of the model for the index. Any ARMA-GARCH type of process can be used to model the index. The basis risk structure is specified through the conditional distribution of innovations for individual house prices. Second, it is easy to use the model for simulation. One can fix an index first and then simulate house prices under different levels of basis risk. Third, the simulated house prices are reasonable. The paths look random but do not fluctuate wildly; their averages are close to the index and the standard deviation increases over time. We calibrate our model using historical regional house price indices by treating these indices as house prices and their average as the index.

Since basis risk is considered as a problem for PILM, we use simulated house prices to analyze the effect of basis risk on a PILM portfolio. In particular, we compare the PILM portfolio default rate and value with the corresponding FRM portfolio under different levels of basis risk. We fix a depressing scenario where the index falls significantly in the first five years. When there is no basis risk, all mortgage loans within a portfolio has the same probability to default and the same contract value. When basis risk is present, contract values of different mortgage loans have different means and variances. Our results show that when the level of basis risk increases, default rates increase and portfolio values decrease for both FRM and PILM portfolios, but the PILM portfolio always has lower default rates and higher values than the FRM portfolio. This implies that PILM could be preferred to FRM despite the existence of basis risk.

Our basis risk model can also be used to analyze other index related contracts. For example, the effectiveness of hedging house price basis risk using index derivatives. The model can be incorporated into Chapter 3 by assuming all borrowers behave optimally according to the prices of their own houses. We then examine the cash flow and default rate of a portfolio. When estimating the basis risk parameters we may assume different levels of basis risk within different time periods and identify the periods with high/low basis risk. We can further try to improve our estimation of basis risk parameters, and/or our identification of basis risk structure, by using real estate transaction data when they are available.

Appendix A

A Numerical Procedure for Transition Probabilities Calculation

To price a policy using formula (1.3) or (1.5), one needs to evaluate $\lambda_{14}(t)$, which involves $p_{11}(0,t)$. Equations (1.1) & (1.2) form a system of integral equation for $p_{11}(x,t)$ and $p_{21}(x,0,t)$. In general it is very difficult to find analytic solutions to integral equations. Instead numerical methods that approximate analytic solutions are used widely in application. They are easy to implement and provide acceptable results if used appropriately.

In this appendix, we try to solve equations (1.1) & (1.2) simultaneously by trapezoidal rule, which approximates an integral by the values of integrand at both ends,

$$\int_0^h f(x)dx \approx \frac{h}{2}\big[f(0) + f(h)\big].$$

For a large integration interval, we may divide the interval into subintervals and use the above rule repeatedly,

$$\int_0^t f(x)dx \approx \frac{h}{2}\left[f(0) + 2\sum_{j=1}^{N-1} f(jh) + f(Nh)\right],$$

where the interval $[0,t]$ is divided into N equal subintervals with cutting points $0, h, 2h, \cdots, (N-1)h, Nh = t$. This is also called composite trapezoidal rule.[1] For convenience, we shall

[1] The absolute value of the difference between the true and approximated value is

$$\frac{t^3 K}{12N^2},$$

replace the approximation sign($\approx$) by equal sign($=$) in the following derivations.

Assume transition intensities $\mu_{12}(x)$, $\mu_{21}(x,u)$, and "stay" probabilities $p_{\overline{11}}(x,t)$, $p_{\overline{22}}(x,u,t)$ are known for all x, t, u. Divide the interval $[0,T]$ into N equal subintervals of length h such that $0, h, 2h, \cdots, (N-1)h, Nh = T$ are cutting points. Our goal is to find function values $p_{11}(x,t)$ and $p_{21}(x,0,t)$ at each point on the square $\{0, h, 2h, \cdots, (N-1)h, Nh\} \times \{0, h, 2h, \cdots, (N-1)h, Nh\}$, where $Nh = T$.[2] Suppose we are at point (x, ih), expand (1.1) by trapezoidal rule, we have

$$\begin{aligned}
p_{11}(x,ih) = \frac{h}{2}\Bigg[&p_{\overline{11}}(x,0)\mu_{12}(x)p_{21}(x,0,ih) \\
&+ 2\sum_{j=1}^{i-1} p_{\overline{11}}(x,jh)\mu_{12}(x+jh)p_{21}(x+jh,0,(i-j)h) \\
&+ p_{\overline{11}}(x,ih)\mu_{12}(x+ih)p_{21}(x+ih,0,0)\Bigg] + p_{\overline{11}}(x,ih) \\
= \frac{h}{2}\Big[&\mu_{12}(x)p_{21}(x,0,ih) + m(i)\Big] + p_{\overline{11}}(x,ih),
\end{aligned}$$

where

$$m(i) = 2\sum_{j=1}^{i-1} p_{\overline{11}}(x,jh)\mu_{12}(x+jh)p_{21}(x+jh,0,(i-j)h)$$

for $i \geq 2$ and $m(1) = 0$. The initial conditions

$$p_{11}(x,0) = 1 \quad \text{and} \quad p_{21}(x,0,0) = 0 \quad \forall x \geq 0$$

where K is an upper bound for $f''(x)$ $x \in [0,t]$, see[81][10].

[2]Note that the transition probabilities are not used in policy pricing when $x+t > T$ but mathematically they can be calculated.

have been substituted in to simplify the expression. Similarly,

$$
\begin{aligned}
p_{21}(x,0,ih) = \frac{h}{2}\Big[& p_{\overline{22}}(x,0,0)\mu_{21}(x,0)p_{11}(x,ih) \\
& + 2\sum_{j=1}^{i-1} p_{\overline{22}}(x,0,jh)\mu_{21}(x+jh,jh)p_{11}(x+jh,(i-j)h) \\
& + p_{\overline{22}}(x,0,ih)\mu_{21}(x+ih,ih)p_{11}(x+ih,0)\Big] \\
= \frac{h}{2}\Big[& \mu_{21}(x,0)p_{11}(x,ih) + n(i)\Big],
\end{aligned}
$$

where

$$
n(i) = 2\sum_{j=1}^{i-1} p_{\overline{22}}(x,0,jh)\mu_{21}(x+jh,jh)p_{11}(x+jh,(i-j)h) + p_{\overline{22}}(x,0,ih)\mu_{21}(x+ih,ih)
$$

for $i \geq 2$ and

$$
n(1) = p_{\overline{22}}(x,0,h)\mu_{21}(x+h,h).
$$

Note that $m(i)$ and $n(i)$ only involve $p_{11}(x,t)$, $p_{21}(x,0,t)$ for $t < ih$. So our calculation sequence is $[p_{11}(x,h), p_{21}(x,0,h)]$, $[p_{11}(x,2h), p_{21}(x,0,2h)]$, $\cdots$, $[p_{11}(x,Nh), p_{21}(x,0,Nh)]$, and both $m(i)$ and $n(i)$ are known values when solving for $[p_{11}(x,ih), p_{21}(x,0,ih)]$:[3]

$$
\begin{aligned}
p_{11}(x,ih) &= \frac{n(i)\mu_{12}(x)h^2 + 2m(i)h + 4p_{\overline{11}}(x,ih)}{4 - \mu_{12}(x)\mu_{21}(x,0)h^2} \\
p_{21}(x,0,ih) &= \frac{m(i)\mu_{21}(x,0)h^2 + 2n(i)h + 2p_{\overline{11}}(x,ih)\mu_{21}(x,0)h}{4 - \mu_{12}(x)\mu_{21}(x,0)h^2}.
\end{aligned}
$$

[3]The calculation sequence for different x with fixed ih does not matter.

Appendix B

Numerical Detail for the Optimization Problem in Chapter 3

Discretization of the state space
To numerically solve the optimization problem posed in chapter 3 using dynamic programming, the first step is to discretize the state space $(t, H_t, \mathrm{MR}_t, \mathrm{ER}_t, M_t)$. We then calculate the action value I_t, contract value J_t and the associated optimal action at all grid points. The space is divided into hypercubes with coordinates in each dimension given by

- $t \in \{0, 1, 2, \cdots, T\}$;
- $H \in S_H = \{e^{3+0.092i}, i = 0, 1, 2, \cdots, 49\}$;
- $\mathrm{MR}/\mathrm{ER} \in S_R = \{e^{-4.6+0.194i}, i = 0, 1, 2, \cdots, 17\}$;
- $M \in S_M = \{2i, i = 0, 1, 2, \cdots, 24\}$.

Any combination of these values is a grid point in the state space that represents a particular state.

Calculation of expectations
Since contract values J_t are only known at finite number of states, the expectation $E[J_t]$ taken over (H_t, MR_t) must be approximated numerically. We choose the simplest approximation by assuming transition can only occur between grid points of (H_t, MR_t). We further assume the transition probability follows a two dimensional discrete distribution

with probability mass function $\mathrm{P}(\cdot,\cdot)$ proportional to the bivariate normal density defined by (3.2). Hence,

$$E_t[J_{t+1}] \approx \sum_{(H_{t+1},\mathrm{MR}_{t+1})\in\{S_H\times S_R\}} J(t+1,H_{t+1},\mathrm{MR}_{t+1})\mathrm{P}(H_{t+1},\mathrm{MR}_{t+1}|H_t,\mathrm{MR}_t).$$

The largest and smallest values in S_H and S_R become bounds for property value and mortgage rate. According to the original definition (3.2), the marginal distribution of $(\log H_{t+1}, \log \mathrm{MR}_{t+1})$ is centered at $(\log H_t, \log \mathrm{MR}_t)$. However this is not the case under our discrete transition probability unless $(\log H_t, \log \mathrm{MR}_t)$ is at the center of $S_H \times S_R$, since the bounds of the state space are fixed and independent of time-t state. This should not be a serious problem because we divide each dimension equally in log scale within a wide range and normal distribution has a thin tail. The discrete transition distribution is not much biased as long as the time-t state is away from the boundary.

Interpolation

We simulate (H_t, MR_t) process in the continuous space so their values are not constrained by discretization. This means (ER, M) can be at any point between the grid points, hence we need to interpolate the value of J_t using its values at nearby grid points. By the nature of contract value, we require the interpolation method to be monotonicity preserving. For example J_t must be decreasing in M_t when other variables being fixed. We choose to use the Fritsch-Butland piecewise cubic interpolation[1] along the dimension M_t and the linear interpolation along ER, in order to achieve a relative high degree of accuracy and meanwhile control for complexity and computational burden.[2]

[1]See Hyman[49] for the formulas.

[2]The monotonicity preserving bicubic algorithm is much more complicated than the univariate one, see Carlson & Fritsch[24].

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